STUDIA UNIVERSITATIS BABEȘ-BOLYAI
Engineering 65(1) 2020 doi:10.24193/subbeng.2020.1.16

Deflection of a beam with a transverse crack under dead load

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This paper proposes an analytic relation for the deflection of a cantilever beam with a transverse crack subjected to dead load. The mathematical relation is deduced involving the decreased capacity of the beam to store energy, which is in direct relation with the crack position and depth. Eventually, the validity of the relation is proved by means of the finite element method.

Keywords: *cantilever beam, deflection, dead load, transverse crack*

1. Introduction

Cantilever beams are subject of dead weights and operational loads. Mathematical relations to calculate the free end deflection of such structures with constant cross-section [1]-[3] and even for the tapered beams [4] are known. The deflection is calculated by considering the distribution of the bending moment along the beam, respectively the equivalent bending moment for the tapered beams. An estimation of the deflection of stepped beams can be made involving relations proposed in [5]-[6], but the results accuracy is affected by the disturbance of the state of stress in the region where the cross-section changes suddenly [7]. The bigger the cross-section decrease, the bigger the error in estimating the free end deflection. The real deflection for stepped beams is always bigger as the calculation because a region of the beam in the proximity of the sudden cross-section increase or decrease does not contribute to the stiffness of the beam. Thus, from calculus, we always obtain an under-evaluated deflection.

As far as we know, there is no relation that allows calculating the deflection of a cantilever beam with transverse cracks. In this paper we propose a relation to calculate the deflection of a cantilever beam with a transverse crack, when the load consists of dead weight. The energy method is considered and a severity coefficient, similar to that found for the case of vibrating beams [8] is used. We prove by means of the finite element method (FEM) that the proposed mathematical relation provides accurate results.

2. Analytical approach

The analysis is performed for a clamped beam (fig. 1) loaded with its own weight $p=\rho Ag$ [N/m], where g [m/s^2] is the earth gravity. The beam is made of structural steel with density ρ [kg/m^3], Young modulus E [N/m^2] and having length L [m]. The beam is of rectangular section having area $A=bh$ [m^2] and moment of inertia I_U [m^4]. Under the action of its own mass, the maximum deflection of the beam at the free end is d [m].

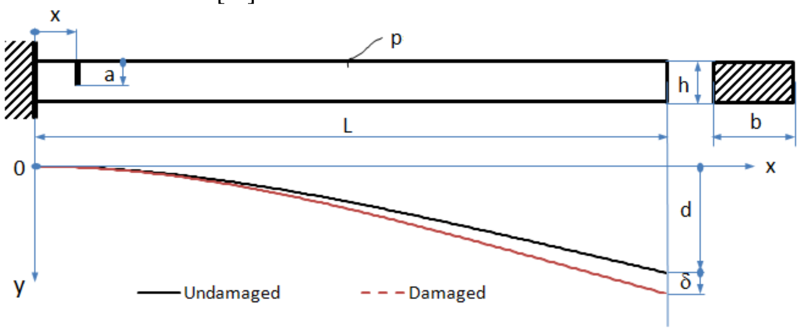

Figure 1. Clamped beam with transversal damage at the fixed end

From the strength of the materials it is known that the deflection at the free end of the beam can be determined with the relation:

$$d = \frac{pL^4}{8EI_U} \tag{1}$$

and the moment of inertia from relation (1) becomes:

$$I_U = \frac{pL^4}{8Ed} \tag{2}$$

Let's considering a crack near de clamped end (fig. 1) at the distance x [m]. The crack is along the whole width b [m] of the beam and has the depth a [m]. For this case, the moment of inertia of the equivalent beam (with reduced constant cross-section that exhibit the same free end deflection as the damaged beam) becomes I_{ech} [m^4], and, consequently, the deflection of the beam with a crack can be written as:

$$d+\delta=\frac{pL^4}{8EI_{ech}} \tag{3}$$

From relation (3), the moment of inertia of the damaged beam becomes:

$$I_{ech}=\frac{pL^4}{8E(d+\delta)} \tag{4}$$

It is known that the fundamental natural frequencies of the beam with constant cross-section is computed with the relation:

$$f_U=\frac{\lambda^2}{2\pi}\sqrt{\frac{EI_U}{\rho AL^4}}=\frac{\lambda^2}{2\pi}\sqrt{\frac{E\frac{pL^4}{8Ed}}{\rho AL^4}}=\frac{\lambda^2}{2\pi}\sqrt{\frac{g}{8d}} \tag{5}$$

where, λ is the dimensionless wave number for the first vibration mode.

Taking into consideration relations (5) and (4), the natural frequency for the beam with a crack is:

$$f_D=\frac{\lambda^2}{2\pi}\sqrt{\frac{EI_{ech}}{\rho AL^4}}=\frac{\lambda^2}{2\pi}\sqrt{\frac{E\frac{pL^4}{8E(d+\delta)}}{\rho AL^4}}=\frac{\lambda^2}{2\pi}\sqrt{\frac{g}{8(d+\delta)}} \tag{6}$$

By dividing f_D to f_U it will be obtained:

$$\frac{f_D}{f_U}=\sqrt{\frac{d}{d+\delta}} \tag{7}$$

From the literature it is known that [9]:

$$\frac{f_D}{f_U}=1-\gamma(\bar{a})\cdot\left[\bar{\phi}''(\bar{x})\right]^2 \tag{8}$$

In the relation (8), we denote with $\gamma(\bar{a})$ the severity of the crack that has the relative depth $\bar{a}=a/h$. We consider two beams, both having h/L=1/200, and the edges of the cross-section in the relation h/b=1 and h/b=0.1, respectively. The severity, according to [10], is:

$$\gamma(\bar{a})=0.4464(\bar{a})^4-0.2483(\bar{a})^3+0.1376(\bar{a})^2-0.0037(\bar{a}) \tag{9}$$

Alternatively, the values for the severity can be obtained from figure 2, by choosing the severity for a given damage depth [11].

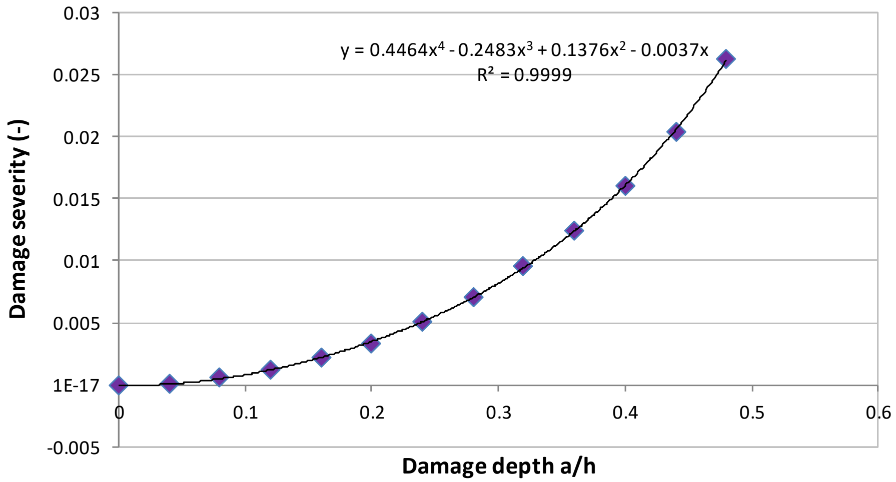

Figure 2. Values for the severity when the deep of the crack is between 0 and 0.5

One can observe in figure 2 that the severity does not depend on the dimensions of the rectangular cross-section of the beam, i.e. the ratio *h/b*.

The second term in the right part of relation (8), namely $\bar{\phi}''(\bar{x})$, represents the normalized bending moment or the normalized beam curvature at the normalized distance $\bar{x} = x/L$. The bending moment is calculated as the second derivative of the vibration mode shape of the beam. For the clamped beam, it is:

$$\bar{\phi}''(x) = \frac{1}{2}\left(\cos(\lambda x) + \cosh(\lambda x) - \frac{\cos(\lambda) + \cosh(\lambda)}{\sin(\lambda) + \sinh(\lambda)}\left[\sin(\lambda x) + \sinh(\lambda x)\right]\right) \qquad (10)$$

From the mathematical relations (7) and (8), which both reflect the frequency ratio f_D/f_U, the deflection of the beam with a transverse crack results in:

$$(d + \delta) = \frac{1}{\left\{1 - \gamma(\bar{a})\left[\bar{\phi}''(x)\right]^2\right\}^2} \cdot d = \kappa \cdot d \qquad (11)$$

Denoting the fraction in the right part of relation (11) as κ, and replacing the deflection of the beam with constant cross-section *d* with its expression defined in relation (1), we obtain the deflection of the cracked beam:

$$(d + \delta) = \frac{1}{\left\{1 - \gamma(\bar{a})\left[\bar{\phi}''(x)\right]^2\right\}^2} d = \kappa \cdot d = \kappa \frac{pL^4}{8EI_U} \qquad (12)$$

We represent in figure 3 the evolution of the coefficient κ for two cases of the crack depth: *a/h*=0.25 and *a/h*=0.50. The crack is relocated along the whole length of the beam $x \in (0, L)$.

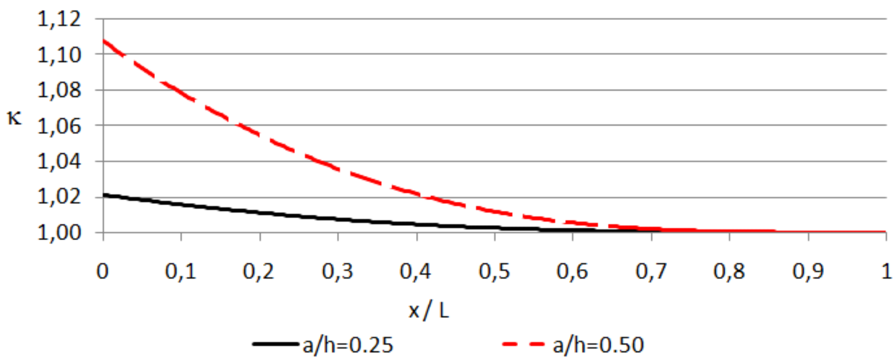

Figure 3. Deflection coefficient for *a/h*=0.25 and *a/h*=0.50 when crack is moving along the whole length of the beam

One can observe from figure 3 that the coefficient κ equals the unit at the free end of the beam. The proportionality between the amplitudes of the two curves, imposed by the crack depth is obvious.

3. Experimental research

In this section we prove that relation (12) derived for the deflection of a cracked beam works. To this aim, we involve the FEM and perform simulations for the beam with a uniform cross-section that has the dimensions and mechanical properties given in table 1.

Table 1. Properties of the beam with a uniform cross-section

Length L [mm]	Width b [mm]	Thickness h [mm]	Young modulus E [N/m^2]	Density ρ [kg/m^3]
1000	50	5	2×10^{11}	7850

The damaged beam has the same geometry as the healthy beam, but cracks with different positions and depths. The damage scenarios are described in table 2. For all cases, the crack width is w=0.04 mm. The analysis is performed with the ANSYS software, the beam getting a mesh with the maximum edge of the hexahedral elements 2 mm. In consequence, the beam with a constant cross-section is discretized with 37500 elements and 193282 nodes.

For the beams with discontinuities, the number of elements and nodes increase because around the discontinuity the mesh needs to be finer in order to make the transition from the constant cross-section to the reduced cross-section. The finest mesh is generated at the top of the crack.

Table 2. Damage scenarios

Scenario no.	Crack depth a [mm]	Crack position x [mm]	Curvature ϕ	Severity γ	Coefficient κ
S1	0.5	150	0.79377	0.000802	1.001011
S2	0.5	290	0.60417	0.000802	1.000586
S3	0.5	470	0.37493	0.000802	1.000225
S4	1	150	0.79377	0.003492	1.004414
S5	1	290	0.60417	0.003492	1.002554
S6	1	470	0.37493	0.003492	1.000982
S7	1.5	150	0.79377	0.008186	1.010395
S8	1.5	290	0.60417	0.008186	1.006002
S9	1.5	470	0.37493	0.008186	1.002305

In table 3, we compare the deflection obtained directly from Fem simulation with those calculated involving the deflection of the beam with uniform cross-section multiplied with the coefficient κ. One can observe in the mentioned table that the differences are extremely small, the error being less than 0.1%. even if the deflection increase is not big at all.

Table 3. Deflection achieved by calculus and simulation

Scenario no.	FEM	Calculus		FEM	Error ε [%]
	Deflection d [mm]	Coefficient κ	Deflection $d+\delta$ [mm]	Deflection $d+\delta$ [mm]	
S1	22.948	1.001011	22.97122	22.969	0.009662
S2	22.948	1.000586	22.96145	22.95959	0.008077
S3	22.948	1.000225	22.95318	22.95256	0.002671
S4	22.948	1.004414	23.04931	23.03289	0.071239
S5	22.948	1.002554	23.00661	22.99679	0.042704
S6	22.948	1.000982	22.97054	22.96793	0.011376
S7	22.948	1.010395	23.18656	23.14839	0.164593
S8	22.948	1.006002	23.08575	23.06658	0.083044
S9	22.948	1.002305	23.0009	22.99626	0.020181

An overview of the results presented in table 3 is given in figure 4. One can again observe the good fit of the results achieved by simulation and calculus.

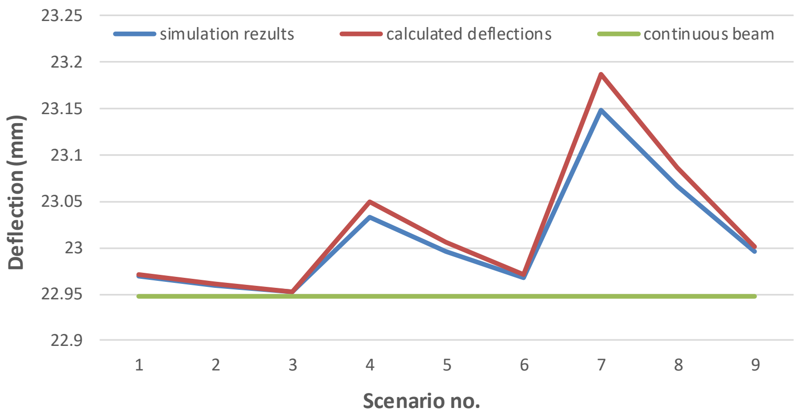

Figure 4. Deflection of the beam for the considered damage scenarios

Since we considered numerous damage scenarios, chosen randomly, for which the calculated deflections fit these obtained by simulation, we can conclude that relation (12) is precise and can be used to determine the deflection of cracks with discontinuities. Note that, this mathematical relation base on the capacity of the cracked beam to store energy, but does not use the equivalent bending moment [12] because the interval on which the cross-section is reduced is zero or infinitesimally small.

4. Conclusion

We introduce a mathematical relation to calculate the deflection of cantilever beams discontinuities of the cross-section under dead load. The relation involves a severity coefficient deduced for the crack at the fixed end, which is adjusted with the local effect when it is located elsewhere. The local effect considers the bending moment at the crack location, which is proportional with the energy stored at that location. Further research will focus on extending the theory to beams that have cross-sections with a different shape, and will also consider the cases of the beams with other support types and/or multiple cracks.

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