STUDIA UBB. PHILOSOPHIA, Vol. 67 (2022), Sp. Iss., pp. 63-75
(RECOMMENDED CITATION)
DOI:10.24193/subbphil.2022.sp.iss.04

BETWEEN PATHOLOGY AND WELL-BEHAVIOUR – A POSSIBLE FOUNDATION FOR TAME MATHEMATICS

Angelo-Vlad MOLDOVAN*

ABSTRACT. An in-depth examination of the foundations of mathematics reveals how its treatment is centered around the topic of "unique foundation vs. no need for a foundation" in a traditional setting. In this paper, I show that by applying Shelah's stability procedures to mathematics, we confine ourselves to a certain section that manages to escape the Gödel phenomenon and can be classified. We concentrate our attention on this mainly because of its tame nature. This result makes way for a new approach in foundations through model-theoretic methods. We then cover Penelope Maddy's "foundational virtues" and what it means for a theory to be foundational. Having explored what a tame foundation can amount to, we argue that it can fulfil some of Maddy's foundational qualities. In the last part, we will examine the consequences of this new paradigm – some philosophical in nature – on topics like philosophy of mathematical practice, the incompleteness theorems and others.

Keywords: foundations of mathematics, tame mathematics, clarity-based knowledge, philosophy of mathematical practice, incompleteness theorems

* PhD candidate, Doctoral School of Philosophy, Faculty of Philosophy, University of Bucharest, Bucharest, Romania. E-mail: moldovanangelo@hotmail.com

Introduction

Since the beginning of the twentieth century, mathematicians and philosophers have tried to assemble a foundation for all mathematics by reducing it to a finite number of axioms. This attempt, however, proved unsuccessful and forced researchers to either argue that mathematics does not need a foundation or to suggest other possible foundational theories. In this paper I will propose a nonstandard strategy to develop a foundation only for the well-behaved parts of mathematics through a different approach called model-theoretic local foundations. In the first part we introduce Penelope Maddy's concept of foundational virtue based on which we establish what is the foundational role of a theory and elaborate the basics of the local foundation project, then we examine how our local foundation for tame mathematics fulfills Maddy's foundational virtues and its impact on incompleteness and epistemology.

Penelope Maddy's foundational virtues

Aware of the foundational debate, Penelope Maddy's strategy is to dissect the mainstream approach of finding an appropriate foundation for mathematics with the intention of pinning down the foundational character of a theory by analyzing its nature and listing the so-called *foundational virtues* that a certain theory must have in order to be a suitable foundation. The traditional framework for foundation is made possible by set theory which is a remarkable case because almost all mathematical objects "can be modeled as sets and all standard mathematical theorems [can be] proved from its axioms".[1] The fundamental question concerning set theory is: "what's the point of this exercise? What goal, properly thought of as 'foundational', is served by this 'embedding'?".[2] From Maddy's perspective,

[1] Maddy, „What do we want a foundation to do?", p. 294

[2] Maddy, "Set-theoretic foundations", p. 290

> "Set theory hopes to provide a foundation for classical mathematics, that is, to provide a dependable and perspicuous mathematical theory that is ample enough to include (surrogates for) all the objects of classical mathematics and strong enough to imply all the classical theorems about them. [...]Thus set theory aims to provide a single arena in which the objects of classical mathematics are all included, where they can be compared and contrasted and manipulated and studied side-by-side".[3]

The first step towards identifying the foundational virtues concerns the dangers of inconsistencies. There must be some kind of apparatus to assess how risky a particular new construction is. Despite being plagued by paradoxes since its inception, the development of the iterative conception and the constructible universe diminished the hazards significantly for set theory. The introduction of large cardinal axioms facilitated the measurement of the consistency strength of various theories. This ability to expose these risks is essential for mathematicians: we must know how much mathematics can a certain foundational theory capture. The first foundational role set theory provides is called Risk Assessment.[4] Now, let's just ask ourselves the following question: if each domain is described by its own list of axioms, then how can we transfer information from one domain to another? This common ground occurs when these distinct mathematical areas are embedded into a single set theory, where every theorem is interpreted as a theorem of the same system. The Von Neumann universe – or the V universe - represents this final court where all mathematics takes place, where we study all structures and objects, their relations, interpretations and methods from different areas of mathematics. Thus, the discipline has a "Generous Arena" where all mathematical entities are located and a "Shared Standard of Proof" where set-theoretic proofs are the standard way of proving in mathematics. The aforementioned embedding also has the purpose of converting mathematics into a list of formal sentences. This makes possible the application of meta-

[3] Maddy, *Second Philosophy - A Naturalistic Method*, p. 354

[4] Maddy, "What do we want a foundation to do?", p. 298

mathematical tools to prove theorems about the system itself. Therefore, we have a “Meta-mathematical Corral” through which we trace the origin of all mathematical life forms to a list of straightforward axioms.[5] The last virtue is about establishing a foundation that encapsulates the fundamental nature of mathematics that guides mathematicians “toward the truly important concepts and structures, without getting bogged-down in irrelevant details”.[6] Maddy proposes an “Essential Guidance” in the hope of highlighting two paramount features of foundation: revealing the essence of the founded mathematics and guiding the progress along its essential features.

Model-theoretic foundation

Understanding what caused the abandonment of the set-theoretic foundations in our project is crucial. Besides being riddled with issues and prone to lose meaning, the global framework Zermelo-Fraenkel + Axiom of Choice set theory gave us seems far away from what we envisioned for mathematics and it based philosophy of mathematics on the myth that we can reduce all mathematics to certain foundations. Furthermore, the complications brought about by undecidability, incompleteness and the unsolvability of different mathematical problems from the axioms of ZFC made this foundational theory extremely problematic. The solution is to leave the traditional structure, replace it with a more suitable candidate - model-theoretic local foundations – and apply it only to the well-behaved parts of mathematics. Set theory is unsuitable for philosophical work and as a foundation for mathematics:

[5] Ibidem, p. 301

[6] Maddy, “Set-theoretic foundations”, p. 305

> "we view the practice-based philosophy of mathematics as a broad inquiry into and critical analysis of the conceptual foundations of actual mathematical work. [...]The foundationalist goal of justifying mathematics is a part of this study. But the study we envision cannot be carried out by interpreting the theory into an über theory such as ZFC; too much information is lost. The coding does not reflect the ethos of the particular subject area of mathematics. The intuition behind fundamental ideas such as homomorphism or manifold disappears when looking at a complicated definition of the notion in a language whose only symbol is ε. Tools must be developed for the analysis and comparison of distinct areas of mathematics in a way that maintains meaning; a simple truth preserving transformation into statements of set theory is inherently inadequate".[7]

Influenced by modern model theory, our project takes the model-theoretic procedures introduced by Saharon Shelah's classification theory and develops the mechanism behind the local-foundation enterprise. Firstly, formalization[8] of specific mathematical areas is made possible by model theory and could be used to investigate both mathematical and philosophical problems. Secondly, if we have local formalizations for each theory, then we could systematically compare them. The last point concerns how geometrical properties of tame theories play an important role in analyzing models and solving problems in mathematics.[9] Here we need to briefly describe Shelah's classification project. It was originally designed to help mathematicians capture the pathological behavior exhibited by first-order theories with numberless non-isomorphic models for every uncountable cardinality. Intuitively, if I(T,k)[10] = 2^{k} for all uncountable cardinals k, then the number of models is too big and our theory cannot be classified. The apparatus consists

[7] Baldwin, "Model Theoretic Perspectives on the Philosophy of Mathematics", p. 2

[8] By *formalization* we mean choosing the right vocabulary, the right logic and the right axioms for our theory.

[9] Baldwin, "Model Theoretic Perspectives on the Philosophy of Mathematics", p. 3

[10] I(k,T) = T's number of unique models of cardinality k.

in a system of invariants that assigns a dimension to each structure in order to point up its geometric characteristics.[11] This makes way for a distinction between classifiable (or stable) theories and nonclassifiable (or unstable) theories. But there is also another relevant distinction at play: the one between tame and wild theories. We can make sense of this distinction just in a mathematical setting. For example, first-order arithmetic – i.e. $T(\mathbb{N}, +, \cdot)$ – is a wild theory because it lacks an effective axiomatization[12] and admits a pairing function.[13] The ring of integers $(Z,+, \cdot)$ and the field of rationals $(Q,+, \cdot)$ are also wild structures[14] because we can define an isomorphic copy of $T(\mathbb{N}, +, \cdot)$ in them.[15] These privileges do not extend to the field of complex numbers, whose uncountable domain cannot be definably interpreted in $T(\mathbb{N}, +, \cdot)$[16] and whose theory is finitely axiomatizable.[17] The field of real numbers,[18] algebraically closed fields and algebraically closed valued fields are other examples of tame structures.[19] Here lies the essence of model theory as van den Dries himself describes it: "a lot of model theory is concerned with discovering and charting the 'tame' regions of mathematics, where wild phenomena like space filling curves and Gödel incompleteness are absent, or at least under control. As Hrushovski put it recently: Model Theory = Geography of Tame Mathematics".[20] We find this behaviour in theories that are characterized by properties like superstability**,** stability**,** o-minimality, simpleness and so on.[21] In John Baldwin's opinion, through non-gödelian

[11] Morales, "Around logical perfection", p. 6
[12] van den Dries, "Classical model theory of fields", p. 38
[13] Baldwin, *Model Theory and the Philosophy of Mathematical Practice*, p. 10
[14] Buss, "The Prospects for Mathematical Logic in the Twenty-First Century", p. 17
[15] Roman, *Mathematical Logic: On Numbers, Sets, Structures, and Symmetry*, p. 149
[16] Ibidem, p. 161
[17] van den Dries, "Classical model theory of fields", p. 38
[18] Buss, "The Prospects for Mathematical Logic in the Twenty-First Century", p. 17
[19] Baldwin, *Model Theory and the Philosophy of Mathematical Practice*, p. 314
[20] van den Dries, "O-minimal Structures and Real Analytic Geometry", p. 106
[21] Baldwin, "The Reasonable Effectiveness of Model Theory in Mathematics"

formal systems a better comprehension of modern mathematics can be achieved.[22] Our plan was to escape the ZFC-based global framework which is known for its "specter of undecidability"[23] and for the many fundamental questions that are formally unsolvable from its axioms.[24] Through model-theoretic procedures, we managed to do this by isolating the well-behaved part of mathematics that is characterized by a lot of interesting properties and is not subjected to incompleteness.

Taming the foundational virtues

In this third section I am going to apply the foundational virtues proposed by Penelope Maddy to this well-behaved part of mathematics in order to find out if our project has foundational character. The first essential feature concerns the testing of the proposed foundational theories' levels of consistency via the hierarchy of large cardinals. Model theory cannot provide risk assessment and it leaves the justification of its tools to the traditional system.[25] As Vladimir Voevodsky himself said, "Set theory will remain the most important benchmark of consistency".[26] A common framework where all mathematical areas are embedded into a unique set-theoretic universe and all theorems are theorems of the same system is provided by a generous arena.[27] This Von Neumann universe – which seems to be a mathematician's *promised land* – is far away from what mathematicians dreamed of. The hope for such a place in the V universe was shattered when Cohen demonstrated that the continuum hypothesis cannot be proved from the

---

[22] Baldwin, *Model Theory and Philosophy of Mathematical Practice*, p. 14
[23] Woodin, "Strong Axioms of Infinity and the Search for V", p. 526
[24] Woodin, "The Transfinite Universe", p. 449
[25] Baldwin, "Entanglement of Set Theory and Model Theory Eventual Behavior and Noise"
[26] Voevodsky, "Univalent foundations and set theory"
[27] Maddy, "What do we want a foundation to do?", 298

ZFC axioms.[28] In our model-theoretic worldview, the universe consists only of tame theories and MT "provides a different organization of mathematical topics which better preserves the methods and ethos of various areas than set theory does".[29] The following foundational feature concerns the "Shared Standard of what counts as a proof".[30] This means the axioms of ZFC are setting the standard of proof in mathematics and model theory relies, once again, on set theory.[31] The meta-mathematical corral involves "tracing the vast reaches of mathematics to a set of axioms so simple that they can then be studied formally with remarkable success".[32] Through classification theory we could provide a nicer meta-mathematical construction: if we have an Essential Guidance for general mathematical research, then we have one for set theory. Hence, it guides set-theoretic research towards new meta-mathematical territories where "various instances of model-theoretic problems engendering new animals in the corral".[33]

The last foundational virtue guides mathematicians towards the really fundamental concepts and structures without focusing on irrelevant details. Named Essential Guidance by Maddy, its main task is zeroing in on the following details: such a foundation aims to reveal the fundamental aspects on which mathematics is based without being distracted by other developments, and guide mathematical progress with the help of these aspects. Unfortunately, set theory cannot provide such a virtue.[34] Model-theoretic Essential Guidance, by contrast, establishes a formal framework suitable for every mathematical subject in order to clarify arguments in that area and reveals via stability how combinatorial principles forge connections

[28] Woodin, "Strong Axioms of Infinity and the Search for V", p. 504

[29] Baldwin, "Entanglement of Set Theory and Model Theory Eventual Behavior and Noise"

[30] Maddy, "Set-theoretic foundations", p. 296

[31] Baldwin, "Entanglement of Set Theory and Model Theory Eventual Behavior and Noise"

[32] Maddy, "What do we want a foundation to do?", p. 301

[33] Ibidem, p. 378

[34] Maddy, "Set-theoretic foundations", p. 305

between different subjects.[35] In conclusion, model theory can provide a Generous Arena, a Meta-mathematical Corral and Essential Guidance for its tame foundations, but it cannot acquire the set-theoretic giants: Shared Standard of Proof and Risk Assessment. The model-theoretic tame foundation, despite being a much safer approach to foundations than the global framework, is in some ways still dependent on its set-theoretic host.

Main outcomes

First and foremost, the basic idea behind tameness is that a theory characterized by it does not have enough power to formulate a Gödel-sentence. Through model-theoretic tools we establish that many mathematical theories of general interest are tractable, but not foundational. Both ZFC and Peano arithmetic are equally unruly. Avoiding the gödelian phenomenon means formalizing topics locally by "axioms which catch the relevant data but avoid accidentally encoding arithmetic and, more generally, pairing functions".[36] Consequently, the tame areas of mathematics escape incompleteness and simultaneously could open the door to new approaches in philosophy of mathematics and new methods of testing and researching theories in mathematics. Our most philosophically charged subject concerns the relationship between epistemology and model theory in the context of the traditional reliability-based approach represented by the set-theoretical foundations. Model theory breaks away from this tradition and emphasizes instead the notion of clarification as a salient feature of knowledge. This undertaking concerns why philosophy of knowledge addresses exclusively problems of reliability and does not deal with the problem of the nature of clarity, especially when a close "look at achievements in mathematics shows that genuine mathematical accomplishment consists primarily in making

[35] Baldwin, "The dividing line methodology: Model theory motivating set theory", p. 378

[36] Baldwin, *Model Theory and the Philosophy of Mathematical Practice*, p. 148

clear by using new concepts”.[37] Reliability represents a necessary feature of knowledge, but our obsession with it has led to a poorly distorted theory of knowledge. In consequence, any provable mathematical sentence, with or without an intelligible proof, is now an item of knowledge and mathematical progress has been reduced to a hierarchy of theorems.[38] In Manders’ own words, “proof by itself is insufficient for comprehension”.[39] Concomitantly, he was aware of the universality of set-theoretic language - if everything is expressible in a language, nothing important follows. Model theory, on the other hand, formalizes “one object at a time in a language no richer than absolutely necessary to do so, carefully chosen to display the relevant ‘underlying structure’“.[40] The primacy of MT is motivated by our need “to be able to emphasize special features of a given mathematical area and its relationships to others, rather than how it fits into an absolutely general pattern”.[41] In order to make mathematical problems more accessible, Manders develops a syntactic theory for information transfer between theories and wants to discover which structural properties of the new theory are simplifying the transferred information.[42] These relationships are called *reconceptualization relationships* because they render contents from the original setting comprehensible. Our model-theoretic formalization of all tame theories allows the transfer of information and the examination of all mathematical properties. The idea of transferring mathematical problems from a theory to a new one where they are more tractable is based on the successful applications of model-theoretic methods in other mathematical fields: the Ax-Grothendieck theorem which is hard to solve using the tools of algebraic geometry, but easy to solve in model theory and the Ax-Kochen theorem

[37] Manders, “Logic and Conceptual Relationships in Mathematics”, p. 193
[38] Manders, „Logic and Conceptual Relationships in Mathematics”, p. 194
[39] Manders, „Logic and Conceptual Relationships in Mathematics”, p. 199
[40] Ibidem, p. 200
[41] Ibidem, p. 193
[42] Ibidem, p. 203

from number theory, characterized by Bruno Poizat as "the first witness to the maturity of model theory, its first indisputable application outside the narrow scope of logic".[43] Regarding his must-have properties, he calls them *accessibility properties*[44] since they seem to elucidate why some statements become accessible to inquiry once transferred to theories characterized by them. Philosophically speaking, mathematical understanding is located outside of proof because in a reliability-based framework increasing the precision of proofs is achieved at the expense of understanding, this is why "fully formalized proofs are usually unintelligible. Whatever goes into clarity of mathematical ideas can be obscured by the way those ideas are represented in reliability theoretic mathematical foundations".[45] Since formalized proofs are settling questions of reliability and they are simultaneously inimical to understanding, then we cannot obtain mathematical understanding from proofs.[46] Our clarity-based theory of knowledge keeps in touch with mathematical practice, the epistemological goal of clarity is obtained by a change of framework and proofs are an expression of success, but not its essence.[47]

REFERENCES

1. Baldwin, John T., "Entanglement of Set Theory and Model Theory - Eventual Behavior and Noise", in *Methodological Approaches in the Study of Recent Mathematics: Mathematical Philosophy and Mathematical Practic*e, 2018, http://homepages.math.uic.edu/~jbaldwin/pub/konsetangsli.pdf.

---

[43] Poizat, *A Course in Model Theory*, p. 105-106

[44] Model completeness and quantifier elimination are some examples of accessible properties.

[45] Manders, "Logic and Conceptual Relationships in Mathematics", p. 202

[46] Macbeth, "Proof and Understanding in Mathematical Practice", p. 35

[47] Manders, "Logic and Conceptual Relationships in Mathematics, p. 195

2. Baldwin, John T., “Model Theoretic Perspectives on the Philosophy of Mathematics”, Unpublished Draft, 2010, 1-14, http://homepages.math.uic.edu/~jbaldwin/pub/amstersh.pdf.
3. Baldwin, John T., “The dividing line methodology: Model theory motivating set theory”, in *Theoria*, 87.2/2021, 361-393.
4. Baldwin, John T., “The Reasonable Effectiveness of Model Theory in Mathematics”, 2019, http://homepages.math.uic.edu/~jbaldwin/pub/NDreaseffNov619.pdf.
5. Baldwin, John T., *Model Theory and the Philosophy of Mathematical Practice: Formalization without Foundationalism,* Cambridge University Press, 2018.
6. Buss, Samuel R., et al., “The prospects for mathematical logic in the twenty-first century”, in *Bulletin of Symbolic Logic,* 7.2/2001, 169-196.
7. Macbeth, Danielle, “Proof and understanding in mathematical practice”, in *Philosophia Scientiæ. Travaux d’histoire et de philosophie des sciences,* 16-1/2012, 29-54.
8. Maddy, Penelope, “Set-theoretic foundations”, in Andrés Eduardo Caicedo et al., *Foundations of Mathematics, American Mathematical Society,* 2017, 289-322.
9. Maddy, Penelope, “What do we want a foundation to do?”, in Stefania Centrone et al., *Reflections on the Foundations of Mathematics - Univalent Foundations, Set Theory and General Thoughts,* Springer Nature, 2019, 293-315.
10. Maddy, Penelope, *Second Philosophy: A Naturalistic Method,* Oxford University Press, 2007.
11. Manders, Kenneth, “Logic and conceptual relationships in mathematics”, in *The Paris Logic Group, Logic Colloquium ‘85,* Elsevier Science Publishers B.V., 1987, 193-211.
12. Morales, J.A. Cruz, et al., “Around logical perfection”. arXiv preprint, 1803.04909/2018, 1-15.
13. Poizat, Bruno, *A Course in Model Theory - An Introduction to Contemporary Mathematical Logic,* Springer Verlag, 2000.
14. Roman, Kossak, *Mathematical Logic: On Numbers, Sets, Structures, and Symmetry,* Spinger, 2018.
15. Van den Dries, Lou, “O-minimal structures and real analytic geometry”, in *Current developments in mathematics,* 1998.1/1998, 105-152.

16. Van Den Dries, Lou, “Classical model theory of fields”, in Deirdre Haskell et al., *Model Theory, Algebra, and Geometry*, Cambridge University Press, 2000, 37-52.
17. Voevodsky, Vladimir, “Univalent Foundations and Set Theory”, *Talk at the Institute for Advanced Study in Princeton,* 2013, https://www.math.ias.edu/vladimir/sites/math.ias.edu.vladimir/files/2013_ASL.pdf.
18. Woodin, Hugh W., “Strong Axioms of Infinity and the Search for V”, in Rajendra Bhatia, *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010)* (in 4 Volumes) Vol. I: Plenary Lectures and Ceremonies, Hindustan Book Agency, 2010, 504-531.
19. Woodin, Hugh W., “The transfinite universe”, in Matthias Baaz et al., *Kurt Gödel and the Foundations of Mathematics - Horizons of Truth,* Cambridge University Press, 2011, 449-475.