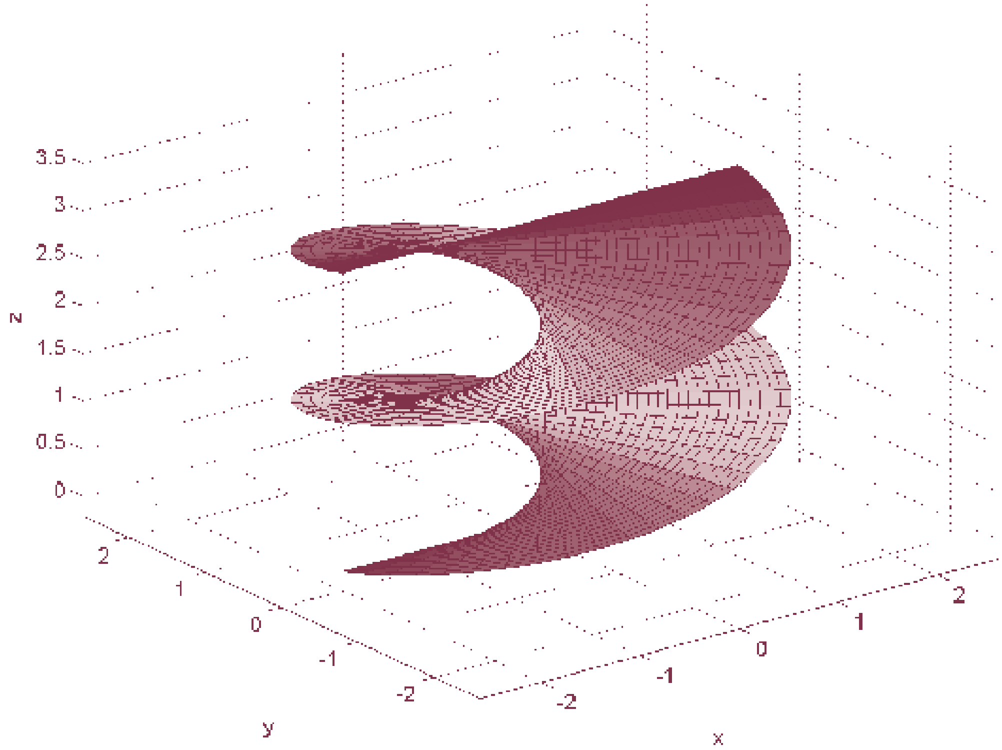
STUDIA UNIVERSITATIS
BABEŞ-BOLYAI



MATHEMATICA

2/2016

STUDIA
UNIVERSITATIS BABEŞ-BOLYAI
MATHEMATICA

2/2016

EDITORIAL BOARD OF
STUDIA UNIVERSITATIS BABEŞ-BOLYAI MATHEMATICA

YEAR	**(LXI) 2016**
MONTH	**JUNE**
ISSUE	**2**

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
2

Redacţia: 400084 Cluj-Napoca, str. M. Kogălniceanu nr. 1
Telefon: 0264 405300

CONTENTS

Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 2, 127–135

On certain class of meromorphic univalent functions with positive coefficients defined by Dziok-Srivastava operator

Mohamed K. Aouf, Adela O. Mostafa, Abd-Elmoem Y. Lashen and Basheer M. Munassar

Abstract. In this paper, we introduce a new class of meromorphic univalent functions defined by using Dziok-Srivastava operator and obtain some results including coefficient inequality, growth and distortion theorems and modified Hadamard products.

Mathematics Subject Classification (2010): 30C45.

Keywords: Meromorphic functions, univalent functions, growth and distortion theorem, Hadamard product, Dziok-Srivastava operator.

1. Introduction

Let Σ_m denote the class of functions f of the form:

$$f(z)=\frac{1}{z}+\sum_{k=m}^{\infty} a_k z^k \;\; \left(m\in\mathbb{N}=\{1,2,...\}\right), \tag{1.1}$$

which are analytic and univalent in the punctured unit disc $U^*=\{z : z\in\mathbb{C}$ and $0<|z|<1\}=U\backslash\{0\}$. For $g\in\Sigma_m$, given by

$$g\left(z\right)=\frac{1}{z}+\sum_{k=m}^{\infty} b_k z^k, \tag{1.2}$$

the Hadamard product (or convolution) of f and g is given by

$$\left(f*g\right)\left(z\right)=\frac{1}{z}+\sum_{k=m}^{\infty} a_k b_k z^k=\left(g*f\right)\left(z\right). \tag{1.3}$$

A function $f\in\Sigma_m$ is said to be meromorphically starlike of order λ if

$$-\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\lambda \;\; (z\in U;\; 0\leq\lambda<1). \tag{1.4}$$

Denote by $\Sigma S_m^*(\lambda)$ the class of all meromorphically starlike functions of order λ. A function $f \in \Sigma_m$ is said to be meromorphically convex of order λ if

$$-\text{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \lambda \ (z \in U;\ 0 \leq \lambda < 1). \tag{1.5}$$

Denote by $\Sigma K_m(\lambda)$ the class of all meromorphically convex functions of order λ. We note that

$$f(z) \in \Sigma K_m(\lambda) \Longleftrightarrow -zf'(z) \in \Sigma S_m^*(\lambda).$$

The classes $\Sigma S_m^*(\lambda)$ and $\Sigma K_m(\lambda)$ were introduced by Owa et al. [8]. Various subclasses of the class Σ_m when $m = 1$ were considered earlier by Pommerenke [9], Miller [6] and others.

For complex parameters

$$\alpha_1, ..., \alpha_q \text{ and } \beta_1, ..., \beta_s \quad \left(\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\};\ j = 1, 2, ..., s\right),$$

the generlized hypergeometric function ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ is defined by

$${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k, ..., (\alpha_q)_k}{(\beta_1)_k, ..., (\beta_s)_k} \cdot \frac{z^k}{k!}$$

$$(q \leq s+1;\ q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ z \in U), \tag{1.6}$$

where $(\theta)_v$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_v = \frac{\Gamma(\theta+v)}{\Gamma(\theta)} = \begin{cases} 1 & \text{if } (v = 0;\ \theta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}), \\ \theta(\theta+1)(\theta+2)...(\theta+v-1) & \text{if } (v \in \mathbb{N};\ \theta \in \mathbb{C}). \end{cases} \tag{1.7}$$

Corresponding to the function $h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$, defined by

$$h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z^{-1}\ {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z), \tag{1.8}$$

we consider the linear operator

$$H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) : \Sigma_m \to \Sigma_m,$$

which is defined by means of the following Hadamard product (or convolution):

$$H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) f(z) = h_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z). \tag{1.9}$$

We observe that, for a function f of the form (1.1), we have

$$H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) f(z) = z^{-1} + \sum_{k=m}^{\infty} \frac{(\alpha_1)_{k+1}, ..., (\alpha_q)_{k+1}}{(\beta_1)_{k+1}, ..., (\beta_s)_{k+1}} \cdot \frac{a_k}{(k+1)!} z^k. \tag{1.10}$$

For convenience, we write

$$H_{q,s}(\alpha_1) = H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s). \tag{1.11}$$

The linear operator $H_{q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [5, with $p = 1$] and Aouf [2, with $p = 1$].

For fixed parameters A, B, β and λ $(0 < \beta \leq 1,\ -1 \leq A < B \leq 1,\ 0 \leq \lambda < 1)$, we say that a function $f \in \Sigma_m$ is in the class $\Sigma^m_{q,s}(\alpha_1; A, B, \lambda, \beta)$ of meromorphically univalent functions in if it satisfies the inequality:

$$\left| \frac{\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} + 1}{B\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} + [B + (A - B)(1 - \lambda)]} \right| < \beta \quad (z \in U^*). \tag{1.12}$$

A function f in Σ_m is said to belong to the class $C^m_{q,s}(\alpha_1; A, B, \lambda, \beta)$ if and only if $-zf'(z) \in \Sigma^m_{q,s}(\alpha_1; A, B, \lambda, \beta)$ that is

$$f \in C^m_{q,s}(\alpha_1; A, B, \lambda, \beta) \Longleftrightarrow -zf' \in \Sigma^m_{q,s}(\alpha_1; A, B, \lambda, \beta). \tag{1.13}$$

We note that:

(i) $\Sigma^m_{2,1}(1; -1, 1, \lambda, 1) = \Sigma S^*_m(\lambda)$ and
$C^m_{2,1}(1; -1, 1, \lambda, 1) = \Sigma K_m(\lambda)$ $(0 \leq \lambda < 1, m \in \mathbb{N})$.

(ii) $\Sigma^1_{2,1}(1; A, B, \lambda, \beta) = \Sigma^*(A, B, \lambda, \beta)$ was studied by Aouf [1];

(iii) $\Sigma^1_{2,1}(1; -1, 1, \lambda, \beta) = \Sigma^*(\lambda, \beta)$ and $C^1_{2,1}(1; -1, 1, \lambda, \beta) = C(\lambda, \beta)$ (Mogra et al.[7]);

(iv) $\Sigma^m_{2,1}(1; A, B, \lambda, \beta) = \Sigma_m(A, B, \lambda, \beta)$ (Aouf et al. [6]).

We note also that:

$$\Sigma^1_{q,s}(\alpha_1; \beta, -\beta, \lambda, 1) = \Sigma^+_{q,s}(\alpha_1; \lambda, \beta)$$

$$= \left\{ f(z) \in \Sigma_m : \left| \frac{\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} + 1}{\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} - 1 + 2\lambda} \right| < \beta \ (z \in U,\ 0 < \beta \leq 1,\ 0 \leq \lambda < 1) \right\}.$$

2. Coefficient inequality

Unless otherwise mentioned, we shall assume in the reminder of this paper that, the parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ are positive real numbers, $0 < \beta \leq 1$, $-1 \leq A < B \leq 1$, $0 \leq \lambda < 1, m \in \mathbb{N}$, $\Gamma_{k+1}(\alpha_1)$ is defined by (2.2) and $z \in U^*$.

In order to prove our results we need the following lemma for the class $\Sigma^m_{q,s}(\alpha_1; A, B, \lambda, 1)$ given by Aouf [3, with $p = 1$].

Lemma 2.1. Let a function f defined by (1.1) be in the class Σ_m. If

$$\sum_{k=m}^{\infty} \{(k+1) + \beta[(Bk + A) + (B - A)\lambda]\} \Gamma_{k+1}(\alpha_1) |a_k| \leq (B - A)\beta(1 - \lambda) \tag{2.1}$$

then $f \in \Sigma^m_{q,s}(\alpha_1; A, B, \lambda, \beta)$, where

$$\Gamma_{k+1}(\alpha_1) = \frac{(\alpha_1)_{k+1}, ..., (\alpha_q)_{k+1}}{(\beta_1)_{k+1}, ..., (\beta_s)_{k+1}} . \frac{1}{(k+1)!}. \tag{2.2}$$

From Lemma 2.1 and (1.13), we have the following lemma.

Lemma 2.2. Let a function f defined by (1.1) be in the class Σ_m. If

$$\sum_{k=m}^{\infty} k\{(k+1) + \beta[(Bk + A) + (B - A)\lambda]\} \Gamma_{k+1}(\alpha_1) |a_k| \leq (B - A)\beta(1 - \lambda) \tag{2.3}$$

then $f \in C_{q,s}^{m}(\alpha_1; A, B, \lambda, \beta)$.

3. Growth and distortion theorems

Theorem 3.1. If the function f defined by (1.1) is in the class $\Sigma_{q,s}^{m}(\alpha_1; A, B, \lambda, \beta)$, then

$$\frac{1}{|z|} - \frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)}|z|^m \le |f(z)|$$

$$\le \frac{1}{|z|} + \frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)}|z|^m, \tag{3.1}$$

and

$$\frac{1}{|z|^2} - \frac{m(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)}|z|^{m-1} \le \left|f'(z)\right|$$

$$\le \frac{1}{|z|^2} + \frac{m(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)}|z|^{m-1}. \tag{3.2}$$

The bounds in (3.1) and (3.2) are attained for the function f given by

$$f(z) = \frac{1}{z} + \frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)}z^m. \tag{3.3}$$

Proof. First of all, for $\Sigma_{q,s}^{m}(\alpha_1; A, B, \lambda, \beta)$, it follows from (2.1) that

$$\sum_{k=m}^{\infty} a_k \le \frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)}, \tag{3.4}$$

which, in view of (1.1), yields

$$\begin{aligned} |f(z)| &\ge \frac{1}{|z|} - |z|^m \sum_{k=m}^{\infty} |a_k| \qquad (3.5)\\ &\ge \frac{1}{|z|} - \frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)}|z|^m, \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\le \frac{1}{|z|} + |z|^m \sum_{k=m}^{\infty} |a_k| \qquad (3.6)\\ &\le \frac{1}{|z|} + \frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)}|z|^m. \end{aligned}$$

Next, we see from (2.1) that

$$\frac{\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}{m}\sum_{k=m}^{\infty}k\left|a_k\right| \tag{3.7}$$

$$\leq \sum_{k=m}^{\infty}\{(k+1)+\beta\left[(Bk+A)+(B-A)\lambda\right]\}\Gamma_{k+1}(\alpha_1)\left|a_k\right|$$

$$\leq (B-A)\beta(1-\lambda)$$

then

$$\sum_{k=m}^{\infty}k\left|a_k\right| \leq \frac{m(B-A)\beta(1-\lambda)}{\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}.$$

which, again in view of (1.1) , yields

$$\left|f'(z)\right| \geq \frac{1}{|z|^2}-|z|^{m-1}\sum_{k=m}^{\infty}k\left|a_k\right| \tag{3.8}$$

$$\geq \frac{1}{|z|^2}-\frac{m(B-A)\beta(1-\lambda)}{\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}|z|^{m-1},$$

and

$$\left|f'(z)\right| \leq \frac{1}{|z|^2}+|z|^{m-1}\sum_{k=m}^{\infty}k\left|a_k\right| \tag{3.9}$$

$$\leq \frac{1}{|z|^2}+\frac{m(B-A)\beta(1-\lambda)}{\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}|z|^{m-1}.$$

Finally, it is easy to see that the bounds in (3.1) and (3.2) are attained for the function f given by (3.3) .

Corollary 3.1. If the function f defined by (1.1) is in the class $C_{q,s}^{m}(\alpha_1; A, B, \lambda, \beta)$, then

$$\frac{1}{|z|}-\frac{(B-A)\beta(1-\lambda)}{m\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}|z|^m \leq |f(z)|$$

$$\leq \frac{1}{|z|}+\frac{(B-A)\beta(1-\lambda)}{m\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}|z|^m, \tag{3.10}$$

and

$$\frac{1}{|z|^2}-\frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}|z|^{m-1} \leq \left|f'(z)\right|$$

$$\leq \frac{1}{|z|^2}+\frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}|z|^{m-1}. \tag{3.11}$$

The bounds in (3.1) and (3.2) are attained for the function f given by

$$f(z)=\frac{1}{z}+\frac{(B-A)\beta(1-\lambda)}{m\{(m+1)+\beta\left[(Bm+A)+(B-A)\lambda\right]\}\Gamma_{m+1}(\alpha_1)}z^m. \tag{3.12}$$

4. Modified Hadamard product

Let each of the functions f_1 and f_1 defined by

$$f_j(z) = \frac{1}{z} + \sum_{k=m}^{\infty} a_{k,j} z^k \quad (j = 1,\ 2) \tag{4.1}$$

belong to the class Σ_m. We denote by $(f_1 * f_2)$ the modified Hadamard product (or convolution) of the functions f_1 and $f_{2,}$ that is,

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=m}^{\infty} a_{k,1} a_{k,2} z^k. \tag{4.2}$$

Theorem 4.1. Let the functions f_j $(j = 1,\ 2)$ defined by (4.1) be in the class $\Sigma_{q,s}^m(\alpha_1; A, B, \lambda, \beta)$. Then $(f_1 * f_2)(z) \in \Sigma_{q,s}^m(\alpha_1; A, B, \gamma, \beta)$,where

$$\gamma = 1 - \frac{(B-A)\beta(1-\lambda)^2(1+\beta B)(m+1)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}^2 \Gamma_{m+1}(\alpha_1) + (B-A)^2 \beta^2 (1-\lambda)^2}. \tag{4.3}$$

The result is sharp for the functions f_j $(j = 1,\ 2)$ given by

$$f_j(z) = \frac{1}{z} + \frac{(B-A)\beta(1-\lambda)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}\Gamma_{m+1}(\alpha_1)} z^m \quad (j = 1, 2). \tag{4.4}$$

Proof. Employing the technique used ealier by Schild and Silverman [10], we need to find the largest γ such that

$$\sum_{k=m}^{\infty} \frac{\{(k+1)+\beta[(Bk+A)+(B-A)\gamma]\}\Gamma_{k+1}(\alpha_1)}{(B-A)\beta(1-\gamma)} |a_{k,1}||a_{k,2}| \le 1 \tag{4.5}$$

for $(f_1 * f_2)(z) \in \Sigma_{q,s}^m(\alpha_1; A, B, \gamma, \beta)$. Indeed, since each of the functions f_j $(j = 1, 2)$ belongs to the class $\Sigma_{q,s}^m(\alpha_1; A, B, \lambda, \beta)$, then

$$\sum_{k=m}^{\infty} \frac{\{(k+1)+\beta[(Bk+A)+(B-A)\lambda]\}\Gamma_{k+1}(\alpha_1)}{(B-A)\beta(1-\lambda)} |a_{k,j}| \le 1 \quad (j = 1, 2). \tag{4.6}$$

Now, by the Cauchy-Schwarz inequality, we find from (4.6) that

$$\sum_{k=m}^{\infty} \frac{\{(k+1)+\beta[(Bk+A)+(B-A)\lambda]\}\Gamma_{k+1}(\alpha_1)}{(B-A)\beta(1-\lambda)} \sqrt{|a_{k,1}||a_{k,2}|} \le 1. \tag{4.7}$$

Equation (4.7) implies that we need only to show that

$$\begin{aligned} &\frac{\{(k+1)+\beta[(Bk+A)+(B-A)\gamma]\}}{(1-\gamma)} |a_{k,1}||a_{k,2}| \\ &\le \frac{\{(k+1)+\beta[(Bk+A)+(B-A)\lambda]\}}{(1-\lambda)} \sqrt{|a_{k,1}||a_{k,2}|} \quad (k \ge m), \end{aligned} \tag{4.8}$$

that is, that

$$\sqrt{|a_{k,1}||a_{k,2}|} \le \frac{\{(k+1)+\beta[(Bk+A)+(B-A)\lambda]\}(1-\gamma)}{\{(k+1)+\beta[(Bk+A)+(B-A)\gamma]\}(1-\lambda)} \quad (k \ge m). \tag{4.9}$$

Hence, by the inequality (4.7) it is sufficient to prove that

$$\frac{(B-A)\,\beta\,(1-\lambda)}{\{(k+1)+\beta\,[(Bk+A)+(B-A)\,\lambda]\}\,\Gamma_{k+1}(\alpha_1)} \le \frac{\{(k+1)+\beta\,[(Bk+A)+(B-A)\,\lambda]\}\,(1-\gamma)}{\{(k+1)+\beta\,[(Bk+A)+(B-A)\,\gamma]\}\,(1-\lambda)} \quad (k \ge m). \tag{4.10}$$

It follows from (4.10) that

$$\gamma \le 1 - \frac{(B-A)\beta(1+\beta B)(k+1)(1-\lambda)^2}{\{(k+1)+\beta[(Bk+A)+(B-A)\lambda]\}^2\Gamma_{k+1}(\alpha_1)+(B-A)^2\beta^2(1-\lambda)^2} \quad (k \ge m). \tag{4.11}$$

Defining the function $\Phi\,(k)$ by

$$\Phi\,(k) = 1 - \frac{(B-A)\beta(1+\beta B)(k+1)(1-\lambda)^2}{\{(k+1)+\beta[(Bk+A)+(B-A)\lambda]\}^2\Gamma_{k+1}(\alpha_1)+(B-A)^2\beta^2(1-\lambda)^2} \quad (k \ge m), \tag{4.12}$$

we see that $\Phi\,(k)$ is an increasing function of k $(k \ge m)$. Therefore, we conclude from (4.11) that

$$\gamma \le \Phi\,(m) = 1 - \frac{(B-A)\beta(1+\beta B)(m+1)(1-\lambda)^2}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}^2\Gamma_{m+1}(\alpha_1)+(B-A)^2\beta^2(1-\lambda)^2}, \tag{4.13}$$

which completes the proof of the main assertion of Theorem 4.1.

Corollary 4.1. Let the functions f_j $(j=1,\ 2)$ defined by (4.1) be in the class $C^m_{q,s}\,(\alpha_1; A, B, \lambda, \beta)$. Then $(f_1 * f_2)\,(z) \in C^m_{q,s}\,(\alpha_1; A, B, \mu, \beta)$, where

$$\mu = 1 - \frac{(B-A)\beta(1-\lambda)^2(1+\beta B)(m+1)}{m\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}^2\Gamma_{m+1}(\alpha_1)+(B-A)^2\beta^2(1-\lambda)^2}. \tag{4.14}$$

The result is sharp for the functions f_j $(j=1,\ 2)$ given by

$$f_j\,(z) = \frac{1}{z} + \frac{(B-A)\,\beta\,(1-\lambda)}{m\,\{(m+1)+\beta\,[(Bm+A)+(B-A)\,\lambda]\}\,\Gamma_{m+1}(\alpha_1)} z^m \quad (j=1,2). \tag{4.15}$$

Theorem 4.2. Let the functions f_j $(j=1,\ 2)$ defined by (4.1) be in the class $\Sigma^m_{q,s}\,(\alpha_1; A, B, \lambda, \beta)$. Then the function $h(z)$ defined by

$$h\,(z) = \frac{1}{z} + \sum_{k=m}^{\infty} \left(a^2_{k,1} + a^2_{k,2}\right) z^k \tag{4.16}$$

belongs to the class $\Sigma^m_{q,s}\,(\alpha_1; A, B, \xi, \beta)$, where

$$\xi = 1 - \frac{2\,(B-A)\,\beta\,(1-\lambda)^2\,(1+\beta B)\,(m+1)}{\{(m+1)+\beta\,[(Bm+A)+(B-A)\,\lambda]\}^2\,\Gamma_{m+1}(\alpha_1)+2\,(B-A)^2\,\beta^2\,(1-\lambda)^2}. \tag{4.17}$$

The result is sharp for the functions f_j $(j=1,\ 2)$ given by (4.4).

Proof. Noting that

$$\sum_{k=m}^{\infty} \left[\frac{\{(k+1)+\beta\,[(Bk+A)+(B-A)\,\lambda]\}\,(\Gamma_{k+1}(\alpha_1))}{(B-A)\,\beta\,(1-\lambda)}\right]^2 |a_{k,j}|^2 \le \left[\sum_{k=m}^{\infty} \frac{\{(k+1)+\beta\,[(Bk+A)+(B-A)\,\lambda]\}\,(\Gamma_{k+1}(\alpha_1))}{(B-A)\,\beta\,(1-\lambda)}\,|a_{k,j}|\right]^2 \le 1, \tag{4.18}$$

for $f_j \in \Sigma_{q,s}^m(\alpha_1; A, B, \lambda, \beta)$ $(j = 1,\ 2)$, we have

$$\sum_{k=m}^{\infty} \frac{\{(k+1) + \beta[(Bk + A) + (B - A)\lambda]\}^2 (\Gamma_{k+1}(\alpha_1))^2}{2(B - A)^2 \beta^2 (1 - \lambda)^2} \left(|a_{k,1}|^2 + |a_{k,2}|^2\right) \leq 1. \tag{4.19}$$

Thus we need to find the largest ξ such that

$$\frac{\{(k+1) + \beta[(Bk + A) + (B - A)\xi]\}}{(1 - \xi)} \tag{4.20}$$

$$\leq \frac{\{(k+1) + \beta[(Bk + A) + (B - A)\lambda]\}^2 (\Gamma_{k+1}(\alpha_1))}{2(B - A)\beta(1 - \lambda)^2} \quad (k \geq m),$$

that is, that

$$\xi \leq 1 - \frac{2(B-A)\beta(1-\lambda)^2(1+\beta B)(k+1)}{\{(k+1)+\beta[(Bk+A)+(B-A)\lambda]\}^2\Gamma_{k+1}(\alpha_1)+2(B-A)^2\beta^2(1-\lambda)^2} \quad (k \geq m). \tag{4.21}$$

Defining the function $\Theta(k)$ by

$$\Theta(k) = 1 - \frac{2(B-A)\beta(1-\lambda)^2(1+\beta B)(k+1)}{\{(k+1)+\beta[(Bk+A)+(B-A)\lambda]\}^2\Gamma_{k+1}(\alpha_1)+2(B-A)^2\beta^2(1-\lambda)^2} \quad (k \geq m), \tag{4.22}$$

we observe that $\Theta(k)$ is an increasing function of k $(k \geq m)$. Therefore, we conclude from (4.21) that

$$\xi \leq \Theta(m) = 1 - \frac{2(B-A)\beta(1-\lambda)^2(1+\beta B)(m+1)}{\{(m+1)+\beta[(Bm+A)+(B-A)\lambda]\}^2\Gamma_{m+1}(\alpha_1)+2(B-A)^2\beta^2(1-\lambda)^2}, \tag{4.23}$$

which completes the proof of Theorem 4.2.

Corollary 4.2. Let the functions f_j $(j = 1,\ 2)$ defined by (4.1) be in the class $C_{q,s}^m(\alpha_1; A, B, \lambda, \beta)$. Then the function $h(z)$ defined by (4.18) belongs to the class $C_{q,s}^m(\alpha_1; A, B, \rho, \beta)$, where

$$\rho = 1 - \frac{2(B - A)\beta(1 - \lambda)^2(1 + \beta B)(m + 1)}{m\{(m+1) + \beta[(Bm + A) + (B - A)\lambda]\}^2 \Gamma_{m+1}(\alpha_1) + 2(B - A)^2 \beta^2 (1 - \lambda)^2}. \tag{4.24}$$

The result is sharp for the functions f_1 and f_2 given by (4.15).

Remarks. (i) Putting $q = 2$ and $s = \alpha_1 = \alpha_2 = \beta_1 = 1$ in the above results, we get the results obtained by Aouf et al. [4, Lemmas 1 and 2 and Corollaries 1, 2, 3, 4, 7 and 8, respectively];

(ii) Putting $q = 2$, $s = \alpha_1 = \alpha_2 = \beta_1 = B = 1$ and $A = -1$, in Theorems 4.1, 4.2 and Corollaries 4.1, 4.2, we get the results obtained by Aouf et al. [4, Corollaries 5, 9, 6 and 10, respectively].

Acknowledgment. The authors thank the referees for their valuable suggestions which led to the improvement of this paper.

References

[1] Aouf, M.K., *A certain subclass of meromorphic starlike functions with positive coefficients*, Rend Mat. Appl., **9**(1989), no. 7, 225-235.

[2] Aouf, M.K., *Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function*, Comput. Math. Appl., **55**(2008), no. 3, 494-509.

[3] Aouf, M.K., *The generalized hypergeometric function and associated familes of meromorphically multivalent functions*, J. Math. Ineq., **3**(2009), no. 1, 43-62.

[4] Aouf, M.K., Darwish H.E., Srivastava, H.M., *A general class of meromorphically univalent functions*, PanAmer. Math. J., **7**(1997), no. 3, 69-84.

[5] Liu, J.L., Srivastava, H.M., *Classes of meromorphically multivalent functions associated with the generalized hypergeometric function*, Math. Comput. Modelling, **39**(2004), 21-34.

[6] Miller, J.E., *Convex meromorphic mapping and related functions*, Proc. Amer. Math. Soc., **25**(1970), 220-228.

[7] Mogra, M.L., Reddy, T.R., Juneja, O.P., *Meromorphic univalent functions with positive coefficients*, Bull. Austral. Math. Soc., **32**(1985), 161-176.

[8] Owa, S., Nunokawa, M., Saitoh, H., *Generalization of certain subclasses of meromorphic univalent functions*, J. Fac. Sci. Tech. Kinki Univ., **25**(1989), 21-27.

[9] Pommerenke, Ch., *On meromorphic starlike functions*, Pacific J. Math., **13**(1963), 221-235.

[10] Schild, A., Silverman, H., *Convolution of univalent functions with negative coeficients*, Ann. Math Curie-Sklodowska Sect. A, **29**(1975), 99-107.

Mohamed K. Aouf
Department of Mathematics, Faculty of Science
University of Mansoura, Mansoura 35516, Egypt
e-mail: `mkaouf127@yahoo.com`

Adela O. Mostafa
Department of Mathematics, Faculty of Science
University of Mansoura, Mansoura 35516, Egypt
e-mail: `adelaeg254@yahoo.com`

Abd-Elmoem Y. Lashen
Department of Mathematics, Faculty of Science
University of Mansoura, Mansoura 35516, Egypt
e-mail: `aylashin@mans.edu.eg`

Basheer M. Munassar
Department of Mathematics, Faculty of Science
University of Mansoura, Mansoura 35516, Egypt
e-mail: `bmunassar@yahoo.com`

Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 2, 137–146

On a subclass of convex functions

Olga Engel and Róbert Szász

Abstract. In this paper we study a subclass of convex functions. Among others we prove an interesting property regarding the composition of functions from this class. The basic tool of the proof is the theory of differential subordination.

Mathematics Subject Classification (2010): 30C45.

Keywords: Starlikeness, univalent function, composition of functions.

1. Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$. We denote by $\mathcal{A}$ the class of the functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

defined in U. We say that f is starlike in U if $f : U \to \mathbb{C}$ is univalent and $f(U)$ is a starlike domain in $\mathbb{C}$ with respect to 0.

It is well-known that $f \in \mathcal{A}$ is starlike in U if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad \text{for all } z \in U.$$

The function $f \in \mathcal{A}$ is convex in U if and only if $f : U \to \mathbb{C}$ is univalent and $f(U)$ is a convex domain in $\mathbb{C}$. The function $f \in \mathcal{A}$ is convex if and only if

$$\mathrm{Re}\,\frac{zf''(z)}{f'(z)} + 1 > 0, \quad z \in U.$$

The subclass of $\mathcal{A}$ which contain convex functions will be denoted by $\mathcal{K}$.
We define the class S^{***} by the equality

$$S^{***} = \left\{ f \in \mathcal{A} : \left|1 - \frac{zf''(z)}{f'(z)}\right| < \sqrt{\frac{5}{4}},\ z \in U \right\}. \tag{1.1}$$

We will prove in the followings that $S^{***} \subset \mathcal{K}$, we will determine the order of starlikeness of the class S^{***} and we will show that if

$$f, g \in S^{***}, \text{ then } f \circ g \text{ is starlike in the disk } U(r_0),$$

where $r_0 = \sup\{r > 0 | g(U(r)) \subset U\}$.

2. Preliminaries

In order to prove the Main Result, we need the following results. These lemmas can be found in [1], p.24-25, and [2], p. 201-203.
Let Q be the class of analytic functions q in U which has the property that are analytic and injective on $\overline{U} \setminus E(q)$, where

$$E(q) = \left\{ \zeta \in \partial U : \lim_{z \longrightarrow \zeta} q(z) = \infty \right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.

Lemma 2.1. [Miller-Mocanu] *Let $q \in Q$, with $q(0) = a$, and let $p(z) = a + a_n z^n + \dots$ be analytic in U with $p(z) \not\equiv a$ and $n \geq 1$. If $f \not\prec q$, then there are two points $z_0 = r_0 e^{i\theta_0} \in U$, and $\zeta_0 \in \partial U \setminus E(q)$ and a real number $m \in [n, \infty)$ for which $p(U_{r_0}) \subset q(U)$,*
(i) $p(z_0) = q(\zeta_0)$
(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$
(iii) $\operatorname{Re} \frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m \operatorname{Re}\left(\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right).$

The following result is a particular case of Lemma 2.1.

Lemma 2.2. [Miller-Mocanu] *Let $p(z) = 1 + a_n z^n + \dots$ be analytic in U with $p(z) \not\equiv 1$ and $n \geq 1$.*
If $\operatorname{Re} p(z) \not> 0$, $z \in U$, then there is a point $z_0 \in U$, and there are two real numbers $x, y \in \mathbb{R}$ such that
(i) $p(z_0) = ix$
(ii) $z_0 p'(z_0) = y \leq -\frac{n(x^2+1)}{2}$,
(iii) $\operatorname{Re} z_0^2 p''(z_0) + z_0 p'(z_0) \leq 0.$

We also need the following result, which is a particular case of the Theorem 3.2d. from [1]. The next result is Theorem 3.2i. from [1].

Lemma 2.3. *Let h be convex in U, with $h(0) = 1$ and let n be a positive integer. If q is the analytic solution of*

$$q(z) + \frac{zq'(z)}{q(z)} = h(z), \quad q(0) = 1,$$

and if $\operatorname{Re} q(z) > 0$, $z \in U$, then q is univalent. If $p(z) = 1 + a_n z^n + a_{n+1} z^{n+1} + \dots$ is an analytic function in U, and

$$p(z) + \frac{zp'(z)}{p(z)} \prec h(z),$$

then $p \prec q$, *and* q *is the best dominant.*

We also need the following result, which is a particular case of the Theorem 3.2d. from [1].

Lemma 2.4. *Let* $\beta, \gamma \in \mathbb{C}$ *and let* n *be a positive integer. Let* $R_{\beta a+\gamma,n}$ *be given by*

$$R_{c,n}(z) = 2C_n \frac{(z+b)(1+\overline{b}z)}{(1+\overline{b}z)^2-(z+b)^2}, \ C_n = \frac{n}{\operatorname{Re} c}[|c|\sqrt{1+2\operatorname{Re}(c/n)}+\operatorname{Im} c].$$

Let h *be analytic in* U, *with* $h(0)=a$, *and let* $\operatorname{Re}[\beta a+\gamma]>0$. *If*

$$\beta h(z)+\gamma \prec R_{\beta a+\gamma,n(z)},$$

then the solution q *of the equation*

$$q(z)+\frac{nzq'(z)}{\beta q(z)+\gamma}=h(z),$$

with $q(0)=a$ *is analytic in* U *and satisfies* $\operatorname{Re}[\beta q(z)+\gamma]>0,\ z\in U$.
If $a \neq 0$, *then the solution is given by*

$$q(z) = z^{\gamma/n}[H(z)]^{\beta a/n}\left((\beta/n)\int_0^z [H(t)]^{\beta a/n} t^{(\gamma/n)-1}dt\right)^{-1} - \gamma/\beta,$$

where

$$H(z) = z\exp\int_0^z [(h(t)-a)/at]dt.$$

Lemma 2.5. *If* $x>0$, *and* $y\in\mathbb{R}$, *then*

$$\operatorname{Re}(x+iy)^{\frac{1}{4}} \geq x^{\frac{1}{4}}.$$

Proof. We have

$$\operatorname{Re}(x+iy)^{\frac{1}{4}} = (x^2+y^2)^{\frac{1}{8}}\cos\left(\frac{1}{4}\arctan\frac{y}{x}\right)$$

and in order to prove the lemma we have to show that

$$(x^2+y^2)^{\frac{1}{2}}\cos^4\left(\frac{1}{4}\arctan\frac{y}{x}\right) \geq x. \tag{2.1}$$

Since

$$\cos^4\left(\frac{1}{4}\arctan\frac{y}{x}\right) = \frac{1}{4}\left(1+\sqrt{\frac{1+\frac{x}{\sqrt{x^2+y^2}}}{2}}\right)^2$$

the inequality (2.1) is equivalent to

$$\frac{1}{4}\left(1+\sqrt{\frac{1+\frac{x}{\sqrt{x^2+y^2}}}{2}}\right)^2 \geq \frac{x}{\sqrt{x^2+y^2}}.$$

Since $\frac{x}{\sqrt{x^2+y^2}} \in [0,1]$ it follows that there is a real number $\alpha \in [0, \frac{\pi}{2}]$ such that $\frac{x}{\sqrt{x^2+y^2}} = \cos\alpha$, and the previous inequality can be rewritten as follows

$$\cos^4 \frac{\alpha}{4} \geq \cos\alpha, \quad \alpha \in \left[0, \frac{\pi}{2}\right].$$

This inequality is equivalent to

$$\left(1 - \cos^2 \frac{\alpha}{4}\right)\left(7\cos^2 \frac{\alpha}{4} - 1\right) \geq 0, \ \alpha \in \left[0, \frac{\pi}{2}\right],$$

and the proof is done taking into account that $7\cos^2 \frac{\alpha}{4} \geq 7\cos^2 \frac{\pi}{8} > 1$. □

We need also the following lemma which can be found in [2], p.271.

Lemma 2.6. *Let* $g : [-\pi, \pi][0,1] \to \mathbb{C}$ *a function such that* $g(e^{i\theta}, .)$ *is integrable on* $[0,1]$, *for each* $\theta \in [-\pi, \pi]$. *If* $\alpha : [0,1] \to (0, \infty)$ *ia also integrable and*

$$\text{Re}\, \frac{1}{g(e^{\theta}, t)} \geq \frac{1}{\alpha(t)}, \ \ \theta \in [-\pi, \pi], t \in [0,1],$$

then

$$\text{Re}\, \frac{1}{\displaystyle\int_0^1 g(e^{i\theta}, t)dt} \geq \frac{1}{\displaystyle\int_0^1 \alpha(t)dt}, \ \ \theta \in [-\pi, \pi].$$

3. Main results

Theorem 3.1. *If* $f \in \mathcal{A}$ *and*

$$\left|1 - \frac{zf''(z)}{f'(z)}\right| < \sqrt{7}, \ \ z \in U,$$

then it follows that $f \in S^*$.

Proof. We will prove that $p(z) = \frac{zf'(z)}{f(z)} > 0, \quad z \in U$.
It is easily seen that

$$1 - \frac{zf''(z)}{f'(z)} = 2 - p(z) - \frac{zp'(z)}{p(z)},$$

and consequently the following equivalence holds

$$\left|1 - \frac{zf''(z)}{f'(z)}\right| < \sqrt{7}, \ \ z \in U \Leftrightarrow \left|2 - p(z) - \frac{zp'(z)}{p(z)}\right| < \sqrt{7}, \ \ z \in U. \tag{3.1}$$

If the condition $p(z) = \dfrac{zf'(z)}{f(z)} > 0, \quad z \in U$, does not hold, then according to the Miller-Mocanu lemma (Lemma 2.2) there is a point $z_0 \in U$, and there are two real numbers $x, y \in \mathbb{R}$ such that

$$p(z_0) = ix,$$

and

$$z_0 p'(z_0) = y \leq -\frac{1+x^2}{2}.$$

These equalities imply

$$\left|2 - p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right| = \left|2 - i\left(x - \frac{y}{x}\right)\right| = \sqrt{4 + \frac{(x^2 - y)^2}{x^2}}$$
$$\geq \sqrt{4 + \left(\frac{3x}{2} + \frac{1}{2x}\right)^2} \geq \sqrt{7}.$$

This inequality contradicts (3.1), and consequently $p(z) = \dfrac{zf'(z)}{f(z)} > 0,\ z \in U$ holds. □

Remark 3.2. The result of Theorem 3.1 shows that the following inclusion holds $S^{***} \subset S^*$.

We will determine the exact order of starlikeness of the class S^{***} in the followings.

Theorem 3.3. *If $f \in S^{***}$, then*

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{1}{\displaystyle\int_0^1 \left(\frac{\sqrt{\frac{5}{4}} - t}{\sqrt{\frac{5}{4}} - 1}\right)^{\frac{1}{4}} dt} = \frac{5}{4} \frac{\left(\sqrt{\frac{5}{4}} - 1\right)^{\frac{1}{4}}}{\left(\sqrt{\frac{5}{4}}\right)^{\frac{5}{4}} - \left(\sqrt{\frac{5}{4}} - 1\right)^{\frac{5}{4}}},\ z \in U.$$

The result is sharp.

Proof. The inequality $\left|1 - \frac{zf''(z)}{f'(z)}\right| < \sqrt{\frac{5}{4}},\ z \in U$ is equivalent to the subordination

$$1 - \frac{zf''(z)}{f'(z)} \prec M \frac{Mz + 1}{M + z},$$

where $M = \sqrt{\frac{5}{4}}$. Denoting $p(z) = \frac{zf'(z)}{f(z)}$ the subordination can be rewritten in the following form

$$2 - p(z) - \frac{zp'(z)}{p(z)} \prec M \frac{Mz + 1}{M + z},$$

and this is equivalent to

$$p(z) + \frac{zp'(z)}{p(z)} \prec h(z) = 2 - M \frac{Mz + 1}{M + z}. \tag{3.2}$$

If we denote by q the solution of the equation

$$q(z) + \frac{zq'(z)}{q(z)} = 2 - M \frac{Mz + 1}{M + z} = h(z)$$

then $\operatorname{Re} q(z) > 0,\ \ z \in U$. Indeed if the inequality $\operatorname{Re} q(z) > 0,\ \ z \in U$ does not holds, then there is a point $z_0 \in U$, and there are two real numbers $x, y \in \mathbb{R}$ such that

$$q(z_0) = ix,$$

and

$$z_0 q'(z_0) = y \leq -\frac{1 + x^2}{2}.$$

These equalities imply

$$2<\left|2-ix-\frac{y}{ix}\right|=\left|2-q(z_0)-\frac{z_0q'(z_0)}{q(z_0)}\right|=\left|M\frac{Mz_0+1}{M+z_0}\right|<\sqrt{\frac{5}{4}},$$

which is a contradiction. Thus $\operatorname{Re} q(z)>0,\quad z\in U$, and h is a convex function, consequently Lemma 2.3 is applicable and we get $p(z)\prec q(z)$.
According to Lemma 2.4 we have

$$q(z)=\frac{H(z)}{\displaystyle\int_0^z H(t)t^{-1}dt},$$

where $H(z)=\displaystyle\int_0^z\frac{h(t)-1}{t}dt=z\left(1+\frac{z}{M}\right)^{M^2-1}$.
The subordination $p\prec q$ implies that

$$\operatorname{Re} p(z)>\inf_{|z|<1} q(z)=\inf_{|z|<1}\frac{H(z)}{\displaystyle\int_0^z H(t)t^{-1}dt}=\inf_{\theta\in[-\pi,\pi]}\frac{H(e^{i\theta})}{\displaystyle\int_0^{e^{i\theta}} H(s)s^{-1}ds}. \tag{3.3}$$

On the other hand we have

$$\inf_{\theta\in[-\pi,\pi]}\frac{H(e^{i\theta})}{\displaystyle\int_0^{e^{i\theta}} H(s)s^{-1}ds}=\inf_{\theta\in[-\pi,\pi]}\frac{1}{\displaystyle\int_0^1\left(\frac{M+te^{i\theta}}{M+e^{i\theta}}\right)^{M^2-1}dt}. \tag{3.4}$$

A simple calculation leads to

$$\operatorname{Re}\frac{1}{\dfrac{M+te^{i\theta}}{M+e^{i\theta}}}=\operatorname{Re}\frac{M+e^{i\theta}}{M+te^{i\theta}}\ge\frac{M-1}{M-t},\ t\in[0,1],\ \theta\in[-\pi,\pi].$$

This inequality implies

$$\left(\operatorname{Re}\frac{1}{\dfrac{M+te^{i\theta}}{M+e^{i\theta}}}\right)^{\frac14}\ge\left(\frac{M-1}{M-t}\right)^{\frac14},\ t\in[0,1],\ \theta\in[-\pi,\pi].$$

Putting $x+iy=\dfrac{1}{\dfrac{M+te^{i\theta}}{M+e^{i\theta}}}$ in Lemma 2.5, we infer

$$\operatorname{Re}\frac{1}{\left(\dfrac{M+te^{i\theta}}{M+e^{i\theta}}\right)^{\frac14}}\ge\left(\operatorname{Re}\frac{1}{\dfrac{M+te^{i\theta}}{M+e^{i\theta}}}\right)^{\frac14}\ge\left(\frac{M-1}{M-t}\right)^{\frac14},\ t\in[0,1],\ \theta\in[-\pi,\pi].$$

Since $M^2-1=\frac{1}{4}$, we get

$$\operatorname{Re}\frac{1}{\left(\dfrac{M+te^{i\theta}}{M+e^{i\theta}}\right)^{M^2-1}} \geq \left(\frac{M-1}{M-t}\right)^{M^2-1}, \ t\in[0,1],\ \theta\in[-\pi,\pi].$$

Now we can apply Lemma 2.6 and it follows that

$$\operatorname{Re}\frac{1}{\displaystyle\int_0^1\left(\frac{M+te^{i\theta}}{M+e^{i\theta}}\right)^{M^2-1}dt} \geq \frac{1}{\displaystyle\int_0^1\left(\frac{M-t}{M-1}\right)^{M^2-1}dt},\ \theta\in[-\pi,\pi]. \tag{3.5}$$

Finally (3.3), (3.4) and (3.5) imply

$$\operatorname{Re}p(z) \geq \frac{1}{\displaystyle\int_0^1\left(\frac{M-t}{M-1}\right)^{M^2-1}dt} = \frac{5}{4}\,\frac{\left(\sqrt{\frac{5}{4}}-1\right)^{\frac{1}{4}}}{\left(\sqrt{\frac{5}{4}}\right)^{\frac{5}{4}}-\left(\sqrt{\frac{5}{4}}-1\right)^{\frac{5}{4}}}. \qquad \square$$

Theorem 3.4. *We have $S^{***}\subset\mathcal{K}$.*

Proof. Let f be a function from the class S^{***}.
We will prove that $p(z)=1+\frac{zf''(z)}{f'(z)}>0,\quad z\in U$.
It is easily seen that

$$1-\frac{zf''(z)}{f'(z)}=2-p(z),$$

and consequently the following equivalence holds

$$\left|1-\frac{zf''(z)}{f'(z)}\right|<\sqrt{\frac{5}{4}},\ z\in U \Leftrightarrow |2-p(z)|<\sqrt{\frac{5}{4}},\ z\in U. \tag{3.6}$$

If the condition $p(z)=1+\frac{zf''(z)}{f'(z)}>0,\quad z\in U$, does not hold, then according to the Miller-Mocanu lemma (Lemma 2.2) there is a point $z_0\in U$, and there are two real numbers $x,y\in\mathbb{R}$ such that

$$p(z_0)=ix,$$

and

$$z_0p'(z_0)=y\leq-\frac{1+x^2}{2}.$$

These equalities imply

$$|2-p(z_0)|=|2-ix|=\sqrt{4+x^2}>\sqrt{\frac{5}{4}}.$$

This inequality contradicts (3.6), and consequently

$$\operatorname{Re}\,p(z)=\operatorname{Re}\,\left(1+\frac{zf''(z)}{f'(z)}\right)>0,\quad z\in U$$

holds. $\square$

Theorem 3.5. *If* $f \in S^{***}$, *then* $\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4}$, $z \in U$.

Proof. The inequality $\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4}$, $z \in U$ is equivalent to

$$p(z) = \frac{zf'(z)}{f(z)} \prec \sqrt{\frac{1+z}{1-z}} = q(z), \quad z \in U. \tag{3.7}$$

We will prove the subordination (3.7) using again the Miller-Mocanu lemma.
If the subordination (3.7) does not hold, then according to Lemma 2.1 there are two points $z_0 \in U$ and $\zeta_0 = e^{i\theta} \in \partial U$, and a real number $m \in [1, \infty)$, such that

$$p(z_0) = q(\zeta_0) = q(e^{i\theta}) = \sqrt{\frac{1+e^{i\theta}}{1-e^{i\theta}}} = \left(\cos\frac{\pi}{4} \pm i \sin\frac{\pi}{4}\right) x,$$

where $x = \sqrt{|\cot \frac{\theta}{2}|}$, and

$$\frac{z_0 p'(z_0)}{p(z_0)} = m\frac{\zeta_0 q'(\zeta_0)}{q(\zeta_0)} = m\frac{e^{i\theta}}{1-e^{2i\theta}}.$$

According to (3.6) the function f belongs to the class S^{***} if and only if

$$\left|2 - p(z) - \frac{zp'(z)}{p(z)}\right| < \sqrt{\frac{5}{4}}, \ z \in U. \tag{3.8}$$

On the other hand we have

$$\left|2 - p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right| = \left|2 - q(\zeta_0) - m\frac{\zeta_0 q'(\zeta_0)}{q(\zeta_0)}\right| = \left|2 - \sqrt{\frac{1+e^{i\theta}}{1-e^{i\theta}}} - m\frac{e^{i\theta}}{1-e^{2i\theta}}\right|$$
$$= \left|2 - \sqrt{\frac{1+e^{i\theta}}{1-e^{i\theta}}} - m\frac{i}{2\sin\theta}\right| = \left|2 - \sqrt{i\cot\frac{\theta}{2}} - m\frac{i}{2\sin\theta}\right|, \ \theta \in [-\pi, \pi].$$

Denoting $x = \sqrt{|\cot \frac{\theta}{2}|}$, it follows that $x \in (0, \infty)$, and in case $\theta \in [-\pi, 0]$, we have

$$\left|2 - p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right| = \left|2 - \left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) x + im\frac{x^4+1}{4x^2}\right|$$
$$= \sqrt{\left(2 - \frac{x}{\sqrt{2}}\right)^2 + \left(\frac{x}{\sqrt{2}} + m\frac{x^4+1}{4x^2}\right)^2}. \tag{3.9}$$

If $\theta \in [0, \pi]$, then

$$\left|2 - p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right| = \left|2 - \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) x - im\frac{x^4+1}{4x^2}\right|$$
$$= \sqrt{\left(2 - \frac{x}{\sqrt{2}}\right)^2 + \left(\frac{x}{\sqrt{2}} + m\frac{x^4+1}{4x^2}\right)^2}. \tag{3.10}$$

Thus we get

$$\left|2 - p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right| = \sqrt{\left(2 - \tfrac{x}{\sqrt{2}}\right)^2 + \left(\tfrac{x}{\sqrt{2}} + m\tfrac{x^4+1}{4x^2}\right)^2}$$
$$\geq \sqrt{\left(2 - \tfrac{x}{\sqrt{2}}\right)^2 + \left(\tfrac{x}{\sqrt{2}} + \tfrac{x^4+1}{4x^2}\right)^2}. \quad (3.11)$$

The inequality between the arithmetic and geometric means implies

$$\frac{x}{\sqrt{2}} + \frac{x^4+1}{4x^2} = \frac{x}{2\sqrt{2}} + \frac{x}{2\sqrt{2}} + \frac{x^2}{4} + \frac{1}{8x^2} + \frac{1}{8x^2} \geq \frac{5}{2^{\frac{6}{5}}} > 2. \quad (3.12)$$

Finally (3.11) and (3.12) imply that

$$\left|2 - p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right| > \sqrt{\frac{5}{4}}.$$

This inequality contradicts (3.8) and consequently

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4}, \ z \in U.$$ □

Theorem 3.6. *If $f \in S^{***}$, then $|\arg f'(z)| < \frac{\pi}{4},\ z \in U$.*

Proof. The inequality $|\arg f'(z)| < \frac{\pi}{4},\ z \in U$ is equivalent to

$$f'(z) \prec \sqrt{\frac{1+z}{1-z}} = q(z), \quad z \in U. \quad (3.13)$$

If the subordination (3.13) does not hold, then according to Lemma 2.1 there are two points $z_0 \in U$ and $\zeta_0 = e^{i\theta} \in \partial U$, and a real number $m \in [1, \infty)$, such that

$$f'(z_0) = q(\zeta_0) = q(e^{i\theta}) = \sqrt{\frac{1+e^{i\theta}}{1-e^{i\theta}}},$$

$$\frac{z_0 f''(z_0)}{f'(z_0)} = m\frac{\zeta_0 q'(\zeta_0)}{q(\zeta_0)} = m\frac{e^{i\theta}}{1-e^{2i\theta}} = \frac{im}{2\sin\theta}.$$

Thus we get

$$\left|1 - \frac{z_0 f''(z_0)}{f'(z_0)}\right| = \left|1 - \frac{im}{2\sin\theta}\right| = \sqrt{1 + \left(\frac{m}{2\sin\theta}\right)^2} \geq \sqrt{1 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}}.$$

This inequality contradicts $f \in S^{***}$. The contradiction implies that the subordination (3.13) holds, and the proof is done. □

Now we are able to prove the result proposed in the Introduction regarding the composition of functions.

Theorem 3.7. *If $f, g \in S^{***}$, and $r_0 = \sup\{r \in (0,1] \big| f(U(r)) \subset U\}$, then $f \circ g$ will be starlike in $U(r_0)$.*

Proof. We have
$$\frac{z(f\circ g)'(z)}{(f\circ g)(z)} = \frac{zf'(g(z))}{f(g(z))}f'(z). \tag{3.14}$$
If $f, g \in S^{***}$, then Theorem 3.5 and Theorem 3.6 imply the inequalities
$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4},\ z\in U,$$
and
$$|\arg f'(z)| < \frac{\pi}{4},\ z\in U.$$
The equality (3.14) implies that
$$\arg\frac{z(f\circ g)'(z)}{(f\circ g)(z)} = \arg\frac{zf'(g(z))}{f(g(z))} + \arg f'(z).$$
Thus we get
$$\left|\arg\frac{z(f\circ g)'(z)}{(f\circ g)(z)}\right| \le \left|\arg\frac{zf'(g(z))}{f(g(z))}\right| + |\arg f'(z)| \le \frac{\pi}{2},\quad z\in U(r_0).$$
This inequality means that
$$\operatorname{Re}\frac{z(f\circ g)'(z)}{(f\circ g)(z)} > 0,\ z\in U(r_0),$$
and consequently $f\circ g$ is starlike in $U(r_0)$. □

References

[1] Miller, S.S., Mocanu, P.T., *Differential Subordinations Theory and Applications*, Marcel Dekker, New York, Basel, 2000.

[2] Mocanu, P.T., Bulboacă, T., Sălăgean, G.S., *Teoria Geometrică a Funcţiilor Univalente,* Ed. a II-a, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2006, 460+9 pag., ISBN 973-686-959-8 (Romanian only).

Olga Engel
Babeş-Bolyai University
Faculty of Mathematics and Computer Sciences
1, Kogălniceanu Street
400084 Cluj-Napoca, Romania
e-mail: engel_olga@hotmail.com

Róbert Szász
Sapientia Hungarian University of Transilvania
Faculty of Tehnical and Human Sciences
1C, Corunca, 540485 Târgu-Mureş, Romania
e-mail: szasz_robert2001@yahoo.com

Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 2, 147–153

On a subclass of analytic functions for operator on a Hilbert space

Sayali Joshi, Santosh B. Joshi and Ram Mohapatra

Abstract. In this paper we introduce and study a subclass of analytic functions for operators on a Hilbert space in the open unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$. We have established coefficient estimates, distortion theorem for this subclass, and also an application to operators based on fractional calculus for this class is investigated.

Mathematics Subject Classification (2010): 30C45.

Keywords: Univalent function, coefficient estimates, distortion theorem.

1. Introduction

Let A denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n \; z^n \tag{1.1}$$

in the open unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S denote the subclass of A, consisting of functions of the form (1.1) which are normalised and univalent in U.

A function $f \in A$ is said to be starlike of order δ $(0 \leq \delta < 1)$ if and only if

$$\mathrm{Re}\, \frac{zf^{'}(z)}{f(z)} > \delta, \;\; z \in \mathrm{U}. \tag{1.2}$$

Also, a function $f \in A$ is said to be convex of order δ $(0 \leq \delta < 1)$ if and only if

$$\mathrm{Re} \left[1 + \frac{zf^{''}(z)}{f'(z)} \right] > \delta, \;\; z \in \mathrm{U}. \tag{1.3}$$

We denote by $S^*(\delta)$ and $K(\delta)$ respectively the classes of functions in S, which are starlike and convex of order δ in U. The subclass $S^*(\delta)$ was introduced by Robertson [7] and studied further by Schild [8], MacGregor [4], and others.

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n \ z^n, \ a_n \geq 0. \tag{1.4}$$

We begin by setting

$$F_\lambda(z) = (1-\lambda)f(z) + \lambda z f'(z), \quad 0 \leq \lambda \leq 1, \quad f \in T, \tag{1.5}$$

so that

$$F_\lambda(z) = z - \sum_{n=2}^{\infty} [1 + \lambda(n-1)] a_n \ z^n. \tag{1.6}$$

A function $f \in S$ is said to be in the class $S_\lambda(\alpha, \beta, \mu)$ if it satisfies

$$\left| \frac{\dfrac{zF_\lambda'(z)}{F_\lambda(z)} - 1}{\mu \dfrac{zF_\lambda'(z)}{F_\lambda(z)} + 1 - (1+\mu)\alpha} \right| < \beta \ , \ z \in \mathrm{U}, \tag{1.7}$$

where $0 \leq \alpha < 1$, $0 < \beta \leq 1$ and $0 \leq \mu \leq 1$.

Let us define

$$S_\lambda^*(\alpha, \beta, \mu) = S_\lambda(\alpha, \beta, \mu) \cap T. \tag{1.8}$$

The study of various subclasses of S and other related work has been done by Silverman [9], Gupta and Jain [3], Owa and Aouf [6].

Let H be a complex Hilbert space and A be an operator on H. For an analytic function f defined on U, we denote by $f(A)$ the operator on H defined by the well known *Riesz-Dunford integral*

$$f(A) = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z)(zI - A)^{-1} dz \ \ , \tag{1.9}$$

where I is the identity operator on H, $\mathcal{C}$ is a positively oriented simple closed contour lying in U and containing the spectrum of A on the interior of the domain. The conjugate operator of A is denoted by A^*.

A function given by (1.4) is in the class $S_\lambda^*(\alpha, \beta, \mu; A)$ if it satisfies the condition

$$||AF_\lambda'(A) - F_\lambda(A)|| < \beta \, ||\mu \ A \ F_\lambda'(A) + F_\lambda(A) - (1+\mu)\alpha F_\lambda(A)|| \tag{1.10}$$

with the same constraints as α, β and μ, given in (1.7) and for all A with $||A|| < 1, \ A \neq \theta$, where θ is the zero operator on H. Such type of work was earlier done by Fan [2], Xiaopei [10], etc.

In the present paper we have established coefficient estimates, distortion theorem for $S_\lambda^*(\alpha, \beta, \mu; A)$ and further we consider application to a class of operators defined through fractional calculus.

2. Main Results

Theorem 2.1. *A function f be given by* (1.4) *is in the class $S^*_\lambda(\alpha,\beta,\mu;A)$ for all proper contraction A with $A\neq\theta$ if and only if*

$$\sum_{n=2}^{\infty}\{(n-1)+\beta[1+\mu n-(1+\mu)\alpha]\}a_n \leq \beta(1+\mu)(1-\alpha), \tag{2.1}$$

for $0\leq\alpha<1,\ 0<\beta\leq 1,\ 0\leq\mu\leq 1.$

The result is best possible for

$$f(z)=z-\frac{\beta(1+\mu)(1-\alpha)}{(n-1)+\beta[1+\mu\ n-(1+\mu)\alpha]}\ z^n, \quad n\in\mathbb{N}\setminus\{1\} \tag{2.2}$$

Proof. Assuming that (2.1) holds, we deduce that

$$||AF'_\lambda(A)-F_\lambda(A)||-\beta\ ||\mu\ A\ F'_\lambda(A)+F_\lambda(A)-(1+\mu)\alpha\ F_\lambda(A)||$$

$$=\|\sum_{n=2}^{\infty}(n-1)a_nA^n\|-\beta\|(1+\mu)(1-\alpha)A^n-\sum_{n=2}^{\infty}\{1+\mu\ n-(1+\mu)\alpha\}a_n\ A^n\|$$

$$\leq\sum_{n=2}^{\infty}\{(n-1)+\beta\ [1+\mu\ n-(1+\mu)\alpha]\}a_n-\beta(1+\mu)(1-\alpha)\leq 0\ ,$$

hence, f is in the class $S^*_\lambda(\alpha,\beta,\mu;A)$.

Conversely, if we suppose that f belongs to $S^*_\lambda(\alpha,\beta,\mu;A)$, then

$$||AF'_\lambda(A)-F_\lambda(A)||<\beta\ ||\mu\ A\ F'_\lambda(A)+F_\lambda(A)-(1+\mu)\alpha\ F_\lambda(A)||\ ,$$

therefore

$$||\sum_{n=2}^{\infty}(n-1)a_nA^n||\leq\beta\ ||(1+\mu)(1-\alpha)-\sum_{n=2}^{\infty}\{\mu\ n+1-(1+\mu)\alpha\}a_n\ A^n||\ .$$

Selecting $A=eI$ $(0<e<1)$ in the above inequality, we get

$$\frac{\sum_{n=2}^{\infty}(n-1)a_ne^n}{(1+\mu)(1-\alpha)-\sum_{n=2}^{\infty}\{\mu\ n+1-(1+\mu)\alpha\}}<\beta. \tag{2.3}$$

Upon clearing denominator in (2.3) and letting $e\to 1$ $(0<e<1)$, we get

$$\sum_{n=2}^{\infty}(n-1)a_n\leq\beta(1+\mu)(1-\alpha)-\beta\sum_{n=2}^{\infty}\{\mu\ n+1-(1+\mu)\alpha\}a_n,$$

which implies that

$$\sum_{n=2}^{\infty}\{(n-1)+\beta[1+\mu n-(1+\mu)\alpha]\}a_n\leq\beta(1+\mu)(1-\alpha),$$

and this completes the proof of our theorem. □

Corollary 1.1. *If a function f given by (1.4) is in the class $S^*_\lambda(\alpha,\beta,\mu;A)$, then*

$$a_n \leq \frac{\beta(1+\mu)(1-\alpha)}{(n-1)+\beta[1+\mu\, n-(1+\mu)\alpha]}\ , \quad n=2,3,4,\ldots \tag{2.4}$$

Theorem 2.2. *If the function f given by (1.4) is in the class $S^*_\lambda(\alpha,\beta,\mu;A)$ for $0\leq \alpha<1,\ 0<\beta\leq 1,\ 0\leq\mu\leq 1,\ ||A||<1$ and $A\neq\theta$, then*

$$||A|| - \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}\ \ ||A||^2 \leq ||f(A)||$$

$$\leq ||A|| + \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}\ \ ||A||^2. \tag{2.5}$$

The result is sharp for the function

$$f(z)=z-\frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}\ \ z^n. \tag{2.6}$$

Proof. In view of Theorem 2.1, we have

$$1+\beta[(1+2\mu)-(1+2\mu)\alpha]\sum_{n=2}^{\infty}a_n$$

$$\leq \sum_{n=2}^{\infty}\{(n-1)+\beta[1+\mu n-(1+2\mu)\alpha]\}a_n \leq \beta(1+2\mu)(1-\alpha),$$

which gives us

$$\sum_{n=2}^{\infty}a_n \leq \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}\ . \tag{2.7}$$

Hence, we have

$$||f(A)|| \geq ||A|| - ||A||^2\sum_{n=2}^{\infty}a_n$$

$$\geq ||A|| - \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}||A||^2,$$

and

$$||f(A)|| \leq ||A|| + ||A||^2\sum_{n=2}^{\infty}a_n$$

$$\leq ||A|| + \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}||A||^2,$$

which completes our proof. □

Theorem 2.3. *Let $f_1(z)=z$, and*

$$f_n(z)=z-\frac{\beta(1+\mu)(1-\alpha)}{(n-1)+\beta[(1+\mu\ n)\ -(1+\mu)\alpha]}\ \ z^n, \qquad n\geq 2. \tag{2.8}$$

Then, any function f of the form (1.4) *is in the class* $S_\lambda^*(\alpha,\beta,\mu;A)$ *if and only if it can be expressed as,*

$$f(z)=\sum_{n=1}^{\infty}\lambda_n\ f_n(z)\ ,\quad with\quad \lambda_n\geq 0\ ,\quad \sum_{n=1}^{\infty}\lambda_n=1. \tag{2.9}$$

Proof. First, let us assume that

$$f(z)=\sum_{n=1}^{\infty}\lambda_n\ f_n(z)=z-\sum_{n=2}^{\infty}\frac{\beta(1+\mu)(1-\alpha)}{(n-1)+\beta[(1+\mu\ n)\ -(1+\mu)\alpha]}\ \lambda_n\ z^n.$$

Then, we have

$$\sum_{n=2}^{\infty}\frac{(n-1)+\beta[(1+\mu\ n)\ -(1+\mu)\alpha]}{\beta(1+\mu)(1-\alpha)}\ \lambda_n\ \frac{\beta(1+\mu)(1-\alpha)}{(n-1)+\beta[(1+\mu\ n)\ -(1+\mu)\alpha]}$$

$$=\sum_{n=2}^{\infty}\lambda_n=1-\lambda_1\leq 1\ ,$$

hence $f\in S_\lambda^*(\alpha,\beta,\mu;A)$.

Conversely, let us assume that the function f given by (1.4) is in the class $S_\lambda^*(\alpha,\beta,\mu;A)$. Then, from Corollary 1.1 we get

$$a_n\leq\frac{\beta(1+\mu)(1-\alpha)}{(n-1)+\beta[1+\mu\ n-(1+\mu)\alpha]}\ .$$

We may set

$$\lambda_n=\frac{(n-1)+\beta[1+\mu\ n-(1+\mu)\alpha]}{\beta(1+\mu)(1-\alpha)}\ a_n\ ,$$

and

$$\lambda_1=1-\sum_{n=2}^{\infty}\lambda_n\ ,$$

hence it is easy to check that f can be expressed by (2.9), and this completes the proof of Theorem 2.3. □

3. Distortion Theorem involving Fractional Calculus

In this section we shall prove distortion theorem for function belonging to the class $S_\lambda^*(\alpha,\beta,\mu;A)$, and each of these results would involve operators of fractional calculus which are defined as follows (for details, see [5]).

Definition 3.1. *The fractional integral operator of order k associated with a function f is defined by*

$$D_A^{-k}f(A)=\frac{1}{\Gamma(k)}\int_0^1 A^k\ f(tA)\ (1-t)^{k-1}dt\ ,$$

where $k>0$ and f is an analytic function in a simply connected region of the complex plane containing the origin.

Definition 3.2. *The fractional derivative operator of order k associated with a function f is defined by*

$$D_A^k f(A) = \frac{1}{\Gamma(1-k)} g'(A),$$

where

$$g(A) = \int_0^1 A^{(1-k)}\ f(tA)\ (1-t)^{-k} dt\ ,\quad 0 < k < 1\ ,$$

and f is an analytic function in a simply connected region of the complex plane containing the origin.

Theorem 3.3. *If the function f given by (1.4) is in the class $S_\lambda^*(\alpha,\beta,\mu;A)$ for $0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ 0 \leq \mu \leq 1$, then*

$$||D_A^{-k} f(A)|| \geq \frac{||A||^k}{\Gamma(k+2)} - \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}\ \frac{||A||^{k+2}}{\Gamma(k+2)}\ ,$$

and

$$||D_A^{-k} f(A)|| \leq \frac{||A||^k}{\Gamma(k+2)} + \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}\ \frac{||A||^{k+2}}{\Gamma(k+2)}\ .$$

Proof. If we consider

$$F(A) = \Gamma(k+2)A^{-k}D_A^{-k} f(A)$$

$$= A - \sum_{n=1}^{\infty} \frac{\Gamma(n+2)\ \Gamma(k+2)}{\Gamma(n+k+2)}\ a_{n+1}A^{n+1} = A - \sum_{n=2}^{\infty} B_n A^n\ ,$$

where $B_n = \dfrac{\Gamma(n+1)\ \Gamma(k+2)}{\Gamma(n+k+1)}\ a_n$, then we obtain that

$$\sum_{n=2}^{\infty}\{(n-1)+\beta[1+\mu\ n-(1+\mu)\alpha]\}B_n$$

$$\leq \sum_{n=2}^{\infty}\{(n-1)+\beta[1+\mu\ n-(1+\mu)\alpha]\}a_n \leq \beta(1+\mu)(1-\alpha),$$

as $0 < \dfrac{\Gamma(n+1)\ \Gamma(k+2)}{\Gamma(n+k+1)} < 1$, hence F belongs to $S_\lambda^*(\alpha,\beta,\mu;A)$.

Therefore, by Theorem 2.2 we deduce that

$$||D_A^{-k} f(A)|| \leq \frac{||A^{k+1}||}{\Gamma(k+2)} + \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}\ \frac{||A^{k+2}||}{\Gamma(k+2)}\ .$$

and

$$||D_A^{-k} f(A)|| \geq \frac{||A^{k+1}||}{\Gamma(k+2)} - \frac{\beta(1+2\mu)(1-\alpha)}{1+\beta[(1+2\mu)\ -(1+2\mu)\alpha]}\ \frac{||A^{k+2}||}{\Gamma(k+2)}.$$

Note that $(A^{\frac{1}{q}}) * A^{\frac{1}{q}} = A^{\frac{1}{q}}(A^{\frac{1}{q*}}); q \in N$ and by Corollary 3.8 [11] we have $||A^m|| = ||A||^m$, where m is rational number and ‘ * ’ is the *Hadamard product* or convolution product of two analytic functions. When s is any irrational number, we choose a single-valued branch of z^s and a single valued branch of z^{k_n} (k_n is a sequence of rational numbers) such that $k_n \to s$, as $||A^{k_n}|| = ||A||^{k_n}$, and Lemma 13 [1] allows us to have $||A^{k_n}|| \to ||A^s||$, $||A^{k_n}|| = ||A||^{k_n} \to ||A^s||,\ k_n \to s$.

That is $||A^s|| = ||A||^s$, hence $||A^k|| = ||A||^k,\ k > 0$. □

Acknowledgments. The authors wish to express their sincere thanks to the referee of this paper for several useful comments and suggestions.

References

[1] Dunford, N., Schwartz, J.T., *Linear operators part I. General Theory*, New York-London, 1958.

[2] Fan, K.,*Analytic functions of a proper contraction*, Math. Zeitschr., **160**(1978), 275-290.

[3] Gupta, V.P., Jain, P.K., *A certain classes of univalent functions with negative coefficients*, Bull. Austral. Math. Soc., **14**(1976), 409-416.

[4] MacGregor, T.H., *The radius of convexity for starlike functions of order* $\frac{1}{2}$, Proc. Amer. Math. Soc., **14**(1963), 71-76.

[5] Owa, S., *On the distortion theorem*, Kyungpook Math. J., **18**(1978), 53-59.

[6] Owa, S., Aouf, M.K., *Some applications of fractional calculus operators to classes of univalent functions negative coefficients*, Integral Trans. and Special Functions, **3**(1995), no. 3, 211-220.

[7] Robertson, M.S., *A characterization of the class of starlike univalent functions*, Michigan Math. J., **26**(1979), 65-69.

[8] Schild, A., *On a class of functions schlicht in the unit circle*, Proc. Amer. Math. Soc., **5**(1954), 115-120.

[9] Silverman, H., *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., **51**(1975), 109-116.

[10] Xiaopei, Y., *A subclass of analytic p-valent functions for operator on Hilbert space*, Math. Japon., **40**(1994), no. 2, 303-308.

[11] Xia, D., *Spectral theory of hyponormal operators*, Sci. Tech. Press, Shanghai, 1981, Birkhauser Verlag, Basel-Boston-Stuttgart, 1983, 1-241.

Sayali Joshi
Department of Mathematics, Sanjay Bhokare Group of Institutes, Miraj
Miraj 416410, India
e-mail: joshiss@sbgimiraj.org

Santosh B. Joshi
Department of Mathematics, Walchand College of Engineering
Sangli 416415, India
e-mail: joshisb@hotmail.com

Ram Mohapatra
Department of Mathematics, University of Central Florida
Orlando, F.L. U.S.A.
e-mail: ramm1627@gmail.com

Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 2, 155–162

The norm of pre-Schwarzian derivatives of certain analytic functions with bounded positive real part

Hormoz Rahmatan, Shahram Najafzadeh and Ali Ebadian

Abstract. For real numbers $0 \leq \alpha < 1$ and $\beta > 1$ we define the univalent function in the unit disk Δ which maps Δ on to the strip domain ω with $\alpha < Re\,\omega < \beta$. In this paper we give the best estimates for the norm of the pre-Schwarzian derivative $T_f(z) = \frac{f''(z)}{f'(z)}$ where $\|T_f\| = \sup_{|z|<1} (1-|z|^2) \left|\frac{f''(z)}{f'(z)}\right|$.

Mathematics Subject Classification (2010): 30C45.

Keywords: Univalent functions, starlike functions, subordination, pre-Schwarzian derivatives.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The subclass of A, consisting of all univalent functions f in Δ is denoted by S. In [5] the authors introduced a new class for certain analytic functions, and they denote by $S(\alpha, \beta)$ the class of functions $f \in A$ which satisfy the inequality

$$\alpha < Re\,\frac{zf'(z)}{f(z)} < \beta, \ (z \in \Delta). \tag{1.2}$$

for some real number $0 \leq \alpha < 1$ and some real number $\beta > 1$. Also, the authors introduced the class $\nu(\alpha, \beta)$ of functions $f \in A$ which satisfy the inequality

$$\alpha < Re \left\{ \left(\frac{z}{f(z)}\right)^2 f'(z) \right\} < \beta, \ (z \in \Delta). \tag{1.3}$$

where $0 \leq \alpha < 1$ and $\beta > 1$.
Let f and g be analytic in Δ. The function f is called to be *subordinate* to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function ω such that $\omega(0) = 0$, $|\omega(z)| < 1$, and $f(z) = g(\omega(z))$ on Δ. The pre-Schwarzian derivative of f is denoted by

$$T_f(z) = \frac{f''(z)}{f'(z)},$$

and we define the norm of T_f by

$$\|T_f\| = \sup_{|z|<1} \left(1 - |z|^2\right) \left|\frac{f''(z)}{f'(z)}\right|.$$

This norm have a significant meaning in the theory of Teichmuller spaces. For a univalent function f, it is well known that $\|T_f\| < 6$, and this estimate is the best possible [3,6]. On the other hand the following result is important to be noted:

Theorem 1.1. *Let f be analytic and locally univalent in Δ. Then,*
(i) if $\|T_f\| \leq 1$ then f is univalent, and
(ii) if $f \in S^*(\alpha)$, then $\|T_f\| \leq 6 - 4\alpha$.

The part (i) is due to Becker [1], and the sharpness of the constants is due to Becker and Pommerenke [2]. The part (ii) is due to Yamashita [8]. The norm estimates for typical subclasses of univalent functions are investigated by many authors like [4,7,8].
In this paper we shall give the best estimate for the norm of pre-Schwarzian derivatives of the class $S(\alpha, \beta)$ and $\nu(\alpha, \beta)$.

2. Main Results

To prove our main results we shall need the Schwartz' lemma.
Now, we define an analytic function $P : \Delta \to \mathbb{C}$ by

$$P(z) = 1 + \frac{\beta - \alpha}{\pi}\, i\, \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z}\right),$$

due to Kuroki and Owa [5]. They proved that p maps conformally Δ onto a convex domain ω with $\alpha < Re\,\omega < \beta$. Using this fact and the definition of subordination, we can directly obtain the following lemmas:

Lemma 2.1. *Let $f \in A$ and $0 \leq \alpha < \alpha < 1 < \beta$. Then, $f \in S(\alpha, \beta)$ if and only if*

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{\beta - \alpha}{\pi}\, i\, \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z}\right),$$

Lemma 2.2. *Let $f \in A$ and $0 \leq \alpha < 1 < \beta$. Then, $f \in \nu(\alpha,\beta)$ if and only if*

$$\left(\frac{z}{f(z)}\right)^2 f^{'}(z) \prec 1 + \frac{\beta-\alpha}{\pi}\, i\, \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z}\right).$$

In this work, first we find norm estimate of the pre-Schwarzian derivative for $f \in S(\alpha,\beta)$, and then we find the norm estimate of the pre-Schwarzian derivative for $f \in \nu(\alpha,\beta)$.

Theorem 2.3. *For $0 \leq \alpha < 1 < \beta$, if $f \in S(\alpha,\beta)$, then*

$$\|T_f\| \leq \frac{2(\beta-\alpha)}{\pi}\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}\right).$$

Proof. For an arbitrary function $f \in S(\alpha,\beta)$, set $g(z) = \dfrac{zf^{'}(z)}{f(z)}$. Then, g is a holomorphic function on Δ satisfying $g(0) = 1$ and

$$g(\Delta) \subset \{\omega \in \mathbb{C} : \alpha < Re\,\omega < \beta\} := H(\alpha,\beta).$$

The univalent map $P(z) = 1 + \dfrac{\beta-\alpha}{\pi}\, i\, \log\left(\dfrac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z}\right)$ on Δ satisfies $P(0) = 1$ and $P(z) = H(\alpha,\beta)$, therefore g is subordinate to P. Thus, there exists a holomorphic function $\omega = \omega_f : \Delta \to \Delta$ with $\omega(0) = 0$ such that,

$$g(z) = (P \circ \omega)(z) = 1 + \frac{\beta-\alpha}{\pi}\, i\, \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} \omega(z)}{1-\omega(z)}\right). \tag{2.1}$$

By the logarithmic differentation of (2.1), we have

$$\log\frac{zf^{'}(z)}{f(z)} = \log\left\{1 + \frac{\beta-\alpha}{\pi}\, i\, \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} \omega(z)}{1-\omega(z)}\right)\right\},$$

and consequently

$$\log z + \log f^{'}(z) - \log f(z) = \log\left\{1 + \frac{\beta-\alpha}{\pi}\, i\, \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} \omega(z)}{1-\omega(z)}\right)\right\}.$$

Hence,

$$\frac{1}{z}+\frac{f''(z)}{f'(z)}-\frac{f^{'}(z)}{f(z)}=$$

$$=\frac{\beta-\alpha}{\pi}\,i\,\frac{-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega^{'}(z)\,(1-\omega(z))+\omega^{'}(z)\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega(z)\right)}{(1-\omega(z))\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega(z)\right)},$$

Then,

$$\frac{f''(z)}{f'(z)}=\frac{\beta-\alpha}{\pi}\,i\,\left(\frac{1}{z}\log\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega(z)}{1-\omega(z)}+\frac{-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega^{'}(z)}{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega(z)}+\frac{\omega^{'}(z)}{1-\omega(z)}\right),$$

and therefore,

$$T_f(z)=\frac{f''(z)}{f'(z)}=$$

$$=\frac{\beta-\alpha}{\pi}\,i\,\left(\frac{1}{z}\log\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega(z)}{1-\omega(z)}+\frac{\omega^{'}(z)\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right)}{(1-\omega(z))\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega(z)\right)}\right).$$

Setting $\omega=id_{\Delta}$, we also have

$$T_{f_{\alpha,\beta}}(z)=\frac{\beta-\alpha}{\pi}\,i\,\left(\frac{1}{z}\log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-z}\right)+\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}}{(1-z)\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z\right)}\right),$$

and we conclude by using of Schwartz' lemma that,

$$\left(1-|z|^2\right)|T_f(z)|\leq\left(1-|z|^2\right)\left|T_{f_{\alpha,\beta}}(z)\right|. \tag{2.2}$$

Thus, we can estimate as follows

$$\left(1-|z|^2\right)|T_f(z)| \leq \frac{\beta-\alpha}{\pi}\left(\frac{1-|z|^2}{|z|}\left|\log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-z}\right)\right|\right.$$

$$\left.+\left(1-|z|^2\right)\left|\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}}{(1-z)\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z\right)}\right|\right).$$

By using of maximum principle we can obtain upper bound of $\|T_f\|$, therefore

$$\lim_{z\to 0}\left(1-|z|^2\right)\left|\frac{\log\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-z}}{z}\right|$$

$$=\lim_{z\to 0}\left(1-|z|^2\right)\cdot\lim_{z\to 0}\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}}{(1-z)\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z\right)}$$

$$=1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}} \tag{2.3}$$

Also, we have

$$\lim_{z\to 0}\left(1-|z|^2\right)\left|\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}}{(1-z)\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z\right)}\right|=1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}, \tag{2.4}$$

hence, by (2.2) and (2.3) combined with (2.4), we conclude

$$\sup\left(1-|z|^2\right)|T_f(z)| \leq \frac{2(\beta-\alpha)}{\pi}\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right),$$

and this completes our proof. □

Theorem 2.4. *For* $0 \leq \alpha < 1 < \beta$, *if* $f \in \nu(\alpha, \beta)$, *then*

$$\|T_f\| \leq \frac{3(\beta-\alpha)}{\pi}\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}\right).$$

Proof. Let $f \in \nu(\alpha, \beta)$, and set $g(z) = \left(\frac{z}{f(z)}\right)^2 f^{'}(z)$. Then, the functiong is a holomorphic function on Δ satisfying $g(0) = 1$ and

$$g(\Delta) \subset \{\omega \in \mathbb{C} : \alpha < Re\,\omega < \beta\} := H(\alpha, \beta).$$

The univalent map $P(z) = 1 + \frac{\beta-\alpha}{\pi}\, i\, \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z}\right)$ on Δ satisfies $P(0) = 1$ and $P(z) = H(\alpha, \beta)$, hence g is subordinate to P. So, there exists a holomorphic function $\omega = \omega_f : \Delta \to \Delta$ with $\omega(0) = 0$ such that

$$g(z) = (P \circ \omega)(z) = 1 + \frac{\beta-\alpha}{\pi}\, i\, \log \frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} \omega(z)}{1-\omega(z)}. \tag{2.5}$$

By the logarithmic differentiation of (2.5) and using the same method as proof of Theorem 2.3, we have

$$2\left(\frac{1}{z} - \frac{f^{'}(z)}{f(z)}\right) + \frac{f''(z)}{f'(z)} =$$

$$= \frac{\beta-\alpha}{\pi}\, i\, \frac{-e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} \omega^{'}(z)\,(1-\omega(z)) + \omega^{'}(z)\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} \omega(z)\right)}{(1-\omega(z))\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} \omega(z)\right)}. \tag{2.6}$$

With (2.1) we have,

$$\frac{zf^{'}(z)}{f(z)} = 1 + \frac{\beta-\alpha}{\pi}\, i\, \log \frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} \omega(z)}{1-\omega(z)},$$

therefore

$$T_f(z) = \frac{f''(z)}{f'(z)} =$$

$$= \frac{\beta-\alpha}{\pi}\, i \left(\frac{2}{z} \log \frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} \omega(z)}{1-\omega(z)} + \frac{\omega'(z)\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}\right)}{(1-\omega(z))\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} \omega(z)\right)} \right)$$

Setting $\omega = id_{\Delta}$, we also have

$$T_{f_{\alpha,\beta}}(z) = \frac{\beta-\alpha}{\pi}\, i \left(\frac{2}{z} \log \frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z} + \frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}}{(1-z)\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z\right)} \right).$$

Therefore,

$$\left(1-|z|^2\right)|T_f(z)| \le \left(1-|z|^2\right)\left|T_{f_{\alpha,\beta}}(z)\right|,$$

hence we have

$$\sup \left(1-|z|^2\right)|T_f(z)| \le \frac{3(\beta-\alpha)}{\pi}\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}\right).$$

This completes the proof of our theorem. □

References

[1] Becker, J., *Lownersche differentialgleichung and quasikonform fortsetzbare schlichte funktionen*, J. Reine Angew. Math, **255**(1972), 23–43.

[2] Becker, J. Pommerenke, Ch., *Schlichtheit-skriterien und jordangebiete*, J. Reine Angew. Math, **354**(1984), 74–94.

[3] Duren, P.L., *Univalent functions*, Springer, New York, 1978.

[4] Kim, Y.C. Sugawa, T., *Norm estimates of the pre-Schwarzian derivatives for certain classes of univalent functions*, Proc. Edinburgh Math. Soc, **49**(2006), 131–143.

[5] Kuroki, K. Owa, S., *Notes on new class for certain analytic functions*, Advances in Mathematics: Scientific. Journal 1, **2**(2012), 127–131.

[6] Miller, S.S, Mocanu, P.T., *Differential subordinations, theory and applications*, Marcel Dekker, 2000.

[7] Okuyama, Y., *The norm estimates of pre-Schwarzian derivatives of spiral-like functions*, Complex Var. Theory Appl, **42**(2000), 225–239.

[8] Yamashita, S., *Norm estimates for function starlike or convex of order alpha*, Hokkaido Mathematical Journal, **28**(1999), 217–230.

Hormoz Rahmatan
Department of Mathematics
Payame Noor University
P. O. Box 19395-3697 Tehran, Iran
e-mail: h.rahmatan@gmail.com

Shahram Najafzadeh
Department of Mathematics
Payame Noor University
P. O. Box 19395-3697 Tehran, Iran
e-mail: najafzadeh1234@yahoo.ie

Ali Ebadian
Department of Mathematics
Payame Noor University
P. O. Box 19395-3697 Tehran, Iran
e-mail: ebadian.ali@gmail.com

Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 2, 163–168

On the starlikeness of iterative integral operators

Parviz Arjomandinia and Rasoul Aghalary

Abstract. The main object of the present paper is to investigate starlikeness of certain integral operators, which are defined here by means of iterative in the open disk $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$ with $R \geq 1$. Also we prove that these result are best possible.

Mathematics Subject Classification (2010): 30C45, 30C80.

Keywords: Integral operator, iterative, univalent and starlike functions.

1. Introduction

Let $\mathcal{A}$ denote the class of normalized analytic functions $f(z)$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ which are in the form

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots .$$

Also, let S and S^* denote the subclasses of $\mathcal{A}$ consisting of the univalent and starlike functions respectively. Studying the geometric properties of certain integral operators were considered by many authors during the last years. For example, some results of integral operator $F_\alpha(z) = \int_0^z (f(t)/t)^\alpha dt$ were obtained by Merkes and Wright [3]. Other type of integral operator such as $G_\alpha(z) = \int_0^z (f'(t))^\alpha dt$ was studied by the authors in [3] and [6]. Recently, the authors in [5] defined integral operators $L^k f(z)$ and $L_k f(z)$ which are iterative and take normalized analytic functions into the class S when restricted to $\mathbb{D}$. In this note, we define two new iterative integral operators $F^n(\gamma)(f(z)), F_n(\gamma)(f(z))$ and investigate the starlikeness of them in $\mathbb{D}$.

2. Integral Operators $F^n(\gamma)(f(z))$ and $F_n(\gamma)(f(z))$

Suppose that $\mathcal{A}_R$ denote the class of normalized analytic functions $f(z)$ in $\mathbb{D}_R$ with radius of convergence R and $R \geq 1$. We recall the generalized Bernardi integral operator $F(\gamma) : \mathcal{A} \to \mathcal{A}$, with $\gamma > -1$ as following (see [4])

$$F(\gamma)(f(z)) = \frac{1+\gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t)dt; \;\; (z \in \mathbb{D}, f \in \mathcal{A}). \tag{2.1}$$

Note that all powers in (2.1) are principal ones.

We now introduce the following two operators defined on $\mathcal{A}_R$ with $R \geq 1$.

Definition 2.1. *For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}_R$, let*

$$F^1(\gamma)(f(z)) = F(\gamma)(f(z)) = z + \sum_{k=2}^{\infty} \frac{1+\gamma}{\gamma+k} a_k z^k$$

$$F^2(\gamma)(f(z)) = F(\gamma)(F(\gamma)(f(z))) = z + \sum_{k=2}^{\infty} \left(\frac{1+\gamma}{\gamma+k}\right)^2 a_k z^k.$$

In general, for $n \in \mathbb{N}$ we define

$$F^n(\gamma)(f(z)) = F(\gamma)(F^{n-1}(\gamma)(f(z))) = z + \sum_{k=2}^{\infty} \left(\frac{1+\gamma}{\gamma+k}\right)^n a_k z^k. \tag{2.2}$$

Definition 2.2. *For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}_R$ we define*

$$F_1(\gamma)(f(z)) = F(\gamma)(f(z)) = z + \sum_{k=2}^{\infty} \frac{1+\gamma}{\gamma+k} a_k z^k,$$

$$\begin{aligned} F_2(\gamma)(f(z)) &= \frac{(1+\gamma)(2+\gamma)}{z^{\gamma+1}} \int_0^z \int_0^{t_2} t_1^{\gamma-1} f(t_1) dt_1 dt_2 \\ &= z + \sum_{k=2}^{\infty} \frac{(1+\gamma)(2+\gamma)}{(\gamma+k)(\gamma+k+1)} a_k z^k \end{aligned} \tag{2.3}$$

and in general we have

$$\begin{aligned} F_n(\gamma)(f(z)) &= \frac{(1+\gamma)(2+\gamma)...(n+\gamma)}{z^{\gamma+n-1}} \int_0^z \int_0^{t_n} \int_0^{t_{n-1}} ... \int_0^{t_2} t_1^{\gamma-1} f(t_1) dt_1 dt_2 ... dt_n \\ &= z + \sum_{k=2}^{\infty} \frac{(1+\gamma)(2+\gamma)...(n+\gamma)}{(\gamma+k)(\gamma+k+1)...(\gamma+k+n-1)} a_k z^k. \end{aligned} \tag{2.4}$$

The aim of this note is to show that for $f \in \mathcal{A}_R$ with $R > 1$ there exists a positive integer N such that for $n \geq N$, $F^n(\gamma)(f(z))$ and $F_n(\gamma)(f(z))$ are starlike. Also, we show that these results are sharp.

To prove our main results, we need each of the following lemmas.

Lemma 2.3. ([2]) If $f \in \mathcal{A}$ satisfies

$$\Re(f'(z) + \alpha z f''(z)) > \frac{\frac{-1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1} \frac{1-t}{1+t}\, dt}{1 - \frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1} \frac{1-t}{1+t}\, dt}; \ (z \in \mathbb{D})$$

for $\alpha \geq \frac{1}{3}$, then $f \in S^*$. The result is sharp.

Theorem 2.4. (Bieberbach's Theorem [1]) If $f \in S$, then $|a_n| \leq n$. The equality holds if and only if f is a rotation of the Koebe function.

3. Main Results

Theorem 3.1. Suppose that $f \in \mathcal{A}_R$, where $R > 1$. There exists a positive integer N such that for every $n \geq N$, $F^n(\gamma)(f(z))$ when restricted to $\mathbb{D}$ is starlike.

Proof. Let $n \in \mathbb{N}, \alpha \geq \frac{1}{3}$ and $f \in \mathcal{A}_R$. From (2.2) we obtain

$$(F^n(\gamma)(f(z)))' = 1 + \sum_{k=2}^{\infty} k \left(\frac{1+\gamma}{\gamma+k}\right)^n a_k z^{k-1}$$

and

$$\alpha z(F^n(\gamma)(f(z)))'' = \sum_{k=2}^{\infty} \frac{\alpha k(k-1)(1+\gamma)^n}{(\gamma+k)^n} a_k z^{k-1}.$$

So we obtain

$$Re\{(F^n(\gamma)(f(z)))' + \alpha z(F^n(\gamma)(f(z)))''\} := 1 + G(z)$$

where

$$G(z) = \sum_{k=2}^{\infty} \frac{k(1+\alpha(k-1))(1+\gamma)^n}{(\gamma+k)^n} Re(a_k z^{k-1}).$$

From last equality we observe that

$$|G(z)| \leq \sum_{k=2}^{\infty} \frac{k(1+\alpha(k-1))(1+\gamma)^n}{(\gamma+k)^n} |a_k|; \ (|z| < 1).$$

Since the radius of convergence of f (i.e. R) is greater than one, so there exists an $\epsilon > 0$ such that $C := \frac{1}{R} + \epsilon < 1$. In view of $R = \frac{1}{\limsup |a_k|^{\frac{1}{k}}}$ and the property of limit superior, there exists $N_1 \in \mathbb{N}, N_1 \geq 3$ such that for every $k \geq N_1$ we have $|a_k| < C^k$. Let $C_1 = \max\{|a_2|, |a_3|, ..., |a_{N_1-1}|\}$, then we obtain

$$\begin{aligned} |G(z)| &\leq (1+\gamma)^n \left(\sum_{k=2}^{N_1-1} \frac{k(1+\alpha(k-1))}{(\gamma+k)^n} C_1 + \sum_{k=N_1}^{\infty} \frac{k(1+\alpha(k-1))}{(\gamma+k)^n} C^k \right) \\ &\leq C_1 M_1 \left(\frac{1+\gamma}{2+\gamma}\right)^n + M_2 \left(\frac{1+\gamma}{\gamma+N_1}\right)^n \end{aligned}$$

where

$$M_1 := \sum_{k=2}^{N_1-1} k(1+\alpha(k-1)), M_2 := \sum_{k=N_1}^{\infty} k(1+\alpha(k-1))C^k < \infty.$$

Now from the last inequality we observe that there exists $N \in \mathbb{N}$ such that for $n \geq N$ we have $|G(z)| < \frac{1}{1-\beta}$, where

$$0 < \beta = \frac{1}{\alpha} \int_0^1 t^{\frac{1}{\alpha}-1} \frac{1-t}{1+t} \, dt < 1.$$

With this N we see that, $1 + G(z) > \frac{-\beta}{1-\beta}$, and by lemma 2.3 we conclude that $F^n(\gamma)(f(z))$ is starlike in $\mathbb{D}$ whenever $n \geq N$. □

Theorem 3.2. Let $f \in \mathcal{A}_R$ with $R > 1$. There exists $N \in \mathbb{N}$ such that for every $n \geq N, F_n(\gamma)(f(z))$ is starlike in $\mathbb{D}$.

Proof. Let $n \in \mathbb{N}$ and $f \in \mathcal{A}_R$. From (2.4) we have

$$(F_n(\gamma)(f(z)))' = 1 + \sum_{k=2}^{\infty} \frac{(1+\gamma)(2+\gamma)...(n+\gamma)ka_k}{(\gamma+k)(\gamma+k+1)...(\gamma+k+n-1)} z^{k-1}$$

and

$$\alpha z(F_n(\gamma)(f(z)))'' = \sum_{k=2}^{\infty} \frac{\alpha(1+\gamma)(2+\gamma)...(n+\gamma)k(k-1)a_k}{(\gamma+k)(\gamma+k+1)...(\gamma+k+n-1)} z^{k-1}.$$

So we obtain

$$\Re\{(F_n(\gamma)(f(z)))' + \alpha z(F_n(\gamma)(f(z)))''\} = 1 + H(z),$$

where

$$H(z) = \sum_{k=2}^{\infty} \frac{(1+\gamma)(2+\gamma)...(n+\gamma)k(1+\alpha(k-1))}{(\gamma+k)(\gamma+k+1)...(\gamma+k+n-1)} Re(a_k z^{k-1}).$$

Now the last equality implies that

$$|H(z)| \leq \sum_{k=2}^{\infty} \frac{(1+\gamma)(2+\gamma)...(n+\gamma)k(1+\alpha(k-1))}{(\gamma+k)(\gamma+k+1)...(\gamma+k+n-1)} |a_k|; \ (z \in \mathbb{D}).$$

Since the radius of convergence of f is greater than one, hence there exists an $\epsilon > 0$ such that $B = \frac{1}{R} + \epsilon < 1$. Now using $\limsup |a_k|^{\frac{1}{k}} = \frac{1}{R}$ and the property of limit superior, there exists $N_1 \in \mathbb{N}, N_1 \geq 3$ such that for $k \geq N_1$ we have $|a_k| < B^k$. Let $C_1' = \max\{|a_2|, |a_3|, ..., |a_{N_1-1}|\}$, then we obtain

$$\begin{aligned} |H(z)| &\leq C_1' \sum_{k=2}^{N_1-1} \frac{(1+\gamma)k(1+\alpha(k-1))}{\gamma+n+1} + \sum_{k=N_1}^{\infty} A_n k(1+\alpha(k-1))B^k \\ &= C_1' M_1' \left(\frac{1+\gamma}{n+\gamma+1}\right) + A_n M_2', \end{aligned} \tag{3.1}$$

where

$$M_1' := \sum_{k=2}^{N_1-1} k(1+\alpha(k-1)), M_2' := \sum_{k=N_1}^{\infty} k(1+\alpha(k-1))B^k < \infty$$

and

$$A_n = \frac{(1+\gamma)(2+\gamma)...(n+\gamma)}{(\gamma+N_1)(\gamma+N_1+1)...(\gamma+N_1+n-1)}.$$

It is easy to see that $\lim_{n\to\infty} A_n = 0$. Using this fact, the relation (3.1) shows that there exists $N \in \mathbb{N}$ such that for $n \geq N$ we have $|H(z)| < \frac{1}{1-\beta}$, where

$$0 < \beta = \frac{1}{\alpha} \int_0^1 t^{\frac{1}{\alpha}-1} \frac{1-t}{1+t}\, dt < 1.$$

With this N we see that, $1 + H(z) > \frac{-\beta}{1-\beta}$ and $F_n(\gamma)(f(z))$ is starlike in $\mathbb{D}$ for $n \geq N$. $\square$

Now we shall see that the radius of convergence of f (*i.e.* $R > 1$) is best possible. To this end, let

$$L(z) = z + \sum_{k=2}^{\infty} a_k z^k, \text{ where } a_k = \begin{cases} (l+\gamma)^l, & \text{if } k = (3+\lfloor\gamma\rfloor)^l, l \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Since $\limsup |a_k|^{\frac{1}{k}} = 1$, so the radius of convergence of $L(z)$ is one. In fact, this shows that $L(z) \in \mathcal{A}$. We then show that $F^n(\gamma)(L(z))$ and $F_n(\gamma)(L(z))$ are not starlike in $\mathbb{D}$ for every positive integer n.

Theorem 3.3. $F^n(\gamma)(L(z))$ is not starlike in $\mathbb{D}$ for every $n \in \mathbb{N}$.

Proof. For fixed $n \in \mathbb{N}$, we have

$$F^n(\gamma)(L(z)) = z + \sum_{k=2}^{\infty} b_k z^k, \text{ with } b_k = \begin{cases} \frac{(1+\gamma)^n(l+\gamma)^l}{(k+\gamma)^n}, & \text{if } k = (3+\lfloor\gamma\rfloor)^l, l \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\gamma > -1$ and $k = (3+\lfloor\gamma\rfloor)^l$, then we obtain

$$b_k = \frac{(1+\gamma)^n(l+\gamma)^l}{((3+\lfloor\gamma\rfloor)^l+\gamma)^n} > \left(\frac{1+\gamma}{2}\right)^n \left(\frac{l+\gamma}{(3+\lfloor\gamma\rfloor)^n}\right)^l.$$

Since

$$\lim_{l\to\infty} \left(\frac{1+\gamma}{2}\right)^{\frac{n}{l}} \frac{l+\gamma}{(3+\lfloor\gamma\rfloor)^n} = \infty,$$

there is $N \in \mathbb{N}$ such that for $l \geq N$ we have

$$\left(\frac{1+\gamma}{2}\right)^{\frac{n}{l}} \frac{l+\gamma}{(3+\lfloor\gamma\rfloor)^n} > 3+\lfloor\gamma\rfloor,$$

or equivalently

$$\left(\frac{1+\gamma}{2}\right)^n \left(\frac{l+\gamma}{(3+\lfloor\gamma\rfloor)^n}\right)^l > (3+\lfloor\gamma\rfloor)^l = k.$$

Therefore we conclude that $b_k > k$, and by theorem 2.4 $F^n(\gamma)(L(z))$ is not starlike in $\mathbb{D}$. Since $n \in \mathbb{N}$ is arbitrary, the proof is complete. □

Theorem 3.4. $F_n(\gamma)(L(z))$ is not starlike in $\mathbb{D}$ for every $n \in \mathbb{N}$.

Proof. For a fixed $n \in \mathbb{N}$ we obtain $F_n(\gamma)(L(z)) = z + \sum_{k=2}^{\infty} c_k z^k$, where

$$c_k = \begin{cases} \frac{(1+\gamma)(2+\gamma)\dots(n+\gamma)(l+\gamma)^l}{(\gamma+k)(\gamma+k+1)\dots(\gamma+k+n-1)}, & \text{if } k = (3+\lfloor\gamma\rfloor)^l, l \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

There is $N_1 \in \mathbb{N}$ such that for $l \geq N_1$ we have

$$0 < \gamma + (3+\lfloor\gamma\rfloor)^l + n - 1 < 2\,(3+\lfloor\gamma\rfloor)^l.$$

Now for $\gamma > -1, k = (3+\lfloor\gamma\rfloor)^l$ and $l \geq N_1$ we obtain

$$\begin{aligned} c_k &\geq \frac{(1+\gamma)(2+\gamma)\dots(n+\gamma)(l+\gamma)^l}{(\gamma+(3+\lfloor\gamma\rfloor)^l+n-1)^n} \\ &> \frac{(1+\gamma)(2+\gamma)\dots(n+\gamma)}{2^n} \left(\frac{l+\gamma}{(3+\lfloor\gamma\rfloor)^n}\right)^l. \end{aligned} \tag{3.3}$$

As in the proof of theorem 3.3, it is easy to see that there is $N \in \mathbb{N}, N \geq N_1$ such that for $l \geq N$ we have

$$\frac{(1+\gamma)(2+\gamma)...(n+\gamma)}{2^n}\left(\frac{l+\gamma}{(3+\lfloor\gamma\rfloor)^n}\right)^l > (3+\lfloor\gamma\rfloor)^l = k.$$

Hence for $l \geq N$ we have $c_k > k$, and $F_n(\gamma)(L(z))$ is not starlike in $\mathbb{D}$. This completes the proof. □

References

[1] Branges, L. de, *A proof of the Bieberbach conjecture*, Acta Math., **154**(1985), 137-152.

[2] Fournier, R., Ruscheweyh, S., *On two extremal problems related to univalent functions*, The Rocky Mountain J. Math., **24**(1994), no. 2, 529-538.

[3] Merkes, E.P., Wright, D.J., *On the univalence of a certain integral*, Proc. Amer. Math. Soc., **27**(1971), 97-100.

[4] Miller, S.S., Mocanu, P.T., *Differential subordinations, Theory and Applications*, Marcel Dekker, New York, 2000.

[5] Ong, K.W., Tan, S.L., Tu, Y.E., *Integral operators and univalent functions*, Tamk. J. Math., **43**(2012), no. 2, 215-221.

[6] Pfaltzgraff, J.A., *Univalence of the integral of $f'(z)^\lambda$*, Bull. Lond. Math. Soc., **7**(1975), 254-256.

Parviz Arjomandinia
"Payame Noor" University
Department of Mathematics
P.O. BOX 19395-3697, Tehran, Iran
e-mail: p.arjomandinia@gmail.com

Rasoul Aghalary
"Urmia" University
Faculty of Sciences, Department of Mathematics
Urmia, Iran
e-mail: raghalary@yahoo.com

Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 2, 169–176

Complex operators generated by q-Bernstein polynomials, $q \geq 1$

Gülen Başcanbaz–Tunca, Nursel Çetin and Sorin G. Gal

Dedicated to the memory of Akif D. Gadjiev

Abstract. By using a univalent and analytic function τ in a suitable open disk centered in origin, we attach to analytic functions f, the complex Bernstein-type operators of the form $B_{n,q}^{\tau}(f) = B_{n,q}\left(f \circ \tau^{-1}\right) \circ \tau$, where $B_{n,q}$ denote the classical complex q-Bernstein polynomials, $q \geq 1$. The new complex operators satisfy the same quantitative estimates as $B_{n,q}$. As applications, for two concrete choices of τ, we construct complex rational functions and complex trigonometric polynomials which approximate f with a geometric rate.

Mathematics Subject Classification (2010): 30E10, 41A35, 41A25.

Keywords: q-Bernstein-type operator, Voronovskaja's theorem, quantitative estimates, complex rational operators, complex trigonometric polynomials.

1. Introduction

Starting from the classical Bernstein polynomials defined for $f \in C[0,1]$ by

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right),$$

a new sequence of Bernstein-type operators of real variable is introduced in [1] by the formula

$$B_n^{\tau} f := B_n\left(f \circ \tau^{-1}\right) \circ \tau,$$

where τ is a real-valued function on $[0,1]$ which satisfies the following conditions:

(τ_1) τ is differentiable of any order on $[0,1]$,

(τ_2) $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ on $[0,1]$.

Specifically, $B_n^{\tau}(f)$ in [1] is given by

$$B_n^{\tau}(f)(x) = \sum_{k=0}^{n} \binom{n}{k} \tau^k(x)\left(1-\tau(x)\right)^{n-k} (f \circ \tau^{-1})(\tfrac{k}{n}), \ x \in [0,1].$$

According to [1], the sequence $B_n^{\tau}(f)$, $n \in \mathbb{N}$, converges uniformly to $f \in C[0,1]$.

In [6]-[7] and [2], the complex form of the q-Bernstein polynomials, $q \geq 1$, given by

$$B_{n,q}(f)(z) = \sum_{k=0}^{n} \binom{n}{k}_q z^k \cdot \Pi_{s=0}^{n-k-1}(1-q^s z) f\left(\frac{[k]_q}{[n]_q}\right), \ n \in \mathbb{N},$$

were intensively studied. Here f is a complex-valued analytic function in an open disk of radius ≥ 1 and centered in origin. Also, above we have

$$[n]_q = (q^n - 1)/(q-1),$$
$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!},$$
$$[n]_q! = [1]_q \cdot [2]_q \cdot \ldots \cdot [n]_q, \ [0]_q! = 1.$$

Note that for $q = 1$, $B_{n,q}(f)$ reduce to the classical Bernstein polynomials.

Inspired by the real case in [1], in this paper we consider the idea in the complex setting and introduce the complex operators defined by

$$B^{\tau}_{n,q}(f)(z) = B_{n,q}(f \circ \tau^{-1})(\tau(z)), n \in \mathbb{N}, z \in \mathbb{C}, q \geq 1,$$

where denoting $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$, now τ satisfies the following properties:

$$\tau : \mathbb{D}_R \to \mathbb{C}, R > 1, \text{ is analytic, univalent, } \tau(0) = 0, \tau(1) = 1,$$
$$\text{and there exists } R' > 1 \text{ such that } \mathbb{D}_{R'} \subset \tau(\mathbb{D}_R). \tag{1.1}$$

By using the approach in [2], for the complex operators $B^{\tau}_{n,q}$ we prove upper and lower estimates and a quantitative Voronovskaja-type result in some compact subsets generated by τ.

Also, two important examples for τ are considered, which generate sequences of complex rational operators and of trigonometric polynomials of complex variable, approximating for $q > 1$ the function f with the geometric rate $\frac{1}{q^n}$ in some compact disks centered in origin.

2. Approximation results

In this section, we present the main approximation properties of the operators $B^{\tau}_{n,q}$. Firstly, we consider the case when $q = 1$. We have:

Theorem 2.1. *Let τ be satisfying the conditions in (1.1) and $f : \mathbb{D}_R \to \mathbb{C}$ be analytic in $\mathbb{D}_R$, $R > 1$. Since $g : \mathbb{D}_{R'} \to \mathbb{C}$ defined by $g(w) = (f \circ \tau^{-1})(w)$ is analytic on the disk $\mathbb{D}_{R'}$, $R' > 1$, let us write $g(w) = \sum_{k=0}^{\infty} c_k w^k$, for all $w \in \mathbb{D}_{R'}$.*

Let $1 \leq r' < R'$ be arbitrary fixed. Then, for all $z \in \mathbb{D}_R$ with $|\tau(z)| \leq r'$ and for all $n \in \mathbb{N}$, we have:

(i) (Upper estimate)

$$\left|B^{\tau}_{n,1}(f)(z) - f(z)\right| \leq \frac{C^{\tau}_{r'}}{n}, \tag{2.1}$$

where $C^{\tau}_{r'} = \frac{3r'(r'+1)}{2} \sum_{k=2}^{\infty} |c_k| k(k-1)(r')^{k-2} < \infty.$

(ii) (Voronovskaja-type result)

$$\left| B_{n,1}^{\tau}(f)(z) - f(z) - \frac{\tau(z)(1-\tau(z))}{2n} D_{\tau}^{2}(f)(z) \right| \leq \frac{5(1+r')^2 M_{r'}^{\tau}}{2n^2} \tag{2.2}$$

where $D_{\tau}^{2} f(z) := \left(f \circ \tau^{-1}\right)''(\tau(z)) = g''(\tau(z))$ *is detailed by*

$$D_{\tau}^{2}(f)(z) = \frac{f''(z)}{(\tau'(z))^2} - \frac{\tau''(z) f'(z)}{(\tau'(z))^3} = \frac{1}{\tau'(z)} \left(\frac{f'(z)}{\tau'(z)} \right)'$$

and

$$M_{r'}^{\tau} = \sum_{k=3}^{\infty} |c_k| k(k-1)(k-2)^2 \cdot (r')^{k-2} < \infty.$$

(iii) If f *is not a polynomial in* τ *of degree* ≤ 1, *then*

$$\left\| B_{n,1}^{\tau}(f) - f \right\|_{r',\tau} \sim \frac{1}{n},$$

where $\|F\|_{r',\tau} = \sup\{|F(z)|; |z| < R, |\tau(z)| \leq r'\}$ *and the constants in the equivalence depend only on* f, τ *and* r'.

Proof. Let $g(w) = \sum_{k=0}^{\infty} c_k w^k$ be an analytic function in a disk $\mathbb{D}_{R'}$ with $R' > 1$. Also, for simplicity, denote the classical Bernstein polynomials $B_{n,1}(g)(w)$ by $B_n(g)(w)$.

(i) According to Theorem 1.1.2, (i), page 6 in [2], for all $1 \leq r' < R'$, $n \in \mathbb{N}$ and $|w| \leq r'$, we have

$$|B_n(g)(w) - g(w)| \leq \frac{C_{r'}}{n},$$

where $C_{r'} = \frac{3r'(1+r')}{2} \sum_{k=2}^{\infty} k(k-1)|c_k|(r')^{k-2}$.

Now, if above we replace g by $f \circ \tau^{-1}$ and w by $\tau(z)$, then we easily arrive at the required estimate (2.1).

(ii) According to Theorem 1.1.3, (ii), page 9 in [2], for all $1 \leq r' < R'$, $n \in \mathbb{N}$ and $|w| \leq r'$, we have

$$\left| B_n(g)(w) - g(w) - \frac{w(1-w)}{2n} g''(w) \right| \leq \frac{5(1+r')^2 M_{r'}}{2n^2},$$

where $M_{r'} = \sum\limits_{k=3}^{\infty} |c_k| k(k-1)(k-2)^2 \cdot (r')^{k-2}$. Take $g(w) = (f \circ \tau^{-1})(w) = f[\tau^{-1}(w)]$. Since

$$g'(w) = f'[\tau^{-1}(w)] \cdot (\tau^{-1}(w))' = f'[\tau^{-1}(w)] \cdot \frac{1}{\tau'(\tau^{-1}(w))},$$

differentiating once again, we easily get

$$g''(w) = \frac{f''(\tau^{-1}(w))}{[\tau'(\tau^{-1}(w))]^2} - \frac{f'(\tau^{-1}(w)) \cdot \tau''(\tau^{-1}(w))}{[\tau'(\tau^{-1}(w))]^3}.$$

Now, replacing in the above estimate g by $f \circ \tau^{-1}$ and w by $\tau(z)$, we immediately get (2.2).

(iii) According to Corollary 1.1.5, page 14 in [2], it follows that for all $1 \leq r' < R'$ we have

$$\|B_n(g) - g\|_{r'} = \sup\{|B_n(g)(w) - g(w)|; |w| \leq r'\} \sim \frac{1}{n}.$$

But

$$\|B_n(g)-g\|_{r'} \geq \sup\{|B_n(g)(\tau(z))-g(\tau(z))|; |z|<R, |\tau(z)|\leq r'\}$$
$$= \|B^{\tau}_{n,1}(f)-f\|_{r',\tau},$$

which does not imply the required equivalence in the statement.

For this reason, we have to use here the standard method in [2] and the estimates (2.1) and (2.2). Thus, for all $z \in \mathbb{D}_R$ with $|\tau(z)| \leq r'$ and $n \in \mathbb{N}$ we can write

$$B^{\tau}_{n,1}(f)(z)-f(z) = \frac{1}{n}\left\{\frac{\tau(z)(1-\tau(z))}{2}D^2_{\tau}(f)(z)\right.$$
$$\left.+\frac{1}{n}\left[n^2\left(B^{\tau}_{n,1}(f)(z)-f(z)-\frac{\tau(z)(1-\tau(z))}{2n}D^2_{\tau}(f)(z)\right)\right]\right\}.$$

Then, the obvious inequality $\|F+G\|_{r',\tau} \geq \|F\|_{r',\tau} - \|G\|_{r',\tau}$ implies

$$\|B^{\tau}_{n,1}(f)-f\|_{r',\tau} \geq \frac{1}{n}\left\{\left\|\frac{\tau(1-\tau)}{2}D^2_{\tau}(f)\right\|_{r',\tau}\right.$$
$$\left.-\frac{1}{n}\left[n^2\left(\left\|B^{\tau}_{n,1}(f)-f-\frac{\tau(1-\tau)}{2n}D^2_{\tau}(f)\right\|_{r',\tau}\right)\right]\right\}.$$

By the hypothesis on f we immediately get that $g(\tau(z))$ is not a polynomial in $\tau(z)$ of degree ≤ 1. Then, by the formula $D^2_{\tau}(f)(z) = g''(\tau(z))$ we easily get

$$\left\|\frac{\tau(1-\tau)}{2}D^2_{\tau}(f)\right\|_{r',\tau} > 0.$$

Indeed, supposing the contrary, it follows the obvious contradiction $g''(\tau(z)) = 0$, for all $z \in \mathbb{D}_R$.

Since by (2.2) there exists a constant $C > 0$ with

$$n^2\left(\left\|B^{\tau}_{n,1}(f)-f-\frac{\tau(1-\tau)}{2n}D^2_{\tau}(f)\right\|_{r',\tau}\right) \leq C,$$

it is clear that there exists $n_0 \in \mathbb{N}$ such that

$$\|B^{\tau}_{n,1}(f)-f\|_{r',\tau} \quad \geq \frac{1}{2n}\left\|\frac{\tau(1-\tau)}{2}D^2_{\tau}(f)\right\|_{r',\tau}, \text{ for all } n \geq n_0.$$

Then, for $1 \leq n \leq n_0 - 1$ we obviously have

$$\|B^{\tau}_{n,1}(f)-f\|_{r',\tau} \geq \frac{M_{r',n,\tau}(f)}{n},$$

with $M_{r',n,\tau}(f) = n \cdot \|B^{\tau}_{n,1}(f)-f\|_{r',\tau} > 0$, which finally leads to

$$\|B^{\tau}_{n,1}(f)-f\|_{r',\tau} \geq \frac{C_{r',\tau}(f)}{n}, \text{ for all } n \in \mathbb{N},$$

where

$$C_{r',\tau}(f) = \min\left\{M_{r',1,\tau}, M_{r',2,\tau}(f), ..., M_{r',n_0-1,\tau}(f), \left\|\frac{\tau(1-\tau)}{4}D^2_{\tau}(f)\right\|_{r',\tau}\right\}.$$

Combining now with the estimate (2.1) from the point (i), we get the required equivalence. □

In the case $q > 1$ we have the following upper estimate of the geometric order $\frac{1}{q^n}$.

Theorem 2.2. *Let $f : \mathbb{D}_R \to \mathbb{C}$ be analytic in $\mathbb{D}_R$, $R > q$ and τ satisfying the conditions in (1.1). Denote*

$$g(w) = (f \circ \tau^{-1})(w) = \sum_{k=0}^{\infty} c_k w^k, \ w \in \mathbb{D}_{R'}.$$

For all $q \in (1, R')$, $1 \leq r' < \frac{R'}{q}$, $n \in \mathbb{N}$ and $z \in \mathbb{D}_R$ with $|\tau(z)| \leq r'$, we have

$$\left|B_{n,q}^{\tau}(f)(z) - f(z)\right| \leq \frac{M_{r',q}^{\tau}}{[n]_q} \leq \frac{q \cdot M_{r',q}^{\tau}}{q^n},$$

where $M_{r',q}^{\tau} = 2 \sum\limits_{k=2}^{\infty} |c_k| (k-1) [k-1]_q (r')^k < \infty$.

Proof. According to Theorem 1.5.1, page 51 in [2] we have

$$|B_{n,q}(g)(w) - g(w)| \leq \frac{M_{r',q}}{[n]_q} \leq \frac{q \cdot M_{r',q}^{\tau}}{q^n}, \text{ for all } 1 \leq r' < R', n \in \mathbb{N}, |w| \leq r',$$

where $M_{r',n} = \frac{3r'(1+r')}{2} \sum_{k=2}^{\infty} k(k-1)|c_k|(r')^{k-2}$.

Now, if above we replace g by $f \circ \tau^{-1}$ and w by $\tau(z)$, then we easily arrive at the required estimate. □

Remark 2.3. In a similar manner with Theorem 2.1, (ii), applying the results in, e.g., [10], for $B_{n,q}^{\tau}(f)$ we may deduce a quantitative Voronovskaja-type result of order $\frac{1}{q^{2n}}$.

3. Applications

In this section we apply the previous results to the cases of two concrete examples for τ. As consequences, we construct sequences of complex rational functions and complex trigonometric polynomials, convergent to f with a geometric rate. The first result is the following.

Theorem 3.1. *Let $f : \mathbb{D}_R \to \mathbb{C}$ be analytic in $\mathbb{D}_R$ with $R > 1 + \sqrt{2}$ and denote*

$$\tau(z) = \frac{Rz}{R+1-z}, \ |z| < R.$$

Then, with the notations in Theorems 2.1 and 2.2 we have:

(i) $B_{n,1}^{\tau}(f)(z)$ and $B_{n,q}^{\tau}(f)(z)$, $q > 1$, are complex rational functions on $\mathbb{D}_R$;

(ii) τ satisfies the conditions in (1.1) with $R' = \frac{R^2}{2R+1} > 1$;

(iii) if $1 \leq r' < R'$ then $1 \leq \frac{r'(R+1)}{R+r'} < R$ and for all $|z| \leq r = \frac{r'(R+1)}{R+r'}$, the upper estimates (2.1), (2.2) in Theorem 2.1, (i)-(ii) and the equivalence $\|B_{n,1}^{\tau}(f) - f\|_r \sim \frac{1}{n}$ hold.

(iv) If $1 < q < R'$ and $1 \leq r' < \frac{R'}{q}$, then the estimate in Theorem 2.2 holds for all $|z| \leq r = \frac{r'(R+1)}{R+r'}$.

Proof. (i) It is clear that both kinds of operators $B^{\tau}_{n,1}(f)(z)$ and $B^{\tau}_{n,q}(f)$, $q>1$, are complex rational functions on $\mathbb{D}_R$.

(ii) We are interested on the image of $\mathbb{D}_R$ through the analytic and univalent mapping τ. Writing $w=\frac{Rz}{R+1-z}$, we get $z=\frac{(R+1)w}{w+R}$, so that $|z|<R$ is equivalent to

$$\left|\frac{(R+1)w}{w+R}\right|<R.$$

Denoting now $w=u+iv$, the previous inequality is equivalent to

$$\frac{(R+1)\sqrt{u^2+v^2}}{\sqrt{(u+R)^2+v^2}}<R,$$

which is equivalent to the inequality $(R+1)^2(u^2+v^2)<R^2[(u+R)^2+v^2]$. Simple calculations lead this last inequality to the following list of equivalent inequalities:

$$u^2[(R+1)^2-R^2]+v^2[(R+1)^2-R^2]<2R^3u+R^4,$$

$$u^2-2u\frac{R^3}{2R+1}+v^2<\frac{R^4}{2R+1},$$

$$\left(u-\frac{R^3}{2R+1}\right)^2+v^2<\left[\frac{R^2(R+1)}{2R+1}\right]^2.$$

This last inequality represents a disk of center $(R^3/(2R+1),0)$ and of radius

$$R^2(R+1)/(2R+1).$$

Now, simple geometric reasonings lead to the fact that the above disk includes the disk of center in origin and of radius

$$\left|\frac{R^3}{2R+1}-\frac{R^2(R+1)}{2R+1}\right|=\frac{R^2}{2R+1},$$

where by the hypothesis $R>1+\sqrt{2}$ we immediately get $R^2/(2R+1)>1$. Concluding, since also we have $\tau(0)=0$ and $\tau(1)=1$, it follows that τ satisfies (1.1) with $R'=R^2/(2R+1)$.

(iii) Let $1\le r'<R'$. Evidently that $\frac{r'(R+1)}{R+r'}\ge 1$ and since the function

$$F(x)=\frac{(R+1)x}{R+x}$$

is strictly increasing as function of $x\ge 0$, it follows

$$\frac{r'(R+1)}{R+r'}<\frac{R'(R+1)}{R+R'}=\frac{R^3+R^2}{3R^2+R}<R.$$

Then, since $\frac{R|z|}{R+1-|z|}\le r'$ is equivalent with the inequality $|z|\le r=\frac{r'(R+1)}{R+r'}$, by the obvious inequality $|\tau(z)|=\frac{R|z|}{|R+1-z|}\le\frac{R|z|}{R+1-|z|}$, $|z|<R$, it follows that the inequality $|z|\le\frac{r'(R+1)}{R+r'}$ implies $|\tau(z)|\le r'$ and therefore Theorem 2.1, (i), (ii) holds for these z.

In order to prove the equivalence, we use exactly the same reasonings as in the proof of Theorem 2.1, (iii), taking into account that (2.1) and (2.2) hold for all $|z|\le r=\frac{r'(R+1)}{R+r'}$.

(iv) If $1 < q < R'$ and $1 \leq r' < \frac{R'}{q}$, then reasoning as in the previous case $q = 1$, we immediately get the desired conclusion. □

Theorem 3.2. *Let $f : \mathbb{D}_{\pi/2} \to \mathbb{C}$ be analytic in $\mathbb{D}_{\pi/2}$ and $\tau(z) = \frac{\sin(z)}{\sin(1)}$, $|z| < \frac{\pi}{2}$. Then, with the notations in Theorems 2.1 and 2.2 we have:*

(i) $B^{\tau}_{n,1}(f)(z)$ and $B^{\tau}_{n,q}(f)(z)$, $q > 1$, are trigonometric polynomials of complex variable on $\mathbb{D}_{\pi/2}$;

(ii) τ satisfies the conditions in (1.1) with $R = \frac{\pi}{2}$ and $R' = \frac{1}{\sin(1)} > 1$;

(iii) for any $1 \leq r' < \frac{1}{\sin(1)}$ and for all $|z| \leq r := \frac{\pi r' \sin(1)}{2\cosh(\pi/2)} < \frac{\pi}{2}$, the upper estimates (2.1), (2.2) in Theorem 2.1, (i)-(ii) and the equivalence $\|B^{\tau}_{n,1}(f) - f\|_r \sim \frac{1}{n}$ hold.

(iv) If $1 < q < R'$ and $1 \leq r' < \frac{R'}{q}$, then the estimate in Theorem 2.2 holds for all $|z| \leq r = \frac{\pi r' \sin(1)}{2\cosh(\pi/2)}$.

Proof. (i) It is clear that both kinds of operators $B^{\tau}_{n,1}(f)(z)$ and $B^{\tau}_{n,q}(f)$, $q > 1$, are trigonometric polynomials of complex variable on $\mathbb{D}_{\pi/2}$.

(ii) From the well-known facts that $\sin(z)$ is univalent in $\mathbb{D}_{\pi/2}$ and that its inverse $\arcsin(z)$ exists in $\mathbb{C} \setminus ((-\infty, 1) \cup (1, +\infty))$ (see, e.g., [3], p. 164 and [8], pp. 90-91), it is immediate that $\tau(z)$ satisfies (1.1) with $R = \pi/2$ and $R' = \frac{1}{\sin(1)} > 1$.

(iii) For any $r' \in [1, R')$, we are interested to find a disk centered in origin and contained in the set $\{z \in \mathbb{D}_{\pi/2}; |\tau(z)| \leq r'\}$.

Firstly, we observe that for all $|z| < \pi/2$ we have

$$|\tau(z)| = \frac{|\sin z|}{\sin(1)} = \left| \frac{e^{iz} - e^{-iz}}{2i\sin(1)} \right| \leq \frac{1}{\sin(1)} \frac{e^{-y} + e^{y}}{2} = \frac{1}{\sin(1)} \cosh y < \frac{\cosh\frac{\pi}{2}}{\sin(1)}.$$

Now, we will use the following version of the Schwarz's lemma (see, e.g., [9], p. 218): if f is analytic in $\mathbb{D}_R$, $f(0) = 0$ and $|f(z)| < M$ for all $|z| < R$, then $|f(z)| \leq \frac{M}{R}|z|$, for all $|z| < R$.

Taking above $R = \frac{\pi}{2}$ and $M = \frac{\cosh\frac{\pi}{2}}{\sin(1)}$, we immediately get that for all $|z| < \frac{\pi}{2}$ we have $|\tau(z)| \leq \frac{2}{\pi} \frac{\cosh\frac{\pi}{2}}{\sin(1)} |z|$.

Now, if we put the condition $\frac{2}{\pi} \frac{\cosh\frac{\pi}{2}}{\sin(1)} |z| \leq r'$, then we easily obtain that for all $|z| \leq r = \frac{\pi r' \sin(1)}{2\cosh(\pi/2)}$ it follows $|\tau(z)| \leq r'$ and therefore Theorem 2.1, (i) and (ii) hold for these values of z.

Note here that for any $1 \leq r' < \frac{1}{\sin(1)}$, we still have $\frac{\pi r' \sin(1)}{2\cosh(\pi/2)} < \frac{\pi}{2}$.

The equivalence is immediate from Theorem 2.1, (iii).

(iv) If $1 < q < R'$ and $1 \leq r' < \frac{R'}{q}$, then reasoning as in the previous case $q = 1$, we easily get the desired conclusion. □

Remark 3.3. The hypothesis $\tau(0) = 0$ and $\tau(1) = 1$ in (1.1) imply that the new defined τ-operators coincide with the function f at the points 0 and 1.

Remark 3.4. Evidently that the considerations in this paper can be applied to other choices of the mapping τ and to other complex q-Benstein-type operators like, for example, those studied in [4]-[5].

References

[1] Cárdenas-Morales, D., Garrancho, P., Raşa I., *Bernstein-type operators which preserve polynomials*, Comput. Math. Appl., **62**(2011), no. 1, 158–163.

[2] Gal, S.G., *Approximation by Complex Bernstein and Convolution Type Operators,* Series on Concrete and Applicable Mathematics, vol. 8, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2009.

[3] Goodman, A.W., *An invitation to the study of univalent and multivalent functions*, Internat. J. Math. Math. Sci., **2**(1979), no. 2, 163–186.

[4] Mahmudov, N., Kara, M., *Approximation theorems for generalized complex Kantorovich-type operators,* J. Appl. Math., (2012), Article Number: 454579.

[5] Mahmudov, N., *Approximation by q-Durrmeyer type polynomials in compact disks in the case* $q > 1$, Appl. Math. Comp., **237**(2014), 293–303.

[6] Ostrovskii, I., Ostrovska, S., *On the analyticity of functions approximated by their q-Bernstein polynomials when* $q > 1$, Appl. Math. Comp., **217**(2010), no. 1, 65–72.

[7] Ostrovska, S., *q-Bernstein polynomials and their iterates,* J. Approx. Theory, **123**(2003), 232-255.

[8] Sveshnikov, A.G., Tikhonov, A.N., *The Theory of Functions of a Complex Variable*, Mir Publishers, Moscow, 1971.

[9] Silverman, H., *Complex Variables*, Houghton Mifflin Co., Boston, 1975.

[10] Wang, H., Wu, X.Z., *Saturation of convergence for q-Bernstein polynomials in the case* $q \geq 1$, J. Math. Anal. Appl., **337**(2008), 744–750.

Gülen Başcanbaz–Tunca
Ankara University, Faculty of Science
Department of Mathematics
06100, Tandogan, Ankara, Turkey
e-mail: tunca@science.ankara.edu.tr

Nursel Çetin
Turkish State Meteorological Service
Research Department
06120, Kecioren, Ankara, Turkey
e-mail: nurselcetin07@gmail.com

Sorin G. Gal
Department of Mathematics and Computer Sciences
University of Oradea
Oradea, Romania
e-mail: galso@uoradea.ro

Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 2, 177–194

On a system of nonlinear partial functional differential equations of different types

László Simon

Abstract. We consider a system of a semilinear hyperbolic functional differential equation (where the lower order terms contain functional dependence on the unknown function) and a quasilinear parabolic functional differential equation with initial and boundary conditions. Existence of weak solutions for $t \in (0, T)$ and for $t \in (0, \infty)$ will be shown and some qualitative properties of the solutions in $(0, \infty)$ will be formulated.

Mathematics Subject Classification (2010): 35M33.

Keywords: Semilinear hyperbolic equation, quasilinear parabolic equation, partial functional differential equation.

1. Introduction

In the present paper we consider weak solutions of the following system of equations:

$$u''(t) + Q(u(t)) + \varphi(x)h'(u(t)) + H(t, x; u, z) + \psi(x)u'(t) = F_1(t, x; z), \tag{1.1}$$

$$z'(t) - \sum_{j=1}^{n} D_j[a_j(t, x, Dz(t), z(t); u, z)] + a_0(t, x, Dz(t), z(t); u, z) = F_2(t, x; u) \tag{1.2}$$

$$(t, x) \in Q_T = (0, T) \times \Omega$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and we use the notations $u(t) = u(t, x)$, $z(t) = z(t, x)$ $u' = D_t u$, $z' = D_t z$ $u'' = D_t^2 u$, $Dz = (D_1 z, \dots, D_n z)$, Q may be e.g. a linear second order symmetric elliptic differential operator in the variable x; h is a C^2 function having certain polynomial growth, H contains nonlinear functional (nonlocal) dependence on u and z, with some polynomial growth and F_1 contains some functional dependence on z. Further, the functions a_j define a quasilinear elliptic differential operator in x (for fixed t) with functional dependence on u and z. Finally,

This paper was presented at the International Conference on Nonlinear Operators, Differential Equations and Applications (ICNODEA), July 14-17, 2015, Cluj-Napoca, Romania.

F_2 may non-locally depending on u. (The system (1.1), (1.2) consists of a semilinear hyperbolic functional equation and a parabolic functional equation.)

This paper was motivated by some problems which were modelled by systems consisting of (functional) differential equations of different types. In [4] S. Cinca investigated a model, consiting of an elliptic, a parabolic and an ordinary nonlinear differential equation, which arise when modelling diffusion and transport in porous media with variable porosity. In [6] J.D. Logan, M.R. Petersen and T.S. Shores considered and numerically studied a similar system which describes reaction-mineralogy-porosity changes in porous media with one-dimensional space variable. J. H. Merkin, D.J. Needham and B.D. Sleeman considered in [7] a system, consisting of a nonlinear parabolic and an ordinary differential equation, as a mathematical model for the spread of morphogens with density dependent chemosensitivity. In [3], [8], [9] the existence of solutions of such systems were studied.

In Section 2 the existence of weak solutions will be proved for $t \in (0,T)$, in Section 3 some examples will be shown and in Section 4 we shall prove existence and certain properties of solutions for $t \in (0,\infty)$.

2. Solutions in $(0,T)$

Denote by $\Omega \subset \mathbb{R}^n$ a bounded domain having the uniform C^1 regularity property (see [1]), $Q_T = (0,T) \times \Omega$. Denote by $W^{1,p}(\Omega)$ the Sobolev space of real valued functions with the norm

$$\|u\| = \left[\int_\Omega \left(\sum_{j=1}^n |D_j u|^p + |u|^p\right) dx\right]^{1/p} \qquad (2 \le p < \infty).$$

The number q is defined by $1/p + 1/q = 1$. Further, let $V_1 \subset W^{1,2}(\Omega)$ and $V_2 \subset W^{1,p}(\Omega)$ be closed linear subspaces containing $C_0^\infty(\Omega))$, $V_j^\star$ the dual spaces of V_j, the duality between $V_j^\star$ and V_j will be denoted by $\langle \cdot, \cdot \rangle$, the scalar product in $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$. Finally, denote by $L^p(0,T;V_j)$ the Banach space of the set of measurable functions $u : (0,T) \to V_j$ with the norm

$$\|u\|_{L^p(0,T;V_j)} = \left[\int_0^T \|u(t)\|_{V_j}^p dt\right]^{1/p}$$

and $L^\infty(0,T;V_j)$, $L^\infty(0,T;L^2(\Omega))$ the set of measurable functions $u : (0,T) \to V_j$, $u : (0,T) \to L^2(\Omega)$, respectively, with the $L^\infty(0,T)$ norm of the functions $t \mapsto \|u(t)\|_{V_j}$, $t \mapsto \|u(t)\|_{L^2(\Omega)}$, respectively.

Now we formulate the assumptions on the functions in (1.1), (1.2).

(A_1). $Q : V_1 \to V_1^\star$ is a linear continuous operator such that

$$\langle Qu, v\rangle = \langle Qv, u\rangle, \quad \langle Qu, u\rangle \ge c_0 \|u\|_{V_1}^2$$

for all $u, v \in V_1$ with some constant $c_0 > 0$.

(A_2). $\varphi, \psi : \Omega \to \mathbb{R}$ are measurable functions satisfying

$$c_1 \le \varphi(x) \le c_2, \quad c_1 \le \psi(x) \le c_2 \text{ for a.a. } x \in \Omega$$

with some positive constants c_1, c_2.

(A_3). $h : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function satisfying

$$h(\eta) \geq 0, \quad |h''(\eta)| \leq \text{const}|\eta|^{\lambda-1} \text{ for } |\eta| > 1 \text{ where}$$
$$1 < \lambda \leq \lambda_0 = \frac{n}{n-2} \text{ if } n \geq 3, \quad 1 < \lambda < \infty \text{ if } n = 2.$$

(A_4). $H : Q_T \times L^2(Q_T) \times L^p(Q_T) \to \mathbb{R}$ is a function for which $(t,x) \mapsto H(t,x;u,z)$ is measurable for all fixed $u \in L^2(\Omega)$, $z \in L^p(Q_T)$, H has the Volterra property, i.e. for all $t \in [0,T]$, $H(t,x;u,z)$ depends only on the restriction of u and z to $(0,t)$. Further, the following inequality holds for all $t \in [0,T]$ and $u \in L^2(\Omega)$, $z \in L^p(Q_T)$:

$$\int_\Omega |H(t,x;u,z)|^2 dx \leq \text{const}\left[\|z\|^2_{L^p(Q_T)} + 1\right]\left[\int_0^t \int_\Omega h(u(\tau))dxd\tau + \int_\Omega h(u)dx\right];$$
$$\int_0^t \left[\int_\Omega |H(\tau,x;u_1,z) - H(\tau,x;u_2,z)|^2 dx\right] d\tau \leq M(K,z) \int_0^t \left[\int_\Omega |u_1 - u_2|^2 dx\right] d\tau$$
$$\text{if } \|u_j\|_{L^\infty(0,T;V_1)} \leq K$$

where for all fixed number $K > 0$, $z \mapsto M(K,z) \in \mathbb{R}^+$ is a bounded (nonlinear) operator.

Finally, $(z_k) \to z$ in $L^p(Q_T)$ implies

$$H(t,x;u_k,z_k) - H(t,x;u_k,z) \to 0 \text{ in } L^2(Q_T) \text{ uniformly if } \|u_k\|_{L^2(Q_T)} \leq \text{const}.$$

(A_5). $F_1 : Q_T \times L^p(Q_T) \to \mathbb{R}$ is a function satisfying $(t,x) \mapsto F_1(t,x;z) \in L^2(Q_T)$ for all fixed $z \in L^p(Q_T)$ and $(z_k) \to z$ in $L^p(Q_T$ implies that $F_1(t,x;z_k) \to F_1(t,x;z)$ in $L^2(Q_T)$.

Further,

$$\int_0^T \|F_1(\tau,x;z)\|^2_{L^2(\Omega)} d\tau \leq \text{const}\left[1 + \|z\|^{\beta_1}_{L^p(Q_T)}\right]$$

with some constant $\beta_1 > 0$.

(B_1) The functions

$$a_j : Q_T \times \mathbb{R}^{n+1} \times L^2(Q_T) \times L^p(Q_T) \to \mathbb{R} \quad (j = 0, 1, \dots n),$$

are measurable in $(t,x) \in Q_T$ for all fixed $\xi = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, $u \in L^2(Q_T)$, $z \in L^p(Q_T)$ and continuous in $\xi \in \mathbb{R}^{n+1}$ for all fixed $u \in L^2(Q_T)$, $z \in L^p(Q_T)$ and a.a. fixed $(t,x) \in Q_T$.

Further, if $(u_k) \to u$ in $L^2(Q_T)$ and $(z_k) \to z$ in $L^p(Q_T)$ then for all $\xi \in \mathbb{R}^{n+1}$, a.a. $(t,x) \in Q_T$, for a subsequence

$$a_j(t,x,\xi;u_k,z_k) \to a_j(t,x,\xi;u,z) \quad (j = 0, 1, \dots, n).,$$

(B_2) For $j = 0, 1, \dots, n$

$$|a_j(t,x,\xi;u,z)| \leq g_1(u,z)|\xi|^{p-1} + [k_1(u,z)](t,x),$$

where $g_1 : L^2(Q_T) \times L^p(Q_T) \to \mathbb{R}^+$ is a bounded, continuous (nonlinear) operator,

$$k_1 : L^2(Q_T) \times L^p(Q_T) \to L^q(Q_T) \text{ is continuous and}$$
$$\|k_1(u,z)\|_{L^q(Q_T)} \leq \text{const}(1 + \|u\|^\gamma_{L^2(Q_T)} + \|z\|^{p_1}_{L^p(Q_T)})$$

with some constants $\gamma > 0$, $0 < p_1 < p - 1$.

(B_3) The following inequality holds for all $t \in [0,T]$ with some constants $c_2 > 0$, $c_3 \geq 0$, $\beta \geq 0$, $\gamma_1 \geq 0$ (not depending on t, u, z):

$$\sum_{j=0}^{n} [a_j(t,x,\xi;u,z) - a_j(t,x,\xi^\star;u,z)](\xi_j - \xi_j^\star) \geq$$

$$\frac{c_2}{1 + \|u\|^{\beta}_{L^2(Q_T)} + \|z\|^{\gamma_1}_{L^p(Q_T)}} |\xi - \xi^\star|^p - c_3|\xi_0 - \xi_0^\star|^2.$$

(B_4) For all fixed $u \in L^2(Q_T)$ the function

$$F_2 : Q_T \times L^2(Q_T) \to \mathbb{R} \text{ satisfies } (t,x) \mapsto F_2(t,x;u) \in L^q(Q_T),$$

$$\|F_2(t,x;u)\|_{L^q(Q_T)} \leq \text{const}\left[1 + \|u\|^{\gamma}_{L^2(Q_T)}\right]$$

(see (B_2)) and

$$(u_k) \to u \text{ in } L^2(Q_T) \text{ implies } F_2(t,x;u_k) \to F_2(t,x;u) \text{ in } L^q(Q_T).$$

Finally,

$$\max\{(\beta_1\beta)/2, \gamma_1\} + \max\{(\beta_1\gamma)/2, p_1\} < p - 1.$$

Theorem 2.1. *Assume (A_1) – (A_5) and (B_1) – (B_4). Then for all $u_0 \in V_1$, $u_1 \in L^2(\Omega)$, $z_0 \in L^2(\Omega)$ there exists $u \in L^\infty(0,T;V_1)$ such that*

$$u' \in L^\infty(0,T;L^2(\Omega)), \quad u'' \in L^2(0,T;V_1^\star) \text{ and } z \in L^p(0,T;V_2), \quad z' \in L^q(0,T;V_2^\star)$$

such that u satisfies (1.1) in the sense: for a.a. $t \in [0,T]$, all $v \in V_1$

$$\langle u''(t), v\rangle + \langle Q(u(t)), v\rangle + \int_\Omega \varphi(x)h'(u(t))v dx + \int_\Omega H(t,x;u,z)v dx + \tag{2.1}$$

$$\int_\Omega \psi(x)u'(t)v dx = \int_\Omega F_1(t,x;z)v)dx$$

and the initial conditions

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2.2}$$

Further, u, z satisfy (1.2) in the sense: for a.a. $t \in (0,T)$, all $w \in V_2$

$$\langle z'(t), w\rangle + \int_\Omega \left[\sum_{j=1}^{n} a_j(t,x,Dz(t),z(t);u,z)\right] D_j w dx + \tag{2.3}$$

$$\int_\Omega a_0(t,x,Dz(t),z(t);u,z)w dx = \int_\Omega F_2(t,x;u)w dx \text{ and}$$

$$z(0) = z_0. \tag{2.4}$$

Proof. The proof is based on the results of [11], the theory of monotone operators (see, e.g. [13]) and Schauder's fixed point theorem as follows.

Consider the problem (2.1), (2.2) for u with an arbitrary fixed $z = \tilde{z} \in L^p(Q_T)$. According to [11] assumptions (A_1) – (A_5) imply that there exists a unique solution $u = \tilde{u} \in L^\infty(0,T;V_1)$ with the properties $\tilde{u}' \in L^\infty(0,T;L^2(\Omega))$, $\tilde{u}'' \in L^2(0,T;V_1^\star)$ satisfying (2.1) and the initial condition (2.2). Then consider problem (2.3) (2.4) for

z with the above $u = \tilde{u}$ and with $z = \tilde{z}$ functional terms (see (2.6)). According to the theory of monotone operators (see, e.g., [13]) there exists a unique solution $z \in L^p(0,T;V_2)$ of (2.3), (2.4) such that $z' \in L^q(0,T;V_2^\star)$. By using the notation $S(\tilde{z}) = z$, we shall show that the operator $S : L^p(Q_T) \to L^p(Q_T)$ satisfies the assumptions of Schauder's fixed point theorem: it is continuous, compact and there exists a closed ball $B_0(R) \subset L^p(Q_T)$ such that

$$S(B_0(R)) \subset B_0(R). \tag{2.5}$$

Then Schauder's fixed point theorem will imply that S has a fixed point $z^\star \in L^p(0,T;V_2)$. Defining $u^\star$ by the solution of (2.1), (2.2) with $z = z^\star$, functions $u^\star$, $z^\star$ satisfy (2.1) – (2.4).

Lemma 2.2. *Consider problem (2.1), (2.2) for u with an arbitrary fixed $z = \tilde{z} \in L^p(Q_T)$. Assumptions (A_1) – (A_5) imply that there exists a unique $u = \tilde{u} \in L^\infty(0,T;V_1)$ such that $\tilde{u}' \in L^\infty(0,T;L^2(\Omega))$, $\tilde{u}'' \in L^2(0,T;V_1^\star)$ and (2.1), (2.2) are satisfied.*

Lemma 2.2 directly follows from Theorem 4.1 of [11].

Lemma 2.3. *Consider the following modification of problem (2.3), (2.4) with arbitrary fixed $\tilde{u} \in L^2(Q_T)$, $\tilde{z} \in L^p(Q_T)$: find $z \in L^p(0,T;V_2)$ such that $z' \in L^q(0,T;V_2^\star)$ and for a.a. $t \in [0,T]$, all $w \in V_2$*

$$\langle z'(t), w\rangle + \int_\Omega \left[\sum_{j=1}^n a_j(t,x,Dz(t),z(t);\tilde{u},\tilde{z})\right] D_j w dx + \tag{2.6}$$

$$\int_\Omega a_0(t,x,Dz(t),z(t);\tilde{u},\tilde{z}) w dx = \int_\Omega F_2(t,x;\tilde{u}) w dx,$$

$$z(0) = z_0. \tag{2.7}$$

Assumptions (B_1) – (B_4) imply that there exists a unique solution of (2.6), (2.7).

Proof. Let $a > 0$ be a fixed constant. A function z is a solution of (1.2), (2.4) if and only if $\hat{z}(t) = \mathrm{e}^{-at} z(t)$ satisfies

$$\hat{z}'(t) - \mathrm{e}^{-at} \sum_{j=1}^n D_j[a_j(t,x,\mathrm{e}^{at} D\hat{z}(t), \mathrm{e}^{at}\hat{z}(t);\tilde{u},\tilde{z})] + \tag{2.8}$$

$$\mathrm{e}^{-at} a_0(t,x,\mathrm{e}^{at} D\hat{z}(t), \mathrm{e}^{at}\hat{z}(t);\tilde{u},\tilde{z}) + a\hat{z}(t) = \mathrm{e}^{-at} F_2(t,x;\tilde{u}),$$

$$\hat{z}(0) = z_0. \tag{2.9}$$

We shall apply the theory of monotone operators to (2.8), (2.9) with sufficiently large $a > 0$.

Define (with fixed $\tilde{u} \in L^2(Q_T)$, $\tilde{z} \in L^p(Q_T)$, $t \in [0,T]$) operator $\hat{A}_{\tilde{u},\tilde{z}}$ by

$$\langle \hat{A}_{\tilde{u},\tilde{z}}(\hat{z}), w\rangle = \int_\Omega \mathrm{e}^{-at} \sum_{j=1}^n a_j(t,x,\mathrm{e}^{at} D\hat{z}(t), \mathrm{e}^{at}\hat{z}(t);\tilde{u},\tilde{z}) D_j w dx +$$

$$\int_\Omega \mathrm{e}^{-at} a_0(t,x,\mathrm{e}^{at} D\hat{z}(t), \mathrm{e}^{at}\hat{z}(t);\tilde{u},\tilde{z}) w dx + a \int_\Omega \hat{z} w dx,$$

$$\hat{z} \in L^p(0,T;V_2), \quad w \in V_2.$$

By (B_1), (B_2) operator $\hat{A}_{\tilde{u},\tilde{z}} : L^p(0,T;V_2) \to L^q(0,T;V_2^\star)$ is bounded and demi-continuous (see, e.g. [13]). Further, it is uniformly monotone if $a > 0$ is sufficiently large.

Indeed, by (B_3), for arbitrary $\hat{z}_1, \hat{z}_2 \in L^p(0,T;V_2)$

$$\int_0^T \langle \hat{A}_{\tilde{u},\tilde{z}}(\hat{z}_1) - \hat{A}_{\tilde{u},\tilde{z}}(\hat{z}_2), \hat{z}_1 - \hat{z}_2 \rangle dt = \tag{2.10}$$

$$\int_{Q_T} \mathrm{e}^{-2at} \sum_{j=1}^n [a_j(t,x,\mathrm{e}^{at}D\hat{z}_1(t), \mathrm{e}^{at}\hat{z}_1(t);\tilde{u},\tilde{z}) -$$
$$a_j(t,x,\mathrm{e}^{at}D\hat{z}_2(t), \mathrm{e}^{at}\hat{z}_2(t);\tilde{u},\tilde{z})]\mathrm{e}^{at}D_j(\hat{z}_1 - z_2)dtdx +$$
$$\int_{Q_T} \mathrm{e}^{-2at}[a_0(t,x,\mathrm{e}^{at}D\hat{z}_1(t), \mathrm{e}^{at}\hat{z}_1(t);\tilde{u},\tilde{z}) -$$
$$a_0(t,x,\mathrm{e}^{at}D\hat{z}_2(t), \mathrm{e}^{at}\hat{z}_2(t);\tilde{u},\tilde{z})]\mathrm{e}^{at}(\hat{z}_1 - \hat{z}_2)dtdx \geq$$
$$\frac{c_2}{1 + \|\tilde{u}\|^\beta_{L^2(Q_T)} + \|\tilde{z}\|^{\gamma_1}_{L^p(Q_T)}} \int_{Q_T} \mathrm{e}^{-2at}[\mathrm{e}^{at}|D\hat{z}_1 - D\hat{z}_2|^p + \mathrm{e}^{at}|\hat{z}_1 - \hat{z}_2|^p]dtdx -$$
$$c_3 \int_{Q_T} |\hat{z}_1 - \hat{z}_2|^2 dtdx + a\int_{Q_T} |\hat{z}_1 - \hat{z}_2|^2 dtdx \geq$$
$$\frac{c_2'}{1 + \|\tilde{u}\|^\beta_{L^2(Q_T)} + \|\tilde{z}\|^{\gamma_1}_{L^p(Q_T)}} \int_{Q_T} [|D\hat{z}_1 - D\hat{z}_2|^p + |\hat{z}_1 - \hat{z}_2|^p]dtdx$$

with some constant $c_2' > 0$ (depending on T) if $a > 0$ is sufficiently large.

Consequently, according to the theory of monotone operators (see, e.g. [13]) problem (2.8), (2.9) for $\hat{z}$ has a unique weak solution, thus (2.6), (2.7) has a unique solution.

By using Lemmas 2.2, 2.3 we may define operator $S : L^p(Q_T) \to L^p(Q_T)$ as follows. Let $\tilde{z} \in L^p(Q_T)$ be an arbitrary element. By Lemma 2.2 there exists a unique solution $\tilde{u}$ of (2.1), (2.2). According to Lemma 2.3 there exists a unique solution z of (2.6), (2.7). Operator S is defined by $S(\tilde{z}) = z$.

Lemma 2.4. *The operator $S : L^p(Q_T) \to L^p(Q_T)$ is compact.*

Proof. Let $(\tilde{z}_k)$ be a bounded sequence in $L^p(Q_T)$ and consider the (unique) solution $\tilde{u}_k$ of (2.1), (2.2) with fixed $z = \tilde{z}_k$. We show that $(\tilde{u}_k)$ is bounded in $L^\infty(0,T;V_1)$ and $(\tilde{u}_k')$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Indeed, applying the arguments in the proof of Theorem 2.1 in [11], one gets the solutions $\tilde{u}_k$ of (2.1), (2.2) as the (weak) limit of Galerkin approximations

$$\tilde{u}_{mk}(t) = \sum_{l=1}^m g^k_{lm}(t) w_l \text{ where } g^k_{lm} \in W^{2,2}(0,T)$$

and $w_1, w_2, \ldots$ is a linearly independent system in V_1 such that the linear combinations are dense in V_1, further, the functions $\tilde{u}_{mk}$ satisfy (for $j = 1,\ldots,m$)

$$\langle \tilde{u}''_{mk}(t), w_j \rangle + \langle Q(\tilde{u}_{mk}(t)), w_j \rangle + \int_\Omega \varphi(x) h'(\tilde{u}_{mk}(t)) w_j dx + \tag{2.11}$$

$$\int_\Omega H(t,x;\tilde u_{mk},\tilde z_k)w_j dx + \int_\Omega \psi(x)\tilde u'_{mk}(t)w_j dx = \int_\Omega F_1(t,x;\tilde z_k)w_j dx,$$

$$\tilde u_{mk}(0) = u_{m0}, \quad \tilde u'_{mk}(0) = u_{m1} \tag{2.12}$$

where u_{m0}, u_{m1} $(m = 1,2,\dots)$ are linear combinations of $w_1, w_2, \dots w_m$, satisfying $(u_{m0}) \to u_0$ in V_1 and $(u_{m1}) \to u_1$ in $L^2(\Omega)$ as $m \to \infty$.

Multiplying (2.11) by $(g^k_{lm})'(t)$, summing with respect to j and integrating over $(0,t)$, by Young's inequality we find

$$\frac{1}{2}\|\tilde u'_{mk}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\langle Q(\tilde u_{mk}(t)), \tilde u_{mk}(t)\rangle + \int_\Omega \varphi(x)h(\tilde u_{mk}(t))dx + \tag{2.13}$$

$$\int_0^t \left[\int_\Omega H(\tau,x;\tilde u_{mk},\tilde z_k)\tilde u'_{mk}(\tau)dx\right]d\tau + \int_0^t \left[\int_\Omega \psi(x)|\tilde u'_{mk}(\tau)|^2 dx\right]d\tau =$$

$$\int_0^t \left[\int_\Omega F_1(\tau,x;\tilde z_k)\tilde u'_{mk}(\tau)dx\right]d\tau + \frac{1}{2}\|\tilde u'_{mk}(0)\|^2_H + \frac{1}{2}\langle Q(\tilde u_{mk}(0)), \tilde u_{mk}(0)\rangle +$$

$$\int_\Omega \varphi(x)h(\tilde u_{mk}(0))dx \le \frac{1}{2}\int_0^T \|F_1(\tau,x;\tilde z_k)\|^2_{L^2(\Omega)}d\tau + \frac{1}{2}\int_0^T \|\tilde u'_{mk}(\tau)\|^2_{L^2(\Omega)} + \text{const}$$

where the constant is not depending on m,k,t. (See [11].)

By using (A_2), (A_4), (A_5) and the Cauchy-Schwarz inequality, we obtain from (2.13)

$$\frac{1}{2}\|\tilde u'_{mk}(t)\|^2_{L^2(\Omega)} + \frac{c_0}{2}\|\tilde u_{mk}(t))\|^2_{V_1} + c_1\int_\Omega h(\tilde u_{mk}(t))dx \le \tag{2.14}$$

$$\int_0^T \|F_1(\tau,x;\tilde z_k)\|^2_{L^2(\Omega)}d\tau +$$

$$\text{const}\left\{1 + \int_0^t \|\tilde u'_{mk}(\tau)\|^2_{L^2(\Omega)}d\tau + \int_0^t \left[\int_\Omega h(\tilde u_{mk}(\tau))dx\right]d\tau\right\}.$$

Consequently,

$$\|\tilde u'_{mk}(t)\|^2_{L^2(\Omega)} + \int_\Omega h(\tilde u_{mk}(t))dx \le$$

$$\text{const}\left\{1 + \int_0^t [\|\tilde u'_{mk}(\tau)\|^2_{L^2(\Omega)} + \int_\Omega h(\tilde u_{mk}(\tau))dx]\right\}$$

where the constant is not depending on k,m,t. Thus by Gronwall's lemma

$$\|\tilde u'_{mk}(t)\|^2_{L^2(\Omega)} + \int_\Omega h(\tilde u_{mk}(t))dx \le \text{const} \tag{2.15}$$

and so by (A_1) and (2.14)

$$\|\tilde u_{mk}(t)\|_{V_1} \le \text{const} \tag{2.16}$$

where the constants are not depending on k,m,t. The inequalities (2.15), (2.16) imply that the weak limits $\tilde u_k$, $\tilde u'_k$ of $(\tilde u_{mk})$ and $(\tilde u'_{mk})$, respectively, are bounded in $L^\infty(0,T;V_1)$, $L^\infty(0,T;L^2(\Omega))$, respectively.

Consequently, by the well known compact imbedding theorem (see [5]) there is a subsequence of $(\tilde u_k)$, again denoted by $(\tilde u_k)$, for simplicity, which is convergent in $L^2(Q_T)$ to some $\tilde u$ and $(\tilde u_k) \to \tilde u$ a.e. in Q_T.

Consider the sequence of solutions z_k of (2.6) (2.7) with $\tilde{u} = \tilde{u}_k$, $\tilde{z} = \tilde{z}_k$. We show that the sequence z_k is bounded in $L^p(0,T;V_2)$. Indeed, for the functions $\hat{z}_k = \mathrm{e}^{-at} z_k$ we have

$$\langle \hat{z}_k', w \rangle + \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}_k), w \rangle = \langle \mathrm{e}^{-at} F_2(t,x;\tilde{u}_k), w \rangle, \tag{2.17}$$

thus, integrating (2.17) over $(0,T)$ with $w = \hat{z}_k$ one obtains

$$\frac{1}{2}\|\hat{z}_k(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\hat{z}_k(0)\|^2_{L^2(\Omega)} + \int_0^T \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}_k), \hat{z}_k \rangle dt = \tag{2.18}$$

$$\int_0^T \langle \mathrm{e}^{-at} F_2(t,x;\tilde{u}_k), w \rangle dt.$$

Applying the inequality (2.10) to $\hat{z}_1 = \hat{z}_k$ and $\hat{z}_2 = 0$, we obtain

$$\frac{\text{const}}{1 + \|\tilde{u}_k\|^{\beta}_{L^2(Q_T)} + \|\tilde{z}_k\|^{\gamma_1}_{L^p(Q_T)}} \int_{Q_T} [|D\hat{z}_k|^p + |\hat{z}_k|^p] dt \le \tag{2.19}$$

$$\int_0^T \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}_k) - \hat{A}_{\tilde{u}_k, \tilde{z}_k}(0), \hat{z}_k - 0 \rangle dt =$$

$$\int_0^T \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}_k), \hat{z}_k \rangle dt - \int_0^T \hat{A}_{\tilde{u}_k, \tilde{z}_k}(0), \hat{z}_k \rangle dt.$$

By (2.18)

$$\left| \int_0^T \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}_k), \hat{z}_k \rangle dt \right| \le \left| \int_0^T \langle \mathrm{e}^{-at} F_2(t,x;\tilde{u}_k), w \rangle dt \right| + \text{const} \le \tag{2.20}$$

$$\text{const}\|F_2(t,x;\tilde{u}_k)\|_{L^q(Q_T)} \|\hat{z}_k\|_{L^p(Q_T)}$$

and by (B_2)

$$\left| \int_0^T \hat{A}_{\tilde{u}_k, \tilde{z}_k}(0), \hat{z}_k \rangle dt \right| \le \text{const}\|\hat{z}_k\|_{L^p(Q_T)} \tag{2.21}$$

Hence by (2.19), (2.20), (B_4), $(\hat{z}_k)$ is bounded in $L^p(0,T;V_2)$ (as $p > 1$ and $\|\tilde{u}_k\|_{L^2(Q_T)}$, $\|\tilde{z}_k\|_{L^p(Q_T)}$ are bounded).

Further, the equality (2.17) implies that $(\hat{z}_k')$ is bounded in $L^q(0,T;V_2^\star)$. So by the well known compact imbedding theorem (see [5]) there is a subsequence of $(\hat{z}_k)$ which is convergent in $L^p(Q_T)$. Therefore, the corresponding subsequence of (z_k) is convergent, too in $L^p(Q_T)$.

Lemma 2.5. *The operator $S : L^p(Q_T) \to L^p(Q_T)$ is continuous.*

Proof. Assume that

$$(\tilde{z}_k) \to \tilde{z} \text{ in } L^p(Q_T). \tag{2.22}$$

Now we show that for the solutions $\tilde{u}_k$ of (2.1), (2.2) with $z = \tilde{z}_k$

$$(\tilde{u}_k) \to \tilde{u} \text{ in } L^2(Q_T) \tag{2.23}$$

and a.e. in Q_T for a subsequence where $\tilde{u}$ is the solution of (2.1), (2.2) with $z = \tilde{z}$.

In the proof of (2.23) we use the (uniqueness) Theorem 4.1 of [11]. Since $(\tilde{z}_k)$ is bounded in $L^p(0,T;V_2)$, $(\tilde{u}_k)$ is bounded in $L^2(Q_T)$ (see the proof of Lemma 2.4).

Further, $\tilde{u}$ and $\tilde{u}_k$ are weak solutions of (1.1) (i.e. of (2.1) with $z = \tilde{z}$ and $z = \tilde{z}_k$, respectively and satisfy the initial conditions (2.2), thus

$$\tilde{u}''(t) + Q(\tilde{u}(t)) + \varphi(x)h'(\tilde{u}(t)) + H(t, x; \tilde{u}, \tilde{z}) + \psi(x)\tilde{u}'(t) = F_1(t, x; \tilde{z}), \tag{2.24}$$

$$\tilde{u}_k''(t) + Q(\tilde{u}_k(t)) + \varphi(x)h'(\tilde{u}_k(t)) + H(t, x; \tilde{u}_k, \tilde{z}) + \psi(x)\tilde{u}_k'(t) = F_1(t, x; \tilde{z}_k) + H(t, x; \tilde{u}_k, \tilde{z}) - H(t, x; \tilde{u}_k, \tilde{z}_k). \tag{2.25}$$

Theorem 4.1 of [11] implies that for the solutions $\tilde{u}$ of (2.24) and $\tilde{u}_k$ of (2.25) we have for any $s \in [0, T]$ an estimation of the form

$$\|\tilde{u}_k(s) - \tilde{u}(s)\|^2_{L^2(\Omega)} \le \text{const} \int_{Q_T} \left| \int_0^t [F_1(\tau, x; \tilde{z}_k) - F_1(\tau, x; \tilde{z})] d\tau \right|^2 dt dx +$$

$$\text{const} \int_{Q_T} \left| \int_0^t [H(\tau, x; \tilde{u}_k, \tilde{z}_k) - H(\tau, x; \tilde{u}_k, \tilde{z})] d\tau \right|^2 dt dx$$

where the right hand side is converging to 0 as $k \to \infty$ by (A_4), (A_5).

So we have proved (2.23).

Now we show that (2.22), (2.23) imply:

$$(z_k) \to z \text{ in } L^p(Q_T), \text{ i.e. } (\hat{z}_k) \to \hat{z} \text{ in } L^p(Q_T) \tag{2.26}$$

for the solutions of (2.6), (2.7) and (2.8), (2.9), respectively (in the case of $z_k, \hat{z}_k$, instead of $\tilde{u}, \tilde{z}$ we have $\tilde{u}_k, \tilde{z}_k$). Since

$$\langle (\hat{z}_k - \hat{z})', \hat{z}_k - \hat{z} \rangle + \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}_k) - \hat{A}_{\tilde{u}, \tilde{z}}(\hat{z}), \hat{z}_k - \hat{z} \rangle =$$
$$\langle \mathrm{e}^{-at} F_2(t, x; \tilde{u}_k) - \mathrm{e}^{-at} F_2(t, x; \tilde{u}), \hat{z}_k - \hat{z} \rangle,$$

integrating over $(0, T)$ with respect to t, we find

$$\frac{1}{2}\|\hat{z}_k(T) - \hat{z}(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\hat{z}_k(0) - \hat{z}(0)\|^2_{L^2(\Omega)} + \int_0^T \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}_k) - \hat{A}_{\tilde{u}, \tilde{z}}(\hat{z}), \hat{z}_k - \hat{z} \rangle dt = \int_0^T \langle \mathrm{e}^{-at} F_2(t, x; \tilde{u}_k) - \mathrm{e}^{-at} F_2(t, x; \tilde{u}), \hat{z}_k - \hat{z} \rangle dt \tag{2.27}$$

where by (2.10)

$$\int_0^T \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}_k) - \hat{A}_{\tilde{u}, \tilde{z}}(\hat{z}), \hat{z}_k - \hat{z} \rangle dt = \tag{2.28}$$

$$\int_0^T \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}_k) - \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}), \hat{z}_k - \hat{z} \rangle dt + \int_0^T \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}) - \hat{A}_{\tilde{u}, \tilde{z}}(\hat{z}), \hat{z}_k - \hat{z} \rangle dt \ge$$

$$\frac{c_2'}{1 + \|\tilde{u}_k\|^{\beta}_{L^2(Q_T)} + \|\tilde{z}_k\|^{\gamma_1}_{L^p(Q_T)}} \int_{Q_T} [|D\hat{z}_k - D\hat{z}|^p + |\hat{z}_k - \hat{z}|^p] dt dx +$$

$$\int_0^T \langle \hat{A}_{\tilde{u}_k, \tilde{z}_k}(\hat{z}) - \hat{A}_{\tilde{u}, \tilde{z}}(\hat{z}), \hat{z}_k - \hat{z} \rangle dt.$$

By (2.22), (B_1), (B_2), Vitali's theorem and Hölder's inequality

$$\lim_{k\to\infty}\int_0^T \langle \hat{A}_{\tilde{u}_k,\tilde{z}_k}(\hat{z}) - \hat{A}_{\tilde{u},\tilde{z}}(\hat{z}), \hat{z}_k - \hat{z}\rangle dt = 0 \tag{2.29}$$

as $\|\hat{z}_k - \hat{z}\|_{L^p(Q_T)}$ is bounded. Similarly, the right hand side of (2.27) is coverging to 0 by (B_4). Therefore, (2.27) – (2.29) imply (2.26).

Lemma 2.6. *There is a closed ball*

$$\overline{B_R(0)} = \{z \in L^p(Q_T) : \|z\|_{L^p(Q_T)} \le R\}$$

such that $S(\overline{B_R(0)}) \subset \overline{B_R(0)}$.

Proof. According to (2.14) we have for the sequence $(\tilde{u}_m)$ of Galerkin approximation of the solution of (2.1), (2.2) (with $z = \tilde{z}$)

$$\frac{1}{2}\|\tilde{u}_m'(t)\|^2_{L^2(\Omega)} + \frac{c_0}{2}\|\tilde{u}_m(t)\|^2_{V_1} + c_1\int_\Omega h(\tilde{u}_m(t))dx \le \tag{2.30}$$

$$\frac{1}{2}\int_0^T \|F_1(\tau,x;\tilde{z})\|^2_{L^2(\Omega)}d\tau + \text{const}\int_0^t \|\tilde{u}_m'(\tau)\|^2_{L^2(\Omega)}d\tau +$$

$$\int_0^t \left[\int_\Omega h(\tilde{u}_m(\tau))dx\right] d\tau + \text{const}$$

where the constants are not depending on $m, t, \tilde{z}$. Hence, by Gronwall's lemma one obtains

$$\|\tilde{u}_m'(t)\|^2_H + \int_\Omega h(\tilde{u}_m(t))dx \le \text{const}\left[1 + \int_0^T \|F_1(\tau,x;\tilde{z})\|^2_{L^2(\Omega)}d\tau\right] + \tag{2.31}$$

$$\text{const}\int_0^t \left[1 + \int_0^T \|F_1(\tau,x;\tilde{z})\|^2_{L^2(\Omega)}d\tau \cdot \mathrm{e}^{t-s}\right] ds =$$

$$\text{const}\left[1 + \int_0^T \|F_1(\tau,x;\tilde{z})\|^2_{L^2(\Omega)}d\tau\right]$$

where the constants are independent of $m, t, \tilde{z}$. Thus by (2.30) and (A_5) we find

$$\|\tilde{u}_m(t)\|^2_{V_1} \le \text{const}\left[1 + \int_0^T \|F_1(\tau,x;\tilde{z})\|^2_{L^2(\Omega)}d\tau\right] \le \text{const}\left[1 + \|\tilde{z}\|^{\beta_1}_{L^p(0,T;V_2)}\right]$$

which implies (for the solution $\tilde{u}$ of (2.1), (2.2), the limit of $(\tilde{u}_m)$)

$$\|\tilde{u}\|^2_{L^2(Q_T)} \le \text{const}\left[1 + \|\tilde{z}\|^{\beta_1}_{L^p(Q_T)}\right]. \tag{2.32}$$

On the other hand, similarly to (2.19) – (2.21), by (B_2), (B_4) we have for $\hat{z}(t) = \mathrm{e}^{-at}z(t)$ (where z is the solution of (2.3), (2.4))

$$\frac{\text{const}}{1 + \|\tilde{u}\|^{\beta}_{L^2(Q_T)} + \|\tilde{z}\|^{\gamma_1}_{L^p(Q_T)}}\int_{Q_T}[|D\hat{z}|^p + |\hat{z}|^p]dt \le$$

$$\int_0^T \langle \hat{A}_{\tilde{u},\tilde{z}}(\hat{z}), \hat{z}\rangle dt - \int_0^T \langle \hat{A}_{\tilde{u},\tilde{z}}(0), \hat{z}\rangle dt \le$$

$$\text{const} + \text{const}\|F_2(t,x;\tilde{u})\|_{L^q(Q_T)}\|\hat{z}\|_{L^p(Q_T)} + \text{const}\|k_1(\tilde{u},\tilde{z})\|_{L^q(Q_T)}\|\hat{z}\|_{L^p(Q_T)} \le$$
$$\text{const} + \text{const}\left(1 + \|\tilde{u}\|^{\gamma}_{L^2(Q_T)} + \|\tilde{z}\|^{p_1}_{L^p(Q_T)}\right)\|\hat{z}\|_{L^p(Q_T)} \le$$
$$\text{const} + \text{const}\left(1 + \|\tilde{z}\|^{\beta_1\gamma/2}_{L^p(Q_T)} + \|\tilde{z}\|^{p_1}_{L^p(Q_T)}\right)\|\hat{z}\|_{L^p(Q_T)} \le$$
$$\tilde{c}_1 + \tilde{c}_2\left(1 + \|\tilde{z}\|^{\max\{(\beta_1\gamma)/2,p_1\}}_{L^p(Q_T)}\right)\|\hat{z}\|_{L^p(Q_T)}.$$

Thus for $\|\hat{z}\|_{L^p(Q_T)} \ge \tilde{c}_1/\tilde{c}_2$

$$\|\hat{z}\|^{p-1}_{L^p(Q_T)} \le \text{const}\left[1 + \|\tilde{u}\|^{\beta}_{L^2(Q_T)} + \|\tilde{z}\|^{\gamma_1}_{L^p(Q_T)}\right]\left[1 + \|\tilde{z}\|^{\max\{(\beta_1\gamma)/2,p_1\}}_{L^p(Q_T)}\right] \le \tag{2.33}$$
$$\text{const}\left[\left(1 + \|\tilde{z}\|^{\beta_1}_{L^p(Q_T)}\right)^{\beta/2} + \|\tilde{z}\|^{\gamma_1}_{L^p(Q_T)}\right]\cdot\left[1 + \|\tilde{z}\|^{\max\{(\beta_1\gamma)/2,p_1\}}_{L^p(Q_T)}\right] \le$$
$$\text{const}[1 + \|\tilde{z}\|^{\delta}_{L^p(Q_T)}]$$

where

$$\delta = \max\{(\beta_1\beta)/2, \gamma_1\} + \max\{(\beta_1\gamma)/2, p_1\}. \tag{2.34}$$

By (B_4) $\delta < p-1$, thus for sufficiently large R

$$\tilde{z} \in \overline{B_R(0)} = \left\{\tilde{z} \in L^p(Q_T), \quad \|\tilde{z}\|_{L^p(Q_T)} \le R\right\}$$

implies

$$\|z\|_{L^p(Q_T)} \le R, \text{ i.e. } z \in \overline{B_R(0)}.$$

(The norm of $\|z\|_{L^p(Q_T)}$ can be estimated by $\|\hat{z}\|_{L^p(Q_T)}$, multiplied by a constant.) So the proof of Lemma 2.6 is completed.

Finally, Lemmas 2.4 - 2.6 and Schauder's fixed point theorem imply that S has a fixed point and, consequently, there exists a solution of (2.1) – (2.4).

3. Examples

Let the operator Q be defined by

$$\langle Qu, v\rangle = \int_\Omega\left[\sum_{j,l=1}^{n} a_{jl}(x)(D_l u)(D_j v) + d(x)uv\right]dx +$$

where $a_{jl}, d \in L^\infty(\Omega)$, $a_{jl} = a_{lj}$, $\sum_{j,l=1}^n a_{jl}(x)\xi_j\xi_l \ge c_0|\xi|^2$, $d \ge c_0$ with some positive constant c_0. Then, clearly, assumption (A_1) is satisfied.

If h is a C^2 function such that $h(\eta) = |\eta|^{\lambda+1}$ if $|\eta| > 1$ then $(A_3)$is satisfied.

The condition (A_4) is satisfied e.g. if

$$H(t,x;u,z) = \chi(t,x)g_1(L_1 z)g_2(L_2 u) \text{ where } \chi \in L^\infty(Q_T),$$
$$L_1 : L^p(0,T;V_2) \to L^2(Q_T), \quad L_2 : L^2(Q_T) \to L^2(Q_T)$$

are continuous linear operators (with the Volterra property); g_1 is a globally Lipschitz bounded function, g_2 is a globally Lipschitz function. In the particular case when

$$L_2 \text{ is an } L^2(Q_T) \to L^\infty(Q_T) \text{ bounded linear operator} \tag{3.1}$$

then g_2 may be a locally Lipschitz function satisfying

$$|g_2(\eta)| \le \text{const}|\eta|^{(\lambda+1)/2} \text{ for } |\eta| > 1.$$

The operator L_2 has the property (3.1) e.g. if

$$(L_2 u)(t,x) = \int_{Q_t} \tilde{K}(t,x;\tau,\xi)u(\tau,\xi)d\tau d\xi \text{ where}$$

$$\int_{Q_T} |\tilde{K}(t,x;\tau,\xi)|^2 d\tau d\xi \le \text{const for all } (t,x) \in Q_T.$$

The operator $F_1 : Q_T \times L^p(0,T;V_2) \to \mathbb{R}$ may have the form

$$F_1(t,x;z) = f_1(t,x,L_3 z)$$

where $f_1(t,x,\mu)$ is measurable in (t,x), continuous in μ and

$$|f_1(t,x,\mu)| \le \text{const}|\mu|^{\beta_1/2} + \tilde{f}_1(t,x) \text{ where}$$

$$0 \le \beta_1 \le 2, \quad \tilde{f}_1 \in L^2(Q_T), \quad L_3 : L^p(0,T;V_2) \to L^2(Q_T)$$

is a linear continuous operator. Then (A_5) is fulfilled. In the particular case when

$$L_3 \text{ is } L^p(0,T;V_2) \to L^\infty(Q_T)$$

linear and continuous then $\beta_1 \le 2$ is not assumed.

Now we formulate examples for a_j satisfying (B_1) – (B_3):

$$a_j(t,x,\xi;u,z) = \alpha(t,x,L_4 u, L_5 z)\xi_j|\zeta|^{p-2}, \quad j = 1,\dots,n \text{ where } \zeta = (\xi_1,\dots\xi_n),$$

$\alpha(t,x,\nu_1,\nu_2)$ is measurable in (t,x), continuous in ν_1,ν_2 and satisfies

$$\frac{\text{const}}{1+|\nu_1|^\beta + |\nu_2|^{\gamma_1}} \le \alpha(t,x,\nu_1,\nu_2) \le \text{const}(1+|\nu_1|^\gamma + |\nu_2|^{p_1})$$

with some positive constants, $L_4, L_5 : L^2(Q_T) \to L^\infty(Q_T)$ are continuous linear operators,

$$a_0(t,x,\xi;u,z) = \alpha_0(t,x,L_6 u, L_7 z)\xi_0|\xi_0|^{p-2} + \alpha_1(z),$$

where $\alpha_0(t,x,\nu_1,\nu_2)$ is measurable in (t,x), continuous in ν_1,ν_2,

$$\frac{\text{const}}{1+|\nu_1|^\beta + |\nu_2|^{\gamma_1}} \le \alpha_0(t,x,\nu_1,\nu_2) \le \text{const}(1+|\nu_1|^\gamma + |\nu_2|^{p_1})$$

with some positive constants, $L_6, L_7 : L^2(Q_T) \to L^\infty(Q_T)$ are continuous linear operators and α_1 is a globally Lipschitz function. If the values of α, α_0 are between two positive constants then $L_4 - L_7$ may be $L^2(Q_T) \to L^2(Q_T)$ continuous linear operators.

Finally, the function $F_2 : Q_T \times L^2(Q_T) \to \mathbb{R}$ may have the form

$$F_2(t,x;u) = f_2(t,x,L_8 u)$$

where $f_2(t,x,\mu)$ is measurable in (t,x), continuous in μ and

$$|f_2(t,x,\mu)| \le \text{const}|\mu|^\gamma + \tilde{f}_2(t,x),$$

$$0 \le \gamma \le 1, \quad \tilde{f}_2 \in L^2(Q_T) \text{ and } L_8 : L^2(Q_T) \to L^2(Q_T)$$

is a continuous linear operator. Then (B_4) is satisfied. In the particular case when

$$L_8 \text{ is an } L^2(Q_T) \to L^\infty(Q_T) \text{ bounded linear operator}$$

then $\gamma \leq 1$ is not assumed.

4. Solutions in $(0, \infty)$

Now we formulate an existence theorem with respect to solutions for $t \in (0, \infty)$. Denote by $L^p_{loc}(0, \infty; V_1)$ the set of functions $u : (0, \infty) \to V_1$ such that for each fixed finite $T > 0$, their restrictions to $(0, T)$ satisfy $u|_{(0,T)} \in L^p(0, T; V_1)$ and let $Q_\infty = (0, \infty) \times \Omega$, $L^\alpha_{loc}(Q_\infty)$ the set of functions $u : Q_\infty \to \mathbb{R}$ such that $u|_{Q_T} \in L^\alpha(Q_T)$ for any finite T.

Now we formulate assumptions on H, F_1, a_j, F_2.

$(\tilde{A}_4)$ The function $H : Q_\infty \times L^2_{loc}(Q_\infty) \times L^p_{loc}(Q_\infty) \to \mathbb{R}$ is such that for all fixed $u \in L^2_{loc}(Q_\infty)$, $z \in L^p_{loc}(Q_\infty)$ the function $(t, x) \mapsto H(t, x; u, z)$ is measurable, H has the Volterra property (see (A_4)) and for each fixed finite $T > 0$, the restriction H_T of H to $Q_T \times L^2(Q_T) \times L^p(Q_T)$ satisfies (A_4).

Remark. Since H has the Volterra property, this restriction H_T is well defined by the formula

$$H_T(t, x; \tilde{u}, \tilde{z}) = H(t, x; u, z), \quad (t, x) \in Q_T \quad \tilde{u} \in L^2(Q_T), \tilde{z} \in L^p(Q_T)$$

where $u \in L^2_{loc}(Q_\infty)$, $z \in L^p_{loc}(Q_\infty)$ may be any function satisfying $u(t, x) = \tilde{u}(t, x)$, $z(t, x) = \tilde{z}(t, x)$ for $(t, x) \in Q_T$.

$(\tilde{A}_5)$ $F_1 : Q_\infty \times L^p_{loc}(Q_\infty) \to \mathbb{R}$ has the Volterra property and for each fixed finite $T > 0$, the restriction of F_1 to $(0, T)$ satisfies (A_5).

$(\tilde{B})$ $a_j : Q_\infty \times \mathbb{R}^{n+1} \times L^2_{loc}(Q_\infty) \times L^p_{loc}(Q_\infty) \to \mathbb{R}$ $(j = 0, 1, \ldots, n)$ have the Volterra property and for each finite $T > 0$, their restrictions to $(0, T)$ satisfy (B_1) – (B_3).

$(\tilde{B}_4)$ $F_2 : Q_\infty \times L^2_{loc}(Q_\infty) \to \mathbb{R}$ has the Volterra property and for each fixed finite $T > 0$, the restriction of F_2 to $(0, T)$ satisfies (B_4).

Theorem 4.1. *Assume (A_1) – (A_3), $(\tilde{A}_4)$, $(\tilde{A}_5)$, $(\tilde{B})$, $(\tilde{B}_4)$. Then for all $u_0 \in V_1$, $u_1 \in L^2(\Omega)$ there exist*

$$u \in L^\infty_{loc}(0, \infty; V_1), \quad z \in L^p_{loc}(0, \infty; V_2) \text{ such that}$$

$$u' \in L^\infty_{loc}(0, \infty; L^2(\Omega)), \quad u'' \in L^2_{loc}(0, \infty; V_1^\star), \quad z' \in L^q_{loc}(0, \infty; V_2^\star),$$

(2.1) – (2.4) hold for a.a. $t \in (0, \infty)$ and the initial condition (2.2) is fulfilled.

Assume that the following additional conditions are satisfied: there exist H^∞, $F_1^\infty \in L^2(\Omega)$, $u_\infty \in V_1$, a bounded function $\tilde{\beta}$, belonging to $L^2(0, \infty; L^2(\Omega))$ such that

$$Q(u_\infty) = F_1^\infty - H^\infty, \tag{4.1}$$

$$|H(t, x; u, z) - H^\infty| \leq \tilde{\beta}(t, x), \quad |F_1(t, x; z) - F_1^\infty(x)| \leq \tilde{\beta}(t, x) \tag{4.2}$$

for all fixed $u \in L^2_{loc}(Q_\infty)$, $z \in L^p_{loc}(Q_\infty)$). Further, there exist functions

$$a_j^\infty : \Omega \times \mathbb{R}^{n+1} \times L^2(\Omega) \to \mathbb{R}, \quad j = 1, \ldots, n \qquad F_2^\infty : \Omega \times L^2(\Omega) \to \mathbb{R}$$

such that for each fixed $z_0 \in V_2$, $z \in L^p_{loc}(Q_\infty)$ *and* $u \in L^2_{loc}(Q_\infty)$, $w_0 \in V_1$ *with the property*

$$\lim_{t\to\infty} \|u(t) - w_0\|_{L^2(\Omega)} = 0$$

for the functions

$$\varphi_j(t) = \|a_j(t, x, Dz_0, z_0; u, z) - a_j^\infty(x, Dz_0, z_0; w_0)\|_{L^q(\Omega)}, \quad j = 0, 1, \dots, n, \tag{4.3}$$

$$\psi(t) = \|F_2(t, x; u) - F_2^\infty(x; w_0)\|_{L^q(\Omega)} \tag{4.4}$$

we have

$$\lim_{t\to\infty} \varphi_j(t) = 0, \quad \lim_{t\to\infty} \psi(t) = 0. \tag{4.5}$$

Finally, (B_3) *is satisfied such that the following inequalities hold for all* $t > 0$ *with constants* $c_2 > 0$, $\beta > 0$, *not depending on* t:

$$\sum_{j=0}^{n} [a_j(t, x, \xi; u, z) - a_j(t, x, \xi^\star; u, z)][\xi_j - \xi_j^\star] \tag{4.6}$$

$$\frac{c_2}{1 + \|u\|^\beta_{L^2(Q_t \setminus Q_{t-a})}} |\xi - \xi^\star|^p$$

with some fixed $a > 0$ *(finite delay).*

Then for the above solutions u, z *we have*

$$u \in L^\infty(0, \infty; V_1), \tag{4.7}$$

$$\|u'(t)\|_{L^2(\Omega)} \le const e^{-c_1 t} \tag{4.8}$$

where c_1 *is given in* (A_2) *and there exists* $w_0 \in V_1$ *such that*

$$u(T) \to w_0 \text{ in } L^2(\Omega) \text{ as } T \to \infty, \quad \|u(T) - w_0\|_{L^2(\Omega)} \le const e^{-c_1 T} \tag{4.9}$$

and w_0 *satisfies*

$$Q(w_0) + \varphi h'(w_0) = F_1^\infty - H^\infty. \tag{4.10}$$

Finally, there exists a unique solution $z_0 \in V_2$ *of*

$$\sum_{j=1}^{n} \int_\Omega a_j^\infty(x, Dz_0, z_0; w_0) D_j v dx + \int_\Omega a_0^\infty(x, Dz_0, z_0; w_0) v dx = \tag{4.11}$$

$$\int_\Omega F_2^\infty(x; w_0) v dx \text{ for all } v \in V_2$$

(where w_0 *is the solution of (4.10)) and*

$$\lim_{t\to\infty} \|z(t) - z_0\|_{L^2(\Omega)} = 0, \quad \lim_{T\to\infty} \int_{T-b}^{T+b} \|z(t) - z_0\|^p_{V_2} dt = 0 \tag{4.12}$$

for arbitrary fixed $b > 0$. *If*

$$\varphi_j, \psi \in L^q(0, \infty) \text{ then } z \in L^p(0, \infty; V_2). \tag{4.13}$$

Proof. Let $(T_k)_{k\in\mathbb{N}}$ be a monotone increasing sequence, converging to $+\infty$. According to Theorem 2.1, there exist solutions u_k, z_k of (2.1) – (2.4) for $t \in (0, T_k)$. The Volterra property of H, F_1, a_j, F_2 implies that the restrictions of u_k, z_k to $t \in (0, T_l)$ with $T_l < T_k$ satisfy (2.1) – (2.4) for $t \in (0, T_l)$.

Now consider the restrictions $u_k|_{(0,T_1)}$, $z_k|_{(0,T_1)}$, $k = 2, 3, \ldots$. Applying (2.33), (2.34) and $\delta < p - 1$ to $T = T_1$ and $\tilde{z} = z_k|_{(0,T_1)}$ we obtain that the sequence

$$\left(z_k|_{(0,T_1)}\right)_{k\in\mathbb{N}} \text{ is bounded in } L^p(Q_{T_1}) \tag{4.14}$$

thus by Lemma 2.4 there is a subsequence $(z_{1k})_{k\in\mathbb{N}}$ of $(z_k)_{k\in\mathbb{N}}$ such that the sequence of restrictions $(z_{1k}|_{(0,T_1)})_{k\in\mathbb{N}}$ is convergent in $L^p(Q_{T_1})$.

Now consider the restrictions $z_{1k}|_{(0,T_2)}$ By using the above arguments, we find that there exists a subsequence $(z_{2k})_{k\in\mathbb{N}}$ of $(z_{21})_{k\in\mathbb{N}}$ such that $(z_{2k}|_{(0,T_2)})_{k\in\mathbb{N}}$ is convergent in $L^p(Q_{T_2})$.

Thus for all $l \in \mathbb{N}$ we obtain a subsequence $(z_{lk})_{k\in\mathbb{N}}$ of $(z_k)_{k\in\mathbb{N}}$ such that $(z_{lk}|_{(0,T_l)})_{k\in\mathbb{N}}$ is convergent in $L^p(Q_{T_l})$. Then the diagonal sequence $(z_{kk})_{k\in\mathbb{N}}$ is a subsequence of $(z_k)_{k\in\mathbb{N}}$ such that for all fixed $l \in \mathbb{N}$, $(z_{kk}|_{(0,T_l)})_{k\in\mathbb{N}}$ is convergent in $L^p(Q_{T_l})$ to some $z^\star \in L^p_{loc}(Q_\infty)$. Since z_{ll} is a fixed point of $S = S_l : L^p(Q_{T_l}) \to L^p(Q_{T_l})$ and S_l is continuous thus the limit $z^\star|_{(0,T_l)}$ in $L^p(Q_{T_l})$ of $(z_{kk}|_{(0,T_l)})_{k\in\mathbb{N}}$ is a fixed point of $S = S_l$.

Consequently, the solutions $u_l^\star$ of (2.1), (2.2) when z is the restriction of $z^\star$ to $(0, T_l)$ and the restriction of $z^\star$ to $(0, T_l)$ satisfy (2.1) – (2.4) for $t \in (0, T_l)$. Since for $m < l$, $u_l^\star|_{(0,T_m)} = u_m^\star$ (by the Volterra property of H, F_1, a_j, F_2), we obtain $u^\star \in L^2_{loc}(Q_\infty)$ such that for all fixed l, $u^\star|_{(0,T_l)}$, $z^\star|_{(0,T_l)}$ satisfy (2.1) – (2.4) for $t \in (0, T_l)$, so the first part of Theorem 4.1 is proved.

Now assume that the additional conditions (4.1) - (4.6) are satisfied. Then we obtain (4.7) – (4.10) for $u = u^\star$, $z = z^\star$ by using the arguments of the proof of Theorem 3.2 in [11]. For convenience we formulate the main steps of the proof.

The sequence $(z_{kk})|_{k\in\mathbb{N}}$ is bounded in $L^p(0, T_l; V_2)$ for each fixed l by (2.19) – (2.21), (B_4), (4.14)), consequently, from (2.13) (with $\tilde{z}_k = z_{kk}$) we obtain for the solutions u_{kk} of (2.1), (2.2) with $\tilde{z} = z_{kk}$ (since u_{kk} is the limit of the Galerkin approximations $\tilde{u}_{mk}$)

$$\frac{1}{2}\|u'_{kk}(t)\|^2_H + \frac{1}{2}\langle Q(u_{kk}(t)), u_{kk}(t)\rangle + \int_\Omega \varphi(x)h(u_{kk}(t))dx + \tag{4.15}$$

$$\int_0^t \left[\int_\Omega \psi(x)|u'_{kk}(\tau)|^2 dx\right] d\tau + \int_0^t \left[\int_\Omega H(\tau, x; u_{kk}, z_{kk})u'_{kk}(\tau)dx\right] d\tau =$$

$$\int_0^t \left[\int_\Omega F_1(\tau, x; z_{kk})u'_{kk}(\tau)dx\right] d\tau + \frac{1}{2}\|u'_{kk}(0)\|^2_H + \frac{1}{2}\langle Q(u_{kk}(0)), u_{kk}(t)\rangle +$$

$$\int_\Omega \varphi(x)h(u_{kk}(0))dx$$

for all $t > 0$. Hence we find by (4.1), (4.2) and Young's inequality for $w_{kk} = u_{kk} - u_\infty$

$$\frac{1}{2}\|w'_{kk}(t)\|^2_{L^2(\Omega)} + \frac{c_0}{2}\|u_{kk}(t))\|^2_{V_1} + c_1\int_\Omega h(u_{kk}(t))dx + \text{const}\int_0^t \left[\int_\Omega |w'_{kk}|^2 dx\right] d\tau \leq \tag{4.16}$$

$$\text{const}\left\{\int_0^t \|F_1(\tau,x;z_{kk}) - F_1^\infty\|_H^2 d\tau + \int_0^t \|H(\tau,x;u_{kk}z_{kk}) - H^\infty\|_H^2 d\tau\right\} +$$

$$\varepsilon\int_0^t\left[\int_\Omega |w'_{kk}|^2dx\right]d\tau + \frac{1}{2}\|u'_{kk}(0)\|_H^2 + \frac{1}{2}\langle Q(u_{kk}(0)), u_{kk}(0)\rangle + c_2\int_\Omega h(u_{kk}(0))dx \le$$

$$\varepsilon\int_0^t\left[\int_\Omega |w'_{kk}|^2dx\right]d\tau + \text{const} + C(\varepsilon)\|\tilde\beta\|^2_{L^2(0,\infty;H)}.$$

Choosing sufficiently small $\varepsilon > 0$, we obtain

$$\int_0^t\left[\int_\Omega |w'_{kk}|^2dx\right]d\tau \le \text{const} \tag{4.17}$$

and thus by (4.16)

$$\|u'_{kk}(t)\|^2_{L^2(\Omega)} + \tilde c\int_0^t \|u'_{kk}(\tau)\|^2_{L^2(\Omega)}d\tau \le c^\star$$

with some positive constants $\tilde c$ and $c^\star$ not depending on k and $t \in (0,\infty)$. Hence by Gronwall's lemma we obtain (4.8) and by (4.16) we find (4.7).

It is not difficult to show that

$$\|u(T_2) - u(T_1)\|_H \le \int_{T_1}^{T_2} \|u'(t)\|_H dt \tag{4.18}$$

(see [11]), thus (4.8) implies (4.9) and by $u \in L^\infty(0,\infty;V_1)$, the limit w_0 of $u(t)$ as $t \to \infty$ must belong to V_1.

In order to prove (4.10) we apply equation (1.1) to $v\chi_{T_k}(t)$ with arbitrary fixed $v \in V_1$ where $\lim_{k\to\infty}(T_k) = +\infty$ and

$$\chi_{T_k}(t) = \chi(t - T_k), \quad \chi \in C_0^\infty, \quad supp\chi \subset [0,1], \quad \int_0^1 \chi(t)dt = 1.$$

Then by (4.8) one obtains (4.10) as $k \to \infty$.

Now we show that there exists a unique solution $z_0 \in V_2$ of (4.11). This statement follows from the fact that the operator (applied to $z_0 \in V_2$) on the left hand side of (4.11) is bounded, demicontinuous and uniformly monotone (see, e.g. [13]) by (B_1), (B_2), (4.9), (4.5), (4.6).

Finally, we show (4.12). By (4.6) we have

$$\frac{1}{2}\frac{d}{dt}\|z(t) - z_0\|_H^2 + \frac{c_2}{1 + \|u\|_{L^2(Q_t\setminus Q_{t-a})}}\|z(t) - z_0\|^p_{V_2} \le \tag{4.19}$$

$$\int_\Omega \sum_{j=1}^n [a_j(t,x,Dz,z;u,z) - a_j(t,x,Dz_0,z_0;u,z)](D_jz - D_jz_0)dx +$$

$$\int_\Omega [a_0(t,x,Dz,z;u,z) - a_0(t,x,Dz_0,z_0;u,z)](z - z_0)dx =$$

$$\int_\Omega [F_2(t,x;u) - F_2^\infty(x,w_0)](z - z_0)dx -$$

$$\int_\Omega \sum_{j=1}^n [a_j(t,x,Dz_0,z_0;u,z) - a_j^\infty(x,Dz_0,z_0;w_0)](D_jz - D_jz_0)dx -$$

$$\int_\Omega [a_0(t,x,Dz_0,z_0;u,z) - a_0^\infty(t,x,Dz_0,z_0;w_0)](z-z_0)dx \le$$
$$C(\varepsilon)\|F_2(t,x;u) - F_2^\infty(x,w_0)\|_{L^q(\Omega)} + \varepsilon\|z(t)-z_0\|_{L^p(\Omega)} +$$
$$C(\varepsilon)\sum_{j=1}^n \|a_j(t,x,Dz_0,z_0;u,z) - a_j^\infty(x,Dz_0,z_0;w_0)\|^q_{L^q(\Omega)} + \varepsilon\|D_j z(t) - D_j z_0\|^p_{L^p(\Omega)} +$$
$$C(\varepsilon)\|a_0(t,x,Dz_0,z_0;u,z) - a_0^\infty(x,Dz_0,z_0;w_0)\|^q_{L^q(\Omega)} + \varepsilon\|z(t)-z_0\|^p_{L^p(\Omega)}.$$

Since $\|u\|^\beta_{L^2(Q_t\setminus Q_{t-a})}$ is bounded for $t\in(0,\infty)$ by (4.9) and

$$\|z(t)-z_0\|_{V_2} \ge \text{const}\|z(t)-z_0\|_{L^2(\Omega)}$$

with some positive constant, thus by (4.3) – (4.5), (4.19) with sufficiently small $\varepsilon > 0$ we obtain for

$$y(t) = \|z(t)-z_0\|^2_H$$

the inequality

$$y'(t) + c^\star[y(t)]^{p/2} \le g(t) \tag{4.20}$$

where $c^\star$ is a positive constant and $\lim_\infty g = 0$.

The inequality (4.20) implies the first part of (4.12):

$$\lim_\infty y = 0 \tag{4.21}$$

(see [10]). Integrating (4.19) with respect to t over $(T-b, T+b)$ we obtain the second part of (4.12) by (4.21). Integrating (4.19) with respect to t over $(0,T)$, by (4.21) we obtain (4.13) as $T\to\infty$.

Acknowledgement. This work was supported by the Hungarian National Foundation for Scientific Research under grant OTKA K 115926.

References

[1] Adams, R.A., *Sobolev spaces*, Academic Press, New York - San Francisco - London, 1975.

[2] Berkovits, J., Mustonen, V., *Topological degreee for perturbations of linear maximal monotone mappings and applications to a class of parabolic problems*, Rend. Mat. Ser. VII, **12**(1992), 597-621.

[3] Besenyei, A., *On nonlinear systems containing nonlocal terms*, PhD Thesis, Eötvós Loránd University, Budapest, 2008.

[4] Cinca, S., *Diffusion und Transport in porösen Medien bei veränderlichen Porosität*, Diplomawork, Univ. Heidelberg, 2000.

[5] Lions, J.L., *Quelques méthodes de résolution des problemes aux limites non linéaires*, Dunod Gauthier-Villars, Paris, 1969.

[6] Logan, J.D., Petersen, M.R., Shores, T.S., *Numerical study of reaction-mineralogy-porosity changes in porous media*, Appl. Math. Comput., **127**(2002), 149-164.

[7] Merkin, J.H., Needham, D.J., Sleeman, B.D., *A mathematical model for the spread of morphogens with density dependent chemosensitivity*, Nonlinearity, **18**(2005), 2745-2773.

[8] Simon, L., *On some singular systems of parabolic functional equations*, Math. Bohem., **135**(2010), 123-132.

[9] Simon, L., *On singular systems of parabolic functional equations*, Operator Theory: Advances and Applications, **216**(2011), 317-330.

[10] Simon, L., *Application of monotone type operators to parabolic and functional parabolic PDE's*, Handbook of Differential Equations. Evolutionary Equations, Vol. 4, Elsevier, 2008, 267-321.

[11] Simon, L., *Semilinear hyperbolic functional equations*, Banach Center Publications, **101**(2014), 205-222.

[12] Wu, J., *Theory and Applications of Partial Functional Differential Equations*, Springer, 1996.

[13] Zeidler, E., *Nonlinear functional analysis and its applications II A and II B*, Springer, 1990.

László Simon
L. Eötvös University of Budapest, Hungary
e-mail: `simonl@cs.elte.hu`

Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 2, 195–220

Inner amenable hypergroups, invariant projections and Hahn-Banach extension theorem related to hypergroups

Nazanin Tahmasebi

Abstract. Let K be a hypergroup with a Haar measure. In the present paper we initiate the study of inner amenable hypergroups extending amenable hypergroups and inner amenable locally compact groups. We also provide characterizations of amenable hypergroups by hypergroups having the Hahn-Banach extension or monotone projection property. Finally we focus on weak*-invariant complemented subspaces of $L_\infty(K)$.

Mathematics Subject Classification (2010): 43A07.

Keywords: Hypergroup, inner amenability, quasi central, approximate identity, asymptotically central, semidirect product hypergroup, strong ergodicity, Hahn-Banach extension property, monotone extension property, partially ordered real Banach space; amenability, real version of amenability, multiplicative left invariant mean.

1. Introduction

The classified theory of topological hypergroups have been well established in the 1970's by the works of Dunkl [6], Jewett [12] and Spector [29] independently. The history then observed a good interest in the study of this object in diverse areas of mathematics such as compact quantum hypergroups [2] weighted hypergroups [8, 9], amenable [13, 15, 31, 32] and commutative hypergroups [14, 24, 25]. A complete history of hypergroups can be found in [26].

Inner amenable locally compact groups G are ones possessing a mean m on $L_\infty(G)$ such that $m(R_g L_{g^{-1}} f) = m(f)$, for all $f \in L_\infty(G)$ and $g \in G$. This concept was introduced by Effros in 1975 for discrete groups and was studied by several authors [3, 4, 7, 17, 19, 21, 22]. It has been shown by Losert and Rindler that the existence of an inner invariant mean on $L_\infty(G)$ is equivalent to the existence of an asymptotically central net in $L_1(G)$ which is in the case of groups equivalent to the existence of a quasi central net in $L_1(G)$.

In section 3 we define the notion of inner amenable hypergroups extending amenable hypergroups and inner amenable locally compact groups. We say that a hypergroup K is inner amenable and m is an inner invariant mean if m is a mean on $L_\infty(K)$ and $m(L_g f) = m(R_g f)$ for all $f \in L_\infty(K)$ and all $g \in K$. An inner invariant mean m on a discrete hypergroup K is nontrivial if $m(f) \neq f(e)$ for $f \in l_\infty(K)$. In the process of constructing a discrete hypergroup with no nontrivial inner invariant mean we also define the concept of strong ergodicity of an action of a locally compact group on a hypergroup. Then we prove a relation between nontrivial inner invariant means on bounded functions of the semidirect product $K \rtimes_\tau G$ of a discrete hypergroup K and a discrete group G and strong ergodicity of the action τ. If K is commutative and τ is not strongly ergodic, then $l_\infty(K \rtimes_{\tau|_S} S)$ possesses a nontrivial inner invariant mean for each subgroup S of G, however, if τ is strongly ergodic and $l_\infty(G)$ has no nontrivial inner invariant mean, then $l_\infty(K \rtimes_\tau G)$ has no nontrivial inner invariant mean (Theorem 3.5).

Then we prove that inner amenability is an asymptotic property; there is a positive norm one net $\{\phi_\alpha\}$ in $L_1(K)$ such that $||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha||_1 \to 0$, for all $g \in K$ if and only if K is inner amenable (Lemma 3.2), while the existence of a positive norm one net $\{\phi_\alpha\}$ in $L_2(K)$ such that $||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2 \to 0$, for all $g \in K$ only implies the inner amenability of K (Lemma 3.6) and implies the existence of a state m on $B(L_2(K))$ such that $m(L_g) = m(\Delta^{\frac{1}{2}}(g)R_g)$, for all $g \in K$ (Theorem 3.8). Furthermore, in Corollary 3.14 we characterize inner amenability of a hypergroup K in terms of compact operators; K is inner amenable if and only if there is a non-zero positive compact operator T in $B(L_\infty(K))$ such that $TL_g = TR_g$, for all $g \in K$.

Classical Hahn-Banach extension theorem and monotone extension property are well known and are widely used in several areas of mathematics. As one deals with (positive normalized) anti-actions of a semigroup on a real (partially ordered) topological vector space (with a topological vector unit), it is also interesting to know the condition under which the extension of an invariant (monotonic) linear functional is also invariant (and monotonic). In 1974 Lau characterized left amenable semigroups with these properties ([16], Theorems 1 and 2).

In section 4 we shall be concerned about hypergroup version of Hahn-Banach extension and monotone extension properties and we prove in Theorem 4.1 that $RUC(K)$ has a right invariant mean if and only if whenever $\{T_g \in B(E) \mid g \in K\}$ is a separately continuous representation of K on a Banach space E and F is a closed T_K-invariant subspace of E. If p is a continuous seminorm on E such that $p(T_g x) \leq p(x)$ for all $x \in E$ and $g \in K$ and Φ is a continuous T_K-invariant linear functional on F such that $|\Phi(x)| \leq p(x)$, then there is a continuous T_K-invariant linear functional $\tilde{\Phi}$ on E extending Φ such that $|\tilde{\Phi}(x)| \leq p(x)$, for all $x \in E$, if and only if for any positive normalized separately continuous linear representation $\mathscr{T}$ of K on a partially ordered real Banach space E with a topological order unit 1, if F is a closed $\mathscr{T}$-invariant subspace of E containing 1, and Φ is a $\mathscr{T}$-invariant monotonic linear functional on F, then there exists a $\mathscr{T}$-invariant monotonic linear functional $\tilde{\Phi}$ on E extending Φ.

The three statements above are also equivalent to an algebraic property: for any positive normalized separately continuous linear representation $\mathscr{T}$ of K on a partially ordered real Banach space E with a topological order unit 1, E contains a maximal proper $\mathscr{T}$-invariant ideal. As an application of these important geometric properties we provide a new proof of the known result; if K is a commutative hypergroup, then $UC(K)$ has an invariant mean (Corollary 4.4).

Let X be a weak*-closed left translation invariant subspace of $L_\infty(K)$. The concentration of section 5 is mainly on weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations. It turns out that similar to the locally compact groups ([18], Lemma 5.2), if X is an invariant complemented subspace of $L_\infty(K)$, then there is a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations if and only if $X \cap C_0(K)$ is weak*-dense in X (Theorem 5.1). This theorem has two major consequences; if K is compact, then X is invariantly complemented in $L_\infty(K)$ if and only if there is a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations (Corollary 5.2) and if K is commutative with connected dual, then there is no non-trivial weak*-weak*-continuous projections on $L_\infty(K)$ commuting with left translations (Corollary 5.6). Furthermore, we also characterize compact hypergroups; K is compact if and only if K is amenable and for every weak*-closed left translation invariant, invariant complemented subspace X of $L_\infty(K)$, there exists a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations (Corollary 5.4).

Finally, in section 6 we provide some remarks and related open problems.

2. Preliminaries and some notations

Throughout this manuscript, K denotes a hypergroup with a left Haar measure λ. For basic notations we refer to [12, 1]. The involution on K is denoted by $x \mapsto \check{x}$. Let L_x and R_y denote the left and right translation operators for $x, y \in K$ given by $R_y f(x) = L_x f(y) = \int f(u) d\delta_x * \delta_y(u)$, for any Borel function f on K, if this integral exists. Let $\phi * \mu(g) = \int R_{\check{k}}\phi(g)d\mu(k)$ and $\phi \circledast \mu(g) = \int \Delta(\check{k}) R_{\check{k}}\phi(g)d\mu(k)$, for $\mu \in M(K)$ and $\phi \in L_1(K)$. Then $(\phi \circledast \mu)\lambda = \phi\lambda * \mu$. We note that $\phi \circledast \mu$ is denoted by $\phi * \mu$ in the group setting. A closed subhypergroup N of K is a Weil subhypergroup if the mapping $f \mapsto T_N f$, where $(T_N f)(g * N) = \int R_n f(g) d\lambda_N(n)$ and λ_N is a left Haar measure on N is a well defined map from $C_c(K)$ onto $C_c(K/N)$ [11]. It is well known that any subgroup and any compact subhypergroup is a Weil subhypergroup ([11], p 250). If N is a closed normal subhypergroup, then K/N is a hypergroup if the convolution $\delta_{g*N} * \delta_{k*N}(f) = \int f(u * N) d\delta_g * \delta_k(u)$ $(f \in C_c(K/N))$ is independent of the choice of the representatives $g * N$ and $k * N$ [33]. The locally compact space K/N is a hypergroup if and only if N is a closed normal Weil subhypergroup of K ([33], Theorems 2.3 and 2.6). Let $(K, *)$ and $(J, .)$ be hypergroups. Then a continuous mapping $p : K \to J$ is said to be a hypergroup homomorphism if $\delta_{p(g)}.\delta_{p(k)\check{}} = p(\delta_g * \delta_{\check{k}})$, for all $g, k \in K$. The modular function Δ is defined by $\lambda * \delta_{\check{g}} = \Delta(g)\lambda$, where λ is a left Haar measure on K and $g \in K$.

Let $CB(K)$ denote the space of all bounded continuous complex-valued functions on K and $C_c(K)$ the space of all continuous bounded functions on K with compact support. Let $LUC(K)$ $(RUC(K))$ be the space of all bounded left (right) uniformly continuous functions on K, i.e. all $f \in CB(K)$ such that the map $g \mapsto L_g f$ ($g \mapsto R_g f$) from K into $CB(K)$ is continuous when $CB(K)$ has the norm topology. Then $LUC(K)$ ($RUC(K)$) is a norm closed, conjugate closed, translation invariant subspace of $CB(K)$ containing constant functions.

Let X be a closed translation invariant subspace of $L_\infty(K)$ containing constants. Then a left invariant mean on X is a positive norm one linear functional, which is invariant under left translations and a hypergroup K is said to be amenable if there is a left invariant mean on $L_\infty(K)$. It is known that all compact and commutative hypergroups are amenable [28]. Furthermore, a closed left translation invariant complemented subspace Y of $L_\infty(K)$ is called invariant subspace, if there is a continuous projection P from $L_\infty(K)$ onto Y commuting with left translations. If Y is weak*-closed and P is weak*-weak*-continuous, then we say that Y is weak*-invariant complemented subspace of $L_\infty(K)$.

The representation $\mathscr{T} = \{T_g \mid g \in K\}$ is said to be a separately continuous representation of K on a Banach space X if $T_g : X \to X$, $T_e = I$, $||T_g|| \leq 1$, for each $g \in K$, the mapping $(g, x) \mapsto T_g x$ from $K \times X$ to X is separately continuous, and $T_{g_1}T_{g_2}x = \int T_u x d\delta_{g_1} * \delta_{g_2}(u)$, for $x \in X$ and $g_1, g_2 \in K$. If $\mathscr{T}$ is a continuous representation of K on X, then for $g \in K$, $\mu \in M(K)$, $f \in X^*$ and $\phi \in X$ define $f \,.\, g = M_g f$ by $< f \,.\, g, \phi >=< f, T_g\phi >$ and $f \,.\, \mu = M_\mu f$ by $< f \,.\, \mu, \phi >= \int < f, T_g\phi > d\mu(g)$. Then $f \,.\, \mu \in X^*$, $f \,.\, \delta_g = f \,.\, g$ and $(f \,.\, \mu) \,.\, \nu = f \,.\, (\mu * \nu)$, for $\mu, \nu \in M(K)$. Moreover, let $< N_g m, f >=< m, M_g f >$, $< N_\mu m, f >=< m, f \,.\, \mu >$ and $N_\phi = N_{\phi\lambda}$, for $\mu \in M(K)$, $\phi \in L_1(K)$, $m \in X^{**}$, $f \in X^*$ and $g \in K$. Then $N_\mu N_\nu = N_{\mu*\nu}$ and $N_\phi N_\mu = N_{\phi \circledast \mu}$, for each $\mu, \nu \in M(K)$. In addition, $||M_g|| \leq 1$, $||N_g|| \leq 1$, $||M_\mu|| \leq ||\mu||$ and $||N_\mu|| \leq ||\mu||$, for all $\mu \in M(K)$ and $g \in K$.

3. Inner amenable hypergroups

Let G be a locally compact group. A mean m on $L_\infty(G)$ is called inner invariant and G is called inner amenable if $m(L_g R_{g^{-1}} f) = m(f)$, for all $g \in G$ and $f \in L_\infty(G)$ (see [7] for discrete case) which is equivalent to saying that $L_g^* m = R_g^* m$, for all $g \in G$. However, this equivalence relation breaks down when one deals with hypergroups.

We say that a *hypergroup K is inner amenable* if there exists a mean m on $L_\infty(K)$ such that $m(R_g f) = m(L_g f)$ for all $g \in K$ and $f \in L_\infty(K)$. Of course amenable hypergroups are inner amenable since each invariant mean is also an inner invariant mean. An inner invariant mean m on a non-trivial discrete hypergroup is called non-trivial if $m \neq \delta_e$, the point evaluation function on $l_\infty(K)$. If this is the case, then $m_1 = \frac{m - m(\{e\})\delta_e}{1 - m(\{e\})}$ is an inner invariant mean on $l_\infty(K)$ and $m_1(\{e\}) = 0$. Any invariant mean on $l_\infty(K)$ is a non-trivial inner invariant mean and hence any non-trivial discrete amenable hypergroup possesses a non-trivial inner invariant mean.

Example 3.1. *Let H be a nontrivial discrete amenable hypergroup and J be a discrete non-amenable hypergroup. Then $K = H \times J$ is a non-amenable hypergroup and $l_\infty(K)$ has a non-trivial inner invariant mean.*

Proof. Let H be a discrete nontrivial amenable hypergroup and J be a discrete non-amenable hypergroup. Let $K = H \times J$ with the identity (e_1, e_2). If m is an invariant mean on $l_\infty(H)$ and $f \in l_\infty(K)$, then for each $k \in J$ define a function $f_k \in l_\infty(H)$ via $f_k(g) = f(g,k)$. Furthermore, define a mean m_1 on $l_\infty(K)$ by $m_1(f) = m(f_{e_2})$. Then $m_1(f) = m(f_{e_2}) \neq f_{e_2}(e_1) = f(e_1, e_2)$. In addition, for $(g_1, g_2) \in K$ and $k \in H$ we have

$$\begin{aligned}(L_{(g_1,g_2)}f)_{e_2}(k) &= L_{(g_1,g_2)}f(k,e_2)\\ &= \sum_{(u,v)\in K} f(u,v)\delta_{(g_1,g_2)} * \delta_{(k,e_2)}(u,v)\\ &= \sum_{u\in H}\sum_{v\in J} f(u,v)\delta_{g_1} * \delta_k(u)\delta_{g_2} * \delta_{e_2}(v)\\ &= \sum_{u\in H} f_{g_2}(u)\delta_{g_1} * \delta_k(u)\\ &= L_{g_1}f_{g_2}(k).\end{aligned}$$

Hence, $(L_{(g_1,g_2)}f)_{e_2} = L_{g_1}f_{g_2}$. Similarly, $(R_{(g_1,g_2)}f)_{e_2} = R_{g_1}f_{g_2}$. Thus,

$$\begin{aligned}m_1(L_{(g_1,g_2)}f) &= m((L_{(g_1,g_2)}f)_{e_2})\\ &= m(L_{g_1}f_{g_2})\\ &= m(R_{g_1}f_{g_2})\\ &= m((R_{(g_1,g_2)}f)_{e_2})\\ &= m_1(R_{(g_1,g_2)}f).\end{aligned}$$

□

The following result shows that similar to the locally compact groups ([22], Proposition 1), inner amenability of a hypergroup is also an asymptotic property.

Lemma 3.2. *The following are equivalent:*

1. *K is inner amenable.*
2. *There is a net $\{\phi_\alpha\}$ in $L_1(K)$ with $\phi_\alpha \geq 0$ and $||\phi_\alpha||_1 = 1$ such that*
$$||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha||_1 \to 0,$$
for all $g \in K$.
3. *There is a net $\{\psi_\beta\}$ in $L_1(K)$ with $\psi_\beta \geq 0$ such that*
$$\frac{1}{||\psi_\beta||}||L_g\psi_\beta - \Delta(g)R_g\psi_\beta||_1 \to 0,$$
for all $g \in K$.

Proof. For 3 ⇒ 2 put $\phi_\alpha = \frac{\psi_\alpha}{||\psi_\alpha||}$. We will prove the equivalence of 1 and 2. Let m be a mean on $L_\infty(K)$ such that $m(L_gf) = m(R_gf)$, for $f \in L_\infty(K)$ and $g \in K$. Then there is a net of positive norm one elements $\{q_\gamma\}$ in $L_1(K)$ such that $< L_gq_\gamma - \Delta(g)R_gq_\gamma, f > \to 0$, for each $f \in L_\infty(K)$. Let T be a map from $L_1(K)$ into $L_1(K)^K$ defined by $T\phi(g) = \Delta(g)R_g\phi - L_g\phi$, for $f \in L_\infty(K)$, $\phi \in L_1(K)$ and $g \in K$. Thus, $0 \in \overline{T(P_1(K))}$, where $P_1(K) = \{\phi \in L_1(K) \mid \phi \geq 0,\ ||\phi|| = 1\}$. Therefore, there is a net of positive norm one elements $\{\phi_\alpha\}$ in $L_1(K)$ such that $||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha|| \to 0$. Conversely, let m be any weak*-cluster point of $\{\phi_\alpha\}$ in $L_\infty(K)^*$. Then m is a mean on $L_\infty(K)$ such that $m(R_gf) = m(L_gf)$ for all $g \in K$ and $f \in L_\infty(K)$. □

Corollary 3.3. *Let K be a discrete hypergroup. Then the following are equivalent:*

1. *There is an inner invariant mean m on $l_\infty(K)$ such that $m(\{e\}) = 0$.*
2. *There is a net $\{\phi_\alpha\}$ in $l_1(K)$ with $\phi_\alpha \geq 0$ and $||\phi_\alpha||_1 = 1$ such that $\phi_\alpha(e) = 0$ and that $||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha||_1 \to 0$, for all $g \in K$.*

Let G be a locally compact group and let τ be a continuous group homomorphism from G into the topological group $Aut(K)$ of all hypergroup homomorphisms on K. The semidirect product $K \rtimes_\tau G$ of K and G is the locally compact space $K \times G$ equipped with the product topology, the convolution $\delta_{(k_1,g_1)} * \delta_{(k_2,g_2)} = \delta_{k_1} * \delta_{\tau_{g_1}(k_2)} \otimes \delta_{g_1 g_2}$ and a natural embedding of the tensor product $M(K) \otimes M(G)$ into $M(K \times G)$ [34]. In this case, there is a natural action τ of G on $L_p(K)$ $(1 \leq p \leq \infty)$ defined by $\tau_g f(k) = f(\tau_g k)$ for $f \in L_p(K)$, $g \in G$ and $k \in K$. If G and K are discrete, then we say that τ is strongly ergodic if the condition $||\tau_g \phi_\alpha - \phi_\alpha||_2 \to 0$, for some positive norm one net $\{\phi_\alpha\}$ in $l_2(K)$ and all $g \in G$ implies that $\phi_\alpha(e_1) \to 1$, where e_1 is the identity of K. In addition, a mean m on $l_\infty(K)$ is τ-invariant if $m(\tau_g f) = m(f)$, for all $g \in G$ and $f \in l_\infty(K)$. The trivial τ-invariant mean on $l_\infty(K)$ is given by $\delta_{e_1}(f) = f(e_1)$, for $f \in l_\infty(K)$ (for the corresponding definitions in the countable group setting see [4]).

The following three results are inspired by [4].

Lemma 3.4. *Let G be a discrete group and let τ be a continuous group homomorphism from G into the topological group $Aut(K)$ of all hypergroup homomorphisms on a discrete hypergroup K. Then there is a non-trivial τ-invariant mean m on $l_\infty(K)$ if and only if τ is not strongly ergodic.*

Proof. Let m be a non-trivial τ-invariant mean on $l_\infty(K)$. Without loss of generality assume $m(\delta_e) = 0$, where e is the identity of K. By a standard argument (see the proof of Lemma 3.2 for example) find a positive norm one net $\{\psi_\alpha\}$ in $l_1(K)$ such that $||\tau_g \psi_\alpha - \psi_\alpha|| \to 0$ for all $g \in G$ and $\lim_\alpha \psi_\alpha(e) = 0$. Then $\{\phi_\alpha = \psi_\alpha^{\frac{1}{2}}\}$ is a positive norm one net in $l_2(K)$, $\lim_\alpha \phi_\alpha(e) = 0$ and for $g \in G$

$$||\tau_g \phi_\alpha - \phi_\alpha||_2^2 = ||\tau_g(\psi_\alpha^{\frac{1}{2}}) - \psi_\alpha^{\frac{1}{2}}||_2^2 = ||(\tau_g \psi_\alpha)^{\frac{1}{2}} - \psi_\alpha^{\frac{1}{2}}||_2^2 \leq ||\tau_g \psi_\alpha - \psi_\alpha||_1 \to 0.$$

Therefore, τ is not strongly ergodic. Conversely, let $\{\phi_\alpha\}_{\alpha \in I}$ be a positive norm one net in $l_2(K)$ such that $||\tau_g \phi_\alpha - \phi_\alpha||_2^2 \to 0$ and that $\lim_\alpha \phi_\alpha(e) \neq 1$. Choose $\alpha_0 \in I$ such that $\phi_\alpha(e) \neq 1$ for all $\alpha \geq \alpha_0$ and put $I_1 = \{\alpha \in I \mid \alpha \geq \alpha_0\}$. Then $\{\psi_\alpha = \frac{\phi_\alpha - \phi_\alpha(e)\delta_e}{1-\phi_\alpha(e)}\}_{\alpha \in I_1}$ is a positive norm one net in $l_2(K)$ such that $||\tau_g \psi_\alpha - \psi_\alpha||_2^2 \to 0$ and $\psi_\alpha(e) = 0$ for all $\alpha \in I_1$. Let m be a weak*-cluster point of $\{\psi_\alpha^2\}_{\alpha \in I_1}$ in $l_\infty(K)^*$ and by passing possibly to a subnet assume $m(f) = \lim < \psi_\alpha^2, f >$. Then m is a nontrivial τ-invariant mean on $l_\infty(K)$. □

Theorem 3.5. *Let $K \rtimes_\tau G$ be the semidirect product hypergroup of a discrete hypergroup K and a discrete group G.*

1. *If K is commutative and τ is not strongly ergodic, then for each subgroup S of G, $l_\infty(K \rtimes_{\tau|_S} S)$ possesses a non-trivial inner invariant mean.*
2. *If τ is strongly ergodic and $l_\infty(G)$ has no non-trivial inner invariant mean, then $l_\infty(K \rtimes_\tau G)$ has no non-trivial inner invariant mean.*

Proof. 1. Assume that there exists a subgroup S of G such that $l_\infty(K \rtimes_{\tau|_S} S)$ has no non-trivial inner invariant mean. Let m be a mean on $l_\infty(K)$ such that $m(\tau_g f) = m(f)$, for all $g \in S$ and $f \in l_\infty(K)$. We will show that m is trivial. For $f \in l_\infty(K \rtimes_{\tau|_S} S)$ and $g \in S$ define a function $f_g \in l_\infty(K)$ by $f_g(k) = f(k, g)$,

$(k \in K)$. Let $M(f) = m(f_{e_2})$, for $f \in l_\infty(K \rtimes_{\tau|_S} S)$. Then M is a mean on $l_\infty(K \rtimes_{\tau|_S} S)$. For $f \in l_\infty(K \rtimes_{\tau|_S} S)$, $(k_1, g_1) \in K \rtimes_{\tau|_S} S$ and $k \in K$

$$\begin{aligned}(L_{(k_1,g_1)}f)_{e_2}(k) &= L_{(k_1,g_1)}f(k,e_2)\\ &= \textstyle\sum_{(u,v)} f(u,v)\delta_{(k_1,g_1)} * \delta_{(k,e_2)}(u,v)\\ &= \textstyle\sum_u \sum_v f(u,v)\delta_{k_1} * \delta_{\tau_{g_1}k}(u)\delta_{g_1e_2}(v)\\ &= \textstyle\sum_u f(u,g_1)\delta_{k_1} * \delta_{\tau_{g_1}k}(u)\\ &= \textstyle\sum_u f_{g_1}(u)\delta_{k_1} * \delta_{\tau_{g_1}k}(u)\\ &= L_{k_1}f_{g_1}(\tau_{g_1}k)\\ &= \tau_{g_1}(L_{k_1}f_{g_1})(k).\end{aligned}$$

Moreover,

$$\begin{aligned}(R_{(k_1,g_1)}f)_{e_2}(k) &= R_{(k_1,g_1)}f(k,e_2)\\ &= \textstyle\sum_{(u,v)} f(u,v)\delta_{(k,e_2)} * \delta_{(k_1,g_1)}(u,v)\\ &= \textstyle\sum_u \sum_v f(u,v)\delta_k * \delta_{\tau_{e_2}k_1}(u)\delta_{e_2g_1}(v)\\ &= \textstyle\sum f_{g_1}(u)\delta_k * \delta_{k_1}(u)\\ &= L_{k_1}f_{g_1}(k),\end{aligned}$$

since K is commutative. Hence,

$$\begin{aligned}M(L_{(k_1,g_1)}f) &= m((L_{(k_1,g_1)}f)_{e_2})\\ &= m(\tau_{g_1}(L_{k_1}f_{g_1}))\\ &= m(L_{k_1}f_{g_1})\\ &= m((R_{(k_1,g_1)}f)_{e_2})\\ &= M(R_{(k_1,g_1)}f).\end{aligned}$$

Therefore, M is inner invariant. Then M is trivial, i.e, $M(f) = f(e_1, e_2)$. For $f \in l_\infty(K)$ let $f_1(k,g) = f(k)$ if $g = e_2$ and zero otherwise, $((k,g) \in K \rtimes_{\tau|_S} S)$. Then $(f_1)_{e_2}(k) = f_1(k,e_2) = f(k)$. Thus, $f(e_1) = f_1(e_1,e_2) = M(f_1) = m((f_1)_{e_2}) = m(f)$ which means that m is trivial. Consequently, τ is strongly ergodic by Lemma 3.4.

2. Suppose m is a non-trivial inner invariant mean on $l_\infty(K \rtimes_\tau G)$ and assume without loss of generality that $m(\delta_{(e_1,e_2)}) = 0$, where (e_1, e_2) is the identity of $K \rtimes_\tau G$. Then $m(R_{(e_1,g^{-1})}L_{(e_1,g)}h) = m(h)$, for all $h \in l_\infty(K \rtimes_\tau G)$ and $(e_1, g) \in K \rtimes_\tau G$. For $f \in l_\infty(K)$ let $f_1(k,g) = f(k)$ if $g = e_2$ and zero otherwise, $((k,g) \in K \rtimes_\tau G)$. Then $f_1 \in l_\infty(K \rtimes_\tau G)$. We will show that $m(\chi_{K \rtimes_\tau e_2}) = 0$. If not, then m_1 with

$$m_1(f) = \frac{m(f_1)}{m(\chi_{K \rtimes_\tau e_2})},\ (f \in l_\infty(K))$$

is a mean on $l_\infty(K)$ and $m_1(\delta_{e_1}) = 0$. For $(k_1, g_1), (e_1, g) \in K \rtimes_\tau G$ and $f \in l_\infty(K)$

$$\begin{aligned}R_{(e_1,g)}(\tau_g f)_1(k_1,g_1) &= \textstyle\sum_{(u,v)}(\tau_g f)_1(u,v)\delta_{(k_1,g_1)} * \delta_{(e_1,g)}(u,v)\\ &= \textstyle\sum_u \sum_v (\tau_g f)_1(u,v)\delta_{k_1} * \delta_{e_1}(u)\delta_{g_1g}(v)\\ &= (\tau_g f)_1(k_1, g_1 g)\end{aligned}$$

Hence,

$$R_{(e_1,g)}(\tau_g f)_1(k_1,g_1) = \begin{cases} \tau_g f(k_1) = f(\tau_g(k_1)) & \text{if } g_1 g = e_2, \\ 0 & \text{if } g_1 g \neq e_2. \end{cases} \quad (1)$$

In addition,

$$\begin{aligned} L_{(e_1,g)} f_1(k_1,g_1) &= \textstyle\sum_{(u,v)} f_1(u,v)\delta_{(e_1,g)} * \delta_{(k_1,g_1)}(u,v) \\ &= \textstyle\sum_u \sum_v f_1(u,v)\delta_{e_1} * \delta_{\tau_g(k_1)}(u) d\delta_{gg_1}(v) \\ &= f_1(\tau_g(k_1), gg_1) \end{aligned}$$

Thus,

$$L_{(e_1,g)}(f)_1(k_1,g_1) = \begin{cases} f(\tau_g(k_1)) & \text{if } gg_1 = e_2, \\ 0 & \text{if } gg_1 \neq e_2. \end{cases} \quad (2)$$

Therefore, $R_{(e_1,g)}(\tau_g f)_1 = L_{(e_1,g)} f_1$. In other words

$$(\tau_g f)_1 = R_{(e_1,g^{-1})} L_{(e_1,g)} f_1.$$

Now observe that

$$\begin{aligned} m_1(\tau_g f) &= \tfrac{m((\tau_g f)_1)}{m(\chi_{K \rtimes_\tau e_2})} \\ &= \tfrac{m(R_{(e_1,g^{-1})} L_{(e_1,g)} f_1)}{m(\chi_{K \rtimes_\tau e_2})} \\ &= \tfrac{m(f_1)}{m(\chi_{K \rtimes_\tau e_2})} \\ &= m(f). \end{aligned}$$

A contradiction with the strong ergodicity of τ (Lemma 3.4). Consequently, $m(\chi_{K \rtimes_\tau e_2}) = 0$. For a subset C of G let $m_2(\chi_C) = m(\chi_{K \rtimes_\tau C})$ and let m_3 be an extension of m_2 to a mean on $l_\infty(G)$. Then m_3 is a mean on $l_\infty(G)$ and $m_3(\delta_{e_2}) = m(\chi_{K \rtimes_\tau e_2}) = 0$. Furthermore, m_3 is also inner invariant since m_3 is an extension of m_2 and

$$(K \times gCg^{-1}) = (e_1, g)(K \times C)(e_1, g^{-1})$$

for each $g \in G$ and each subset C of G.

□

Lemma 3.6. *The following conditions hold:*

1. *If there is a net $\{\phi_\alpha\}$ in $L_2(K)$ with $\phi_\alpha \geq 0$ and $||\phi_\alpha||_2 = 1$ such that $||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g) R_g \phi_\alpha||_2 \to 0$, for all $g \in K$, then K is inner amenable.*
2. *If K is unimodular and there is a net $\{V_\alpha\}$ of Borel subsets of K with $0 < \lambda(V_\alpha) < \infty$ such that $||\frac{L_g \chi_{V_\alpha}}{\lambda(V_\alpha)} - \frac{R_g \chi_{V_\alpha}}{\lambda(V_\alpha)}||_1 \to 0$ for all $g \in K$, then there is a net $\{\psi_\alpha\}$ in $L_2(K)$ with $\psi_\alpha \geq 0$ and $||\psi_\alpha||_2 = 1$ such that $||L_g\psi_\alpha - R_g\psi_\alpha||_2 \to 0$, for all $g \in K$.*

Proof. (1): For each α put $\psi_\alpha = \phi_\alpha^2$. Then for $g, k \in K$

$$\begin{aligned} &\textstyle\int\int (\phi_\alpha(u) - \Delta^{\frac{1}{2}}(g)\phi_\alpha(v))^2 d\delta_g * \delta_k(u) d\delta_k * \delta_g(v) \\ &= L_g\phi_\alpha^2(k) + \Delta(g) R_g \phi_\alpha^2(k) - 2\Delta^{\frac{1}{2}}(g) L_g\phi_\alpha(k) R_g\phi_\alpha(k) \\ &= (L_g\phi_\alpha(k) - \Delta^{\frac{1}{2}}(g) R_g\phi_\alpha(k))^2 + L_g\phi_\alpha^2(k) \\ &+\Delta(g) R_g\phi_\alpha^2(k) - (L_g\phi_\alpha)^2(k) - \Delta(g)(R_g\phi_\alpha)^2(k) \end{aligned}$$

Hence,

$$\begin{aligned}
&-[\int\int\int(\phi_\alpha(u)-\Delta^{\frac{1}{2}}(g)\phi_\alpha(v))^2 d\delta_g*\delta_k(u)d\delta_k*\delta_g(v)d\lambda(k)]\\
&=-[\int(L_g\phi_\alpha(k)-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha(k))^2 d\lambda(k)\\
&+\int L_g\phi_\alpha^2(k)d\lambda(k)+\int\Delta(g)R_g\phi_\alpha^2(k)d\lambda(k)\\
&-\int(L_g\phi_\alpha)^2(k)d\lambda(k)-\int\Delta(g)(R_g\phi_\alpha)^2(k)d\lambda(k)]\\
&\leq -||L_g\phi_\alpha(k)-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha(k)||_2^2-||\phi_\alpha||_2^2\\
&-||\phi_\alpha||_2^2+||\phi_\alpha||_2^2+||\phi_\alpha||_2^2\to 0,
\end{aligned}$$

because

$$\begin{aligned}
\int\Delta(g)(R_g\phi_\alpha)^2(k)d\lambda(k) &=<\Delta(g)R_g\phi_\alpha,R_g\phi_\alpha>\\
&=<\phi_\alpha,R_{\check{g}}R_g\phi_\alpha>\\
&\leq ||\phi_\alpha||_2^2
\end{aligned}$$

and each ϕ_α is positive. In addition,

$$\begin{aligned}
&\Delta^{\frac{1}{2}}(g)L_g\phi_\alpha(k)R_g\phi_\alpha(k)-\Delta(g)R_g\phi_\alpha^2(k)\\
&\leq \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha(k)R_g\phi_\alpha(k)-\Delta(g)(R_g\phi_\alpha)^2(k)\\
&=[L_g\phi_\alpha(k)-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha(k)]\ \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha(k),
\end{aligned}$$

by Holder's inequality. Thus,

$$\begin{aligned}
&\int|\Delta^{\frac{1}{2}}(g)L_g\phi_\alpha(k)R_g\phi_\alpha(k)-\Delta(g)R_g\phi_\alpha^2(k)|d\lambda(k)\\
&\leq \Delta^{\frac{1}{2}}(g)||R_g\phi_\alpha||_2\ ||L_g\phi_\alpha(k)-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha(k)||_2\to 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&||L_g\psi_\alpha-\Delta(g)R_g\psi_\alpha||_1\\
&=\int|L_g\phi_\alpha^2(k)-\Delta(g)R_g\phi_\alpha^2(k)|d\lambda(k)\\
&\leq\int|\int\int(\phi_\alpha(u)-\Delta^{\frac{1}{2}}(g)\phi_\alpha(v))^2 d\delta_g*\delta_k(u)d\delta_k*\delta_g(v)|d\lambda(k)\\
&+\int|2\Delta^{\frac{1}{2}}(g)L_g\phi_\alpha(k)R_g\phi_\alpha(k)-2\Delta(g)R_g\phi_\alpha^2(k)|d\lambda(k)\to 0,
\end{aligned}$$

since,

$$\begin{aligned}
&\int\int(\phi_\alpha(u)-\Delta^{\frac{1}{2}}(g)\phi_\alpha(v))^2 d\delta_g*\delta_k(u)d\delta_k*\delta_g(v)\\
&=\int\int[\phi_\alpha^2(u)+\Delta(g)\phi_\alpha^2(v)-2\Delta^{\frac{1}{2}}(g)\phi_\alpha(u)\phi_\alpha(v)]d\delta_g*\delta_k(u)d\delta_k*\delta_g(v)\\
&=L_g\phi_\alpha^2(k)-\Delta(g)R_g\phi_\alpha^2(k)+2\Delta(g)R_g\phi_\alpha^2(k)-2\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha^2(k)L_g\phi_\alpha^2(k).
\end{aligned}$$

By Lemma 3.2 then K is inner amenable. The rest follows by a similar argument as in ([28], Theorem 4.3) if K is unimodular. □

Remark 3.7. *Let K be a discrete hypergroup. If there is a positive norm one net $\{\phi_\alpha\}$ in $l_2(K)$ with $\phi_\alpha(e)\to 0$ such that $||L_g\phi_\alpha-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2\to 0$, for all $g\in K$, $l_\infty(K)$ has a non-trivial inner invariant mean.*

Theorem 3.8. *The following are equivalent:*

1. *There is a net $\{\phi_\alpha\}$ in $L_2(K)$ with $\phi_\alpha\geq 0$ and $||\phi_\alpha||_2=1$ such that $||L_g\phi_\alpha-\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2\to 0$, for all $g\in K$.*

2. There is a net $\{\phi_\alpha\}$ in $L_2(K)$ with $\phi_\alpha \geq 0$ and $||\phi_\alpha||_2 = 1$ such that for each $g \in K$

$$|\ ||L_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)| \to 0$$

and

$$|\ ||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)| \to 0.$$

In this case K is inner amenable and there is a state m on $B(L_2(K))$ such that $m(L_g) = m(\Delta^{\frac{1}{2}}(g)R_g)$, for all $g \in K$, where L_g (R_g) is the left (right) translation operator on $L_2(K)$.

Proof. If (1) holds, then for $g \in K$

$$\begin{aligned}
&|\ ||L_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)\ | \\
&= |\ < L_g\phi_\alpha, L_g\phi_\alpha > - < L_g\phi_\alpha, \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha >\ | \\
&= |\ < L_g\phi_\alpha, L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha >\ | \\
&\leq ||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha|| \to 0.
\end{aligned}$$

Similarly, $|\ ||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)\ | \to 0$, for $g \in K$. Conversely, for each $g \in K$ we have

$$\begin{aligned}
&||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 \\
&=< L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha, L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha > \\
&= ||L_g\phi_\alpha||_2^2 + ||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - 2 < L_g\phi_\alpha, \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha > \\
&= ||L_g\phi_\alpha||_2^2 + ||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - 2\Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e) \\
&\leq |\ ||L_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)\ | \\
&+|\ ||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)\ | \to 0.
\end{aligned}$$

For each $T \in B(L_2(K))$ let $m_\alpha T =< T\phi_\alpha, \phi_\alpha >$ and let m be a weak*-cluster point of the net $\{m_\alpha\}$ in $B(L_2(K))^*$. Without loss of generality assume that $mT = \lim_\alpha m_\alpha(T)$. Then m is a state on $B(L_2(K))$ and for $g \in K$

$$\begin{aligned}
&|m(L_g) - m(\Delta^{\frac{1}{2}}(g)R_g)| \\
&= |\lim_\alpha < L_g\phi_\alpha, \phi_\alpha > - \lim_\alpha < \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha, \phi_\alpha > | \\
&= |\lim_\alpha < L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha, \phi_\alpha > | \\
&\leq \lim_\alpha ||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha|| = 0.
\end{aligned}$$

In addition, K is inner amenable by Lemma 3.6. □

It is known that the amenability of a locally compact group G can be characterized by the existence of a state m on $B(L_2(K))$ with $m(L_g) = 1$, for all $g \in G$ ([3], Theorem 2). By a similar method as in the proof of Theorem 3.8 we have the following:

Remark 3.9. *If K satisfies Reiter's condition P_2, then there is a state m on $B(L_2(K))$ such that $m(L_g) = 1$, for all $g \in K$.*

Let G be a locally compact group. Then G is an $[IN]$-group if and only if G possesses a compact neighborhood V of e with $L_g\chi_V = R_g\chi_V$, for all $g \in G$. However, one may not expect this equivalence relation to hold in the hypergroup setting. A hypergroup K is called $[IN]$-hypergroup if there is a compact neighborhood V of e such that $g * V = V * g$, for all $g \in K$. It is easy to see that each of compact or commutative hypergroups are $[IN]$-hypergroups and possess a compact neighborhood V of e with $L_g\chi_V = R_g\chi_V$, for all $g \in K$. For a discrete hypergroup K the situation is quite different: although K is an $[IN]$-hypergroup, we have that $L_g\delta_e = R_g\delta_e$, for all $g \in K$ if and only if $\delta_g * \delta_{\check{g}}(e) = \delta_{\check{g}} * \delta_g(e)$, for all $g \in K$.

Corollary 3.10. *Let K be a hypergroup possessing a compact neighborhood V of e with $L_g\chi_V = R_g\chi_V$, for all $g \in K$. Let Q_V be the operator on $L_2(K)$ given by $Q_Vf =< f, \chi_V > .\chi_V$ for $f \in L_2(K)$. Then the following are equivalent:*

1. *There is a net $\{\phi_\alpha\}$ in $L_2(K)$ with $\phi_\alpha \geq 0$, $< \phi_\alpha, \chi_V >= 0$ and $||\phi_\alpha||_2 = 1$ such that*
$$||L_g\phi_\alpha - \Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2 \to 0,$$
for all $g \in K$.
2. *There is a net $\{\phi_\alpha\}$ in $L_2(K)$ with $\phi_\alpha \geq 0$, $< \phi_\alpha, \chi_V >= 0$ and $||\phi_\alpha||_2 = 1$ such that for $g \in K$*
$$|\ ||L_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)| \to 0,$$
and
$$||\Delta^{\frac{1}{2}}(g)R_g\phi_\alpha||_2^2 - \Delta^{\frac{1}{2}}(g)L_g\phi_\alpha * L_{\check{g}}\check{\phi}_\alpha(e)| \to 0.$$

In this case

a. *There is an inner invariant mean m on $L_\infty(K)$ with*
$$m(\chi_V) = 0.$$
b. *There is a state m on $B(L_2(K))$ such that $m(Q_V) = 0$ and*
$$m(L_g) = m(\Delta^{\frac{1}{2}}(g)R_g),$$
for all $g \in K$.
c. *The operators $id - Q_V$ and $id + Q_V$ are not in the C^*-algebra generated by $\{L_g - \Delta^{\frac{1}{2}}(g)R_g \mid g \in K\}$.*

Proof. We will show b $\Rightarrow$ c, for all other parts we refer to the proof of Theorem 3.8. Let
$$T = \sum_{i=1}^{n} \lambda_i(L_{g_i} - \Delta^{\frac{1}{2}}(g_i)R_{g_i}).$$
Then $m(T) = 0$ and hence
$$||T - (id - Q_V)|| \geq |m(T) - m(id - Q_V)| = 1.$$
Similarly, $||T - (id + Q_V)|| \geq 1$. Thus, $id - Q_V$ and $id + Q_V$ are not in the C^*-algebra generated by $\{L_g - \Delta^{\frac{1}{2}}(g)R_g \mid g \in K\}$. $\square$

Remark 3.11. *Let K be a unimodular hypergroup possessing a compact neighborhood V of e with $L_g\chi_V = R_g\chi_V$, for all $g \in K$ and let $1 \le p < \infty$. Then there is a compact operator T in $B(L_p(K))$ such that $L_gT = R_gT$, $L_{\check{k}}TL_g = R_{\check{k}}TR_g$ and $TL_g = TR_g$, for all $g, k \in K$.*

Proof. Let $Tf := < \chi_V, f > \chi_V$. Then for $f \in L_p(K)$ and $g, k \in K$,

$$\begin{aligned} L_{\check{k}}TL_gf &= < \chi_V, L_gf > L_{\check{k}}\chi_V \\ &= < L_{\check{g}}\chi_V, f > L_{\check{k}}\chi_V \\ &= < R_{\check{g}}\chi_V, f > R_{\check{k}}\chi_V \\ &= < \chi_V, R_gf > R_{\check{k}}\chi_V \\ &= R_{\check{k}}TR_gf. \end{aligned}$$

Hence, $L_{\check{k}}TL_g = R_{\check{k}}TR_g$, for all $g, k \in K$. Similarly we can prove other parts. □

Example 3.12.
1. *Let $K = H \vee J$ be the hypergroup join of a compact group H and a discrete commutative hypergroup J. Then there is a compact neighborhood V of e with $L_g\chi_V = R_g\chi_V$, for all $g \in K$.*
2. *Let $K = H \vee J$ be the hypergroup join of a finite commutative hypergroup H and a discrete group J. Then $\delta_g * \delta_{\check{g}}(e) = \delta_{\check{g}} * \delta_g(e)$, for all $g \in K$ and hence $L_g\delta_e = R_g\delta_e$, for all $g \in K$. since*

$$\delta_{\check{j}} * \delta_j(e) = \sum_{g \in H} \frac{1}{\delta_{\check{g}} * \delta_g(e)} \delta_g = \delta_j * \delta_{\check{j}}(e),$$

   for $j \in J$.

Lau and Paterson in ([19], Theorem 2) proved that a locally compact group G is inner amenable if and only if there exists a non-zero compact operator in $\mathscr{A}_\infty'$, where

$$\mathscr{A}_\infty' = \{T \in B(L_\infty(G)) \mid L_{g^{-1}}R_gT = TL_{g^{-1}}R_g, \ \forall g \in G\}.$$

We note that

$$\mathscr{A}_\infty' = \{T \in B(L_\infty(G)) \mid R_gTR_{g^{-1}} = L_gTL_{g^{-1}}, \ \forall g \in G\}$$

which is not the case as we step beyond the groundwork of locally compact groups. The following is an extension of ([19], Theorem 2):

Remark 3.13. *The following conditions hold:*

1. *If K is inner amenable, then there is a compact operator T in $B(L_\infty(K))$ such that $T(h) = 1$, for some $h \in L_\infty(K)$,*

$$L_{\check{n}}TL_g = R_{\check{m}}TR_g, \ TL_g = TR_g,$$

   for all $g, n, m \in K$ and $T(f) \ge 0$, for $f \ge 0$.
2. *If there is a non-zero operator T in $B(L_\infty(K))$ such that*

$$TL_g = TR_g,$$

   for all $g \in K$ and $T(f) \ge 0$, for $f \ge 0$, then K is inner amenable and $T(f) \ge 0$, for $f \ge 0$.

Proof. 1. If m is an inner invariant mean on $L_\infty(K)$, then the operator T in $B(L_\infty(K))$ defined by $T(f) = m(f)1$, for $f \in L_\infty(K)$ is the desired operator.

2. Let m be a mean on $L_\infty(K)$. Then $m \circ T$ is an inner invariant positive linear functional on $L_\infty(K)$. Let $f_0 \in L_\infty(K)$ such that $T(f_0) > 0$. Then f_0 can be decomposed into positive elements and if $f \geq 0$, then $T(f) \leq ||f||T(1)$. Hence, $m \circ T(1) \neq 0$ and $\frac{m \circ T}{m \circ T(1)}$ is an inner invariant mean on $L_\infty(K)$.

□

Corollary 3.14. *K is inner amenable if and only if there is a non-zero compact operator T in $B(L_\infty(K))$ such that $TL_g = TR_g$, for all $g \in K$ and $T(f) \geq 0$, for $f \geq 0$.*

Corollary 3.15. *Let G be a locally compact group. Then G is inner amenable if and only if there is a non-zero operator T in $\mathscr{A}_\infty^{'}$ such that $TL_g = TR_g$, for all $g \in G$ and $T(f) \geq 0$, for $f \geq 0$.*

We say that K satisfies *central Reiter's condition P_1*, if there is a net $\{\phi_\alpha\}$ in $L_1(K)$ with $\phi_\alpha \geq 0$ and $||\phi_\alpha||_1 = 1$ such that

$$||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha||_1 \to 0$$

uniformly on compact subsets of K. By Lemma 3.2 if K satisfies central Reiter's condition P_1, then K is inner amenable. Sinclair ([27], page 47) in particular called a net $\{\phi_\alpha\}$ in $L_1(G)$ quasi central if $||\mu * \phi_\alpha - \phi_\alpha * \mu|| \to 0$, for all $\mu \in M(G)$, where G is a locally compact group. We say that the net $\{\phi_\alpha\}$ in $L_1(K)$ is *quasi central* if

$$||\mu * \phi_\alpha - \phi_\alpha \circledast \mu|| \to 0,$$

for all $\mu \in M(K)$.

One note the distinction between the condition $||L_g\phi_\alpha - \Delta(g)R_g\phi_\alpha||_1 \to 0$ uniformly on compacta and the (equivalent for groups, but not for hypergroups) condition $||\phi_\alpha - \Delta(g)L_{\check{g}}R_g\phi_\alpha||_1 \to 0$ uniformly on compacta. For the group case please see ([30], Theorem 4.2).

Remark 3.16. *If the net $\{\phi_\alpha\}$ in $L_1(K)$ satisfies central Reiter's condition P_1, then*

1. *For given $\{\psi_i\}_{i=1}^n \subseteq L_1(K)$ and $\epsilon > 0$, there is an element $\phi \in L_1(K)$ such that $||\psi_i * \phi - \phi * \psi_i|| < \epsilon$, for $i = 1, 2, ..., n$.*
2. *The net $\{\phi_\alpha\}$ is a quasi central net in $L_1(K)$.*

Proof. (1): Let $\epsilon > 0$ be given and let C_i be compact subsets of K such that $\int_{K\setminus C_i} |\psi_i|(g)d\lambda(g) < \epsilon$. Let $C = \bigcup_{i=1}^n C_i$ and let $\alpha \in I$ be such that $||L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha(k)|| < \epsilon$, for all $g \in C$. Then

$$\begin{aligned}
&||\psi_i * \phi_\alpha - \phi_\alpha * \psi_i||_1 \\
&= \int | \int \psi_i(g)L_{\check{g}}\phi_\alpha(k)d\lambda(g) - \int \psi_i(g)\Delta(\check{g})R_{\check{g}}\phi_\alpha(k)d\lambda(g)|d\lambda(k) \\
&\leq \int |\psi_i(g)| \ \int |L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha(k)|d\lambda(k)\ d\lambda(g) \\
&= \int_{K\setminus C} |\psi_i(g)| \ \int |L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha(k)|d\lambda(k)\ d\lambda(g) \\
&+ \int_C |\psi_i(g)| \ \int |L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha(k)|d\lambda(k)\ d\lambda(g) \\
&< \epsilon^2 + \epsilon\ Max_{i=1,\ldots,n}||\psi_i||_1
\end{aligned}$$

(2): Without loss of generality assume that $\mu \in M(K)$ has a compact support C. Let $\epsilon > 0$ be given and let $\alpha \in I$ be such that $||L_{\check{g}}\phi_\alpha - \Delta(\check{g})R_{\check{g}}\phi_\alpha|| < \epsilon$, for all

$g \in C$. Then

$$\begin{aligned} &||\mu * \phi_\alpha - \phi_\alpha \circledast \mu|| \\ &= \int | \int (L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha) d\mu(g)| d\lambda(k) \\ &\leq \int \int_C |L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha| d\mu(g) d\lambda(k) \\ &+ \int \int_{K\setminus C} |L_{\check{g}}\phi_\alpha(k) - \Delta(\check{g})R_{\check{g}}\phi_\alpha| d\mu(g) d\lambda(k) \\ &\leq \epsilon ||\mu||. \end{aligned}$$

□

Losert and Rindler called a net $\{\phi_\alpha\}$ in $L_1(G)$, G is a locally compact group, asymptotically central if $\frac{1}{||\phi_\alpha||}(\Delta(g)R_gL_{g^{-1}}\phi_\alpha - \phi_\alpha) \to 0$ weakly for all $g \in G$ [21].

We say that the net $\{\phi_\alpha\}$ in $L_1(K)$ is *asymptotically central* if

$$\frac{1}{||\phi_\alpha||}(\Delta(g)R_gL_{\check{g}}\phi_\alpha - \phi_\alpha) \to 0$$

weakly for all $g \in K$. In addition, we say that the net $\{\phi_\alpha\}$ in $L_1(K)$ is *hypergroup asymptotically central* if

$$\frac{1}{||\phi_\alpha||}(\Delta(g)R_g\phi_\alpha - L_g\phi_\alpha) \to 0$$

weakly for all $g \in K$. The reason for our definition is that

$$Z(L_1(K)) = \{\phi \in L_1(K) \mid \Delta(g)R_g\phi = L_g\phi,\ \forall g \in K\},$$

where $Z(L_1(K))$ is the algebraic center of the hypergroup algebra $L_1(K)$. Then it is easy to see that if K is discrete and unimodular or commutative, then any approximate identity in $L_1(K)$ is hypergroup asymptotically central and hence $L_1(K)$ is Arens semi-regular (see [10], page 45 for the definition).

Remark 3.17. *If $L_1(K)$ has an asymptotically central bounded approximate identity, then K is an inner amenable locally compact group.*

Proof. Let $\{\phi_\alpha\}$ be an asymptotically central bounded approximate identity for $L_1(K)$ and m be a weak*-cluster point of $\{\phi_\alpha\}$ in $L_\infty(K)^*$. Without loss of generality assume that ϕ_α's are real-valued and $\lim_\alpha < \phi_\alpha, f >=< m, f >$ for each $f \in L_\infty(K)$. Then $m(L_gR_{\check{g}}f) = m(f)$, for each $f \in L_\infty(K)$ and $g \in K$. In addition,

$$m(\phi * f) = \lim < \phi_\alpha, \phi * f >= \lim < \check{\Delta}\check{\phi} * \phi_\alpha, f >=< \check{\Delta}\check{\phi}, f >= \phi * f(e),$$

for $\phi \in L_1(K)$ and $f \in L_\infty(K)$. Thus, $m(f) = f(e)$, for each $f \in C_0(K)$ ([28], Lemma 2.2). Therefore,

$$\delta_g * \delta_{\check{g}}(f) = R_{\check{g}}f(g) = L_gR_{\check{g}}f(e) = m(L_gR_{\check{g}}f) = m(f) = \delta_e(f),$$

for $f \in C_0(K)$. i.e. $\delta_g * \delta_{\check{g}} = \delta_e$, for all $g \in K$ and hence $G(K) = K$. It follows then by the proof of ([21], Theorem 2) that the locally compact group K is also inner amenable. □

In 1991, Lau and Paterson characterized inner amenable locally compact groups G in terms of a fixed point property of an action of G on a Banach space ([17], Theorem 5.1). This characterization can be extended naturally to hypergroups and we have:

Remark 3.18. *The following are equivalent:*

1. *K is inner amenable.*
2. *Whenever $\{T_g \in B(E) \mid g \in K\}$ is a separately continuous representation of K on a Banach space E as contractions, there is some*
$$T \in \overline{\{N_\phi \mid \phi \in L_1(K),\ ||\phi|| = 1,\ \phi \geq 0\}}^{w^*.o.t}$$
such that
$$N_g T = T N_g,$$
for all $g \in K$.

Remark 3.19. *Let N be a closed normal Weil subhypergroup of K. If K is inner amenable, then K/N is also inner amenable.*

Proof. Define a linear isometry ϕ from $L_\infty(K/N)$ to the subspace
$$\{f \in L_\infty(K) \mid R_g f = R_k f,\ g \in k * N,\ k \in K\}$$
of $L_\infty(K)$ by $\phi(f) = f \circ \pi$, where π is the quotient map from K onto K/N. Then
$$\begin{aligned}&\textstyle\int |L_g(\phi f)(k) - \phi(L_{g*N} f)(k)| d\lambda(k)\\ &= \textstyle\int | \int f(u * N) d\delta_g * \delta_k(u) - (L_{g*N} f) \circ \pi(k)| d\lambda(k)\\ &= \textstyle\int | \int f(u * N) d\delta_{g*N} * \delta_{k*N}(u * N) - L_{g*N} f(k * N)| d\lambda(k)\\ &= 0,\end{aligned}$$
since N is a Weil subhypergroup. Thus, $\phi(L_{g*N} f) = L_g(\phi f)$ for $f \in L_\infty(K/N)$ and $g \in K$. Similarly, $\phi(R_{g*N} f) = R_g(\phi f)$ for $f \in L_\infty(K/N)$ and $g \in K$. Let m be an inner invariant mean on $L_\infty(K)$ and define $m_1(f) = m(\phi f)$, $f \in L_\infty(K/N)$. Then m_1 is a mean on $L_\infty(K/N)$. In addition, for $f \in L_\infty(K/N)$ and $g \in K$
$$\begin{aligned}m_1(L_{g*N} f) &= m(\phi(L_{g*N} f))\\ &= m(L_g \phi f)\\ &= m(R_g \phi f)\\ &= m(\phi(R_{g*N} f))\\ &= m_1(R_{g*N} f).\end{aligned}$$
□

4. Hahn-Banach extension and monotone extension properties

It is the purpose of this section to provide a hypergroup version of Hahn-Banach extension property and monotone extension property by which amenable hypergroups can be characterized.

Let E be a partially ordered Banach space over $\mathbb{R}$. An element $1 \in E$ is called a topological order unit if for each $f \in E$ there exists $\lambda > 0$ such that $-\lambda 1 \leq f \leq \lambda 1$ and the set $\{f \in E \mid 1 \leq f \leq 1\}$ is a neighbourhood of E and a proper subspace I of E is said to be a proper ideal if $[0, f] \subseteq I$, for each $f \in E$. Moreover, a separately continuous linear representation $\mathscr{T} = \{T_g \mid g \in K\}$ of K on E is positive if $T_g f \geq 0$ for all $g \in K$ and $f \geq 0$. $\mathscr{T}$ is normalized if $T_g 1 = 1$ for all $g \in K$.

Theorem 4.1. *The following are equivalent:*

1. *$RUC(K)$ has a right invariant mean.*
2. *Let $\{T_g \in B(E) \mid g \in K\}$ be a separately continuous representation of K on a Banach space E and let F be a closed T_K-invariant subspace of E. Let p be a continuous seminorm on E such that $p(T_g x) \leq p(x)$ for all $x \in E$ and $g \in K$ and Φ be a continuous linear functional on F such that $|\Phi(x)| \leq p(x)$ and $\Phi(T_g x) = \Phi(x)$ for $g \in K$ and $x \in F$. Then there is a continuous linear functional $\tilde{\Phi}$ on E such that*
   (a) *$\tilde{\Phi}|_F \equiv \Phi$.*
   (b) *$|\tilde{\Phi}(x)| \leq p(x)$ for each $x \in E$.*
   (c) *$\tilde{\Phi}(T_g x) = \tilde{\Phi}(x)$ for $g \in K$ and $x \in E$.*
3. *For any positive normalized separately continuous linear representation $\mathscr{T}$ of K on a partially ordered real Banach space E with a topological order unit 1, if F is a closed $\mathscr{T}$-invariant subspace of E containing 1, and Φ is a $\mathscr{T}$-invariant monotonic linear functional on F, then there exists a $\mathscr{T}$-invariant monotonic linear functional $\tilde{\Phi}$ on E extending Φ.*
4. *For any positive normalized separately continuous linear representation $\mathscr{T}$ of K on a partially ordered real Banach space E with a topological order unit 1, E contains a maximal proper $\mathscr{T}$-invariant ideal.*

Proof. $1 \Rightarrow 2$: By Hahn-Banach extension theorem there is a continuous linear functional Φ_1 on E such that $|\Phi_1(x)| \leq p(x)$ for each $x \in E$ and $\Phi_1|_F \equiv \Phi$. For each $f \in E$ define a continuous bounded function $h_{\Phi_1,f}$ on K via $h_{\Phi_1,f}(g) = \Phi_1(T_g f)$. Let $\{g_\alpha\}$ be a net in K converging to e. Then

$$\begin{aligned}||R_{g_\alpha} h_{\Phi_1,f} - h_{\Phi_1,f}|| &= \sup_{g\in K} |R_{g_\alpha} h_{\Phi_1,f}(g) - h_{\Phi_1,f}(g)| \\ &= \sup_{g\in K} |\textstyle\int \Phi_1(T_u f) d\delta_g * \delta_{g_\alpha}(u) - \Phi_1(T_g f)| \\ &= \sup_{g\in K} |\Phi_1(T_g T_{g_\alpha} f) + \Phi_1(-T_g f)| \\ &\leq \sup_{g\in K} p(T_g T_{g_\alpha} f - T_g f) \\ &\leq p(T_{g_\alpha} f - f) \to 0,\end{aligned}$$

since $\Phi_1 \in E^*$. Hence, $h_{\Phi_1,f} \in RUC(K)$ ([28], Remark 2.3). Let m be a right invariant mean on $RUC(K)$ and let $\tilde{\Phi}(f) = m(h_{\Phi_1,f})$, for $f \in E$. Then $\tilde{\Phi}|_F \equiv \Phi$ since $h_{\Phi_1,f}(g) = \Phi_1(T_g f) = \Phi(f)$, for $f \in F$. Furthermore, $|\tilde{\Phi}(f)| \leq \sup_{g\in K} |\Phi_1(T_g f)| \leq p(f)$, for $f \in E$ and

$$\begin{aligned}h_{\Phi_1,T_g f}(k) &= \Phi_1(T_k T_g f) \\ &= \textstyle\int \Phi_1(T_u f) d\delta_k * \delta_g(u) \\ &= \textstyle\int h_{\Phi_1,f}(u) d\delta_k * \delta_g(u) \\ &= R_g h_{\Phi_1,f}(k).\end{aligned}$$

Thus,

$$\tilde{\Phi}(T_g f) = m(h_{\Phi_1,T_g f}) = m(R_g h_{\Phi_1,f}) = m(h_{\Phi_1,f}) = \tilde{\Phi}(f).$$

$2 \Rightarrow 1$: Let $E = RUC(K)$, $F = \mathbb{C}.1$ and consider the continuous representation $\{R_g \mid g \in K\}$ of K on $RUC(K)$. Define a seminorm p on E by $p(f) = ||f||$. Then $p(R_g f) \leq p(f)$, for $f \in E$ and $g \in K$. In addition, δ_a is a left invariant mean on F for a given $a \in K$ with $|\delta_a(f)| \leq p(f)$. Therefore, there is some $m \in RUC(K)^*$ such that $m|_F \equiv \delta_a$, $m(f) \leq ||f||$ and $m(R_g f) = m(f)$, for $f \in E$ and $g \in K$. Then m is a right invariant mean on $RUC(K)$ because $m(1) = \delta_a(1) = 1 = ||m||$.

For all other parts we refer to ([16], Theorem 2) and a similar argument as above. □

Let $CB_{\mathbb{R}}(K)$ denote all bounded continuous real-valued functions on K and $UC_{\mathbb{R}}(K)$ ($RUC_{\mathbb{R}}(K)$) denote all functions in $CB_{\mathbb{R}}(K)$ which are (right) uniformly continuous. It is easy to see that $UC_{\mathbb{R}}(K)$ and $RUC_{\mathbb{R}}(K)$ are norm-closed translation invariant subspace of $CB_{\mathbb{R}}(K)$ containing constants. However, in contrast to the group case, $RUC_{\mathbb{R}}(K)$ need not be a Banach lattice in general. The following result is a consequence of Theorem 4.1 and the proof of ([16], Theorem 1).

Remark 4.2. *Let K be a hypergroup such that $RUC_{\mathbb{R}}(K)$ is a Banach lattice. Then the following are equivalent:*

1. *$RUC(K)$ has a right invariant mean.*
2. *For any linear action $\mathscr{T}$ of K on a Banach space E, if U is a $\mathscr{T}$-invariant open convex subset of E containing a $\mathscr{T}$-invariant element, and M is a $\mathscr{T}$-invariant subspace of E which does not meet U, then there exists a closed $\mathscr{T}$-invariant hyperplane H of E such that H contains M and H does not meet U.*
3. *For any contractive action $\mathscr{T} = \{T_g \in B(E) \mid g \in K\}$ of K on a Hausdorff Banach space E, any two points in $\{f \in E \mid T_g f = f,\ \forall g \in K\}$ can be separated by a continuous $\mathscr{T}$-invariant linear functional on E.*

Example 4.3. 1. *Let K be a hypergroup such that the maximal subgroup $G(K)$ is open. Then $RUC_{\mathbb{R}}(K)$ is a Banach lattice.*

2. *Let $K = H \vee J$ be the hypergroup join of a compact hypergroup H and a discrete hypergroup J. Then $RUC_{\mathbb{R}}(K) = CB_{\mathbb{R}}(K)$ is a Banach lattice.*

Proof. To see 1, let $f, h \in RUC_{\mathbb{R}}(K)$ and $\{g_\alpha\}$ be a net in K converging to e. Then $g_\alpha \in G(K)$, for some α_0 and all $\alpha \geq \alpha_0$ since $G(K)$ is open. Thus, $R_{g_\alpha}(f \vee h) = R_{g_\alpha} f \vee R_{g_\alpha} h$ for $\alpha \geq \alpha_0$. Therefore, the mapping

$$g \mapsto (R_g f, R_g h) \mapsto R_g f \vee R_g h$$

from K to $CB_{\mathbb{R}}(K)$ is continuous at e and hence $f \vee h \in RUC_{\mathbb{R}}(K)$. □

Next we use Theorem 4.1 to prove that $UC(K)$ has an invariant mean, for any commutative hypergroup K.

Corollary 4.4. *Let K be a commutative hypergroup. Then $UC(K)$ has an invariant mean.*

Proof. Let $\mathscr{T} = \{T_g \in B(E) \mid g \in K\}$ be a separately continuous representation of K on a real Banach space E and let F be a closed $\mathscr{T}$-invariant subspace of E. Let p be a continuous sublinear map on E such that $p(T_g x) \leq p(x)$ for all $x \in E$ and $g \in K$ and ϕ be a continuous $\mathscr{T}$-invariant linear functional on F such that $\phi(x) \leq p(x)$ for $x \in F$. Define a representation $\{T_\mu \in B(E) \mid \mu \in M_1^c(K)\}$ of $M_1^c(K)$, the probability measures with compact support on K, on E via

$$T_\mu x = \int T_g x d\mu(g).$$

Then $T_{\mu*\nu}=T_\mu T_\nu$, for $\mu,\nu\in M_1^c(K)$. In addition,

$$p(T_\mu x)=p(\int T_g x d\mu(g))\leq\int p(T_g x)d\mu(g)\leq p(x).$$

Define a real valued function q on E via

$$q(x)=\inf\{\frac{1}{m}p(T_{\mu_1}x+...+T_{\mu_m}x)\},$$

where the inf is taken over all finite collection of probability measures with compact support $\{\mu_1,...,\mu_m\}$ on K. Then $q(x)\leq p(x)$ for $x\in E$ since for each $m\in\mathbb{N}$,

$$\tfrac{1}{m}p(T_{\mu_1}x+...+T_{\mu_m}x)\quad\leq\tfrac{1}{m}\big[p(T_{\mu_1}x)+...+p(T_{\mu_m}x)\big]\leq p(x).$$

Moreover, q is sublinear. In fact for $m\in\mathbb{N}$, $\alpha\in\mathbb{R}^+$ and $x\in E$,

$$\tfrac{1}{m}p(T_{\mu_1}(\alpha x)+...+T_{\mu_m}(\alpha x))=\tfrac{1}{m}\alpha p(T_{\mu_1}x+...+T_{\mu_m}x).$$

Thus, $q(\alpha x)=\alpha q(x)$ for $\alpha\in\mathbb{R}^+$ and $x\in E$. To see that $q(x+y)\leq q(x)+q(y)$, let $x,y\in E$ and $\epsilon>0$ be given. Choose probability measures $\mu_1,...,\mu_m,\nu_1,...,\nu_n$ on K with compact support such that

$$\frac{1}{m}p(T_{\mu_1}x+...+T_{\mu_m}x)\leq q(x)+\epsilon,$$

and

$$\frac{1}{n}p(T_{\nu_1}x+...+T_{\nu_n}x)\leq q(y)+\epsilon.$$

Consider the set $\mathcal{K}=\{\nu_j*\mu_i\mid j=1,...,n,\ i=1,...,m\}$. Then

$$\begin{aligned}\tfrac{1}{nm}p\Big[\textstyle\sum_{j=1}^n\sum_{i=1}^m T_{\nu_j*\mu_i}x\Big] &=\tfrac{1}{nm}p\big[\textstyle\sum_{j=1}^n T_{\nu_j}(\sum_{i=1}^m T_{\mu_i}x)\big]\\ &\leq\tfrac{1}{nm}\textstyle\sum_{j=1}^n p\big[T_{\nu_j}(\sum_{i=1}^m T_{\mu_i}x)\big]\\ &\leq\tfrac{1}{nm}\textstyle\sum_{j=1}^n p\big[\sum_{i=1}^m T_{\mu_i}x\big]\\ &=\tfrac{1}{m}p\big[\textstyle\sum_{i=1}^m T_{\mu_i}x\big]\\ &\leq q(x)+\epsilon,\end{aligned}$$

and similarly, $\tfrac{1}{nm}p\Big[\sum_{j=1}^n\sum_{i=1}^m T_{\nu_j*\mu_i}y\Big]\leq q(y)+\epsilon$. Hence,

$$\begin{aligned}&\tfrac{1}{nm}p\Big[\textstyle\sum_{j=1}^n\sum_{i=1}^m T_{\nu_j*\mu_i}(x+y)\Big]\\ &=\tfrac{1}{nm}p\Big[\textstyle\sum_{j=1}^n\sum_{i=1}^m T_{\nu_j*\mu_i}x+\sum_{j=1}^n\sum_{i=1}^m T_{\nu_j*\mu_i}y\Big]\\ &\leq\tfrac{1}{nm}p\Big[\textstyle\sum_{j=1}^n\sum_{i=1}^m T_{\nu_j*\mu_i}x\Big]+\tfrac{1}{nm}p\Big[\sum_{j=1}^n\sum_{i=1}^m T_{\nu_j*\mu_i}y\Big]\\ &\leq q(x)+q(y)+2\epsilon.\end{aligned}$$

Therefore,

$$q(x+y)\leq q(x)+q(y).$$

For $\mu\in M_1^c(K)$, $x\in E$ and $m\in\mathbb{N}$,

$$\begin{aligned}&\tfrac{1}{m}p(T_{\mu_1}T_\mu x+...+T_{\mu_m}T_\mu x)\\ &=\tfrac{1}{m}p(T_\mu T_{\mu_1}x+...+T_\mu T_{\mu_m}x)\\ &\leq\tfrac{1}{m}p(T_{\mu_1}x+...+T_{\mu_m}x).\end{aligned}$$

Hence, $q(T_\mu x) \leq q(x)$. Furthermore, for each $m \in \mathbb{N}$

$$\tfrac{1}{m}p(T_{\mu_1}x + ... + T_{\mu_m}x) \quad \leq \tfrac{1}{m}[p(T_{\mu_1}x) + ... + p(T_{\mu_m}x)] \leq p(x).$$

Thus, $q(x) \leq p(x)$. By Hahn-Banach extension theorem there is a continuous linear functional $\tilde{\phi}$ on E such that $\tilde{\phi}(x) \leq q(x)$ for each $x \in E$ and $\tilde{\phi}|_F \equiv \phi$. For $x \in E$, $n \in \mathbb{N}$ and $\mu \in M_1^c(K)$

$$\begin{aligned}
&q(x - T_\mu x) \\
&\leq \tfrac{1}{n+1}p\Big[\Big(T_e(x - T_\mu x) + T_\mu(x - T_\mu x) \\
&+T_\mu T_\mu(x - T_\mu x) + ... + \underbrace{T_\mu T_\mu ... T_\mu}_{\text{n times}}(x - T_\mu x)\Big)\Big] \\
&= \tfrac{1}{n+1}p(x + \underbrace{T_\mu T_\mu ... T_\mu}_{\text{n+1 times}}(-x)) \\
&\leq \tfrac{1}{n+1}[p(x) + p(-x)] \to 0.
\end{aligned}$$

Therefore, $\tilde{\phi}(x - T_\mu x) \leq q(x - T_\mu x) \leq 0$. Since $\tilde{\phi}$ is linear By replacing x by $-x$, one has $\tilde{\phi}(T_\mu x) = \tilde{\phi}(x)$. In particular, $\tilde{\phi}(T_g x) = \tilde{\phi}(x)$ for $g \in K$ and $x \in E$. Therefore, $UC(K)$ has an invariant mean (Theorem 4.1). □

5. Weak*-invariant complemented subspaces of $L_\infty(K)$

Let X be a weak*-closed left translation invariant, invariant complemented subspace of $L_\infty(K)$. Then this section provides a connection between X being invariantly complemented in $L_\infty(K)$ by a weak*-weak*-continuous projection and the behavior of $X \cap C_0(K)$.

Theorem 5.1. *Let X be a weak*-closed, left translation invariant, invariant complemented subspace of $L_\infty(K)$. Then the following are equivalent:*

1. *There exists a weak*-weak*-continuous projection Q from $L_\infty(K)$ onto X commuting with left translations.*
2. *$X \cap C_0(K)$ is weak* dense in X.*

Proof. Let P be a continuous projection from $L_\infty(K)$ onto X commuting with left translations. We first observe that $P(LUC(K)) \subseteq LUC(K)$. In fact if $f \in LUC(K)$ and $\{g_\alpha\}$ is a net in K such that $g_\alpha \to g \in K$, then

$$||L_{g_\alpha}Pf - L_gPf|| = ||P(L_{g_\alpha}f - L_gf)|| \leq ||P||\ ||L_{g_\alpha}f - L_gf|| \to 0.$$

Thus, $P|_{C_0(K)}$ is a bounded operator from $C_0(K)$ into $CB(K)$. Define a bounded linear functional on $C_0(K)$ by $\psi_1(f) := (P\check{f})(e)$. Let $\mu \in M(K)$ be such that $(Pf)(e) = \int \check{f}(x)d\mu(x)$, for each $f \in C_0(K)$. Then for $x \in K$ and $f \in C_0(K)$,

$$(Pf)(x) = L_xPf(e) = PL_xf(e) = \textstyle\int L_xf(\check{y})d\mu(y) = f * \mu(x).$$

Hence, $P(f) = f * \mu$, for $f \in C_0(K)$. Define an operator $T : L_1(K) \to L_1(K)$ via $T(h) := h * \check{\mu}$.
Then $Q = T^*$ is weak*-weak*-continuous and $< Qf, h >=< f, h * \check{\mu} >=< f * \mu, h >$,

for $h \in L_1(K)$ and $f \in C_0(K)$. Thus, $Q(f) = f * \mu$ for $f \in C_0(K)$. In addition, Q commutes with left translations on $L_\infty(K)$, since for $h \in L_1(K)$ and $f \in L_\infty(K)$

$$\begin{aligned} < QL_x f, h > &=< L_x f, h * \check{\mu} > \\ &=< f, (L_{\check{x}} h) * \check{\mu} > \\ &=< Q(f), L_{\check{x}} h > \\ &=< L_x Q(f), h > . \end{aligned}$$

We will show that Q is a projection. For $f \in C_0(K) \cap X$, and $h \in L_1(K)$,

$$\begin{aligned} < f * \mu, h > &= [(f * \mu) * \check{h}](e) \\ &= [f * (h * \check{\mu})\check{\ }](e) \\ &= [(h * \check{\mu}) * \check{f}](e) \\ &= \int (h * \check{\mu})(x) \check{f}(\check{x}) dx \\ &=< f, h * \check{\mu} > . \end{aligned}$$

Hence,

$$< Q(f), h > \quad =< f, h * \check{\mu} >=< f * \mu, h >=< P(f), h >=< f, h > .$$

If $X \cap C_0(K)$ is weak* dense in X, let $\{f_\alpha\}$ be a net in $X \cap C_0(K)$ such that $f_\alpha \to f$ in the weak*-topology of $L_\infty(K)$. Then, $Q(f) = f$ since Q is weak*-continuous.

Moreover, for $f \in C_0(K)$ and $h \in X^\perp$,

$$< Q(f), h > \quad =< f, h * \check{\mu} >=< f * \mu, h >=< P(f), h >= 0.$$

Thus, $< Q(f), h >= 0$, for each $f \in L_\infty(K)$ and $h \in X^\perp$, since $C_0(K)$ is weak*-dense in $L_\infty(K)$. i.e. $Q(f) \in X$.

Conversely, if Q is a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations, then there exists some $\mu \in M(K)$ such that $Q^*|_{L_1(K)}(h) = h * \mu$, for $h \in L_1(K)$ ([1], Theorem 1.6.24). Hence, for $f \in C_0(K)$ we have $Q(f) = f * \check{\mu}$ which is in $C_0(K) \cap X$ ([1], Theorem 1.2.16, iv). Then $C_0(K) \cap X$ is weak*-dense in $X = \{Q(f) \mid f \in L_\infty(K)\}$ since $C_0(K)$ is weak*-dense in $L_\infty(K)$ and Q is weak*-weak*-continuous. □

As a direct consequence of Theorem 5.1 we have the following result:

Corollary 5.2. *Let K be a compact hypergroup and let X be a weak*-closed left translation invariant subspace of $L_\infty(K)$. Then X is invariantly complemented if and only if there is a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations.*

Corollary 5.3. *Let K be a compact hypergroup and let X be a left translation invariant w^*-subalgebra of $L_\infty(K)$ such that $X \cap CB(K)$ has the local translation property TB. Then X is the range of a weak*-weak*-continuous projection commuting with left translations.*

Proof. This follows from ([31], Corollary 3.13, Lemma 3.9) and Theorem 5.1. □

Corollary 5.4. *The following are equivalent:*

1. *K is compact.*

2. *K is amenable and for every weak*-closed left translation invariant, invariant complemented subspace X of $L_\infty(K)$, there exists a weak*-weak*-continuous projection from $L_\infty(K)$ onto X commuting with left translations.*

Proof. If K is compact, then item 2 follows from ([31], Lemma 3.9) and ([28], Example 3.3). Conversely, consider the one-dimensional subspace $X = \mathbb{C}.1$. Then X is a weak*-closed left translation invariant, invariant complemented subspace of $L_\infty(K)$, since K is amenable. If P is a weak*-weak*-continuous projection from $L_\infty(K)$ onto $\mathbb{C}.1$ commuting with left translations, then there is some $\phi \in L_1(K)$ such that $P(f) = \delta_\phi(f)$ for $f \in L_\infty(K)$. Hence, $\delta_\phi(1) = 1$ and $< \delta_\phi, L_g f >=< \delta_\phi, f >$. i.e., $L_g\phi = \phi$, for $g \in K$. In particular, $L_g\phi(e) = \phi(g) = \phi(e)$, for all $g \in K$. Therefore, $1 = \delta_\phi(1) = \int_K \phi(g)d\lambda(g) = \phi(e)\lambda(K)$ which means that K is compact. □

Commutative hypergroups with connected dual can be found in the study of hypergroups constructed on $\mathbb{R}_+$. In fact any Sturm-Liouville hypergroup on $\mathbb{R}_+$ associated with a function $A : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying certain conditions falls in this range ([36], Theorem 4.4). If K is a commutative hypergroup, then $\widehat{K}$ carries a dual hypergroup structure if $\widehat{K}$ can be equipped with a hypergroup structure such that the functions δ_g with $\delta_g(\xi) = \xi(g)$, for $\xi \in \widehat{K}$ are characters of $\widehat{K}$ for all $g \in K$. In addition, K is said to be a Pontryagin hypergroup if $\widehat{K}$ carries a dual hypergroup structure and $\widehat{\widehat{K}}$ can be identified with K. One knows that all Bessel-Kingman hypergroups are Pontryagin hypergroup. ([35], p 483). Let $M_0(K)$ denote the class of all closed subsets of K which contain a support of a non-zero measure in $M(K)$ with the Fourier-Stieltjes transform vanishing at infinity and let $\Delta(X) = \widehat{K} \cap X$.

Lemma 5.5. *Let K be a commutative hypergroup such that the dual space $\widehat{K}$ is connected and let X be a weak*-closed translation invariant, invariant complemented subspace of $L_\infty(K)$. Then $X = L_\infty(K)$ or $C_0(K) \cap X = \{0\}$.*

Proof. Let P be a continuous projection from $L_\infty(K)$ onto X commuting with left translations. Then it follows from the proof of Theorem 5.1 that $P|_{C_0(K)}(f) = f * \mu \in C_0(K)$, for some $\mu \in M(K)$. Hence, $\hat{\mu} = (\mu * \mu)\hat{} = \hat{\mu}.\hat{\mu}$ ([12], 7.3.E). Therefore, $\hat{\mu}(\xi) = 0$ or 1, for $\xi \in \widehat{K}$. Then $\hat{\mu} \equiv 0$ or $\hat{\mu} \equiv 1$, since $\xi \mapsto \hat{\mu}(\xi)$ is continuous on $\widehat{K}$ ([12], 7.3.E) and $\widehat{K}$ is connected. Consequently, $X \cap C_0(K) = \{0\}$ or $X = L_\infty(K)$. □

Corollary 5.6. *Let K be a commutative hypergroup such that $\widehat{K}$ is connected. Then there is no non-trivial weak*-weak*-continuous projection from $L_\infty(K)$ into $L_\infty(K)$ commuting with translations.*

Proof. This follows from Theorem 5.1 and Lemma 5.5. □

Corollary 5.7. *Let K be a commutative Pontryagin hypergroup such that $\widehat{K}$ is connected. Then there is no proper weak*-closed translation invariant, invariant complemented subspace X of $L_\infty(K)$ with $\Delta(X) \in M_0(\widehat{K})$.*

Proof. This follows from Lemma 5.5. □

Corollary 5.7 has the following immediate consequence:

Corollary 5.8. *Let K be a commutative Pontryagin hypergroup such that $\widehat{K}$ is connected. Then there is no non-trivial, invariant complemented ideal I of $L_1(K)$ with $\Delta(I^{\perp}) \in M_0(\widehat{K})$.*

6. Miscellaneous Remarks and Open Problems

Let A be a closed translation invariant subalgebra of $L_\infty(K)$ containing constant functions. In what follows we provide an equivalent condition for A to possess a multiplicative left invariant mean. This equivalence is given in terms of a fixed point property which is a generalization of Mitchell fixed point theorem ([23], Theorem 1).

Definition 6.1. *Let A be a closed translation-invariant subalgebra of $L_\infty(K)$ containing constant functions. Let E be a separated locally convex topological vector space and Y be a compact subset of E. Let X be the space of all probability measures on Y. Let $\{T_g \mid g \in K\}$ be a continuous representation of K on X. Suppose that $B := \{y \in Y \mid T_g y \in Y, \ \forall g \in K\} \neq \emptyset$ and for each $y \in B$, define $h_{y,\phi}(g) = \phi(T_g y)$, for $g \in K$ and $\phi \in CB(Y)$. It is easy to see that $h_{y,\phi}$ is continuous and $||h_{y,\phi}|| \leq ||\phi||$. Therefore, $h_y : \phi \mapsto h_{y,\phi}$ is a bounded linear operator from $CB(Y)$ into $CB(K)$. Let $Y_1 := \{y \in B \mid h_y(CB(Y)) \subseteq A\}$.*

The family $\mathscr{T}$ is an $E-E$-representation of (K, A) on X if $B \neq \emptyset$ and $Y_1 \neq \emptyset$,

Definition 6.2. *The pair (K, A) has the common fixed point property on compacta with respect to $E-E$-representations if, for each compact subset Y of a separated locally convex topological vector space E and for each $E-E$-representation of K, A on X, there is in Y a common fixed point of the family $\mathscr{T}$.*

Remark 6.3. *Let A be a closed translation-invariant subalgebra of $L_\infty(K)$ containing constant functions. Then the following are equivalent:*

1. *A has a multiplicative left invariant mean.*
2. *The pair (K, A) has the common fixed point property on compacta with respect to $E-E$-representations.*

Proof. Let $\mathscr{T}$ be an $E-E$-representation of (K, A) on X. Then there exists an element $y \in Y$ such that $h_y(CB(Y)) \subseteq A$ and $T_g y \in Y$ for all $g \in K$. Let h_y^* be the adjoint of h_y and let m be a multiplicative left invariant mean on A. Then $< h_y^* m, 1 >= 1$, where 1 is the constant 1 function on Y. Also $h_y(f_1 f_2) = (h_{y,f_1})(h_{y,f_2})$, for $f_1, f_2 \in CB(Y)$ and $g \in K$. In addition, since m is multiplicative, $h_y^* m$ is a nonzero multiplicative linear functional on $CB(Y)$ and $< h_y^*(m), \bar{h} >= \overline{< h_y^*(m), h >}$, Thus, there exists an element $x_y \in Y$ such that $f(x_y) =< h_y^* m, f >=< m, h_{y,f} >$, for all $f \in CB(Y)$.

For each $g \in K$, define a map $\Psi_g : E^* \to CB(Y)$ via $(\Psi_g f)(z) =< f, T_g z >$, for $f \in E^*$, $z \in Y$. Then $h_{y,\Psi_g f} = L_g[h_{y,f}]$ since $f \in E^*$. Hence, $T_g x_y = x_y$, for each $g \in K$ since m is left translation invariant and E^* separates point of E.

Conversely, let $E = A^*$ and Y be the set of all multiplicative means on A. Then $X = Mean(A)$. Define $(g, m) \mapsto L_g^* m$ from $K \times Mean(A)$ into $Mean(A)$, where $Mean(A)$ has the weak*-topology of A^*. Then $\mathscr{T} = \{L_g^* \mid g \in K\}$ is a separately continuous representation of K on X. We note that each $\phi \in CB(Y)$ corresponds to an element $f_\phi \in A$ such that $\phi(m) = m(f_\phi)$, for $m \in Y$. Let $P(K) = \{g \in K \mid \delta_k * \delta_g$ is a point mass measuse, $\delta_{kg},\ \forall k \in K\}$, $g \in P(K)$ and $k \in K$. Then

$$\delta_{g_{L_K}}\phi(k) = \phi(L_k^*\delta_g) = \phi(\delta_{kg}) = \delta_{kg}(f_\phi) = R_g f_\phi(k).$$

Hence, $\delta_{g_{L_K}}\phi \in A$, since A is right translation invariant. i.e, $\delta_{g_{L_K}}(CB(Y)) \subseteq A$, for $g \in P(K)$. Thus, $\mathscr{T}$ is an $E-E$-representation of K, A on X. Therefore, there is some $m_0 \in Y$ such that $L_g^* m_0 = m_0$, for all $g \in K$. □

Let T be a bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$. Then T commutes with convolution from the left if $T(\phi * f) = \phi * T(f)$, for all $\phi \in L_1(K)$ and $f \in L_\infty(K)$. The following can be proved by a similar argument as in ([20], Theorem 2).

Remark 6.4. *The following are equivalent:*

1. *K is compact.*
2. *Any bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$ which commutes with convolution from the left is weak*-weak* continuous.*

Using bounded approximate identity of $L_1(K)$, one can show that any bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$ which commutes with convolution from the left also commutes with left translations. However, the converse is not true in general. For instance, if K is a direct product $G \times J$ of any locally compact non-discrete group G which is amenable as a discrete group and a finite hypergroup J, then for any left invariant mean m on $L_\infty(K)$ which is not topological left invariant, the operator $T(f) := m(f).1$ commutes with left translations but not with convolutions from the left.

It is important to note that in contrast to the group case, there is a class of compact commutative hypergroups for which any bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$ commuting with convolution is weak*-weak* continuous:

Example 6.5. *Fix $0 < a \leq \frac{1}{2}$ and let H_a be the hypergroup on $\mathbb{Z}_+ \cup \{\infty\}$ given by $\delta_m * \delta_n = \delta_{min(n,m)}$, for $m \neq n \in \mathbb{Z}_+$, $\delta_\infty * \delta_m = \delta_m * \delta_\infty = \delta_m$ and $\delta_n * \delta_n = \frac{1-2a}{1-a}\delta_n + \sum_{k=n+1}^{\infty} a^k \delta_k$ [5]. Then any bounded linear operator from $L_\infty(H_a)$ into $L_\infty(H_a)$ commuting with translations is weak*-weak* continuous.*

Proof. Let T be a bounded linear operator from $L_\infty(H_a)$ into $L_\infty(H_a)$ commuting with translations. For each $\phi \in L_1(K)$ and $n \in \mathbb{Z}_+$ define a function ϕ_n on K which coincide with ϕ on $\{0, 1, ..., n\}$ and zero otherwise. Then $||\phi_n - \phi||_1 \to 0$. In addition, for each $f \in L_\infty(K)$ we have $||T(\phi_n * f) - T(\phi * f)|| \to 0$ and $||\phi_n * Tf - \phi * Tf|| \to 0$ ([12], 6.2 C). For each $f \in L_\infty(K)$

$$\begin{aligned} T(\phi_n * f) &= T(\textstyle\sum_{k=0}^{n} \phi(k)(1-a)a^k L_{\check{k}} f) \\ &= \textstyle\sum_{k=0}^{n} \phi(k)(1-a)a^k T(L_{\check{k}} f) \\ &= \textstyle\sum_{k=0}^{n} \phi(k)(1-a)a^k L_{\check{k}} Tf \\ &= \phi_n * Tf \end{aligned}$$

we have that $T(\phi * f) = \phi * Tf$. Now the result follows from Remark 6.4. □

The following problems are still open:

Question 6.6. *Let K be a compact hypergroup such that $L_\infty(K)$ has a unique left invariant mean. Let T be a bounded linear operator from $L_\infty(K)$ into $L_\infty(K)$ which commutes with left translations. Can we conclude that T commutes with convolution from the left?*

Question 6.7. *Let G be a locally compact group. Then $L_1(G)$ is Arens semi-regular if and only if G is abelian or discrete (*[21]*, Theorem 1). Can we characterize hypergroups for which $L_1(K)$ is Arens semi-regular?*

Question 6.8. *Is there any non-inner amenable hypergroup K such that $Z(L_1(K))$ is non-trivial?*

Question 6.9. *Let K be a hypergroup such that $L_1(K)$ has a positive non-trivial center. Is there a compact neighbourhood V of the identity with $\Delta(g)R_g\chi_V = L_g\chi_V$?*

Question 6.10. *Let K be a connected, inner amenable hypergroup. Is K amenable?*

We say that a hypergroup K is topologically inner amenable if there exists a mean m on $L_\infty(K)$ such that $m((\Delta\phi)\check{} * f) = m(f * \check{\phi})$ for any positive norm one element ϕ in $L_1(K)$ and any $f \in L_\infty(K)$. It is easy to see that any inner invariant mean on $UC(K)$ is topologically inner invariant since

$$\begin{aligned} m(f * \check{\phi}) &= \int < m, R_g f\phi(g) > d\lambda(g) \\ &= \int < m, L_g f\phi(g) > d\lambda(g) \\ &=< m, \int L_g f\phi(g) d\lambda(g) > \\ &=< m, \int L_g f\phi(g)\Delta(g) d\check{\lambda}(g) > \\ &= m((\Delta\phi)\check{} * f). \end{aligned}$$

. However, on the space $L_\infty(K)$ the relation between topological inner invariant means and inner invariant means is not clear.

Question 6.11. *Let m be a topological inner invariant mean on $L_\infty(K)$. Is m also an inner invariant mean?*

Question 6.12. *Let K be an inner amenable hypergroup. Is there any topological inner invariant mean on $L_\infty(K)$?*

Question 6.13. *Let K be an inner amenable hypergroup. Does K satisfy central Reiter's condition P_1? (see (*[22]*, Remark) for the group case).*

Question 6.14. *Let K be a compact hypergroup. Can we have an exact description of weak*-closed left translation invariant complemented subspaces of $L_\infty(K)$?*

Acknowledgments. The author is pleased to express her appreciation to Professor Lau for his motivation and very helpful suggestions. The author would also like to thank the referee for his or her very helpful suggestions.

References

[1] Bloom, W., Heyer, H., *Harmonic analysis of probability measures on hypergroups*, de Gruyter Studies in Mathematics, 20, Berlin, 1995.

[2] Chapovsky, Y., Vainerman, L.I., *Compact quantum hypergroups*, J. Operator Theory, **41**(1999), no. 2, 261-289.

[3] Choda, H., Choda, M., *Fullness, simplicity and inner amenability*, Math. Japon., **24**(1979), no. 2, 235-246.

[4] Choda, M., *Effect of inner amenability on strong ergodicity*, Math. Japon., **28**(1983), no. 1, 109-115.

[5] Dunkl, C.F., Ramirez, D.E., *A family of countably compact P*-hypergroups*, Trans. Amer. Math. Soc., **202**(1975), 339-356.

[6] Dunkl, C.F., *The measure algebra of a locally compact hypergroup*, Trans. Amer. Hath. Soc., **179**(1973), 331-348.

[7] Effros, E.G., *Property* Γ *and inner amenability*, Proc. Amer. Math. Soc., **47**(1975), 483-486.

[8] Ghahramani, F., Medgalchi, A.R., *Compact multipliers on weighted hypergroup algebras*, Math. Proc. Cambridge Philos. Soc., **98**(1985), no. 3, 493-500.

[9] Ghahramani, F., Medgalchi, A.R., *Compact multipliers on weighted hypergroup algebras. II*, Math. Proc. Cambridge Philos. Soc., **100**(1986), no. 1, 145-149.

[10] Grosser, M., *Arens semiregular Banach algebras*, Monatsh. Math., **98**(1984), no. 1, 41-52.

[11] Hermann, P., *Induced representations and hypergroup homomorphisms*, Monatsh. Math., **116**(1993), 245-262.

[12] Jewett, R.I., *Spaces with an abstract convolution of measures*, Advances in Math., **18**(1975), 1-101.

[13] Lasser, R., *Amenability and weak amenability of* l_1*-algebras of polynomial hypergroups*, Studia Math., **182**(2007), no. 2, 183-196.

[14] Lasser, R., *Orthogonal polynomials and hypergroups. II. The symmetric case*, Trans. Amer. Math. Soc., **341**(1994), no. 2, 749-770.

[15] Lasser, R., Skantharajah, M., *Reiter's condition for amenable hypergroups*, Monatsh. Math., **163**(2011), no. 3, 327-338.

[16] Lau, A. T.-M., *Amenability and invariant subspaces*, J. Austral. Math. Soc., **18**(1974), no. 2, 200-204.

[17] Lau, A. T.-M., Paterson, A.L.T., *Inner amenable locally compact groups*, Trans. Amer. Math. Soc., **325**(1991), no. 1, 155-169.

[18] Lau, A. T.-M., *Invariantly complemented subspaces of* $L_\infty(G)$ *and amenable locally compact groups*, Illinois J. Math., **26**(1982), 226-235.

[19] Lau, A. T.-M., Paterson, A.L.T., *Operator theoretic characterizations of* $[IN]$*-groups and inner amenability*, Proc. Amer. Math. Soc., **102**(1988), no. 4, 893-897.

[20] Lau, A. T.-M., *Operators which commute with convolutions on subspaces of* $L_\infty(G)$, Colloq. Math., **39**(1978), no. 2, 351-359.

[21] Losert, V., Rindler, H., *Asymptotically central functions and invariant extensions of Dirac measure*, Probability measures on groups, VII, Oberwolfach, 1983, 368-378.

[22] Losert, V., Rindler, H., *Conjugation-invariant means*, Colloq. Math., **51**(1987), 221-225.

[23] Mitchell, T., *Function algebras, means, and fixed points*, Trans. Amer. Math. Soc., **130**(1968), 117-126.

[24] Ross, K.A., *Centers of hypergroups*, Trans. Amer. Math. Soc., **243**(1978), 251-269.

[25] Ross, K.A., *LCA hypergroups*, Topology Proc., **24**(1999), 533-546.

[26] Ross, K.A., *Hypergroups and centers of measure algebras*, Symposia Mathematica, **XXII**(1997), 189-203.

[27] Sinclair, A.M., *Continuous semigroups in Banach algebras*, London Mathematical Society Lecture Note Series, 63, Cambridge University Press, Cambridge-New York, 1982.

[28] Skantharajah, M., *Amenable hypergroups*, Illinois J. Math., **36**(1992), no. 1, 15-46.

[29] Spector, R., *Apercu de la théorie des hypergroupes*, Lecture Notes in Math., Vol. 497, Springer, Berlin, 1975, 643-673.

[30] Stokke, R., *Quasi-central bounded approximate identities in group algebras of locally compact groups*, Illinois J. Math., **48**(2004), no. 1, 151-170.

[31] Tahmasebi, N., *Hypergroups and invariant complemented subspaces*, J. Math. Anal. Appl., **414**(2014), 641-655.

[32] Tahmasebi, N., *Fixed point properties, invariant means and invariant projections related to hypergroups*, J. Math. Anal. Appl., **437**(2016), 526-544.

[33] Voit, M., *Duals of subhypergroups and quotients of commutative hypergroups*, Math. Z., **210**(1992), no. 2, 289-304.

[34] Willson, B., *Configuration and invariant nets for amenable hypergroups and related algebras*, Trans. Amer. Math. Soc., **366**(2014), no. 10, 5087-5112.

[35] Zeuner, H., *Duality of commutative hypergroups*, Probability measures on groups, Oberwolfach, **10**(1990), 467-488.

[36] Zeuner, H., *Moment functions and laws of large numbers on hypergroups*, Math. Z., **211**(1992), no. 3, 369-407.

Nazanin Tahmasebi
Department of Mathematical and Statistical Sciences, University of Alberta, AB, Canada,
Department of Computing Sciences, University of Alberta, AB, Canada,
Servier Virtual Cardiac Centre, Mazankowski Alberta Heart Institute, AB, Canada
e-mail: ntahmase@ualberta.ca

Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 2, 221–237

Local C-semigroups and complete second order abstract Cauchy problems

Chung-Cheng Kuo

Abstract. Let $C : X \to X$ be an injective bounded linear operator on a Banach space X over the field $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$ and $0 < T_0 \leq \infty$. Under some suitable assumptions, we deduce some relationship between the generation of a local (or an exponentially bounded) $\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$-semigroup on $X \times X$ with subgenerator (resp., the generator) $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ and one of the following cases: (i) the well-posedness of a complete second-order abstract Cauchy problem ACP(A, B, f, x, y): $w''(t) = Aw'(t)+Bw(t)+f(t)$ for a.e. $t \in (0, T_0)$ with $w(0) = x$ and $w'(0) = y$; (ii) a Miyadera-Feller-Phillips-Hille-Yosida type condition; (iii) B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local C-cosine function on X for which A may not be bounded; (iv) A is a subgenerator (resp., the generator) of a local C-semigroup on X for which B may not be bounded.

Mathematics Subject Classification (2010): 47D60, 47D62.

Keywords: Integrated C-semigroups, generator, subgenerator, abstract Cauchy problem.

1. Introduction

Let X be a Banach space over the field $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$ with norm $\|\cdot\|$, and let L(X) denote the family of all bounded linear operators from X into itself. For each $0 < T_0 \leq \infty$, we consider the following two abstract Cauchy problems:

$$\text{ACP}(A, f, x) \qquad \begin{cases} u'(t) = Au(t) + f(t) & \text{for a.e. } t \in (0, T_0) \\ u(0) = x \end{cases}$$

and

$$\text{ACP}(A, B, f, x, y) \qquad \begin{cases} w''(t) = Aw'(t) + Bw(t) + f(t) & \text{for a.e. } t \in (0, T_0) \\ w(0) = x, w'(0) = y, \end{cases}$$

where $x, y \in X$, $A : \mathrm{D}(A) \subset X \to X$ and $B : \mathrm{D}(B) \subset X \to X$ are closed linear operators, and $f \in \mathrm{L}^1_{loc}([0, T_0), X)$ (the family of all locally Bochner integrable functions from $[0, T_0)$ into X). A function u is called a (strong) solution of ACP(A, f, x) if $u \in \mathrm{C}([0, T_0), X)$ satisfies ACP(A, f, x) (that is $u(0) = x$ and for a.e. $t \in (0, T_0)$, $u(t)$ is differentiable and $u(t) \in \mathrm{D}(A)$, and $u'(t)=Au(t)+f(t)$ for a.e. $t \in (0, T_0)$). For each $\alpha > 0$ and each injection $C \in L(X)$, a subfamily $S(\cdot)(= \{S(t)|0 \le t < T_0\})$ of $L(X)$ is called a local α-times integrated C-semigroup on X if it is strongly continuous, $S(\cdot)C = CS(\cdot)$ and satisfies

$$(1.1)\quad S(t)S(s)x = \frac{1}{\Gamma(\alpha)}\left[\int_0^{t+s} - \int_0^t - \int_0^s\right](t+s-r)^{\alpha-1}S(r)Cx\,dr$$

for all $x \in X$ and $0 \le t, s \le t + s < T_0$ (see [1-2,12-14,18-21,28,30,32,35]) or called a local (0-times integrated) C-semigroup on X if it is strongly continuous, $S(\cdot)C = CS(\cdot)$ and satisfies

$$(1.2)\quad S(t)S(s)x = S(t+s)Cx$$

for all $x \in X$ and $0 \le t, s \le t + s < T_0$ (see [4,6,13,23,29]), where $\Gamma(\cdot)$ denotes the Gamma function. Moreover, we say that $S(\cdot)$ is

(1.3) locally Lipschitz continuous, if for each $0 < t_0 < T_0$ there exists a $K_{t_0} > 0$ such that $\|S(t+h) - S(t)\| \le K_{t_0}h$ for all $0 \le t, h, t + h \le t_0$;
(1.4) exponentially bounded, if $T_0 = \infty$ and there exist $K, \omega \ge 0$ such that $\|S(t)\| \le Ke^{\omega t}$ for all $t \ge 0$;
(1.5) nondegenerate, if $x = 0$ whenever $S(t)x = 0$ for all $0 \le t < T_0$.

A nondegenerate local α-times integrated C-semigroup $S(\cdot)$ on X implies that $S(0) = C$ if $\alpha = 0$, and $S(0) = 0$ (zero operator on X) otherwise, and the (integral) generator $A : \mathrm{D}(A) \subset X \to X$ of $S(\cdot)$ is a closed linear operator in X defined by $\mathrm{D}(A) = \{x \in X \mid S(\cdot)x - j_\alpha(\cdot)Cx = \widetilde{S}(\cdot)y_x$ on $[0, T_0)$ for some $y_x \in X\}$ and $Ax = y_x$ for all $x \in \mathrm{D}(A)$ (see [6,13-14,23]), which is also equal to the linear operator A in X defined by $D(A) = \{x \in X \mid \lim_{h\to 0^+}(S(h)x - Cx)/h \in R(C)\}$ and $Ax = C^{-1}\lim_{h\to 0^+}(S(h)x - Cx)/h$ for $x \in D(A)$ when $\alpha = 0$ (see [4,23,27]). Here $j_\beta(t) = \frac{t^\beta}{\Gamma(\beta+1)}$ and $\widetilde{S}(t)z = \int_0^t S(s)z\,ds$. In general, a local C-semigroup is called a C-semigroup if $T_0 = \infty$(see [2,4,14,26,32]) or a C_0-semigroup if $C = I$ (identity operator on X) (see [1,5]). It is known that the theory of local C-semigroup is related to another family in $\mathrm{L}(X)$ which is called a local C-cosine function (see [2,4,8-9,24,28-29,32]). Perturbation of local (integrated) C-semigroups has been extensively studied by many authors appearing in [1,6-7,10-12,15-16,22,30-32]. Some interesting applications of this topic are also illustrated there. The well-posedness of ACP(A, B, f, x, y) had been studied by many authors when $f = 0$ (see [3,6,9,17-18,20,25,32-34]). Some relationship between the well-posedness of ACP$(A, B, 0, x, y)$, a Miyadera-Feller-Phillips-Hille-Yosida type condition (see (1.6) below), and the generation of a C_0-semigroup on $X \times X$ with generator $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ have been established in [25] when A and B are commutable on $\mathrm{D}(B) \cap \mathrm{D}(A)$, in [20] and [32] for $A \in L(X)$, in [32] for $B \in L(X)$, and in [17] for the general case. In particular, Xiao and Liang [32, Theorems 2.6.1, 2.5.2 and 2.5.1] show that $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ generates a C_0-semigroup on $X \times X$ (if and) only if $B \in L(X)$ (and A generates a C_0-semigroup on X), but the conclusion may not be true when C_0-semigroups are replaced by local C-semigroups; and the well-posedness of $\mathrm{ACP}(A, B, 0, x, y)$ is equivalent to A generates a C_0-semigroup on X if $B \in L(X)$, and equivalent to B generates a cosine function on X if $A \in L(X)$. Unfortunately, a local C-semigroup may not be exponentially bounded and is not necessarily extendable to the half line $[0, \infty)$, and $\begin{pmatrix} 0 & 0 \\ B & A \end{pmatrix}$ may not be the generator of a local $\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$-semigroup on $X \times X$ whenever $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is. Moreover, $\lambda \in \rho_C(A, B)$ may not imply that $\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{\mathrm{D}(B)\cap\mathrm{D}(A)})}$ and $\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}}$ are bounded even though $\mathrm{D}(B) \cap \mathrm{D}(A)$ is dense in X and $C = I$, and $\lambda \in \rho_{\mathcal{C}}(\mathcal{T})$ implies that $\lambda \in \rho_C(A, B)$, $\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{\mathrm{D}(B)\cap\mathrm{D}(A)})}$ and $\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}}$ are bounded, but may not be bounded on X even though $C = I$. In particular, they are bounded on X when the assumption of $\mathrm{D}(B) \cap \mathrm{D}(A)$ is dense in X is added (see [17] for the case $C = I$). In this paper, we will extend the aforementioned results to the case of local C-semigroup by different methods (see Theorems 2.2 and 2.3 below). We show that for each $(x, y) \in \mathcal{D}$ $\mathrm{ACP}(A, B, 0, Cx, Cy)$ has a unique solution z which depends continuously differentiable on (x, y) and satisfies $Bz + Az' \in C([0, T_0), X)$ if and only if $\mathcal{T}$ is a subgenerator of a local $\mathcal{C}$-semigroup on $X \times X$ if and only if for each $(x, y) \in \mathcal{D}$ $\mathrm{ACP}(A, B, CBx, 0, Cy)$ has a unique solution w which depends continuously differentiable on (x, y) and satisfies $Bw + Aw' \in C([0, T_0), X)$ (see Theorem 2.5 below). Here $\mathcal{T} = \begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$, $\mathcal{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$, and $\mathcal{D}$ is a fixed subspace of $\mathrm{D}(B) \times \mathrm{D}(A)$ that is dense in $X \times X$. We then prove two important lemmas (see Lemmas 2.7 and 2.8 below) which can be used to show that there exist $M, \omega > 0$ so that for each pair $x, y \in \mathrm{D}(B) \cap \mathrm{D}(A)$ $\mathrm{ACP}(A, B, CBx, 0, Cy)$ has a unique solution w with $\|w(t)\|, \|w'(t)\| \le Me^{\omega t}(\|x\| + \|y\|)$ for all $t \ge 0$ and $Bw + Aw' \in C([0, \infty), X)$ if and only if $\mathcal{T}$ is a subgenerator of an exponentially bounded $\mathcal{C}$-semigroup on $X \times X$ if and only if there exist $M, \omega > 0$ so that $\lambda \in \rho_C(A, B)$ and

$$(1.6) \quad \|[\lambda(\lambda^2 - \lambda A - B)^{-1}C]^{(k)}\|, \|[\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}}]^{(k)}\| \le \frac{Mk!}{(\lambda-\omega)^{k+1}}$$

for all $\lambda > \omega$ and $k \in \mathbb{N} \cup \{0\}$ if and only if there exist $M, \omega > 0$ so that for each pair $x, y \in \mathrm{D}(B) \cap \mathrm{D}(A)$ $\mathrm{ACP}(A, B, 0, Cx, Cy)$ has a unique solution z with $\|z(t)\|, \|z'(t)\| \le Me^{\omega t}(\|x\| + \|y\|)$ for all $t \ge 0$ and satisfies $Bz + Az' \in C([0, \infty), X)$ (see Corollary 2.6 and Theorem 2.9 below). Here $\rho_C(A, B) = \{\lambda \in \mathbb{F} \mid \lambda^2 - \lambda A - B$ is injective, $R(C) \subset R(\lambda^2 - \lambda A - B)$, and $(\lambda^2 - \lambda A - B)^{-1}C \in L(X)\}$. When $\rho(\mathcal{T})$ (resolvent set of $\mathcal{T}$) is nonempty, we can combine Lemma 2.4 with [23, Corollary 3.6] to show that for each $(x, y) \in \mathrm{D}(B) \times \mathrm{D}(A)$ $\mathrm{ACP}(A, B, CBx, 0, Cy)$ has a unique solution w such that $Bw + Aw' \in C([0, T_0), X)$ if and only if $\mathcal{T}$ is the generator of a local $\mathcal{C}$-semigroup $\mathcal{S}(\cdot)$ on $X \times X$ if and only if for each $(x, y) \in \mathrm{D}(B) \times \mathrm{D}(A)$ $\mathrm{ACP}(A, B, 0, Cx, Cy)$ has a unique solution z such that $Bz + Az' \in C([0, T_0), X)$ (see

Theorem 2.11 below). We then apply the modifications of [12, Theorem 2.12 and Theorem 3.2] concerning the bounded and unbounded perturbations of a locally Lipschitz continuous local once integrated C-semigroup on X (see Theorem 2.12 below) and a basic property of local C-cosine function (see [6, Theorem 2.1.11]) to obtain two new equivalence relations concerning the generations of a local $\mathcal{C}$-semigroup on $X \times X$ with subgenerator (resp., the generator) $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ and either a locally Lipschitz continuous local C-cosine function on X with subgenerator (resp., the generator) B for which A may not be bounded (see Theorem 2.13 below) or a local C-semigroup on X with subgenerator (resp., the generator) A for which B may not be bounded (see Theorem 2.16 below). Under some suitable assumptions, which can be used to show those preceding equivalence conditions which are equivalent to B is the generator of a locally Lipschitz continuous local C-cosine function on X for which A may not be bounded (see Corollaries 2.14 and 2.15 below), and are also equivalent to A is the generator of a local C-semigroup on X for which B may not be bounded (see Corollaries 2.17 and 2.18 below).

2. Abstract Cauchy problems

In this section, we consider the existence of solutions of the abstract Cauchy problem ACP(A, B, f, x, y). A function u is called a (strong) solution of ACP(A, B, f, x, y) if $u \in \mathrm{C}^1([0, T_0), X)$ satisfies ACP(A, B, f, x, y) (that is $u(0) = x$, $u'(0) = y$, and for a.e. $t \in (0, T_0)$, $u'(t)$ is differentiable and $u'(t) \in \mathrm{D}(A)$, and $u''(t)=Au'(t)+Bu(t)+f(t)$ for a.e. $t \in (0, T_0)$). A linear operator A in X is called a subgenerator of a local α-times integrated C-semigroup $S(\cdot)$ if $S(t)x - j_\alpha(t)Cx = \int_0^t S(r)Axdr$ for all $x \in \mathrm{D}(A)$ and $0 \le t < T_0$, and $\int_0^t S(r)xdr \in \mathrm{D}(A)$ and $A\int_0^t S(r)xdr = S(t)x - j_\alpha(t)Cx$ for all $x \in X$ and $0 \le t < T_0$. Moreover, a subgenerator A of $S(\cdot)$ is called the maximal subgenerator of $S(\cdot)$ if it is an extension of each subgenerator of $S(\cdot)$ to $D(A)$. We next note some basic properties of a local C-semigroup, and then deduce some results about connections between the unique existence of solutions of ACP$(A, B, CBx, 0, Cy)$, ACP$\left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}\right)$, and ACP$(A, B, 0, Cx, Cy)$.

Proposition 2.1. (see [4,13,23]) *Let A be the generator of a local C-semigroup $S(\cdot)$ on X. Then*

(2.1) $S(t)S(s) = S(s)S(t)$ *for* $0 \le t, s, t + s < T_0$;
(2.2) A *is closed and* $C^{-1}AC = A$;
(2.3) $S(t)x \in D(A)$ *and* $S(t)Ax = AS(t)x$ *for* $x \in D(A)$ *and* $0 \le t < T_0$;
(2.4) $\int_0^t S(r)xdr \in D(A)$ *and* $A\int_0^t S(r)xdr = S(t)x - Cx$ *for* $x \in X$ *and* $0 \le t < T_0$;
(2.5) $R(S(t)) \subset \overline{D(A)}$ *for* $0 \le t < T_0$;
(2.6) A *is the maximal subgenerator of* $S(\cdot)$;
(2.7) $C^{-1}\overline{A_0}C$ *is the generator of* $S(\cdot)$ *for each subgenerator* A_0 *of* $S(\cdot)$.

Theorem 2.2. (see [13,23]) *Let A be a closed linear operator in X such that $CA \subset AC$. Then A is a subgenerator of a local C-semigroup $S(\cdot)$ on X if and only if for each*

$x \in X$ *ACP*$(A, Cx, 0)$ *has a unique (strong) solution* $u(\cdot, x)$ *in* $C^1([0, T_0), X)$. *In this case, we have* $u(t, x) = j_0 * S(t)x (= \int_0^t S(s)xds)$ *for all* $x \in X$. By slightly modifying the proof of [23, Corollary 3.5], the next theorem concerning the well-posedness of ACP$(A, f, x)$is attained, and so its proof is omitted.

Theorem 2.3. *Let* A *be a closed linear operator in* X *such that* $CA \subset AC$ *and* D *dense in* X *for some subspace* D *of* $D(A)$. *Then the following are equivalent:*

(i) A *is a subgenerator of a nondegenerate local* C*-semigroup* $S(\cdot)$ *on* X;
(ii) *For each* $x \in D$ *ACP*$(A, 0, Cx)$ *has a unique solution* $u(\cdot; Cx)$ *in* $C([0, T_0), [D(A)])$ *which depends continuously on* x. *That is, if* $\{x_n\}_{n=1}^{\infty}$ *is a Cauchy sequence in* $(D, \|\cdot\|)$, *then* $\{u(\cdot; Cx_n)\}_{n=1}^{\infty}$ *converges uniformly on compact subsets of* $[0, T_0)$.

In this case, $u(\cdot, Cx) = S(\cdot)x$.

In the following, we always assume that A and B are biclosed linear operators in X such that $CA \subset AC$ and $CB \subset BC$.

Lemma 2.4. *Assume that* $\mathcal{D}$ *is a subspace of* $D(B) \times D(A)$. *Then the following are equivalent:*

(i) *For each* $(x, y) \in \mathcal{D}$ *ACP*$(A, B, CBx, 0, Cy)$ *has a unique solution* w *such that* $Bw + Aw' \in C([0, T_0), X)$;
(ii) *For each* $(x, y) \in \mathcal{D}$ *ACP*$\left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}\right)$ *has a unique solution* $\begin{pmatrix} u \\ v \end{pmatrix}$ *in* $C([0, T_0), [\mathcal{T}])$;
(iii) *For each* $(x, y) \in \mathcal{D}$ *ACP*$(A, B, 0, Cx, Cy)$ *has a unique solution* z *such that* $Bz + Az' \in C([0, T_0), X)$.

In this case, $w = j_0 * v$ *and* $z = u$.
In particular, $z, w \in C^1([0, T_0), [D(A)]) \cap C([0, T_0), [D(B)])$ *if either* A *or* B *is bounded.*
Here $\mathcal{T} = \begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ *and* $\mathcal{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$.

Proof. Since the biclosedness of A and B with $CA \subset AC$ and $CB \subset BC$ implies that $\mathcal{T}$ is a closed linear operator in $X \times X$ so that $\mathcal{CT} \subset \mathcal{TC}$. Suppose that $(x, y) \in \mathcal{D}$ and $\begin{pmatrix} u \\ v \end{pmatrix}$ denotes the unique solution of ACP$\left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}\right)$ in C$([0, T_0), [(\mathcal{T}])$. Then v and $Bu + Av$ are continuous on $[0, T_0)$, and $u' = v$ and $v' = Bu + Av+$ a.e. on $[0, T_0)$, so that $u = j_0 * v + Cx$ on $[0, T_0)$, $j_0 * v(t) \in$ D(B) for all $t \in [0, T_0)$, and $v' = Bj_0 * v + CBx$ a.e. on $[0, T_0)$. Hence, $w = j_0 * v$ is a solution of ACP$(A, B, CBx, 0, Cy)$ such that $Bw + Aw' \in$ C$([0, T_0), X)$. The uniqueness of solutions of ACP$(A, B, CBx, 0, Cy)$ follows from the fact that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the unique solution of ACP $\left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$ in C$([0, T_0), [\mathcal{T}])$. Conversely, suppose that $(x, y) \in \mathcal{D}$ and w denotes the unique solution of ACP$(A, B, CBx, 0, Cy)$ such that $Bw + Aw' \in$ C$([0, T_0), X)$. We set $u = w + Cx$ and $v = w'$ on $[0, T_0)$. Then

$$\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} Cx \\ Cy \end{pmatrix}, \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \in \mathrm{D}(B) \times \mathrm{D}(A) = \mathrm{D}(\mathcal{T})$$

for all $t \in [0, T_0)$ and $\mathcal{T}\begin{pmatrix} u \\ v \end{pmatrix}$ is continuous on $[0, T_0)$, and for a.e. $t \in (0, T_0)$ $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ is differentiable and

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} w'(t) \\ w''(t) \end{pmatrix} = \begin{pmatrix} w'(t) \\ Aw'(t) + Bw(t) + CBx \end{pmatrix}$$

$$= \begin{pmatrix} v(t) \\ Av(t) + Bu(t) \end{pmatrix} = \mathcal{T}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix},$$

and so $\begin{pmatrix} u \\ v \end{pmatrix}$ is a solution of ACP $\left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}\right)$ in $C([0, T_0), [\mathcal{T}])$. The uniqueness of solutions follows from the fact that 0 is the unique solution of ACP$(A, B, 0, 0, 0)$. Similarly, we can show that (ii) and (iii) are equivalent. □

Just as an application of Theorem 2.3, the next theorem concerning the well-posedness of ACP(A, B, f, x, y) is also attained.

Theorem 2.5. *Assume that $\mathcal{D}$ is dense in $X \times X$ for some subspace $\mathcal{D}$ of $D(B) \times D(A)$. Then the following are equivalent:*

(i) *For each $(x, y) \in \mathcal{D}$ ACP$(A, B, CBx, 0, Cy)$ has a unique solution w which depends continuously differentiable on (x, y) (that is, if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(D(B), \|\cdot\|)$ and $\{y_n\}_{n=1}^{\infty}$ a Cauchy sequence in $(D(A), \|\cdot\|)$, and w_n denotes the unique solution of ACP$(A, B, CBx_n, 0, Cy_n)$, then $\{w_n(\cdot)\}_{n=1}^{\infty}$ and $\{w_n'(\cdot)\}_{n=1}^{\infty}$ both converge uniformly on compact subsets of $[0, T_0)$) and satisfies $Bw + Aw' \in C([0, T_0), X)$;*

(ii) *$\mathcal{T}$ is a subgenerator of a local $\mathcal{C}$-semigroup $\mathcal{S}(\cdot)$ on $X \times X$;*

(iii) *For each $(x, y) \in \mathcal{D}$ ACP$(A, B, 0, Cx, Cy)$ has a unique solution z which depends continuously differentiable on (x, y) and satisfies $Bz + Az' \in C([0, T_0), X)$.*

Here $\mathcal{T} = \begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ *and* $\mathcal{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$.

Proof. Since for each $(x, y) \in \mathcal{D}$ $\begin{pmatrix} u \\ v \end{pmatrix}$ is the unique solution of

$$\text{ACP}\left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}\right)$$

in $C([0, T_0), [\mathcal{T}])$ if and only if for each $(x, y) \in \mathcal{D}$ $u = w + Cx$ and $v = w'$ on $[0, T_0)$, and w is the unique solution of ACP$(A, B, CBx, 0, Cy)$ such that $Bw + Aw' \in C([0, T_0), X)$. By Theorem 2.3, we also have $\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{S}(\cdot)\begin{pmatrix} x \\ y \end{pmatrix}$. Consequently, $\mathcal{T}$ is a subgenerator of a local $\mathcal{C}$-semigroup on $X \times X$ if and only if for each $(x, y) \in \mathcal{D}$ ACP$(A, B, CBx, 0, Cy)$ has a unique solution w which depends continuously differentiable on (x, y). Similarly, we can show that (ii) and (iii) are equivalent. □

Corollary 2.6. *Assume that $\mathcal{D}$ is dense in $X \times X$ for some subspace $\mathcal{D}$ of $D(B) \times D(A)$. Then the following are equivalent:*

(i) *There exist* $M,\omega > 0$ *such that for each* $(x,y)\in \mathcal{D}$ *ACP*$(A,B,CBx,0,Cy)$ *has a unique solution* w *with* $\|w(t)\|,\|w'(t)\| \le Me^{\omega t}(\|x\|+\|y\|)$ *for all* $t\ge 0$ *and* $Bw + Aw' \in C([0,\infty),X)$;

(ii) $\mathcal{T}$ *is a subgenerator of an exponentially bounded* $\mathcal{C}$*-semigroup on* $X\times X$;

(iii) *There exist* $M,\omega > 0$ *such that for each* $(x,y)\in \mathcal{D}$ *ACP*$(A,B,0,Cx,Cy)$ *has a unique solution* z *with* $\|z(t)\|,\|z'(t)\| \le Me^{\omega t}(\|x\|+\|y\|)$ *for all* $t\ge 0$ *and satisfies* $Bz + Az' \in C([0,T_0),X)$.

Lemma 2.7. *Assume that* $\lambda \in \rho_{\mathcal{C}}(\mathcal{T})$ *(*$\mathcal{C}$*-resolvent set of* $\mathcal{T}$*). Then*

(i) $\lambda\in\rho_C(A,B)$;

(ii) $(\lambda^2-\lambda A-B)^{-1}C(\lambda - A_{D(B)\cap D(A)})$ *and* $(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}$ *are closable, and their closures are bounded and have the same domain;*

(iii) $(\lambda-\mathcal{T})^{-1}\mathcal{C} = \begin{pmatrix} \overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda - A_{D(B)\cap D(A)})} & (\lambda^2-\lambda A-B)^{-1}C \\ \overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}} & \lambda(\lambda^2-\lambda A-B)^{-1}C \end{pmatrix}$ *on* $D(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda - A_{D(B)\cap D(A)})})\times X$, *and on* $X\times X$ *if* $D(B)\cap D(A)$ *is dense in* X.

Proof. To show that $\lambda^2-\lambda A-B$ is closed. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathrm{D}(B)\cap\mathrm{D}(A)$ which converges to x in X and $\{(\lambda^2-\lambda A-B)x_n\}_{n=1}^{\infty}$ converges to y in X. Then $\begin{pmatrix} x_n \\ \lambda x_n\end{pmatrix}\in \mathrm{D}(\mathcal{T})$, $\begin{pmatrix} x_n \\ \lambda x_n\end{pmatrix}\to\begin{pmatrix} x \\ \lambda x\end{pmatrix}$, and

$$(\lambda-\mathcal{T})\begin{pmatrix} x_n \\ \lambda x_n\end{pmatrix} = \begin{pmatrix} 0 \\ (\lambda^2-\lambda A-B)x_n\end{pmatrix}\to\begin{pmatrix} 0 \\ y\end{pmatrix}.$$

By the closedness of $\lambda-\mathcal{T}$, we have $\begin{pmatrix} x \\ \lambda x\end{pmatrix}\in\mathrm{D}(\mathcal{T})$ and

$$\begin{pmatrix} 0 \\ (\lambda^2-\lambda A-B)x\end{pmatrix} = (\lambda-\mathcal{T})\begin{pmatrix} x \\ \lambda x\end{pmatrix} = \begin{pmatrix} 0 \\ y\end{pmatrix},$$

and so $(\lambda^2-\lambda A-B)x=y$. Hence, $\lambda^2-\lambda A-B$ is closed. To show that $\lambda^2-\lambda A-B$ is injective. Suppose that $(\lambda^2-\lambda A-B)x=0$. Then

$$(\lambda-\mathcal{T})\begin{pmatrix} x \\ \lambda x\end{pmatrix} = \begin{pmatrix} 0 \\ (\lambda^2-\lambda A-B)x\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix},$$

and so $\begin{pmatrix} x \\ \lambda x\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix}$. Hence, $x=0$, which implies that $\lambda^2-\lambda A-B$ is injective. To show that $\mathrm{R}(C)\subset\mathrm{R}(\lambda^2-\lambda A-B)$. Suppose that $z\in X$ is given. Then

$$(\lambda-\mathcal{T})\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} 0 \\ Cz\end{pmatrix}$$

for some $(x,y)\in\mathrm{D}(\mathcal{T})=\mathrm{D}(B)\times\mathrm{D}(A)$, so that $\lambda x-y=0$ and $-Bx+(\lambda-A)y=Cz$. Hence, $x\in\mathrm{D}(B)\cap\mathrm{D}(A)(=\mathrm{D}(\lambda^2-\lambda A-B))$ and $(\lambda^2-\lambda A-B)x=Cz$, which implies that $\mathrm{R}(C)\subset\mathrm{R}(\lambda^2-\lambda A-B)$. Consequently, $\lambda\in\rho_C(A,B)$.
To show that $(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{\mathrm{D}(B)\cap\mathrm{D}(A)})$ and $(\lambda^2-\lambda A-B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}$ are closable, we need only to show that $(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{\mathrm{D}(B)\cap\mathrm{D}(A)})$ or $(\lambda^2-\lambda A-B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}$ is closable. We will show that

$$(2.8)\quad (\lambda-\mathcal{T})^{-1}\mathcal{C}=\begin{pmatrix} (\lambda^2-\lambda A-B)^{-1}C(\lambda-A) & (\lambda^2-\lambda A-B)^{-1}C \\ (\lambda^2-\lambda A-B)^{-1}CB & \lambda(\lambda^2-\lambda A-B)^{-1}C \end{pmatrix}$$

on $D(B)\cap D(A)$ first or equivalently,

$$(\lambda-\mathcal{T})\begin{pmatrix} (\lambda^2-\lambda A-B)^{-1}C(\lambda-A) & (\lambda^2-\lambda A-B)^{-1}C \\ (\lambda^2-\lambda A-B)^{-1}CB & \lambda(\lambda^2-\lambda A-B)^{-1}C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} Cx \\ Cy \end{pmatrix}$$
$$=\mathcal{C}\begin{pmatrix} x \\ y \end{pmatrix}$$

for all $x,y\in D(B)\cap D(A)$. Suppose that $x,y\in D(B)\cap D(A)$ are given. Then by the fact $B(\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x{=}(\lambda-A)(\lambda^2-\lambda A-B)^{-1}CBx$ that we have

$$\begin{aligned}&(\lambda-\mathcal{T})\begin{pmatrix} (\lambda^2-\lambda A-B)^{-1}C(\lambda-A) & (\lambda^2-\lambda A-B)^{-1}C \\ (\lambda^2-\lambda A-B)^{-1}CB & \lambda(\lambda^2-\lambda A-B)^{-1}C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\\
&=\begin{pmatrix} \lambda & -I \\ -B & \lambda-A \end{pmatrix}\begin{pmatrix} (\lambda^2-\lambda A-B)^{-1}C(\lambda-A) & (\lambda^2-\lambda A-B)^{-1}C \\ (\lambda^2-\lambda A-B)^{-1}CB & \lambda(\lambda^2-\lambda A-B)^{-1}C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\\
&=\begin{pmatrix} \lambda & -I \\ -B & \lambda-A \end{pmatrix}\begin{pmatrix} (\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x+(\lambda^2-\lambda A-B)^{-1}Cy \\ (\lambda^2-\lambda A-B)^{-1}CBx+\lambda(\lambda^2-\lambda A-B)^{-1}Cy \end{pmatrix}\\
&=\begin{pmatrix} Cx \\ Cy \end{pmatrix}.\end{aligned}$$

Suppose that $x_n\in D(B)\cap D(A)$, $x_n\to 0$ in X, and $(\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n\to y$ in X. Then

$$\begin{aligned}(\lambda^2-\lambda A-B)^{-1}CB)x_n&=(\lambda^2-\lambda A-B)^{-1}C(B+\lambda A-\lambda^2))x_n\\
&\quad+(\lambda^2-\lambda A-B)^{-1}C(\lambda^2-\lambda A)x_n\\
&=Cx_n+(\lambda^2-\lambda A-B)^{-1}C(\lambda^2-\lambda A)x_n\\
&\to\lambda y,\end{aligned}$$

and so

$$\begin{aligned}(\lambda-\mathcal{T})^{-1}C\begin{pmatrix} x_n \\ 0 \end{pmatrix}&=\begin{pmatrix} (\lambda^2-\lambda A-B)^{-1}C(\lambda-A) & (\lambda^2-\lambda A-B)^{-1}C \\ (\lambda^2-\lambda A-B)^{-1}CB & \lambda(\lambda^2-\lambda A-B)^{-1}C \end{pmatrix}\begin{pmatrix} x_n \\ 0 \end{pmatrix}\\
&=\begin{pmatrix} (\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n \\ (\lambda^2-\lambda A-B)^{-1}CBx_n \end{pmatrix}\\
&\to\begin{pmatrix} y \\ \lambda y \end{pmatrix}=(\lambda-\mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} 0 \\ 0 \end{pmatrix}.\end{aligned}$$

Hence, $y=0$, which implies that $(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})$ is closable.
To show that $\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}$ is bounded.
Let $x\in D(\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}})$ be given.
Then $(x_n,(\lambda^2-\lambda A-B)^{-1}CBx_n)\to(x,\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}x)$ for some $x_n\in D(B)\cap D(A)$, and so

$$(\lambda-\mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} x_n \\ 0 \end{pmatrix}=\begin{pmatrix} (\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n \\ (\lambda^2-\lambda A-B)^{-1}CBx_n \end{pmatrix}\to(\lambda-\mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Hence, $\{(\lambda^2-\lambda A-B)^{-1}(\lambda-A)x_n\}_{n=1}^{\infty}$ and $\{(\lambda^2-\lambda A-B)^{-1}Bx_n\}_{n=1}^{\infty}$ both converge. By the closedness of $\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})}$ and $\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}$, we have $x\in D(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})})$ and

$$(\lambda-\mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} x \\ 0 \end{pmatrix}=\begin{pmatrix} \overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})}x \\ \overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}x \end{pmatrix},$$

which implies that $\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}$ is bounded and

$$D(\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}})\subset D(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})}).$$

Similarly, we can show that $\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})}$ is bounded and

$$D(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})})\subset D(\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}),$$

which implies that

$$\begin{aligned}(\lambda-\mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} x \\ y \end{pmatrix}&=(\lambda-\mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} x \\ 0 \end{pmatrix}+(\lambda-\mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} 0 \\ y \end{pmatrix}\\ &=\begin{pmatrix} \overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})} & (\lambda^2-\lambda A-B)^{-1}C \\ \overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}} & \lambda(\lambda^2-\lambda A-B)^{-1}C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\end{aligned}$$

for all $(x,y)\in D(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})})\times X$. Combining this with the closedness of $\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}$ and the denseness of $D(B)\cap D(A)$ in X, we have

$$\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}\in L(X).$$

□

Lemma 2.8. *Assume that $\lambda\in\rho_C(A,B)$. Then*

(i) *$\lambda-\mathcal{T}$ is injective;*

(ii) *$(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})$ and $(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}$ are closable and their closures have the same domain, and*

$$(\lambda-\mathcal{T})\begin{pmatrix} \overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})} & (\lambda^2-\lambda A-B)^{-1}C \\ \overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}} & \lambda(\lambda^2-\lambda A-B)^{-1}C \end{pmatrix}=\mathcal{C}$$

on $D(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})})\times X$;

(iii) *$\lambda\in\rho_{\mathcal{C}}(\mathcal{T})$ and*

$$(\lambda-\mathcal{T})^{-1}\mathcal{C}=\begin{pmatrix} \overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})} & (\lambda^2-\lambda A-B)^{-1}C \\ \overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}} & \lambda(\lambda^2-\lambda A-B)^{-1}C \end{pmatrix},$$

if $\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})}\in L(X)$.

In particular, the conclusion of (iii) holds when A or B in $L(X)$, or $D(B)\cap D(A)$ is dense in X with $AB=BA$ on $D(B)\cap D(A)$.

Proof. To show that $\lambda-\mathcal{T}$ is injective. Suppose that $(\lambda-\mathcal{T})\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then $\lambda x-y=0$ and $-Bx+(\lambda-A)y=0$, so that $\lambda x=y$ and $-Bx+(\lambda^2-\lambda A)x=0$. Hence, $x=0=y$, which implies that $\lambda-\mathcal{T}$ is injective. Just as in the proof of Lemma 2.7, we will apply (2.8) to show that $(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})$ and

$(\lambda^2-\lambda A-B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}$ are closable. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathrm{D}(B)\cap\mathrm{D}(A)$ which converges to 0 in X and $\{(\lambda^2-\lambda A-B)(\lambda-A)x_n\}_{n=1}^{\infty}$ converges to y in X. Then

$$(\lambda^2-\lambda A-B)^{-1}CBx_n=-Cx_n+(\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n\to\lambda y,$$

and so $\begin{pmatrix}(\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n\\(\lambda^2-\lambda A-B)^{-1}CBx_n\end{pmatrix}\to\begin{pmatrix}y\\\lambda y\end{pmatrix}$. Hence,

$$(\lambda-\mathcal{T})\begin{pmatrix}(\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n\\(\lambda^2-\lambda A-B)^{-1}CBx_n\end{pmatrix}=\begin{pmatrix}Cx_n\\0\end{pmatrix}\to\begin{pmatrix}0\\0\end{pmatrix}.$$

By the closedness of $\mathcal{T}$, we have $\begin{pmatrix}y\\\lambda y\end{pmatrix}\in\mathrm{D}(\mathcal{T})$ and $(\lambda-\mathcal{T})\begin{pmatrix}y\\\lambda y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, which together with the injectivity of $\lambda-\mathcal{T}$ implies that $y=0$.
Consequently, $(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{\mathrm{D}(B)\cap\mathrm{D}(A)})$ is closable. Similarly, we can show that $(\lambda^2-\lambda A-B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}$ is closable. Just as in the proof of Lemma 2.7, we will show that

$$\mathrm{D}(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{\mathrm{D}(B)\cap\mathrm{D}(A)})})=\mathrm{D}(\overline{(\lambda^2-\lambda A-B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}}),$$

and for each $x\in\mathrm{D}(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{\mathrm{D}(B)\cap\mathrm{D}(A)})})$

$$\begin{pmatrix}\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{\mathrm{D}(B)\cap\mathrm{D}(A)})}x\\\overline{(\lambda^2-\lambda A-B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}}x\end{pmatrix}\in\mathrm{D}(\mathcal{T}).$$

Suppose that $x\in\mathrm{D}(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{\mathrm{D}(B)\cap\mathrm{D}(A)})})$ is given. Then $x_n\to x$ and $(\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n\to\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{\mathrm{D}(B)\cap\mathrm{D}(A)})}x$ for some sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathrm{D}(B)\cap\mathrm{D}(A)$, and so

$$(\lambda^2-\lambda A-B)^{-1}CBx_n\to-Cx+\lambda\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{\mathrm{D}(B)\cap\mathrm{D}(A)})}x.$$

Hence, $x\in\mathrm{D}(\overline{(\lambda^2-\lambda A-B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}})$, which implies that

$$\mathrm{D}(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{\mathrm{D}(B)\cap\mathrm{D}(A)})})\subset\mathrm{D}(\overline{(\lambda^2-\lambda A-B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}}).$$

Similarly, we can show that

$$\mathrm{D}(\overline{(\lambda^2-\lambda A-B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}})\subset\mathrm{D}(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{\mathrm{D}(B)\cap\mathrm{D}(A)})}).$$

Since

$$\begin{pmatrix}(\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n\\(\lambda^2-\lambda A-B)^{-1}CBx_n\end{pmatrix}\to\begin{pmatrix}\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{\mathrm{D}(B)\cap\mathrm{D}(A)})}x\\\overline{(\lambda^2-\lambda A-B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}}x\end{pmatrix}$$

and

$$(\lambda-\mathcal{T})\begin{pmatrix}(\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n\\(\lambda^2-\lambda A-B)^{-1}CBx_n\end{pmatrix}=\begin{pmatrix}Cx_n\\0\end{pmatrix}\to\begin{pmatrix}Cx\\0\end{pmatrix}.$$

By the closedness of $\lambda-\mathcal{T}$, we have

$$(\lambda-\mathcal{T})\begin{pmatrix}\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{\mathrm{D}(B)\cap\mathrm{D}(A)})}&(\lambda^2-\lambda A-B)^{-1}C\\\overline{(\lambda^2-\lambda A-B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}}&\lambda(\lambda^2-\lambda A-B)^{-1}C\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix}=\mathcal{C}\begin{pmatrix}x\\0\end{pmatrix}.$$

Consequently,

$$(\lambda - \mathcal{T})\begin{pmatrix} \overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{\mathrm{D}(B)\cap \mathrm{D}(A)})} & (\lambda^2 - \lambda A - B)^{-1}C \\ \overline{(\lambda^2 - \lambda A - B)^{-1}CB_{\mathrm{D}(B)\cap \mathrm{D}(A)}} & \lambda(\lambda^2 - \lambda A - B)^{-1}C \end{pmatrix} = \mathcal{C}$$

on $\mathrm{D}(\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{\mathrm{D}(B)\cap \mathrm{D}(A)})}) \times X$. □

Since $\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{\mathrm{D}(B)\cap \mathrm{D}(A)})} = [\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{\mathrm{D}(B)\cap \mathrm{D}(A)}}]\frac{1}{\lambda}+\frac{1}{\lambda}C$ and $(\lambda^2 - \lambda A - B)^{-1}C=[\lambda(\lambda^2 - \lambda A - B)^{-1}C]\frac{1}{\lambda}$, we can combine Lemma 2.7 with Lemma 2.8 and [1, Theorem 2.4.1] or [32, Theorem 1.2.1] to obtain the next new Miyadera-Feller-Phillips-Hille-Yosida type theorem concerning the generation of an exponentially bounded $\mathcal{C}$-semigroup on $X \times X$.

Theorem 2.9. *Assume that $D(B) \cap D(A)$ is dense in X. Then $\mathcal{T}$ is a subgenerator of an exponentially bounded $\mathcal{C}$-semigroup on $X \times X$ if and only if there exist $M, \omega > 0$ such that $\lambda \in \rho_C(A, B)$ and (1.6) holds for all $\lambda > \omega$ and $k \in \mathbb{N} \cup \{0\}$.*

Just as a result in [17, Theorem 2] for the case of C_0-semigroup, we can combine Corollary 2.6 with Theorem 2.9 to obtain the next corollary.

Corollary 2.10. *Assume that $D(B) \cap D(A)$ is dense in X. Then the following statements are equivalent:*

- (i) *There exist $M, \omega > 0$ such that for each pair $x, y \in D(B) \cap D(A)$, $ACP(A, B, CBx, 0, Cy)$ has a unique solution w with $\|w(t)\|, \|w'(t)\| \leq Me^{\omega t}(\|x\| + \|y\|)$ for all $t \geq 0$ and $Bw + Aw' \in C([0, \infty), X)$;*
- (ii) *$\mathcal{T}$ is a subgenerator of an exponentially bounded $\mathcal{C}$-semigroup on $X \times X$;*
- (iii) *There exist $M, \omega > 0$ such that $\lambda \in \rho_C(A, B)$ and (1.6) holds for all $\lambda > \omega$ and $k \in \mathbb{N} \cup \{0\}$;*
- (iv) *There exist $M, \omega > 0$ such that for each pair $x, y \in D(B) \cap D(A)$, $ACP(A, B, 0, Cx, Cy)$ has a unique solution z with $\|z(t)\|, \|z'(t)\| \leq Me^{\omega t}(\|x\|+ \|y\|)$ for all $t \geq 0$ and satisfies $Bz + Az' \in C([0, T_0), X)$.*

Combining Lemma 2.4 with [23, Corollary 3.6], the next theorem is also attained.

Theorem 2.11. *Assume that $\rho(\mathcal{T})$ (resolvent set of $\mathcal{T}$) is nonempty. Then the following are equivalent:*

- (i) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, CBx, 0, Cy)$ has a unique solution w such that $Bw + Aw' \in C([0, T_0), X)$;*
- (ii) *$\mathcal{T}$ is the generator of a local $\mathcal{C}$-semigroup $\mathcal{S}(\cdot)$ on $X \times X$;*
- (iii) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, 0, Cx, Cy)$ has a unique solution z such that $Bz + Az' \in C([0, T_0), X)$.*

By modifying slightly the proofs of [12, Theorem 2.12 and Theorem 3.2], the next theorem is also attained, and so its proof is omitted.

Theorem 2.12. *Let B be a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated C-semigroup on X. Assume that A is a bounded linear operator from $\overline{D(B)}$ into $R(C)$ or a bounded linear operator from $[D(B)]$ into $R(C)$ so that $R(C^{-1}A) \subset D(B)$ and $A + B$ is closed. Then $A + B$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated C-semigroup on X.*

Since B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local C-cosine function on X if and only if $\begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated $\mathcal{C}$-semigroup on $X \times X$ (see [6, Theorem 2.1.11]); and A is a bounded linear operator from $[D(B)]$ into $\mathrm{R}(C)$ so that $\mathrm{R}(C^{-1}A) \subset D(B)$ implies that

$$\mathrm{R}\left(\mathcal{C}^{-1}\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}\right) = \mathrm{R}\left(\begin{pmatrix} 0 & 0 \\ 0 & C^{-1}A \end{pmatrix}\right) \subset D\left(\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}\right) = D(B) \times D(A),$$

we can apply Theorem 2.12 to obtain the next new result concerning the generations of a locally Lipschitz continuous local C-cosine function on X with subgenerator (resp., the generator) B and a local $\mathcal{C}$-semigroup on $X \times X$ with subgenerator (resp., the generator) $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ for which A may not be bounded.

Theorem 2.13. *Assume that A is a bounded linear operator from $\overline{D(B)}$ into $R(C)$ or a bounded linear operator from $[D(B)]$ into $R(C)$ so that $R(C^{-1}A) \subset D(B)$. Then $\mathcal{T}$ is a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup on $X \times X$ only if B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local C-cosine function on X. The "if part" is also true when the assumption of $D(B)$ is dense in X is added.*

Proof. Suppose that $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup on $X \times X$. Then it is also a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated $\mathcal{C}$-semigroup on $X \times X$, and so $\begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated $\mathcal{C}$-semigroup on $X \times X$. Hence, B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local C-cosine function on X. Conversely, suppose that $\mathrm{D}(B)$ is dense in X and B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local C-cosine function on X. Then $\begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated $\mathcal{C}$-semigroup on $X \times X$, and so $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated $\mathcal{C}$-semigroup on $X \times X$. Hence, it is also a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup on $X \times X$. □

Combining Theorem 2.11 with Theorem 2.13, we can obtain the next two corollaries.

Corollary 2.14. *Assume that $\rho(A, B)$ is nonempty and $A \in L(X)$. Then the following are equivalent:*

(i) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, CBx, 0, Cy)$ has a unique solution w in $C([0, T_0), [D(B)])$;*

(ii) *$\mathcal{T}$ is the generator of a local $\mathcal{C}$-semigroup on $X \times X$;*

(iii) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, 0, Cx, Cy)$ has a unique solution z in $C([0, T_0), [D(B)])$.*

Moreover, (i)-(iii) imply

(iv) *B is the generator of a locally Lipschitz continuous local C-cosine function on X*

if $R(A) \subset R(C)$, and (i)-(iv) are equivalent if the assumption of $D(B)$ is dense in X is also added. Here $[D(B)]$ denotes the Banach space $D(B)$ with norm $|\cdot|$ defined by $|x| = \|x\| + \|Bx\|$ for $x \in D(B)$.

Corollary 2.15. *Assume that $D(B) \cap D(A)$ is dense in X, $\rho(A, B)$ nonempty, and $AB = BA$ on $D(B) \cap D(A)$. Then the following are equivalent:*

(i) *For each $(x, y) \in D(B) \times D(A)$ ACP$(A, B, CBx, 0, Cy)$ has a unique solution w such that $Bw + Aw' \in C([0, T_0), X)$;*
(ii) *$\mathcal{T}$ is the generator of a local $\mathcal{C}$-semigroup on $X \times X$;*
(iii) *For each $(x, y) \in D(B) \times D(A)$ ACP$(A, B, 0, Cx, Cy)$ has a unique solution z such that $Bz + Az' \in C([0, T_0), X)$.*

Moreover, (i)-(iii) are equivalent to

(iv) *B is the generator of a locally Lipschitz continuous local C-cosine function on X*

if A is a bounded linear operator from $[D(B)]$ into $R(C)$ so that $R(C^{-1}A) \subset D(B)$.

Since B is a bounded linear operator from $[D(A)]$ into $\mathrm{R}(C)$ so that $\mathrm{R}(C^{-1}B) \subset D(A)$ implies that

$$\mathrm{R}\left(\mathcal{C}^{-1}\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}\right) = \mathrm{R}\left(\begin{pmatrix} 0 & 0 \\ C^{-1}B & 0 \end{pmatrix}\right) \subset D\left(\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}\right) = D(B) \times D(A),$$

we can combine Theorem 2.11 with Theorem 2.13 to obtain the next new result concerning the generations of a local C-semigroup on X with subgenerator (resp., the generator) A and a local $\mathcal{C}$-semigroup on $X \times X$ with subgenerator (resp., the generator) $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ for which B may not be bounded.

Theorem 2.16. *Assume that B is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ or a bounded linear operator from $[D(A)]$ into $R(C)$ so that $R(C^{-1}B) \subset D(A)$. Then $\mathcal{T}$ is a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup on $X \times X$ if and only if A is a subgenerator (resp., the generator) of a local C-semigroup on X.*

Proof. Clearly, $\mathcal{C}\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\mathcal{C}$ on $X \times \mathrm{D}(A)$ (resp., $\mathcal{C}^{-1}\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\mathcal{C} = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$) is equivalent to $CA = AC$ on $\mathrm{D}(A)$ (resp., $C^{-1}AC = A$). Suppose that $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup on $X \times X$. Then $\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup $\mathcal{S}(\cdot)$ on $X \times X$. For each pair $x, y \in X$, we set

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = j_0 * \mathcal{S}(t)\begin{pmatrix} x \\ y \end{pmatrix}$$

for all $0 \le t < T_0$. Then

$$\begin{pmatrix} u \\ v \end{pmatrix} \in \mathrm{C}^1([0,T_0),X\times X)\cap \mathrm{C}([0,T_0),[\mathcal{T}]),\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix}=\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}+\begin{pmatrix} Cx \\ Cy \end{pmatrix}=\begin{pmatrix} v(t) \\ Av(t) \end{pmatrix}+\begin{pmatrix} Cx \\ Cy \end{pmatrix}$$

for all $0 \le t < T_0$, so that $u(0) = 0 = v(0)$, $u'(t) = v(t) + Cx$ and $v'(t) = Av(t) + Cy$ for all $0 \le t < T_0$. Hence, v is a solution of $\mathrm{ACP}(A, Cy, 0)$ in $\mathrm{C}^1([0,T_0),X)\cap \mathrm{C}([0,T_0),[\mathrm{D}(A)])$, $u(0) = 0$, and $u' = v$ on $[0,T_0)$. To show that A is a subgenerator (resp., the generator) of a local C-semigroup on X, we remain only to show that 0 is the unique solution of $\mathrm{ACP}(A,0,0)$ in $\mathrm{C}^1([0,T_0),X)\cap\mathrm{C}([0,T_0),[\mathrm{D}(A)])$ (see Theorem 2.2). To this end. Suppose that v is a solution of $\mathrm{ACP}(A,0,0)$ in $\mathrm{C}^1([0,T_0),X)\cap \mathrm{C}([0,T_0),[\mathrm{D}(A)])$. We set $u = j_0 * v$, then $u(0) = 0 = v(0)$ and

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix}=\begin{pmatrix} v(t) \\ Av(t) \end{pmatrix}=\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

for all $0 \le t < T_0$. The uniqueness of solutions of $\mathrm{ACP}(A,0,0)$ follows from the uniqueness of solutions of $\mathrm{ACP}\left(\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix},\begin{pmatrix} 0 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$. Conversely, suppose that A is a subgenerator (resp., the generator) of a local C-semigroup $S(\cdot)$ on X. To show that $\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup on $X\times X$, we need only to show that for each pair $x, y \in X$, $\mathrm{ACP}\left(\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix},\begin{pmatrix} Cx \\ Cy \end{pmatrix},\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$ has a unique solution in $\mathrm{C}^1([0,T_0),X\times X)\cap \mathrm{C}\left([0,T_0),\left[\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\right]\right)$. To do this. For each pair $x, y \in X$, we set $v(t) = j_0 * S(t)y$ and $u(t) = j_0 * v(t) + tCx$ for all $0 \le t < T_0$. Then $u(0) = 0 = v(0)$, and $v'(t)=S(t)y=Av(t)+Cy$ and $u'(t) = v(t)+Cx$ for all $0 \le t < T_0$, so that

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix}=\begin{pmatrix} v(t)+Cx \\ Av(t)+Cy \end{pmatrix}=\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}+\begin{pmatrix} Cx \\ Cy \end{pmatrix}$$

for all $0 \le t < T_0$. Hence, $\begin{pmatrix} u \\ v \end{pmatrix}$ is a solution of $\mathrm{ACP}\left(\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix},\begin{pmatrix} Cx \\ Cy \end{pmatrix},\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$ in $\mathrm{C}^1([0,T_0),X\times X)\cap \mathrm{C}\left([0,T_0),\left[\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\right]\right)$. The uniqueness of solutions of $\mathrm{ACP}\left(\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix},\begin{pmatrix} 0 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$ in $\mathrm{C}^1([0,T_0),X\times X)\cap \mathrm{C}\left([0,T_0),\left[\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\right]\right)$ follows from the uniqueness of solutions of $\mathrm{ACP}(A,0,0)$. Consequently, $\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup on $X\times X$, which implies that $\mathcal{T}$ is a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup on $X \times X$. □

Corollary 2.17. *Assume that $\rho(A,B)$ is nonempty and $B \in L(X)$. Then the following are equivalent:*

(i) *For each $(x,y) \in D(B) \times D(A)$ $ACP(A,B,CBx,0,Cy)$ has a unique solution w in $C^1([0,T_0),[D(A)])$;*

(ii) *$\mathcal{T}$ is the generator of a local $\mathcal{C}$-semigroup on $X \times X$;*

(iii) *For each $(x,y) \in D(B) \times D(A)$ $ACP(A,B,0,Cx,Cy)$ has a unique solution z in $C^1([0,T_0),[D(A)])$.*

Moreover, (i)-(iii) are equivalent to

(vi) *A is the generator of a local C-semigroup on X,*

if $R(B) \subset R(C)$.

Corollary 2.18. *Assume that $D(B) \cap D(A)$ is dense in X, $\rho(A,B)$ nonempty, and $AB = BA$ on $D(B) \cap D(A)$. Then the following are equivalent:*

(i) *For each $(x,y) \in D(B) \times D(A)$ $ACP(A,B,CBx,0,Cy)$ has a unique solution w such that $Bw + Aw' \in C([0,T_0),X)$;*

(ii) *$\mathcal{T}$ is the generator of a local $\mathcal{C}$-semigroup on $X \times X$;*

(iii) *For each $(x,y) \in D(B) \times D(A)$ $ACP(A,B,0,Cx,Cy)$ has a unique solution z such that $Bz + Az' \in C([0,T_0),X)$.*

Moreover, (i)-(iii) are equivalent to

(iv) *A is the generator of a local C-semigroup on X,*

if B is a bounded linear operator from $[D(A)]$ into $R(C)$ so that $R(C^{-1}B) \subset D(A)$.

We end this paper with a simple illustrative example. Let $S(\cdot)(= \{S(t)|0 \le t < 1\})$ be a family of bounded linear operators on c_0(, family of all convergent sequences in $\mathbb{F}$ with limit 0,) defined by $S(t)x = \{e^{-n}e^{nt}x_n\}_{n=1}^{\infty}$, then $S(\cdot)$ is a local C-semigroup on c_0 with generator A defined by $Ax = \{nx_n\}_{n=1}^{\infty}$ for all $x = \{x_n\}_{n=1}^{\infty} \in c_0$ with $\{nx_n\}_{n=1}^{\infty} \in c_0$. Here $C = S(0)$. Let $\{p_n\}_{n=1}^{\infty} \in l^{\infty}$ with $\{e^n p_n\}_{n=1}^{\infty} \in l^{\infty}$, and B be a bounded linear operator from $[D(A)]$ into $R(C)$ defined by $Bx = \{nx_n p_n\}_{n=1}^{\infty}$ for all $x = \{x_n\}_{n=1}^{\infty} \in D(A)$, then $R(C^{-1}B) \subseteq D(A)$, $CB = BC$ on $D(A)$, and $B : D(A) \subset c_0 \to c_0$ can be extended to a bounded linear operator on $\overline{D(A)} = c_0$. Applying Corollary 2.17, we get that $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is the generator of a local $\mathcal{C}$-semigroup on $c_0 \times c_0$.

Acknowledgements. The author is grateful to the referee for his or her corrections and valuable suggestions. This research was partially supported by the National Science Council of Taiwan.

References

[1] Arendt, W., Batty, C.J.K., Hieber, H., Neubrander, F., *Vector-Valued Laplace Transforms and Cauchy Problems*, 96, Birkhauser Verlag, Basel-Boston-Berlin, 2001.

[2] DeLaubenfuls, R., *Integrated Semigroups, C-Semigroups and the Abstract Cauchy Problem*, Semigroup Forum, **41**(1990), 83-95.

[3] Fattorini, H.O., *The Cauchy Problem*, Addison-Wesley, Reading, Mass., 1983.

[4] Gao, M.C., *Local C-Semigroups and C-Cosine Functions*, Acta Math. Sci., **19**(1999), 201-213.

[5] Goldstein, J.A., *Semigroups of Linear Operators and Applications*, Oxford, 1985.

[6] Kostic, M., *Generalized Semigroups and Cosine Functions*, Mathematical Institute Belgrade, 2011.

[7] Kostic, M., *Abstract Volterra Integro-Differential Equations*, Taylor and Francis Group, 2015.

[8] Kostic, M., *Convoluted C-Cosine Functions and C-Semigroups. Relations with Ultradistribution and Hyperfunction Sines*, J. Math. Anal. Appl., **338**(2008), 1224-1242.

[9] Kuo, C.-C., Shaw, S.-Y., *C-Cosine Functions and the Abstract Cauchy Problem I, II*, J. Math. Anal. Appl., **210**(1997), 632-646, 647-666.

[10] Kuo, C.-C., *Additive Perturbations of Local C-Semigroups*, Acta Math. Sci., **35B**(2015), no. 6, 1566-1576.

[11] Kuo, C.-C., *Multiplicative Perturbations of Local C-Semigroups*, Proc. Mathematical Sci., **125**(2015), no. 1, 45-55.

[12] Kuo, C.-C., *Perturbation Theorems for Local Integrated Semigroups*, Studia Math., **197**(2010), 13-26.

[13] Kuo, C.-C., *Local K-Convoluted C-Semigroups and Abstract Cauchy Problems*, Taiwanese J. Math., **19**(2015), no. 4, 1227-1245.

[14] Kuo, C.-C., Shaw, S.-Y., *On α-Times Integrated C-Semigroups and the Abstract Cauchy Problem*, Studia Math., **142**(2000), 201-217.

[15] Liang, J., Xiao, T.-J., Li, F., *Multiplicative Perturbations of Local C-Regularized Semigroups*, Semigroup Forum, **72**(2006), no. 3, 375-386.

[16] Li, Y.-C., Shaw, S.-Y., *Perturbation of Nonexponentially-Bounded α-Times Integrated C-Semigroups*, J. Math. Soc. Japan, **55**(2003), 1115-1136.

[17] Mel'nikova, I.-V., *The Method of Integrated Semigroups for the Cauchy Problems in Banach spaces*, Siberian Math. J., **40**(1999), 100-109.

[18] Mel'nikova, I.-V., Filinkov, A.I., *Abstract Cauchy Problems: Three Approaches*, Chapman Hill/CRC, 2001.

[19] Neubrander, F., *Integrated Semigroups and their Applications to the Abstract Cauchy Problem*, Pacific J. Math., **135**(1988), 111-155.

[20] Neubrander, F., *Integrated Semigroups and their Applications to Complete Second Order Cauchy Problems*, Semigroup Forum, **38**(1989), 233-251.

[21] Nicasie, S., *The Hille-Yosida and Trotter-Kato Theorems for Integrated Semigroups*, J. Math. Anal. Appl., **180**(1993), 303-316.

[22] Shaw, S.-Y., Kuo, C.-C., Li, Y.-C., *Perturbation of Local C-Semigroups*, Nonlinear Analysis, **63**(2005), 2569-2574.

[23] Shaw, S.-Y., Kuo, C.-C., *Generation of Local C-Semigroups and Solvability of the Abstract Cauchy Problems*, Taiwanese J. Math., **9**(2005), 291-311.

[24] Shaw, S.-Y., Li, Y.-C., *Characterization and generator of local C-Cosine and C-Sine Functions*, Inter. J. Evolution Equations, **1**(2005), no. 4, 373-401.

[25] Sova, M., *Linear Differential Equations in Banach spaces*, Rozprovy Ceskoslovenske Acad. Vd. Rada. Mat. Prirod. Ved., **85**(1975), 1-150.

[26] Tanaka, N., Miyadera, I., *C-semigroups and the Abstract Cauchy Problem*, J. Math. Anal. Appl., **170**(1992), 196-206.

[27] Tanaka, N., Okazawa, N., *Local C-Semigroups and Local Integrated Semigroups*, Proc. London Math. Soc., **61**(1990), no. 3, 63-90.

[28] Takenaka, T., Piskarev, S., *Local C-Cosine Families and N-Times Integrated Local Cosine Families*, Taiwanese J. Math., **8**(2004), 515-546.

[29] Wang, S.-W., Gao, M.-C., *Automatic Extensions of Local Regularized Semigroups and Local Regularized Cosine Functions*, Proc. London Math. Soc., **127**(1999), 1651-1663.

[30] Wang, S.-W., Wang, M.-Y., Shen, Y., *Perturbation Theorems for Local Integrated Semigroups and Their Applications*, Studia Math., **170**(2005), 121-146.

[31] Xiao, T.-J., Liang, J., *Multiplicative Perturbations of C-Regularized Semigroups*, Comput. Math. Appl., **41**(2001), 1215-1221.

[32] Xiao, T.-J., Liang, J., *The Cauchy Problem for Higher-Order Abstract Differential Equations*, Lectures Notes in Math., 1701, Springer, 1998.

[33] Xiao, T.-J., Liang, J., *On Complete Second Order Linear Differential Equations in Banach spaces*, Pacific J. Math., **142**(1990), 175-195.

[34] Xiao, T.-J., Liang, J., *Differential Operators and C-Wellposedness of Complete Second Order Abstract Cauchy Problems*, Pacific J. Math., **186**(1998), 167-200.

[35] Xiao, T.-J., Liang, J., *Approximations of Laplace Transforms and Integrated Semigroups*, J. Funct. Anal., **172**(2000), 202-220.

Chung-Cheng Kuo
Department of Mathematics
Fu Jen University
New Taipei City, Taiwan 24205
e-mail: `cckuo@math.fju.edu.tw`

Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 2, 239–250

Weingarten tube-like surfaces in Euclidean 3-space

Adel H. Sorour

Abstract. In this paper, we study a special kind of tube surfaces, so-called tube-like surface in 3-dimensional Euclidean space $\mathbf{E}^3$. It is generated by sweeping a space curve along another central space curve. This study investigates a tube-like surface satisfying some equations in terms of the Gaussian curvature, the mean curvature, the second Gaussian curvature and the second mean curvature. Furthermore, some important theorems are obtained. Finally, an example of tube-like surface is used to demonstrate our theoretical results and graphed.

Mathematics Subject Classification (2010): 53A04, 56B34, 53B25.

Keywords: Tube-like surface, Gaussian curvature, mean curvature, second Gaussian curvature, second mean curvature, Weingarten surfaces, linear Weingarten surfaces, Euclidean 3-space.

1. Introduction

Weingarten surfaces are surfaces whose Gaussian and mean curvatures satisfy a functional relationship (of class C^0 at least). The class of Weingarten surfaces contains already mentioned surfaces of constant curvatures K or H. Furthermore, a C^r-surface, $r > 3$, is Weingarten if and only if $K_s H_t - K_t H_s = 0$. On the other hand, let A and B be smooth functions on a surface $M(s,t)$ in Euclidean 3-space $\mathbf{E}^3$. The Jacobi function $\Phi(A,B)$ formed with A and B is defined by:

$$\Phi(A,B) = \det \begin{pmatrix} A_s & A_t \\ B_s & B_t \end{pmatrix},$$

where $A_s = \dfrac{\partial A}{\partial s}$ and $A_t = \dfrac{\partial A}{\partial t}$.

For the pair (A,B) of curvatures K, H and K_{II} of M in $\mathbf{E}^3$, if M satisfies $\Phi(A,B) = 0$ and $aA + bB = c$, then we call (A,B)-Weingarten surface (*W-surface*) and (A,B)-linear Weingarten surface (*LW-surface*), respectively, where $a, b, c \in \mathbb{R}$, $(a,b,c) \neq (0,0,0)$.

The classification of the Weingarten surfaces in Euclidean space is almost completely open today. These surfaces were introduced by J. Weingarten [21, 22] in the context of the problem of finding all surfaces isometric to a given surface of revolution. Applications of Weingarten surfaces on computer aided design and shape investigation can seen in [19].

The authors in [9, 25] have investigated ruled Weingarten surfaces and ruled linear Weingarten surfaces in $\mathbf{E}^3$. Besides, a classification of ruled Weingarten surfaces and ruled linear Weingarten surfaces in a Minkowski 3-space $\mathbf{E}^3_1$ is given in [4, 7, 16]. Munteanu and Nistor [13] studied polynomial translation linear Weingarten surfaces in Euclidean 3-space. Also, Lopez [10, 11] studied cyclic linear Weingarten surface in Euclidean 3-space. In [12] Lopez classified all parabolic linear Weingarten surfaces in hyperbolic 3-space. Ro and Yoon [15] studied a tube of Weingarten types in Euclidean 3-space satisfying some equation in terms of the Gaussian curvature, mean curvature and second Gaussian curvature. Kim and Yoon [8] classified quadric surfaces in Euclidean 3-space in terms of the Gaussian curvature and the mean curvature. In addition to, Yoon and Jun [26] classified non-degenerate quadric surfaces in Euclidean 3-space in terms of the isometric immersion and the Gauss map. Furthermore in [1, 2], Weingarten timelike tube surfaces around spacelike and timelike curves were studied in Minkowski 3-space $\mathbf{E}^3_1$.

Several geometers [15, 1, 18] have studied tubes in Euclidean 3-space and Minkowski 3-space satisfying some equations in terms of the Gaussian curvature K, the mean curvature H and the second Gaussian curvature K_{II}. Following the Jacobi function and the linear equation with respect to the Gaussian curvature K, the mean curvature H and the second Gaussian curvature K_{II} an interesting geometric question is raised: Classify all surfaces in Euclidean 3-space satisfying the conditions

$$\Phi(A, B) = 0, \tag{1.1}$$

$$a\,A + b\,B = c, \tag{1.2}$$

where $A, B \in \{K, H, K_{II}\}$, $A \neq B$ and $(a, b, c) \neq (0, 0, 0)$.

In this paper, we investigate the tube-like surfaces in 3-dimensional Euclidean space satisfying the Jacobi condition and the linear equation with respect to their curvatures have been studied. Furthermore, we obtained some theorems.

2. Preliminaries

Let $\mathbf{E}^3$ be a Euclidean 3-space with the scalar product given by [5]

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbf{E}^3$. In particular, the norm of a vector $X \in \mathbf{E}^3$ is given by $\|X\| = \sqrt{\langle X, X \rangle}$. If $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ are arbitrary vectors in $\mathbf{E}^3$, the vector product of X and Y is given by

$$X \wedge Y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)\,. \tag{2.1}$$

Let $M : \Phi = \Phi(s,t)$ be a surface in Euclidean 3-space. The unit normal vector field of M can be defined by

$$N = \frac{\Phi_s \wedge \Phi_t}{\|\Phi_s \wedge \Phi_t\|}, \ \ \Phi_s = \frac{\partial \Phi}{\partial s}, \ \ \Phi_t = \frac{\partial \Phi}{\partial t}, \tag{2.2}$$

where $\wedge$ stands the vector product of $\mathbf{E}^3$. The first fundamental form I of the surface M is

$$I = Eds^2 + 2Fdsdt + Gdt^2, \tag{2.3}$$

with coefficients

$$E = \langle \Phi_s, \Phi_s \rangle, \quad F = \langle \Phi_s, \Phi_t \rangle, \quad G = \langle \Phi_t, \Phi_t \rangle. \tag{2.4}$$

The second fundamental form of the surface M is given by

$$II = eds^2 + 2fdsdt + gdt^2. \tag{2.5}$$

From which the components of the second fundamental form e, f and g are expressed as

$$e = \langle \Phi_{ss}, N \rangle, \quad f = \langle \Phi_{st}, N \rangle, \quad g = \langle \Phi_{tt}, N \rangle. \tag{2.6}$$

Under this parametrization of the surface M, the Gaussian curvature K and the mean curvature H have the classical expressions, respectively [14]

$$K = \frac{eg - f^2}{EG - F^2}, \tag{2.7}$$

$$H = \frac{Eg + Ge - 2Ff}{2\left(EG - F^2\right)}. \tag{2.8}$$

From Brioschi's formula in a Euclidean 3-space, we are able to compute K_{II} of a surface by replacing the components of the first fundamental form E, F and G by the components of the second fundamental form e, f and g respectively. Consequently, the second Gaussian curvature K_{II} of a surface is defined by [3]

$$\mathbf{K}_{II} = \frac{1}{\left(eg - f^2\right)^2} \left\{ \begin{vmatrix} -\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\ \frac{1}{2}e_t & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix} \right\}. \tag{2.9}$$

Having in mind the usual technique for computing the second mean curvature H_{II} by using the normal variation of the area functional for the surfaces in $\mathbf{E}^3$ one gets [20]

$$H_{II} = H + \frac{1}{4}\Delta_{II}\ln(K),$$

where H and K denote the mean, respectively Gaussian curvatures of surface and Δ_{II} is the Laplacian for functions computed with respect to the second fundamental form II as metric. The second mean curvature H_{II} can be equivalently expressed as

$$H_{II} = H + \frac{1}{2\sqrt{\det(II)}} \sum_{i,j} \frac{\partial}{\partial u^i}\left[\sqrt{\det(II)}h^{ij}\frac{\partial}{\partial u^j}(\ln\sqrt{K})\right], \tag{2.10}$$

where (h_{ij}) denotes the associated matrix with its inverse (h^{ij}), the indices i, j belong to $\{1, 2\}$ and the parameters u^1, u^2 are s, t respectively.

Now, we can write the following important definition [23]:

Definition 2.1. *(1): A regular surface is flat (developable) if and only if its Gaussian curvature vanishes identically.*

(2): A regular surface for which the mean curvature vanishes identically is called a minimal surface.

(3): A non-developable surface is called II-flat if the second Gaussian curvature vanishes identically.

(4): A non-developable surface is called II-minimal if the second mean curvature vanishes identically.

Remark 2.2. [24] It is well known that: a minimal surface has a vanishing second Gaussian curvature but that a surface with the vanishing second Gaussian curvature need not to be minimal.

3. Tube-like surface in $\mathbf{E}^3$

The aim of this section, we will obtain the tube-like surface from the tube surface. Since the tube surfaces are special kinds of the canal surfaces in Euclidean 3-space. If we find the canal surface with taking variable radius $r(s)$ as constant, then the tube surface can be found, since the canal surface is a general case of the tube surface.

A canal surface is the envelope of a moving sphere with varying radius, defined by the trajectory $C(s)$ (center curve) of its center and a radius function $r(s)$. If the center curve $C(s)$ is a helix and the radius function $r(s)$ is a constant, then the surface is called helical canal surface. If the radius function $r(s)$ is a constant, this time the canal surface is called a tube [6]. Canal surface around the center curve $C(s)$ is parametrized as

$$K(s,t) = C(s) - r(s)r'(s)e_1(s) \mp r(s)\sqrt{1 - r'^2(s)}\Big(\cos[t]e_2(s) + \sin[t]e_3(s)\Big),\ 0 \leqslant t \leqslant 2\pi,$$

where s is arclength parameter and $e_1(s), e_2(s), e_3(s)$ Frenet vectors of $C(s)$. If the radius function $r(s) = r$ is a constant, then, the canal surface is called a tube (pipe) surface and it parametrized as

$$\text{Tube}(s,t) = C(s) + r\Big(\cos[t]e_2(s) + \sin[t]e_3(s)\Big).$$

The aim of this work is to introduce a simple method for parametrization of tube-like surface in Euclidean 3-space. Given a space curve $\alpha(t) = \Big(x(t), y(t), z(t)\Big)$, at each point, there are three directions associated with it, the tangent, normal and binormal directions. The unit tangent vector is denoted by e_1, i.e., $e_1(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}$, the unit normal vector is denoted by e_2, i.e., $e_2(t) = \frac{e_1'(t)}{\|e_1'(t)\|}$, the unit binormal vector is denoted by e_3, i.e., $e_3(t) = e_1(t) \wedge e_2(t)$ (cross product). With $\alpha(t), e_1(t), e_2(t)$ and $e_3(t)$, a tube-like surface can be expressed as follows

$$M : \Phi(s,t) = \alpha(t) + r\Big(\cos[s]e_2(t) - \sin[s]e_3(t)\Big), \tag{3.1}$$

where r is a parameter corresponding to the radius of the rotation (In general r can be a function of t). For fixed t, when s runs from 0 to 2π, we have a circle around the point $\alpha(t)$ in the e_1, e_2 plane. As we change t, this circle moves along the space curve α, and we will generate a tube-like surface along α (a special kind of tube surfaces defined by (3.1)). The Frenet-Serret equations, express the reat of change of the moving orthonormal tried $\{e_1(t), e_2(t), e_3(t)\}$ along the curve α are given by [17]

$$\begin{bmatrix} e_1'(t) \\ e_2'(t) \\ e_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}, \tag{3.2}$$

where the prime denotes the differentiation with respect to t and we denote by κ, τ the curvature and the torsion of the curve α. We can know that e_1, e_2, e_3 are mutually orthogonal vector fields satisfying equations

$$\begin{aligned} &\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \langle e_3, e_3\rangle = 1, \\ &\langle e_1, e_2\rangle = \langle e_2, e_3\rangle = \langle e_3, e_1\rangle = 0, \\ &\det(e_1, e_2, e_3) = 1. \end{aligned}$$

Calculating the partial derivative of (3.1) with respect to s and t respectively, we get

$$\begin{aligned} \Phi_s &= -r\Big[\sin[s]e_2 + \cos[s]e_3\Big], \\ \Phi_t &= Qe_1 + r\tau\Big[\sin[s]e_2 + \cos[s]e_3\Big], \end{aligned} \tag{3.3}$$

where $Q = 1 - r\kappa\cos[s]$. From which, the components of the first fundamental form are

$$E = r^2, \quad F = -r^2\tau, \quad G = Q^2 + r^2\tau^2. \tag{3.4}$$

Using equations (2.1) and (2.2) the unit normal vector on Φ takes the form

$$N = -\cos[s]e_2 + \sin[s]e_3. \tag{3.5}$$

The second order partial differentials of M are found

$$\begin{aligned} \Phi_{ss} &= -r\Big[\cos[s]e_2 - \sin[s]e_3\Big], \\ \Phi_{st} &= r\Big[\kappa\sin[s]e_1 + \tau(\cos[s]e_2 - \sin[s]e_3)\Big], \\ \Phi_{tt} &= -r(\kappa\tau\sin[s] + \kappa'\cos[s])e_1 + (\kappa - r(\kappa^2 + \tau^2)\cos[s] \\ &\quad + r\tau'\sin[s])e_2 + r(\tau^2\sin[s] + \tau'\cos[s])e_3. \end{aligned}$$

From the equation (3.5) and the last equations, we find the second fundamental form coefficients as follows

$$e = r, \quad f = -r\tau, \quad g = -Q\kappa\cos[s] + r\tau^2, \tag{3.6}$$

Theorem 3.1. *M is a regular tube-like surface if and only if* $1 - r\kappa\cos[s] \neq 0$.

Proof. For a regular surface, $EG - F^2 \neq 0$. From (3.6), we get

$$EG - F^2 = r^2\Big(1 - r\kappa\cos[s]\Big)^2,$$

where $EG - F^2 \neq 0$ and $r > 0$, M is a regular tube-like surface if and only if

$$1 - r\kappa\cos[s] \neq 0.$$

Based on the above calculations, the Gaussian curvature K and the mean curvature H of (3.1) are given by

$$K = -\frac{\kappa \cos[s]}{rQ}, \tag{3.7}$$

$$H = \frac{1 - 2r\kappa \cos[s]}{2rQ}. \tag{3.8}$$

If the second fundamental form of Φ is non-degenerate, i.e., $eg - f^2 \neq 0$. In this case, we can define formally the second Gaussian K_{II} and second mean H_{II} curvatures on $\Phi(s,t)$ as follows

$$K_{II} = \frac{1}{4rQ^4 \cos^2[s]}\Big[1 + \cos^2[s] - 6r\kappa \cos^3[s] + 4r^2\kappa^2 \cos^4[s]\Big], \tag{3.9}$$

$$H_{II} = \frac{-1}{64rQ^3\kappa^3 \cos^2[s]}\Big[A_0 + \sum_{i=1}^{6} A_i \cos[is] + \sum_{j=1}^{3} B_j \sin[js]\Big],$$

where the coefficients A_i and B_j are

$$\begin{aligned}
A_0 &= -r\Big[\kappa^2[33\kappa^2 + 20\kappa^2(r^2\kappa^2 - \tau^2)] - 4(3\kappa'^2 - 2\kappa\kappa'')\Big],\\
A_1 &= 2\kappa\Big[\kappa^2[5 - 4r^2(3\tau^2 - 11\kappa^2)] - 6r^2(3\kappa'^2 - \kappa\kappa'')\Big],\\
A_2 &= -2r\Big[\kappa^2[3\kappa^2(8 + 5r^2\kappa^2) + 2\tau^2] - 2(3\kappa'^2 - 2\kappa\kappa'')\Big],\\
A_3 &= 2\kappa\Big[\kappa^2[3 + r^2(23\kappa^2 + 4\tau^2)] - 2r^2(3\kappa'^2 - \kappa\kappa'')\Big],\\
A_4 &= -3r\kappa^4\Big[5 + 4r^2\kappa^2\Big], \quad A_5 = 10r^2\kappa^5, \quad A_6 = -2r^3\kappa^6
\end{aligned}$$

and

$$B_1 = 4r^2\kappa^2\Big[4\kappa'\tau - \kappa\tau'\Big], \quad B_2 = -8r\kappa\Big[\kappa'\tau - \kappa\tau'\Big], \quad B_3 = 4r^2\kappa^2\Big[4\kappa'\tau - \kappa\tau'\Big].$$

Under the previous calculations, one can formulate the following theorems:

Theorem 3.2. *If the Gaussian curvature K is zero, then M is generated by a moving sphere with the radius $r = 1$.*

Proof. At $\kappa = 0$, from the equation (3.7) $\cos[s] = 0$, i.e., $s = \frac{\pi}{2}(2n+1)$, $n = 0, \pm1, \pm2, \pm3, ...$, and the unit normal vector on M takes the form

$$\begin{aligned}
N(s,t) &= -\cos[s]e_2(t) + \sin[s]e_3(t)\\
&= \pm e_3(t).
\end{aligned}$$

Again, when $\cos[s] = 0$, i.e., $s = \frac{\pi}{2}(2n+1)$, $n = 0, \pm1, \pm2, \pm3, ...$, implies that

$$\begin{aligned}
\Phi(s,t) - \alpha(t) &= r\Big(\cos[s]e_2(t) - \sin[s]e_3(t)\Big)\\
N(s,t) &= -\Big(\cos[s]e_2(t) - \sin[s]e_3(t)\Big)\\
\pm e_3(t) &= \mp r e_3(t).
\end{aligned}$$

From the last equation, we get $r = 1$.

Theorem 3.3. *The surface* (3.1) *is a developable surface if and only if it is an open part of a circular-like cylinder.*

Theorem 3.4. *There are no minimal tube-like surfaces in Euclidean 3-space* $\mathbf{E}^3$.

Theorem 3.5. *Let M be a tube-like surface with non-degenerate second fundamental form in the Euclidean 3-space $\mathbf{E}^3$, then M is not II-flat as well as not II-minimal.*

3.1. Weingarten tube-like surfaces

In the following, we study the tube-like surface Φ in $\mathbf{E}^3$ satisfying the Jacobi equation $\Phi(X,Y)=0$, $X\neq Y$, of the curvatures K, H and K_{II} of Φ and we formulate the main results in the next theorems.

Theorem 3.6. *Let M be a tube-like surface in $\mathbf{E}^3$ defined by (3.1). Then M is a (K,H)-Weingarten surface.*

Proof. Let M be a tube-like surface in $\mathbf{E}^3$. Differentiating K and H with respect to s and t respectively, then we obtain

$$K_s=\frac{\kappa\sin[s]}{rQ^2},\quad K_t=-\frac{\kappa'\cos[s]}{rQ^2}, \tag{3.10}$$

$$H_s=\frac{\kappa\sin[s]}{rQ^2},\quad H_t=-\frac{\kappa'\cos[s]}{rQ^2}. \tag{3.11}$$

By using (3.10) and (3.11), M satisfies identically the Jacobi equation

$$\Phi(K,H)=K_sH_t-K_tH_s=0.$$

Therefore M is a *Weingarten surface.*

Theorem 3.7. *Let M be a tube-like surface in the Euclidean 3-space $\mathbf{E}^3$ parametrized by (3.1) with non-degenerate second fundamental form. If M is a (K,K_{II})-Weingarten surface, then $\kappa'=0$. Then, the curvature of $\alpha(t)$ is a non-zero constant.*

Proof. Let M be a tube-like surface in $\mathbf{E}^3$ parametrized by (3.1). If we take derivative of K_{II} given by (3.9) with respect to s and t respectively, then we have

$$\begin{cases} (K_{II})_s=\frac{1}{2rQ^3\cos^3[s]}\Big[1-r\kappa(2\sin^2[s]+r\kappa\cos^3[s])\cos[s]\Big]\sin[s], \\ \\ (K_{II})_t=\frac{\kappa'}{2Q^3\cos[s]}\Big[1-2\cos^2[s]+r\kappa\cos^3[s]\Big]. \end{cases} \tag{3.12}$$

We consider a tube-like surface (3.1) in $\mathbf{E}^3$ satisfying the Jacobi equation

$$\Phi(K,K_{II})=K_s(K_{II})_t-K_t(K_{II})_s=0, \tag{3.13}$$

with respect to the Gaussian curvature K and the second Gaussian curvature K_{II}. Then, substituting from (3.10) and (3.12) into (3.13), we get

$$\kappa'\sin[s]=0.$$

Since this polynomial is equal to zero for every s, its coefficient must be zero. Therefore, we conclude that $\kappa'=0$.

Theorem 3.8. *Let M be a tube-like surface in the Euclidean 3-space $\mathbf{E}^3$ parametrized by (3.1) with non-degenerate second fundamental form. If M is a (H,K_{II})-Weingarten surface, then $\kappa'=0$. Then, the curvature of $\alpha(t)$ is a non-zero constant.*

Proof. We assume that a tube-like surface parametrized by (3.1) with non-degenerate second fundamental form in $\mathbf{E}^3$ is (H, K_{II})-Weingarten surface. Then, it satisfies the Jacobi equation

$$\Phi(H, K_{II}) = H_s(K_{II})_t - H_t(K_{II})_s = 0, \tag{3.14}$$

which implies

$$\kappa' \sin[s] = 0. \tag{3.15}$$

From (3.15),*one can get* $\kappa' = 0$. Thus, the curvature of $\alpha(t)$ is a non-zero constant.

4. Linear Weingarten tube-like surfaces

Now, to examine the linear Weingarten property of the tube-like surface Φ defined along the space curve $\alpha(t)$. Let us analyze the following theorems.

Theorem 4.1. *Suppose that a tube-like surface defined by (*3.1*) in* $\mathbf{E}^3$ *is a linear Weingarten surface satisfying* $aK+bH = c$*. Then* $\kappa = 0$*.* M *is an open part of a circular-like cylinder.*

Proof. Consider the parametrization (3.1) with K and H given by (3.7) and (3.8) respectively, we have

$$aK + bH = c,$$

implies

$$2\kappa\Big[a + br - cr^2\Big]\cos[s] - b + 2cr = 0. \tag{4.1}$$

Since $\cos[s]$ and 1 are linearly independent, we have

$$2\kappa\Big[a + br - cr^2\Big] = 0, \qquad b = 2cr,$$

which imply

$$\kappa(a + cr^2) = 0.$$

If $a + cr^2 \neq 0$, then $\kappa = 0$. Thus, M is an open part of a circular-like cylinder.

Theorem 4.2. *Let* $(A, B) \in \{(K, K_{II}), (H, K_{II})\}$*. Then, there are no* (A, B)*-linear Weingarten tube-like surfaces in Euclidean 3-space* $\mathbf{E}^3$*.*

Proof. Firstly, we suppose that a tube-like surface (3.1) with non-degenerate second fundamental form in $\mathbf{E}^3$ satisfies the equation

$$aK + bK_{II} = c. \tag{4.2}$$

By using (3.7) and (3.9), the equation (4.2) takes the form

$$\begin{aligned}\tfrac{1}{4rQ^2}\Big[&4r\kappa^2(a + br - cr^2)\cos^4[s] - 2\kappa(2a + 3br - 4cr^2)\cos^3[s]\\ &+(b - 4cr)\cos^2[s] + b\Big] = 0.\end{aligned} \tag{4.3}$$

Since the identity holds for every s, all the coefficients must be zero. Therefore, we obtain

$$\begin{cases} 4r\kappa^2(a+br-cr^2)=0, \\ 2\kappa(2a+3br-4cr^2)=0, \\ b-4cr=0, \\ b=0. \end{cases}$$

Thus, we get $b=0,\ c=0$ and $\kappa=0$. In this case, the second fundamental form of M is degenerate. Thus, this completes proof.

Secondly, let a tube-like surface (3.1) with non-degenerate second fundamental form in $\mathbf{E}^3$ satisfy the relation

$$aH+bK_{II}=c. \tag{4.4}$$

From equations. (3.8), (3.9) and (4.4), we get

$$\tfrac{1}{4rQ^2}\Big[4r^2\kappa^2(a+b-cr)\cos^4[s]-2r\kappa(3a+3br-4cr)\cos^3[s]$$

$$+(2a+b-4cr)\cos^2[s]+b\Big]=0.$$

From which, one can obtain $b=0,\ c=0$ and $\kappa=0$. Also, the second fundamental form of tube-like is degenerate. Then, there are no (H,K_{II})-linear Weingarten tube-like surfaces in $\mathbf{E}^3$.

5. Applications

Here, we consider an example to illustrate the main results that we have presented in our paper.

Example 5.1. Let us consider a surface

$$\Phi(s,t)=\alpha(t)+r\Big(\cos[s]e_2(t)-\sin[s]e_3(t)\Big), \tag{5.1}$$

where $\alpha(t)$ is

$$\alpha(t)=(\cos[t],\sin[t],0),$$

and the Frenet's frame is

$$e_1(t)=(-\sin[t],\cos[t],0),\ \ e_2(t)=-(\cos[t],\sin[t],0),\ \ e_3(t)=(0,0,1).$$

Thus, we obtained tube-like surface as follows

$$\Phi(s,t)=\Big((1-r\cos[s])\cos[t],(1-r\cos[s])\sin[t],-r\sin[s]\Big). \tag{5.2}$$

The components of the first and second fundamental forms of the surface (5.2) are given by, respectively

$$\begin{aligned} &E=r^2, \quad F=0, \quad G=(1-r\cos[s])^2, \\ &e=r, \quad f=0, \quad g=-(1-r\cos[s])\cos[s]. \end{aligned}$$

The unit normal vector of the surface (5.2) takes the form

$$N=-\cos[s]e_2(t)+\sin[s]e_3(t). \tag{5.3}$$

For this surface, the Gaussian curvature K and the mean curvature H are defined by, respectively

$$K = -\frac{\cos[s]}{r(1 - r\cos[s])}, \tag{5.4}$$

$$H = \frac{1 - 2r\cos[s]}{2r(1 - r\cos[s])}. \tag{5.5}$$

As $cos[s] = 0$, Eqs. (5.4) and (5.5) lead to

$$K = 0, \quad H = \frac{1}{2r},$$

i.e., the surface (5.2) is a developable and not minimal.

Since $eg - f^2 \neq 0$, then we can get the second Gaussian curvature K_{II} and the second mean curvature H_{II} on $\Phi(s,t)$ as follows

$$K_{II} = \frac{1 + \cos^2[s] - 6r\cos^3[s] + 4r^2\cos^4[s]}{4r(1 - r\cos[s])^2\cos^2[s]}, \tag{5.6}$$

$$H_{II} = \frac{-1 + 2r\cos[s] + 3\cos^2[s] - 12r\cos^3[s] + 8r^2\cos^4[s]}{8r(1 - r\cos[s])^2\cos^2[s]}. \tag{5.7}$$

From aforementioned data, one can deduce that the Weingarten and linear Weingarten on Φ corresponding to the induced metric form satisfies the above theorems.

One can see the graph of $\Phi(s,t)$ in Figure 1.

Under the previous, we consider the following remark:

Remark 5.1. (1): It easily seen that, the vector $e_3(t) = (0,0,1)$ is a constant vector, then the surface (5.1) is a circular-like cylinder surface.

(2): The tube-like surface defined by (5.2) is a torus.

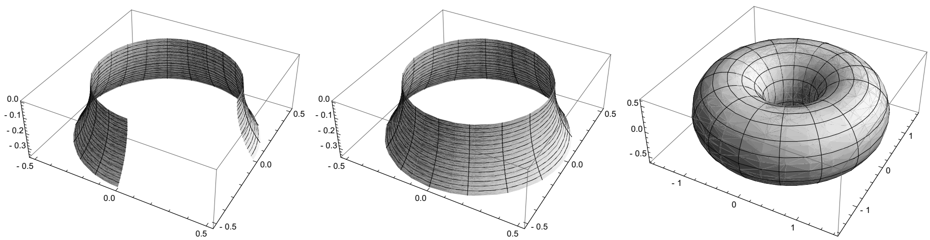


Figure 1. Some tube-like surfaces generated by circle with $r = \frac{1}{2}$,
Left (half circular-like cylinder): $s,t \in [0, \frac{6}{5}\pi]$,
Middle (circular-like cylinder): $s,t \in [0, \frac{3}{2}\pi]$ and
Right (torus): $s,t \in [0, 2\pi]$.

Acknowledgements. The author would like to thank the referees for the helpful suggestions.

References

[1] Abdel-Aziz, H.S., Khalifa Saad, M., *Weingarten timelike tube surfaces around a spacelike curve,* Int. J. of Math. Analysis, **25(5)**(2011), 1225-1236.

[2] Abdel-Aziz, H.S., Khalifa Saad, M., Kiziltug, S., *Parallel Surfaces of Weingarten Type in Minkowski 3-Space,* Int. Math. Forum, **4(7)**(2012), 2293-2302.

[3] Baikoussis, C., Koufogiorgos, T., *On the inner curvature of the second fundamental form of helicoidal surfaces*, Arch. Math., **68(2)**(1997), 169–176.

[4] Dillen, F., Kühnel, W., *Ruled Weingarten surfaces in a Minkowski 3-space*, Manuscripta Math. **98**(1999), 307–320.

[5] do Carmo, M., *Differential geometry of curves and surfaces,* Prentice-Hall, Englewood Cliffs, 1976.

[6] Doğan, F., *A note on tubes,* Int. J. of Phys. and Math. Sciences, **3(1)**(2013), 98-105.

[7] Kimm, H.Y., Yoon, D.W., *Classification of ruled surfaces in a Minkowski 3-space*, J. Geom. Phys., **49**(2004), 89–100.

[8] Kim, Y.H., Yoon, D.W., *Weingarten quadric surfaces in a Euclidean 3-space*, Turk. J. Math., **35**(2011), 479–485.

[9] Kühnel, W., *Ruled W-surfaces*, Arch. Math. **62**(1994), 475–480.

[10] Lopez, R., *On linear Weingarten surfaces*, International J. Math., **19**(2008), 439–448.

[11] Lopez, R., *Special Weingarten surfaces foliated by circles*, Monatsh. Math. **154**(2008), 289–302.

[12] Lopez, R., *Parabolic Weingarten surfaces in hyperbolic space*, Publ. Math. Debrecen, **74**(2009), 59–80.

[13] Munteanu, M.I., Nistor, A.I., *Polynomial translation Weingarten surfaces in 3-dimensional Euclidean space*, Differential geometry, 316-320, Worled Sci. Publ., Hackensack, NJ, 2009.

[14] O'Neill, B., *Elementary Differential Geometry*, Academic press, New York, 1966.

[15] Ro, J., Yoon, D.W., *Tubes of Weingarten types in a Euclidean 3-space*, J. Chungcheong Math. Soc., **22(3)**(2009), 359–366.

[16] Sodsiri, W., *Ruled surfaces of Weingarten type in Minkowski 3-space*, PhD. thesis, Katholieke Universiteit Leuven, Belgium, 2005.

[17] Struik, D.J., *Lectures on classical differential geometry*, Addison-Wesley Publishing Company, Inc., 1961.

[18] Tuncer, Y., Yoon, D.W., Karacan, M.K., *Weingarten and linear Weingarten type tubular surfaces in* $\mathbf{E}^3$, Mathematical Problems in Engineering, Article ID 191849, 2011.

[19] Van-Brunt, Grant, K., *Potential applications of Weingarten surfaces in CAGD, Part I: Weingarten surfaces and surface shape investigation*, Comput. Aided Geom. Des., **13**(1996), 569–582.

[20] Verpoort, S., *The Geometry of the second fundamental form: Curvature properties and variational aspects*, PhD. Thesis, Katholieke Universiteit Leuven, Belgium, 2008.

[21] Weingarten, J.,*Ueber eine Klasse auf einander abwickelbarer Flaachen*, J. Reine Angew. Math., **59**(1861), 382–393.

[22] Weingarten, J.,*Ueber die Flachen derer Normalen eine gegebene Flache beruhren*, J. Reine Angew. Math., **62**(1863), 61–63.

[23] Yilmaz, T., Murat, K.K., *Differential Geometry of three dimensions*, Syndic of Cambridge University Press, 1981.

[24] Yoon, D.W., *On non-developable ruled surfaces in Euclidean 3-spaces*, Indian J. Pure Appl. Math., **38**(2007), 281-289.

[25] Yoon, D.W., *Some properties of the helicoid as ruled surfaces*, JP J. Geom. Topology, **2(2)**(2002), 141–147.

[26] Yoon, D.W., Jun, J.S., *Non-degenerate quadric surfaces in Euclidean 3-space*, Int. J. Math. Anal., **6(52)**(2012), 2555–2562.

Adel H. Sorour
Department of Mathematics, Faculty of Science
Sohag University, 82524, Sohag, Egypt
e-mail: adel7374@yahoo.com

Stud. Univ. Babeş-Bolyai Math. 61(2016), No. 2, 251–253

Book reviews

Ronald S. Calinger, Leonhard Euler. Mathematical genius in the Enlightenment, Princeton, NJ: Princeton University Press, 2016, xvii+669 p,
(ISBN 978-0-691-11927-4/hbk; 978-0-691-11927-4/ebook).

As writes the author in the Preface, Leonhard Euler (1707-1783) ranks among the four greatest mathematicians of all time, the other three being Archimedes, Isaac Newton and Karl Friederic Gauss. Although there exist some previous books containing short biographies of Euler, this is the first detailed and comprehensive account on Euler's life, research, computations and professional interactions. The presentation was possible due to the almost completion of more than eighty large volumes of Euler's *Opera Omnia.*

The presentation focusses on the life of Euler and his achievements in calculus and analytical mechanics. As it is well known, Euler was an encyclopedic mind, his publications are written in five languages - most in Latin and French, and some in German, Russian, English. His interests covered a large area of human knowledge (including music, the theory of light and colors, letters to a German princess, construction of ships), so that an analysis of Euler's contributions would require experts from different areas with skills in several languages, working together under the direction of an editor to strengthen a coherent perspective. The author mentions in this direction Clifford Truesdell (the founder of the journal *Archive of History for Exact Sciences*), a master of six languages, including Greek and Latin, who edited five volumes of Opera Omnia and wrote a critical consideration of Euler's writings, especially on his contributions to theoretical physics.

The book present a synoptic study of the full scope of his research, the character of his colleagues and rivals, and the sources of problems, presented in a chronological order, starting with his Swiss years and formation (1707-1727), then his work in Sankt Petersburg Academy (1727-1641), Berlin Academy (1741-1760), and again in Russia, Sankt Petersburg (1760-1783), where he died. In spite of the fact that in the last years he lost his sight he continued to work, thanks to his prodigious memory. A special attention is paid to some rivalries and disputes - Euler, d'Alambert and Clairaut, Maupertuis and König. The polemics around Maupertuis' principle of minimal action and on other of his writings, in which were involved great personalities of the eighteen century, including Voltaire and King Frederick II of Prusia, is discussed at large in Sections 10 and 11 of the book.

Undoubtedly that the present monograph is an important contribution on Euler's life and on his achievements in various areas of human knowledge, being of interest to all people interested in the development of science in historical perspective.

S. Cobzaş

Aref Jeribi and Bilel Krichen; Nonlinear functional analysis in Banach spaces and Banach algebras. Fixed point theory under weak topology for nonlinear operators and block operator matrices with applications, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2016, xvi+355 p, ISBN: 978-1-4822-9909-0/hbk; 978-1-4822-9910-6/ebook.

The book is dedicated to fixed point theory in the weak topology setting and its applications to block operator matrices.

The book discusses various aspects of fixed point theory in Banach spaces and Banach algebras with nice applications to Mathematical Physics and Mathematical Biology. The structure of the book is the following: two main parts: I. *Fixed Point Theory*, II.*Applications to Mathematical Physics and Biology*, preceded by a *Preface* and followed by a consistent References list with 154 titles.

The main topics of the first part are:

I.1. *Fundamentals* (normed spaces, weak topology, measures of weak noncompactness (MNWC), basic tools in Banach algebras, elementary fixed point theorems, positivity and cones);

I.2. *Fixed Point Theory under Weak Topology* (fixed point theorems in DP (Dunford-Pettis) spaces and weak compactness, Banach spaces and weak compactness, fixed point theorems and MNWC, fixed point theorems for multi-valued mappings, some Leray-Schauder's alternatives);

I.3. *Fixed Point Theory in Banach Algebras* (fixed point theorems involving three operators, WC-Banach algebras, Leray-Schauder's alternatives in Banach algebras involving three operators, convex-power condensing operators, ws-compact and ω-convex-power condensing maps);

I.4. *Fixed Point Theory for BOM (Block Operator Matrix) on Banach Spaces and Banach Algebras* (some variants of Schauder's and Krasnoselskii's fixed point theorem for BOM, fixed point theory under weak topology features, fixed point theorems for BOM in Banach algebras, fixed point results in regular cones, BOM with multi-valued inputs);

The focus of the second part is on the applications of the above mentioned theory to:

II.5. *Existence of Solutions for Transport Equations*
II.6. *Existence of Solutions for Nonlinear Integral Equations*
II.7. *Two-Dimensional Boundary Value Problems.*

The book is interesting, clearly written and contains many important results (most of them obtained by the authors of this book) in the field of applied nonlinear analysis. Of course, the focus is on fixed point theory in Banach spaces and Banach algebras under the weak topology structure. The sources of the presented results are carefully mentioned and interesting open questions are pointed out for further investigations. The book will be an important reference tool for researchers working in fixed point theory and related topics, as well as, for those interested in applications of this theory in other areas, such as physics and biology.

Adrian Petruşel