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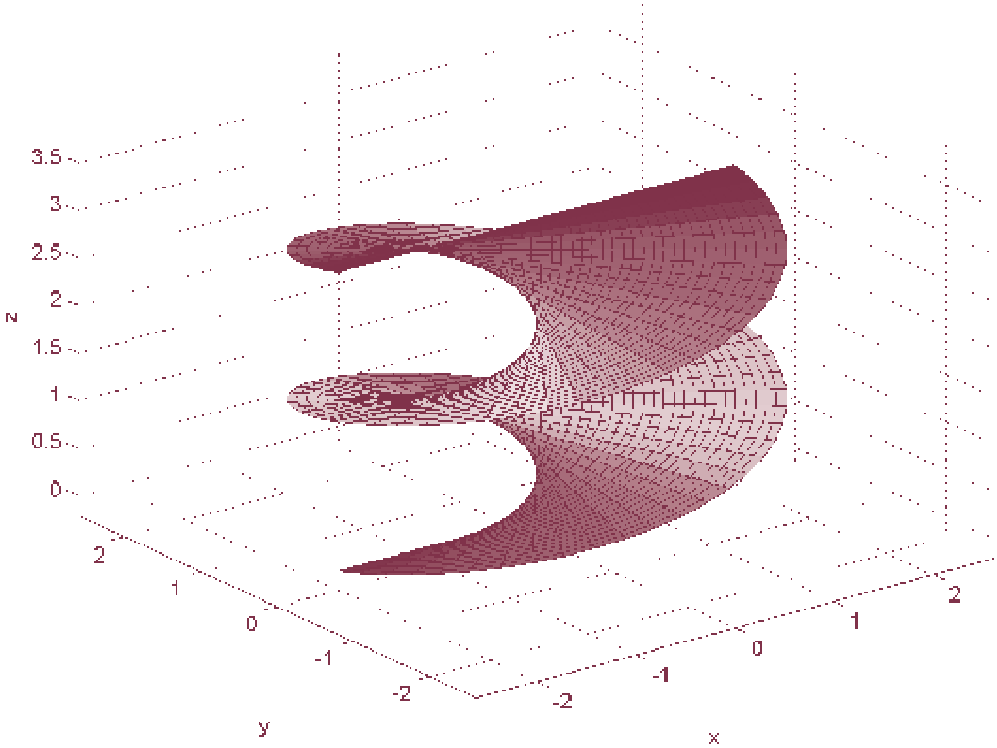

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CONTENTS

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Application of the multi-step homotopy analysis method to solve nonlinear differential algebraic equations

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Abstract. In this paper, a differential algebraic equations (DAE's) is studied and its approximate solution is presented using a multi-step homotopy analysis method (MHAM). The method is only a simple modification of the homotopy analysis method (HAM), in which it is treated as an algorithm in a sequence of small intervals (i.e. time step) for finding accurate approximate solutions to the corresponding systems. The solutions obtained are also presented graphically. Figurative comparisons between the MHAM and the exact solution reveal that this modified method is very effective and convenient.

Mathematics Subject Classification (2010): 11Y35, 65L05.

Keywords: Differential algebraic equations, multi-step homotopy analysis method, numerical solutions.

1. Introduction

Differential algebraic equations can be found in a wide variety of scientific and engineering applications, including circuit analysis, computer-aided design and real-time simulation of mechanical systems, power systems, chemical process simulation, and optimal control. Many important mathematical models can be expressed in terms of DAEs. In resent years, much research has been focused on the numerical solution of systems of DAEs. For the solutions of nonlinear differential equation, some numerical methods have been developed such as Pade approximation method [6, 7], homotopy perturbation method [11, 10], Adomain decomposition method [15, 5, 4] and variation iteration method [12]. Homotopy analysis method was first introduced by Liao [8], who solved linear and nonlinear problems. The new algorithm, MHAM presented in this paper,accelerates the convergence of the series solution over a large region and improve the accuracy of the HAM. The validity of the modified technique is verified through illustrative examples. The paper is organized as follows. In section 2, the proposed method is described. In section 3, the method is applied to our problem and numerical simulations are presented graphicallly. Finally, the conclusions are given in Section 4.

2. Multi-step homotopy analysis method algorithm

Although the MHAM is used to provide approximate solutions for nonlinear problem in terms of convergent series with easily computable components, it has been shown that the approximated solution obtained are not valid for large t for some systems [13, 9, 14, 1, 2]. Therefore we use the MHAM, which is offers accurate solution over a longer time frame compared to the HAM [16, 3, 17]. For this purpose, we consider the following initial value problem for systems of algebraic differential equations

$$u_i^{'}(t) = f_i(t, u_1, ..., u_n, u_1^{'}, ..., u_n^{'}), \quad t \geq 0, \quad i = 1, 2, ..., n-1,$$
$$0 = g(t, u_1, ..., u_n), \tag{2.1}$$

subject to the initial conditions

$$u_i(0) = c_i, \quad i = 1, 2, ..., n, \tag{2.2}$$

where ($f_i(t)$, $i = 1, 2, ..., n-1$) and g are known analytical functions. Let $[0, T]$ be the interval over which we want to find the solution of the initial value problem (2.1) and (2.2). Assume that the interval $[0, T]$ is divided into M subintervals $[t_{j-1}, t_j]$, $j = 1, 2, ..., M$ of equal step size $h = \frac{T}{M}$ by using the nodes $t_j = j\ h$. The main ideas of the MHAM are as follows: Apply the HAM to the initial value problem (2.1) and (2.2) over the interval $[t_0, t_1]$, we will obtain the approximate solution $u_{i,1}$, $t \in [t_0, t_1]$, using the intial condition ($u_i(t_0) = c_i$, $i = 1, 2, ..., n$). For $j \geq 2$ and at each subinterval $[t_{j-1}, t_j]$ we will use the initial condition $u_{i,j}(t_{j-1}) = u_{i,j-1}(t_{j-1})$ and apply the HAM to the initial value problem (2.1) and (2.2) over the interval $[t_{j-1}, t_j]$. The process is repeated and generates a sequence of approximate solutions $u_{i,j}(t)$, $i = 1, 2, ..., n$, $j = 1, 2, ..., M$. Now, we can construct the so-called zeroth-order deformation equations of the system (2.1) by

$$(1-q)L[\phi_{i,j}(t;q) - u_{i,j}(t^*)] = qh[\frac{d}{dt}\phi_{i,j}(t;q) - f_i(t, \phi_{1,j}(t;q), \ldots,$$
$$\phi_{n,j}(t;q), \frac{\partial}{\partial t}\phi_{1,j}(t;q), \ldots, \frac{\partial}{\partial t}\phi_{n,j}(t;q))], \quad i = 1, 2, \ldots, n-1, \tag{2.3}$$
$$(1-q)[\phi_{n,j}(t;q) - u_{n,j}(t^*)] = -qh\ g(t, \phi_{1,j}(t;q), \ldots, \phi_{n,j}(t;q)), \quad j = 1, 2, ..., M,$$

where t^* be the initial value for each subintervals $[t_{j-1}, t_j]$, $q \in [0, 1]$ is an embedding parameter, L is an auxiliary linear operator, $h \neq 0$ is an auxiliary parameter and $\phi_{i,j}(t;q)$ are unknown functions. Obviously, when $q = 0$ and $q = 1$, we have

$$\phi_{i,j}(t;0) = u_{i,j}(t^*), \phi_{i,j}(t;1) = u_{i,j}(t), \quad i = 1, 2, \ldots, n, \quad j = 1, 2, ..., M,$$

respectively. Expanding $\phi_{i,j}(t;q)$, $i = 1, 2, ..., n$, $j = 1, 2, ..., M$, in Taylor series with respect to q, we get

$$\phi_{i,j}(t;q) = u_{i,j}(t^*) + \sum_{m=1}^{\infty} u_{i,j,m}(t)q^m, i = 1, 2, \ldots, n, \quad j = 1, 2, ..., M, \tag{2.4}$$

where

$$u_{i,j,m}(t) = \frac{1}{m!}\frac{\partial^m \phi_{i,j}(t;q)}{\partial q^m}|_{q=0}.$$

If the initial guesses $u_{i,j}(t^*)$, the auxiliary linear operator L and the nonzero auxiliary parameter h are properly chosen so that the power series (2.4) converges at $q = 1$, one has

$$u_{i,j}(t) = \phi_{i,j}(t;1) = u_{i,j}(t^*) + \sum_{m=1}^{\infty} u_{i,j,m}(t),$$

For brevity, define the vector

$$\overrightarrow{u}_{i,j,m}(t) = \{u_{i,j,0}(t), u_{i,j,1}(t), \ldots, u_{i,j,m}(t)\},$$

Differentiating the zero-order deformation equation (2.3) m times with respective to q and then dividing by $m!$ and finally setting $q = 0$, we have the so-called high-order deformation equations

$$\begin{aligned} L[u_{i,j,m}(t) - \chi_m u_{i,j,m-1}(t)] &= h\, \Re_{i,j,m}(\overrightarrow{u}_{i,j,m-1}(t)), \\ u_{n,j,m}(t) &= \chi_m u_{n,j,m-1}(t) + h\, \Re_{n,j,m}(\overrightarrow{u}_{n,j,m-1}(t)) \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \Re_{i,j,m}(\overrightarrow{u}_{i,j,m-1}(t)) &= u_{i,j,m-1}(t) - \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}[f_i(t, \phi_{1,j}(t;q), \ldots, \phi_{n,j}(t;q), \\ &\quad \frac{\partial}{\partial t}\phi_{1,j}(t;q), \ldots, \frac{\partial}{\partial t}\phi_{n,j}(t;q))])]|_{q=0}, \\ \Re_{n,j,m}(\overrightarrow{u}_{n,j,m-1}(t)) &= \frac{-1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}[g(t, \phi_{1,j}(t;q), \ldots, \phi_{n,j}(t;q))]|_{q=0}, \\ i &= 1, 2, ..., n-1, \qquad j = 1, 2, \ldots, M, \end{aligned} \tag{2.6}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}.$$

Select the auxiliary linear operator $L = \frac{d}{dt}$, then the mth-order deformation equations (2.5) can be written in the form

$$\begin{aligned} u_{i,j,m}(t) &= \chi_m u_{i,j,m-1}(t) + h \int_{t_{j-1}}^{t} \Re_{i,j,m}(\overrightarrow{u}_{i,j,m-1}(\tau))d\tau, \\ u_{n,j,m}(t) &= \chi_m u_{n,j,m-1}(t) + h\, \Re_{n,j,m}(\overrightarrow{u}_{n,j,m-1}(t)), \\ i &= 1, 2, ..., n-1, \quad j = 1, 2, ..., M, \end{aligned} \tag{2.7}$$

and a power series solution has the form

$$u_{i,j}(t) = \sum_{m=0}^{\infty} u_{i,j,m}(t),\ i = 1, 2, \ldots, n, \quad j = 1, 2, ..., M. \tag{2.8}$$

Finally, the solutions of system (2.1) has the form

$$u_i(t) = \begin{cases} u_{i,1}(t), & t \in [t_0, t_1] \\ u_{i,2}(t), & t \in [t_1, t_2] \\ & \vdots \\ u_{i,M}(t), & t \in [t_{M-1}, t_M] \end{cases}, \qquad i = 1, 2, ..., n.$$

3. Numerical results

In order to assess both the accuracy and the convergence order of the MHAM presented in this paper for system of differential algebraic equations, we have applied it to the following three problems.

Example 3.1. Consider the following system of differential algebraic equations

$$\begin{aligned} u_1^{'}(t) &= tu_2^{'}(t) - u_1(t) + (1+t)u_2(t), \\ u_2(t) &= \sin t, \end{aligned} \tag{3.1}$$

subject to the initial condition

$$u_1(0) = 1, \;\; u_2(0) = 0. \tag{3.2}$$

The exact solutions of this system are $(u_1(t) = e^{-t} + t \sin t, \;\; u_2(t) = \sin t)$. In this example, we apply the proposed algorithm on the interval $[0, 50]$. We choose to divide the interval $[0, 50]$ to subintervals with time step $\triangle t = 0.1$. In general, we do not have these information at our clearance except at the initial point $t^* = t_0 = 0$, but we can obtain these values by assuming that the new initial condition is the solution in the previous interval. (i.e. If we need the solution in interval $[t_{j-1}, t_j]$, then the initial conditions of this interval will be as

$$\begin{aligned} u_{1,1}(t^*) &= 1, \;\; u_{1,j}(t^*) = u_{1,j-1}(t_{j-1}) = a_j, \\ u_{2,1}(t^*) &= 0, \;\; u_{2,j}(t^*) = u_{2,j-1}(t_{j-1}) = b_j, \qquad j = 2, 3, ..., M. \end{aligned} \tag{3.3}$$

Where t^* is the initial value for each subinterval $[t_{j-1}, t_j]$ and a_j, b_j are the initial conditions in the subinterval $[t_{j-1}, t_j]$, $j = 1, 2, ..., M$). In view of the algorithm presented in the previous section, we have the mth-order deformation equation (2.7), where

$$\begin{aligned} \Re_{1,j,m}(\overrightarrow{u}_{1,j,m-1}(t)) &= u_{1,j,m-1}^{'}(t) - tu_{2,j,m-1}' + u_{1,j,m-1}(t) - (1+t)u_{2,j,m-1}(t), \\ \Re_{2,j,m}(\overrightarrow{u}_{2,j,m-1}(t)) &= u_{2,j,m-1}(t) - \sin(t)(1 - \chi_m), \\ j &= 1, 2, ..., M, \quad m = 1, 2, 3, \ldots, \end{aligned} \tag{3.4}$$

and the series solution for system (3.1) is given by

$$
\begin{aligned}
u_{1,j,1}(t) &= h\ \left((a_j - b_j)(t-t^*) - \frac{b_j}{2}(t-t^*)^2\right),\\
u_{1,j,2}(t) &= h\ \left((a_j - b_j)(t-t^*) - \frac{b_j}{2}(t-t^*)^2\right) + h^2((a_j - 2b_j)(t-t^*)\\
&\quad + \frac{1}{2}(a_j-b_j)(t-t^*)^2 - \frac{b_j}{6}(t-t^*)^3\\
&\quad - \sin(t-t^*) - (t-t^*)(\cos(t-t^*) + \sin(t-t^*))),\\
&\ \ \vdots
\end{aligned}
\tag{3.5}
$$

Then the 10-term of the approximate solutions of system (3.1) are

$$u_{1,j}(t) = a_j + \sum_{m=1}^{9} u_{1,j,m}(t-t^*).$$

Fig. 1 shows the displacement of the MHAM when $h=-1$ and the exact solution of the system (3.1). It can be seen that the results from the MHAM match the results of the exact solution very well, therefore, the proposed method is very efficient and accurate method that can be used to provide analytical solutions for linear systems of differential algebraic equations.

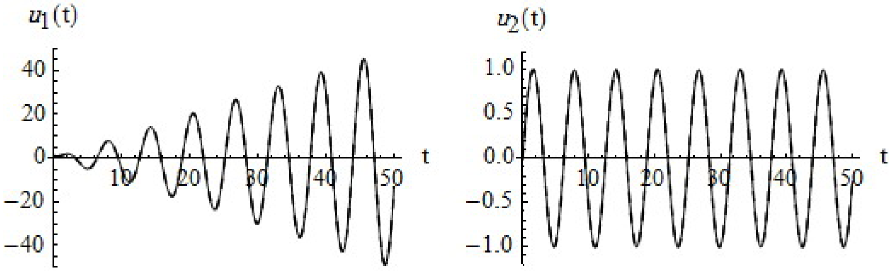

Figure 1. Plots of solution of system (3.1).
Solid line: Exact solution, dashed line: MHAM solution.

Example 3.2. Consider the following nonlinear system of differential algebraic equations

$$
\begin{aligned}
u_1'(t) &= u_1(t)u_2(t) + u_2'(t)(1-u_2'(t)) - t^2 + 2,\\
u_1^2(t) + u_2^2(t) + 2u_1(t)u_2(t) &= 4t^2,
\end{aligned}
\tag{3.6}
$$

subject to the initial condition

$$u_1(0) = 0,\ u_2(0) = 0. \tag{3.7}$$

The exact solutions of this system are $(u_1(t) = t + \sin t,\ \ u_2(t) = t - \sin t)$. Apply the proposed algorithm on the interval $[0, 50]$. We choose to divide the interval $[0, 50]$ to subintervals with time step $\triangle t = 0.1$. So we start with initial approximation

$$
\begin{aligned}
u_{1,1}(t^*) &= 0,\ u_{1,j}(t^*) = u_{1,j-1}(t_{j-1}) = a_j,\\
u_{2,1}(t^*) &= 0,\ u_{2,j}(t^*) = u_{2,j-1}(t_{j-1}) = b_j, \qquad j = 2,3,\dots,M.
\end{aligned}
\tag{3.8}
$$

In view of the algorithm presented in the previous section, we have the mth-order deformation equation (2.7), where

$$\begin{aligned}
\Re_{1,j,m}(\overrightarrow{u}_{1,j,m-1}(t)) &= u'_{1,j,m-1}(t) - u'_{2,j,m-1}(t) - \sum_{i=0}^{m-1} u_{1,j,i}(t)u_{2,j,m-i-1}(t) \\
&\quad + \sum_{i=0}^{m-1} u'_{2,j,i}(t)u'_{2,j,m-i-1}(t) + (t^2-2)(1-\chi_m), \\
\Re_{2,j,m}(\overrightarrow{u}_{2,j,m-1}(t)) &= \sum_{i=0}^{m-1} u_{1,j,i}(t)u_{1,j,m-i-1}(t) + \sum_{i=0}^{m-1} u_{2,j,i}(t)u_{2,j,m-i-1}(t) \\
&\quad -2\sum_{i=0}^{m-1} u_{1,j,i}(t)u_{2,j,m-i-1}(t) - 4t^2(1-\chi_m), \qquad (3.9)
\end{aligned}$$

and the series solution for system (3.6) is given by

$$\begin{aligned}
u_{1,j,1}(t) &= -h\ \left((a_jb_j+2)(t-t^*) - \frac{1}{3}(t-t^*)^3\right), \\
u_{2,j,1}(t) &= h\ \left((a_j+b_j)^2 - 4(t-t^*)^2\right), \\
u_{1,j,2}(t) &= -h\ \Big((1+h)(a_jb_j+2) + h(a_j^3+2a_j^2b_j+a_jb_j^2))(t-t^*) \\
&\quad -\frac{h}{2}(a_jb_j^2-2b_j-8)(t-t^*)^2 \\
&\quad -\frac{1}{3}(h(4a_j+1)+1)(t-t^*)^3 + \frac{h}{12}b_j(t-t^*)^4\Big), \qquad (3.10) \\
u_{2,j,2}(t) &= h\ \Big((1+2h(a_j+b_j))(a_j+b_j)^2 - 2h(a_jb_j+2)(a_j+b_j)(t-t^*) \\
&\quad -4(1+2h(a_j+b_j))(t-t^*)^2 + \frac{2h}{3}(a_j+b_j)(t-t^*)^3\Big), \\
&\ \ \vdots
\end{aligned}$$

So, the solution of system (3.6) will be as follows:

$$\begin{aligned}
u_{1,j}(t) &= a_j + \sum_{m=1}^{9} u_{1,j,m}(t-t^*), \\
u_{2,j}(t) &= b_j + \sum_{m=1}^{9} u_{2,j,m}(t-t^*).
\end{aligned}$$

Fig. 2 shows the displacement of the MHAM when $h=-1$ and the exact solution of the system (3.6). The results of our computations are in excellent agreement with the results obtained by the exact solution.

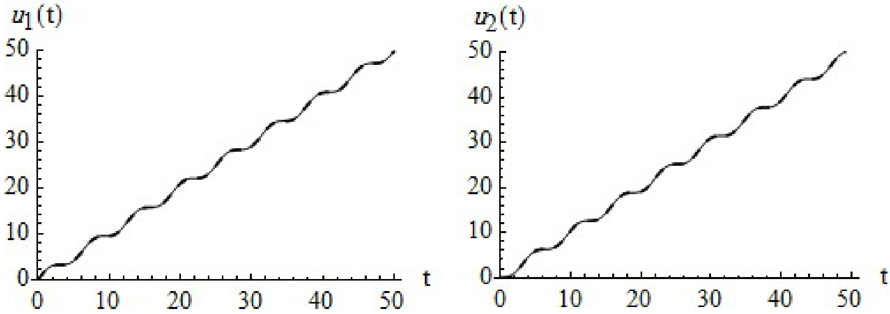

Figure 2. Plots of solution of system (3.6).
Solid line: Exact solution, dashed line: MHAM solution.

Example 3.3. Consider the following nonlinear system of differential algebraic equations

$$\begin{aligned} u_1'(t) &= u_2(t)u_3'(t) - u_2'(t)u_3(t) + u_3'(t) - u_1(t)) + 1 + \sin t, \\ u_2'(t) &= u_3(t)u_1'(t) + u_3(t)u_1(t) + \cos t, \\ & \quad u_1(t)u_2(t)u_3(t) - e^{-t}\sin t \ \cos t, \end{aligned} \tag{3.11}$$

subject to the initial condition

$$u_1(0) = 1,\ u_2(0) = 0,\ u_3(0) = 1 \tag{3.12}$$

The exact solutions of this system are $(u_1(t) = e^{-t},\ \ u_2(t) = \sin t,\ \ u_3(t) = \cos t)$. Apply the proposed algorithm on the interval $[0, 50]$. We choose to divide the interval $[0, 50]$ to subintervals with time step $\triangle t = 0.1$. To solve system (3.11) by means of MHAM, we start with initial approximations

$$\begin{aligned} u_{1,1}(t^*) &= 1,\ u_{1,j}(t^*) = u_{1,j-1}(t_{j-1}) = a_j, \\ u_{2,1}(t^*) &= 0,\ u_{2,j}(t^*) = u_{2,j-1}(t_{j-1}) = b_j, \\ u_{3,1}(t^*) &= 1,\ u_{3,j}(t^*) = u_{3,j-1}(t_{j-1}) = c_j, \quad j = 2, 3, ..., M. \end{aligned} \tag{3.13}$$

In view of the formula (2.7), we can construct the homotopy as follows

$$\begin{aligned} \Re_{1,j,m}(\overrightarrow{u}_{1,j,m-1}(t)) &= u_{1,j,m-1}'(t) - \sum_{i=0}^{m-1} u_{2,j,i}(t)u_{3,j,m-i-1}'(t) \\ &\quad + \sum_{i=0}^{m-1} u_{2,j,i}'(t)u_{3,j,m-i-1}(t) \\ &\quad + u_{1,j,m-1}(t) - u_{3,j,m-1}'(t) - (1 + \sin t)(1 - \chi_m), \end{aligned}$$

$$\begin{aligned}
\Re_{2,j,m}(\overrightarrow{u}_{2,j,m-1}(t)) &= u'_{2,j,m-1}(t) \\
&\quad - \sum_{i=0}^{m-1} u_{1,j,i}(t)u_{3,j,m-i-1}(t) - \sum_{i=0}^{m-1} u'_{1,j,i}(t)u_{3,j,m-i-1}(t) \\
&\quad - \cos t\ (1-\chi_m), \\
\Re_{3,j,m}(\overrightarrow{u}_{3,j,m-1}(t)) &= \sum_{i=0}^{m-1} u_{1,j,m-i-1}(t) \sum_{n=0}^{i} u_{2,j,n}(t)u_{3,j,i-n}(t) \\
&\quad -e^{-t}\sin t\ \cos t(1-\chi_m).
\end{aligned}$$

When $h = -1$, the MHAM solution for the system (3.11) in each subinterval $[t_{j-1}, t_j]$ has the form

$$\begin{aligned}
u_{1,j,1}(t) &= 1-(a_j-1)(t-t^*)-\cos(t-t^*), \\
u_{2,j,1}(t) &= a_jc_j(t-t^*)+\sin(t-t^*), \\
u_{3,j,1}(t) &= -a_jb_jc_j+\sin(t-t^*)\cos(t-t^*)\ e^{-(t-t^*)}, \\
u_{1,j,2}(t) &= -(1+a_jc_j^2)(t-t^*)-\frac{1}{2}(1-a_j)(t-t^*)^2+(1-c_j)\sin(t-t^*) \\
&\quad +\frac{1}{2}(1+b_j)\sin(2(t-t^*))\ e^{-(t-t^*)}, \\
u_{2,j,2}(t) &= c_j+\frac{a_j}{5}+c_j(2-a_j-a_j^2b_j)(t-t^*)+\frac{c_j}{2}(1-a_j)(t-t^*)^2 \\
&\quad -c_j(\sin(t-t^*)+\cos(t-t^*))-\frac{a_j}{10}(\sin(2(t-t^*)) \\
&\quad +2\cos(2(t-t^*)))\ e^{-(t-t^*)}, \\
u_{3,j,2}(t) &= b_jc_j(a_j^2b_j-a_j-1)-c_j(a_j^2c_j+a_jb_j-b_j)(t-t^*) \\
&\quad +c_j(b_j\cos(t-t^*)-a_j\sin(t-t^*)) \\
&\quad +(1-a_jb_j)\sin(t-t^*)\cos(t-t^*)\ e^{-(t-t^*)}. \\
&\ \ \vdots
\end{aligned} \tag{3.14}$$

So, the solution of system (3.11) will be as follows:

$$\begin{aligned}
u_{1,j}(t) &= a_j+\sum_{m=1}^{9} u_{1,j,m}(t-t^*), \\
u_{2,j}(t) &= b_j+\sum_{m=1}^{9} u_{2,j,m}(t-t^*), \\
u_{3,j}(t) &= c_j+\sum_{m=1}^{9} u_{3,j,m}(t-t^*).
\end{aligned}$$

Fig. 3 shows the displacement of the MHAM and the exact solution of the system (3.11). From the numerical results in all Figures it is clear that the numerical results obtained using MHAM is in excellent agreement with the exact solutions.

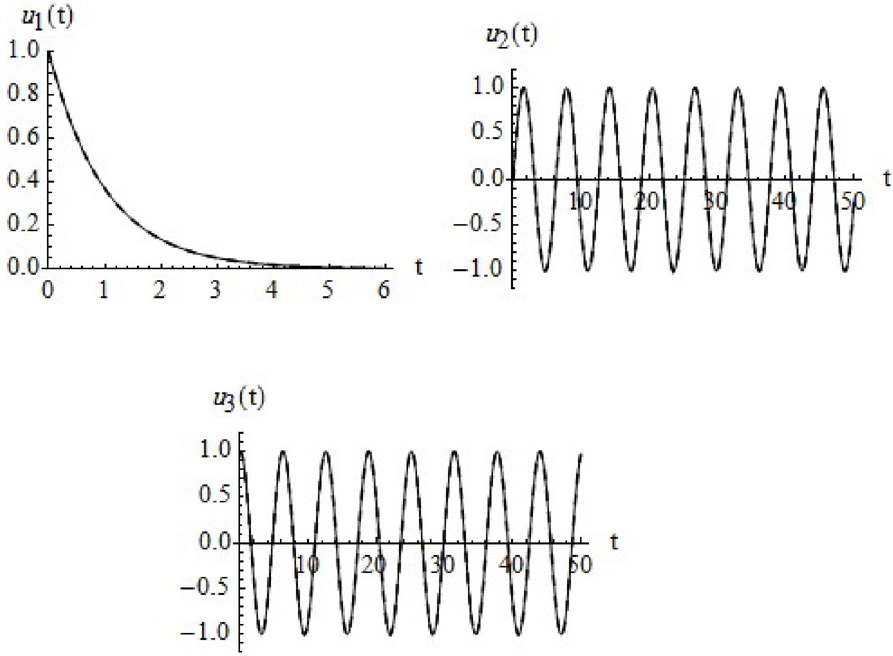

Figure 3. Plots of solution of system (3.11).
Solid line: Exact solution, dashed line: MHAM solution.

4. Conclusions

The purpose of this paper is to construct the multi-step homotopy analysis method to nonlinear systems of differential algebraic equations. The MHAM is that the solution expressed as an infinite series converges very fast to exact solutions. Results have been found very accurate when they are compared with analytical solutions. The approximate solutions obtained by MHAM are highly accurate and valid for a long time. In practice, the utilization of the method is straightforward if some symbolic software as Mathematica is used to implement the calculations. The proposed approach can be further implemented to solve other nonlinear problems.

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On products of self-small abelian groups

Josef Dvořák

Abstract. An abelian group A is called self-small if direct sums of copies of A commute with the covariant $\mathrm{Hom}\,(A,-)$ functor. The paper presents an elementary example of a non-self-small countable product of self-small abelian groups without non-zero homomorphisms between different ones. A criterion of self-smallness of a finite product of modules is given.

Mathematics Subject Classification (2010): 16D10, 16S50, 16D70.

Keywords: Self-small abelian group.

1. Introduction

The notion of self-small module as a generalization of the finitely generated module appears as a useful tool in the study of splitting properties [1], groups of homomorphisms of graded modules [10] or representable equivalences between subcategories of module categories [8].

The paper [4] in which the topic of self-small modules is introduced contains a mistake in the proof of [4, Corollary 1.3], which states when the product of (infinite) system $(A_i \mid i \in I)$ of self-small modules is self-small. A counterexample and correct version of the hypothesis were presented in [12] for a system of modules over a non-steady abelian regular ring. In the present paper an elementary counterexample in the category of $\mathbb{Z}$-modules, i.e. abelian groups, is constructed and as a consequence, an elementary example of two self-small abelian groups such that their product is not self-small is presented.

Throughout the paper a *module* means a right module over an associative ring with unit. If A and B are two modules over a ring R, $\mathrm{Hom}_R(A,B)$ denotes the abelian group of all R-homomorphisms $A \to B$. The set of all prime numbers is denoted by $\mathbb{P}$, for given $p \in \mathbb{P}$, $\mathbb{Z}_p$ means the cyclic group of order p and $\mathbb{Q}$ is the group of rational numbers. $E(A)$ denotes the injective envelope of the module A. Recall that injective $\mathbb{Z}$-modules, i.e. abelian groups, are precisely the divisible ones. For non-explained terminology we refer to [9].

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Definition 1.1. *An R-module A is self-small, if for arbitrary index set I and each $f \in \mathrm{Hom}_R(A, \bigoplus_{i\in I} A_i)$, where $A_i \cong A$, there exists a finite $I' \subseteq I$ such that $f(A) \subseteq \bigoplus_{i\in I'} A_i$.*

Properties of self-small modules and mainly of self-small groups are thoroughly investigated in [2], [3], [4], [5] and [6] revealing several characterizations of self-small groups and discussing the properties of the category of self-small groups and modules.

For our purpose the following notation will be of use:

Definition 1.2. *For an R-module A and $B \subseteq A$ we define the annihilator of B*

$$B^* := \{f \mid f \in End_R(A), f(a) = 0 \ for\ each\ a \in B\}\ .$$

The first (negative) characterization of self-small modules is given in [4] and it describes non-self-small modules via annihilators and chains of submodules:

Theorem 1.3. [4, Proposition 1.1] *For an R-module A the following conditions are equivalent:*

1. *A is not self-small*
2. *there exists a chain $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq \dots \subsetneq A$ of proper submodules in A such that $\bigcup_{n=1}^{\infty} A_n = A$ and for each $n \in \mathbb{N}$ we have $A_n^* \neq \{0\}$.*

2. Examples

The key tool for constructions of this paper is the following well-known lemma:

Lemma 2.1. $\prod_{p\in\mathbb{P}} \mathbb{Z}_p / \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p \cong \mathbb{Q}^{(2^\omega)}$.

Proof. Since $\bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$ is the torsion part of $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$, the group $\prod_{p\in\mathbb{P}} \mathbb{Z}_p / \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$ is torsion-free. Now the assertion follows from [7, Exercises S 2.5 and S 2.7]. □

Let $B = \prod_{p\in\mathbb{P}} \mathbb{Z}_p / \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$. By the previous lemma it is easy to see that there exists an infinite countable chain of subgroups $B_i \subseteq B_{i+1}$ of B such that $B = \bigcup_n B_n$ and $\mathrm{Hom}_{\mathbb{Z}}(B/B_n, \mathbb{Q}) \neq 0$ for each n.

Recall that $\mathbb{Q}$ is torsion-free of rank 1 and each nontrivial factor of $\mathbb{Q}$ is a torsion group, hence there is no nonzero non-injective endomorphism $\mathbb{Q}$, which by Theorem 1.3 implies well-known fact that $\mathbb{Q}$ is self-small.

Using the previous observations, the counterexample to [4, Corollary 1.3] can be constructed:

Example 2.2. Since $\mathbb{Z}_p$ is finite for every $p \in \mathbb{P}$, it is a self-small group. Now, all homomorphisms between $\mathbb{Z}_p$'s for different $p \in \mathbb{P}$, or $\mathbb{Q}$ are trivial:

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Q}) = \{0\}$, since $\mathbb{Z}_p$ is a torsion group, whereas $\mathbb{Q}$ is torsion-free. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_p) = \{0\}$, since every factor of $\mathbb{Q}$ is divisible and 0 is the only divisible subgroup of $\mathbb{Z}_p$. Obviously, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_q, \mathbb{Z}_p) = \{0\}$.

Let $A = \mathbb{Q} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p$ and $B = A/\left(\mathbb{Q} \times \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p\right) \cong \prod_{p\in\mathbb{P}} \mathbb{Z}_p / \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$. Then by Lemma 2.1 there exists a countable chain of subgroups $B_i \subseteq B_{i+1}$ of B, $i < \omega$, such that $B = \bigcup_{i<\omega} B_i$ and $\mathrm{Hom}_{\mathbb{Z}}(B/B_n, \mathbb{Q}) \neq 0$ for each n, where $\mathbb{Q}$ may be viewed as a subgroup of A. Now put A_n to be the preimage of B_n in A under factorization by $\mathbb{Q} \times \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$. Then the subgroups $A_n, n \in \mathbb{N}$ form a chain of subgroups and $A = \bigcup_{n\in\mathbb{N}} A_n$. At the same time the composition of the factorization by $\mathbb{Q} \times \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$ and $\bar{\nu_n}$ is an endomorphism φ_n of the group A such, that $A_n \subseteq Ker\,\varphi_n$. Therefore the condition of Theorem 1.3 is satisfied, hence the group A is not self-small.

The previous example shows that for two different primes p, q

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_q, \mathbb{Z}_p) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_p) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Q}) = \{0\},$$

all the groups $\mathbb{Z}_p$, $p \in \mathbb{P}$ and $\mathbb{Q}$ are self-small, but the group $\mathbb{Q} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is not self-small.

Finally, as a consequence of Example 2.2 an elementary example of two self-small abelian groups such that their product is not self-small may be constructed. It illustrates that the assumption $\mathrm{Hom}_{\mathbb{Z}}(M_j, M_i) = 0$ for each $i \neq j$ cannot be omitted even in the category of $\mathbb{Z}$-modules.

Example 2.3. By [12, Example 2.7] the group $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is self-small as well as the group $\mathbb{Q}$. Moreover, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \prod_{p\in\mathbb{P}} \mathbb{Z}_p) = \prod_{p\in\mathbb{P}} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_p) = 0$. Nevertheless, the product $\mathbb{Q} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is not self-small by Example 2.2. Note that it is not surprising in view of Corollary 2.6 that the structure of $\mathrm{Hom}_{\mathbb{Z}}(\prod_{p\in\mathbb{P}} \mathbb{Z}_p, \mathbb{Q})$ is quite rich as shown in Lemma 2.1.

Recall that classes of small modules, i.e. modules over which the covariant Hom-functor commutes with all direct sums, are closed under homomorphic images and extensions [11, Proposition 1.3]. Obviously, self-small modules do not satisfy this closure property and, moreover, although any class of self-small modules is closed under direct summands, the last example illustrates that it is not closed under finite direct sums.

Proposition 2.4. *The following conditions are equivalent for a finite system of self-small R-modules $(M_i|\ i \leq k)$:*

1. $\prod_{i\leq k} M_i$ *is not self-small*
2. *there exist $i, j \leq k$ and a chain $N_1 \subseteq N_2 \subseteq ... \subseteq N_n \subseteq ...$ of proper submodules of M_i such that $\bigcup_{n=1}^{\infty} N_n = M_i$ and $\mathrm{Hom}_R(M_i/N_n, M_j) \neq 0$ for each $n \in \mathbb{N}$.*

Proof. Put $M = \prod_{i\leq k} M_i$.

(1)→(2) If M is not self-small, there exists a chain $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq \dots$ of proper submodules of M for which $\bigcup_{n=1}^{\infty} A_n = M$ and $\mathrm{Hom}_R(M/A_n, M) \neq 0$ for each $n \in \mathbb{N}$. Put $A_n^i = M_i \cap A_n$ for each $i \leq k$ and $n \in \mathbb{N}$. Then $M_i = \bigcup_n A_n^i$ for each $i \leq k$ and there exists at least one index i such that the chain $A_1^i \subseteq A_2^i \subseteq \dots \subseteq A_n^i \subseteq \dots$ consists of proper submodules of M_i (or else the condition on the original chain is broken) and further on we consider only such i's.

Since for each $n \in \mathbb{N}$ there exist $0 \neq b_n \in M \setminus A_n$ and $f_n : M/A_n \to M$ such that $f_n(b_n + A_n) \neq 0$, for each n we can find an index $i(n) \leq k$ with $f_n \pi_{A_n} \nu_{i(n)} \pi_{i(n)}(b_n) \neq 0$ (where $\pi_{i(n)}$, resp. $\nu_{i(n)}$ are the natural projection, resp. injection and π_{A_n} is is the natural projection $M \to M/A_n$). Now, by pigeonhole principle, there must exist at least one index i_0 such that $S := \{n \in \mathbb{N} \,|\, i(n) = i_0\}$ is infinite. By the same principle, there must exist at least one index j_0 such that $T := \{n \in S \mid \pi_{j_0} f_n \pi_{A_n} \nu_{i_0} \pi_{i_0} (b_n) \neq 0\}$ is infinite. The couple i_0, j_0 proves the implication.

(2)→(1) Put $A_n = \pi_i^{-1}(N_n)$ where $\pi_i : M \to M_i$ is the natural projection, so $\bigcup_n A_n = M$. If $0 \neq f_n \in \mathrm{Hom}_R(M_i, M_j)$ such that $N_n \subseteq \ker f_n$ and $f_n(m_n) \neq 0$ for some suitable $m_n \in M_i$, then $\nu_j f_n \pi_i \in \mathrm{Hom}_R(M, M)$, where $\nu_j : M_j \to M$ is the natural injection, $A_n \subseteq \ker \nu_j f_n \pi_i$ and the nonzero element having m_n on the i-th position show that the condition of the Theorem 1.3 holds. □

Corollary 2.5. *Let $(M_i|\ i \leq k)$ be a finite system of R-modules. Then $\prod_{i\leq n} M_i$ is self-small if and only if for every i, j and every chain $N_1 \subseteq N_2 \subseteq \dots \subseteq N_n \subseteq \dots$ of proper submodules of M_i such that $\bigcup_{n=1}^{\infty} N_n = M_i$ there exist n for which $\mathrm{Hom}_R(M_i/N_n, M_j) = 0$.*

In consequence we see that the "finite version" of [4, Corollary 1.3] remains true:

Corollary 2.6. *Let $(M_i|\ i \leq n)$ be a finite system of self-small modules satisfying the condition $\mathrm{Hom}_R(M_j, M_i) = 0$ for each $i \neq j$. Then $\prod_{i\leq n} M_i$ is a self-small module.*

The previous results motivates the formulation of the following open problem.

Question. Let $0 \to S_0 \to S_1 \to S_2 \to 0$ be a short exact sequence in the category of abelian groups (or more generally right modules over a ring). If $i \neq j \neq k \neq i$ and S_i, S_j are self-small, can the condition that S_k is self-small be characterized by properties of the groups S_i, S_j and the corresponding homomorphisms?

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A-Whitehead groups

Ulrich Albrecht

Abstract. This paper investigates various extensions of the notion of Whitehead modules. An Abelian group G is an A-Whitehead group if there exists an exact sequence $0 \to U \to \oplus_I A \to G \to 0$ such that $S_A(U) = U$ with respect to which A is injective. We investigate the structure of A-Whitehead groups.

Mathematics Subject Classification (2010): 20K20, 20K40.

Keywords: Whitehead modules, endomorphism rings, adjoint functors.

1. Introduction

A right R-module M is a *Whitehead module* if $\mathrm{Ext}^1_R(M, R) = 0$. It is the goal of this paper to investigate Whitehead modules in the context of A-projective and A-solvable Abelian groups. The class of *A-projective groups*, which consists of all groups P which are isomorphic to a direct summand of $\oplus_I A$ for some index-set I, was introduced by Arnold, Lady and Murley ([6] and [7]). An A-projective group P has *finite A-rank* if I can be chosen to be finite. A-projective groups are usually investigated using the adjoint pair (H_A, T_A) of functors between the category Ab of Abelian groups and the category M_E of right E-modules defined by $H_A(G) = \mathrm{Hom}(A, G)$ and $T_A(M) = M \otimes_E A$ for all $G \in Ab$ and all $M \in M_E$. Here, $E = E(A)$ denotes the endomorphism ring of A. These functors induce natural maps $\theta_G : T_A H_A(G) \to G$ and $\phi_M : M \to H_A T_A(M)$ defined by $\theta_G(\alpha \otimes a) = \alpha(a)$ and $[\phi_M(x)](a) = x \otimes a$. An Abelian group G is *A-solvable* if θ_G is an isomorphism. If A is self-small, then all A-projective groups are A-solvable. Here, A is self-small if the natural map $H_A(\oplus_I A) \to \Pi_I E$ actually maps into $\oplus_I E$ for all index-sets I [7].

An Abelian group G is *(finitely, κ-) A-generated* if it is an epimorphic image of $\oplus_I A$ for some index-set I (with $|I| < \infty$, $|I| < \kappa$ respectively). It is easy to see that G is A-generated iff $S_A(G) = G$ where $S_A(G) = im(\theta_G)$. The group G is *A-presented* if there exists an exact sequence $0 \to U \to F \to G \to 0$ in which F is A-projective and U is A-generated. A sequence $0 \to G \to H \to L \to 0$ is *A-cobalanced (A-balanced)* if A is injective (projective) with respect to it. For a self-small group A, the A-solvable groups can be described as those groups G for which we can find an A-balanced exact sequence $0 \to U \to F \to G \to 0$ in which F is A-projective and U is A-generated [4].

The functor Ext^1_R can be defined either in terms of equivalence classes of exact sequences or via projective resolutions. We thus call an A-generated group W an *A-Whitehead splitter* if every exact sequence $0 \to A \to G \to W \to 0$ with $S_A(G) = G$ splits. On the other hand, a group W is an *A-Whitehead group* if it admits an A-cobalanced resolution $0 \to U \to F \to W \to 0$ in which F is A-projective and U is A-generated. Section 2 investigates how A-Whitehead groups and A-Whitehead splitters are related. While all A-presented A-Whitehead splitters are A-Whitehead groups, the converse surprisingly fails in general. Several examples demonstrate the differences between the classic concepts and our more general situation. We show that all A-Whitehead groups are A-Whitehead-splitters if E has injective dimension at most 1 as a right and left E-module. In particular, all countably A-generated A-Whitehead groups are A-projective if A has a right and left Noetherian, hereditary endomorphism ring. By [10], strongly κ-projective and Whitehead modules are closely related. The last results of this paper show that this relation extends to A-Whitehead groups.

2. A-Whitehead Groups

An Abelian group A is *(faithfully) flat* if it is flat (and faithful) as a left E-module. Since every exact sequence $0 \to U \to G \to A \to 0$ with $S_A(G) = G$ splits if A is faithfully flat [2], A is an A-Whitehead splitter in this case. However, this may not be true without the faithfulness condition as the next result shows.

Example 2.1. There exists a flat torsion-free Abelian group A of finite rank such that A is not an A-Whitehead splitter.

Proof. Let p, q, and r be distinct primes, and select subgroups A_1, A_2, and A_3 of $\mathbb{Q}$ such that A_1 is divisible by all primes except p and q, A_2 is divisible by all primes except p and r, and A_3 is divisible by all primes except q and r. By [8, Section 2], there exists a strongly indecomposable subgroup G of $\mathbb{Q} \oplus \mathbb{Q}$ which is generated by $A_1(1,0)$, $A_2(0,1)$, and $A_3(1,1)$. Moreover, $A_4 = G/A_1(1,0)$ is a subgroup of $\mathbb{Q}$ which is divisible by all primes except q. The group $A = \mathbb{Z} \oplus A_1 \oplus A_2 \oplus A_3 \oplus A_4$ is flat as a left E-module by Ulmer's Theorem [16]. Since $A_1 + A_3 = A_4$, A is not faithful. However, the exact sequence $0 \to A \to G \oplus A \oplus A_2 \oplus A_3 \oplus \mathbb{Z} \oplus \mathbb{Z} \to A \to 0$ cannot split since otherwise G would be completely decomposable. Because G is A-generated, A is not an A-Whitehead splitter. □

Proposition 2.2. *Let A be a self-small Abelian group. If W is an A-presented A-Whitehead splitter, then W is an A-Whitehead group.*

Proof. Consider an exact sequence $0 \to U \overset{\alpha}{\to} F \overset{\beta}{\to} W \to 0$, where F is A-projective and $U = S_A(U)$. For $\psi \in \mathrm{Hom}(U, A)$, we obtain the push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & W & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle 1_W} & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha_1} & X & \longrightarrow & W & \longrightarrow & 0. \end{array}$$

As a push-out, X is A-generated being an epimorphic image of $A \oplus F$. Since W is an A-Whitehead splitter, the bottom sequence splits, say $\delta\alpha_1 = 1_A$. Now it is easy to see that $\delta\psi_1\alpha = \psi$. □

However the converse of the last result fails in general:

Example 2.3. There exists a self-small faithfully flat Abelian group A for which we can find an A-Whitehead group G which is not an A-Whitehead splitter.

Proof. Let $\mathcal{P}$ be the set of primes, and consider the groups $A = \Pi_{\mathcal{P}}\mathbb{Z}_p$ and $U = \oplus_{\mathcal{P}}\mathbb{Z}_p$. Then, A is a self-small [18, Proposition 1.6], faithfully flat Abelian group, and U is an A-generated subgroup of A such that $A/U \cong \mathbb{Q}^{(2^{\aleph_0})}$. The sequence $0 \to U \to A \to A/U \to 0$ is A-cobalanced since each $\mathbb{Z}_p$ is fully invariant in A and U. Therefore, A/U is an A-Whitehead group and $S_A(X_p) = X_p$.

Fix a a prime p, and choose a group X_p with $E(X_p) = \mathbb{Z}_p$ and $X_p/\mathbb{Z}_p \cong \mathbb{Q}$. This is possible by Corner's Theorem [12]. Then, the induced sequence $0 \to \mathbb{Z}_p^{(2^{\aleph_0})} \to X_p^{(2^{\aleph_0})} \to \mathbb{Q}^{(2^{\aleph_0})} \to 0$ does not split although $A/U \cong \mathbb{Q}^{(2^{\aleph_0})}$ is an A-Whitehead group. □

Moreover, A-Whitehead splitters need not be A-presented. To see this, let p be a prime. If A is any torsion-free Abelian group with $pA = A$, then $\mathbb{Z}(p^\infty)$ is an epimorphic image of A. Moreover, $\mathrm{Ext}(\mathbb{Z}(p^\infty), A) = 0$ because $pA = A$ [12]. Therefore, $\mathbb{Z}(p^\infty)$ is an A-Whitehead splitter. However, no p-group can be A-presented since all A-generated groups are p-divisible.

If A is faithfully flat, then every exact sequence $0 \to U \to G \to H \to 0$ with G and H A-solvable is A-balanced and $S_A(U) = U$ [2]. If U is a submodule of $H_A(G)$, let $UA = \langle \phi(A) | \phi \in U \rangle$.

Lemma 2.4. *If A is a faithfully flat Abelian group, then the following hold for an A-solvable group G:*

a) *If U is a submodule of $H_A(G)$, then the evaluation map $\theta : T_A(U) \to UA$ defined by $\theta(u \otimes a) = u(a)$ is an isomorphism.*
b) *If U and V are submodules of $H_A(G)$ with $UA = VA$, then $U = V$.*

Proof. a) Clearly, θ is onto. To see that it is one-to-one, consider the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & T_A(U) & \longrightarrow & T_AH_A(G) \\ & & \Big\downarrow \theta & & \wr\Big\downarrow \theta_G \\ 0 & \longrightarrow & UA & \longrightarrow & G \end{array}$$

whose top-row is exact since A is flat.

b) Since $UA = VA = (U+V)A$, it suffices to consider the case $U \subseteq V$. By a), the evaluation maps $T_A(U) \to UA$ and $T_A(V) \to VA$ in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_A(U) & \longrightarrow & T_A(V) & \longrightarrow & T_A(V/U) & \longrightarrow & 0 \\ & & \wr\Big\downarrow & & \wr\Big\downarrow & & & & \\ 0 & \longrightarrow & UA & \xrightarrow{=} & VA & \longrightarrow & 0 & & \end{array}$$

are isomorphisms. Thus, $T_A(V/U) = 0$ which yields $V/U = 0$ since A is faithfully flat. □

Theorem 2.5. *Let A be a self-small faithfully flat Abelian group. The following are equivalent for an A-generated Abelian group W:*

a) *W is an A-Whitehead group.*
b) *There exists a Whitehead-module M with $W \cong T_A(M)$.*

Proof. *a)* $\Rightarrow$ *b)*: Consider an A-cobalanced exact sequence $0 \to U \overset{\alpha}{\to} F \overset{\beta}{\to} W \to 0$ in which U is A-generated and F is A-projective. It induces the sequence $0 \to H_A(U) \overset{H_A(\alpha)}{\longrightarrow} H_A(F) \overset{H_A(\beta)}{\longrightarrow} M \to 0$ where $M = Im(H_A(\beta))$ is a submodule of $H_A(W)$. We obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_AH_A(U) & \xrightarrow{T_AH_A(\alpha)} & T_AH_A(F) & \xrightarrow{T_AH_A(\beta)} & T_A(M) & \longrightarrow & 0 \\ & & \wr\downarrow\theta_U & & \wr\downarrow\theta_F & & \downarrow\theta & & \\ 0 & \longrightarrow & U & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & W & \longrightarrow & 0. \end{array}$$

By the 3-Lemma, the induced map θ is an isomorphism, and it remains to show that M is a Whitehead-module.

For $\psi \in \mathrm{Hom}_E(H_A(U), E)$, consider $T_A(\psi) : T_AH_A(U) \to T_A(E)$. Let $\sigma : T_A(E) \to A$ be an isomorphism. By a), there is $\lambda : F \to A$ with $\lambda\alpha = \sigma T_A(\psi)\theta_U^{-1}$. An application of H_A gives

$$\begin{aligned} H_A(\sigma^{-1}\lambda\theta_F)H_AT_AH_A(\alpha)) &= H_A(\sigma^{-1}\lambda\theta_F T_AH_A(\alpha)) \\ &= H_A(\sigma^{-1}\lambda\alpha)\theta_U = H_AT_A(\psi). \end{aligned}$$

Since $H_AT_A(\psi)\phi_{H_A(U)} = \phi_E\psi$, we have

$$\begin{aligned} \phi_E^{-1}H_A(\sigma^{-1}\lambda\theta_F)\phi_{H_A(F)}H_A(\alpha) &= \phi_E^{-1}H_A(\sigma^{-1}\lambda\theta_F)H_AT_AH_A(\alpha)\phi_{H_A(U)} \\ &= \phi_E^{-1}H_AT_A(\psi)\phi_{H_A(U)} = \psi, \end{aligned}$$

and M is a Whitehead-module.

b) $\Rightarrow$ *a)*: Consider an exact sequence $0 \to U \overset{\alpha}{\to} F \overset{\beta}{\to} M \to 0$ in which F is a free right E-module. Since A is faithfully flat, ϕ_U is an isomorphism by [4]. It remains to show that the induced sequence $0 \to T_A(U) \overset{T_A(\alpha)}{\longrightarrow} F \overset{T_A(\beta)}{\longrightarrow} T_A(M) \to 0$ is A-cobalanced. For this, consider a map $\psi \in \mathrm{Hom}(T_A(U), A)$. Because $\mathrm{Ext}_E^1(M, E) = 0$, there exists $\lambda : F \to E$ with $H_A(\psi)\phi_U = \lambda\alpha$. Then,

$$\begin{aligned} \theta_AT_A(\lambda)T_A(\alpha) &= \theta_AT_AH_A(\psi)T_A(\phi_U) \\ &= \psi\theta_{T_A(U)}T_A(\phi_U) = \psi \end{aligned}$$

since $\theta_{T_A(U)}T_A(\phi_U)(u \otimes a) = \theta_{T_A(U)}(\phi_U(u) \otimes a) = u \otimes a$ for all $u \in U$ and $a \in A$. □

Example 2.6. There exists a self-small faithfully flat Abelian group A and a A-Whitehead group W such that $W \cong T_A(M)$ for some right E-module M with $\mathrm{Ext}_R^1(M, E) \neq 0$.

Proof. Let A and U be as in Example 2.3, and consider the A-Whitehead-group $W = A/U$. In view of the proof of Theorem 2.5, it suffices to construct an exact sequence $0 \to V \to P \to W \to 0$ such that P is A-projective and V is A-generated which is not A-cobalanced.

Since A/U is a $\mathbb{Z}_p$-module, there are index-sets I and J and an exact sequence $0 \to \oplus_I \mathbb{Z}_p \to \oplus_J \mathbb{Z}_p \to A/U \to 0$. Because of $\mathrm{Ext}_{\mathbb{Z}_p}(\mathbb{Q}, \mathbb{Z}_p) \neq 0$, this sequence cannot be A-cobalanced. It is easy to see that it cannot be A-balanced either. □

If G and H are A-solvable, and A is a self-small faithfully flat Abelian group, then the equivalence classes of exact sequences $0 \to H \to X \to G \to 0$ with $S_A(X) = X$ form a subgroup of $Ext(G, H)$ denoted by $A - Bext(G, H)$ [3].

Theorem 2.7. *Let A be a self-small faithfully flat Abelian group. The following are equivalent for an A-generated group W:*

a) *W is an A-solvable A-Whitehead splitter.*
b) *W is an A-solvable A-Whitehead group.*
c) *W is A-solvable and $H_A(W)$ is a Whitehead module.*
d) *There exists an exact sequence $0 \to U \to \oplus_I F \to W \to 0$ with $S_A(U) = U$ which is A-balanced and A-cobalanced.*
e) *W is an A-solvable group with $A - Bext(W, A) = 0$.*

Proof. Since *a)* $\Rightarrow$ *b)* holds by Proposition 2.2, we consider an A-solvable A-Whitehead group W. As in the proof of Theorem 2.5, there exists a submodule M of $H_A(W)$ with $\mathrm{Ext}^1_E(M, E) = 0$ such that the evaluation map $\theta : T_A(M) \to W$ is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_A(M) & \longrightarrow & T_AH_A(A) & \longrightarrow & T_A(H_A(W)/M) & \longrightarrow & 0 \\ & & \wr\big\downarrow\theta & & \wr\big\downarrow\theta_W & & & & \\ & & W & \xrightarrow{1_W} & W & & & & \end{array}$$

which yields $T_A(H_A(W)/M) = 0$. Since A is faithfully flat, $H_A(W) = M$ is a Whitehead-module.

c) $\Rightarrow$ *d)*: Since W is A-solvable where exists an A-balanced sequence $0 \to U \to F \to W \to 0$ with $S_A(U) = U$ and F A-projective. By the Adjoint-Functor-Theorem, there exists an isomorphism $\lambda_G : \mathrm{Hom}(G, A) \to \mathrm{Hom}_E(H_A(G), E)$ for all A-solvable groups G. We therefore obtain the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_E(H_A(F), E) & \longrightarrow & \mathrm{Hom}_E(H_A(U), E) & \longrightarrow & \mathrm{Ext}^1_E(H_A(W), E) = 0 \\ \wr\big\uparrow\lambda_F & & \wr\big\uparrow\lambda_U & & \\ \mathrm{Hom}(F, A) & \longrightarrow & \mathrm{Hom}(U, A) & & \end{array}$$

whose top-row is exact since the original sequence is A-balanced.

d) $\Rightarrow$ *a)*: Since there exists an A-balanced sequence $0 \to U \to F \to W \to 0$ with $S_A(U) = U$ and F A-projective, we know that W is A-solvable. Using the maps λ_G

as before, we obtain the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_E(H_A(F),E) & \longrightarrow & \mathrm{Hom}_E(H_A(U),E) & \longrightarrow & \mathrm{Ext}^1_E(H_A(W),E) & \longrightarrow & 0 \\
\wr\uparrow\lambda_F & & \wr\uparrow\lambda_U & & & & \\
\mathrm{Hom}(F,A) & \longrightarrow & \mathrm{Hom}(U,A) & \longrightarrow & 0 & &
\end{array}$$

from which it follows that $H_A(W)$ is a Whitehead module. Since A is faithfully flat, an exact sequence $0 \to A \to G \to W \to 0$ with $S_A(G) = G$ is A-balanced. Therefore, it induces the exact sequence $0 \to H_A(A) \to H_A(G) \to H_A(W) \to 0$ which splits because $H_A(W)$ is a Whitehead module. We therefore obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_AH_A(A) & \longrightarrow & T_AH_A(G) & \longrightarrow & T_AH_A(W) & \longrightarrow & 0 \\
 & & \wr\downarrow\theta_A & & \downarrow\theta_G & & \wr\downarrow\theta_W & & \\
0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & W & \longrightarrow & 0
\end{array}$$

whose top-row splits. Since θ_G is an isomorphism by the 3-Lemma, the bottom row splits too.

Since $A{-}Bext(G,H) \cong Ext^1_E(H_A(G), H_A(H)$ whenever G and H are A-solvable [3], c) and e) are equivalent. □

3. Groups with Endomorphism Rings of Injective Dimension 1

We now discuss the Abelian groups A for which all A-Whitehead groups are A-Whitehead splitters. The nilradical of a ring R is denoted by $N = N(R)$. If A is a torsion-free Abelian group whose endomorphism ring has finite rank, then $N(E) = 0$ if and only if its quasi-endomorphism ring $\mathbb{Q}E$ is semi-simple Artinian. Moreover, $E(A)$ is right and left Noetherian in this case [8, Section 9]. An Abelian group G is *locally A-projective* if every finite subset of G is contained in an A-projective direct summand of G which has finite A-rank [7]. If $E(A)$ has finite rank, then H_A and T_A give a category equivalence between the categories of locally A-projective groups and locally projective right E-modules [7]. We want to remind the reader that *the A-radical of a group G* is $R_A(G) = \cap\{\mathrm{Ker}\ \phi \mid \phi \in \mathrm{Hom}(G,A)\}$. Clearly, $R_A(G) = 0$ if and only if G can be embedded into A^I for some index-set I.

Theorem 3.1. *The following are equivalent for a faithfully flat Abelian group A such that $\mathbb{Q}E$ is a finite-dimensional semi-simple $\mathbb{Q}$-algebra:*

a) $id(E_E) = 1$.
b) *A-generated subgroups of torsion-free A-Whitehead groups are A-Whitehead groups.*

For such an A, every A-Whitehead groups W satisfies $R_A(W) = 0$ and is A-solvable. In particular, W is an A-Whitehead splitter.

Proof. a) ⇒ b): If V is a submodule of a Whitehead module X, then we obtain an exact sequence $0 = \mathrm{Ext}^1_E(X,E) \to \mathrm{Ext}^1_E(V,E) \to \mathrm{Ext}^2_E(X/V,E) = 0$ because $id(E_E) \le 1$. Thus, V is a Whitehead module.

Let W be a torsion-free A-Whitehead group. To see $R_A(W) = 0$, observe that there is a Whitehead module M with $W \cong T_A(M)$ by Theorem 2.5. Since A is flat, the sequence $0 \to T_A(tM) \to T_A(M) \cong W$ is exact. Hence, $T_A(tM) = 0$, which yields $tM = 0$ because A is a faithful E-module. The submodule $U = \cap\{\text{Ker } \phi \mid \phi \in \text{Hom}_E(M, E)\}$ of M is a Whitehead module by the first paragraph.

We consider the exact sequence

$$\begin{aligned} 0 \to \text{Hom}_E(M/U, E) &\stackrel{\pi^*}{\longrightarrow} \text{Hom}_E(M, E) \to \text{Hom}_E(U, E) \\ &\to \text{Ext}^1_E(M/U, E) \to \text{Ext}^1_E(M, E) = 0. \end{aligned}$$

Since π^* is onto, $\text{Hom}_E(U, E) \cong \text{Ext}^1_E(M/U, E)$. Because U is pure in M as an Abelian group, multiplication by a non-zero integer n induces an exact sequence $\text{Ext}^1_E(M/U, E) \stackrel{n\times}{\longrightarrow} \text{Ext}^1_E(M/U, E) \to \text{Ext}^2_E(., E) = 0$, from which we obtain that $\text{Ext}^1_E(M/U, E) \cong \text{Hom}_E(U, E)$ is divisible. However, this is only possible if $\text{Hom}_E(U, E) = 0$ since $\text{Hom}_E(U, E)$ is reduced.

Let D be the injective hull of U. Since $\mathbb{Q}E$ is semi-simple Artinian, $D \cong \mathbb{Q} \otimes_{\mathbb{Z}} U$ by [15]. Hence, D/U is torsion as an Abelian group, and we can find an index-set I, non-zero integers $\{n_i \mid i \in I\}$, and an exact sequence $0 \to X \to \oplus_I E/n_iE \to D/U \to 0$. It induces

$$\begin{aligned} 0 = \text{Hom}_E(X, E) &\to \text{Ext}^1_E(D/U, E) \\ &\to \text{Ext}^1_E(\oplus_I E/n_iE, E) \cong \Pi_I \text{Ext}^1_E(E/n_iE, E). \end{aligned}$$

Therefore, $\text{Ext}^1_E(D/U, E)$ is reduced since the exact sequence $\text{Hom}_E(E, E) \stackrel{n_i\times}{\longrightarrow} \text{Hom}_E(E, E) \to \text{Ext}^1_E(E/n_iE, E) \to 0$ yields $\text{Ext}^1_E(E/n_iE, E) \cong E/n_iE$. On the other hand, we have the induced sequence $0 = \text{Hom}_E(U, E) \to \text{Ext}^1_E(D/U, E) \to \text{Ext}^1_E(D, E) \to \text{Ext}^1_E(U, E) = 0$ where the last Ext-group vanishes since U is a Whitehead module. Since D is torsion-free and divisible, the same holds for $\text{Ext}^1_E(D/U, E)$. However, this is only possible if $\text{Ext}^1_E(D/U, E) \cong \text{Ext}^1_E(D, E) = 0$.

If $D \neq 0$, then it has a direct summand S which is simple as a $\mathbb{Q}E$-module since $\mathbb{Q}E$ is semi-simple Artinian. In particular, $\text{Ext}^1_E(S, E) = 0$. Using Corner's Theorem [12], we can find a reduced Abelian group B with $End(B) \cong E^{op}$ which fits into an exact sequence $0 \to E^{op} \to B \to \mathbb{Q}E^{op} \to 0$ as a left E^{op}-module. Then, B can be viewed as a right E-module fitting into an exact sequence $0 \to E \to B \to \mathbb{Q}E \to 0$. We can find an E-submodule $E \subseteq V$ of B with $V/E \cong S$. Since $\text{Ext}^1_E(S, E) = 0$, we have $V \cong E \oplus S$. However, S is divisible as an Abelian group, while V is reduced, a contradiction. Therefore, $D = 0$; and $M \subseteq E^J$ for some index-set J. Since E is Noetherian as mentioned before, E^J is locally projective [1]. In particular, ϕ_{E^J} is an isomorphism by [7]. Because A is faithfully flat, ϕ_M has to be an isomorphism too by [4]. Therefore, $W \cong T_A(M)$ is A-solvable as a subgroup of the locally A-projective group $T_A(E^J)$ and $R_A(W) = 0$. By Theorem 2.7, W is an A-Whitehead splitter, and $H_A(W)$ is a Whitehead module.

An A-generated subgroup C of W is A-solvable since A is flat. By Theorem 2.7, C is an A-Whitehead group if the can show that $H_A(C)$ is a Whitehead module. However, this holds because the class of Whitehead modules is closed with respect to submodule if $id(E_E) = 1$ by the first paragraph.

b) $\Rightarrow$ *a)*: Clearly, $id(E_E) = 1$ if and only if $\mathrm{Ext}^1_E(E/I, \mathbb{Q}E/E) = 0$ for all right ideals I of E. Standard homological arguments show $\mathrm{Ext}^1_E(E/I, \mathbb{Q}E/E) \cong \mathrm{Ext}^1_E(I, E)$. To see that I is a Whitehead module, observe that $IA \cong T_A(I)$ is an A-solvable A-Whitehead module by b) because A is an A-Whitehead splitter since A is faithfully flat. By Theorem 2.7, IA is an A-Whitehead splitter, and $H_A(IA)$ is a Whitehead module. But, $I \cong H_A T_A(I) \cong H_A(IA)$ since A is faithfully flat. □

Corollary 3.2. *Let A be an Abelian group such that $\mathbb{Q}E$ is a finite dimensional semi-simple $\mathbb{Q}$-algebra and $id({}_E E) = id(E_E) = 1$. Every A-Whitehead group is torsion-free, A-solvable and an A-Whitehead splitter.*

Proof. If p is a prime with $pA = A$, then $(E/J)_p = 0$ for every essential right ideal J of E since E/J is bounded and p-divisible. By Theorem 3.1, it remains to show that every A-Whitehead group W is torsion-free. Suppose that W is not torsion-free, and select a Whitehead module M with $W \cong T_A(M)$. Since A is faithfully flat, $tW \cong T_A(tM)$. Select a cyclic submodule U of M with U^+ torsion. Because $id(E_E) = 1$, U is a Whitehead module. There is a right ideal I of E with $E/I \cong U$ which is a reflexive E-module by [14]. The exact sequence $0 = \mathrm{Hom}_E(U, E) \to \mathrm{Hom}_E(E, E) \to \mathrm{Hom}_E(I, E) \to \mathrm{Ext}^1_E(U, E) = 0$ yields $\mathrm{Hom}_E(I, E) \cong E$. Hence, $I \cong \mathrm{Hom}_E(\mathrm{Hom}_E(I, E), E) \cong E$. Thus, U fits into an exact sequence $0 \to E \to E \to U \to 0$, from which we get $E \cong E \oplus U$, which is a contradiction unless $U = 0$. □

Moreover, if E is right and left Noetherian and hereditary, then A is self-small and faithfully flat, and E is semi-prime [4].

Corollary 3.3. *Let A be a self-small faithfully flat Abelian group such that E is a right and left Noetherian, hereditary ring with $r_0(E) < \infty$. If W is an A-Whitehead group, then W is locally A-projective. In particular, every countably A-generated A-Whitehead group is A-projective.*

Proof. Select a finite subset X of $H_A(W)$ and a finitely generated submodule U of $H_A(W)$ containing X. The $\mathbb{Z}$-purification V of U in $H_A(W)$ is countable. Since E is hereditary, $\mathrm{Ext}_E(H_A(W)/V, E)$ is divisible as an Abelian group. On the other hand, we have an exact sequence $\mathrm{Hom}_E(H_A(W), E) \to \mathrm{Hom}_E(V, E) \to \mathrm{Ext}_E(H_A(W)/V, E) \to Ext_E(H_A(W), E) = 0$, because $H_A(W)$ is a Whitehead module. Since $\mathrm{Hom}_E(V, E)$ is a finitely generated right E-module, the same holds for $\mathrm{Ext}_E(H_A(W)/V, E)$. Thus, $\mathrm{Ext}_E(H_A(W)/V, E) \cong P' \oplus T$ where P' is projective and T^+ is bounded. Because A is reduced, $\mathrm{Ext}_E(H_A(W)/V, E)$ is reduced, which is not possible unless $\mathrm{Ext}_E(H_A(W)/V, E) = 0$.

Since $R_A(W) = 0$ by Theorem 3.1, $H_A(W) \subseteq E^I$ for some index-set I. Because E is left Noetherian, E^I is a locally projective module. Thus, its countable submodule V has to be projective. Since V contains a finitely generated essential submodule, it is finitely generated by Sandomierski's Lemma [9]. But then, there is $n < \omega$ such that $\mathrm{Ext}_E(H_A(W)/V, V) = 0$ since $\mathrm{Ext}_E(H_A(W)/V, E) = 0$. Consequently, V is a finitely generated projective direct summand of $H_A(W)$, and $H_A(W)$ is locally projective. By [7], $W \cong T_A H_A(W)$ is locally A-projective.

If G is an epimorphic image of $\oplus_\omega A$, then $H_A(G)$ is an image of $\oplus_\omega E$ since G is A-solvable. However, a countably generated locally projective module is projective. □

4. κ-A-Projective Groups

Let κ be an uncountable cardinal, and assume that A is a torsion-free Abelian with $|A| < \kappa$ whose endomorphism ring is right and left Noetherian and hereditary. An A-generated group G is *κ-A-projective* if every κ-A-generated subgroup of G is A-projective. Since every finitely A-generated subgroup of G is A-projective in this case, κ-A-projective groups are A-solvable. An A-projective subgroup U of an $\aleph_0$-A-projective group G is κ-A-closed if $(U+V)/U$ is A-projective for all κ-A-generated subgroups V of G. If $|U| < \kappa$, then this is equivalent to the condition that W/U is A-projective for all κ-A-generated subgroups W of G with $U \subseteq W$. Finally, G is *strongly κ-A-projective* if it is κ-A-projective and every κ-A-generated subgroup of G is contained in a κ-A-generated, κ-A-closed subgroup of G. Our first result reduces the investigation of strongly κ-A-projective groups to that of strongly κ-projective modules.

Proposition 4.1. *Let κ be a regular uncountable cardinal. If A is a torsion-free Abelian group with $|A| < \kappa$ whose endomorphism ring is right and left Noetherian and hereditary, then the following are equivalent for a κ-A-projective group G with $|G| \geq \kappa$:*

a) *G is strongly κ-A-projective.*
b) *$H_A(G)$ is a strongly κ-projective right E-module.*

Proof. Consider an exact sequence $0 \to U \to \oplus_I A \overset{\beta}{\to} G \to 0$ with $|I| \geq \kappa$. Since A is faithfully flat, the sequence is A-balanced and $S_A(U) = U$. Thus, $H_A(G)$ is an epimorphic image of $\oplus_I E$. Moreover, $G \cong T_A H_A(G)$ yields $|H_A(G)| = |G| \geq \kappa$.

a) $\Rightarrow$ *b*): Suppose that U is a submodule of $H_A(G)$ with $|U| < \kappa$. By Lemma 2.4, the evaluation map $\theta : T_A(U) \to UA$ is an isomorphism since G is A-solvable and A is faithfully flat. Then, $|UA| < \kappa$, and there is a κ-A-generated κ-closed subgroup V of G with $UA \subseteq V$. Observe that V is A-projective. Therefore, $H_A(UA) \subseteq H_A(V)$ is projective since E is right hereditary. However, $U \cong H_A T_A(U) \cong H_A(UA)$ since $U \subseteq H_A(G)$ and $\phi_{H_A(G)}$ is an isomorphism by [4]. Thus, $H_A(G)$ is κ-projective.

We now show that $H_A(V)$ is κ-closed in $H_A(G)$. Let W be a submodule of $H_A(G)$ with $|W| < \kappa$ which contains $H_A(V)$. Since $|WA| < \kappa$ and $V \subseteq WA$, we obtain that WA/V is A-projective. Hence, V is a direct summand of WA by [2] since E is right and left Noetherian and hereditary. Applying the functor H_A yields that $H_A(V)$ is a direct summand of $H_A(WA)$. By Lemma 2.4, $H_A(WA) = W$, and we are done.

b) $\Rightarrow$ *a*): For a κ-A-generated subgroup U of G, choose an exact sequence $\oplus_I A \overset{\pi}{\to} U \to 0$. Since G is A-solvable, the same holds for U, and the last sequence is A-balanced. Therefore, $H_A(U)$ is a κ-generated submodule of $H_A(G)$. We can find a κ-closed submodule W of $H_A(G)$ containing $H_A(U)$ with $|W| < \kappa$. Then, $U = H_A(U)A \subseteq WA$ has cardinality less than κ, and it remains to show that WA is κ-A-closed. For this, let V be a κ-A-generated subgroup of G containing WA. Since $W = H_A(WA)$ by Lemma 2.4, $H_A(V)/H_A(WA)$ is projective. Consider the commutative diagram

$$\begin{array}{ccccccc} T_AH_A(WA) & \longrightarrow & T_AH_A(V) & \longrightarrow & T_A(H_A(V)/H_A(WA)) & \longrightarrow & 0 \\ \wr\uparrow\theta_{WA} & & \wr\uparrow\theta_V & & \uparrow & & \\ WA & \longrightarrow & V & \longrightarrow & V/WA & \longrightarrow & 0. \end{array}$$

Since WA and V are A-solvable, V/WA is A-projective . □

We now can prove the main result of this section.

Theorem 4.2. *Let κ be a regular, uncountable cardinal which is not weakly compact, and suppose that A is a torsion-free Abelian group with $|A| < \kappa$ such that E is right and left Noetherian and hereditary.*

a) *If we assume $V = L$, then there exists a strongly κ-A-projective group G with* $\mathrm{Hom}(G, A) = 0$.

b) *Let $\kappa = \aleph_1$, and assume $MA + \aleph_1 < 2^{\aleph_0}$. Every strongly $\aleph_1$-A-projective group G with $|G| < 2^{\aleph_0}$ is an A-Whitehead splitter.*

Proof. a) By [13], there exists strongly κ-free left E^{op}-module M of cardinality κ with $End_{\mathbb{Z}}(M) = E^{op}$. Therefore, $End_{E^{op}}(M) = C(E)$, the center of E. Viewing M as an E-module yields a strongly κ-free right E-module M with $End_E(M) = C(E)$. We consider $G = T_A(M)$. If $\phi_1, \ldots, \phi_n \in H_AT_A(M)$, then there is a κ-generated submodule U of M such that $\phi_1(A) + \ldots + \phi_n(A) \subseteq T_A(U)$ since $|A| < \kappa$. However, since U is contained in a free submodule P of M, we obtain that $\phi_1(A) + \ldots + \phi_n(A)$ is A-projective. Thus, G is A-solvable, and $\phi_{H_AT_A(M)}$ is an isomorphism. By [4], ϕ_M is an isomorphism since A is faithfully flat. Consequently, $M \cong H_A(G)$ is strongly κ-projective. By Proposition 4.1, G is strongly κ-A-projective. Moreover, every subset of G of cardinality less than κ is contained in an A-free subgroup of G.

Since E is Noetherian, it does not have any infinite family of orthogonal idempotent, and the same holds for $C(E)$. By the Adjoint-Functor-Theorem, we have $End_{\mathbb{Z}}(T_A(M)) \cong End_E(M) = C(E)$ since $T_A(M)$ is A-solvable. Therefore, $End_{\mathbb{Z}}(G)$ is commutative, and $G = G_1 \oplus \ldots \oplus G_m$ where each G_j is indecomposable and $\mathrm{Hom}(G_i, G_j) = 0$ for $i \neq j$. Since G_i is A-generated and indecomposable, G_i is either A-projective of finite A-rank, or $\mathrm{Hom}(G_i, A) = 0$ since $E(A)$ is right and left Noetherian and hereditary. Consequently, $G = B \oplus C$ where C is A-projective of finite A-rank, and $\mathrm{Hom}(B, A) = \mathrm{Hom}(B, C) = \mathrm{Hom}(C, B) = 0$.

Since $|A| < \kappa$, G contains a subgroup U isomorphic to $\oplus_\omega A$. We can find a subgroup V of G which is A-free and contains C and U, say $V \cong \oplus_I A$ for some infinite index-set I. Since A is discrete in the finite topology, it is self-small. Therefore, we can find a finite subset J of I such that $\alpha(A) \subseteq \oplus_J A$. Since C is a direct summand of G, we have $V = C \oplus (B \cap V)$ and $B \cap V \cong (\oplus_J A)/C \oplus (\oplus_{I\setminus J} A)$. But then, $\mathrm{Hom}(C, B) \neq 0$, which results in a contradiction unless $C = 0$. This shows, $\mathrm{Hom}(G, A) = 0$.

b) If G is a strongly $\aleph_1$-projective group with $\aleph_1 \leq |G| < 2^{\aleph_0}$, then G is A-solvable. By Proposition 4.1, $H_A(G)$ is a strongly $\aleph_1$-projective right E-module. Arguing as in the case $A = \mathbb{Z}$ (e.g. see [10, Chapter 12] or [11]), we obtain that $H_A(G)$ is a Whitehead-module. By Theorem 2.7, G is an A-Whitehead splitter. □

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Rodrigues formula for the Cayley transform of groups $\mathbf{SO}(n)$ and $\mathbf{SE}(n)$

Dorin Andrica and Oana Liliana Chender

Abstract. In Theorem 3.1 we present, in the case when the eigenvalues of the matrix are pairwise distinct, a direct way to determine the Rodrigues coefficients of the Cayley transform for the special orthogonal $\mathbf{SO}(n)$ by reducing the Rodrigues problem in this case to the system (3.2). The similar method is discussed for the Euclidean group $\mathbf{SE}(n)$.

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Keywords: Lie group, Lie algebra, exponential map, Cayley transform, special orthogonal group $\mathbf{SO}(n)$, Euclidean group $\mathbf{SE}(n)$, Rodrigues coefficients.

1. Introduction

The Cayley transform of the group of rotations $\mathbf{SO}(n)$ of the Euclidean space $\mathbb{R}^n$ is defined by $\mathrm{Cay} : \mathfrak{so}(n) \to \mathbf{SO}(n), \mathrm{Cay}(A) = (I_n + A)(I_n - A)^{-1}$, where $\mathfrak{so}(n)$ is the Lie algebra of $\mathbf{SO}(n)$. Because the inverse of the matrix $I_n - A$ can be written as $(I_n - A)^{-1} = I_n + A + A^2 + \dots$ on a sufficiently small neighborhood of O_n, from the well-known Hamilton-Cayley Theorem, it follows that $\mathrm{Cay}(A)$ has the polynomial form

$$\mathrm{Cay}(A) = b_0(A)I_n + b_1(A)A + \cdots + b_{n-1}(A)A^{n-1},$$

where the coefficients $b_0, b_1, \dots, b_{n-1}$ depend on the matrix A and are uniquely defined. By analogy with the case of the exponential map (see [1] and [2]), they are called *Rodrigues coefficients* of A with respect to the Cayley transform.

Using the main idea in the articles [3] (see also [4]), in this paper we present a method to derive the Rodrigues coefficients for the Cayley transform of the group $\mathbf{SO}(n)$. The case of the Euclidean group $\mathbf{SE}(n)$ is also discussed.

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2. Cayley transform of the group $\mathbf{SO}(n)$

The matrices of the $\mathbf{SO}(n)$ group describe the rotations as movements in the space $\mathbb{R}^n$. If the matrix A belongs to the Lie Algebra $\mathfrak{so}(n)$ of the Lie group $\mathbf{SO}(n)$, then the matrix $I_n - A$ is invertible.

Indeed, the eigenvalues $\lambda_1, ..., \lambda_n$ of the matrix A are 0 or purely imaginary, so eigenvalues of the matrix $I_n - A$ are $1-\lambda_1, ..., 1-\lambda_n$. They are clearly different from 0, therefore we have $\det(I_n - A) = (1-\lambda_1)...(1-\lambda_n) \neq 0$, so $I_n - A$ is invertible.

The map Cay : $\mathfrak{so}(n) \to \mathbf{SO}(n)$, defined by

$$\mathrm{Cay(A) = (I_n + A)(I_n - A)^{-1}}$$

is called the *Cayley transform* of the group $\mathbf{SO}(n)$. Let show that this map is well defined. Let be $\mathrm{Cay}(A) = R$. We have

$$\begin{aligned} R^t R &= (I_n + A)(I_n - A)^{-1t}[(I_n + A)(I_n - A)^{-1}] \\ &= (I_n + A)(I_n - A)^{-1t}[(I_n - A)^{-1}]^t (I_n + A) \\ &= (I_n + A)(I_n - A)^{-1}(I_n -^t A)^{-1}(I_n +^t A) \\ &= (I_n + A)(I_n - A)^{-1}(I_n + A)^{-1}(I_n - A) = I_n, \end{aligned}$$

because matrices and their inverses commute. Therefore $R \in \mathbf{SO}(n)$. The map Cay is obviously continuous and we have $\mathrm{Cay}(O_n) = I_n \in \mathbf{SO}(n)$, hence necessarily we have $R \in \mathbf{SO}(n)$.

Denote by $\sum$ the set of the group $\mathbf{SO}(n)$ containing the matrices with eigenvalue -1. Clearly, we have $R \in \sum$ if and only if the matrix $I_n + R$ is singular.

Theorem 2.1. *The map* Cay : $\mathfrak{so}(n) \to \mathbf{SO}(n) \setminus \sum$ *is bijective and its inverse is* $\mathrm{Cay}^{-1} : \mathbf{SO}(n) \setminus \sum \to \mathfrak{so}(n)$, *where* $\mathrm{Cay}^{-1}(R) = (R + I_n)^{-1}(R - I_n)$.

Proof. If $R \in \mathbf{SO}(n) \setminus \sum$ then, the relation $\mathrm{Cay}(A) = R$ is equivalent to

$$R = (I_n + A)(I_n - A)^{-1} = (2I_n - (I_n - A))(I_n - A)^{-1} = 2(I_n - A)^{-1} - I_n.$$

Because $R \in \mathbf{SO}(n) \setminus \sum$, it follows that the matrix $R + I_n$ is invertible and from above relation we obtain that its inverse is $(R + A)^{-1} = \frac{1}{2}(I_n - A)$. Using this relation we have

$$(R + I_n)^{-1}(R - I_n) = \frac{1}{2}(I_n - A)(2(I_n - A)^{-1} - 2I_n) = I_n - I_n + A = A,$$

so $\mathrm{Cay}^{-1}(R) = (R + I_n)^{-1}(R - I_n)$.

In addition, a simple computation shows that if the matrix R is orthogonal, then the matrix $A = (R + I_n)^{-1}(R - I_n)$ is antisymmetric. Indeed, we have

$$\begin{aligned} {}^t A &= ({}^t R - I_n)({}^t R + I_n)^{-1} = (R^{-1} - I_n)(R^{-1} + I_n)^{-1} \\ &= (I_n - R)R^{-1}R(I_n + R)^{-1} = -(R + I_n)^{-1}(R - I_n) = -A, \end{aligned}$$

because the matrices $R - I_n$ and $(R + I_n)^{-1}$ commute. $\square$

3. Rodrigues type formulas for Cayley transform

Because the inverse of the matrix $I_n - A$ can be written in the form

$$(I_n - A)^{-1} = I_n + A + A^2 + ...$$

for a sufficiently small neighborhood of O_n, from Hamilton-Cayley theorem, it follows that the Cayley transform of A can be written in the polynomial form

$$\mathrm{Cay}(A) = b_0(A)I_n + b_1(A)A + ... + b_{n-1}(A)A^{n-1} \tag{3.1}$$

where the coefficients $b_0, ..., b_{n-1}$ are uniquely determined and depend on the matrix A. We will call these numbers, by analogy with the situation of the exponential map, Rodrigues coefficients of A with respect to the application Cay.

As in the case of the exponential map, an important property of the Rodrigues coefficients is the invariance with respect to equivalent matrices, i.e. for any invertible matrix U, the following relations hold

$$b_k(UAU^{-1}) = b_k(A), k = 0, ..., n-1.$$

This property is obtained from the uniqueness of the Rodrigues coefficients and from the following property of the Cayley transform

$$U\mathrm{Cay}(A)U^{-1} = \mathrm{Cay}(UAU^{-1}).$$

To justify the last relation just observe that we have successively

$$U\mathrm{Cay}(A)U^{-1} = U(I_n + A)(I_n - A)^{-1}U^{-1} = U(I_n + A)U^{-1}U(+I_n - A)^{-1}U^{-1}$$
$$= (I_n + UAU^{-1})(U^{-1})^{-1}(I_n - A)^{-1}U^{-1}(I_n + UAU^{-1})(U(I_n - A)U^{-1})^{-1}$$
$$= (I_n + UAU^{-1})(I_n + UAU^{-1})^{-1} = \mathrm{Cay}(UAU^{-1}).$$

Theorem 3.1. *Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of the matrix $A \in \mathfrak{so}(\mathfrak{n})$.*

1) Rodrigues coefficients of A relative to the application Cay *are solutions of the system*

$$\sum_{k=0}^{n-1} S_{k+j} b_k = \sum_{s=1}^{n} \lambda_s^j \frac{1+\lambda_s}{1-\lambda_s}, j = 0, ..., n-1, \tag{3.2}$$

where $S_j = \lambda_1^j + ... + \lambda_n^j$.

2) If the eigenvalues $\lambda_1, ..., \lambda_n$ of the matrix A are pairwise distinct, then the Rodrigues coefficients $b_0, ..., b_{n-1}$ are perfectly determined by this system and are rational functions of $\lambda_1, ..., \lambda_n$.

Proof. 1) By multiplying the relation (3.1) by the power $A^j, j = 0, ..., n-1$, we obtain the matrix relations

$$A^j\mathrm{Cay}(A) = \sum_{k=0}^{n-1} b_k A^{k+j}, j = 0, ..., n-1.$$

Now, considering the trace in both sides of the above relations, it follows

$$\sum_{k=0}^{n-1} tr(A^{k+j})b_k = tr(A^j\mathrm{Cay}(A)), j = 0, ..., n-1. \tag{3.3}$$

The matrix A^{k+j} has the eigenvalues $\lambda_1^{k+j}, ..., \lambda_n^{k+j}$, and the matrix $A^j\mathrm{Cay}(A)$ has the eigenvalues $\lambda_1^j\frac{1+\lambda_1}{1-\lambda_1}, ..., \lambda_n^j\frac{1+\lambda_n}{1-\lambda_n}$ and the system (3.3) is equivalent to the system (3.2).

2) For the second statement, observe that the determinant of the system (3.2) can be written as

$$D_n = \det\begin{pmatrix} S_0 & S_1 & \dots & S_{n-1} \\ S_1 & S_2 & \dots & S_n \\ \dots & \dots & \dots & \dots \\ S_{n-1} & S_n & \dots & S_{2n-1}, \end{pmatrix}$$

where $S_l = S_l(\lambda_1, \dots, \lambda_n) = \lambda_1^l + \dots + \lambda_n^l, l = 0, \dots, 2n-1$.

It is clear that

$$D_n = \det\begin{pmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_n \\ \dots & \dots & \dots \\ \lambda_1^{n-1} & \dots & \lambda_n^{n-1} \end{pmatrix} \cdot \det\begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & \lambda_n & \dots & \lambda_n^{n-1} \end{pmatrix}$$

$$= V_n^2(\lambda_1, ..., \lambda_n) = \prod_{1\le i<j\le n} (\lambda_i - \lambda_j)^2,$$

where $V_n = V_n(\lambda_1, \dots, \lambda_n)$ is the Vandermonde determinant of order n. According to the well-known formulas giving the solution $b_0, \dots, b_{n-1}$ to the system (3.2), the conclusion follows. □

We will continue to illustrate the particular cases $n = 2$ and $n = 3$. If $A = O_n$, then $\mathrm{Cay}(A) = I_n$ and so $b_0(O_n) = 1, b_1(O_n) = ... = b_{n-1}(O_n) = 0$.

In the case $n = 2$, consider the antisymmetric matrix $A \neq O_2$, where

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, a \in \mathbb{R}^*,$$

with eigenvalues $\lambda_1 = ai, \lambda_2 = -ai$. System (3.2) becomes in this case

$$\begin{cases} 2b_0 = \frac{1+ai}{1-ai} + \frac{1-ai}{1+ai} \\ -2a^2b_1 = ai\frac{1+ai}{1-ai} - ai\frac{1-ai}{1+ai} \end{cases}$$

and we obtain

$$b_0 = \frac{1-a^2}{1+a^2}, b_1 = \frac{1}{1+a^2}.$$

Thus, the Rodrigues type formula for the Cayley transform is

$$\mathrm{Cay}(A) = \frac{1-a^2}{1+a^2}I_2 + \frac{2}{1+a^2}A. \tag{3.4}$$

For $n = 3$ any real antisymmetric matrix is of the form

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

with the characteristic polynomial $p_A(t) = t^3 + \theta^2 t$, where $\theta = \sqrt{a^2+b^2+c^2}$. The eigenvalues of the matrix A are $\lambda_1 = \theta i, \lambda_2 = -\theta i, \lambda_3 = 0$. We have $A = O_3$ if and only if $\theta = 0$, so it is enough to consider only the situation in which $\theta \neq 0$. The system 3.2 becomes

$$\begin{cases} 3b_0 - 2\theta^2 b_2 = \frac{1+\theta i}{1-\theta i} + \frac{1-\theta i}{1+\theta i} + 1 \\ -2\theta^2 b_1 = \theta i \frac{1+\theta i}{1-\theta i} - \theta i \frac{1-\theta i}{1+\theta i} \\ -2\theta^2 b_0 + \theta^4 b_2 = -\theta^2 \left(\frac{1+\theta i}{1-\theta i} + \frac{1-\theta i}{1+\theta i} \right) \end{cases}$$

with the solution

$$b_0 = 1, b_1 = \frac{2}{1+\theta^2}, b_2 = \frac{2}{1+\theta^2}.$$

It follows the Rodrigues type formula for the Cayley transform of group $\mathbf{SO}(3)$

$$\mathrm{Cay}(A) = I_3 + \frac{2}{1+\theta^2} A + \frac{2}{1+\theta^2} A^2. \tag{3.5}$$

Formula (3.5) offers the possibility to obtain another formula for the inverse of Cayley transform. Let be $R \in \mathbf{SO}(3)$ such that

$$R = I_3 + \frac{2}{1+\theta^2} A + \frac{2}{1+\theta^2} A^2,$$

where A is an antisymmetric matrix. Considering the matrix transpose in both sides of the above relation and taking into account that ${}^tA = -A$, we obtain

$$R -^t R = \frac{4}{1+\theta^2} A. \tag{3.6}$$

On the other hand, we have

$$tr(R) = 3 - \frac{4\theta^2}{1+\theta^2} = -1 + \frac{4}{1+\theta^2},$$

and by replacing in the relation (3.6), we get the formula

$$\mathrm{Cay}^{-1}(R) = \frac{1}{1+tr(R)} (R -^t R). \tag{3.7}$$

Formula (3.7) makes sense for rotations $R \in \mathbf{SO}(3)$ for which $1 + tr(R) \neq 0$. If R is a rotation of angle α, then we have $tr(R) = 1 + 2\cos\alpha$, so application Cay^{-1} is not defined for the rotations of angle $\alpha = \pm\pi$. Because in the domain where is defined, the application Cay is bijective, it follows that the antisymmetric matrices from $\mathfrak{so}(3)$ can be used as coordinates for rotations. Considering the Lie algebra isomorphism "$\widehat{\ }$" between $(\mathbb{R}^3, \times)$ and $(\mathfrak{so}(3), [\cdot,\cdot])$, where " $\times$ " denote the vector product, defined by $v \in \mathbb{R}^3 \to \widehat{v} \in \mathfrak{so}(3)$, where

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and

$$\widehat{v} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix},$$

by composing the applications

$$\mathbb{R}^3 \hookrightarrow \mathfrak{so}(3) \xrightarrow{\text{Cay}} \mathbf{SO}(3)$$

we get a vectorial parameterization of rotations from $\mathbf{SO}(3)$.

4. The Cayley transform for Euclidean group $\mathbf{SE}(n)$

In this subparagraph we will define a Cayley type transformation for the special Euclidean group $\mathbf{SE}(n)$. By analogy with the special orthogonal group $\mathbf{SO}(n)$, we define the application $\text{Cay}_{n+1} : \mathfrak{se}(n) \to \mathbf{SE}(n)$, where

$$\text{Cay}_{n+1}(S) = (I_{n+1} + S)(I_{n+1} - S)^{-1}. \tag{4.1}$$

We will call this application *Cayley transform* of the group $\mathbf{SE}(n)$. First we show that it is well defined. Let be $S \in \mathfrak{se}(n)$, a matrix defined in blocks

$$S = \begin{pmatrix} A & u \\ 0 & 0 \end{pmatrix},$$

where $A \in \mathfrak{so}(n)$ and $u \in \mathbb{R}^n$. A simple computation shows that we have the formula

$$(I_{n+1} + S)(I_{n+1} - S)^{-1} = \begin{pmatrix} R & (R + I_n)u \\ 0 & 1 \end{pmatrix},$$

where $R = (I_n + A)(I_n - A)^{-1} = \text{Cay}(A) \in \mathbf{SO}(n)$, that is the desired formula.

The connection between the transform $\text{Cay} : \mathfrak{so}(n) \to \mathbf{SO}(n)$ and $\text{Cay}_{n+1} : \mathfrak{se}(n) \to \mathbf{SE}(n)$ is given by the formula

$$\text{Cay}_{n+1}(S) = \begin{pmatrix} \text{Cay}(A) & (R + I_n)u \\ 0 & 1 \end{pmatrix}.$$

As for the classical transform $\text{Cay} : \mathfrak{so}(n) \to \mathbf{SO}(n)$ we can get effective Rodrigues type formulas for transform $\text{Cay}_{n+1} : \mathfrak{se}(n) \to \mathbf{SE}(n)$, for small values of n. Using the observation from section 5.1 in the paper of R.-A. Rohan [5], we obtain that for a matrix $S \in \mathfrak{se}(n)$ defined in blocks as above, its characteristic polynomial p_S satisfy the relation $p_S(t) = tp_A(t)$. The Rodrigues formula for the transform $\text{Cay}_{n+1} : \mathfrak{se}(n) \to \mathbf{SE}(n)$ is of the form

$$\text{Cay}_{n+1}(S) = c_0 I_{n+1} + c_1 S + \ldots + c_n S^n,$$

where the coefficients $c_0, c_1, \ldots, c_n$ depend on the matrix S.

For $n = 2$, consider the antisymmetric matrix $A \neq O_2$, where

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, a \in \mathbb{R}^*.$$

Using the above observation, it follows that the matrix $S \in \mathfrak{se}(2)$ has eigenvalues $\lambda_1 = ai, \lambda_2 = -ai, \lambda_3 = 0$, and the corresponding Rodrigues formula has the form

$$\text{Cay}_3(S) = c_0 I_3 + c_1 S + c_2 S^2.$$

We have a result analogous to that of Theorem 3.1, which is reduced to the system

$$\begin{cases} S_0c_0 + S_1c_1 + S_2c_2 = 1 + \frac{1+\lambda_1}{1-\lambda_1} + \frac{1+\lambda_2}{1-\lambda_2} \\ S_1c_0 + S_2c_1 + S_3c_2 = \lambda_1\frac{1+\lambda_1}{1-\lambda_1} + \lambda_2\frac{1+\lambda_2}{1-\lambda_2} \\ S_2c_0 + S_3c_1 + S_4c_2 = \lambda_1^2\frac{1+\lambda_1}{1-\lambda_1} + \lambda_2^2\frac{1+\lambda_2}{1-\lambda_2} \end{cases}$$

where in our case we have $S_0 = 3, S_1 = 0, S_2 = -2a^2, S_3 = 0, S_4 = 2a^2$. This system is equivalent to

$$\begin{cases} 3c_0 - 2a^2c_2 = 1 + \frac{2(1-a^2)}{1+a^2} \\ -2a^2c_1 = -\frac{4a^2}{1+a^2} \\ -2a^2c_0 + 2a^4c_2 = -2a^2\frac{1-a^2}{1+a^2} \end{cases}$$

with solution

$$c_0 = 1, c_1 = \frac{1}{1+a^2}, c_2 = \frac{1}{1+a^2}.$$

So Rodrigues formula for transformation Cay_3 is

$$\mathrm{Cay}_3(S) = I_3 + \frac{1}{1+a^2}S + \frac{1}{1+a^2}S^2. \tag{4.2}$$

For $n = 3$ we consider an antisymmetric matrix of the form

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

with the characteristic polynomial $p_A(t) = t^3 + \theta^2 t$, where $\theta = \sqrt{a^2+b^2+c^2}$. The matrix $S \in \mathfrak{se}(3)$ has the characteristic polynomial $p_S(t) = tp_A(t) = t^4 + \theta^2t^2$, and the eigenvalues of its are $\lambda_1 = \theta i, \lambda_2 = -\theta i, \lambda_3 = 0, \lambda_4 = 0$. Rodrigues formula has the form

$$\mathrm{Cay}_4(S) = c_0I_4 + c_1S + c_2S^2 + c_3S^3.$$

After a similar computation, we obtain the formula

$$\mathrm{Cay}_3(S) = I_3 + 2S + \frac{2}{1+\theta^2}S^2 + \frac{2}{1+\theta^2}S^3. \tag{4.3}$$

As for Cayley transform of the group $\mathbf{SO}(n)$, denote by $\sum_{n+1}$ the set of matrices from $\mathbf{SE}(n)$ that has -1 as eigenvalue. Clearly we have $M \in \mathbf{SE}(n)$ if and only if the matrix $I_{n+1} + M$ is singular. With a similar proof as in Theorem 3.1, we get

Theorem 4.1. *The map* $\mathrm{Cay}_{n+1} : \mathfrak{se}(n) \to SE(n) \setminus \sum_{n+1}$ *is bijective and its inverse is given by*

$$\mathrm{Cay}_{n+1}^{-1}(M) = \begin{pmatrix} \mathrm{Cay}^{-1}(M) & (R+I_n)^{-1}\mathbf{t} \\ 0 & 0 \end{pmatrix},$$

where the matrix M *is defined in blocks by*

$$S = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix}.$$

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Quantitative uniform approximation by generalized discrete singular operators

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Abstract. Here we study the approximation properties with rates of generalized discrete versions of Picard, Gauss-Weierstrass, and Poisson-Cauchy singular operators. We treat both the unitary and non-unitary cases of the operators above. We establish quantitatively the pointwise and uniform convergences of these operators to the unit operator by involving the uniform higher modulus of smoothness of a uniformly continuous function.

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Keywords: Discrete singular operator, modulus of smoothness, uniform convergence.

1. Introduction

This article is motivated mainly by [4], where J. Favard in 1944 introduced the discrete version of Gauss-Weierstrass operator

$$\left(F_n f\right)(x)=\frac{1}{\sqrt{\pi n}}\sum_{\nu=-\infty}^{\infty} f\left(\frac{\nu}{n}\right)\exp\left(-n\left(\frac{\nu}{n}-x\right)^2\right), \tag{1.1}$$

$n\in\mathbb{N}$, which has the property that $\left(F_n f\right)(x)$ converges to $f(x)$ pointwise for each $x\in\mathbb{R}$, and uniformly on any compact subinterval of $\mathbb{R}$, for each continuous function f $(f\in C(\mathbb{R}))$ that fulfills $|f(t)|\leq Ae^{Bt^2}$, $t\in\mathbb{R}$, where A, B are positive constants.

The well-known Gauss-Weierstrass singular convolution integral operators is

$$\left(W_n f\right)(x)=\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} f(u)\exp\left(-n\left(u-x\right)^2\right)du. \tag{1.2}$$

We are also motivated by [1], [2], and [3] where the authors studied extensively the approximation properties of particular generalized singular integral operators such as Picard, Gauss-Weierstrass, and Poisson-Cauchy as well as the general cases of singular integral operators.These operators are not necessarily positive linear operators.

In this article, we define the discrete versions of the operators mentioned above and we study quantitatively their uniform approximation properties regarding convergence to the unit. We examine thoroughly the unitary and non-unitary cases and their interconnections.

2. Background

In [3] p.271-279, the authors studied smooth general singular integral operators $\Theta_{r,\xi}(f,x)$ defined as follows. Let $\xi > 0$ and let μ_ξ be Borel probability measures on $\mathbb{R}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ they defined

$$\alpha_j = \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 1, ..., r, \\ 1 - \sum_{i=1}^{r} (-1)^{r-i}\binom{r}{i} i^{-n}, & j = 0 \end{cases} \tag{2.1}$$

that is $\sum_{j=0}^{r} \alpha_j = 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable, they defined

$$\Theta_{r,\xi}(f,x) := \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x + jt) \right) d\mu_\xi(t) \tag{2.2}$$

for $x \in \mathbb{R}$.

The operators $\Theta_{r,\xi}$ are not necessarily positive linear operators. Indeed we have:

Let $r = 2$, $n = 3$. Then $\alpha_0 = \frac{23}{8}$, $\alpha_1 = -2$, $\alpha_2 = \frac{1}{8}$. Consider $f(t) = t^2 \geq 0$ and $x = 0$. Then

$$\begin{aligned} \Theta_{2,\xi}\left(t^2; 0\right) &= \int_{-\infty}^{\infty} \left(\sum_{j=0}^{2} \alpha_j j^2 t^2 \right) d\mu_\xi(t) \\ &= -\frac{3}{2} \left(\int_{-\infty}^{\infty} t^2 d\mu_\xi(t) \right) \leq 0, \end{aligned}$$

given that $\int_{-\infty}^{\infty} t^2 d\mu_\xi(t) < \infty$.

Authors assumed that $\Theta_{r,\xi}(f,x) \in \mathbb{R}$ for all $x \in \mathbb{R}$.

In [3] p.272, the rth modulus of smoothness finite given as

$$\omega_r(f^{(n)}, h) := \sup_{|t| \leq h} \|\Delta_t^r f^{(n)}(x)\|_{\infty,x} < \infty, \quad h > 0, \tag{2.3}$$

where$\|.\|_{\infty,x}$ is the supremum norm with respect to x, $f \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$, and

$$\Delta_t^r f^{(n)}(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f^{(n)}(x + jt). \tag{2.4}$$

They introduced also

$$\delta_k := \sum_{j=1}^{r} \alpha_j j^k, \quad k = 1, \dots, n \in \mathbb{N}, \tag{2.5}$$

and the even function

$$G_n(t) := \int_0^{|t|} \frac{(|t| - w)^{n-1}}{(n-1)!} \omega_r(f^{(n)}, w) dw, \quad n \in \mathbb{N} \tag{2.6}$$

with

$$G_0(t) := \omega_r(f, |t|), \quad t \in \mathbb{R}. \tag{2.7}$$

In [3] p.273, they proved

Theorem 2.1. *The integrals $c_{k,\xi} := \int_{-\infty}^{\infty} t^k d\mu_\xi(t)$, $k = 1, \dots, n$, are assumed to be finite. Then*

$$\left| \Theta_{r,\xi}(f; x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi} \right| \leq \int_{-\infty}^{\infty} G_n(t) d\mu_\xi(t). \tag{2.8}$$

Moreover, they showed ([3], p.274)

Corollary 2.2. *Suppose $\omega_r(f, \xi) < \infty, \xi > 0$. Then it holds for $n = 0$ that*

$$|\Theta_{r,\xi}(f; x) - f(x)| \leq \int_{-\infty}^{\infty} \omega_r(f, |t|) \, d\mu_\xi(t) \, . \tag{2.9}$$

Furthermore, by using the inequalities

$$G_n(t) \leq \frac{|t|^n}{n!} \omega_r(f^{(n)}, |t|) \tag{2.10}$$

and

$$\omega_r(f, \lambda t) \leq (\lambda + 1)^r \omega_r(f, t), \lambda, t > 0, \tag{2.11}$$

they obtained

$$\begin{aligned} K_1 &:= \left\| \Theta_{r,\xi}(f; x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi} \right\|_{\infty,x} \\ &\leq \frac{\omega_r(f^{(n)}, \xi)}{n!} \int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} K_2 &:= \|\Theta_{r,\xi}(f; x) - f(x)\|_\infty \\ &\leq \omega_r(f, \xi) \int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t). \end{aligned} \tag{2.13}$$

Additionally, they demonstrated ([3], p.279)

Theorem 2.3. *Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$. Set $c_{k,\xi} := \int_{-\infty}^{\infty} t^k d\mu_\xi(t)$, $k = 1, \ldots, n$. Suppose also $\omega_r(f^{(n)}, h) < \infty, \forall h > 0$. It is also assumed that*

$$\int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) < \infty. \tag{2.14}$$

Then

$$\left\| \Theta_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi} \right\|_{\infty,x} \tag{2.15}$$
$$\leq \frac{\omega_r(f^{(n)}, \xi)}{n!} \int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t).$$

When $n = 0$, the sum in L.H.S (2.15) collapses.

3. Main Results

Here we study important special cases of $\Theta_{r,\xi}$ operators for discrete probability measures μ_ξ.

Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$, $0 < \xi \leq 1$, $x \in \mathbb{R}$.

i) When

$$\mu_\xi(\nu) = \frac{e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}, \tag{3.1}$$

we define the generalized discrete Picard operators as

$$P_{r,\xi}^*(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x + j\nu) \right) e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}. \tag{3.2}$$

ii) When

$$\mu_\xi(\nu) = \frac{e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}, \tag{3.3}$$

we define the generalized discrete Gauss-Weierstrass operators as

$$W_{r,\xi}^*(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x + j\nu) \right) e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}. \tag{3.4}$$

iii) Let $\alpha \in \mathbb{N}$, and $\beta > \frac{1}{\alpha}$. When

$$\mu_\xi(\nu) = \frac{\left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}, \tag{3.5}$$

we define the generalized discrete Poisson-Cauchy operators as

$$\Theta^*_{r,\xi}(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x+j\nu)\right) \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}. \tag{3.6}$$

Observe that for c constant we have

$$P^*_{r,\xi}(c;x) = W^*_{r,\xi}(c;x) = \Theta^*_{r,\xi}(c;x) = c. \tag{3.7}$$

We assume that the operators $P^*_{r,\xi}(f;x)$, $W^*_{r,\xi}(f;x)$, and $\Theta^*_{r,\xi}(f;x) \in \mathbb{R}$, for $x \in \mathbb{R}$. This is the case when $\|f\|_{\infty,\mathbb{R}} < \infty$.

iv) Let $f \in C_u(\mathbb{R})$ (uniformly continuous functions) or $f \in C_b(\mathbb{R})$ (continuous and bounded functions). When

$$\mu_\xi(\nu) := \mu_{\xi,P}(\nu) := \frac{e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}}, \tag{3.8}$$

we define the generalized discrete non-unitary Picard operators as

$$P_{r,\xi}(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x+j\nu)\right) e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}}. \tag{3.9}$$

Here $\mu_{\xi,P}(\nu)$ has mass

$$m_{\xi,P} := \frac{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}}. \tag{3.10}$$

We observe that

$$\frac{\mu_{\xi,P}(\nu)}{m_{\xi,P}} = \frac{e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}, \tag{3.11}$$

which is the probability measure (3.1) defining the operators $P^*_{r,\xi}$.

v) Let $f \in C_u(\mathbb{R})$ or $f \in C_b(\mathbb{R})$.When

$$\mu_\xi(\nu) := \mu_{\xi,W}(\nu) := \frac{e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1}, \tag{3.12}$$

with $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, $\operatorname{erf}(\infty) = 1$, we define the generalized discrete non-unitary Gauss-Weierstrass operators as

$$W_{r,\xi}(f;x) := \frac{\sum_{\nu=-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+j\nu) \right) e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi}} \right) \right) + 1}. \tag{3.13}$$

Here $\mu_{\xi,W}(\nu)$ has mass

$$m_{\xi,W} := \frac{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi}} \right) \right) + 1}. \tag{3.14}$$

We observe that

$$\frac{\mu_{\xi,W}(\nu)}{m_{\xi,W}} = \frac{e^{\frac{-\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}, \tag{3.15}$$

which is the probability measure (3.3) defining the operators $W^*_{r,\xi}$.

Clearly, here $P_{r,\xi}(f;x)$, $W_{r,\xi}(f;x) \in \mathbb{R}$, for $x \in \mathbb{R}$.

We present our first result.

Proposition 3.1. *Let $n \in \mathbb{N}$. Then, there exists $K_1 > 0$ such that*

$$\frac{\sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi} \right)^r e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \leq K_1 < \infty \tag{3.16}$$

for all $\xi \in (0,1]$.

Proof. We observe that

$$\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}} > 1,$$

then

$$\frac{1}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} < 1.$$

Therefore, we obtain

$$\frac{\sum\limits_{\nu=-\infty}^{\infty}|\nu|^n\left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-|\nu|}{\xi}}} \tag{3.17}$$

$$< \sum_{\nu=-\infty}^{\infty}|\nu|^n\left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}$$

$$:= R_1.$$

We notice that

$$\begin{aligned} R_1 &= 2\sum_{\nu=1}^{\infty}\nu^n\left(1+\frac{\nu}{\xi}\right)^r e^{\frac{-\nu}{\xi}} \\ &= 2\sum_{\nu=1}^{\infty}\left(\nu^n e^{\frac{-\nu}{2\xi}}\right)\left(\left(1+\frac{\nu}{\xi}\right)^r e^{\frac{-\nu}{2\xi}}\right). \end{aligned} \tag{3.18}$$

Since we have $\frac{\nu}{\xi} \geq 1$ for $\nu \geq 1$, we get

$$\left(1+\frac{\nu}{\xi}\right)^r e^{\frac{-\nu}{2\xi}} \leq \frac{2^r\nu^r}{\xi^r e^{\frac{\nu}{2\xi}}} = \frac{2^r z^r}{e^{\frac{z}{2}}} \tag{3.19}$$

where $z := \frac{\nu}{\xi}$. Additionally, since

$$e^{\frac{z}{2}} = \sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^k}{k!} \geq \frac{z^r}{2^r r!}, \tag{3.20}$$

we obtain

$$\frac{z^r}{e^{\frac{z}{2}}} \leq 2^r r!. \tag{3.21}$$

Hence, by (3.18) , (3.19) , and (3.21) , we have

$$\begin{aligned} R_1 &\leq 2^{2r+1}r!\sum_{\nu=1}^{\infty}\nu^n e^{\frac{-\nu}{2\xi}} \\ &\leq 2^{2r+1}r!\sum_{\nu=1}^{\infty}\nu^n e^{\frac{-\nu}{2}}. \end{aligned} \tag{3.22}$$

Now, we define the function $f(\nu) = \nu^n e^{\frac{-\nu}{2}}$ for $\nu \geq 1$. Then, we have

$$f'(\nu) = \nu^{n-1}e^{\frac{-\nu}{2}}\left(n-\frac{\nu}{2}\right).$$

Thus, $f(\nu)$ is positive, continuous, and decreasing for $\nu > 2n$. Hence, by shifted triple inequality similar to [5], we get

$$\begin{aligned} & \sum_{\nu=1}^{\infty} \nu^n e^{\frac{-\nu}{2}} \qquad (3.23) \\ = & \sum_{\nu=1}^{2n} \nu^n e^{\frac{-\nu}{2}} + \sum_{\nu=2n+1}^{\infty} \nu^n e^{\frac{-\nu}{2}} \\ \leq & \sum_{\nu=1}^{2n} \nu^n e^{\frac{-\nu}{2}} + \int_{2n+1}^{\infty} \nu^n e^{\frac{-\nu}{2}} d\nu + f(2n+1) \\ \leq & \sum_{\nu=1}^{2n} \nu^n e^{\frac{-\nu}{2}} + \int_{0}^{\infty} \nu^n e^{\frac{-\nu}{2}} d\nu + (2n+1)^n e^{-\frac{(2n+1)}{2}} \\ = & \lambda_n + (2n+1)^n e^{-\frac{(2n+1)}{2}} + \int_{0}^{\infty} \nu^n e^{\frac{-\nu}{2}} d\nu, \end{aligned}$$

where

$$\lambda_n := \sum_{\nu=1}^{2n} \nu^n e^{\frac{-\nu}{2}} < \infty \qquad (3.24)$$

for all $\xi \in (0,1]$. Furthermore, by the integral calculation in [3], p.86, we obtain

$$\int_{0}^{\infty} \nu^n e^{\frac{-\nu}{2}} d\nu = n! 2^{n+1}. \qquad (3.25)$$

Thus, by (3.22), (3.23), and (3.25), we get

$$\begin{aligned} R_1 & \leq 2^{2r+1} r! \left(\lambda_n + (2n+1)^n e^{-\frac{(2n+1)}{2}} + n! 2^{n+1} \right) \qquad (3.26) \\ & < \infty \end{aligned}$$

for all $\xi \in (0,1]$. Let $K_1 := 2^{2r+1} r! \left(\lambda_n + (2n+1)^n e^{-\frac{(2n+1)}{2}} + n! 2^{n+1} \right)$. Then, by (3.17) and (3.26), the proof is done. ☐

Theorem 3.2. *The sums*

$$c^*_{k,\xi} := \frac{\sum\limits_{\nu=-\infty}^{\infty} \nu^k e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}, \; k = 1, ..., n, \qquad (3.27)$$

are finite for all $\xi \in (0,1]$. *Moreover,*

$$\left| P^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c^*_{k,\xi} \right| \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} G_n(\nu) e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}. \qquad (3.28)$$

Clearly the operators $P_{r,\xi}^{*}$ are not necessarily positive operators.

Proof. We observe that

$$\sum_{\nu=-\infty}^{\infty} \nu^k e^{\frac{-|\nu|}{\xi}} = \begin{cases} 0, & k \text{ is odd} \\ 2\sum\limits_{\nu=1}^{\infty} \nu^k e^{\frac{-\nu}{\xi}}, & k \text{ is even} \end{cases}. \tag{3.29}$$

Assume that k is even. Then, since

$$|\nu|^k \leq |\nu|^n$$

and

$$1+\frac{|\nu|}{\xi} > 1,$$

we obtain

$$\begin{aligned} &\sum_{\nu=-\infty}^{\infty} \nu^k e^{\frac{-|\nu|}{\xi}} \\ =& \sum_{\nu=-\infty}^{\infty} |\nu|^k e^{\frac{-|\nu|}{\xi}} \\ \leq& \sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}. \end{aligned} \tag{3.30}$$

Thus, by (3.30) and *Proposition 3.1*, we have

$$c_{k,\xi}^{*} \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} < \infty$$

for all $\xi \in (0,1]$. Therefore, by *Theorem 2.1*, we derive (3.28). □

For $n=0$, we have the following result

Corollary 3.3. *Let $f \in C_u(\mathbb{R})$. Then*

$$\left|P_{r,\xi}^{*}(f;x) - f(x)\right| \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} \omega_r(f,|\nu|) e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}. \tag{3.31}$$

Proof. By *Corollary 2.2*. □

Remark 3.4. Inequalities (3.28) and (3.31) give us the uniform estimates

$$\left\| P_{r,\xi}^{*}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi}^{*} \right\|_{\infty,x} \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} G_n(\nu) e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \tag{3.32}$$

and

$$\left\| P^*_{r,\xi}(f;x) - f(x) \right\|_{\infty,x} \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} \omega_r(f,|\nu|) e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \tag{3.33}$$

for $n = 0$.

Remark 3.5. By (2.12) and (2.13), we obtain

$$K_1^* := \left\| P^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c^*_{k,\xi} \right\|_{\infty,x} \tag{3.34}$$

$$\leq \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \right),$$

and

$$K_2^* := \left\| P^*_{r,\xi}(f;x) - f(x) \right\|_{\infty,x} \tag{3.35}$$

$$\leq \omega_r(f,\xi) \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \right).$$

Hence, by *Proposition 3.1*, for $f^{(n)} \in C_u(\mathbb{R})$, we have $K_1^* \to 0$ as $\xi \to 0^+$ and since

$$\frac{\sum\limits_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}},$$

by *Proposition 3.1*, for $f \in C_u(\mathbb{R})$, we get $K_2^* \to 0$ as $\xi \to 0^+$.

Based on *Remark 3.5, we have*

Theorem 3.6. *Let* $f \in C^n(\mathbb{R})$ *with* $f^{(n)} \in C_u(\mathbb{R})$, $n \in \mathbb{N}$. *Then*

$$\left\| P^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c^*_{k,\xi} \right\|_{\infty,x} \tag{3.36}$$

$$\leq \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \right).$$

Proof. By *Proposition 3.1* and *Remark 3.5.* □

Next, we present our results for generalized discrete Gauss-Weierstrass operators.

Proposition 3.7. *Let $n \in \mathbb{N}$. Then, there exists $K_2 > 0$ such that*

$$\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \leq K_2 < \infty \tag{3.37}$$

for all $\xi \in (0,1]$.

Proof. We observe that

$$\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}} > 1.$$

Thus

$$\frac{1}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} < 1.$$

Therefore, we have

$$\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} < \sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}. \tag{3.38}$$

On the other hand, since

$$\frac{\nu^2}{\xi} \geq \frac{|\nu|}{\xi},$$

we have

$$e^{\frac{-\nu^2}{\xi}} \leq e^{\frac{-|\nu|}{\xi}}. \tag{3.39}$$

Therefore, by (3.26), (3.38), and (3.39), we have

$$\begin{aligned} &\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \qquad (3.40) \\ < &\sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}} \\ = &\; R_1 < \infty \end{aligned}$$

for all $\xi \in (0,1]$. □

Theorem 3.8. *The sums*

$$p^*_{k,\xi} := \frac{\sum\limits_{\nu=-\infty}^{\infty} \nu^k e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}, \;\; k = 1, ..., n, \tag{3.41}$$

are finite. Furthermore,

$$\left|W_{r,\xi}^{*}\left(f;x\right)-f(x)-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_{k}p_{k,\xi}^{*}\right|\leq\frac{\sum\limits_{\nu=-\infty}^{\infty}G_{n}(\nu)e^{\frac{-\nu^{2}}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-\nu^{2}}{\xi}}}. \quad (3.42)$$

Clearly the operators $W_{r,\xi}^{*}$ *are not necessarily positive operators.*

Proof. We observe that

$$\sum_{\nu=-\infty}^{\infty}\nu^{k}e^{\frac{-\nu^{2}}{\xi}}=\begin{cases}0, & k \text{ is odd}\\ 2\sum\limits_{\nu=1}^{\infty}\nu^{k}e^{\frac{-\nu^{2}}{\xi}}, & k \text{ is even}\end{cases}. \quad (3.43)$$

Assume that k is even. Then, since

$$|\nu|^{k}\leq|\nu|^{n}$$

and

$$1+\frac{|\nu|}{\xi}>1,$$

we obtain

$$\begin{aligned}&\sum_{\nu=-\infty}^{\infty}\nu^{k}e^{\frac{-\nu^{2}}{\xi}} \quad (3.44)\\ =&\sum_{\nu=-\infty}^{\infty}|\nu|^{k}e^{\frac{-\nu^{2}}{\xi}}\\ \leq&\sum_{\nu=-\infty}^{\infty}|\nu|^{n}\left(1+\frac{|\nu|}{\xi}\right)^{r}e^{\frac{-\nu^{2}}{\xi}}.\end{aligned}$$

Thus, by (3.44) and *Proposition 3.7*, we have

$$p_{k,\xi}^{*}\leq\frac{\sum\limits_{\nu=-\infty}^{\infty}|\nu|^{n}\left(1+\frac{|\nu|}{\xi}\right)^{r}e^{\frac{-\nu^{2}}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-\nu^{2}}{\xi}}}<\infty$$

for all $\xi\in(0,1]$. Therefore, by *Theorem 2.1*, we derive (3.42). □

For $n=0$, we have the following result.

Corollary 3.9. *Suppose* $f\in C_{u}(\mathbb{R})$. *Then*

$$\left|W_{r,\xi}^{*}\left(f;x\right)-f(x)\right|\leq\frac{\sum\limits_{\nu=-\infty}^{\infty}\omega_{r}(f,|\nu|)e^{\frac{-\nu^{2}}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-\nu^{2}}{\xi}}}. \quad (3.45)$$

Proof. By *Corollary 2.2*. □

Remark 3.10. Inequalities (3.42) and (3.45) give us the uniform estimates

$$\left\| W_{r,\xi}^{*}\left(f;x\right)-f(x)-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_{k}p_{k,\xi}^{*}\right\|_{\infty,x}\leq\frac{\sum\limits_{\nu=-\infty}^{\infty}G_{n}(\nu)e^{\frac{-\nu^{2}}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-\nu^{2}}{\xi}}} \tag{3.46}$$

and

$$\left\| W_{r,\xi}^{*}\left(f;x\right)-f(x)\right\|_{\infty,x}\leq\frac{\sum\limits_{\nu=-\infty}^{\infty}\omega_{r}(f,|\nu|)e^{\frac{-\nu^{2}}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-\nu^{2}}{\xi}}}. \tag{3.47}$$

Remark 3.11. By (2.12) and (2.13), we obtain

$$\begin{aligned} M_{1}^{*} &:= \left\| W_{r,\xi}^{*}\left(f;x\right)-f(x)-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_{k}p_{k,\xi}^{*}\right\|_{\infty,x} \\ &\leq\frac{\omega_{r}(f^{(n)},\xi)}{n!}\left(\frac{\sum\limits_{\nu=-\infty}^{\infty}|\nu|^{n}\left(1+\frac{|\nu|}{\xi}\right)^{r}e^{\frac{-\nu^{2}}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-\nu^{2}}{\xi}}}\right), \end{aligned} \tag{3.48}$$

and

$$\begin{aligned} M_{2}^{*} &:= \left\| W_{r,\xi}^{*}\left(f;x\right)-f(x)\right\|_{\infty,x} \\ &\leq\omega_{r}(f,\xi)\left(\frac{\sum\limits_{\nu=-\infty}^{\infty}\left(1+\frac{|\nu|}{\xi}\right)^{r}e^{\frac{-\nu^{2}}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-\nu^{2}}{\xi}}}\right). \end{aligned} \tag{3.49}$$

Hence, by *Proposition 3.7*, for $f^{(n)}\in C_u(\mathbb{R})$, we have $M_1^*\to 0$ as $\xi\to 0^+$ and since

$$\frac{\sum\limits_{\nu=-\infty}^{\infty}\left(1+\frac{|\nu|}{\xi}\right)^{r}e^{\frac{-\nu^{2}}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-\nu^{2}}{\xi}}}\leq\frac{\sum\limits_{\nu=-\infty}^{\infty}|\nu|^{n}\left(1+\frac{|\nu|}{\xi}\right)^{r}e^{\frac{-\nu^{2}}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-\nu^{2}}{\xi}}},$$

by *Proposition 3.7*, for $f\in C_u(\mathbb{R})$, we get $M_2^*\to 0$ as $\xi\to 0^+$.

By previous *Remark 3.11*, we have

Theorem 3.12. *Let $f \in C^n(\mathbb{R})$ with $f^{(n)} \in C_u(\mathbb{R})$, $n \in \mathbb{N}$. Then*

$$\left\| W_{r,\xi}^{*}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k p_{k,\xi}^{*} \right\|_{\infty,x} \tag{3.50}$$
$$\leq \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \right).$$

Proof. By *Proposition 3.7* and *Remark 3.11.* □

Now, we present our results for generalized discrete Poisson-Cauchy operators.

Proposition 3.13. *Let $n \in \mathbb{N}$, $\beta > \frac{n+r+1}{2\alpha}$, and $\alpha \in \mathbb{N}$. Then, there exists $K_3 > 0$ such that*

$$\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \leq K_3 < \infty \tag{3.51}$$

for all $\xi \in (0,1]$.

Proof. We have

$$\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta} \tag{3.52}$$
$$= \xi^{-2\alpha\beta} + 2\sum_{\nu=1}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}$$
$$\geq \xi^{-2\alpha\beta}.$$

Therefore

$$\frac{1}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \leq \xi^{2\alpha\beta}. \tag{3.53}$$

Hence, we get

$$\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \tag{3.54}$$
$$\leq \xi^{2\alpha\beta} \left[\sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta} \right]$$
$$= 2\sum_{\nu=1}^{\infty} \nu^n \left(\xi^{\frac{2\alpha\beta}{r}} + \nu\xi^{\frac{2\alpha\beta}{r}-1}\right)^r \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}.$$

We notice that

$$\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta} \leq \nu^{-2\alpha\beta}. \tag{3.55}$$

Thus, by (3.54) and (3.55), we obtain

$$\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \tag{3.56}$$

$$\leq 2\sum_{\nu=1}^{\infty} \nu^{n-2\alpha\beta}\left(\xi^{\frac{2\alpha\beta}{r}}+\nu\xi^{\frac{2\alpha\beta}{r}-1}\right)^r \leq 2\sum_{\nu=1}^{\infty} \nu^{n-2\alpha\beta}\left(1+\nu\right)^r$$

$$\leq 2\sum_{\nu=1}^{\infty} \frac{2^r\nu^r}{\nu^{2\alpha\beta-n}} \leq 2^{r+1}\sum_{\nu=1}^{\infty}\left(\frac{1}{\nu}\right)^{2\alpha\beta-n-r} < \infty$$

for all $\xi \in (0,1]$. □

Theorem 3.14. *The sums*

$$q_{k,\xi}^{*} := \frac{\sum\limits_{\nu=-\infty}^{\infty} \nu^k \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}},\ k=1,...,n, \tag{3.57}$$

are finite where $\beta > \frac{n+r+1}{2\alpha}$ *and* $\alpha \in \mathbb{N}$. *Moreover,*

$$\left|\Theta_{r,\xi}^{*}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k q_{k,\xi}^{*}\right| \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} G_n(\nu)\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}. \tag{3.58}$$

Clearly the operators $\Theta_{r,\xi}^{*}$ *are not necessarily positive operators.*

Proof. We observe that

$$\sum_{\nu=-\infty}^{\infty} \nu^k \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta} = \begin{cases} 0, & k \text{ is odd} \\ 2\sum\limits_{\nu=1}^{\infty} \nu^k\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}, & k \text{ is even} \end{cases}. \tag{3.59}$$

Assume that k is even. Then, since

$$|\nu|^k \leq |\nu|^n \quad \text{and} \quad 1+\frac{|\nu|}{\xi} > 1,$$

we obtain

$$\sum_{\nu=-\infty}^{\infty} \nu^k \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta} = \sum_{\nu=-\infty}^{\infty} |\nu|^k \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta} \tag{3.60}$$

$$\leq \sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}.$$

Thus, by (3.60) and *Proposition 3.13*, we have

$$q_{k,\xi}^{*} \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^{n} \left(1+\frac{|\nu|}{\xi}\right)^{r} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} < \infty$$

for all $\xi \in (0,1]$. Therefore, by *Theorem 2.1*, we derive (3.58). □

For $n=0$, we have following result.

Corollary 3.15. *Suppose $f \in C_u(\mathbb{R})$. Then*

$$\left|\Theta_{r,\xi}^{*}(f;x) - f(x)\right| \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} \omega_r(f,|\nu|) \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}. \tag{3.61}$$

Proof. By *Corollary 2.2*. □

Remark 3.16. Inequalities (3.58) and (3.61) give us the uniform estimates

$$\left\|\Theta_{r,\xi}^{*}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k q_{k,\xi}^{*}\right\|_{\infty,x} \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} G_n(\nu) \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \tag{3.62}$$

and

$$\left\|\Theta_{r,\xi}^{*}(f;x) - f(x)\right\|_{\infty,x} \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} \omega_r(f,|\nu|) \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}. \tag{3.63}$$

Remark 3.17. By (2.12) and (2.13), we obtain

$$\begin{aligned} F_1^{*} &:= \left\|\Theta_{r,\xi}^{*}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k q_{k,\xi}^{*}\right\|_{\infty,x} \qquad (3.64) \\ &\leq \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^{n} \left(1+\frac{|\nu|}{\xi}\right)^{r} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}\right), \end{aligned}$$

and

$$\begin{aligned} F_2^{*} &:= \left\|\Theta_{r,\xi}^{*}(f;x) - f(x)\right\|_{\infty,x} \qquad (3.65) \\ &\leq \omega_r(f,\xi) \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^{r} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}\right). \end{aligned}$$

Hence, by *Proposition 3.13*, for $f^{(n)} \in C_u(\mathbb{R})$, we have $F_1^* \to 0$ as $\xi \to 0^+$ and since

$$\frac{\sum\limits_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \leq \frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}},$$

by *Proposition 3.13*, for $f \in C_u(\mathbb{R})$, we get $F_2^* \to 0$ as $\xi \to 0^+$.

As a conclusion, we state

Theorem 3.18. *Let $f \in C^n(\mathbb{R})$ with $f^{(n)} \in C_u(\mathbb{R})$, $n \in \mathbb{N}$, and $\beta > \frac{n+r+1}{2\alpha}$. Then, we have*

$$\left\| \Theta_{r,\xi}^*\left(f;x\right) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k q_{k,\xi}^* \right\|_{\infty,x} \tag{3.66}$$

$$\leq \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \right).$$

Proof. By *Proposition 3.13* and *Remark 3.17.* □

Remark 3.19. Let μ be a positive finite Borel measure on $\mathbb{R}$ with mass m, i.e. $\mu(\mathbb{R}) = m$. And let $f, g_1, g_2 : \mathbb{R} \to \mathbb{R}$ be Borel measurable functions, $x \in \mathbb{R}$. We observe that

$$\int_{\mathbb{R}} g_1 d\mu + \int_{\mathbb{R}} g_2 d\mu - f(x) \tag{3.67}$$

$$= \int_{\mathbb{R}} g_1 d\mu + \int_{\mathbb{R}} g_2 d\mu - f(x) - mf(x) + mf(x)$$

$$= \int_{\mathbb{R}} g_1 d\mu + \int_{\mathbb{R}} g_2 d\mu - mf(x) + f(x)\left(m-1\right).$$

Hence, it holds

$$\left| \int_{\mathbb{R}} g_1 d\mu + \int_{\mathbb{R}} g_2 d\mu - f(x) \right| \tag{3.68}$$

$$\leq \left| \int_{\mathbb{R}} g_1 d\mu + \int_{\mathbb{R}} g_2 d\mu - mf(x) \right| + |f(x)|\,|m-1|$$

$$\leq m \left| \int_{\mathbb{R}} g_1 \frac{d\mu}{m} + \int_{\mathbb{R}} g_2 \frac{d\mu}{m} - f(x) \right| + |f(x)|\,|m-1|.$$

That is

$$\left|\int_{\mathbb{R}} g_1 d\mu + \int_{\mathbb{R}} g_2 d\mu - f(x)\right| \tag{3.69}$$

$$\leq \quad m\left|\int_{\mathbb{R}} g_1 \frac{d\mu}{m} + \int_{\mathbb{R}} g_2 \frac{d\mu}{m} - f(x)\right| + |f(x)|\,|m-1|,$$

where now $\frac{\mu}{m}$ is a probability measure on $\mathbb{R}$.

We prove that $m_{\xi,P} \to 1$ and $m_{\xi,W} \to 1$ as $\xi \to 0^+$. We observe that the function $g(\nu) = e^{-\frac{\nu}{\xi}}$ is positive, continuous,and decreasing for $\nu \geq 1$. Thus, by [5], we have

$$\int_1^\infty e^{-\frac{\nu}{\xi}} d\nu \leq \sum_{\nu=1}^\infty e^{-\frac{\nu}{\xi}} \leq e^{-\frac{1}{\xi}} + \int_1^\infty e^{-\frac{\nu}{\xi}} d\nu. \tag{3.70}$$

Thus,

$$1 + 2\int_1^\infty e^{-\frac{\nu}{\xi}} d\nu \leq \sum_{\nu=-\infty}^\infty e^{-\frac{|\nu|}{\xi}} \leq 1 + 2e^{-\frac{1}{\xi}} + 2\int_1^\infty e^{-\frac{\nu}{\xi}} d\nu. \tag{3.71}$$

Since $\int_1^\infty e^{-\frac{\nu}{\xi}} d\nu = \xi e^{-\frac{1}{\xi}}$, we obtain

$$1 + 2\xi e^{-\frac{1}{\xi}} \leq \sum_{\nu=-\infty}^\infty e^{-\frac{|\nu|}{\xi}} \leq 1 + 2e^{-\frac{1}{\xi}} + 2\xi e^{-\frac{1}{\xi}}. \tag{3.72}$$

We have $1 + 2\xi e^{-\frac{1}{\xi}} \to 1$ and $1 + 2e^{-\frac{1}{\xi}} + 2\xi e^{-\frac{1}{\xi}} \to 1$ as $\xi \to 0^+$. Therefore,

$$\sum_{\nu=-\infty}^\infty e^{-\frac{|\nu|}{\xi}} \to 1 \text{ as } \xi \to 0^+. \tag{3.73}$$

Thus,

$$m_{\xi,P} = \frac{\sum_{\nu=-\infty}^\infty e^{-\frac{|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \to 1 \text{ as } \xi \to 0^+. \tag{3.74}$$

Now, define the function $h(\nu) = e^{-\frac{\nu^2}{\xi}}$ for $\nu \geq 1$. Observe that $h(\nu)$ is positive, continuous, and decreasing for $\nu \geq 1$. Then, by [5], we have

$$\int_1^\infty e^{-\frac{\nu^2}{\xi}} d\nu \leq \sum_{\nu=1}^\infty e^{-\frac{\nu^2}{\xi}} \leq e^{-\frac{1}{\xi}} + \int_1^\infty e^{-\frac{\nu^2}{\xi}} d\nu. \tag{3.75}$$

Thus,

$$1 + 2\int_1^\infty e^{-\frac{\nu^2}{\xi}} d\nu \leq \sum_{\nu=-\infty}^\infty e^{-\frac{\nu^2}{\xi}} \leq 1 + 2e^{-\frac{1}{\xi}} + 2\int_1^\infty e^{-\frac{\nu^2}{\xi}} d\nu. \tag{3.76}$$

As in [2], we have

$$2\int_{1}^{\infty} e^{-\frac{\nu^2}{\xi}}\,d\nu = \sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right). \tag{3.77}$$

Therefore,

$$\begin{aligned} 1+\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) &\le \sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}} \\ &\le 1+2e^{-\frac{1}{\xi}}+\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right). \end{aligned} \tag{3.78}$$

We have $1+\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)\to 1$ and $1+2e^{-\frac{1}{\xi}}+\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)\to 1$ as $\xi\to 0^+$. Hence,

$$\sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}} \to 1 \text{ as } \xi\to 0^+. \tag{3.79}$$

Thus,

$$m_{\xi,W} = \frac{\sum\limits_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}}}{1+\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)} \to 1 \text{ as } \xi\to 0^+. \tag{3.80}$$

We define the following error quantities:

$$\begin{aligned} E_{0,P}(f,x) &:= P_{r,\xi}(f;x)-f(x) \\ &= \frac{\sum\limits_{\nu=-\infty}^{\infty}\left(\sum\limits_{j=0}^{r}\alpha_j f(x+j\nu)\right)e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}} - f(x), \end{aligned} \tag{3.81}$$

$$\begin{aligned} E_{0,W}(f,x) &:= W_{r,\xi}(f;x)-f(x) \\ &= \frac{\sum\limits_{\nu=-\infty}^{\infty}\left(\sum\limits_{j=0}^{r}\alpha_j f(x+j\nu)\right)e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} - f(x). \end{aligned} \tag{3.82}$$

Furthermore, we define the errors ($n\in\mathbb{N}$):

$$E_{n,P}(f,x) := P_{r,\xi}(f;x)-f(x)-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_k\frac{\sum\limits_{\nu=-\infty}^{\infty}\nu^k e^{-\frac{|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}} \tag{3.83}$$

and

$$E_{n,W}(f,x) := W_{r,\xi}(f;x)-f(x)-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_k\frac{\sum\limits_{\nu=-\infty}^{\infty}\nu^k e^{-\frac{\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}. \tag{3.84}$$

Next, working as in inequality (3.69) to the errors $E_{0,P}$, $E_{0,W}$, $E_{n,P}$, and $E_{n,W}$, we obtain

$$|E_{0,P}(f,x)| \leq m_{\xi,P} \left| P^*_{r,\xi}(f;x) - f(x) \right| + |f(x)| \, |m_{\xi,P} - 1| \tag{3.85}$$

and

$$|E_{0,W}(f,x)| \leq m_{\xi,W} \left| W^*_{r,\xi}(f;x) - f(x) \right| + |f(x)| \, |m_{\xi,W} - 1| \,. \tag{3.86}$$

Furthermore, we obtain $(n \in \mathbb{N})$:

$$\begin{aligned} & |E_{n,P}(f,x)| \\ \leq \;& m_{\xi,P} \left| P^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c^*_{k,\xi} \right| + |f(x)| \, |m_{\xi,P} - 1| \end{aligned} \tag{3.87}$$

and

$$\begin{aligned} & |E_{n,W}(f,x)| \\ \leq \;& m_{\xi,W} \left| W^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k p^*_{k,\xi} \right| + |f(x)| \, |m_{\xi,W} - 1| \,. \end{aligned} \tag{3.88}$$

Based on *Remark 3.19*, we derive

Theorem 3.20. *It holds*

$$|E_{n,P}(f,x)| \leq \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} G_n(\nu) e^{-\frac{|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \right) + |f(x)| \, |m_{\xi,P} - 1| \,. \tag{3.89}$$

Clearly, the operators $P_{r,\xi}(f;x)$ are not necessarily positive operators.

Proof. By (3.28) and (3.87). □

For $n = 0$,we have the following result

Corollary 3.21. *Let $f \in C_u(\mathbb{R})$. Then*

$$|E_{0,P}(f,x)| \leq \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \omega_r(f, |\nu|) e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \right) + |f(x)| \, |m_{\xi,P} - 1| \,. \tag{3.90}$$

Proof. By (3.31) and (3.85). □

We have also the following result

Theorem 3.22. *Let $f \in C^n(\mathbb{R})$ with $f^{(n)} \in C_u(\mathbb{R})$, $n \in \mathbb{N}$,and $\|f\|_{\infty,\mathbb{R}} < \infty$. Then*

$$\begin{aligned} & \|E_{n,P}(f,x)\|_{\infty,x} \\ \leq \;& \frac{\omega_r(f^{(n)}, \xi)}{n!} \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \right) + \|f\|_{\infty,\mathbb{R}} \, |m_{\xi,P} - 1| \,. \end{aligned} \tag{3.91}$$

Proof. By (3.36) and (3.87). □

Next, we present our results for $E_{0,W}(f,x)$ and $E_{n,W}(f,x)$.

Theorem 3.23. *It holds*

$$|E_{n,W}(f,x)| \leq \left(\frac{\sum_{\nu=-\infty}^{\infty} G_n(\nu) e^{-\frac{\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right) + |f(x)| \, |m_{\xi,W}-1| . \tag{3.92}$$

Clearly, the operators $W_{r,\xi}(f;x)$ are not necessarily positive operators.

Proof. By (3.42) and (3.88). □

For $n=0$, we have following result

Corollary 3.24. *Let $f \in C_u(\mathbb{R})$. Then*

$$|E_{0,W}(f,x)| \leq \left(\frac{\sum_{\nu=-\infty}^{\infty} \omega_r(f,|\nu|) e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right) + |f(x)| \, |m_{\xi,W}-1| . \tag{3.93}$$

Proof. By (3.45) and (3.86). □

We have also the following result

Theorem 3.25. *Let $f \in C^n(\mathbb{R})$ with $f^{(n)} \in C_u(\mathbb{R})$, $n \in \mathbb{N}$, and $\|f\|_{\infty,\mathbb{R}} < \infty$. Then*

$$\|E_{n,W}(f,x)\|_{\infty,x} \tag{3.94}$$
$$\leq \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right) + \|f\|_{\infty,\mathbb{R}} \, |m_{\xi,W}-1| .$$

Proof. By (3.50) and (3.88). □

Conclusion. All of our results presented above imply the higher order of approximation with rates of discrete singular linear operators $P^*_{r,\xi}$, $W^*_{r,\xi}$, $\Theta^*_{r,\xi}$, $P_{r,\xi}$, and $W_{r,\xi}$ to the unit operator I, as $\xi \to 0^+$. Our convergences are pointwise and uniform.

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Partial hyperbolic implicit differential equations with variable times impulses

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Abstract. In this paper we investigate the existence of solutions for the initial value problems (IVP for short), for a class of functional hyperbolic impulsive implicit differential equations with variable time impulses involving the mixed regularized fractional derivative. Our works will be considered by using Schaefer's fixed point.

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Keywords: Partial hyperbolic differential equation, fractional order, left-sided mixed Riemann-Liouville integral, mixed regularized derivative, impulse, variable times, solution, fixed point.

1. Introduction

The subject of fractional calculus is as old as the differential calculus since, starting from some speculations of G.W. Leibniz (1697) and L. Euler (1730), it has been developed up to nowadays. Fractional calculus techniques are widely used in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc. [18, 23]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [6, 7], Kilbas *et al.* [19], Miller and Ross [22], Samko *et al.* [24], the papers of Abbas and Benchohra [1, 2, 3, 4], Abbas *et al.* [5, 8, 9], Benchohra *et al.* [13], Vityuk and Golushkov [26], and the references therein.

The theory of impulsive differential equations have become important in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulse theory in recent years, especially in the area of impulsive differential equations and inclusions with fixed moments; see the monographs of Benchohra *et al.* [12], Lakshmikantham *et al.* [20], the papers of Abbas *et al.* [2, 3, 5] and the references therein. The theory of impulsive differential equations with variable time is relatively less developed due to the difficulties created by the state-dependent impulses. Some interesting extensions to impulsive differential equations with variable times have been done by Bajo and Liz [10], Abbas and Benchohra [1, 2], Benchohra *et al.* [12], Frigon and O'Regan [14, 15, 16], Lakshmikantham *et al.* [21], Vityuk [25], Vityuk and Golushkov [26], Vityuk and Mykhailenko [27, 28] and the references cited therein.

In the present article we are concerning by the existence of solutions to fractional order IVP for the system

$$\overline{D}_{\theta_k}^{r} u(x,y) = f(x,y,u(x,y),\overline{D}_{\theta_k}^{r} u(x,y)); \quad \text{if } (x,y) \in J_k, \\ x \neq x_k(u(x,y)),\ k = 0,\dots,m, \tag{1.1}$$

$$u(x^+,y) = I_k(u(x,y)); \quad \text{if } (x,y) \in J,\ x = x_k(u(x,y)),\ k = 1,\dots,m, \tag{1.2}$$

$$\begin{cases} u(x,0) = \varphi(x);\ x \in [0,a], \\ u(0,y) = \psi(y);\ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{1.3}$$

where $a,b > 0$, $J := [0,a] \times [0,b]$, $J_0 = [0,x_1] \times (0,b]$, $J_k := (x_k, x_{k+1}] \times (0,b]$; $k = 1,...,m$, $\theta_k = (x_k, 0)$, $\overline{D}_{\theta_k}^{r}$ is the mixed regularized derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$, $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = a$, $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1,\dots,m$, $\varphi \in AC([0,a])$ and $\psi \in AC([0,b])$.
In the present article, we present an existence result based on Schaefer's fixed point.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J)$ we denote the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|w\|_\infty = \sup_{(x,y)\in J} \|w(x,y)\|,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$.
As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into $\mathbb{R}^n$ and $L^1(J)$ is the space of Lebegue-integrable functions $w : J \to \mathbb{R}^n$ with the norm

$$\|w\|_1 = \int_0^a \int_0^b \|w(x,y)\| dy dx.$$

Definition 2.1. [19, 24] *Let $\alpha \in (0,\infty)$ and $u \in L^1(J)$. The partial Riemann-Liouville integral of order α of $u(x,y)$ with respect to x is defined by the expression*

$$I_{0,x}^{\alpha} u(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s,y) ds, \ \textit{for a.a. } x \in [0,a] \textit{ and all } y \in [0,b],$$

where $\Gamma(.)$ is the (Euler's) Gamma function defined by

$$\Gamma(\varsigma) = \int_0^\infty t^{\varsigma-1} e^{-t} dt;\ \varsigma > 0.$$

Analogously, we define the integral

$$I_{0,y}^{\alpha}u(x,y)=\frac{1}{\Gamma(\alpha)}\int_0^y (y-s)^{\alpha-1}u(x,s)ds, \text{ for a.a. } x\in[0,a] \text{ and a.a. } y\in[0,b].$$

Definition 2.2. [19, 24] *Let $\alpha\in(0,1]$ and $u\in L^1(J)$. The Riemann-Liouville fractional derivative of order α of $u(x,y)$ with respect to x is defined by*

$$(D_{0,x}^{\alpha}u)(x,y)=\frac{\partial}{\partial x}I_{0,x}^{1-\alpha}u(x,y), \text{ for a.a. } x\in[0,a] \text{ and a.a. } y\in[0,b].$$

Analogously, we define the derivative

$$(D_{0,y}^{\alpha}u)(x,y)=\frac{\partial}{\partial y}I_{0,y}^{1-\alpha}u(x,y), \text{ for a.a. } x\in[0,a] \text{ and a.a. } y\in[0,b].$$

Definition 2.3. [19, 24] *Let $\alpha\in(0,1]$ and $u\in L^1(J)$. The Caputo fractional derivative of order α of $u(x,y)$ with respect to x is defined by the expression*

$${}^cD_{0,x}^{\alpha}u(x,y)=I_{0,x}^{1-\alpha}\frac{\partial}{\partial x}u(x,y), \text{ for a.a. } x\in[0,a] \text{ and a.a. } y\in[0,b].$$

Analogously, we define the derivative

$${}^cD_{0,y}^{\alpha}u(x,y)=I_{0,y}^{1-\alpha}\frac{\partial}{\partial y}u(x,y), \text{ for a.a. } x\in[0,a] \text{ and a.a. } y\in[0,b].$$

Definition 2.4. [26] *Let $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, $\theta=(0,0)$ and $u\in L^1(J)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by*

$$(I_{\theta}^{r}u)(x,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}u(s,t)dtds.$$

In particular,

$$(I_{\theta}^{\theta}u)(x,y)=u(x,y),\ (I_{\theta}^{\sigma}u)(x,y)=\int_0^x\int_0^y u(s,t)dtds; \text{ for a.a. } (x,y)\in J,$$

where $\sigma=(1,1)$.
For instance, $I_{\theta}^{r}u$ exists for all $r_1,r_2\in(0,\infty)$, when $u\in L^1(J)$. Note also that when $u\in C(J)$, then $(I_{\theta}^{r}u)\in C(J)$, moreover

$$(I_{\theta}^{r}u)(x,0)=(I_{\theta}^{r}u)(0,y)=0;\ x\in[0,a],\ y\in[0,b].$$

Example 2.5. Let $\lambda,\omega\in(0,\infty)$ and $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, then

$$I_{\theta}^{r}x^{\lambda}y^{\omega}=\frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)}x^{\lambda+r_1}y^{\omega+r_2}, \text{ for a.a. } (x,y)\in J.$$

By $1-r$ we mean $(1-r_1,1-r_2)\in[0,1)\times[0,1)$. Denote by $D_{xy}^2:=\frac{\partial^2}{\partial x\partial y}$, the mixed second order partial derivative.

Definition 2.6. [26] *Let $r\in(0,1]\times(0,1]$ and $u\in L^1(J)$. The mixed fractional Riemann-Liouville derivative of order r of u is defined by the expression $D_{\theta}^{r}u(x,y)=(D_{xy}^2I_{\theta}^{1-r}u)(x,y)$ and the Caputo fractional-order derivative of order r of u is defined by the expression ${}^cD_{\theta}^{r}u(x,y)=(I_{\theta}^{1-r}D_{xy}^2u)(x,y)$.*

The case $\sigma = (1,1)$ is included and we have

$$(D_\theta^\sigma u)(x,y) = ({}^cD_\theta^\sigma u)(x,y) = (D_{xy}^2 u)(x,y), \text{ for a.a. } (x,y) \in J.$$

Example 2.7. Let $\lambda, \omega \in (0,\infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$, then

$$D_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2}, \text{ for a.a. } (x,y) \in J.$$

Definition 2.8. [28] *For a function $u : J \to \mathbb{R}^n$, we set*

$$q(x,y) = u(x,y) - u(x,0) - u(0,y) + u(0,0).$$

By the mixed regularized derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$ of a function $u(x,y)$, we name the function

$$\overline{D}_\theta^r u(x,y) = D_\theta^r q(x,y).$$

The function

$$\overline{D}_{0,x}^{r_1} u(x,y) = D_{0,x}^{r_1}[u(x,y) - u(0,y)],$$

is called the partial r_1-order regularized derivative of the function $u : J \to \mathbb{R}^n$ with respect to the variable x. Analogously, we define the derivative

$$\overline{D}_{0,y}^{r_2} u(x,y) = D_{0,y}^{r_2}[u(x,y) - u(x,0)].$$

Let $a_1 \in [0,a],\ z^+ = (a_1, 0) \in J,\ J_z = [a_1, a] \times [0,b],\ r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $u \in L^1(J_z, \mathbb{R}^n)$, the expression

$$(I_{z^+}^r u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} u(s,t)\,dtds,$$

is called the left-sided mixed Riemann-Liouville integral of order r of u.

Definition 2.9. [26]. *For $u \in L^1(J_z, \mathbb{R}^n)$ where $D_{xy}^2 u$ is Lebesque integrable on $J_k;\ k = 0, \ldots, m$, the Caputo fractional-order derivative of order r of u is defined by the expression $({}^cD_{z^+}^r f)(x,y) = (I_{z^+}^{1-r} D_{xy}^2 f)(x,y)$. The Riemann-Liouville fractional-order derivative of order r of u is defined by $(D_{z^+}^r f)(x,y) = (D_{xy}^2 I_{z^+}^{1-r} f)(x,y)$.*

Analogously, we define the derivatives

$$\overline{D}_{z^+}^r u(x,y) = D_{z^+}^r q(x,y),$$

$$\overline{D}_{a_1,x}^{r_1} u(x,y) = D_{a_1,x}^{r_1}[u(x,y) - u(0,y)],$$

and

$$\overline{D}_{a_1,y}^{r_2} u(x,y) = D_{a_1,y}^{r_2}[u(x,y) - u(x,0)].$$

3. Existence of solutions

To define the solutions of problems (1.1)-(1.3), we shall consider the space

$$\Omega = \big\{u : J \to \mathbb{R}^n : \text{ there exist } 0 = x_0 < x_1 < x_2 < ... < x_m < x_{m+1} = a$$
$$\text{such that } x_k = x_k(u(x_k, .)), \text{ and } u(x_k^-, .),\ u(x_k^+, .) \text{ exist with}$$
$$u(x_k^-, .) = u(x_k, .);\ k = 0, \ldots, m, \text{ and } u \in C(J_k); k = 0, \ldots, m\big\}.$$

This set is a Banach space with the norm

$$\|u\|_\Omega = \max\{\|u_k\|;\ k = 0, ..., m\},$$

where u_k is the restriction of u to $J_k;\ k = 0, ..., m$.

Definition 3.1. *A function $u \in \Omega \cap (\cup_{k=0}^m AC(J_k))$ such that $u(x,y)$, $\overline{D}_{x_k,x}^{r_1} u(x,y)$, $\overline{D}_{x_k,y}^{r_2} u(x,y)$, $\overline{D}_{z_k^+}^{r} u(x,y)$; $k = 0, \ldots, m$, are continuous for $(x,y) \in J_k$ and $I_{z^+}^{1-r} u(x,y) \in AC(J_k)$ is said to be a solution of (1.1)-(1.3) if u satisfies equation (1.1) on J_k, and conditions (1.2), (1.3) are satisfied.*

For the existence of solutions for the problem (1.1)-(1.3) we need the following lemmas

Lemma 3.2. [28] *Let a function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Then problem*

$$\overline{D}_\theta^r u(x,y) = f(x,y,u(x,y),\overline{D}_\theta^r u(x,y));\ \text{if } (x,y) \in J := [0,a] \times [0,b], \tag{3.1}$$

$$\begin{cases} u(x,0) = \varphi(x);\ x \in [0,a], \\ u(0,y) = \psi(y);\ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{3.2}$$

is equivalent to the equation

$$g(x,y) = f(x,y,\mu(x,y) + I_\theta^r g(x,y), g(x,y)), \tag{3.3}$$

and if $g \in C(J)$ is the solution of (3.3), then $u(x,y) = \mu(x,y) + I_\theta^r g(x,y)$, where

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

Lemma 3.3. [2] *Let $0 < r_1, r_2 \le 1$ and let $h : J \to \mathbb{R}^n$ be continuous. A function u is a solution of the fractional integral equation*

$$u(x,y) = \begin{cases} \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds; \\ \quad \text{if } (x,y) \in J_0, \\ \varphi(x) + I_k(u(x_k,y)) - I_k(u(x_k,0)) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds; \\ \quad \text{if } (x,y) \in J_k;\ k = 1, \ldots, m, \end{cases} \tag{3.4}$$

if and only if u is a solution of the fractional IVP

$$^cD_{\theta_k}^r u(x,y) = h(x,y); \quad (x,y) \in J_k;\ k = 0, \ldots, m, \tag{3.5}$$

$$u(x_k^+, y) = I_k(u(x_k,y)); \quad y \in [0,b];\ k = 1, \ldots, m. \tag{3.6}$$

By Lemmas 3.2 and 3.3, we conclude the following Lemma

Lemma 3.4. *Let a function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Then problem (1.1)-(1.3) is equivalent to the equation*

$$g(x,y) = f(x,y,\xi(x,y),g(x,y)), \tag{3.7}$$

where

$$\xi(x,y) = \begin{cases} \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}h(s,t)dtds; \\ \quad \text{if } (x,y) \in J_0, \\ \varphi(x) + I_k(u(x_k,y)) - I_k(u(x_k,0)) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}h(s,t)dtds; \\ \quad \text{if } (x,y) \in J_k;\ k = 1,\ldots,m, \end{cases}$$

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

And if $g \in C(J)$ is the solution of (3.7), then $u(x,y) = \xi(x,y)$.

Theorem 3.5. (Schaefer) [17] *Let X be a Banach space and $N : X \to X$ completely continuous operator. If the set*

$$E(N) = \{u \in X : \ u = \lambda N(u) \text{ for some } \lambda \in [0,1]\}$$

is bounded, then N has fixed points.

Further, we present conditions for the existence of solutions of problem (1.1)-(1.3).

Theorem 3.6. *Assume*

(H_1) *The function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous,*
(H_2) *For any $u,v,w,z \in \mathbb{R}^n$ and $(x,y) \in J$, there exist constants $M > 0$ such that*

$$\|f(x,y,u,z)\| \leq M(1 + \|u\| + \|z\|),$$

(H_3) *The function $x_k \in C^1(\mathbb{R}^n, \mathbb{R})$ for $k = 1,\ldots,m$. Moreover,*

$$0 = x_0(u) < x_1(u) < \ldots < x_m(u) < x_{m+1}(u) = a; \quad \text{for all } u \in \mathbb{R}^n,$$

(H_4) *There exists a constant $M^* > 0$ such that $\|I_k(u)\| \leq M^*$; for each $u \in \mathbb{R}^n$ and $k = 1,\ldots,m$,*
(H_5) *For all $u \in \mathbb{R}^n$, $x_k(I_k(u)) \leq x_k(u) < x_{k+1}(I_k(u))$; for $k = 1,\ldots,m$,*
(H_6) *For all $(s,t,u) \in J \times \mathbb{R}^n$ and $k = 1,\ldots,m$, we have*

$$x_k'(u)\Big[\varphi'(s) + \frac{r_1 - 1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^s \int_0^t (s-\theta)^{r_1-2}(t-\eta)^{r_2-1} g(\theta,\eta)d\eta d\theta\Big] \neq 1,$$

where

$$g(x,y) = f(x,y,u(x,y),g(x,y));\ (x,y) \in J.$$

If

$$M + \frac{Ma^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{3.8}$$

then (1.1)-(1.3) has at least one solution on J.

Proof. The proof will be given in several steps.
Step 1. Consider the following problem

$$\overline{D}_\theta^r u(x,y) = f(x,y,u(x,y),\overline{D}_\theta^r u(x,y)); \text{ if } (x,y) \in J, \tag{3.9}$$

$$u(x,0) = \varphi(x),\ u(0,y) = \psi(y);\ x \in [0,a],\ y \in [0,b], \varphi(0) = \psi(0). \tag{3.10}$$

Transform problem (3.9)-(3.10) into a fixed point problem. Consider the operator $N : C(J) \to C(J)$ defined by

$$(Nu)(x,y) = \mu(x,y) + I_\theta^r g(x,y),$$

where $g \in C(J)$ such that

$$g(x,y) = f(x,y,u(x,y),g(x,y)).$$

Lemma 3.2 implies that the fixed points of operator N are solutions of problem (3.9)-(3.10). We shall show that the operator N is continuous and completely continuous.
Claim 1. *N is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in $C(J)$. Let $\eta > 0$ be such that $\|u_n\| \le \eta$. Then

$$\|(Nu_n)(x,y) - (Nu)(x,y)\| \le \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \times \|g_n(s,t) - g(s,t)\| dtds, \tag{3.11}$$

where $g_n, g \in C(J)$ such that

$$g_n(x,y) = f(x,y,u_n(x,y),g_n(x,y))$$

and

$$g(x,y) = f(x,y,u(x,y),g(x,y)).$$

Since $u_n \to u$ as $n \to \infty$ and f is a continuous function, we get

$$g_n(x,y) \to g(x,y) \text{ as } n \to \infty, \text{ for each } (x,y) \in J.$$

Hence, (3.11) gives

$$\|(Nu_n) - (Nu)\|_\infty \le \frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\|g_n - g\|_\infty \to 0 \text{ as } n \to \infty.$$

Claim 2. *N maps bounded sets into bounded sets in $C(J)$.*
Indeed, it is enough show that for any $\eta^* > 0$, there exists a positive constant $\ell^* > 0$ such that, for each

$$u \in B_{\eta^*} = \{u \in C(J) : \|u\|_\infty \le \eta^*\},$$

we have $\|N(u)\|_\infty \le \ell^*$. For $(x,y) \in J$, we have

$$\|(Nu)(x,y)\| \le \|\mu(x,y)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|g(s,t)\|dtds, \tag{3.12}$$

where $g \in C(J)$ such that

$$g(x,y) = f(x,y,u(x,y),g(x,y)).$$

By (H_2) we have for each $(x,y)\in J$,

$$\begin{aligned}\|g(x,y)\| &\leq M(1+\|\mu(x,y)+I_\theta^r g(x,y)\|+\|g(x,y)\|)\\ &\leq M\Big(1+\|\mu\|_\infty+\frac{a^{r_1}b^{r_2}\|g(x,y)\|}{\Gamma(1+r_1)\Gamma(1+r_2)}\Big)+M\|g(x,y)\|.\end{aligned}$$

Then, by (3.8) we get

$$\|g(x,y)\| \leq \frac{M(1+\|\mu\|_\infty)}{1-M-\frac{Ma^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}:=\ell.$$

Thus, (3.12) implies that

$$\|N(u)\|_\infty\leq\|\mu\|_\infty+\frac{\ell a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}:=\ell^*.$$

Claim 3. *N maps bounded sets into equicontinuous sets in $C(J)$.*
Let $(x_1,y_1),(x_2,y_2)\in J$, $x_1<x_2$, $y_1<y_2$, B_{η^*} be a bounded set of $C(J)$ as in Claim 2, and let $u\in B_{\eta^*}$. Then

$$\begin{aligned}&\|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)\|\\ \leq\ &\|\mu(x_2,y_2)-\mu(x_1,y_1)\|\\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{x_1}\int_0^{y_1}[(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\\ &-(x_1-s)^{r_1-1}(y_1-t)^{r_2-1}]\|g(s,t)\|dtds\\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|dtds\\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{x_1}\int_{y_1}^{y_2}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|dtds\\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_0^{y_1}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|dtds,\end{aligned}$$

where $g\in C(J)$ such that

$$g(x,y)=f(x,y,u(x,y),g(x,y)).$$

But $\|g\|_\infty\leq\ell$. Thus

$$\begin{aligned}&\|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)\|\\ \leq\ &\|\mu(x_2,y_2)-\mu(x_1,y_1)\|\\ +\ &\frac{\ell}{\Gamma(1+r_1)\Gamma(1+r_2)}[2y_2^{r_2}(x_2-x_1)^{r_1}+2x_2^{r_1}(y_2-y_1)^{r_2}\\ +\ &x_1^{r_1}y_1^{r_2}-x_2^{r_1}y_2^{r_2}-2(x_2-x_1)^{r_1}(y_2-y_1)^{r_2}].\end{aligned}$$

As $x_1\to x_2$, $y_1\to y_2$ the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that N is continuous and completely continuous.

Claim 4. *A priori bounds.*
Now it remains to show that the set

$$\mathcal{E} = \{u \in C(J) : u = \lambda N(u) \ \text{ for some } \ 0 < \lambda < 1\}$$

is bounded. Let $u \in \mathcal{E}$, then $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus, for each $(x, y) \in J$, we have

We now show there exists an open set $U \subseteq C(J)$ with $u \neq \lambda N(u)$, for $\lambda \in (0, 1)$ and $u \in \partial U$. Let $u \in C(J)$ and $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus for each $(x, y) \in J$, we have

$$u(x,y) = \lambda\mu(x,y) + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s,t)dtds.$$

This implies by (H_2) and as in Claim 2 that, for each $(x, y) \in J$, we get $\|u\|_\infty \leq \ell^*$. This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 3.5, we deduce that N has a fixed point which is a solution of the problem (3.9)-(3.10). Denote this solution by u_1. Define the function

$$r_{k,1}(x,y) = x_k(u_1(x,y)) - x, \qquad \text{for } x \geq 0, y \geq 0.$$

Hypothesis (H_3) implies that $r_{k,1}(0,0) \neq 0$ for $k = 1, \ldots, m$.
If $r_{k,1}(x, y) \neq 0$ on J for $k = 1, \ldots, m$; i.e.,

$$x \neq x_k(u_1(x,y)), \quad \text{on } J \quad \text{for } k = 1, \ldots, m,$$

then u_1 is a solution of the problem (1.1)-(1.3).
It remains to consider the case when $r_{1,1}(x, y) = 0$ for some $(x, y) \in J$. Now since $r_{1,1}(0, 0) \neq 0$ and $r_{1,1}$ is continuous, there exists $x_1 > 0, y_1 > 0$ such that $r_{1,1}(x_1, y_1) = 0$, and $r_{1,1}(x, y) \neq 0$, for all $(x, y) \in [0, x_1) \times [0, y_1)$.
Thus by (H_6) we have

$$r_{1,1}(x_1,y_1) = 0 \text{ and } r_{1,1}(x,y) \neq 0, \text{ for all } (x,y) \in [0,x_1) \times [0,y_1] \cup (y_1,b].$$

Suppose that there exist $(\bar{x}, \bar{y}) \in [0, x_1) \times [0, y_1] \cup (y_1, b]$ such that $r_{1,1}(\bar{x}, \bar{y}) = 0$. The function $r_{1,1}$ attains a maximum at some point $(s, t) \in [0, x_1) \times [0, b]$.
Since

$$\overline{D}_\theta^r u_1(x,y) = f(x,y,u_1(x,y), \overline{D}_\theta^r u_1(x,y)), \text{ for } (x,y) \in J,$$

then

$$\frac{\partial u_1(x,y)}{\partial x} \text{ exists, and } \frac{\partial r_{1,1}(s,t)}{\partial x} = x_1'(u_1(s,t)) \frac{\partial u_1(s,t)}{\partial x} - 1 = 0.$$

Since

$$\frac{\partial u_1(x,y)}{\partial x} = \varphi'(x) + \frac{r_1 - 1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-2}(y-t)^{r_2-1} g_1(s,t)dtds,$$

where

$$g_1(x,y) = f(x,y,u_1(x,y), g_1(x,y)); \ (x,y) \in J.$$

Then

$$x_1'(u_1(s,t))[\varphi'(s) + \frac{r_1-1}{\Gamma(r_1)\Gamma(r_2)} \int_0^s \int_0^t (s-\theta)^{r_1-2}(t-\eta)^{r_2-1} g_1(\theta,\eta)d\theta d\eta] = 1,$$

witch contradicts (H_6). From (H_3) we have

$$r_{k,1}(x,y) \neq 0 \text{ for all } (x,y) \in [0,x_1) \times [0,b] \text{ and } k = 1,...m.$$

Step 2. In what follows set

$$X_k := [x_k, a] \times [0,b]; \;\; k = 1,...,m.$$

Consider now the problem

$$\overline{D}_{\theta_1}^r u(x,y) = f(x,y,u(x,y),\overline{D}_{\theta_1}^r u(x,y)); \text{ if } (x,y) \in X_1, \tag{3.13}$$

$$u(x_1^+, y) = I_1(u_1(x_1,y)); \;\; y \in [0,b]. \tag{3.14}$$

Consider the operator $N_1 : C(X_1) \to C(X_1)$ defined as

$$\begin{aligned}(N_1 u) &= \varphi(x) + I_1(u_1(x_1,y)) - I_1(u_1(x_1,0)) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s,t)dtds,\end{aligned}$$

where

$$g(x,y) = f(x,y,u(x,y),g(x,y)); \text{ for } (x,y) \in X_1.$$

As in Step 1 we can show that N_1 is completely continuous. Now it remains to show that the set $\mathcal{E}^* = \{u \in C(X_1) : u = \lambda N_1(u) \text{ for some } 0 < \lambda < 1\}$ is bounded.
Let $u \in \mathcal{E}^*$, then $u = \lambda N_1(u)$ for some $0 < \lambda < 1$. Thus, from (H_2), (H_4) and the fact that $\|g\|_\infty \leq \ell$ we get for each $(x,y) \in X_1$,

$$\begin{aligned}\|u(x,y)\| &\leq \|\varphi(x)\| + \|I_1(u_1(x_1,y))\| + \|I_1(u_1(x_1,0))\| \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|g(s,t)\|dtds \\ &\leq \|\varphi\|_\infty + 2M^* + \frac{\ell a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} := \ell^{**}.\end{aligned}$$

This shows that the set $\mathcal{E}^*$ is bounded. As a consequence of of Theorem 3.5, we deduce that N_1 has a fixed point u which is a solution to problem (3.13)-(3.14). Denote this solution by u_2. Define

$$r_{k,2}(x,y) = x_k(u_2(x,y)) - x, \quad \text{for } (x,y) \in X_1.$$

If $r_{k,2}(x,y) \neq 0$ on $(x_1,a] \times [0,b]$ and for all $k = 1,\ldots,m$, then

$$u(x,y) = \begin{cases} u_1(x,y), & \text{if } (x,y) \in J_0, \\ u_2(x,y), & \text{if } (x,y) \in [x_1,a] \times [0,b], \end{cases}$$

is a solution of the problem (1.1)-(1.3). It remains to consider the case when $r_{2,2}(x,y) = 0$, for some $(x,y) \in (x_1,a] \times [0,b]$. By (H_5), we have

$$\begin{aligned} r_{2,2}(x_1^+, y_1) &= x_2(u_2(x_1^+, y_1) - x_1 \\ &= x_2(I_1(u_1(x_1,y_1))) - x_1 \\ &> x_1(u_1(x_1,y_1)) - x_1 \\ &= r_{1,1}(x_1,y_1) = 0. \end{aligned}$$

Since $r_{2,2}$ is continuous, there exists $x_2 > x_1,\ y_2 > y_1$ such that $r_{2,2}(x_2, y_2) = 0$, and $r_{2,2}(x, y) \neq 0$ for all $(x, y) \in (x_1, x_2) \times [0, b]$.
It is clear by (H_3) that

$$r_{k,2}(x, y) \neq 0 \quad \text{for all } (x, y) \in (x_1, x_2)] \times [0, b];\ k = 2, \dots, m.$$

Now suppose that there are $(s, t) \in (x_1, x_2) \times [0, b]$ such that $r_{1,2}(s, t) = 0$. From (H_5) it follows that

$$\begin{aligned} r_{1,2}(x_1^+, y_1) &= x_1(u_2(x_1^+, y_1) - x_1 \\ &= x_1(I_1(u_1(x_1, y_1))) - x_1 \\ &\leq x_1(u_1(x_1, y_1)) - x_1 \\ &= r_{1,1}(x_1, y_1) = 0. \end{aligned}$$

Thus $r_{1,2}$ attains a nonnegative maximum at some point $(s_1, t_1) \in (x_1, a) \times [0, x_2) \cup (x_2, b]$.
Since

$$\overline{D}_{\theta_1}^r u_2(x, y) = f(x, y, u_2(x, y), \overline{D}_{\theta_1}^r u_2(x, y));\ \text{for } (x, y) \in X_1,$$

then we get

$$\begin{aligned} u_2(x, y) \quad &= \quad \varphi(x) + I_1(u_1(x_1, y)) - I_1(u_1(x_1, 0)) \\ &+ \quad \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x} \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} g_2(s, t) dt ds, \end{aligned}$$

where

$$g_2(x, y) = f(x, y, u_2(x, y), g_2(x, y));\ \text{for } (x, y) \in X_1.$$

Hence

$$\frac{\partial u_2}{\partial x}(x, y) = \varphi'(x) + \frac{r_1 - 1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x} \int_0^y (x - s)^{r_1 - 2}(y - t)^{r_2 - 1} g_2(s, t) dt ds,$$

then

$$\frac{\partial r_{1,2}(s_1, t_1)}{\partial x} = x_1'(u_2(s_1, t_1)) \frac{\partial u_2}{\partial x}(s_1, t_1) - 1 = 0.$$

Therefore

$$x_1'(u_2(s_1, t_1))[\varphi'(s_1) + \frac{r_1 - 1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{s_1} \int_0^{t_1} (s_1 - \theta)^{r_1 - 2}(t_1 - \eta)^{r_2 - 1} g_2(\theta, \eta) d\eta d\theta] = 1,$$

which contradicts (H_6).

Step 3. We continue this process and take into account that $u_{m+1} := u\Big|_{X_m}$ is a solution to the problem

$$\begin{aligned} &\overline{D}_{\theta_m}^r u(x, y) = f(x, y, u(x, y), \overline{D}_{\theta_m}^r u(x, y)); \quad \text{a.e. } (x, y) \in (x_m, a] \times [0, b], \\ &u(x_m^+, y) = I_m(u_{m-1}(x_m, y));\ y \in [0, b]. \end{aligned}$$

The solution u of the problem (1.1)-(1.3) is then defined by

$$u(x,y)=\begin{cases} u_1(x,y), & \text{if } (x,y)\in J_0,\\ u_2(x,y), & \text{if } (x,y)\in J_1\\ \dots & \\ u_{m+1}(x,y), & \text{if } (x,y)\in J_m. \end{cases}$$

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Some differential superordination results using a generalized Sălăgean operator and Ruscheweyh operator

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Abstract. In the present paper we establish several differential superordinations regarding the operator DR_λ^m defined by using Ruscheweyh derivative $R^m f(z)$ and the generalized Sălăgean operator $D_\lambda^m f(z)$, $DR_\lambda^m : \mathcal{A}_n \to \mathcal{A}_n$, $DR_\lambda^m f(z) = (D_\lambda^m * R^m) f(z)$, $z \in U$, where $m, n \in \mathbb{N}$, $\lambda \geq 0$ and $f \in \mathcal{A}_n$, $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j, \ z \in U\}$. A number of interesting consequences of some of these superordination results are discussed. Relevant connections of some of the new results obtained in this paper with those in earlier works are also provided.

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1. Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots, \ z \in U\}$ and $\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots, \ z \in U\}$ for $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

Denote by $K = \left\{ f \in \mathcal{A}_n : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U \right\}$, the class of normalized convex functions in U.

If f and g are analytic functions in U, we say that f is superordinate to g, written $g \prec f$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $g(z) = f(w(z))$ for all $z \in U$. If f is univalent, then $g \prec f$ if and only if $f(0) = g(0)$ and $g(U) \subseteq f(U)$.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi(p(z), zp'(z); z)$ are univalent in U and satisfies the (first-order) differential superordination

$$h(z) \prec \psi(p(z), zp'(z); z), \qquad z \in U, \tag{1.1}$$

then p is called a solution of the differential superordination. The analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if $q \prec p$ for all p satisfying (1.1).

An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.1) is said to be the best subordinant of (1.1). The best subordinant is unique up to a rotation of U.

Definition 1.1. (Al Oboudi [5]) *For $f \in \mathcal{A}_n$, $\lambda \geq 0$ and $n, m \in \mathbb{N}$, the operator D_λ^m is defined by $D_\lambda^m : \mathcal{A}_n \to \mathcal{A}_n$,*

$$\begin{aligned} D_\lambda^0 f(z) &= f(z) \\ D_\lambda^1 f(z) &= (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z) \ ,..., \\ D_\lambda^{m+1} f(z) &= (1-\lambda) D_\lambda^m f(z) + \lambda z \left(D_\lambda^m f(z)\right)' = D_\lambda \left(D_\lambda^m f(z)\right), \quad z \in U. \end{aligned}$$

Remark 1.2. If $f \in \mathcal{A}_n$ and $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then

$$D_\lambda^m f(z) = z + \sum_{j=n+1}^{\infty} \left[1 + (j-1)\lambda\right]^m a_j z^j, \ z \in U.$$

For $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator [8].

Definition 1.3. (Ruscheweyh [7]) *For $f \in \mathcal{A}_n$, $n, m \in \mathbb{N}$, the operator R^m is defined by $R^m : \mathcal{A}_n \to \mathcal{A}_n$,*

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \ ,..., \\ (m+1) R^{m+1} f(z) &= z \left(R^m f(z)\right)' + m R^m f(z), \quad z \in U. \end{aligned}$$

Remark 1.4. If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then

$$R^m f(z) = z + \sum_{j=n+1}^{\infty} C_{m+j-1}^m a_j z^j, \ \in U.$$

Definition 1.5. ([1]) *Let $\lambda \geq 0$ and $m, n \in \mathbb{N}$. Denote by $DR_\lambda^m : \mathcal{A}_n \to \mathcal{A}_n$ the operator given by the Hadamard product (the convolution product) of the generalized Sălăgean operator D_λ^m and the Ruscheweyh operator R^m:*

$$DR_\lambda^m f(z) = \left(D_\lambda^m f * R^m f\right)(z),$$

for any $z \in U$ and each nonnegative integer m.

Remark 1.6. If $f \in A$ and $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then

$$DR_\lambda^m f(z) = z + \sum_{j=n+1}^{\infty} \frac{(m+j-1)!}{m!\,(j-1)!} \left[1 + (j-1)\lambda\right]^m a_j^2 z^j, \text{ for } z \in U.$$

Remark 1.7. The operator DR_λ^m was studied in [2], [3], [4].

Definition 1.8. *We denote by Q the set of functions that are analytic and injective on $\overline{U} \backslash E(f)$, where $E(f) = \{\zeta \in \partial U : \lim\limits_{z \to \zeta} f(z) = \infty\}$, and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.*

We will use the following lemmas.

Lemma 1.9. (Miller and Mocanu [6, Th. 3.1.6, p. 71]) *Let h be a convex function with $h(0)=a$ and let $\gamma\in\mathbb{C}\backslash\{0\}$ be a complex number with $Re\ \gamma\geq 0$. If $p\in\mathcal{H}[a,n]\cap Q$, $p(z)+\frac{1}{\gamma}zp'(z)$ is univalent in U and $h(z)\prec p(z)+\frac{1}{\gamma}zp'(z)$, $z\in U$, then $q(z)\prec p(z)$, $z\in U$, where*

$$q(z)=\frac{\gamma}{nz^{\gamma/n}}\int_0^z h(t)t^{\gamma/n-1}dt,\ z\in U.$$

The function q is convex and is the best subordinant.

Lemma 1.10. (Miller and Mocanu [6]) *Let q be a convex function in U and let*

$$h(z)=q(z)+\frac{1}{\gamma}zq'(z),\ z\in U,$$

where $Re\ \gamma\geq 0$. If $p\in\mathcal{H}\left[a,n\right]\cap Q$, $p(z)+\frac{1}{\gamma}zp'(z)$ is univalent in U and

$$q(z)+\frac{1}{\gamma}zq'(z)\prec p(z)+\frac{1}{\gamma}zp'\left(z\right),\ z\in U,$$

then $q(z)\prec p(z)$, $z\in U$, where

$$q(z)=\frac{\gamma}{nz^{\gamma/n}}\int_0^z h(t)t^{\gamma/n-1}dt,\ z\in U.$$

The function q is the best subordinant.

2. Main results

Theorem 2.1. *Let h be a convex function, $h(0)=1$. Let $n,m\in\mathbb{N}$, $\lambda,\delta\geq 0$, $f\in\mathcal{A}_n$ and suppose that $\left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta-1}\left(DR_\lambda^m f(z)\right)'$ is univalent and $\left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta}\in\mathcal{H}\left[1,n\right]\cap Q$. If*

$$h(z)\prec\left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta-1}\left(DR_\lambda^m f(z)\right)',\qquad z\in U,\tag{2.1}$$

then

$$q(z)\prec\left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta},\ z\in U,$$

where

$$q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt.$$

The function q is convex and it is the best subordinant.

Proof. Consider

$$p(z)=\left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta}=\left(\frac{z+\sum_{j=n+1}^{\infty}\left[1+(j-1)\lambda\right]^m\frac{(m+j-1)!}{m!(j-1)!}a_j^2z^j}{z}\right)^{\delta}$$

$$=1+p_nz^n+p_{n+1}z^{n+1}+\ldots,\ z\in U.$$

Differentiating both sides of $p(z)$, we obtain

$$\left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta-1}(DR_\lambda^m f(z))' = p(z) + \frac{1}{\delta}zp'(z),\ z \in U.$$

Then (2.1) becomes $h(z) \prec p(z) + \frac{1}{\delta}zp'(z)$, $z \in U$. By using Lemma 1.9 for $\gamma = \delta$, we have $q(z) \prec p(z)$, $z \in U$, i.e.,

$$q(z) \prec \left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta},\ z \in U,$$

where

$$q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt.$$

The function q is convex and it is the best subordinant. □

Corollary 2.2. *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ be a convex function in U, where $0 \le \beta < 1$. Let $n, m \in \mathbb{N}$, $\lambda, \delta \ge 0$, $f \in \mathcal{A}_n$ and suppose that $\left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta-1}(DR_\lambda^m f(z))'$ is univalent and $\left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta} \in \mathcal{H}[1,n] \cap Q$. If*

$$h(z) \prec \left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta-1}(DR_\lambda^m f(z))', \qquad z \in U, \tag{2.2}$$

then $q(z) \prec \left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta}$, $z \in U$, where q is given by

$$q(z) = 2\beta - 1 + \frac{2(1-\beta)\delta}{nz^{\frac{\delta}{n}}}\int_0^z \frac{t^{\frac{\delta}{n}-1}}{1+t}dt,\ z \in U.$$

The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.1 and considering

$$p(z) = \left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta},$$

the differential superordination (2.2) becomes

$$h(z) = \frac{1+(2\beta-1)z}{1+z} \prec p(z) + \frac{z}{\delta}p'(z),\ z \in U.$$

By using Lemma 1.9 for $\gamma = \delta$, we have $q(z) \prec p(z)$, i.e.,

$$\begin{aligned} q(z) &= \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z t^{\frac{\delta}{n}-1}\frac{1+(2\beta-1)t}{1+t}dt \\ &= \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z \left[(2\beta-1)t^{\frac{\delta}{n}-1} + 2(1-\beta)\frac{t^{\frac{\delta}{n}-1}}{1+t}\right]dt \\ &= (2\beta-1) + \frac{2(1-\beta)\delta}{nz^{\frac{\delta}{n}}}\int_0^z \frac{t^{\frac{\delta}{n}-1}}{1+t}dt \prec \left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta},\ z \in U. \end{aligned}$$

The function q is convex and it is the best subordinant. □

Theorem 2.3. *Let q be convex in U and let h be defined by*

$$h(z) = q(z) + \frac{z}{\delta} q'(z).$$

If $n, m \in \mathbb{N}$, $\lambda, \delta \geq 0$, $f \in \mathcal{A}_n$, suppose that $\left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta-1} (DR_\lambda^m f(z))'$ is univalent and $\left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta} \in \mathcal{H}[1,n] \cap Q$ and satisfies the differential superordination

$$h(z) = q(z) + \frac{z}{\delta} q'(z) \prec \left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta-1} (DR_\lambda^m f(z))', \quad z \in U, \tag{2.3}$$

then

$$q(z) \prec \left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta}, \; z \in U,$$

where

$$q(z) = \frac{\delta}{n z^{\frac{\delta}{n}}} \int_0^z h(t) t^{\frac{\delta}{n}-1} dt.$$

The function q is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.1 and considering

$$p(z) = \left(\frac{DR_\lambda^m f(z)}{z}\right)^{\delta},$$

the differential superordination (2.3) becomes

$$q(z) + \frac{z}{\delta} q'(z) \prec p(z) + \frac{z}{\delta} p'(z), \; z \in U.$$

Using Lemma 1.10 for $\gamma = \delta$, we have $q(z) \prec p(z)$, $z \in U$, i.e.,

$$q(z) = \frac{\delta}{n z^{\frac{\delta}{n}}} \int_0^z h(t) t^{\frac{\delta}{n}-1} dt \prec \left(\frac{RD_{\lambda,\alpha}^m f(z)}{z}\right)^{\delta}, \; z \in U,$$

and q is the best subordinant. □

Theorem 2.4. *Let h be a convex function, $h(0) = 1$. Let $\lambda, \delta \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that*

$$z \frac{\delta+1}{\delta} \frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1} f(z)\right)^2} + \frac{z^2}{\delta} \frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1} f(z)\right)^2} \left[\frac{(DR_\lambda^m f(z))'}{DR_\lambda^m f(z)} - 2 \frac{\left(DR_\lambda^{m+1} f(z)\right)'}{DR_\lambda^{m+1} f(z)}\right]$$

is univalent and $z \frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1} f(z)\right)^2} \in \mathcal{H}[1,n] \cap Q$. If

$$h(z) \prec z \frac{\delta+1}{\delta} \frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1} f(z)\right)^2}$$

$$+ \frac{z^2}{\delta} \frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1} f(z)\right)^2} \left[\frac{(DR_\lambda^m f(z))'}{DR_\lambda^m f(z)} - 2 \frac{\left(DR_\lambda^{m+1} f(z)\right)'}{DR_\lambda^{m+1} f(z)}\right], \; z \in U, \tag{2.4}$$

then

$$q(z) \prec z\frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1} f(z)\right)^2},\ z\in U,$$

where

$$q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}tdt.$$

The function q is convex and it is the best subordinant.

Proof. Consider

$$p(z)=z\frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1} f(z)\right)^2}$$

and we obtain

$$p(z)+\frac{z}{\delta}p'(z)=z\frac{\delta+1}{\delta}\frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1} f(z)\right)^2}$$

$$+\frac{z^2}{\delta}\frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1} f(z)\right)^2}\left[\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}-2\frac{\left(DR_\lambda^{m+1} f(z)\right)'}{DR_\lambda^{m+1} f(z)}\right].$$

Relation (2.4) becomes

$$h(z)\prec p(z)+\frac{z}{\delta}p'(z),\ z\in U.$$

By using Lemma 1.10 for $\gamma=\delta$, we have $q(z)\prec p(z)$, $z\in U$, i.e.,

$$q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}tdt \prec z\frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1} f(z)\right)^2},\ z\in U.$$

The function q is convex and it is the best subordinant. □

Theorem 2.5. *Let q be convex in U and let h be defined by*

$$h(z)=q(z)+\frac{z}{\delta}q'(z).$$

If $n,m\in\mathbb{N}$, $\lambda,\delta\geq 0$, $f\in\mathcal{A}_n$, suppose that

$$z\frac{\delta+1}{\delta}\frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1} f(z)\right)^2}+\frac{z^2}{\delta}\frac{DR_{\lambda\alpha}^m f(z)}{\left(DR_\lambda^{m+1} f(z)\right)^2}\left[\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}-2\frac{\left(DR_\lambda^{m+1} f(z)\right)'}{DR_\lambda^{m+1} f(z)}\right]$$

is univalent and $z\frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1} f(z)\right)^2}\in\mathcal{H}[1,n]\cap Q$ and satisfies the differential superordination

$$h(z)\prec z\frac{\delta+1}{\delta}\frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1} f(z)\right)^2}$$

$$+\frac{z^2}{\delta}\frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1} f(z)\right)^2}\left[\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}-2\frac{\left(DR_\lambda^{m+1} f(z)\right)'}{DR_\lambda^{m+1} f(z)}\right],\ z\in U, \tag{2.5}$$

then

$$q(z)\prec z\frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1} f(z)\right)^2},\ z\in U,$$

where

$$q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}tdt.$$

The function q is the best subordinant.

Proof. Let

$$p(z)=z\frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1}f(z)\right)^2},\ z\in U.$$

Differentiating, we obtain

$$p(z)+\frac{z}{\delta}p'(z)=z\frac{\delta+1}{\delta}\frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1}f(z)\right)^2}$$

$$+\frac{z^2}{\delta}\frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1}f(z)\right)^2}\left[\frac{\left(DR_\lambda^m f(z)\right)''}{DR_\lambda^m f(z)}-2\frac{\left(DR_\lambda^{m+1}f(z)\right)'}{DR_\lambda^{m+1}f(z)}\right],\ z\in U,$$

and (2.5) becomes

$$h(z)=q(z)+\frac{z}{\delta}q'(z)\prec p(z)+\frac{z}{\delta}p'(z),\ z\in U.$$

By using Lemma 1.10 for $\gamma=\delta$, we have $q(z)\prec p(z)$, $z\in U$, i.e.,

$$q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}tdt\prec z\frac{DR_\lambda^m f(z)}{\left(DR_\lambda^{m+1}f(z)\right)^2},\ z\in U,$$

and q is the best subordinant. □

Theorem 2.6. *Let h be a convex function in U with $h(0)=1$ and let $\lambda,\delta\geq 0$, $n,m\in\mathbb{N}$, $f\in\mathcal{A}_n$,*

$$z^2\frac{\delta+2}{\delta}\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}+\frac{z^3}{\delta}\left[\frac{\left(DR_\lambda^m f(z)\right)''}{DR_\lambda^m f(z)}-\left(\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}\right)^2\right]$$

is univalent and $z^2\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}\in\mathcal{H}[0,n]\cap Q$. If

$$h(z)\prec z^2\frac{\delta+2}{\delta}\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}+\frac{z^3}{\delta}\left[\frac{\left(DR_\lambda^m f(z)\right)''}{DR_\lambda^m f(z)}-\left(\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}\right)^2\right],\qquad z\in U,\tag{2.6}$$

then

$$q(z)\prec z^2\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)},\ z\in U,$$

where

$$q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt.$$

The function q is convex and it is the best subordinant.

Proof. Let
$$p(z)=z^2\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)},\ z\in U.$$
Differentiating, we obtain
$$z^2\frac{\delta+2}{\delta}\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}+\frac{z^3}{\delta}\left[\frac{\left(DR_\lambda^m f(z)\right)''}{DR_\lambda^m f(z)}-\left(\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}\right)^2\right]=p(z)+\frac{z}{\delta}p'(z),\ z\in U.$$
Using the notation in (2.6), the differential superordination becomes
$$h(z)\prec p(z)+\frac{z}{\delta}p'(z),\ z\in U.$$

By using Lemma 1.9 for $\gamma=\delta$, we have $q(z)\prec p(z)$, $z\in U$, i.e.,
$$q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt\prec z^2\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)},\ z\in U.$$
The function q is convex and it is the best subordinant. □

Theorem 2.7. *Let q be a convex function in U and*
$$h(z)=q(z)+\frac{z}{\delta}q'(z).$$
Let $\lambda,\delta\geq 0$, $n,m\in\mathbb{N}$, $f\in\mathcal{A}_n$, suppose that
$$z^2\frac{\delta+2}{\delta}\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}+\frac{z^3}{\delta}\left[\frac{\left(DR_\lambda^m f(z)\right)''}{DR_\lambda^m f(z)}-\left(\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}\right)^2\right]$$
is univalent in U and $z^2\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}\in\mathcal{H}[0,n]\cap Q$ and satisfies the differential superordination
$$h(z)\prec z^2\frac{\delta+2}{\delta}\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}+\frac{z^3}{\delta}\left[\frac{\left(DR_\lambda^m f(z)\right)''}{DR_\lambda^m f(z)}-\left(\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}\right)^2\right],\ z\in U,\tag{2.7}$$
then
$$q(z)\prec z^2\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)},\quad z\in U,$$
where
$$q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt.$$
The function q is the best subordinant.

Proof. Let
$$p(z)=z^2\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}.$$
Differentiating, we obtain
$$p(z)+\frac{z}{\delta}p'(z)=z^2\frac{\delta+2}{\delta}\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}+\frac{z^3}{\delta}\left[\frac{\left(DR_\lambda^m f(z)\right)''}{DR_\lambda^m f(z)}-\left(\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}\right)^2\right],\ z\in U.$$

Using the notation in (2.7), the differential superordination becomes

$$h(z) = q(z) + \frac{z}{\delta} q'(z) \prec p(z) + \frac{z}{\delta} p'(z).$$

By using Lemma 1.10 for $\gamma = \delta$ we have $q(z) \prec p(z)$, i.e.,

$$q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}} \int_0^z h(t) t^{\frac{\delta}{n}-1} dt \prec z^2 \frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}, \ z \in U.$$

The function q is the best subordinant. □

Theorem 2.8. *Let h be a convex function, $h(0) = 1$. Let $n, m \in \mathbb{N}$, $\lambda, \delta \geq 0$, $f \in \mathcal{A}_n$ and suppose that $1 - \frac{DR_{\lambda,}^m f(z) \cdot \left(DR_\lambda^m f(z)\right)''}{\left[\left(DR_\lambda^m f(z)\right)'\right]^2}$ is univalent and $\frac{DR_\lambda^m f(z)}{z\left(DR_\lambda^m f(z)\right)'} \in \mathcal{H}[1,n] \cap Q$. If*

$$h(z) \prec 1 - \frac{DR_\lambda^m f(z) \cdot \left(DR_\lambda^m f(z)\right)''}{\left[\left(DR_\lambda^m f(z)\right)'\right]^2}, \qquad z \in U, \tag{2.8}$$

then

$$q(z) \prec \frac{DR_\lambda^m f(z)}{z\left(DR_\lambda^m f(z)\right)'}, \ z \in U,$$

where q is given by

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt, \ z \in U.$$

The function q is convex and it is the best subordinant.

Proof. Let

$$p(z) = \frac{DR_\lambda^m f(z)}{z\left(DR_\lambda^m f(z)\right)'}, \ z \in U.$$

Differentiating, we obtain

$$1 - \frac{DR_{\lambda,}^m f(z) \cdot \left(DR_\lambda^m f(z)\right)''}{\left[\left(DR_\lambda^m f(z)\right)'\right]^2} = p(z) + zp'(z), \ z \in U,$$

and (2.8) becomes $h(z) \prec p(z) + zp'(z)$, $z \in U$.

Using Lemma 1.9 for $\gamma = 1$ we have $q(z) \prec p(z)$, $z \in U$, i.e.,

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt \prec \frac{DR_\lambda^m f(z)}{z\left(DR_\lambda^m f(z)\right)'}, \ z \in U.$$

The function q is convex and it is the best subordinant. □

Corollary 2.9. *Let*

$$h(z) = \frac{1 + (2\beta - 1)z}{1 + z}$$

be a convex function in U, where $0 \leq \beta < 1$. Let $n, m \in \mathbb{N}$, $\lambda, \delta \geq 0$, $f \in \mathcal{A}_n$ and suppose that $1 - \frac{DR_\lambda^m f(z) \cdot \left(DR_\lambda^m f(z)\right)''}{\left[\left(DR_\lambda^m f(z)\right)'\right]^2}$ is univalent and $\frac{DR_\lambda^m f(z)}{z\left(DR_\lambda^m f(z)\right)'} \in \mathcal{H}[1,n] \cap Q$. If

$$h(z) \prec 1 - \frac{DR_\lambda^m f(z) \cdot \left(DR_\lambda^m f(z)\right)''}{\left[\left(DR_\lambda^m f(z)\right)'\right]^2}, \qquad z \in U, \tag{2.9}$$

then

$$q(z) \prec \frac{DR_\lambda^m f(z)}{z\left(DR_\lambda^m f(z)\right)'}, \ z \in U,$$

where q is given by

$$q(z) = (2\beta - 1) + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}, \ z \in U.$$

The function q is convex and it is the best subordinant .

Proof. Following the same steps as in the proof of Theorem 2.8 and considering

$$p(z) = \frac{DR_\lambda^m f(z)}{z\left(DR_\lambda^m f(z)\right)'},$$

the differential subordination (2.9) becomes

$$h(z) = \frac{1 + (2\beta - 1)z}{1+z} \prec p(z) + zp'(z), \ z \in U.$$

By using Lemma 1.9 for $\gamma = 1$, we have $q(z) \prec p(z)$, i.e.,

$$\begin{aligned} q(z) &= \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt = \frac{1}{nz^{\frac{1}{n}}} \int_0^z \frac{1 + (2\beta-1)t}{1+t} t^{\frac{1}{n}-1} dt \\ &= \frac{1}{nz^{\frac{1}{n}}} \int_0^z t^{\frac{1}{n}-1} \left[(2\beta - 1) + \frac{2(1-\beta)}{1+t}\right] dt \\ &= (2\beta - 1) + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} \prec \frac{DR_\lambda^m f(z)}{z\left(DR_\lambda^m f(z)\right)'}, \ z \in U. \end{aligned}$$

□

Theorem 2.10. *Let q be convex in U and let h be defined by $h(z) = q(z) + zq'(z)$. If $n, m \in \mathbb{N}$, $\lambda, \delta \geq 0$, $f \in \mathcal{A}_n$, suppose that $1 - \frac{DR_\lambda^m f(z) \cdot \left(DR_\lambda^m f(z)\right)''}{\left[\left(DR_\lambda^m f(z)\right)'\right]^2}$ is univalent and $\frac{DR_\lambda^m f(z)}{z\left(DR_\lambda^m f(z)\right)'} \in \mathcal{H}[1, n] \cap Q$ and satisfies the differential superordination*

$$h(z) \prec 1 - \frac{DR_\lambda^m f(z) \cdot \left(DR_\lambda^m f(z)\right)''}{\left[\left(DR_\lambda^m f(z)\right)'\right]^2}, \quad z \in U, \tag{2.10}$$

then

$$q(z) \prec \frac{DR_\lambda^m f(z)}{z\left(DR_\lambda^m f(z)\right)'}, \ z \in U,$$

where q is given by

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt, \ z \in U.$$

The function q is the best subordinant.

Proof. Let

$$p(z) = \frac{DR_\lambda^m f(z)}{z\left(DR_\lambda^m f(z)\right)'}.$$

Differentiating, we obtain

$$1-\frac{DR_\lambda^m f(z)\cdot\left(DR_\lambda^m f(z)\right)''}{\left[\left(DR_\lambda^m f(z)\right)'\right]^2}=p(z)+zp'(z),\ z\in U,$$

and (2.10) becomes $h(z)=q(z)+zq'(z)\prec p(z)+zp'(z)$, $z\in U$.

Using Lemma 1.9 for $\gamma=1$, we have $q(z)\prec p(z)$, $z\in U$, i.e.,

$$q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt\prec\frac{DR_\lambda^m f(z)}{z\left(DR_\lambda^m f(z)\right)'},\ z\in U.$$

The function q is the best subordinant. □

Theorem 2.11. *Let h be a convex function, $h(0)=1$ and let $\lambda\geq 0$, $n,m\in\mathbb{N}$, $f\in\mathcal{A}_n$, suppose that $\left[\left(DR_\lambda^m f(z)\right)'\right]^2+DR_\lambda^m f(z)\cdot\left(DR_\lambda^m f(z)\right)''$ is univalent and $\frac{DR_\lambda^m f(z)\cdot\left(DR_\lambda^m f(z)\right)'}{z}\in\mathcal{H}[1,n]\cap Q$. If*

$$h(z)\prec\left[\left(DR_\lambda^m f(z)\right)'\right]^2+DR_\lambda^m f(z)\cdot\left(DR_\lambda^m f(z)\right)'',\ z\in U,\tag{2.11}$$

then

$$q(z)\prec\frac{DR_\lambda^m f(z)\cdot\left(DR_\lambda^m f(z)\right)'}{z},\ z\in U,$$

where

$$q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt.$$

The function q is convex and it is the best subordinant.

Proof. Let

$$p(z)=\frac{DR_\lambda^m f(z)\cdot\left(DR_\lambda^m f(z)\right)'}{z},\ z\in U.$$

Differentiating, we obtain

$$\left[\left(DR_\lambda^m f(z)\right)'\right]^2+DR_\lambda^m f(z)\cdot\left(DR_\lambda^m f(z)\right)''=p(z)+zp'(z),\ z\in U,$$

and (2.11) becomes $h(z)\prec p(z)+zp'(z)$, $z\in U$.

Using Lemma 1.9 for $\gamma=1$, we have $q(z)\prec p(z)$, $z\in U$, i.e.,

$$q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt\prec\frac{DR_\lambda^m f(z)\cdot\left(DR_\lambda^m f(z)\right)'}{z},\ z\in U.$$

The function q is convex and it is the best subordinant. □

Corollary 2.12. *Let*

$$h(z)=\frac{1+(2\beta-1)z}{1+z}$$

be a convex function in U, where $0\leq\beta<1$. Let $\lambda\geq 0$, $n,m\in\mathbb{N}$, $f\in\mathcal{A}_n$, suppose that $\left[\left(DR_\lambda^m f(z)\right)'\right]^2+DR_\lambda^m f(z)\cdot\left(DR_\lambda^m f(z)\right)''$ is univalent and

$$\frac{DR_\lambda^m f(z)\cdot\left(DR_\lambda^m f(z)\right)'}{z}\in\mathcal{H}[1,n]\cap Q.$$

If

$$h(z)\prec\left[\left(DR_\lambda^m f(z)\right)'\right]^2+DR_\lambda^m f(z)\cdot\left(DR_\lambda^m f(z)\right)'',\ z\in U,\tag{2.12}$$

then

$$q(z) \prec \frac{DR_\lambda^m f(z) \cdot (DR_\lambda^m f(z))'}{z}, \ z \in U,$$

where q is given by

$$q(z) = (2\beta - 1) + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}, \ z \in U.$$

The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.11 and considering

$$p(z) = \frac{DR_\lambda^m f(z) \cdot (DR_\lambda^m f(z))'}{z},$$

the differential superordination (2.12) becomes

$$h(z) = \frac{1 + (2\beta - 1)z}{1+z} \prec p(z) + zp'(z), \ z \in U.$$

By using Lemma 1.9 for $\gamma = 1$, we have $q(z) \prec p(z)$, i.e.,

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt = \frac{1}{nz^{\frac{1}{n}}} \int_0^z \frac{1 + (2\beta-1)t}{1+t} t^{\frac{1}{n}-1} dt$$

$$= \frac{1}{nz^{\frac{1}{n}}} \int_0^z t^{\frac{1}{n}-1} \left[(2\beta - 1) + \frac{2(1-\beta)}{1+t} \right] dt$$

$$= (2\beta - 1) + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} \prec \frac{DR_\lambda^m f(z) \cdot (DR_\lambda^m f(z))'}{z}, \ z \in U.$$

The function q is convex and it is the best subordinant. □

Theorem 2.13. *Let q be a convex function in U and h be defined by*

$$h(z) = q(z) + zq'(z).$$

Let $\lambda \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$, suppose that

$$\left[(DR_\lambda^m f(z))' \right]^2 + DR_\lambda^m f(z) \cdot (DR_\lambda^m f(z))''$$

is univalent and $\frac{DR_\lambda^m f(z) \cdot (DR_\lambda^m f(z))'}{z} \in \mathcal{H}[1,n] \cap Q$ *and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec \left[(DR_\lambda^m f(z))' \right]^2 + DR_\lambda^m f(z) \cdot (DR_\lambda^m f(z))'', \ z \in U, \quad (2.13)$$

then

$$q(z) \prec \frac{DR_\lambda^m f(z) \cdot (DR_\lambda^m f(z))'}{z}, \ z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.11 and considering

$$p(z)=\frac{DR_\lambda^m f(z)\cdot\left(DR_\lambda^m f(z)\right)'}{z},$$

the differential superordination (2.13) becomes

$$h(z)=q(z)+zq'(z)\prec p(z)+zp'(z),\ z\in U.$$

By using Lemma 1.10 for $\gamma=1$, we have $q(z)\prec p(z)$, i.e.,

$$q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt\prec\frac{DR_\lambda^m f(z)\cdot\left(DR_\lambda^m f(z)\right)'}{z},\ z\in U.$$

The function q is the best subordinant. □

Theorem 2.14. *Let h be a convex function, $h(0)=1$. Let $\lambda\geq 0$, $\delta\in(0,1)$, $n,m\in\mathbb{N}$, $f\in\mathcal{A}_n$, and suppose that*

$$\left(\frac{z}{DR_\lambda^m f(z)}\right)^\delta\frac{DR_\lambda^{m+1} f(z)}{1-\delta}\left(\frac{\left(DR_\lambda^{m+1} f(z)\right)'}{DR_\lambda^{m+1} f(z)}-\delta\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}\right)$$

is univalent and $\frac{DR_\lambda^{m+1} f(z)}{z}\cdot\left(\frac{z}{DR_\lambda^m f(z)}\right)^\delta\in\mathcal{H}[1,n]\cap Q$. *If*

$$h(z)\prec\left(\frac{z}{DR_\lambda^m f(z)}\right)^\delta\frac{DR_\lambda^{m+1} f(z)}{1-\delta}\left(\frac{\left(DR_\lambda^{m+1} f(z)\right)'}{DR_\lambda^{m+1} f(z)}-\delta\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}\right),\ z\in U, \tag{2.14}$$

then

$$q(z)\prec\frac{DR_\lambda^{m+1} f(z)}{z}\cdot\left(\frac{z}{DR_\lambda^m f(z)}\right)^\delta,\ z\in U,$$

where

$$q(z)=\frac{1-\delta}{nz^{\frac{1-\delta}{n}}}\int_0^z h(t)t^{\frac{1-\delta}{n}-1}dt.$$

The function q is convex and it is the best subordinant.

Proof. Let

$$p(z)=\frac{DR_\lambda^{m+1} f(z)}{z}\cdot\left(\frac{z}{DR_\lambda^m f(z)}\right)^\delta,\ z\in U.$$

Differentiating, we obtain

$$\left(\frac{z}{DR_\lambda^m f(z)}\right)^\delta\frac{DR_\lambda^{m+1} f(z)}{1-\delta}\left(\frac{\left(DR_\lambda^{m+1} f(z)\right)'}{DR_\lambda^{m+1} f(z)}-\delta\frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}\right)$$

$$=p(z)+\frac{1}{1-\delta}zp'(z),\ z\in U,$$

and (2.14) becomes

$$h(z)\prec p(z)+\frac{1}{1-\delta}zp'(z),\ z\in U.$$

Using Lemma 1.9, we have $q(z) \prec p(z)$, $z \in U$, i.e.,

$$q(z) = \frac{1-\delta}{nz^{\frac{1-\delta}{n}}} \int_0^z h(t) t^{\frac{1-\delta}{n}-1} dt \prec \frac{DR_\lambda^{m+1} f(z)}{z} \cdot \left(\frac{z}{DR_\lambda^m f(z)} \right)^\delta, \; z \in U.$$

The function q is convex and it is the best subordinant. □

Theorem 2.15. *Let q be a convex function in U and*

$$h(z) = q(z) + \frac{z}{1-\delta} q'(z).$$

If $\lambda \geq 0$, $\delta \in (0,1)$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$, suppose that

$$\left(\frac{z}{DR_\lambda^m f(z)} \right)^\delta \frac{DR_\lambda^{m+1} f(z)}{1-\delta} \left(\frac{\left(DR_\lambda^{m+1} f(z)\right)'}{DR_\lambda^{m+1} f(z)} - \delta \frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)} \right)$$

is univalent and $\frac{DR_\lambda^{m+1} f(z)}{z} \cdot \left(\frac{z}{DR_\lambda^m f(z)} \right)^\delta \in \mathcal{H}[1,n] \cap Q$ *satisfies the differential superordination*

$$h(z) \prec \left(\frac{z}{DR_\lambda^m f(z)} \right)^\delta \frac{DR_\lambda^{m+1} f(z)}{1-\delta} \left(\frac{\left(DR_\lambda^{m+1} f(z)\right)'}{DR_\lambda^{m+1} f(z)} - \delta \frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)} \right), \; z \in U, \tag{2.15}$$

then

$$q(z) \prec \frac{DR_\lambda^{m+1} f(z)}{z} \cdot \left(\frac{z}{DR_\lambda^m f(z)} \right)^\delta, \; z \in U,$$

where

$$q(z) = \frac{1-\delta}{nz^{\frac{1-\delta}{n}}} \int_0^z h(t) t^{\frac{1-\delta}{n}-1} dt.$$

The function q is the best subordinant.

Proof. Let

$$p(z) = \frac{DR_\lambda^{m+1} f(z)}{z} \cdot \left(\frac{z}{DR_\lambda^m f(z)} \right)^\delta.$$

Differentiating, we obtain

$$\left(\frac{z}{DR_\lambda^m f(z)} \right)^\delta \frac{DR_\lambda^{m+1} f(z)}{1-\delta} \left(\frac{\left(DR_\lambda^{m+1} f(z)\right)'}{DR_\lambda^{m+1} f(z)} - \delta \frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)} \right)$$

$$= p(z) + \frac{1}{1-\delta} z p'(z), \; z \in U.$$

Using the notation in (2.15), the differential superordination becomes

$$h(z) = q(z) + \frac{z}{1-\delta} q'(z) \prec p(z) + \frac{1}{1-\delta} z p'(z).$$

By using Lemma 1.10, we have $q(z) \prec p(z)$, $z \in U$, i.e.,

$$q(z) = \frac{1-\delta}{nz^{\frac{1-\delta}{n}}} \int_0^z h(t) t^{\frac{1-\delta}{n}-1} dt \prec \frac{DR_\lambda^{m+1} f(z)}{z} \cdot \left(\frac{z}{DR_\lambda^m f(z)} \right)^\delta, \; z \in U,$$

and q is the best subordinant. □

Remark 2.16. For $\lambda = 1$ we obtain the same results for the operator SR^n.

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Iterates of increasing linear operators, via Maia's fixed point theorem

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Abstract. Let X be a Banach lattice. In this paper we give conditions in which an increasing linear operator, $A : X \to X$ is weakly Picard operator (see I.A. Rus, *Picard operators and applications*, Sc. Math. Japonicae, 58(2003), No. 1, 191-219). To do this we introduce the notion of "invariant linear partition of X with respect to A" and we use contraction principle and Maia's fixed point theorem. Some applications are also given.

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1. Introduction

There are many techniques to study the iterates of a linear and of increasing linear operators:

(1) for linear operators on a Banach space see: [16], [22], [23], [25], ...
(2) for linear increasing operators on an ordered Banach space see: [4], [8], [11], [12], [21], [23], [38], ...
(3) for some classes of positive linear operators see: [1]-[6], [9], [13]-[15], [17]-[20], [27], [30], [33], [35], ...

In the paper [36] we studied the problem in terms of the following notions:

Definition 1.1. *Let X be a nonempty set and $A : X \to X$ be an operator with $F_A \neq \emptyset$, where $F_A := \{x \in X \mid A(x) = x\}$. By definition, a partition of X, $X = \bigcup_{x^* \in F_A} X_{x^*}$, is a fixed point partition of X with respect to A iff:*

(i) $A(X_{x^*}) \subset X_{x^*}$, $\forall\, x^* \in F_A$;
(ii) $F_A \cap X_{x^*} = \{x^*\}$, $\forall\, x^* \in F_A$.

Definition 1.2. *Let* $(X, +, \mathbb{R})$ *be a linear space and* $A : X \to X$ *be a linear operator with* $F_A \setminus \{\theta\} \neq \emptyset$*. By definition, a fixed point partition,* $X = \bigcup_{x^* \in F_A} X_{x^*}$ *is a* <u>*linear fixed point partition*</u> *of* X *with respect to* A *iff:*

$$X_{x^*} = \{x^*\} + X_\theta, \ \forall \ x^* \in F_A.$$

If there exists a norm on X_θ, $\|\cdot\| : X_\theta \to \mathbb{R}_+$, and $\|A(x)\| \leq l\|x\|$, for all $x \in X_\theta$ with some $l > 0$, then $d_{\|\cdot\|} : X_{x^*} \times X_{x^*} \to \mathbb{R}_+$, $d_{\|\cdot\|}(x, y) := \|x - y\|$ is a metric on X_{x^*} and the restriction of A to X_{x^*}, $A\big|_{X_{x^*}}$, is a Lipschitz operator with constant l. If $l < 1$, in this case we can use the following variant of contraction principle:

Weak contraction principle. *Let* (X, d) *be a metric space and* $A : X \to X$ *be an operator. We suppose that:*

(i) $F_A \neq \emptyset$;
(ii) A *is a* l*-contraction.*

Then:

(a) $F_A = \{x^*\}$;
(b) $A^n(x) \to x^*$ *as* $n \to \infty$, $\forall \ x \in X$, *i.e.,* A *is a Picard operator;*
(c) $d(x, x^*) \leq \frac{1}{1-l} d(x, A(x))$, $\forall \ x \in X$.

In this paper we do not suppose that $F_A \setminus \{\theta\} \neq \emptyset$. So, we introduce the following notion:

Definition 1.3. *Let* $(X, +, \mathbb{R}, \to)$ *be a linear* L*-space (see* [36]*) and* $A : X \to X$ *be a linear operator. By definition, a partition of* X*,* $X = \bigcup_{\lambda \in \Lambda} X_\lambda$*, is an* <u>*invariant linear partition*</u> *(ILP) of* X *with respect to* A *iff:*

(i) *there exists* $\lambda_0 \in \Lambda$ *such that* X_{λ_0} *is a linear subspace of* X *and*

$$X/_{X_{\lambda_0}} = \{X_\lambda \mid \lambda \in \Lambda\};$$

(ii) $A(X_\lambda) \subset X_\lambda$, $\forall \ \lambda \in \Lambda$;
(iii) $X_\lambda = \overline{X}_\lambda$, $\forall \ \lambda \in \Lambda$.

We also need the following fixed point result (see [28], [37], [24], ...):

Maia's fixed point theorem. *Let* X *be a nonempty set,* d *and* ρ *be two metrics on* X *and* $A : X \to X$ *be an operator. We suppose that:*

(i) *there exists* $c > 0$ *such that* $d(x, y) \leq c\rho(x, y)$, $\forall \ x, y \in X$;
(ii) (X, d) *is a complete metric space;*
(iii) $A : (X, d) \to (X, d)$ *is continuous;*
(iv) $A : (X, \rho) \to (X, \rho)$ *is an* l*-contraction.*

Then:

(a) $F_A = \{x^*\}$;
(b) $A : (X, d) \to (X, d)$ *is Picard operator;*
(c) $A : (X, \rho) \to (X, \rho)$ *is Picard operator;*
(d) $\rho(x, x^*) \leq \frac{1}{1-l} \rho(x, A(x))$, $\forall \ x \in X$.

The aim of this paper is to study the iterates of a linear operator and of an increasing linear operator on a Banach lattice in terms of an invariant partition of the space and using contraction principle and Maia's fixed point theorem.

2. Invariant linear partitions

In what follows we shall give some generic examples of ILP of the space.

Let $(X,+,\mathbb{R},\|\cdot\|)$ be a normed space, $A : X \to X$ be a linear operator and $(\Lambda,+,\mathbb{R},\tau)$ be a linear topological space and $\Phi : X \to \Lambda$ be a continuous linear and surjective operator. We suppose that Φ is an invariant operator of A (see [10], [4], [36], [26], ...), i.e., $\Phi(A(x)) = \Phi(x)$, $\forall\ x \in X$. For $\lambda \in \Lambda$, let

$$X_\lambda := \{x \in X \mid \Phi(x) = \lambda\}.$$

We remark that, $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$, is an ILP of X with respect to A. In this case, $\lambda_0 = \theta_\Lambda$.

Here are some examples:

Example 2.1. Let $\mathbb{B}$ be a Banach space, $K \in C([0,1]^2,\mathbb{R})$ and $A : C([0,1],\mathbb{B}) \to C([0,1],\mathbb{B})$ be defined by

$$A(x)(t) := x(0) + \int_0^t K(t,s)x(s)ds,\ \forall\ t \in [0,1].$$

Let $\Lambda := \mathbb{B}$ and $\Phi : C([0,1],\mathbb{B}) \to \mathbb{B}$, be defined by, $\Phi(x) = x(0)$. It is clear that Φ is invariant for A and, $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$, is an ILP of $(C[0,1],\mathbb{B})$ with respect to A. In this case $\lambda_0 = \theta_{\mathbb{B}}$.

Example 2.2. Let $A : C[0,1] \to C[0,1]$ be a continuous linear operator such that $A(x)(0) = x(0)$ and $A(x)(1) = x(1)$ (i.e., 0 and 1 are interpolation points of A (see [34] and the references therein)). Let $\Lambda := \mathbb{R}^2$ and $\Phi : C[0,1] \to \mathbb{R}^2$, $\Phi(x) = (x(0),x(1))$. Then Φ is invariant for A, $\lambda_0 = (0,0)$ and $C[0,1] = \bigcup\limits_{\lambda\in\mathbb{R}^2} X_\lambda$ is an ILP of $C[0,1]$ with respect to A.

Another generic example is the following:

Let $(X,+,\mathbb{R},\to)$ be a linear L-space and $A : X \to X$ be a linear operator. Let us consider the quotient space $X/\overline{(1-A)(X)} = \{X_\lambda \mid \lambda \in \Lambda\}$, with $X_{\lambda_0} := \overline{(1-A)(X)}$. From a remark by Jachymski (see Lemma 1 in [22]), $A(X_\lambda) \subset X_\lambda$. From the definition of quotient space it follows that, $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$ is an ILP of X with respect to A.

Remark 2.3. Let $(X,+,\mathbb{R},\to)$ be a linear L-space and $A : X \to X$ be a linear operator. Let $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$ be an ILP of X. Then $Y = \bigcup\limits_{\lambda\in\Lambda} \overline{A(X_\lambda)}$ is an ILP of Y with respect to the operator $A\big|_Y : Y \to Y$. We remark that, $F_A = F_{A|_Y}$.

3. Main results

Our abstract results are the following:

Theorem 3.1. *Let* $(X, +, \mathbb{R}, \|\cdot\|)$ *be a Banach space and* $A : X \to X$ *be a linear operator. Let* $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ *be an* ILP *of* X *with respect to* A*, with* X_{λ_0} *a linear subspace of* X*. We suppose that there exists* $l \in]0, 1[$ *such that*

$$\|A(x)\| \leq l\|x\|, \ \forall \ x \in X_{\lambda_0}.$$

Then:

(*a*) $F_A \cap X_\lambda = \{x_\lambda^*\}, \ \forall \ \lambda \in \Lambda$;
(*b*) $A^n(x) \to x_\lambda^*$ *as* $n \to \infty$, $\forall \ x \in X_\lambda, \ \lambda \in \Lambda$, *i.e.,* A *is weakly Picard operator* (*WPO*) *on* X *and* $A^\infty(x) = x_\lambda^*, \ \forall \ x \in X_\lambda$;
(*c*) $\|x - A^\infty(x)\| \leq \frac{1}{1-l}\|x - A(x)\|, \ \forall \ x \in X$.

Proof. Let $x, y \in X_\lambda$. Then, $x - y \in X_{\lambda_0}$ and

$$\|A(x) - A(y)\| = \|A(x - y)\| \leq l\|x - y\|.$$

From the contraction principle we have that $F_A \cap X_\lambda = \{x_\lambda^*\}$ and $A : X_\lambda \to X_\lambda$ is Picard operator. We also have that:

$$\|x - x_\lambda^*\| \leq \frac{1}{1-l}\|x - A(x)\|, \ \forall \ x \in X_\lambda.$$

From the definition of A^∞ it follows (c). □

Theorem 3.2. *Let* $(X, +, \mathbb{R}, \|\cdot\|, \leq)$ *be a Banach lattice and* $A : X \to X$ *be an increasing linear operator. We suppose that:*

(*i*) $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ *is an* ILP *of* X *with respect to* A*, with* X_{λ_0} *a linear subspace of* X;
(*ii*) *there exists an order unit element* $u \in X$ *for* X_{λ_0}*, such that*

$$A(u) \leq lu, \ \text{with some } 0 < l < 1.$$

Then:

(*a*) A *is* WPO *with respect to* $\stackrel{\|\cdot\|}{\to}$;
(*b*) $X_\lambda \cap F_A = \{x_\lambda^*\}, \ \forall \ \lambda \in \Lambda$;
(*c*) $A^\infty(x) = x_\lambda^*, \ \forall \ x \in X_\lambda, \ \lambda \in \Lambda$;
(*d*) A *is* WPO *with respect to* $\stackrel{d_{\|\cdot\|_u}}{\to}$, *where* $\|\cdot\|_u$ *is the Minkowski norm on* X_{λ_0} *with respect to* u*, i.e.,* $\|A^n(x) - A^\infty(x)\|_u \to 0$ *as* $n \to +\infty$;
(*e*) $\|x - A^\infty(x)\|_u \leq \frac{1}{1-l}\|x - A(x)\|_u, \ \forall \ x \in X$.

Proof. Let $x \in X_{\lambda_0}$. Since u is order unit for X_{λ_0}, there exists $M(x) > 0$ such that

$$|x| \leq M(x)u.$$

From the definition of Minkowski's norm, $\|\cdot\|_u : X_{\lambda_0} \to \mathbb{R}_+$, we have that

$$|x| \leq \|x\|_u u, \ \forall \ x \in X_{\lambda_0}. \quad (3.1)$$

Since X is a Banach lattice we also have that

$$\|x\| \le \|u\|\ \|x\|_u,\ \forall\ x \in X_{\lambda_0}. \tag{3.2}$$

But A is increasing linear operator. From (3.1) we have

$$|A(x)| \le A(|x|) \le \|x\|_u A(u) \le l\|x\|_u u.$$

From this relations it follows

$$\|A(x)\|_u \le l\|x\|_u,\ \forall\ x \in X_{\lambda_0}. \tag{3.3}$$

Now let $x, y \in X_\lambda$. Then, $x - y \in X_{\lambda_0}$ and from (3.3) we have

$$\|A(x) - A(y)\|_u \le l\|x - y\|_u.$$

On X_λ we have two metrics, $d_{\|\cdot\|}(x,y) := \|x-y\|$ and $d_{\|\cdot\|_u}(x,y) := \|x-y\|_u$. So, by the above considerations, $(X_\lambda, d_{\|\cdot\|}, d_{\|\cdot\|_u})$ and $A\big|_{X_\lambda} : X_\lambda \to X_\lambda$ satisfy the conditions of Maia's fixed point theorem. From this theorem we have, (a)-(e). □

4. Applications

In what follows we present some applications of the above abstract results.

Example 4.1. Let $h > 0$, $b > 0$ and $p, q \in C[0,b]$. We consider the following functional differential equation (see [32])

$$x'(t) = p(t)x(t) + q(t)x(t-h),\ \forall\ t \in [0,b]. \tag{4.1}$$

By a solution of (4.1) we understand a function $x \in C[-h,b] \cap C^1[0,b]$ which satisfies (4.1). The equation (4.1) is equivalent with the following fixed point equation

$$x(t) = \begin{cases} x(t),\ \text{if } t \in [-h,0] \\ x(0) + \int_0^t p(s)x(s)ds + \int_0^t q(s)x(s-h)ds,\ t \in [0,b] \end{cases} \tag{4.2}$$

with $x \in C[-h,b]$.

Let $A : C[-h,b] \to C[-h,b]$ be defined by, $A(x)(t) =$ the second part of (4.2). Let $\Lambda := C[-h,0]$ and $\Phi : C[-h,b] \to C[-h,0]$ be defined by, $\Phi(x) = x\big|_{[-h,0]}$.

We observe that, $\Phi(A(x)) = \Phi(x)$, $\forall\ x \in C[-h,b]$. So, $C[-h,b] = \bigcup\limits_{\lambda \in C[-h,b]} X_\lambda$ is an ILP of $C[-h,b]$ and λ_0 is the constant function $0 \in C[-h,0]$, i.e.,

$$X_0 = \{x \in C[-h,b] \mid x\big|_{[-h,0]} = 0\}.$$

It is clear that there exists $\tau > 0$ such that $A\big|_{X_0} : X_0 \to X_0$ is a contraction with respect to Bielecki norm $\|\cdot\|_\tau$, where

$$\|x\|_\tau := \max_{t \in [-h,b]} |x(t)|e^{-\tau t}.$$

Let us denote by, $\|\cdot\|$, the max norm on $C[-h,b]$. From Theorem 3.1 we have

Theorem 4.2. *In the above considerations we have that:*

(a) *the operator A is WPO with respect to $\xrightarrow{\|\cdot\|}$, i.e., the solution set of (4.1) is $A^\infty(C[-h,b])$;*

(b) $F_A \cap X_\lambda = \{x^*_\lambda\}$, $\lambda \in C[-h,0]$, *i.e.*, x^*_λ *is a unique solution of* (4.1) *which satisfies the condition* $x\big|_{[-h,0]} = \lambda$;
(c) *the operator* $A\big|_{X_\lambda} : X_\lambda \to X_\lambda$ *is PO*, $\forall\ \lambda \in C[-h,0]$.

Remark 4.3. If in addition we suppose that $p \geq 0$, $q \geq 0$, then the operator A is increasing. From the abstract Gronwall lemma (see [31]) we have that if $x \in C[-h,b] \cap C^1[0,b]$ satisfies the inequality

$$x'(t) \leq p(t)x(t) + q(t)x(t-h),\ \forall\ t \in [0,b],$$

then, $x(t) \leq A^\infty(x)(t)$, $\forall\ t \in [-h,b]$.

Example 4.4. Let $(X,+,\mathbb{R},\|\cdot\|)$ be a Banach space and $A : X \to X$ be a linear and continuous operator. We suppose that A is l-graphic contraction, i.e.,

$$\|A(x) - A^2(x)\| \leq l\|x - A(x)\|,\ \forall\ x \in X.$$

This implies that

$$\|Au\| \leq l\|u\|,\ \forall\ u \in (1_X - A)(X).$$

Let us denote, $X_{\lambda_0} := \overline{(1_X - A)(X)}$. We consider the quotient space,

$$X/_{X_{\lambda_0}} = \{X_\lambda \mid \lambda \in \Lambda\}.$$

We remark that, $X = \bigcup\limits_{\lambda \in \Lambda} X_\lambda$ is ILP of X with respect to A. From Theorem 3.1 we have

Theorem 4.5. *In the above considerations we have:*
(a) $A\big|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a* l*-contraction*, $\forall\ \lambda \in \Lambda$;
(b) $F_A \cap X_\lambda = \{x^*_\lambda\}$, $\lambda \in \Lambda$;
(c) *the attraction domain of* x^*_λ, $(AD)_A(x^*_\lambda) = X_\lambda$, $\forall\ \lambda \in \Lambda$.

Example 4.6. Let $\varphi_0, \varphi, \psi_k \in C([0,1],\mathbb{R}_+)$, $k = \overline{1,m}$ and $0 = a_0 < a_1 < \ldots < a_m = 1$. We suppose that the set $\{\varphi_0, \varphi\cdot\psi_1, \ldots, \varphi\cdot\psi_m\}$ is linearly independent. In addition we suppose that $\varphi_0(a_0) = 1$ and $\varphi(a_0) = 0$. Then the following operator

$$A : C[0,1] \to C[0,1],\ A(f) = f(a_0)\varphi_0 + \varphi\sum_{k=1}^{m} f(a_k)\psi_k$$

is increasing and linear, with $A(f)(a_0) = f(a_0)$, for all $f \in C[0,1]$. Let

$$X_\lambda := \{f \in C[0,1] \mid f(a_0) = \lambda\},\ \lambda \in \mathbb{R}.$$

It is clear that, $C[0,1] = \bigcup\limits_{\lambda \in \mathbb{R}} X_\lambda$ is an ILP of $C[0,1]$ with respect to A. From the Theorem 3.2 we have

Theorem 4.7. *In addition to the above conditions we suppose that,* $A(\varphi) \leq l\varphi$, *with* $0 < l < 1$. *Then:*
(a) *the operator* A *is* WPO;
(b) $X_\lambda \cap F_A = \{f^*_\lambda\}$, $\forall\ \lambda \in \mathbb{R}$;
(c) $A^\infty(f) = f^*_\lambda$, $\forall\ \lambda \in \mathbb{R}$.

Proof. We remark that φ is an order unit for $A(X_0)$. □

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Coincidence point and fixed point theorems for rational contractions

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Abstract. The purpose of this work is to present some coincidence point theorems for singlevalued and multivalued rational contractions. A comparative study of different rational contraction conditions is also presented. Our results extend some recent theorems in the literature.

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Keywords: Fixed point, common fixed point, multivalued operator, coincidence point.

1. Introduction

In this first section, for the convenience of the reader, we will recall the standard terminologies and notations in non-linear analysis. See, for example [4], [11], [6], [9].

Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$.

Denote $\widetilde{B}(x_0, r) := \{x \in X | d(x_0, x) \leq r\}$ the closed ball centered at x_0 with radius r.

If $S : X \to X$ is an operator, then we denote by $F(S) := \{x \in X | x = S(x)\}$ the fixed point set of S.

An operator $f : Y \subseteq X \to Y$ is said to be an α-contraction if $\alpha \in [0, 1]$ and $d(f(x), f(y)) \leq \alpha d(x, y)$, for all $x, y \in Y$.

Definition 1.1. *Let $(X, \leq)$ be an partially ordered set and A, B be two nonempty subsets of X. Then we will wrote $A \leq_s B$ if and only for all $a \in A$ exists $b \in B$ satisfying $a \leq b$.*

We denote by $P(X)$ the family of all nonempty subsets of X. Also $P_p(X)$ will denote the family of all nonempty subsets of X having the property "p", where "p"

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could be: b = bounded, cl = closed, cp = compact etc. We consider the following functionals:

$$D : P(X) \times P(X) \to \mathbb{R}_+, \quad D(A, B) = \inf\{d(a, b) | a \in A, b \in B\}$$

$$\rho : P_b(X) \times P_b(X) \to \mathbb{R}_+, \quad \rho(A, B) = \{sup\{D(a, B) | a \in A\}$$

$$H : P_b(X) \times P_b(X) \to \mathbb{R}_+, \quad H(A, B) = \max\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)\}.$$

Definition 1.2. *Let $(X, \preceq)$ be a partially ordered set and $T : X \to P(X)$ be a multivalued mapping, satisfying the following implication*

$$x \preceq y \Rightarrow Tx \preceq_s Ty.$$

Then T is said to be increasing.

Definition 1.3. ([6]) *A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+ := [0, +\infty)$ is called an altering distance function if the following properties are satisfied:*
(Ψ_1) $\psi(t) = 0 \Leftrightarrow t = 0$.
(Ψ_2) ψ is monotonically non-decreasing.
(Ψ_3) ψ is continuous.
By Ψ we denote the set of all altering distance functions.

The following theorem is an result proved by B.K. Das and S Gupta, in 1975.

Theorem 1.4. *Let (X, d) be a metric space and let $S : X \to X$ be a given mapping such that,*

i) there exist $a, b \in \mathbb{R}_+^*$ with $a + b < 1$ for which $d(Sx, Sy) \le ad(x, y) + bm(x, y)$ for all $x, y \in X$ where

$$m(x, y) = d(y, Sy)\frac{1 + d(x, Sx)}{1 + d(x, y)}.$$

ii) there exists $x_0 \in X$, such that the sequence of iterates $(S^n x_0)$ has a subsequence $(S^{n_k} x_0)$ with $\lim_{k\to\infty} (S^{n_k} x_0) = z_0$. Then z_0 is the unique fixed point of S.

Definition 1.5. *Let S be a self mapping of a metric space (M, d) with a nonempty fixed point set $F(S)$. Then S is said to satisfy the property (P) if $F(S) = F(S^n)$ for each $n \in \mathbb{N}$.*

Definition 1.6. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d on X. We say that X is regular if and only if the following hypothesis holds:*
If $\{z_n\}$ is an non-decreasing sequence in X with respect to $\preceq$ such that $\lim_{n\to\infty} z_n = z \in X$ then $z_n \preceq z$ for all $n \in \mathbb{N}$.

Definition 1.7. *Let (X, d) a complete metric space, with $T : X \to P_{cl}(X)$ and $R : X \to X$. Then $C(R, T) = \{x \in X | Rx \in Tx\}$ is called the coincidence point set of S and T. We say that a point $x \in X$ is a coincidence point of R and T if $Rx = Tx$.*

We will denote by $F(T)$ the fixed point set for T and by $SF(T)$ the strict fixed point set of T.

If Y is a nonempty subset of X and $T : Y \to P(X)$ is a multivalued operator, then by definition, an element $x \in Y$ is said to be:

(i) a fixed point of T if and only if $x \in T(x)$;
(ii) a strict fixed point of T if and only if $x = T(x)$.
The following result appeared in [9].

Theorem 1.8. ([9]) *Let $(X, \preceq)$ be a partially ordered set equipped with a metric d on X such that (X, d) is a complete metric space. Let $T, R : X \to X$ be two mappings satisfying(for pair $(x, y) \in X \times X$ where in Rx and Ry are comparable),*

$$d(Tx, Ty) \leq \frac{\alpha d(Rx, Tx) \cdot d(Ry, Ty)}{1 + d(Rx, Ry)} + \beta d(Rx, Ry) \tag{1.1}$$

where α, β are non-negative real numbers with $\alpha + \beta < 1$. Suppose that
a) X is regular and T is weakly increasing with R.
b) the pair (R, T) is commuting and weakly reciprocally continuous.
Then R *and* T *have a coincidence point.*

On the other hand, in [2] the following local fix point theorem for multivalued contraction is given.

Theorem 1.9. *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let $T : \widetilde{B}(x_0; r) \to P_{cl}(X)$ be a multivalued α- contraction such that $D(x_0, T(x_0)) < (1-\alpha)r$. Then $F(T) \neq \emptyset$.*

We also mention that the following fixed point theorem, for the so called multivalued rational contractions was presented in [10], as follows.

Theorem 1.10. *Let (X, d) a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued operator such that exists $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying*

$$H(Tx, Ty) \leq \frac{\alpha D(y, Ty)[1 + D(x, Tx)]}{1 + d(x, y)} + \beta d(x, y), \text{ for all } x, y \in X. \tag{1.2}$$

Then T *has a fixed point.*

The purpose of this paper is twofold. First we will extend Theorem 1.8 for the case of multivalued operators. Secondly, we will present a local fixed point theorem for multivalued rational conractions.

2. Main results

Our first main result is the following coincidence point theorem.

Theorem 2.1. *Let (X, d) be a complete metric space. Let $T : X \to P_{cl}(X)$ and $R : X \to X$ be two operators satisfying*

$$\rho(Tx, Ty) \leq \frac{\alpha D(Ry, Ty)[1 + D(Rx, Tx)]}{1 + d(Rx, Ry)} + \beta d(Rx, Ry), \forall x, y \in X \tag{2.1}$$

where α, β are some non-negative real numbers with $\alpha + \beta < 1$. Suppose that R is continuous and $T(X) \subset R(X)$. Then R and T have a coincidence point.

Proof. Let $x_0 \in X$ be arbitrary. Since $T(x_0) \subset T(X) \subset R(X)$, there exists $x_1 \in X$ such that $R(x_1) \in T(x_0)$. For $R(x_1) \in T(x_0)$ and $T(x_1)$, by well-known property of the functional ρ, for any $q > 1$, there exists $u_1 \in T(x_1)$ such that

$$d(Rx_1, u_1) \leq q\rho(Tx_0, Tx_1).$$

Since $u_1 \in T(x_1) \subset T(X) \subset R(X)$ there exists $x_2 \in X$ such that $u_1 = R(x_2) \in T(x_1)$. Thus

$$d(Rx_1, Rx_2) \leq q\rho(Tx_0, Tx_1) \leq q\left[\frac{\alpha D(Rx_1, Tx_1)[1 + D(Rx_0, Tx_0)]}{1 + d(Rx_0, Rx_1)} + \beta d(Rx_0, Rx_1)\right]$$

$$\leq q\left[\frac{\alpha d(Rx_1, Rx_2)[1 + d(Rx_0, Rx_1)]}{1 + d(Rx_0, Rx_1)} + \beta d(Rx_0, Rx_1)\right].$$

Hence

$$(1 - q\alpha)d(Rx_1, Rx_2) \leq q\beta d(Rx_0, Rx_1)$$

and so

$$d(Rx_1, Rx_2) \leq \frac{q\beta}{1 - q\alpha} d(Rx_0, Rx_1).$$

Now, for $R(x_2) \in T(x_1)$ and $T(x_2)$, for the same arbitrary $q > 1$, there exists $u_2 \in T(x_2)$ such that

$$d(Rx_2, u_2) \leq q\rho(Tx_1, Tx_2).$$

Again, since $u_2 \in T(x_2) \subset T(X) \subset R(X)$ there exists $x_3 \in X$ such that $u_2 = R(x_3) \in T(x_2)$. In this case, by a similar procedure, we obtain

$$d(Rx_2, Rx_3) \leq \frac{q\beta}{1 - q\alpha} d(Rx_1, Rx_2) \leq \left(\frac{q\beta}{1 - q\alpha}\right)^2 d(Rx_0, Rx_1).$$

By this procedure, we obtain a sequence $u_n := R(x_{n+1}) \in T(x_n), n \in \mathbb{N}^*$ such that

$$d(Rx_n, Rx_{n+1}) \leq q\rho(Tx_{n-1}, Rx_n)$$

and

$$d(Rx_n, Rx_{n+1}) \leq \left(\frac{q\beta}{1 - q\alpha}\right)^n d(Rx_0, Rx_1). \tag{2.2}$$

By choosing $1 < q < \frac{1}{\alpha+\beta}$, we obtain thus $r := \frac{q\beta}{1-q\alpha} < 1$.

By (2.2) we get that the sequence $(Rx_n)_{n\in\mathbb{N}^*}$ is Cauchy in the complete metric space (X, d). Thus, there exists x^* such that $Rx_n \to x^*, n \to \infty$. We will show that x^* is a coincidence point for R and T (i.e. $Rx^* \in Tx^*$).

We estimate

$$D(Rx^*, Tx^*) = \inf_{y\in Tx^*} d(Rx^*, y) \leq d(Rx^*, R(Rx_n)) + \inf_{y\in Tx^*} d(R(Rx_n), y)$$

$$\leq d(Rx^*, R(Rx_n)) + D(Rx_{n+1}, Tx^*) \leq d(Rx^*, R(Rx_n)) + \rho(Tx_n, Tx^*)$$

$$\leq d(Rx^*, R(Rx_n)) + \frac{\alpha D(Rx^*, Tx^*)[1 + D(Rx_n, Tx_n)]}{1 + D(Rx_n, Rx^*)} + \beta d(Rx_n, Rx^*)$$

$$\leq d(Rx^*, R(Rx_n)) + \frac{\alpha D(Rx^*, Tx^*)[1 + d(Rx_n, Rx_{n+1})]}{1 + d(Rx_n, Rx^*)} + \beta d(Rx_n, Rx^*)$$

Letting $n \to \infty$ and R continuous, we obtain

$$D(Rx^*, Tx^*) \leq \alpha D(Rx^*, Tx^*)$$
$$(1-\alpha)D(Rx^*, Tx^*) \leq 0.$$

Since $\alpha, \beta > 0$, then T and R has a coincidence point. □

In the next paragraph we will prove Theorem 1.6 using Theorem 1.7 condition.

Theorem 2.2. *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let $T : \widetilde{B}(x_0; r) \to P_{cl}(X)$ be a multivalued operator for which there exist $\alpha, \beta \in \mathbb{R}_+^*$ with $\alpha + \beta < 1$ such that*

$$H(Tx, Ty) \leq \frac{\alpha D(y, Ty)[1 + D(x, Tx)]}{1 + d(x, y)} + \beta d(x, y), \text{ for all } x, y \in X. \tag{2.3}$$

We also suppose that $D(x_0, Tx_0) < \left(\frac{1-\alpha-\beta}{1-\alpha}\right) r$. Then $F(T) \neq \emptyset$.

Proof. We will inductively construct a sequence $x_n \subset \widetilde{B}(x_0; r)$ such that

i) $x_n \in Tx_{n+1}$, $\forall n \in \mathbb{N}^*$

ii) $d(x_n, x_{n-1}) < k^{n-1} r$. We denote by $k = \frac{\beta}{1-\alpha} \in [0, 1)$.

From the condition $D(x_0, Tx_0) < (\frac{1-\alpha-\beta}{1-\alpha}) r$ we have that exists $x_1 \in T(x_0)$ such that $d(x_0, x_1) < (1 - k)r$. Suppose that we construct $x_1, x_2, ..., x_n \in \widetilde{B}(x_0, r)$ with properties i) and ii), now we have to prove the existence of x_{n+1}. We have

$$\begin{aligned} H(Tx_{n-1}, Tx_n) &\leq \frac{\alpha D(x_n, Tx_n)[1 + D(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n) \\ &\leq \frac{\alpha D(x_n, Tx_n)[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n) \\ &= \alpha D(x_n, Tx_n) + \beta d(x_{n-1}, x_n) < \alpha H(Tx_{n-1}, Tx_n) + \beta d(x_{n-1}, x_n) \end{aligned}$$

$$\begin{aligned} H(Tx_{n-1}, Tx_n) &\leq \frac{\beta}{1-\alpha} d(x_{n-1}, x_n) \leq \left(\frac{\beta}{1-\alpha}\right)^n d(x_0, x_1) \\ &< \left(\frac{\beta}{1-\alpha}\right)^n \left(1 - \frac{\beta}{1-\alpha}\right) r. \end{aligned}$$

This proves that $x_{n+1} \in Tx_n$ such that

$$d(x_{n+1}, x_n) < \left(\frac{\beta}{1-\alpha}\right)^n \left(1 - \frac{\beta}{1-\alpha}\right) r,$$

so using k we will have $d(x_{n+1}, x_n) < k^n(1 - k)r$.

Moreover, we have

$$\begin{aligned} d(x_{n+p}, x_n) &\leq (1 + k + ... + k^{p-1}) k^n (1 - k) r \\ &\leq \frac{k^p}{1-k} k^n (1 - k) r \to 0 \quad as \quad n, p \to \infty. \end{aligned} \tag{2.4}$$

Therefore $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, with $\lim\limits_{n\to\infty} x_n = x_0^* \in \widetilde{B}(x_0, r)$. Because T is closed we obtain

$$D(x_0^*, Tx_0^*) \leq d(x_0^*, x_{n+1}) + H(Tx_n, Tx_0^*)$$

$$\leq d(x_0^*, x_{n+1}) + \frac{\alpha D(x_0^*, Tx_0^*)[1 + D(x_n, Tx_n)]}{1 + d(x_n, x_0^*)} + \beta d(x_n, x_0^*)$$

$$\leq d(x_0^*, x_{n+1}) + \frac{\alpha D(x_0^*, Tx_0^*)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x_0^*)} + \beta d(x_n, x_0^*).$$

Letting $n \to \infty$, we have

$$D(x_0^*, Tx_0^*) \leq \alpha D(x_0^*, Tx_0^*).$$

This proves that x_0^* is a fixed point of Tx_0^*. □

The next part of this section, is devoted to generalize Theorem 1.4 to the case of multivalued operators.

Theorem 2.3. *Let (X, d) be a complete metric space, let $\psi \in \Psi$ and $T : X \to P_{cl}(X)$ be a multivalued operator for which there exist $\alpha, \beta \in \mathbb{R}_+^*$ with $\alpha + \beta < 1$ such that*

$$\psi[H(Tx, Ty)] \leq \alpha\psi[m(x, y)] + \beta\psi[d(x, y)], \text{ for all } x, y \in X \tag{2.5}$$

where

$$m(x, y) = D(y, Ty)\frac{1 + D(x, Tx)}{1 + d(x, y)}. \tag{2.6}$$

Then T has a fixed point $x^* \in X$, and there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_0 \in X$ and $x_{n+1} \in T(x_n), n \in \mathbb{N}$ such that $\lim_{n\to\infty} x_n = x^*$.

Proof. Let $x_0 \in X$ be arbitrary chosen and let (x_n) be a sequence defined as follows: $x_{n+1} \in Tx_n \subset T^{n+1}x_0$,for each $n \geq 1$. Now,

$$\psi[d(x_n, x_{n+1})] \leq \psi[qH(Tx_{n-1}, Tx_n)] \leq q\alpha\psi[m(x_{n-1}, x_n)] + q\beta\psi[d(x_{n-1}, x_n] \tag{2.7}$$

using (2.6),

$$m(x_{n-1}, x_n) = D(x_n, Tx_n)\frac{1 + D(x_{n-1}, Tx_{n-1})}{1 + d(x_{n-1}, x_n)}$$

$$\leq d(x_n, x_{n+1})\frac{1 + d(x_{n-1}, x_n)}{1 + d(x_{n-1}, x_n)} = d(x_n, x_{n+1}).$$

Substituting it into (2.7), it follows that,

$$\psi[d(x_n, x_{n+1}) \leq q\alpha\psi[d(x_n, x_{n+1})] + q\beta\psi[d(x_{n-1}, x_n)]$$

so we have,

$$\psi[d(x_n, x_{n+1})] \leq \frac{q\beta}{1 - q\alpha}\psi[d(x_{n-1}, x_n)]$$

$$\leq \left(\frac{q\beta}{1 - q\alpha}\right)^2 \psi[d(x_{n-2}, x_{n-1})] \leq ... \tag{2.8}$$

$$\leq \left(\frac{q\beta}{1 - q\alpha}\right)^n \psi[d(x_0, x_1)] \tag{2.9}$$

Since $r = \frac{q\beta}{1-q\alpha} \in (0, 1)$, from (2.8) we obtain

$$\lim_{n\to\infty} \psi[d(x_n, x_{n+1})] = 0.$$

From the fact that $\psi \in \Psi$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now we will show that (x_n) is a Cauchy sequence. Using (2.9), moreover, for $n < m$, we have

$$\psi[d(x_n, x_m)] \leq \psi[d(x_{n-1}, x_n)] + ... + \psi[d(x_{m-1}, x_m)] \leq (r^n + ... + r^{m-1})\psi[d(x_0, x_1)]$$
$$\leq \frac{r^n}{1-r}\psi[d(x_0, x_1)] \to 0 \quad as \quad n, m \to \infty. \tag{2.10}$$

Therefore (x_n) is a Cauchy sequence. Since (X, d) is a complete metric space, we get that $x \in X \lim_{n\to\infty} x_n = x^*$.

$$\psi[D(x^*, Tx^*)] = \psi[\inf_{y\in Tx^*} d(x^*, y)] \leq \psi[d(x^*, x_{n+1})] + \psi[\inf_{y\in Tx^*} d(x_{n+1}, y)]$$
$$\leq \psi[d(x^*, x_{n+1})] + \psi[H(Tx_n, Tx^*)]$$
$$\leq \psi[d(x^*, x_{n+1})] + \alpha\psi[m(x_n, x^*)] + \beta\psi[d(x_n, x^*)]$$
$$\leq \psi[d(x^*, x_{n+1})] + \alpha\psi[D(x^*, Tx^*)\frac{1 + D(x_n, Tx_n)}{1 + d(x_n, x^*}])] + \beta\psi[d(x_n, x^*)$$
$$\leq \psi[d(x^*, x_{n+1})] + \alpha\psi[D(x^*, Tx^*)\frac{1 + d(x_n, x_{n+1})}{1 + d(x_n, x^*)}] + \beta\psi[d(x_n, x^*).$$

Letting $n \to \infty$ we obtain

$$\psi[D(x^*, Tx^*)](1 - \alpha) \leq 0.$$

Since $\psi \in \Psi$, we have $D(x^*, Tx^*) = 0$. This proves that $x^* \in F_T$. □

As a consequence, we obtain the following fixed point theorem.

Corollary 2.4. *Let (X, d) be a complete metric space and let $T : X \to P_{cl}(X)$ be a multivalued operator. We assume that for each $x, y \in X$,*

$$\int_0^{H(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{D(y,Ty)\frac{1+D(x,Tx)}{1+d(x,y)}} \varphi(t)dt + \beta \int_0^{d(x,y)} \varphi(t)dt \tag{2.11}$$

where $0 < \alpha + \beta < 1$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, is a Lebesque integrable operator which is summable on each compact subset of $[0, +\infty)$, non negative and such that $\int_0^\varepsilon \varphi(t)dt > 0$ for all $\epsilon > 0$. Then T admits a fixed point $x^* \in X$ such that for each $x \in X$

$$\lim_{n\to\infty} x^n = x^*, x_n \in T^n x.$$

Proof. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, be as in the corollary, we define

$$\psi_0(t) = \int_0^t \varphi(t)dt, \ t \in \mathbb{R}_+.$$

ψ_0 is monotonically non decreasing and by hypothesis ψ_0 is continuous. Therefore, $\psi_0 \in \Psi$. So the condition (2.11) becomes

$$\psi_0[H(Tx,Ty)] \leq \alpha\psi_0\left[D(y,Ty)\frac{1+D(x,Tx)}{1+d(x,y)}\right] + \beta\psi_0[d(x,y)]\forall x,y \in X.$$

So, from Theorem 2.3 we have that exists $x^* \in X$ such that for each $x^* \in F(T)$ and there exist a sequence$(x_n)_{n\in\mathbb{N}} \subset X$ with $x_0 \in X$ and $x_{n+1} \in T(x_n), n \in \mathbb{N}$ such that $\lim_{n\to\infty} x_n = x^*$. □

Example 2.5. Let $X = \{(0,0,0),(0,0,1),(1,0,0)\}$ be endowed with the metric d. Consider the multivalued operator $T : X \to P_{cl}(X)$ and a singlevalued operator $R : X \to X$ defined by

$$T(x) = \begin{cases} \{(1,0,0)\}, & \text{if} \quad x = (0,0,1) \\ \{(0,0,0)\}, & \text{if} \quad x = (0,0,0) \\ \{(0,0,0),(1,0,0)\}, & \text{if} \quad x = (1,0,0) \end{cases}$$

$$R(x) = \begin{cases} \{(1,0,0)\}, & \text{if} \quad x = (0,0,1) \\ \{(0,0,0)\}, & \text{if} \quad x = (0,0,0) \\ \{(0,0,1)\}, & \text{if} \quad x = (1,0,0) \end{cases}$$

Then $F_T = \{(0,0,0),(1,0,0)\}$, $F_R = \{(0,0,0)\}$, $C(R,T) = \{(0,0,1),(0,0,0)\}$ and Theorem 2.1 is verified for $\alpha = \frac{1}{9}, \beta = \frac{7}{8}$, $\alpha + \beta < 1$.

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Improved error analysis of Newton's method for a certain class of operators

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Abstract. We present an improved error analysis for Newton's method in order to approximate a locally unique solution of a nonlinear operator equation using Newton's method. The advantages of our approach under the same computational cost – as in earlier studies such as [15, 16, 17, 18, 19, 20] – are: weaker sufficient convergence condition; more precise error estimates on the distances involved and an at least as precise information on the location of the solution. These advantages are obtained by introducing the notion of the center γ_0−condition. A numerical example is also provided to compare the proposed error analysis to the older convergence analysis which shows that our analysis gives more precise error bounds than the earlier analysis.

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1. Introduction

Let $\mathbf{X}$, $\mathbf{Y}$ be Banach spaces. Let $U(x, r)$ and $\overline{U}(x, r)$ stand, respectively, for the open and closed ball in $\mathbf{X}$ with center x and radius $r > 0$. $\mathbb{L}(\mathbf{X}, \mathbf{Y})$ denotes the space of bounded linear operators from $\mathbf{X}$ into $\mathbf{Y}$. In the present paper we are concerned with the problem of approximating a locally unique solution $x^\star$ of nonlinear operator equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet continuously differentiable operator defined on $\overline{U}(x_0, R)$ for some $R > 0$ with values in $\mathbf{Y}$.

Several problems from various disciplines such as Computational Sciences can be brought in the form of equation (1.1) using Mathematical Modelling [13, 14, 21, 5, 7, 17, 18]. The solution of these equations can rarely be found in closed form. That is why the solution methods for these equations are iterative. In particular, the practice of numerical functional analysis and operator theory for finding such solutions is essentially connected to variants of Newton's method [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

The study about convergence matter of Newton methods is usually centered on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of Newton methods; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. We find in the literature several studies on the weakness and/or extension of the hypothesis made on the underlying operators.

There is a plethora on local as well as semi-local convergence results, we refer the reader to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. The most famous among the semi-local convergence of iterative methods is the celebrated Kantorovich theorem for solving nonlinear equations. This theorem provides a simple and transparent convergence criterion for operators with bounded second derivatives F'' or the Lipschitz continuous first derivatives [2, 7, 8, 11, 13, 14, 22]. Another important theorem inaugurated by Smale at the International Conference of Mathematics [17, 18], where the concept of an approximate zero was proposed and the convergence criteria were provided to determine an approximate zero for analytic function, depending on the information at the initial point. Wang [20] generalized Smale's result by introducing the γ-condition (see (1.3)). For more details on Smale's theory, the reader can refer to the excellent Dedieu's book [10, Chapter 3.3]. Newton's method defined by

$$\begin{cases} x_0 \text{ is an initial point} \\ x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \qquad \text{for each} \quad n = 0, 1, 2, \cdots \end{cases} \tag{1.2}$$

is undoubtedly the most popular iterative process for generating a sequence $\{x_n\}$ approximating $x^\star$ [8]. Here, $F'(x) \in \mathbb{L}(\mathbf{X}, \mathbf{Y})$ denotes the Fréchet-derivative of F at $x \in \overline{U}(x_0, R)$.

In the present paper motivated by the works in [9, 15, 16, 17, 18, 19, 20, 21] and optimization considerations, we expand the applicability of Newton's method under the γ-condition by introducing the notion of the center γ_0-condition (to be precised in Definition 3.1) for some $\gamma_0 \leq \gamma$. This way we obtain more precise upper bounds on the norms of $\| F'(x)^{-1} F'(x_0) \|$ for each $x \in \overline{U}(x_0, R)$ (see (1.3), (2.2) and (2.3)) leading to weaker sufficient convergence conditions and a tighter convergence analysis than in earlier studies such as [15, 16, 17, 18, 19, 20, 21]. Our approach of introducing center-Lipschitz condition has already been fruitful for expanding the applicability of Newton's method under the Kantorovich-type theory [2, 3, 4, 5, 6, 7, 13, 14, 22]. Wang [20] used the $\gamma-$Lipschitz condition which is given by

$$\left\| F'(x_0)^{-1} F''(x) \right\| \leq \frac{2\gamma}{\Big(1 - \gamma \|x - x_0\|\Big)^3} \quad \text{for each} \quad x \in U(x_0, r), \; 0 < r \leq R \tag{1.3}$$

where $\gamma > 0$ and x_0 are such that $\gamma \parallel x - x_0 \parallel < 1$ and $F'(x_0)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$ to show the following semi-local convergence result for Newton's method.

Theorem 1.1. [20] *Let* $F : \overline{U}(x_0, R) \subseteq \mathbf{X} \longrightarrow \mathbf{Y}$ *be twice-Fréchet differentiable. Suppose there exists* $x_0 \in U(x_0, R)$ *such that* $F'(x_0)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$ *and*

$$\left\| F'(x_0)^{-1} F(x_0) \right\| \leq \beta; \tag{1.4}$$

condition (1.3) *holds and for* $\alpha = \gamma \beta$

$$\alpha \leq 3 - 2\sqrt{2}; \tag{1.5}$$

$$t^\star \leq R, \tag{1.6}$$

where

$$t^\star = \frac{1 + \alpha - \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma} \leq \left(1 - \frac{1}{\sqrt{2}}\right) \frac{1}{\gamma}. \tag{1.7}$$

Then, sequence $\{x_n\}$ *generated by Newton's method is well defined, remains in* $\overline{U}(x_0, t^\star)$ *for each* $n = 0, 1, \cdots$ *and converges to a unique solution* $x^\star \in \overline{U}(x_0, t^\star)$ *of equation* $F(x) = 0$.

Moreover, the following error estimates hold

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \tag{1.8}$$

and

$$\|x_{n+1} - x^\star\| \leq t^\star - t_n, \tag{1.9}$$

where scalar sequence $\{t_n\}$ *is defined by*

$$\begin{cases} t_0 = 0, \quad t_1 = \beta, \\ t_{n+1} = t_n + \dfrac{\gamma\,(t_n - t_{n-1})^2}{\left(2 - \dfrac{1}{(1-\gamma\, t_n)^2}\right)(1-\gamma t_n)(1-\gamma t_{n-1})^2} = t_n - \dfrac{\varphi(t_n)}{\varphi'(t_n)} \end{cases} \tag{1.10}$$

for each $n = 1, 2, \cdots$, *where*

$$\varphi(t) = \beta - t + \frac{\gamma\, t^2}{1 - \gamma\, t}. \tag{1.11}$$

Notice that $t^\star$ *is the small zero of equation* $\varphi(t) = 0$, *which exists under the hypothesis* (1.5).

The rest of the paper is organized as follows. Section 2 contains the semi-local and local convergence analysis of Newton's method. A numerical example is given in the concluding Section 3.

2. Semi-local convergence of Newton's method

We need some auxiliary results. We shall use the Banach lemma on invertible operators [2, 7, 12].

Lemma 2.1. *Let A, B be bounded linear operators, where A is invertible and $\| A^{-1} \| \| B \| < 1$.*

Then, $A+B$ is invertible and

$$\| (A+B)^{-1} \| \leq \frac{\| A^{-1} \|}{1 - \| A^{-1} \| \| B \|}. \tag{2.1}$$

We shall also use the following definition of Lipschitz and local Lipschitz conditions.

Definition 2.2. *(see [9, p. 634], [22, p. 673]) Let $F : \overline{U}(x_0, R) \longrightarrow \mathbf{Y}$ be Fréchet-differentiable on $U(x_0, R)$. We say that F' satisfies the Lipschitz condition at x_0 if there exists an increasing function $\ell : [0, R] \longrightarrow [0, +\infty)$ such that*

$$\begin{aligned} &\| F'(x_0)^{-1} (F'(x) - F'(y)) \| \leq \ell(r) \| x - y \| \\ &\text{for each } x, y \in \overline{U}(x_0, r),\ 0 < r \leq R. \end{aligned} \tag{2.2}$$

In view of (2.2), there exists an increasing function $\ell_0 : [0, R] \longrightarrow [0, +\infty)$ such that the center-Lipschitz condition

$$\begin{aligned} &\| F'(x_0)^{-1} (F'(x) - F'(x_0)) \| \leq \ell_0(r) \| x - x_0 \| \\ &\text{for each } x \in \overline{U}(x_0, r),\ 0 < r \leq R \end{aligned} \tag{2.3}$$

holds. Clearly,

$$\ell_0(r) \leq \ell(r) \text{ for each } r \in (0, R] \tag{2.4}$$

holds in general and $\ell(r)/\ell_0(r)$ can be arbitrarily large [2, 3, 4, 5, 6, 7].

Lemma 2.3. *(see [9, p. 638]) Let $F : \overline{U}(x_0, R) \longrightarrow \mathbf{Y}$ be Fréchet-differentiable on $U(x_0, R)$. Suppose $F'(x_0)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$ and there exist $0 < \gamma_0 \leq \gamma$ such that $\gamma_0 R < 1$, $\gamma R < 1$. Then, F' satisfies conditions (2.2) and (2.3), respectively, with*

$$\ell(r) := \frac{2\gamma}{(1 - \gamma r)^3} \tag{2.5}$$

and

$$\ell_0(r) := \frac{\gamma_0 (2 - \gamma_0 r)}{(1 - \gamma_0 r)^2}. \tag{2.6}$$

Notice that with preceding choices of functions ℓ and ℓ_0, we have that

$$\ell_0(r) < \ell(r) \quad \text{for each} \quad r \in (0, R]. \tag{2.7}$$

We also need a result by Zabrejko and Nguen.

Lemma 2.4. *(see [22, p. 673]) Let $F : \overline{U}(x_0, R) \longrightarrow \mathbf{Y}$ be Fréchet-differentiable on $U(x_0, R)$. Suppose $F'(x_0)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$ and*

$$\begin{aligned} &\| F'(x_0)^{-1} (F'(x) - F'(y)) \| \leq \lambda(r) \| x - y \| \\ &\text{for each } x, y \in \overline{U}(x_0, r),\ 0 < r \leq R \end{aligned}$$

for some increasing function $\lambda : [0, R] \longrightarrow [0, +\infty)$. Then, the following assertion holds

$$\begin{aligned} &\| F'(x_0)^{-1} (F'(x+p) - F'(x)) \| \leq \Lambda(r + \| p \|) - \Lambda(r) \\ &\text{for each } x \in \overline{U}(x_0, r),\ 0 < r \leq R \text{ and } \| p \| \leq R - r, \end{aligned}$$

where

$$\Lambda(r) = \int_0^R \lambda(t)\,dt.$$

In particular, if

$$\begin{array}{l} \| F'(x_0)^{-1}\,(F'(x) - F'(x_0)) \| \leq \lambda_0(r) \ \| x - x_0 \| \\ for\ each\ x \in \overline{U}(x_0, r),\ 0 < r \leq R \end{array}$$

for some increasing function $\lambda_0 \,:\, [0, R] \longrightarrow [0, +\infty)$*. Then, the following assertion holds*

$$\begin{array}{l} \| F'(x_0)^{-1}\,(F'(x_0 + p) - F'(x_0)) \| \leq \Lambda_0(\| p \|) \\ for\ each\ 0 < r \leq R\ and\ \| p \| \leq R - r, \end{array}$$

where

$$\Lambda_0(r) = \int_0^R \lambda_0(t)\,dt.$$

Using the center-Lipschitz condition and Lemma 2.3, we can show the following result on invertible operators.

Lemma 2.5. *Let* $F \,:\, \overline{U}(x_0, R) \longrightarrow \mathbf{Y}$ *be Fréchet-differentiable on* $U(x_0, R)$*. Suppose* $F'(x_0)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$ *and* $\gamma_0\, R < 1$ *for some* $\gamma_0 > 0$ *and* $x_0 \in \mathbf{X}$*; center-Lipschitz (*2.3*) holds on* $U_0 = U(x_0, r_0)$*, where* $\ell_0(r)$ *is given by (*2.6*) and* $r_0 = (1 - \frac{1}{\sqrt{2}})\,\frac{1}{\gamma_0}$*. Then* $F'(x)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$ *on* U_0 *and satisfies*

$$\| F'(x)^{-1}\,F'(x_0) \| \leq \left(2 - \frac{1}{(1 - \gamma_0\, r)^2}\right)^{-1}. \tag{2.8}$$

Proof. We have by (2.3), (2.6) and $x \in U_0$ that

$$\| F'(x_0)^{-1}\,(F'(x) - F'(x_0)) \| \leq \ell_0(r)\, r \leq \frac{1}{(1 - \gamma_0\, r)^2} - 1 < 1.$$

The result now follows from Lemma 2.1. The proof of Lemma 2.5 is complete. □

Using (1.3) a similar to Lemma 2.1, Banach lemma was given in [20] (see also [9]).

Lemma 2.6. *Let* $F \,:\, \overline{U}(x_0, R) \longrightarrow \mathbf{Y}$ *be twice Fréchet-differentiable on* $U(x_0, R)$*. Suppose* $F'(x_0)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$ *and* $\gamma\, R < 1$ *for some* $\gamma > 0$ *and* $x_0 \in \mathbf{X}$*; condition (*1.3*) holds on* $V_0 = U(x_0, \overline{r_0})$*, where* $\overline{r_0} = (1 - \frac{1}{\sqrt{2}})\,\frac{1}{\gamma}$*. Then* $F'(x)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$ *on* $\overline{V_0}$ *and satisfies*

$$\| F'(x)^{-1}\,F'(x_0) \| \leq \left(2 - \frac{1}{(1 - \gamma\, r)^2}\right)^{-1}. \tag{2.9}$$

Remark 2.7. It follows from (2.8), (2.9) and $\gamma_0 \leq \gamma$ that (2.8) is more precise upper bound on the norm of $F'(x)^{-1}\,F'(x_0)$. This observation leads to a tighter majorizing sequence for $\{x_n\}$ (see Proposition 2.11).

We need an auxiliary result on majorizing sequences for Newton's method (1.2).

Lemma 2.8. *Let $\beta > 0$, $\gamma_0 > 0$, $\gamma > 0$ with $\gamma_0 > \gamma$ be given parameters. Define parameters s_i for $i = 0, 1, 2$ by*

$$s_0 = 0, \quad s_1 = \beta, \quad s_2 = \beta + \frac{\gamma_0\,\beta^2}{\Big[2 - \dfrac{1}{(1-\gamma_0\,\beta)^2}\Big](1-\gamma_0\,\beta)} \tag{2.10}$$

and function Ψ on $[\beta, 1/\gamma_0]$ by

$$\Psi(t) = (t-\beta)\gamma\,\beta\,(1-\gamma_0\,t)^2 - \Big(t - \beta - (s_2 - s_1)\Big)\Big[2(1-\gamma_0\,t)^2 - 1\Big](1-\gamma\,t)^3. \tag{2.11}$$

Suppose that

$$\beta < b := \min\left\{\frac{1}{\gamma}, \frac{0.25331131}{\gamma_0}\right\} \tag{2.12}$$

then, the following hold

$$0 < s_2 - s_1 < \frac{1-\gamma_0\beta}{\gamma_0}, \tag{2.13}$$

$$\gamma_0\beta < 1 - \frac{1}{\sqrt{2}} \tag{2.14}$$

and function ψ has zeros in $(\beta, 1/\gamma_0)$. Denote by ρ the smallest zero of the function Ψ in $(\beta, 1/\gamma_0)$. Moreover suppose that

$$s_2 \le \rho < b \tag{2.15}$$

where s_2 and b are given in (2.10) and (2.12), respectively. Then, for

$$\delta = 1 - \frac{s_2 - s_1}{\rho - \beta} \tag{2.16}$$

the following hold

$$0 < \frac{\gamma\beta(1-\gamma_0\rho)^2}{\Big(2(1-\gamma_0\rho)^2 - 1\Big)(1-\gamma\rho)^3} = \delta < 1, \tag{2.17}$$

$$\frac{s_2 - s_1}{1-\delta} + \beta = \rho \tag{2.18}$$

and the iteration $\{s_n\}$ defined by

$$s_{n+2} = s_{n+1} + \frac{\gamma(s_{n+1} - s_n)^2}{\Big[2 - \dfrac{1}{(1-\gamma_0\,s_{n+1})^2}\Big](1-\gamma\,s_{n+1})(1-\gamma\,s_n)^2} \tag{2.19}$$

for each $n = 1, 2, \ldots$ is strictly increasing, bounded from above by ρ and converges to its unique least upper bound $s^\star$ which satisfies

$$s_2 \le s^\star \le \rho. \tag{2.20}$$

Furthermore, the following estimates hold

$$s_{n+2} - s_{n+1} \le \delta^n (s_2 - s_1) \quad \text{for each} \quad n = 1, 2, \ldots. \tag{2.21}$$

Proof. The left hand side inequality in (2.13) is true by the definition of s_1, s_2 and since $2(1-\gamma_0\beta)^2 - 1 > 0$ and $1-\gamma_0\beta > 0$ by (2.12). The right hand side of (2.13) shall be true, if

$$\frac{\gamma_0(s_1-s_0)^2}{\Big[2(1-\gamma_0\, s_1)^2-1\Big](1-\gamma_0\beta)} < \frac{1-\gamma_0\beta}{\gamma_0}$$

or $(\gamma_0\beta)^2 < (1-\gamma_0\beta)^2\Big(2(1-\gamma_0\,\beta)^2-1\Big)$ or for $z = 1-\gamma_0\beta$ if $2z^4 - 2z^2 + 2z - 1 > 0$, or if $z > 0.74668869$ which is true by (2.12). Estimate (2.14) follows from (2.12) since $1 - 1/\sqrt{2} = 0.292853219 > 0.25331131\ldots$. Using (2.11) we have that

$$\Psi(\beta) = (s_2 - s_1)\Big[2(1-\gamma_2\beta)^2 - 1\Big](1-\gamma\beta)^3 > 0,$$

since $s_2 - s_1 > 0$, $2(1-\gamma_0\beta)^2 - 1 > 0$ and $1 - \gamma\beta > 0$. We also have that

$$\Psi(\frac{1}{\gamma_0}) = \Big[\frac{1}{\gamma_0} - \beta - (s_2 - s_1)\Big]\Big(1-\frac{\gamma}{\gamma_0}\Big)\Big(1-\frac{\gamma}{\gamma_0}\Big)^2 < 0$$

by (2.13) and $\gamma_0 < \gamma$. It follows from the Intermediate mean value theorem applied to function Ψ on the interval $(\beta, 1/\gamma_0)$ that function Ψ has zeros on $(\beta, 1/\gamma_0)$. Denote by ρ the smallest such zero. Then, it follows from the definition of δ, ρ that the equality (2.17) holds. The left hand inequality holds by (2.12) and (2.15). The right hand side inequality in (2.17) holds by (2.13) and (2.15) since $\beta < s_2 \le \rho$. Moreover, we have by (2.16), (2.17) and (2.19) that

$$0 < s_3 \quad \text{and} \quad 0 < s_3 - s_2 \le \delta(s_2 - s_1). \tag{2.22}$$

Then, we also have by (2.22) that

$$\begin{aligned} s_3 &\le s_2 + \delta(s_2 - s_1) - s_1 + s_1 = s_1 + (1+\delta)(s_2 - s_1) \\ &= \beta + \frac{1-\delta^2}{1-\delta}(s_2 - s_1) \le \beta + \frac{s_2 - s_1}{1-\delta} = \rho. \end{aligned}$$

Hence, we deduce that

$$s_3 \le \rho. \tag{2.23}$$

Suppose that

$$0 < s_{n+1}, \quad 0 < s_{n+} - s_n \le \delta^n(s_2 - s_1) \quad \text{and} \quad s_{n+1} \le \rho. \tag{2.24}$$

Then, we have by (2.31), (2.27) and (2.36) that

$$s_{n+2} \ge 0,$$

$$\begin{aligned} s_{n+2} - s_{n+1} &= \frac{\gamma(s_{n+1}-s_n)(s_{n+1}-s_n)}{\Big[2 - \dfrac{1}{(1-\gamma_0\, s_{n+1})^2}\Big](1-\gamma\, s_{n+1})(1-\gamma\, s_n)^2} \\ &\le \frac{\gamma\beta}{\Big[2 - \dfrac{1}{(1-\gamma_0\, \rho)^2}\Big](1-\gamma\rho)^3}(s_{n+1}-s_n) = \delta(s_{n+1}-s_n) \\ &\le \delta^{n+1}(s_2 - s_1) \end{aligned}$$

and

$$\begin{aligned} s_{n+2} &\leq s_{n+1} + \delta^{n+1}(s_2 - s_1) \leq s_n + \delta^n (s_2 - s_1) + \delta^{n+1}(s_2 - s_1) \\ &\leq s_1 + (1+\delta)(s_2 - s_1) + \cdots + \delta^n (s_2 - s_1) + \delta^{n+1}(s_2 - s_1) \\ &= s_1 + (1 + \delta + \cdots + \delta^n + \delta^{n+1})(s_2 - s_1) \\ &= s_1 + \frac{1-\delta^{n+2}}{1-\delta}(s_2 - s_1) \leq \rho. \end{aligned}$$

Hence, by mathematical induction the proof for (2.24) is finished. Hence, sequence $\{s_n\}$ is monotonically increasing, bounded from above by ρ and as such it converges to $s^\star$. □

We can show the main following semi-local convergence theorem for Newton's method.

Theorem 2.9. *Suppose that*

(a) There exist $x_0 \in \mathbf{X}$ and $\beta > 0$ such that

$$F'(x_0)^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X}) \quad \textit{and} \quad \left\| F'(x_0)^{-1} F(x_0) \right\| \leq \beta;$$

(b) Operator F' satisfies Lipschitz and center-Lipschitz conditions (2.2) and (2.3) on $U(x_0, r_0)$ with $\ell(r)$ and $\ell(r)$ are given by (2.5) and (2.6), respectively;

(c) $U_0 \subseteq \overline{U}(x_0, R)$;

(d) Hypotheses of Lemma 2.8 hold for sequence $\{s_n\}$ defined by (2.19).

Then, the following assertions hold: sequence $\{x_n\}$ generated by Newton's method is well defined, remains in xoverlineU$(x_0, s^\star)$ for each $n = 0, 1, \cdots$ and converges to a unique solution $x^\star \in \overline{U}(x_0, s^\star)$ of equation $F(x) = 0$. Moreover, the following estimates hold

$$\|x_{n+1} - x_n\| \leq s_{n+1} - s_n \tag{2.25}$$

and

$$\|x_n - x^\star\| \leq s^\star - s_n \quad \text{for each} \quad n = 0, 1, 2, \cdots . \tag{2.26}$$

Proof. We use Mathematical Induction to prove that

$$\| x_{k+1} - x_k \| \leq s_{k+1} - s_k \tag{2.27}$$

and

$$\overline{U}(x_{k+1}, s^\star - s_{k+1}) \subseteq \overline{U}(x_k, s^\star - s_k) \quad \text{for each} \quad k = 1, 2, \cdots . \tag{2.28}$$

Let $z \in \overline{U}(x_1, s^\star - s_1)$. Then, we obtain that

$$\| z - x_0 \| \leq \| z - x_1 \| + \| x_1 - x_0 \| \leq s^\star - s_1 + s_1 - s_0 = s^\star - s_0,$$

which implies $z \in \overline{U}(x_0, s^\star - s_0)$. Note also that

$$\| x_1 - x_0 \| = \| F'(x_0)^{-1} F(x_0) \| \leq \eta = s_1 - s_0.$$

Hence, estimates (2.27) and (2.28) hold for $k = 0$. Suppose these estimates hold for natural integers $n \leq k$. Then, we have that

$$\| x_{k+1} - x_0 \| \leq \sum_{i=1}^{k+1} \| x_i - x_{i-1} \| \leq \sum_{i=1}^{k+1} (s_i - s_{i-1}) = s_{k+1} - s_0 = s_{k+1}$$

and

$$\| x_k + \theta\,(x_{k+1} - x_k) - x_0 \| \le s_k + \theta\,(s_{k+1} - s_k) \le s^\star \quad \text{for all} \quad \theta \in (0,1).$$

Using (2.2), Lemma 2.1 for $x = x_{k+1}$ and the induction hypotheses we get that

$$\begin{aligned} \| F'(x_0)^{-1}\,(F'(x_{k+1}) - F'(x_0)) \| &\le \frac{1}{(1-\gamma_0 \, \| x_{k+1} - x_0 \|)^2} - 1 \\ &\le \frac{1}{(1-\gamma_0\, s_{k+1})^2} - 1 < 1. \end{aligned} \tag{2.29}$$

It follows from (2.29) and the Banach lemma 2.1 on invertible operators that $F'(x_{k+1})^{-1}$ exists and

$$\| F'(x_{k+1})^{-1}\, F'(x_0) \| \le \left(2 - \frac{1}{(1-\gamma_0\, s_{k+1})^2} \right)^{-1}. \tag{2.30}$$

Using (1.2), we obtain the approximation

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - F'(x_k)\,(x_{k+1} - x_k) \\ &= \int_0^1 (F'(x_k^\tau) - F'(x_k))\, d\tau\, (x_{k+1} - x_k), \end{aligned} \tag{2.31}$$

where $x_k^\tau = x_k + \tau\,(x_{k+1} - x_k)$ and $x_k^{\tau\, s} = x_k + \tau\, s\,(x_{k+1} - x_k)$ for each $0 \le \tau, s \le 1$. Using (2.9) for $k = 0$ we obtain

$$\begin{aligned} \Big\| F'(x_0)^{-1} F(x_1) \Big\| &\le \int_0^1 \Big\| F'(x_0)\Big[F'(x_0 + \tau(x_1 - x_0)) - F'(x_0)\Big] \Big\| \mathrm{d}\tau \|x_1 - x_0\| \\ &\le \int_0^1 \Big[\frac{1}{(1-\gamma_0 \tau \|x_1 - x_0\|)^2} - 1\Big] \mathrm{d}\theta \|x_1 - x_0\| \\ &\le \frac{\gamma_0 \|x_1 - x_0\|^2}{1 - \gamma_0 \|x_1 - x_0\|} \le \frac{\gamma_0 (s_1 - s_0)}{1 - \gamma_0\, s_1}. \end{aligned}$$

Then, using (2.9) for $k = 1, 2, \ldots$, we get

$$\begin{aligned} &\| F'(x_0)^{-1}\, F(x_{k+1}) \| \\ &\le \int_0^1 \| F'(x_0)^{-1}\,(F'(x_k^\tau) - F'(x_k)) \| \; d\tau \; \| x_{k+1} - x_k \| \\ &\le \int_0^1 \int_0^1 \frac{2\,\gamma\,\tau\, ds\, d\tau \; \| x_{k+1} - x_k \|^2}{(1 - \gamma \; \| x_k^{\tau\, s} - x_0 \|)^3} \\ &\le \int_0^1 \int_0^1 \frac{2\,\gamma\,\tau\, ds\, d\tau \; \| x_{k+1} - x_k \|^2}{(1 - \gamma \; \| x_k - x_0 \| - \gamma\,\tau\, s \; \| x_{k+1} - x_k \|)^3} \\ &= \frac{\gamma \; \| x_{k+1} - x_k \|^2}{(1 - \gamma \; \| x_k - x_0 \| - \gamma \; \| x_{k+1} - x_k \|)\,(1 - \gamma \; \| x_k - x_0 \|)^2} \\ &\le \frac{\gamma\,(s_{k+1} - s_k)^2}{(1-\gamma\, s_{k+1})\,(1-\gamma\, s_k)^2} \left(\frac{\| x_{k+1} - x_k \|}{s_{k+1} - s_k} \right)^2 \le \frac{\gamma\,(s_{k+1} - s_k)^2}{(1-\gamma\, s_{k+1})\,(1-\gamma\, s_k)^2}. \end{aligned} \tag{2.32}$$

(see also [16, p. 33, estimate (3.19)]) Then, in view of (1.2), (2.19), (2.30) and (2.32) we obtain that

$$\|x_2 - x_1\| \leq \Big\|F'(x_1)^{-1}F'(x_0)\Big\|\Big\|F'(x_0)^{-1}F(x_1)\Big\| \leq \frac{1}{2 - \dfrac{1}{(1-\gamma_0 s_1)^2}} \frac{\gamma_0 (s_1 - s_0)^2}{1 - \gamma_0 s_1} = s_2 - s_1$$

and furthermore for $k = 1, 2, \ldots$ we obtain

$$\begin{aligned} &\| \; x_{k+2} - x_{k+1} \; \| = \| \; (F'(x_{k+1})^{-1} \, F'(x_0)) \, (F'(x_0)^{-1} \, F(x_{k+1})) \; \| \\ &\leq \| \; F'(x_{k+1})^{-1} \, F'(x_0) \; \| \; \| \; F'(x_0)^{-1} \, F(x_{k+1}) \; \| \\ &\leq \frac{1}{2 - \dfrac{1}{(1 - \gamma_0 \, s_{k+1})^2}} \frac{\gamma \, (s_{k+1} - s_k)^2}{(1 - \gamma \, s_{k+1}) \, (1 - \gamma \, s_k)^2} = s_{k+2} - s_{k+1}. \end{aligned} \tag{2.33}$$

Hence, we showed (2.27) holds for all $k \geq 0$. Furthermore, let $w \in \overline{U}(x_{k+2}, s^\star - s_{k+2})$. Then, we have that

$$\begin{aligned} \| \; w - x_{k+1} \; \| \quad &\leq \quad \| \; w - x_{k+2} \; \| + \| \; x_{k+2} - x_{k+1} \; \| \\ &\leq \quad s^\star - s_{k+2} + s_{k+2} - s_{k+1} = s^\star - s_{k+1}. \end{aligned} \tag{2.34}$$

That is $w \in \overline{U}(x_{k+1}, s^\star - s_{k+1})$. The induction for (2.27) and (2.28) is now completed.

Lemma 2.5 implies that $\{s_n\}$ is a complete sequence. It follows from (2.27) and (2.28) that $\{x_n\}$ is also a complete sequence in a Banach space $\mathbf{X}$ and as such it converges to some $x^\star \in \overline{U}(x_0, s^\star)$ (since $\overline{U}(x_0, s^\star)$ is a closed set).

By letting $k \longrightarrow \infty$ in (2.32) we get $F(x^\star) = 0$. Estimate (2.26) is obtained from (2.25) by using standard majorization techniques [2, 7, 12, 13].

Finally, to show the uniqueness part, let $y^\star \in U(x_0, s^\star)$ be a solution of equation (1.1). Using (2.3) for x replaced by $z^\star = x^\star + \tau \, (y^\star - x^\star)$ and $\mathcal{G} = \displaystyle\int_0^1 F'(z^\star) \, d\tau$ we get as in (2.9) that $\| \; F'(x_0)^{-1} \, (\mathcal{G} - F'(x_0)) \; \| < 1$. That is $\mathcal{G}^{-1} \in \mathbb{L}(\mathbf{Y}, \mathbf{X})$.

Using the identity $0 = F(x^\star) - F(y^\star) = \mathcal{G} \, (x^\star - y^\star)$, we deduce $x^\star = y^\star$. □

Remark 2.10. (a) The convergence criteria in Theorem 2.9 are weaker than in Theorem 1.1. In particular, Theorem 1.1 requires that operator F is twice Fréchet-differentiable but our Theorem 2.9 requires only that F is Fréchet-differentiable.

Notice also that if F is twice Fréchet-differentiable, then (2.2) implies (1.3). Therefore, Theorem 2.9 can apply in cases when Theorem 1.1 cannot.

Notice also in practice the computation of constant γ requires the computation of constant γ_0 as a special case.

(b) Concerning the choice of constants γ and γ_0 let us suppose that the following Lipschitz conditions hold. Operator F satisfies $L-$Lipschitz condition at x_0

$$\Big\|F'(x_0)^{-1}\Big[F'(x) - F'(y)\Big]\Big\| \leq L\|x - y\| \quad \text{for each} \quad x, y \in U(x_0, R_0). \tag{2.35}$$

Operator F satisfies the center L_0-Lipschitz condition at x_0

$$\left\|F'(x_0)^{-1}\Big[F'(x)-F'(x_0)\Big]\right\| \leq L_0\|x-x_0\| \quad \text{for each} \quad x\in U(x_0,R_0). \tag{2.36}$$

Then, (2.35) implies (2.3) for $\gamma_0 = L_0/2$ and $l_0(r)=\gamma_0(2-\gamma_0\, r)/(1-\gamma_0\, r)^2$. Moreover, if F is continuously twice-Fréchet differentiable, the (2.34) implies (2.2)for $l(r) = 2\gamma/(1-\gamma\, r)^3$ and we can set $\gamma = L/2$. Examples where $\gamma_0 < \gamma$ or $L_0 < L$ can be found in [2, 3, 4, 5, 6, 7].

Proposition 2.11. *Let* $F\ :\ \overline{U}(x_0,R)\longrightarrow \mathbf{Y}$ *be twice Fréchet-differentiable on* $U(x_0,R)$. *Suppose that hypotheses of Theorem* 1.1 *and the center-Lipschitz condition (2.3) hold on* $\overline{U}(x_0,r_0)$. *Then, the following assertions hold*

(a) Scalar sequences $\{t_n\}$ *and* $\{s_n\}$ *are increasingly convergent to* $t^\star$, $s^\star$, *respectively.*

(b) Sequence $\{x_n\}$ *generated by Newton's method is well defined, remains in* $\overline{U}(x_0,r_0)$ *for each* $n=0,1,\cdots$ *and converges to a unique solution* $x^\star\in\overline{U}(x_0,r_0)$ *of equation* $F(x)=0$. *Moreover, the following estimates hold for each* $n=0,1,\cdots$

$$s_n \leq t_n, \tag{2.37}$$

$$s_{n+1}-s_n \leq t_{n+1}-t_n, \tag{2.38}$$

$$s^\star \leq t^\star, \tag{2.39}$$

$$\| x_{n+1}-x_n \|\leq s_{n+1}-s_n$$

and

$$\| x_n - x^\star \|\leq s^\star - s_n.$$

Proof. According to Theorems 1.1 and 2.9 we only need to show (2.37)–(2.39). Using the definition of sequences $\{t_n\}$, $\{s_n\}$ and $\gamma_0\leq\gamma$, a simple inductive argument shows (2.37) and (2.38). Finally, (2.39) is obtained by letting $n\longrightarrow\infty$. □

Remark 2.12. (a) In view of (2.37)–(2.39), our error analysis is tighter and new information on the location of the solution $x^\star$ at least as precise as the old one. Notice also that estimates (2.37) and (2.38) hold as strict inequalities for $n>1$ if $\gamma_0<\gamma$ (see also the numerical example) and these advantages hold under the same or less computational cost as before (see Remark 2.10).

(b) If F is an analytic operator, then a possible choice for γ_0(or γ) is given by

$$\gamma_0 = \sup_{n>1}\left\|\frac{F'(x_0)^{-1}F(x_0)^n}{n!}\right\|^{\frac{1}{n-1}}$$

This choice is due to Smale [17] (see also [15, 16, 17, 18, 19, 20]).

We complete this section with an useful and obvious extension.

Theorem 2.13. *Suppose there exists an integer* $N\geq 1$ *such that*

$$s_0 < s_1 < \cdots < s_N < R_0 = \min\left\{\frac{1}{\gamma},\left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{\gamma_0}\right\}.$$

Let $\delta_N = \gamma\,\beta_N$ and $\beta_N = t_N - t_{N-1}$. Conditions of Lemma 2.8 are satisfied for δ_N replacing δ. Then, the conclusions of Theorem 2.9 hold. Notice that if $N = 1$ Theorem 2.13 reduces to Theorem 2.9.

3. Numerical example

We illustrate the theoretical results with a numerical example.

Example 3.1. Let $\mathbf{X} = \mathbf{Y} = \mathbb{R}^2$, $x_0 = (1, 0)$, $D = \overline{U}(x_0, 1-\kappa)$ for $\kappa \in (0, 1)$. Let us define function F on D as follows

$$F(x) = (\zeta_1^3 - \zeta_2 - \kappa, \zeta_1 + 3\,\zeta_2 - \sqrt[3]{\kappa}) \quad \text{with} \quad x = (\zeta_1, \zeta_2). \tag{3.1}$$

Using (3.1) we see that the γ-Lipschitz condition is satisfied for $\gamma = 2 - \kappa$ and γ_0-Lipschitz condition is satisfied for $\gamma_0 = (3-\kappa)/2$. We also have that $\beta = (1-\kappa)/3$.

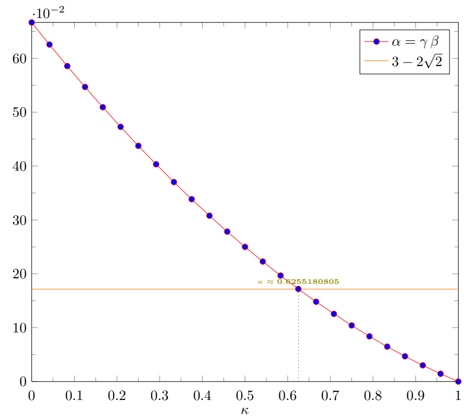

Figure 1. The condition (1.5) of the Theorem 1.1 [20].

The Figure 3.1 plots the condition (1.5) of the Theorem 1.1 [20]. In the Figure 3.1, we notice that for $\kappa \le 3/2 - \left(\sqrt{37 - 24\sqrt{2}}\right)/2$, the condition (1.5) fails. As a consequence the Theorem 1.1 [20] is not applicable. Thus according to the Theorem 1.1 [20] there is no guarantee that the Newton's method starting from x_0 will converge to the solution $x^\star = (\sqrt[3]{\kappa}, 0)$.

To compare the error bounds for the Theorem 1.1 and the Lemma 2.8, we consider $\kappa = 0.7$. From the Figure 3.1, it is clear that the condition (1.5) holds as a result the Theorem 1.1 is applicable. For the hypotheses (2.12) and (2.15) of Lemma 2.8 we obtain

$$0.1000000000 < 0.2202707044,$$
$$0.1179672 < 0.1280403078 < 0.2202707044$$

respectively. Thus our Lemma 2.8 is applicable and the Newton's method starting at $x_0 = (1, 0)$ will converge to the solution $x^\star = (\sqrt[3]{\kappa}, 0)$ for $\kappa = 0.7$. Now we compare the error bounds generated by the sequence $\{t_n\}$ given in (1.10) and the sequence $\{s_n\}$ defined in (2.19).

Table 1. Comparison between the sequences $\{s_n\}$ (2.19) and $\{t_n\}$ (1.10) [20].

n	s_n	t_n	$s_{n+1} - s_n$	$t_{n+1} - t_n$
0	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$	1.000000×10^{-01}	1.000000×10^{-01}
1	1.000000×10^{-01}	1.000000×10^{-01}	1.796716×10^{-02}	2.201246×10^{-02}
2	1.179672×10^{-01}	1.220125×10^{-01}	9.900646×10^{-04}	1.683820×10^{-03}
3	1.189572×10^{-01}	1.236963×10^{-01}	3.196367×10^{-06}	1.069600×10^{-05}
4	1.189604×10^{-01}	1.237070×10^{-01}	3.341751×10^{-11}	4.338887×10^{-10}
5	1.189604×10^{-01}	1.237070×10^{-01}	3.652693×10^{-21}	7.140132×10^{-19}
6	1.189604×10^{-01}	1.237070×10^{-01}	4.364065×10^{-41}	1.933579×10^{-36}
7	1.189604×10^{-01}	1.237070×10^{-01}	6.229418×10^{-81}	1.417992×10^{-71}
8	1.189604×10^{-01}	1.237070×10^{-01}	$1.269287 \times 10^{-160}$	$7.626002 \times 10^{-142}$
9	1.189604×10^{-01}	1.237070×10^{-01}	$5.269686 \times 10^{-320}$	$2.205685 \times 10^{-282}$

In the Table 1, we notice that the error bounds given by the proposed sequence $\{s_n\}$ are tighter than those given by the older sequence $\{t_n\}$ [20].

Conclusions. Using the notion of the center γ_0-Lipschitz condition, we presented a new convergence analysis for Newton's method for approximating a locally unique solution of nonlinear equation in a Banach space setting. Under the same computational cost – as in earlier studies such as [15, 16, 17, 18, 19, 20] – new analysis provide larger convergence domain, weaker sufficient convergence conditions and better error bounds. A numerical example validating the theoretical results is also reported in this study.

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On some generalizations of Nadler's contraction principle

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Abstract. The purpose of this work is to present some generalizations of the well known Nadler's contraction principle. More precisely, using an axiomatic approach of the Pompeiu-Hausdorff metric we will study the properties of the fractal operator generated by a multivalued contraction.

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1. Introduction

Let (X,d) be a metric space and $\mathcal{P}$(X) be the set of all subsets of X. Consider the following families of subsets of X:

P(X):={Y∈ $\mathcal{P}$(X)| Y≠ ∅}, $P_{b,cl}$(X):={Y∈ $\mathcal{P}$(X)|Y is bounded and closed}

The following (generalized) functionals are used in the main sections of the paper.

1. The gap functional generated by d:

$$D_d : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}_+ \cup \{\infty\}, D_d(A, B) = inf\{d(a, b) \mid a \in A, b \in B\}$$

2. The diameter generalized functional:

$$\delta : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}_+ \cup \{\infty\}, \delta(A, B) = \sup\{d(a, b)|a \in A, b \in B\}$$

3. The excess generalized functional:

$$\rho_d : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}_+ \cup \{\infty\}, \rho_d(A, B) = sup\{D(a, B)|a \in A\}$$

4. The Pompeiu-Hausdorff generalized functional:

$$H_d : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}_+ \cup \{\infty\}, H_d(A, B) = \max\{\sup_{a\in A} D_d(a, B), \sup_{b\in B} D_d(b, A)\}$$

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5. The H^+-generalized functional:

$$H^+ : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}_+ \cup \{\infty\}, H^+(A,B) := \frac{1}{2}\{\rho(A,B) + \rho(B,A)\}$$

Let (X,d) be a metric space. If $T : X \to P(X)$ is a multivalued operator, then $x \in X$ is called fixed point for T if and only if $x \in T(x)$. The following concepts are well-known in the literature.

Definition 1.1. [7] *Let (X,d) be a metric space. A mapping $T : X \to P_{b,cl}(X)$ is called a multivalued contraction if there exist a constant $k \in (0,1)$ such that:*

$$H_d(T(x), T(y)) \leq kd(x,y), \; \textit{for all } x, y \in X.$$

Definition 1.2. [5] *Let X be a nonempty set and d, ρ two metrics on X. Then, by definition, d, ρ are called strongly(or Lipschitz) equivalent if there exists $c_1, c_2 > 0$ such that:*

$$c_1\rho(x,y) \leq d(x,y) \leq c_2\rho(x,y), \; \textit{for all } x, y \in X.$$

Definition 1.3. [7] *Let (X,d) be a metric space. Then, by definition, the pair (d, H_d) has the property (p^*) if for $q > 1$, for all $A, B \in P(X)$ and any $a \in A$, there exists $b \in B$ such that:*

$$d(a,b) \leq qH_d(A,B).$$

Definition 1.4. [6] *Let (X,d) be a metric space. $T : X \to P_{b,cl}(x)$ is called H_d- upper semi-continuous in $x_0 \in X$ (H_d-u.s.c) respectively H_d- lower semi-continuous (H_d-l.s.c) if and only if for each sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that*

$$\lim_{n\to\infty} x_n = x_0$$

we have

$$\lim_{n\to\infty} \rho_d(T(x_n), T(x_0)) = 0 \; \textit{respectively} \; \lim_{n\to\infty} \rho_d(T(x_0), T(x_n)) = 0.$$

2. Main results

Concerning the functional H^+ defined below, we have the following properties.

Lemma 2.1. [2] *H^+ is a metric on $P_{b,cl}(X)$.*

Lemma 2.2. [1] *We have the following relations:*

$$\frac{1}{2}H_d(A,B) \leq H^+(A,B) \leq H_d(A,B), \; \textit{for all } A, B \in P_{b,cl}(X) \tag{2.1}$$

(i.e., H_d and H^+ are strongly equivalent metrics).

Proposition 2.3. [2] *Let $(X, ||\cdot||)$ be a normed linear space. For any λ (real or complex), $A, B \in P_{b,cl}(X)$*

1. $H^+(\lambda A, \lambda B) = |\lambda| H^+(A,B)$.
2. $H^+(A + a, B + a) = H^+(A,B)$.

Theorem 2.4. [2] *If $a, b \in X$ and $A, B \in P_{b,cl}(X)$, then the relations hold:*

1. $d(a,b) = H^+(\{a\},\{b\})$.
2. $A \subset \overline{S}(B,r_1), B \subset \overline{S}(A,r_2) \Rightarrow H^+(A,B) \le r$ *where* $r = \frac{r_1+r_2}{2}$.

Theorem 2.5. [2] *If the metric space* (X,d) *is complete, then* $(P_{b,cl}(X), H^+)$ *and* $(P_{b,cl}(X), H_d)$ *are complete too.*

Definition 2.6. [2] *Let (X,d) be a metric space. A multivalued mapping* $T : X \to P_{b,cl}(x)$ *is called* (H^+,k)*-contraction if*

1. there exists a fixed real number k, $0 < k < 1$ *such that for every* $x, y \in X$

$$H^+(T(x),T(y)) \le kd(x,y).$$

2. for every x *in* X, y *in* $T(x)$ *and* $\varepsilon > 0$, *there exists* z *in* $T(y)$ *such that*

$$d(y,z) \le H^+(T(y),T(x)) + \varepsilon.$$

Theorem 2.7. [2] *Let* (X,d) *be a complete metric space,* $T : X \to P_{b,cl}(X)$ *be a multivalued* (H^+,k) *contraction. Then* $FixT \neq \emptyset$.

Remark 2.8. [1] If T is a multivalued k-contraction in the sense of Nadler then T is a multivalued (H^+,k)-contraction but not viceversa.

Example 2.9. Let $X = \{0, \frac{1}{2}, 2\}$ and $d : X \times X \to \mathbb{R}$ be a standard metric. Let $T : X \to P_{b,cl}(X)$ be such that

$$T(x) = \begin{cases} \{0, \frac{1}{2}\}, & \text{for } x = 0 \\ \{0\}, & \text{for } x = \frac{1}{2} \\ \{0,2\}, & \text{for } x = 1 \end{cases}$$

Then T is a (H^+, k) contraction (with $k \in \left[\frac{2}{3},1\right)$) but is not an k- contraction in the sense of Nadler, since

$$H_d(T(0),T(2)) = H_d(\{0,\frac{1}{2}\},\{0,2\}) = 2 \le kd(0,2) = 2k \Rightarrow k \ge 1,$$

which is a contradiction with our assumption that $k < 1$.

Theorem 2.10. [3] *(Nadler) Let* (X,d) *be a metric space and* $T : X \to P_{cp}(X)$ *be a multivalued contraction. Then*

$$H_d(T(A),T(B)) \le kH_d(A,B) \text{ for all } A,B \in P_{cp}(X). \tag{2.2}$$

Lemma 2.11. [4] *Let* (X,d) *be a metric space and* $A, B \in P_{cp}(X)$.
Then for all $a \in A$ *there exists* $b \in B$ *such that*

$$d(a,b) \le H_d(A,B).$$

Theorem 2.12. *Let* (X,d) *be a metric space and* $T : X \to P_{cp}(X)$ *for which there exists* $k > 0$ *such that:*

$$H_d(T(x),T(y)) \le kd(x,y), \text{ for all } x,y \in X$$

Then

$$H^+(T(A),T(B)) \le 2kH^+(A,B) \text{ for all } A,B \in P_{cp}(X).$$

Proof. Let $A, B \in P_{cp}(X)$.

From (2.2) we have $\rho(T(A), T(B)) \leq kH_d(T(A), T(B))$

Combining the previous result and *Lemma*(2.2) we obtain

$$\rho_d(T(A), T(B)) \leq kH_d(A, B) \leq 2kH^+(A, B) \tag{2.3}$$

Interchanging the roles of A and B, we get

$$\rho_d(T(B), T(A)) \leq kH_d(B, A) \leq 2kH^+(B, A) \tag{2.4}$$

Adding (2.3) and (2.4), and then dividing by 2, we get

$$H^+(T(A), T(B)) \leq 2kH^+(A, B).$$

□

Let us recall the relations between u.s.c and $H_d - u.s.c$ of a multivalued operator. If (X, d) is a metric space, then $T : X \to P_{cp}(X)$ is $u.s.c$ on X if and only if T is $H_d - u.s.c$.

Theorem 2.13. *Let (X, d) be a metric space and $T : X \to P_{cp}(X)$ be a multivalued (H^+, k)-contraction. Then*

(a) T is H_d-l.s.c and u.s.c on X.

(b) for all $A \in P_{cp}(X) \Rightarrow T(A) \in P_{cp}(X)$

(c) there exists $k > 0$ such that

$$H^+(T(A), T(B)) \leq 2kH^+(A, B) \ for \ all \ A, B \in P_{cp}(X).$$

Proof. (a) Let $x \in X$ such that $x_n \to x$. We have:

$$\rho_d(T(x), T(x_n)) \leq H_d(T(x), T(x_n)) \leq 2 \cdot H^+(T(x), T(x_n) \leq 2k \cdot d(x, x_n) \to 0$$

In conclusion, T is H_d-l.s.c on X.

Using the relation:

$$\rho_d(T(x_n), T(x)) \leq H_d(T(x_n), T(x)) \leq 2 \cdot H^+(T(x_n), T(x) \leq 2k \cdot d(x, x_n) \to 0$$

we obtain that T is H_d-u.s.c on X.

(b) Let $A \in P_{cp}(X)$. From (a) we obtain the conclusion.

(c) If $u \in T(A)$, then there exists $a \in A$ such that $u \in T(a)$.

From Lemma 2.11 we have that there exists $b \in T(B)$ such that

$$d(a, b) \leq H_d(A, B) \leq 2H^+(A, B).$$

Since

$$D(u, T(B)) \leq D(u, T(b)) \leq \rho_d(T(a), T(b)) \tag{2.5}$$

taking $\sup_{u \in T(A)}$ in (2.5), we have

$$\rho_d(T(A), T(B)) \leq \rho_d(T(a), T(b)) \tag{2.6}$$

Interchanging the roles of A and B, we get

$$\rho_d(T(B), T(A)) \leq \rho_d(T(a), T(b)) \tag{2.7}$$

Adding (2.6) and (2.7), and then dividing by 2, we get for all $A, B \in P_{cp}(X)$ the following result:

$$H^+(T(A), T(B)) \leq H^+(T(a), T(b)) \leq kd(a, b) \leq 2kH^+(A, B).$$

□

As a consequence of the previous result we obtain the following fixed set theorem for a multivalued contraction with respect to H^+.

Theorem 2.14. *Let (X,d) be a complete metric space and $T : X \to P_{cp}(X)$ be a multivalued operator for which there exists $k \in [0, \frac{1}{2})$ such that*

$$H^+(T(x),T(y)) \leq kd(x,y), \ for \ all \ x,y \in X$$

Then, there exists a unique $A^* \in P_{cp}(X)$ such that $T(A^*) = A^*$.

Proof. From $Theorem$ 2.13 we obtain that:

$$H^+(T(A),T(B)) \leq 2kH^+(A,B), \ for \ all \ A,B \in P_{cp}(X)$$

Since $k < \dfrac{1}{2}$ we obtain that T is a $2k$-contraction on the complete metric space $(P_{cp}(X), H^+)$. By Banach contraction principle we get the conclusion. □

In the second part of this section, we will study when the property (p^*) given in $Definition$ 1.3 can be translated between equivalent metrics on a nonempty set X.

Lemma 2.15. *Let X be a nonempty set, d_1, d_2 two Lipschitz equivalent metrics such that there exists $c_1, c_2 > 0$ with $c_1 \leq c_2$ i.e*

$$c_1 d_1(x,y) \leq d_2(x,y) \leq c_2 d_1(x,y), \ for \ all \ x,y \in X \tag{2.8}$$

If the pair (d_1, H_{d_1}) has the property (p^*), then the pair (d_2, H_{d_2}) has the property (p^*).

Proof. Let c_1, c_2 such that

$$c_1 d_1(a,b) \leq d_2(a,b) \leq c_2 d_1(a,b) \ for \ all \ a \in A, b \in B \tag{2.9}$$

and for all $q > 1$, for all $A,B \in P(X)$ and for all $a \in A$, there exists $b^* \in B$ such that

$$d_1(a,b^*) \leq qH_{d_1}(A,B) \tag{2.10}$$

From (2.9) and (2.10) we obtain:

$$d_2(a,b^*) \leq c_2 d_1(a,b^*) \leq c_2 q H_{d_1}(A,B).$$

If, in $c_1 d_1(a,B) \leq d_2(a,B)$ we take $\inf_{b \in B}$, then

$$c_1 D_{d_1}(a,B) \leq D_{d_2}(a,B) \mid \sup_{a \in A} \Leftrightarrow c_1 \rho_{d_1}(A,B) \leq \rho_{d_2}(A,B).$$

In a similar way,

$$c_1 \rho_{d_1}(B,A) \leq \rho_{d_2}(B,A).$$

Taking maximum, we get

$$c_1 H_{d_1}(A,B) \leq H_{d_2}(A,B).$$

Therefore,

$$d_2(a,b^*) \leq \frac{c_2}{c_1} q H_{d_2}(A,B),$$

which means that there exists $b' = b^* \in B$ such that

$$d_2(a,b^*) \leq q_1 H_{d_2}(A,B),$$

where $q_1 := \dfrac{c_2}{c_1} q > 1$. □

Lemma 2.16. *Let X be a nonempty set, d_1, d_2 two metrics on X such that:*

$$\text{there exists } c > 0\text{: } d_2(x,y) \le c d_1(x,y) \quad \text{for all } x, y \in X \tag{2.11}$$

and G_1, G_2 two metrics on $P_{b,cl}(X)$ such that:

$$\text{there exists } e > 0\text{: } eG_{d_1}(A,B) \le G_{d_2}(A,B), \ \text{for all } A, B \in P_{b,cl}(X) \tag{2.12}$$

with $e \le c$. If the pair (d_1, G_1) has the property (p^*) then, the property (p^*) is also true for the pair (d_2, G_2).

Proof. Let $A, B \in P_{b,cl}(X)$. The pair (d_1, G_{d_1}) has the property (p^*) i.e for all $q > 1$ and for all $a \in A$ there exists $b^* \in B$ such that

$$d_1(a, b^*) \le q H_{d_1}(A,B) \tag{2.13}$$

From (2.11), (2.12) and (2.13) we obtain:

$$d_2(a,b') \le c d_1(a,b') \le cqG_{d_1}(A,B) \le \frac{c}{e}\, qG_{d_2}(A,B).$$

Therefore,

$$d_2(a,b') \le \frac{c}{e}\, qG_{d_2}(A,B)$$

which means that there exists $b = b' \in B$ such that

$$d_2(a,b) \le q_1 G_{d_2}(A,B)$$

where $q_1 := \dfrac{c}{e}\, q > 1$ i.e the pair (d_2, G_{d_2}) has the property (p^*). □

Lemma 2.17. *Let X be a nonempty set, d_1, d_2 two metrics on X such that:*

$$\text{there exists } c > 0\text{: } d_2(x,y) \le c d_1(x,y) \ \text{for all } x, y \in X \tag{2.14}$$

and G_1, G_2 two metrics on $P_{b,cl}(X)$ such that:

$$\text{there exists } e > 0\text{: } G_{d_2}(A,B) \le eG_{d_2}(A,B), \ \text{for all } A, B \in P_{b,cl}(X) \tag{2.15}$$

with $c \cdot e < 1$. If the pair (d_1, G_{d_2}) has the property (p^*) then, the property (p^*) is also true for the pair (d_2, G_{d_1}).

Proof. Let $A, B \in P_{b,cl}(X)$. The pair (d_1, G_{d_2}) has the property (p^*) i.e for all $q > 1$ and for all $a \in A$ there exits $b^* \in B$ such that

$$d_1(a, b^*) \le qG_{d_2}(A,B) \tag{2.16}$$

From (2.14), (2.15) and (2.16) we obtain:

$$d_2(a,b') \le c d_1(a,b') \le cqG_{d_2}(A,B) \le c \cdot e \cdot qG_{d_1}(A,B).$$

Therefore,

$$d_2(a,b') \le c \cdot e \cdot qG_{d_2}(A,B)$$

which means that, there exists $b = b' \in B$ such that

$$d_2(a,b) \le q_1 G_{d_2}(A,B)$$

where $q_1 := c \cdot e \cdot q > 1$ i.e the pair (d_2, G_{d_1}) has the property (p^*). □

In the next part of this paper we will give some general abstract results for the metric space $P_{b,cl}(X)$.

Let (X, d) be a metric space, $U \subset P(X)$ and $\Psi : U \to \mathbb{R}_+$. We define some functionals on $U \times U$ as follows:

1. Let $x^* \in X$, $U \subset P_b(X)$

$$G_{\Psi_1}(A, B) = \begin{cases} 0, & A = B \\ \Psi_1(A) + \Psi_1(B), & A \neq B \end{cases}$$

where $\Psi_1(A) := \delta(A, x^*)$.

2. Let $U := P_b(X)$ and $A^* \in P_b(X)$

$$G_{\Psi_2}(A, B) = \begin{cases} 0, & A = B \\ \Psi_2(A) + \Psi_2(B), & A \neq B \end{cases}$$

Where $\Psi_2(A) = H_d(A, A^*)$.

Lemma 2.18. *Let (X, d) be a metric space and $T : X \to P_{cp}(X)$ and $A, B \in P_{cp}(X)$. Let*

$$G_{\Psi_1}(A, B) = \begin{cases} 0, & A = B \\ \Psi_1(A) + \Psi_1(B), & A \neq B \end{cases}$$

Where $\Psi_1(A) = \delta(A, A^*)$, $A^* \in P_{cp}(X)$. Then G_{Ψ_1} is a metric on $P_{cp}(X)$.

Proof. We shall prove that the three axioms of the metric hold:

a) $G_{\Psi_1}(A, B) \geq 0$ for all $A, B \in P_{cp}(X)$

$G_{\Psi_1}(A, B) = \delta(A, A^*) + \delta(B, A^*) \geq 0$

$G_{\Psi_1}(A, B) = 0 \Leftrightarrow A = B$.

This is equivalent to $\Psi_1(A) = 0$ and $\Psi_1(B) = 0$ i.e

$$\delta(A, A^*) = 0 \text{ and } \delta(B, A^*) = 0 \Leftrightarrow A = A^* \text{ and } B = A^* \Rightarrow A = B.$$

b) $G_{\Psi_2}(A, B) = G_{\Psi_2}(B, A)$ is quite obviously.

c) For the third axiom of the metric, let consider $A, B, C \in P_{cp}(X)$. We need to show that:

$$G_{\Psi_1}(A, C) \leq G_{\Psi_1}(A, B) + G_{\Psi}(B, C) \Leftrightarrow$$
$$\Leftrightarrow \Psi_1(A) + \Psi_1(C) \leq \Psi_1(A) + \Psi_1(B) + \Psi_1(B) + \Psi_1(C) \Leftrightarrow$$
$$\Leftrightarrow 0 \leq 2\Psi_1(B) = \delta(B, A^*) \text{ which is true.}$$ □

Lemma 2.19. *If (X, d) is a complete metric space, then $(P_{cp}(X), G_{\Psi_1})$ is complete metric space.*

Proof. We will prove that each Cauchy sequence in $(P_{cp}(X), G_{\Psi_1})$ is convergent. Let $(A_n)_{n \in \mathbb{N}}$, $(A_m)_{m \in \mathbb{N}} \in P_{cp}(X)$, we have:

$$G_{\Psi_1}(A_n, A_m) \to 0,\ m, n \to 0 \Leftrightarrow \delta(A_n, A^*) + \delta(A_m, A^*) \to 0 \Rightarrow$$
$$\Rightarrow \delta(A_n, A^*) \to 0.$$

Therefore,

$$G_{\Psi_1}(A_n, A^*) = \delta(A_n, A^*) + \delta(A^*, A^*) \to 0,\ n \to 0.$$ □

Lemma 2.20. *Let (X, d) be a metric space and $T : X \to P_{cp}(X)$ and $A, B \in P_{cp}(X)$. Let*

$$G_{\Psi_1}(A, B) = \begin{cases} 0, & A = B \\ \Psi_1(A) + \Psi_1(B), & A \neq B \end{cases}$$

where $\Psi_1 : P_{cp}(X) \to \mathbb{R}_+, \Psi_1(A) = \delta(A, A^*)$ with $A^* \in P_{cp}(X)$. Then, the pair (d, G_{Ψ_1}) has the property (p^*).

Proof. We have to show

$$d(a, b) \leq qG_{\Psi_1}(A, B) \iff d(a, b) \leq q(\Psi_1(A) + \Psi_1(B)) \Leftrightarrow$$
$$\Leftrightarrow d(a, b) \leq q(\delta(A, A^*) + \delta(A, A^*))$$

Suppose, by absurdum, that there exists $a \in A$ and there exists $q > 1$ such that for all $b \in B$ we have:

$$d(a, b) > q(\delta(A, A^*) + \delta(B, A^*)).$$

Then, $\delta(A, b) \geq d(a, b) > q(\delta(A, A^*) + \delta(B, A^*))$.
Then, taking $\sup_{b \in B}$, we obtain:

$$\delta(A, A^*) + \delta(A^*, B) \leq \delta(A, B) \geq q(\delta(A, A^*) + \delta(B, A^*))$$

which is a contradiction with $q > 1$. □

Theorem 2.21. *Let (X, d) be a metric space and $T : X \to P_{cp}(X)$ be a multivalued operator for which there exists $k \in (0, 1)$ such that*

$$\delta(T(x), T(y) \leq kd(x, y).$$

For all $A, B \in P_{cp}(X)$ we consider

$$G_{\Psi_1}(A, B) = \begin{cases} 0, & A = B \\ \Psi_1(A) + \Psi_1(B), & A \neq B, \end{cases}$$

where $\Psi_1 : P_{cp}(X) \to \mathbb{R}_+$, $\Psi_1(A) = \delta(A, A^*)$ (with $A^* \in P_{cp}(X)$ is a given set satisfying $A^* = T(A^*)$). Then,

$$G_{\Psi_1}(T(A), T(B)) \leq kG_{\Psi_1}(A, B) \text{ for all } A, B \in P_{cp}(X).$$

Proof. We shall prove that for each $A, B \in P_{cp}(X)$ we have

$$\delta(T(A), A^*) + \delta(T(B), A^*) \leq k(\delta(A, A^*)) + \delta(B, A^*)) \tag{2.17}$$

Since $A^* = T(A^*)$, we have:

$$\delta(A^*, T(A)) + \delta(A^*, T(B)) = \delta(T(A^*), T(A)) + \delta(T(B^*), T(B))$$

Since

$$\delta(T(a), T(b)) \leq kd(a, b) \text{ for all } a \in A \text{ and } b \in B$$

We have (taking $\sup_{a \in A, b \in B}$) that

$$\delta(T(A), T(B)) \leq k\delta(A, B)$$

We obtain:

$$\delta(A^*, T(A)) + \delta(A^*, T(B)) = \delta(T(A^*), T(A)) + \delta(T(A^*), T(B))$$
$$\leq k\delta(A^*, A) + k\delta(A^*, B) = kG_{\psi_1}(A, B)$$

which means:

$$G_{\Psi_1}(T(A),T(B)) \leq kG_{\Psi_1}(A,B) \; for \; all \; A,B \in P_{cp}(X). \qquad \square$$

Lemma 2.22. *Let (X,d) be a metric space and $T : X \to P_{cp}(X)$ and $A,B \in P_{cp}(X)$. Let*

$$G_{\Psi_2}(A,B) = \begin{cases} 0, & A = B \\ \Psi_2(A) + \Psi_2(B), & A \neq B \end{cases}$$

where $\Psi_2 : P_{cp}(X) \to \mathbb{R}_+$, $\Psi_2(A) = H_d(A,A^*)$ with $A^* \in P_{cp}(X)$. Then G_{Ψ_2} is a metric on $P_{cp}(X)$.

Proof. We shall prove that the three axioms of the metric hold:
a) $G_{\Psi_2}(A,B) \geq 0$ for all $A,B \in P_{cp}(X)$
$G_{\Psi_2}(A,B) = H_d(A,A^*) + H_d(B,A^*) \geq 0$
$G_{\Psi_2}(A,B) = 0 \Leftrightarrow A = B$.
This is equivalent to $\Psi_2(A) = 0$ and $\Psi_2(B) = 0$ i.e

$$H_d(A,A^*) = 0 \; and \; H_d(B,A^*) = 0 \Leftrightarrow A = A^* \; and \; B = A^* \Rightarrow A = B.$$

b) $G_{\Psi_2}(A,B) = G_{\Psi_2}(B,A)$ is quite obviously. *c*) For the third axiom of the metric, let consider $A,B,C \in P_{cp}(X)$. We need to show that:

$$G_{\Psi_2}(A,C) \leq G_{\Psi_2}(A,B) + G_{\Psi_2}(B,C) \Leftrightarrow$$

$$\Leftrightarrow \Psi_2(A) + \Psi_2(C) \leq \Psi_2(A) + \Psi_2(B) + \Psi_2(B) + \Psi_2(C) \Leftrightarrow$$

$$\Leftrightarrow 0 \leq 2\Psi_2(B) = 2H_d(B,A^*) \; which \; is \; true. \qquad \square$$

Lemma 2.23. *If (X,d) is a complete metric space, then $(P_{cp}(X), G_{\Psi_2})$ is complete metric space.*

Proof. We will prove that each Cauchy sequence in $(P_{cp}(X), G_{\Psi_2})$ is convergent. Let $(A_n)_{n\in\mathbb{N}}$, $(A_m)_{m\in\mathbb{N}} \in P_{cp}(X)$, we have:

$$G_{\Psi_2}(A_n,A_m) \to 0, \; m,n \to 0 \Leftrightarrow H_d(A_n,A^*) + H_d(A_m,A^*) \to 0 \Leftrightarrow$$

$$\Leftrightarrow H_d(A_n,A^*) \to 0$$

Therefore,

$$G_{\Psi_2}(A_n,A^*) = H_d(A_n,A^*) + H_d(A^*,A^*) \to 0, \; n \to 0. \qquad \square$$

Theorem 2.24. *Let (X,d) be a metric space and $T : X \to P_{cp}(x)$ be a multivalued contraction with respect to H_d and $A,B \in P_{cp}(X)$. Let*

$$G_{\Psi_2}(A,B) = \begin{cases} 0, & A = B \\ \Psi_2(A) + \Psi_2(B), & A \neq B \end{cases}$$

Where $\Psi_2 : P_{cp}(X) \to \mathbb{R}_+$, $\Psi_2(A) = H_d(A,A^*)$ (with $A^* \in P_{cp}(X)$ is a given set satisfying $A^* = T(A^*)$). Then, there exists $k \in (0,1)$ such that

$$G_{\Psi_2}(T(A),T(B)) \leq kG_{\Psi_2}(A,B) \; for \; all \; A,B \in P_{cp}(X).$$

Proof. We shall prove that for each $A, B \in P_{cp}(X)$ we have

$$H_d(T(A), A^*) + H_d(T(B), A^*) \leq k(H_d(A, A^*)) + H_d(B, A^*)).$$

From (2.2) we have $\rho_d(T(A), T(B)) \leq H_d(T(A), T(B))$.
Then

$$\rho_d(T(A), A^*) = \rho_d(T(A), T(A^*)) \leq H_d(T(A), T(A^*)) \leq kH_d(A, A^*).$$

Interchanging the roles of A and B, we get

$$\rho_d(A^*, T(A)) = \rho_d(T(A^*), T(A)) \leq H_d(T(A^*), T(A)) \leq kH_d(A^*, A).$$

Making maximum, we get

$$H_d(T(A), A^*) \leq kH_d(A, A^*). \tag{2.18}$$

Similarly for $B \in P_{cp}(X)$, we have

$$H_d(T(B), A^*) \leq kH_d(B, A^*). \tag{2.19}$$

Adding (2.18) and (2.19) we get:

$$H_d(T(A), A^*) + H_d(T(B), A^*) \leq k(H_d(A, A^*)) + H_d(B, A^*))$$

which means:

$$G_{\Psi_2}(T(A), T(B)) \leq kG_{\Psi_2}(A, B) \; for \; all \; A, B \in P_{cp}(X). \qquad \square$$

Lemma 2.25. *Let (X, d) be a metric space and $T : X \to P_{cp}(X)$ and $A, B \in P_{cp}(X)$. Let*

$$G_{\Psi_2}(A, B) = \begin{cases} 0, & A = B \\ \Psi_2(A) + \Psi_2(B), & A \neq B, \end{cases}$$

where $\Psi_2 : P_{cp}(X) \to \mathbb{R}_+$, $\Psi_2(A) = H_d(A, A^*)$ with $A^* \in P_{cp}(X)$. Then, the pair (d, G_{ψ_2}) has the property (p^*).

Proof. We have to show

$$d(a, b) \leq qG_{\Psi_2}(A, B) \iff d(a, b) \leq q(\Psi_2(A) + \Psi_2(B)) \Leftrightarrow$$

$$\Leftrightarrow d(a, b) \leq q(H_d(A, A^*) + H_d(A, A^*))$$

Supposing again contrary: there exists $q > 1$ and there exists $a \in A$ such that for all $b \in B$ we have:

$$d(a, b) > q(H_d(A, A^*) + H_d(B, A^*)).$$

Then, taking $\inf\limits_{b \in B}$

$$H_d(A, B) \geq \rho_d(A, B) \geq D(a, B) \geq q(H_d(A, A^*) + H_d(B, A^*)).$$

But

$$H_d(A, A^*) + H_d(A^*, B) \geq H_d(A, B) \geq q(H_d(A, A^*) + H_d(B, A^*)).$$

Hence $q \leq 1$, a contradiction. $\square$

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Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 1, 135–135

Erratum: Semilinear operator equations and systems with potential-type nonlinearities, Angela Budescu, Stud. Univ. Babeş-Bolyai Math. 59(2014), No. 2, 199–212

The author thanks Professor Biagio Ricceri for noticing that in Theorem 3.3 the condition (ii) $f(\cdot, 0) = 0$ on Ω is too strong and implies the fact that the unique solution of equation (3.1) is zero. Actually, as Professor Biagio Ricceri remarked, this can be replaced by a similar assumption to that from Theorem 3.4, namely that $f(\cdot, 0) \in L^2(\Omega)$. This is sufficient to guarantee that the Nemytskii operator N_f is well-defined.

Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 1, 137–137

Retraction notice: Exact discrete Morse functions on surfaces, Vasile Revnic, Stud. Univ. Babeş-Bolyai Math., 58(2013), No. 4, 469-476

This article has been retracted at the request of the Editors. The reason is that the article was simultaneously published in Stud. Univ. Babeş-Bolyai Math. and in another journal.

We offer apologies to the readers of the journal for this inconvenience.

Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 1, 139–146

Book reviews

Lance Fortnow, The Golden Ticket: P, NP, and the Search for the Impossible, Princeton University Press, 2013, ISBN 978-0-691-15649-1.

The class P is the class of problems which can be solved in polynomial time on deterministic machines. In Complexity Theory tractable is synonym to having solution algorithm with polynomial runtime. The class NP is the class of problems which can be solved in polynomial time on nondeterministic machines, or equivalently having solutions which can be checked in polynomial time.

The P=NP problem is the most important open problem in computer science, if not all of mathematics. In colloquial language, it asks whether every problem whose solution can be quickly checked by computer can also be quickly solved by computer. The Clay Mathematics Institute offers a million-dollar prize for the solution of this problem.

The title *The Golden Ticket* is inspired from Roald Dahl's book, *Charlie and the Chocolate Factory.*

The first chapter introduces in a nontechnical way the concepts of P and NP and gives some examples: the traveling salesman problem and the partition problem as NP-hard problems; shortest path as an exemplar for P.

In Chapter 2, entitled "The Beautiful World", Fortnow does a fanciful spiritual exercise, analyzing the hypothetical consequences of positive answer to P=NP. The unreal "Urbana algorithm" would leads to world changing: progresses in cancer cure, weather prediction, and so on. As a negative consequence the author mentioned the fall of present-day cryptography methods.

Chapter 3 is dedicated to the introduction of standard problems such as cliques, Hamiltonian path, map coloring, and max-cut. The author starts from "freenemy graph", a graph of friendships (and enemies) in a world where every pair of people is either a friend or an enemy. The author clearly emphasizes the idea that solutions to these problems are easy to check, but difficult to find. He also talks of polynomial algorithms: shortest paths, matching, Eulerian paths, and minimum cut.

The history of "P=NP" problem is presented in Chapters 4 and 5. S. Cook showed in 1971 that the satisfiability problem (SAT) is, in a well-defined sense, as difficult as any problem in NP and if somebody could solve satisfiability in polynomial time, then every other problem in NP could also be solved in polynomial time. Richard Karp then showed that not only satisfiability, but another twenty problems taken from real world had the property that if you solved one of them quickly, all the other could be solved quickly. Earlier (in 1960), Jack Edmonds discussed the polynomial/exponential divide in algorithms and pointed out the need for a formalism of this issue. Besides the western contributions, Fortnow discusses the results of Soviet mathematicians Levin, Yablonsky and Kolmogorov.

Some approaches concerning how to deal with NP-complete problems are covered in Chapter 6. Heuristics and approximations, with examples from problems in Chapter 3 are treated with some attention.

Chapter 7 treats topics related to the proof of "P=NP". Fortnow discusses the "undecidability" and the relationship between circuit complexity and the P versus NP problem; he outlines the difficulties in this direction. The chapter ends in a pessimistic note; Fortnow notes the lack of a path toward resolving the problem and that it is not clear what directions to choose.

The Chapters 8 and 9 are about cryptography and quantum computing, respectively. The quantum chapter in particular is interesting for the effect of quantum computing on complexity classes. Having a workable quantum computer would not, as known so far, allow the solution of NP-complete problems in polynomial time: the best known speedup only changes the complexity of an algorithm by taking the square root of its running time. It would, however, affect problems like factoring, which closely relates the final two chapters.

In the conclusions (Chapter 10), "The Future", Fortnow makes predictions, highlighting parallel computing and big data.

Last, but not least, the book has a nice bibliography of sources which deserve to be consulted.

Intended audience: undergraduates, the general popular science audience (nonspecialists which wish an introduction to subject), lecturers who want to liven their courses.

Radu Trîmbiţaş

Leiba Rodman, Topics in Quaternion Linear Algebra, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, 2014, xii+363 pp, ISBN 978-0-691-16185-3 (hardback).

The algebra $\mathbf{H}$ of quaternions, meaning the space $\mathbf{R^4}$ endowed with a nocommutative multiplication rule, was discovered by William Rowan Hamilton in 1844 (the notation $\mathbf{H}$ comes from his name). As it is known there only four possibilities to endow $\mathbf{R^k}$ with a multiplicative structure: $k = 1$ the real numbers, $k = 2$ the complex numbers, $k = 4$ the quaternions and $k = 8$ Cayley's algebra (or the octonions). The multiplication in $\mathbf{H}$ is noncommutative and in Cayley's algebra is noncommutative and nonassociative.

In spite of their numerous applications in quantum physics, engineering (control systems), computer graphics, chemistry (molecular symmetry), the quaternions were considered only in chapters of some algebra books or in survey articles. The aim of the present book is to fill in this gap, being the first one dedicated entirely to a thorough and detailed presentations of linear algebra over quaternions. Besides classical results it contains new, previously unpublished, results with full proofs as well as results appearing for the first time in book form.

The book can be divided into two parts. The first one (Chapters 2-7), written at upper undergraduate or graduate level, contains the basic properties of quaternions, vector spaces and matrices, matrix decompositions, invariant subspaces and Jordan form, Kroneccker form, Smith form, determinants, numerical ranges. The second one, Chapters 8-14, is concerned with pencils (meaning matrices of the form $A + tB$, for A, B $m \times n$ matrices over $\mathbf{H}$, i.e. first degree matrix polynomials) of Hermitian and skewhermitian matrices and their canonical forms, indefinite inner products and conjugation, involutions for matrix pencils and for inner products. Applications are given to systems of linear differential equations with symmetries and to matrix equations. This part is written at the level of a research monograph.

The book contains also over than 200 exercises and problems of various levels of difficulties, ranging from routine to open research problems. They give opportunity to do original research – concrete, specific problems for undergraduate research and theses, and the research problems for professional mathematicians and PhD theses. The prerequisites are modest - familiarity with linear algebra, complex analysis and some calculus will suffice.

Written in a clear style, with full proofs to almost all included results, the first part of the book can be used for courses in advanced linear algebra, complemented with chapters from the second part. For working mathematicians, interested vector calculus, linear and partial differential equations, as well as practitioners (scientists and engineers) using quaternions in their research, the book is a fairly complete and accessible reference tool.

Cosmin Pelea

Boris Makarov and Anatolii Podkorytov, Real Analysis : Measures, Integrals and Applications, Universitext, Springer, London - Heidelberg - New York - Dordrecht, 2013, ISBN 978-1-4471-5121-0; ISBN 978-1-4471-5121-7 (eBook); DOI 10.1007/978-1-4471-5122-7, xix + 772 pp.

The specific feature of the present book consists in the presentation of abstract measure and integration theory alongside with the Lebesgue and Lebesgue-Stieltjes measure and integral and a lot of applications in analysis and geometry. This approach facilitates reader's access to these applications on a complete rigorous basis, usually lacking in the books treating applications, while most of the books devoted to an abstract development of the theory neglect consistent applications.

The basics of measure theory are developed in the first chapter, starting with measures defined on semi-rings of subsets of a given set and using Carathéodori extension and definition of measure to extend them to σ-algebras. One proves the fundamental properties of the Lebesgue measure – regularity, invariance with respect to rigid motions, behavior under linear maps (used later in the proof of the change of variables). Hausdorff measures and Vitali coverings are considered as well and, as application, a proof of the Brun-Minkowski inequalities and a study of some isoperimetric problems are included.

The definition and basic properties of measurable functions are treated in the third chapter, including various kinds of convergence for sequences of measurable functions and the fundamental theorems relating them, as well as Luzin's theorem on the approximation of Lebesgue measurable functions by continuous functions.

The integration is treated in the fourth chapter, first for positive simple functions and then for positive measurable functions f as the supremum of the integrals of positive simple functions majorized by f. Various theorems on the passage to limit under the integral sign are proved and a detailed study of Lebesgue integral, of functions with bounded variation and of Lebesgue-Stieltjes integral is included. As special topics, we mention the maximal function of Littlewood-Hardy and Lebesgue's theorem on the differentiation of integrals with respect to sets.

Chapter 5, *Product measures*, is concerned mainly with finite products of measure, infinite products being discussed briefly at the end of the chapter. As applications one proves the Cavalieri principle and Gagliardo-Nirenberg-Sobolev inequality relating the integrals of a smooth function and of its gradient.

The delicate problem of the change of variables in multiple Lebesgue integrals is treated in the fifth chapter. This chapter contains also a proof of Poincaré's recurrence theorem for measure preserving transformations and a study of distribution functions and zero-one laws in probability theory. Milnor's proof of Brouwer's fixed point theorem based on the change of variables is also included.

Chapter 7 contains a detailed study of integrals (both proper and improper) depending on parameters with application to the study of Gamma function.

In the seventh chapter, *The surface area*, after a quick introduction to smooth manifolds, one proves the key properties of the k-dimensional surface area in $\mathbb{R}^m$ and Gausss-Ostrogradski formula. The area on Lipschitz manifolds is considered in the last section of the chapter. The theoretical results are applied to harmonic functions.

The last theoretical chapter is Chapter 11. *Charges. The Radon-Nikodym theorem,* devoted to this important result in measure theory. Applications are given to the differentiation of measures and to the differentiability of Lipschitz functions (Rademacher's theorem).

The last of the chapters, 9. *Approximation and convolution in the spaces* $\mathcal{L}^p$, 10. *Fourier series and the Fourier transform,* and 12. *Integral representation of linear functionals,* are devoted to applications.

A consistent chapter (64 pages), *Appendices,* surveys some notions and results (most with proofs) used in the main body of the book – regular measures, extensions of continuous functions, integration of vector functions, smooth mappings and Sard's theorem, convexity.

The book is based on the courses taught by the authors at the Department of Mathematics and Mechanics of St. Petersburg State University, at various levels. Since the volume of the included material exceeds the limits of a course in measure theory, they suggest in the Preface how different chapters (or sections) can be used for introductory courses on measure theory, or for more advanced ones, at Master level: maximal functions and the differentiation of measures, Fourier series and Fourier transform, approximate identities and their applications. Some sections, containing more specialized topics, marked with $\star$, can be skipped at the first reading. A diagram on the dependence of the chapters is also included. Each section ends with a set of exercises of various difficulties.

Written in a didactic style, with clear proofs and intuitive motivations for the abstract notions, the book is a valuable addition to the literature on measure theory and integration and their applications to various areas of analysis and geometry. The numerous nontrivial examples and applications are of great importance for those interested in various domains of modern analysis and geometry, or in teaching.

S. Cobzaş

William Kirk and Naseer Shahzad, Fixed Point Theory in Distance Spaces, Springer, Heidelberg New-York Dordrecht London, 2014, xi + 173 pp, ISBN 978-3-319-10926-8; ISBN 978-3-319-10927-5 (eBook); DOI 10.1007/978-3-319-10927-5.

The book is devoted to various aspects of fixed point theory in metric spaces and their generalizations. Fixed points for contractions, a topic treated in many places, is omitted from this presentation. The book is divided into three parts: I. *Metric spaces*, II. *Length spaces and geodesic spaces*, and III. *Beyond metric spaces.*

The main topic in the first part is Caristi's fixed point theorem and its generalizations. A special attention is paid to the question whether a proof depends on the axiom of choice or on some its weaker forms (Dependent Choice (DC), Countable Choice (CC)), a theme which appears recurrently throughout the book. The fixed point for nonexpansive mappings is proved within the context of metric spaces endowed with a compact and normal convexity structure. The proof is based on Zermelo's fixed point theorem in ordered sets, requiring only ZF+DC (Zermelo-Fraenkel set theory plus the axiom of Dependent Choice). The first part closes with a presentation of fixed points for nonexpansive mappings on hyperconvex metric spaces, with emphasis on hyperconvex ultrametric spaces (metric spaces (X,d) with $d(x,y) \leq \max\{d(x,z), d(z,y)\}$ for all $x,y,z \in X$).

The second part is concerned with spaces which, in addition to their metric structure, have also a geometric structure – length spaces, geodesic spaces, Busemann spaces, CAT(k) spaces, Ptolemaic spaces and $\mathbb{R}$-trees (or metric trees).

A semimetric is a function $d : X \times X \to \mathbb{R}_+$ (X being a nonempty set) such that (i) $d(x,y) = 0 \iff x = y$ and (ii) $d(x,y) = d(y,x)$, for all $x,y \in X$. A b-space is a semimetric space (X,d) such that $d(x,y) \leq s[d(x,y) + d(z,y)]$ for all $x,y \in X$. Obviously, X is a metric space for $s = 1$. One considers also semimetric spaces satisfying a quadrilateral inequality (instead of the triangle inequality) and partial metric spaces. Various aspects of fixed point theory in these generalized metric spaces is examined in the third part of the book.

The book is clearly written and contains a very good selection of results in this rapidly growing area of research – fixed points in metric spaces and their generalizations. The sources of the presented results are carefully mentioned as well as references to related results and further investigation (the bibliography at the end of the book contains 223 items). The book will be an essential reference tool for researchers working in fixed point theory as well as for those interested in applications of metric spaces and their generalizations to other areas – computer science, biology, etc.

S. Cobzaş

Petr Hájek, Michal Johanis, Smooth Analysis in Banach Spaces, De Gruyter Series in Nonlinear Analysis and Applications, Vol. 19, xvi + 465 pp, Walter de Gruyter, Berlin - New York, 2014, ISBN: 978-3-11-025898-1, e-ISBN: 978-3-11-025899-8, ISSN: 0941-813X.

Smoothness is one of the most important and most studied topic in both finite and infinite dimensional analysis. It turns out that in the infinite dimensional case the existence and the properties of smooth mappings between Banach spaces are tightly interconnected withy the structural properties of the underlying spaces. In some cases the existence of a smooth norm forces the Banach space to be isomorphic to a Hilbert one. A decisive role in this study is played by the classical Banach spaces c_0 and ℓ_p (mainly ℓ_1), as well as by other properties as Radon-Nikodym, super-reflexivity, or being Asplund. In fact many new tools in geometric Banach space theory, as e.g. ultraproducts, were devised to solve, among others, problems related to smoothness.

The present book is devoted to a thorough and detailed presentation of various aspects of smoothness in Banach space setting. In this case, like in the finite dimensional one, a prominent role is played by polynomials, via Taylors formula – the most important result concerning smooth mappings. As the authors point out in the Introduction: "In the infinite dimensional setting the role of polynomials is brought even further, as polynomials also provide the vital link with the structure of underlying Banach space." For these reasons three chapters, 2. *Basic properties of polynomials on* $\mathbb{R}^n$, 3. *Weak continuity of polynomials and estimates of coefficients,* and 4. *Asymptotic properties of polynomials,* are entirely devoted to the presentation of various properties of polynomials, both in finite and infinite dimension.

The first chapter, 1. *Fundamental properties of smoothness,* contains a thorough, fairly detailed introduction to smoothness in Banach spaces, including high order smoothness, polynomials, Taylor's formula and converses, power series and analytic mappings. Here both real and complex cases are considered, some real results being transferred to the complex case via complexification techniques.

The rest of the book is devoted to deeper properties of smooth mapping. In Chapter 5, *Smoothness and structure,* the structural properties of Banach spaces admitting smooth functions are studied, via the variational principles of Ekeland, Stegall, , Borwein-Preiss, Fabian-Preiss, Deville-Zizler. Chapter 6, *Structural behavior of smooth mappings,* is concerned with the relations between various classes of smooth mappings involving various notions of weak and strong uniform continuity of the derivatives. An important class of Banach spaces, denoted by $\mathcal{W}$, is introduced here, allowing the extension of some smoothness results from the Banach space $C(K)$ to spaces in the class $\mathcal{W}$.

The last chapter of the book, 7. *Smooth approximation,* is concerned with the uniform approximation of continuous functions by smooth ones, or of C^k-functions by polynomials or by real analytic functions (here only the real case is considered). This line of investigation, having its roots in the pioneering work of Jaroslav Kurzweil from 1954 and 1957, is still in the focus of intense current research, many important problems waiting for solution.

As one of the authors (PH) mentions, a source of inspiration for him (and for many people working in this domain) was a list of 50 problems compiled by V. Zizler in the early 90's, later expanded to 90 in the book by R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman, New York 1993. In the meantime some of these problems were solved, their solutions being reflected in the present book, others, still unsolved, are also mentioned in the book along with new ones posed by the authors.

For reader's convenience, the authors have included (without proofs) auxiliary results (on tensor products, vector holomorphic functions, etc) as paragraphs and sections in the places were they are first used, preventing the reader to jump to appendices or to specialized monographs and leading so to a "smooth" reading of the text.

Written by two eminent specialists in Banach space theory, with important contributions to the field, the book will become an indispensable tool for researchers in Banach space geometry, smoothness and applications. By the detailed presentation of the subject it can be used also by graduate students or by instructors for introduction to the domain. At the same time, the nice and rewarding problems spread trough the text form a valuable source of inspiration for further investigation.

S. Cobzaş

Saleh A. R. Al-Mezel, Falleh R. M. Al-Solamy and Qamrul H. Ansari, Fixed Point Theory, Variational Analysis, and Optimization, CRC Press, Taylor & Francis Group, Boca Raton 2014, xx + 347 pp, ISBN: 13: 978-1-4822-2207-4.

The present volume grew out of an International Workshop on Nonlinear Analysis and Optimization, held at the University of Tabuk, Saudi Arabia, March 16-19, 2013, most of the contributors being participants to this event. It is divided into three parts: I. *Fixed point theory*; II. *Convex analysis and variational analysis*, and III. *Vector optimization.*

The first part contains three papers: Common fixed point in convex metric spaces (by Abdul Rahim Khan and Hafiz Fukhar-ud-din), Fixed points of nonlinear semigroups in modular function spaces (by B. A. Bin Dehaish and M. A. Khamsi), Approximation and selection methods for set-valued maps and fixed point theory (by Hichem Ben-El-Mechaiekh).

The second part consists also of three papers: Convexity, generalized convexity, and applications (by N. Hadjisavvas), New developments in quasiconvex optimization (by D. Aussel), and An introduction to variational-like inequalities (by Qamrul Hasan Ansari).

Two papers – Vector optimization : Basic concepts and solution methods (by Dinh The Luc and Augusta Raţiu) and Multi-objective combinatorial optimization (by Matthias Ehrgott and Xavier Gandibleux) - form the third and the last part of the book.

The papers included in the volume have both an introductory and an advanced character – they contain the basic concepts and results presented with full proofs, and at the same time new results situated at the frontier of current research.

Reporting on basic and new results in these tightly interrelated areas of nonlinear analysis – fixed point theory, variational analysis, and optimization – the survey papers included in this volume, written by renown experts in the domain, are of great interest to researchers in nonlinear analysis, as well as for the novices as a source of a quick and accessible introduction to some problems of great interest in contemporary research.

J. Kolumbán