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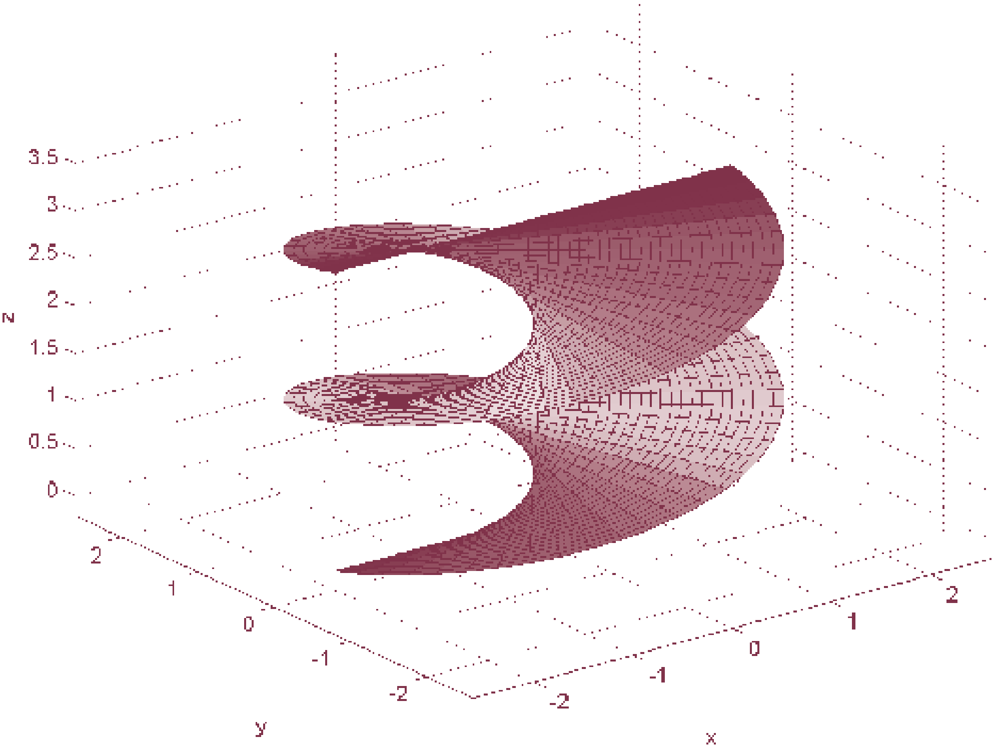

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Power Pompeiu's type inequalities for absolutely continuous functions with applications to Ostrowski's inequality

S. S. Dragomir

Abstract. In this paper, some power generalizations of Pompeiu's inequality for complex-valued absolutely continuous functions are provided. They are applied to obtain some new Ostrowski type results.

Mathematics Subject Classification (2010): 25D10, 25D10.

Keywords: Ostrowski's inequality, Pompeiu's mean value theorem, integral inequalities, special means.

1. Introduction

In 1946, Pompeiu [6] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem* (see also [8, p. 83]).

Theorem 1.1. (Pompeiu, 1946 [6]) *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exists a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \tag{1.1}$$

In 1938, A. Ostrowski [4] proved the following result in the estimating the integral mean:

Theorem 1.2. (Ostrowski, 1938 [4]) *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $|f'(t)| \leq M < \infty$ for all $t \in (a,b)$. Then for any $x \in [a,b]$, we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] M(b-a). \tag{1.2}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it cannot be replaced by a smaller quantity.*

In order to provide another approximation of the integral mean, by making use of the Pompeiu's mean value theorem, the author proved the following result:

Theorem 1.3. (Dragomir, 2005 [3]) *Let* $f : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $[a,b]$ *not containing* 0. *Then for any* $x \in [a,b]$, *we have the inequality*

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{b-a}{|x|}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f-\ell f'\|_\infty, \tag{1.3}$$

where $\ell(t) = t$, $t \in [a,b]$.

The constant $\frac{1}{4}$ *is sharp in the sense that it cannot be replaced by a smaller constant.*

In [7], E. C. Popa using a mean value theorem obtained a generalization of (1.3) as follows:

Theorem 1.4. (Popa, 2007 [7]) *Let* $f : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b). *Assume that* $\alpha \notin [a,b]$. *Then for any* $x \in [a,b]$, *we have the inequality*

$$\left|\left(\frac{a+b}{2}-\alpha\right)f(x)+\frac{\alpha-x}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f-\ell_\alpha f'\|_\infty, \tag{1.4}$$

where $\ell_\alpha(t) = t-\alpha$, $t \in [a,b]$.

In [5], J. Pečarić and S. Ungar have proved a general estimate with the p-norm, $1 \le p \le \infty$ which for $p = \infty$ give Dragomir's result.

Theorem 1.5. (Pečarić & Ungar, 2006 [5]) *Let* $f : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $0 < a < b$. *Then for* $1 \le p,q \le \infty$ *with* $\frac{1}{p}+\frac{1}{q} = 1$ *we have the inequality*

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le PU(x,p)\|f-\ell f'\|_p, \tag{1.5}$$

for $x \in [a,b]$, *where*

$$PU(x,p) \;:\; = (b-a)^{\frac{1}{p}-1}\left[\left(\frac{a^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-a^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{1/q} + \left(\frac{b^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-b^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{1/q}\right].$$

In the cases $(p,q)=(1,\infty),(\infty,1)$ *and* $(2,2)$ *the quantity* $PU(x,p)$ *has to be taken as the limit as* $p\to 1,\infty$ *and* 2, *respectively.*

For other inequalities in terms of the p-norm of the quantity $f-\ell_\alpha f'$, where $\ell_\alpha(t)=t-\alpha$, $t\in[a,b]$ and $\alpha\notin[a,b]$ see [1] and [2].

In this paper, some power Pompeiu's type inequalities for complex valued absolutely continuous functions are provided. They are applied to obtain some new Ostrowski type inequalities.

2. Power Pompeiu's type inequalities

The following inequality is useful to derive some Ostrowski type inequalities.

Corollary 2.1. (Pompeiu's Inequality) *With the assumptions of Theorem* 1.1 *and if* $\|f-\ell f'\|_\infty=\sup_{t\in(a,b)}|f(t)-tf'(t)|<\infty$ *where* $\ell(t)=t$, $t\in[a,b]$, *then*

$$|tf(x)-xf(t)|\le \|f-\ell f'\|_\infty |x-t| \tag{2.1}$$

for any $t,x\in[a,b]$.

The inequality (2.1) was stated by the author in [3].

We can generalize the above inequality for the power function as follows.

Lemma 2.2. *Let* $f:[a,b]\to\mathbb{C}$ *be an absolutely continuous function on the interval* $[a,b]$ *with* $b>a>0$. *If* $r\in\mathbb{R}$, $r\neq 0$, *then for any* $t,x\in[a,b]$ *we have*

$$|t^r f(x)-x^r f(t)| \tag{2.2}$$

$$\le \begin{cases} \frac{1}{|r|}\|f'\ell-rf\|_\infty |t^r-x^r|, \text{ if } f'\ell-rf\in L_\infty[a,b], \\ \\ \|f'\ell-rf\|_p \\ \times\begin{cases} \frac{1}{|1-q(r+1)|}\left|\frac{t^r}{x^{1-q(r+1)-r}}-\frac{x^r}{t^{1-q(r+1)-r}}\right|, \text{ for } r\neq-\frac{1}{p} \\ \\ t^r x^r|\ln x-\ln t|, \text{ for } r=-\frac{1}{p} \end{cases} \\ \text{if } f'\ell-rf\in L_p[a,b], \\ \\ \|f'\ell-rf\|_1\frac{t^r x^r}{\min\{x^{r+1},t^{r+1}\}} \end{cases}$$

or, equivalently

$$\left|\frac{f(x)}{x^r}-\frac{f(t)}{t^r}\right| \tag{2.3}$$
$$\leq\begin{cases}\frac{1}{|r|}\|f'\ell-rf\|_\infty\left|\frac{1}{x^r}-\frac{1}{t^r}\right|, \ \textit{if } f'\ell-rf\in L_\infty[a,b],\\ \\ \|f'\ell-rf\|_p\\ \quad\times\begin{cases}\frac{1}{|1-q(r+1)|}\left|\frac{1}{x^{1-q(r+1)}}-\frac{1}{t^{1-q(r+1)}}\right|, \ \textit{for } r\neq-\frac{1}{p}\\ \\ |\ln x-\ln t|, \ \textit{for } r=-\frac{1}{p}\end{cases}\\ \quad\textit{if } f'\ell-rf\in L_p[a,b],\\ \\ \|f'\ell-rf\|_1\frac{1}{\min\{x^{r+1},t^{r+1}\}},\end{cases}$$

where $p>1, \frac{1}{p}+\frac{1}{q}=1$.

Proof. If f is absolutely continuous, then $f/(\cdot)^r$ is absolutely continuous on the interval $[a,b]$ and

$$\int_t^x\left(\frac{f(s)}{s^r}\right)'ds=\frac{f(x)}{x^r}-\frac{f(t)}{t^r}$$

for any $t,x\in[a,b]$ with $x\neq t$.

Since

$$\int_t^x\left(\frac{f(s)}{s^r}\right)'ds=\int_t^x\frac{f'(s)s^r-rs^{r-1}f(s)}{s^{2r}}ds=\int_t^x\frac{f'(s)s-rf(s)}{s^{r+1}}ds,$$

then we get the following identity

$$t^rf(x)-x^rf(t)=x^rt^r\int_t^x\frac{f'(s)s-rf(s)}{s^{r+1}}ds \tag{2.4}$$

for any $t,x\in[a,b]$.

Taking the modulus in (2.4) we have

$$\begin{aligned}|t^rf(x)-x^rf(t)| &= x^rt^r\left|\int_t^x\frac{f'(s)s-rf(s)}{s^{r+1}}ds\right| \\ &\leq x^rt^r\left|\int_t^x\frac{|f'(s)s-rf(s)|}{s^{r+1}}ds\right|:=I\end{aligned} \tag{2.5}$$

and utilizing Hölder's integral inequality we deduce

$$I \leq x^r t^r \begin{cases} \sup_{s\in[t,x]([x,t])} \left|f'(s)s - rf(s)\right| \left|\int_t^x \frac{1}{s^{r+1}}ds\right|, \\ \left|\int_t^x \left|f'(s)s - rf(s)\right|^p ds\right|^{1/p} \left|\int_t^x \frac{1}{s^{q(r+1)}}ds\right|^{1/q}, \\ \left|\int_t^x \left|f'(s)s - rf(s)\right| ds\right| \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{s^{r+1}}\right\}, \end{cases} \tag{2.6}$$

$$\leq x^r t^r \begin{cases} \frac{1}{|r|} \left\|f'\ell - rf\right\|_\infty \left|\frac{1}{x^r} - \frac{1}{t^r}\right|, \\ \left\|f'\ell - rf\right\|_p \\ \quad \times \begin{cases} \frac{1}{|1-q(r+1)|} \left|\frac{1}{x^{1-q(r+1)}} - \frac{1}{t^{1-q(r+1)}}\right|, r \neq -\frac{1}{p} \\ \left|\ln x - \ln t\right|, r = -\frac{1}{p}, \end{cases} \\ \left\|f'\ell - rf\right\|_1 \frac{1}{\min\{x^{r+1}, t^{r+1}\}}, \end{cases}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and the inequality (2.2) is proved. □

3. Some Ostrowski type results

The following new result also holds.

Theorem 3.1. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b > a > 0$. If $r \in \mathbb{R}$, $r \neq 0$, and $f'\ell - rf \in L_\infty[a,b]$, then for any $x \in [a,b]$ we have*

$$\left|\frac{b^{r+1} - a^{r+1}}{r+1} f(x) - x^r \int_a^b f(t)\,dt\right| \tag{3.1}$$
$$\leq \frac{1}{|r|} \left\|f'\ell - rf\right\|_\infty$$
$$\times \begin{cases} \frac{2rx^{r+1} - x^r(a+b)(r+1) + b^{r+1} + a^{r+1}}{r+1}, & \text{if } r > 0 \\ \frac{x^r(a+b)(r+1) - 2rx^{r+1} - b^{r+1} - a^{r+1}}{r+1}, & \text{if } r \in (-\infty, 0) \setminus \{-1\}. \end{cases}$$

Also, for $r = -1$, we have

$$\left|f(x)\ln\frac{b}{a} - \frac{1}{x}\int_a^b f(t)\,dt\right| \leq 2\left\|f'\ell + f\right\|_\infty \left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2} - x}{x}\right) \tag{3.2}$$

for any $x \in [a,b]$, provided $f'\ell + f \in L_\infty[a,b]$.

The constant 2 in (3.2) is best possible.

Proof. Utilising the first inequality in (2.2) for $r \neq -1$ we have

$$\left|\frac{b^{r+1}-a^{r+1}}{r+1}f(x)-x^{r}\int_{a}^{b}f(t)\,dt\right| \leq \int_{a}^{b}\left|t^{r}f(x)-x^{r}f(t)\right|dt \leq \frac{1}{|r|}\left\|f'\ell-rf\right\|_{\infty}\int_{a}^{b}\left|t^{r}-x^{r}\right|dt. \tag{3.3}$$

Observe that

$$\int_{a}^{b}\left|t^{r}-x^{r}\right|dt = \begin{cases} \int_{a}^{x}\left(x^{r}-t^{r}\right)dt+\int_{x}^{b}\left(t^{r}-x^{r}\right)dt, & \text{if } r>0,\\ \int_{a}^{x}\left(t^{r}-x^{r}\right)dt+\int_{x}^{b}\left(x^{r}-t^{r}\right)dt, & \text{if } r\in(-\infty,0)\setminus\{-1\}. \end{cases}$$

Then for $r>0$ we have

$$\begin{aligned} &\int_{a}^{x}\left(x^{r}-t^{r}\right)dt+\int_{x}^{b}\left(t^{r}-x^{r}\right)dt\\ &= x^{r}(x-a)-\frac{x^{r+1}-a^{r+1}}{r+1}+\frac{b^{r+1}-x^{r+1}}{r+1}-x^{r}(b-x)\\ &= 2x^{r+1}-x^{r}(a+b)+\frac{b^{r+1}+a^{r+1}-2x^{r+1}}{r+1}\\ &= \frac{2rx^{r+1}+2x^{r+1}-x^{r}(a+b)(r+1)+b^{r+1}+a^{r+1}-2x^{r+1}}{r+1}\\ &= \frac{2rx^{r+1}-x^{r}(a+b)(r+1)+b^{r+1}+a^{r+1}}{r+1} \end{aligned}$$

and for $r\in(-\infty,0)\setminus\{-1\}$ we have

$$\begin{aligned} &\int_{a}^{x}\left(t^{r}-x^{r}\right)dt+\int_{x}^{b}\left(x^{r}-t^{r}\right)dt\\ &= -\frac{2rx^{r+1}-x^{r}(a+b)(r+1)+b^{r+1}+a^{r+1}}{r+1}. \end{aligned}$$

Making use of (3.3) we get (3.1).

Utilizing the inequality (2.2) for $r=-1$ we have

$$\left|t^{-1}f(x)-x^{-1}f(t)\right| \leq \left\|f'\ell+f\right\|_{\infty}\left|t^{-1}-x^{-1}\right|$$

if $f'\ell+f\in L_{\infty}[a,b]$.

Integrating this inequality, we have

$$\left|f(x)\ln\frac{b}{a}-x^{-1}\int_{a}^{b}f(t)\,dt\right| \leq \int_{a}^{b}\left|t^{-1}f(x)-x^{-1}f(t)\right|dt \leq \left\|f'\ell+f\right\|_{\infty}\int_{a}^{b}\left|t^{-1}-x^{-1}\right|dt. \tag{3.4}$$

Since

$$\int_a^b \left|\frac{1}{x}-\frac{1}{t}\right| dt = \left[\int_a^x \left(\frac{1}{t}-\frac{1}{x}\right) dt + \int_x^b \left(\frac{1}{x}-\frac{1}{t}\right) dt\right]$$
$$= \left(\ln\frac{x}{a} - \frac{x-a}{x} + \frac{b-x}{x} - \ln\frac{b}{x}\right)$$
$$= \ln\frac{x^2}{ab} + \frac{a+b-2x}{x}$$
$$= 2\left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2}-x}{x}\right),$$

then by (3.4) we get the desired inequality (3.2).

Now, assume that (3.2) holds with a constant $C>0$, i.e.

$$\left|f(x)\ln\frac{b}{a} - x^{-1}\int_a^b f(t)\,dt\right| \le C\left\|f'\ell+f\right\|_\infty \left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2}-x}{x}\right) \tag{3.5}$$

for any $x\in[a,b]$.

If we take in (3.5) $f(t)=1, t\in[a,b]$, then we get

$$\left|\ln\frac{b}{a} - \frac{b-a}{x}\right| \le C\left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2}-x}{x}\right) \tag{3.6}$$

for any for any $x\in[a,b]$.

Making $x=a$ in (3.5) produces the inequality

$$\left|\ln\frac{b}{a} - \frac{b-a}{a}\right| \le C\left(\frac{b-a}{2a} - \frac{1}{2}\ln\frac{b}{a}\right)$$

which implies that $C\ge 2$.

This proves the sharpness of the constant 2 in (3.2). □

Remark 3.2. Consider the r-Logarithmic mean

$$L_r = L_r(a,b) := \left[\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right]^{1/r}$$

defined for $r\in\mathbb{R}\setminus\{0,-1\}$ and the Logarithmic mean, defined as

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

If $A=A(a,b):=\frac{a+b}{2}$, then from (3.1) we get for $x=A$ the inequality

$$\left|L_r^r(b-a)f(A) - A^r\int_a^b f(t)\,dt\right| \tag{3.7}$$
$$\le \frac{2}{|r|}\left\|f'\ell - rf\right\|_\infty \begin{cases} \frac{A\left(b^{r+1},a^{r+1}\right)-A^{r+1}}{r+1}, & \text{if } r>0, \\ \frac{A^{r+1}-A\left(b^{r+1},a^{r+1}\right)}{r+1}, & \text{if } r\in(-\infty,0)\setminus\{-1\}, \end{cases}$$

while from (3.2) we get

$$\left|L^{-1}(b-a)f(A)-A^{-1}\int_a^b f(t)\,dt\right|\le 2\left\|f'\ell+f\right\|_\infty \ln\frac{A}{G}. \tag{3.8}$$

The following related result holds.

Theorem 3.3. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b>a>0$. If $r\in\mathbb{R}$, $r\neq 0$, then for any $x\in[a,b]$ we have*

$$\begin{aligned}&\left|\frac{f(x)}{x^r}(b-a)-\int_a^b\frac{f(t)}{t^r}dt\right| \\ &\le\frac{1}{|r|}\left\|f'\ell-rf\right\|_\infty \\ &\times\begin{cases}\frac{2x^{1-r}-a^{1-r}-b^{1-r}}{1-r}+\frac{1}{x^r}(b+a-2x), & r\in(0,\infty)\diagdown\{1\}\\[2ex] \frac{a^{1-r}+b^{1-r}-2x^{1-r}}{1-r}+\frac{1}{x^r}(2x-a-b), & \textit{if } r<0.\end{cases}\end{aligned} \tag{3.9}$$

Also, for $r=1$, we have

$$\left|\frac{f(x)}{x}(b-a)-\int_a^b\frac{f(t)}{t}dt\right|\le 2\left\|f'\ell-f\right\|_\infty\left(\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x}\right) \tag{3.10}$$

for any $x\in[a,b]$, provided $f'\ell-f\in L_\infty[a,b]$.

The constant 2 is best possible in (3.10).

Proof. From the first inequality in (2.3) we have

$$\left|\frac{f(x)}{x^r}-\frac{f(t)}{t^r}\right|\le\frac{1}{|r|}\left\|f'\ell-rf\right\|_\infty\left|\frac{1}{x^r}-\frac{1}{t^r}\right|, \tag{3.11}$$

for any $t,x\in[a,b]$, provided $f'\ell-rf\in L_\infty[a,b]$.

Integrating over $t\in[a,b]$ we get

$$\begin{aligned}\left|\frac{f(x)}{x^r}(b-a)-\int_a^b\frac{f(t)}{t^r}dt\right|&\le\int_a^b\left|\frac{f(x)}{x^r}-\frac{f(t)}{t^r}\right|dt\\&\le\frac{1}{|r|}\left\|f'\ell-rf\right\|_\infty\int_a^b\left|\frac{1}{x^r}-\frac{1}{t^r}\right|dt\end{aligned} \tag{3.12}$$

for $r\in\mathbb{R}$, $r\neq 0$.

For $r \in (0,\infty)\setminus\{1\}$ we have

$$\begin{aligned}
&\int_a^b \left|\frac{1}{x^r}-\frac{1}{t^r}\right| dt \\
&= \int_a^x \left(\frac{1}{t^r}-\frac{1}{x^r}\right) dt + \int_x^b \left(\frac{1}{x^r}-\frac{1}{t^r}\right) dt \\
&= \frac{x^{1-r}-a^{1-r}}{1-r} - \frac{1}{x^r}(x-a) + \frac{1}{x^r}(b-x) - \frac{b^{1-r}-x^{1-r}}{1-r} \\
&= \frac{2x^{1-r}-a^{1-r}-b^{1-r}}{1-r} + \frac{1}{x^r}(b+a-2x)
\end{aligned}$$

for any $x \in [a,b]$.

For $r<0$, we also have

$$\int_a^b \left|\frac{1}{x^r}-\frac{1}{t^r}\right| dt = \frac{a^{1-r}+b^{1-r}-2x^{1-r}}{1-r} + \frac{1}{x^r}(2x-a-b)$$

for any $x \in [a,b]$.

For $r=1$ we have

$$\int_a^b \left|\frac{1}{x}-\frac{1}{t}\right| dt = 2\left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2}-x}{x}\right)$$

for any $x \in [a,b]$, and the inequality (3.10) is obtained.

The sharpness of the constant 2 follows as in the proof of Theorem 3.1 and the details are omitted. □

Remark 3.4. If we take $x=A$ in Theorem 3.3, then we we have

$$\begin{aligned}
&\left|\frac{f(A)}{A^r}(b-a) - \int_a^b \frac{f(t)}{t^r} dt\right| \qquad (3.13)\\
&\le \frac{2}{|r|}\left\|f'\ell - rf\right\|_\infty \begin{cases} \frac{A^{1-r}-A\left(a^{1-r},b^{1-r}\right)}{1-r}, & r \in (0,\infty)\setminus\{1\}, \\ \frac{A\left(a^{1-r},b^{1-r}\right)-A^{1-r}}{1-r}, & \text{if } r<0. \end{cases}
\end{aligned}$$

Also, for $r=1$, we have

$$\left|\frac{f(A)}{A}(b-a) - \int_a^b \frac{f(t)}{t} dt\right| \le 2\left\|f'\ell - f\right\|_\infty \ln\frac{A}{G}. \qquad (3.14)$$

Remark 3.5. The interested reader may obtain other similar results in terms of the p-norms $\left\|f'\ell - rf\right\|_p$ with $p \ge 1$. However, since some calculations are too complicated, the details are not presented here.

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Semi-φ_h and strongly $\log$-φ convexity

Hira Iqbal and Shaheen Nazir

Abstract. In this note, semi-φ_h-convexity as a generalization of h-convexity and semi φ-convexity, and strongly log-φ convex functions have been introduced and studied. Some properties of semi-φ_h-convex functions are proved. Also, some new results of Hemite-Hadamard type inequalities for semi-φ_h-convex functions, semi log- φ and strongly log-φ convex functions are obtained.

Mathematics Subject Classification (2010): 26A51, 26A15.

Keywords: convex functions, semi-φ-convex functions, Hermite-Hadamard inequalities, semi quasi-φ-convex functions.

1. Introduction

In 1883, Hermite proved an inequality, rediscovered by Hadamard in 1893, that for a convex function f on $[a,b] \in \mathbb{R}$, also continuous at the endpoints, one has that

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

This is known as Hermite-Hadamard inequality. In the literature, many modifications, generalizations and extensions of this inequality has been obtained for last few years.

Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$, is said to be convex on I if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

for all $x, y \in I$ and $t \in (0,1)$.

Let I be an interval in $\mathbb{R}$ and $h : (0,1) \to (0,\infty)$ be a given function. Then a function $f : I \to \mathbb{R}$ is said to be h-convex if

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y),$$

for all $x, y \in I$ and $t \in (0,1)$.

If $h(t) = t^s; s \in (0,1)$, then f is said to be s-convex in second sense [2], if f is non-negative and $h(t) = \frac{1}{t}$ then f is said to be Godunova-Levin function [6] and if f is non-negative with $h(t) = 1$ then f is P-convex function [7].

In [14], Youness introduced a new class of functions called φ-convex functions and he established some results about these sets and functions. Later on, the result by Youness [14] were improved by Yang [13], Duca *et al.* [4] and Chen [3]. Throughout this paper, we assume that $\varphi : I \to I$, where I is a real interval and $h : (0,1) \to (0,\infty)$ are given maps.

Definition 1.1. *A function $f : I \to \mathbb{R}$ is said to be φ-convex on I if*

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)),$$

for all $x, y \in I$ and $t \in (0,1)$.

In [11], Sarikaya has studied φ_h- convexity and obtained some new inequalities.

Definition 1.2. *Let I be an interval in $\mathbb{R}$. We say that a function $f : I \to [0,\infty)$ is a φ_h-convex if*

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)),$$

for all $t \in (0,1)$ and $x, y \in I$.

Theorem 1.3. (Th. 2, [11]) *Let $h : (0,1) \to (0,\infty)$ be a given function. If $f : I \to [0,\infty)$ is Lebesgue integrable on I and φ_h-convex for continuous function $\varphi : [a,b] \to [a,b]$, with $\varphi(a) \neq \varphi(b)$, then the following inequality holds:*

$$\frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)f(\varphi(b) + \varphi(a) - x)dx$$

$$\leq \Big[f^2(\varphi(x)) + f^2(\varphi(y))\Big] \int_0^1 h(t)h(1-t)dt + 2f(\varphi(x))f(\varphi(y)) \int_0^1 h^2(t)dt.$$

Hu at al [8] studied firstly the notion of semi-φ-convexity. Chen in [3] modified their results and defined the following class of functions.

Definition 1.4. *The function $f : I \to \mathbb{R}$ is semi-φ-convex, if for every $x, y \in I$ and $t \in (0,1)$ we have*

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(x) + (1-t)f(y).$$

Toader [12] defined the following function:

Definition 1.5. *Let be $b > 0$ and $m \in (0,1]$. A function $f : [0,b] \to [0,\infty)$ is said to be m-convex if*

$$f(tx + m(1-t)y) \leq tf(x) + (1-t)f(y),$$

for all $x, y \in [0,b]$, $t \in [0,1]$.

In [5], Dragomir and Pečarić showed that the following result holds for m-convex functions.

Theorem 1.6. (Th. 197, [5]) *If $f : [0,\infty) \to [0,\infty)$ is a m-convex function with $m \in (0,1)$ and Lebesgue integrable on $[ma,b]$ where $0 \leq a \leq b$ and $mb \neq a$, then*

$$\frac{1}{m+1}\left[\frac{1}{mb-a}\int_a^{mb} f(x)dx + \frac{1}{b-ma}\int_{ma}^{b} f(x)dx\right] \leq \frac{f(a)+f(b)}{2}.$$

The rest of the paper is organized as follows: In section 2, semi-φ_h-convexity has been defined and some properties are studied. In section 3, some new results of Hadamard type inequalities are proved. In the last section, semi log-φ and strongly log-φ convex functions are discussed and some inequalities are obtained.

2. Semi-φ_h-Convexity

In this section, we define the following function:

Definition 2.1. *Let $\varphi : [a, b] \to [a, b]$ and I be an interval in $\mathbb{R}$ such that $[a, b] \subseteq I$. Let $h : (0, 1) \to (0, \infty)$ be a given function. We say that a function $f : I \to [0, \infty)$ is a semi-φ_h-convex if for all $t \in (0, 1)$ and $x, y \in I$, we have*

$$f(t\varphi(x) + (1 - t)\varphi(y)) \leq h(t)f(x) + h(1 - t)f(y).$$

Remark 2.2. 1. If $h(t) = t$, f is a semi-φ-convex function on I.
2. If $h(t) = t^s$, f is a semi-φ_s-convex function on I.
3. If $h(t) = \frac{1}{t}$, f is a semi-φ Gudunova-Levin convex function on I.
4. If $h(t) = 1$, f is a semi-φP-convex function on I.
5. If $\varphi(x) = x$, f is a h-convex function on I.
6. If $\varphi(x) = x$ and $h(t) = t$, f is a convex function on I.

Example 2.3. [3] Let $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\varphi(x) = \begin{cases} 1, & 1 \leq x \leq 4 \\ 1 + \frac{2}{\pi}\arctan(1 - x), & x < 1 \\ 2 + \frac{\pi}{4}\arctan(x - 4), & x > 4. \end{cases}$$

and $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 7, & x < 1 \text{ or } x > 4 \\ x - 3, & 1 \leq x < 2 \\ 3 - x, & 2 \leq x \leq 3 \\ x - 3, & 3 < x \leq 4. \end{cases}$$

Here f is a semi-φ_h-convex function on $\mathbb{R}$ for $h(t) = t$.

Example 2.4. Let $h(t) = 1$ for all $t \in \mathbb{R}$, $\varphi(x) = -x^2$, for all $x \in \mathbb{R}$, and

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 2, & x \leq 0. \end{cases}$$

Then f is a semi-φ P-convex function on $\mathbb{R}$.

Now we prove some properties of semi-φ_h-convex functions.

Theorem 2.5. *If $f, g : I \to [0, \infty)$ are semi-φ_h-convex functions, where $h : (0, 1) \to (0, \infty)$ is a given function, and $\alpha > 0$ then $f + g$ and αf are semi-φ_h-convex functions.*

Proof. Since f, g are semi-φ_h convex functions then for $x, y \in I$ and $t \in (0, 1)$,

$$\begin{aligned}(f + g)(t\varphi(x) + (1 - t)\varphi(y)) &= f(t\varphi(x) + (1 - t)\varphi(y)) + g(t\varphi(x) + (1 - t)\varphi(y)) \\ &\leq h(t)(f + g)(x) + h(1 - t)(f + g)(y),\end{aligned}$$

and

$$\begin{aligned}(\alpha f)(t\varphi(x)+(1-t)\varphi(y)) &\le \alpha[h(t)f(x)+h(1-t)f(y))] \\ &= h(t)(\alpha f)(x)+h(1-t)(\alpha f)(y).\end{aligned}$$

□

Lemma 2.6. *If $f : I \to [0,\infty)$ is a semi-φ convex function and g is an increasing h-convex function, where range of f is contained in the domain of g and $h : (0,1) \to (0,\infty)$, then $g \circ f$ is a semi-φ_h- convex function.*

Proof. Since f is semi-φ-convex function then for $x, y \in I$ and $t \in (0,1)$,

$$f(t\varphi(x)+(1-t)\varphi(y)) \le tf(x)+(1-t)f(y).$$

Since g is increasing and h-convex we have

$$\begin{aligned}(g\circ f)(t\varphi(x)+(1-t)\varphi(y)) &\le g(tf(x)+(1-t)f(y)) \\ &\le h(t)(g\circ f)(x)+h(1-t)(g\circ f)(y).\end{aligned}$$

This completes the proof. □

Lemma 2.7. *If f is semi-φ-convex and $h(t) \ge t$ then f is semi-φ_h-convex.*

Proof.

$$f(t\varphi(x)+(1-t)\varphi(y)) \le tf(x)+(1-t)f(y) \le h(t)f(x)+h(1-t)f(y).$$

This completes the proof. □

Lemma 2.8. *If f is semi-φ_h convex and $h(t) \le t$ then f is semi-φ-convex.*

Proof.

$$f(t\varphi(x)+(1-t)\varphi(y)) \le h(t)f(x)+h(1-t)f(y) \le tf(x)+(1-t)f(y).$$

This completes the proof. □

Lemma 2.9. *Let $h_1, h_2 : (0,1) \to (0,\infty)$ such that $h_2(t) \le h_1(t)$. If f is semi-φ_{h_2} convex then f is semi-φ_{h_1} convex.*

Proof. Since f is semi-φ_{h_2} convex then for $x, y \in I$ and $t \in (0,1)$ we have

$$f(t\varphi(x)+(1-t)\varphi(y)) \le h_2(t)f(x)+h_2(1-t)f(y) \le h_1(t)f(x)+h_1(1-t)f(y).$$

This completes the proof. □

3. Hermite-Hadamard Type Inequalities

Theorem 3.1. *If $[a,b] \subseteq I$, $\varphi : [a,b] \to [a,b]$ is a continuous function such that $\varphi(a) \ne \varphi(b)$ and the function $f : I \to [0,\infty)$ is Lebesgue integrable on I and semi-φ_h convex, then*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)dx \le \Big(f(a)+f(b)\Big)\int_0^1 h(t)dt.$$

Proof. Since f is semi-φ_h convex, we have for $t \in (0,1)$,

$$f(t\varphi(a) + (1-t)\varphi(b)) \leq h(t)f(a) + h(1-t)f(b).$$

Integrating the above inequality over the interval $(0,1)$,

$$\int_0^1 f(t\varphi(a) + (1-t)\varphi(b))dt \leq (f(a) + f(b)) \int_0^1 h(t)dt.$$

Substituting $x = t\varphi(a) + (1-t)\varphi(b)$ we get the required inequality. □

Corollary 3.2. *Under the assumptions of Theorem* 3.1 *with* $h(t) = t$ *for all* $t \in (0,1)$, *we have*

$$\frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

Corollary 3.3. *Under the assumptions of Theorem* 3.1 *with* $s \in (0,1)$ *and* $h(t) = t^s$ *for all* $t \in (0,1)$, *we have*

$$\frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx \leq \frac{f(a) + f(b)}{s+1}.$$

Corollary 3.4. *Under the assumptions of Theorem* 3.1 *with* $h(t) = 1$ *for* $t \in (0,1)$, *we have*

$$\frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx \leq f(a) + f(b).$$

Remark 3.5. If $h(t) = t$ for $t \in (0,1)$ and $\varphi(x) = x$ we have

$$\frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

Theorem 3.6. *If* $[a,b] \subseteq I$, $\varphi : [a,b] \to [a,b]$ *is a continuous function such that* $\varphi(a) \neq \varphi(b)$ *and the function* $f : I \to [0,\infty)$ *is Lebesgue integrable on* I *and semi-*φ_h *convex, then*

$$\frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)f(\varphi(a) + \varphi(b) - x)dx$$
$$\leq (f^2(a) + f^2(b))(\int_0^1 h(t)h(1-t)dt + 2f(a)f(b) \int_0^1 h^2(t)dt).$$

Proof. Since f is semi-φ_h convex we have for $t \in (0,1)$

$$f(t\varphi(a) + (1-t)\varphi(b)) \leq h(t)f(a) + h(1-t)f(b),$$

and

$$f((1-t)\varphi(a) + (t\varphi(b)) \leq h(1-t)f(a) + h(t)f(b).$$

By multiplying both inequalities, we get

$$f(t\varphi(a) + (1-t)\varphi(b))f((1-t)\varphi(a) + t\varphi(b))$$
$$\leq h(1-t)h(t)(f^2(a) + f^2(b)) + f(a)f(b)(h^2(t) + h^2(1-t)).$$

We obtain

$$\int_0^1 f(t\varphi(a)+(1-t)\varphi(b))f((1-t)\varphi(a)+(t\varphi(b))dt$$
$$\leq (f^2(a)+f^2(b))\int_0^1 h(1-t)h(t)dt + 2f(a)f(b)\int_0^1 h^2(t)dt.$$

Substituting $x = t\varphi(a)+(1-t)\varphi(b)$, we get the required inequality. □

Corollary 3.7. *Under the assumptions of Theorem* 3.6 *with* $h(t) = t$ *for all* $t \in (0,1)$, *we have*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)f(\varphi(b)+\varphi(a)-x)dx$$
$$\leq \frac{f^2(a)+f^2(b)}{6} + \frac{2f(a)f(b)}{3}.$$

Theorem 3.8. *If* $[a,b] \subseteq I$, $\varphi : [a,b] \to [a,b]$ *is a continuous function such that* $\varphi(a) \neq \varphi(b)$ *and the functions* $f, g : I \to [0,\infty)$ *is Lebesgue integrable on* I *and semi-*φ_h *convex, then*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)g(x)dx$$
$$\leq M(a,b)\int_0^1 h^2(t)dt + N(a,b)\int_0^1 h(t)h(1-t)dt.$$

where

$$M(a,b) = f(a)g(a)+f(b)g(b),$$
$$N(a,b) = f(a)g(b)+f(b)g(a).$$

Proof. Since f, g are semi-φ_h-convex we have for $t \in (0,1)$

$$f(t\varphi(a)+(1-t)\varphi(b)) \leq h(t)f(a)+h(1-t)f(b),$$

and

$$g(t\varphi(a)+(1-t)\varphi(b)) \leq h(t)g(a)+h(1-t)g(b).$$

By multiplying both sides, we get

$$f(t\varphi(a)+(1-t)\varphi(b))g(t\varphi(a)+(1-t)\varphi(b))$$
$$\leq h^2(t)f(a)g(a)+h^2(1-t)f(b)g(b)+h(t)h(1-t)f(a)g(b)+h(t)h(1-t)f(b)g(a).$$

Integrating over the interval $(0,1)$, we obtain

$$\int_0^1 f(t\varphi(a)+(1-t)\varphi(b))g(t\varphi(a)+(1-t)\varphi(b))dt$$
$$\leq (f(a)g(a)+f(b)g(b))\int_0^1 h^2(t)dt + (f(a)g(b)+f(b)g(a))\int_0^1 h(t)h(1-t)dt.$$

Replacing $x = t\varphi(a)+(1-t)\varphi(b)$, we get the required inequality. □

Definition 3.9. *Let be* $m \in (0,1]$. *A function* $f : [0,b] \to [0,\infty)$ *is said to be semi-*φ_m*-convex if*

$$f(t\varphi(x)+m(1-t)\varphi(y)) \leq tf(x)+m(1-t)f(y),$$

for all $x, y \in [0,b]$, $t \in [0,1]$.

Remark 3.10. If $m = 1$, then f is semi-φ-convex, and if $m = 1, \varphi(x) = x$ for all $x \in [0, b]$, then f is convex on $[0, b]$.

Theorem 3.11. *If $f : [0,\infty) \to [0,\infty)$ is a semi- φ_m-convex function, with $m \in (0,1)$ such that $m\varphi(b) \neq \varphi(a)$ and $m\varphi(a) \neq \varphi(b)$ and f is Lebesgue integrable on $[ma, b]$ then*

$$\frac{1}{m+1}\Big[\frac{1}{m\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{m\varphi(b)} f(x)dx + \frac{1}{\varphi(b)-m\varphi(a)}\int_{m\varphi(a)}^{\varphi(b)} f(x)dx\Big] \leq \frac{f(a)+f(b)}{2}.$$

Proof. Since f is semi- φ_m-convex we have following inequalities

$$f(t\varphi(a)+m(1-t)\varphi(b)) \leq tf(a)+m(1-t)f(b),$$

$$f((1-t)\varphi(a)+mt\varphi(b)) \leq (1-t)f(a)+mtf(b),$$

$$f(mt\varphi(a)+(1-t)\varphi(b)) \leq mtf(a)+(1-t)f(b),$$

$$f(m(1-t)\varphi(a)+t\varphi(b)) \leq m(1-t)f(a)+tf(b).$$

Adding the above four inequalities, we get

$$f(t\varphi(a)+m(1-t)\varphi(b)) + f((1-t)\varphi(a)+mt\varphi(b))$$
$$+f(mt\varphi(a)+(1-t)\varphi(b)) + f(m(1-t)\varphi(a)+t\varphi(b))$$
$$\leq (m+1)(f(a)+f(b)).$$

Now, integrating over the interval $(0, 1)$, we have

$$\int_0^1 f(t\varphi(a)+m(1-t)\varphi(b))dt + \int_0^1 f((1-t)\varphi(a)+mt\varphi(b))dt +$$
$$\int_0^1 f(mt\varphi(a)+(1-t)\varphi(b))dt + \int_0^1 f(m(1-t)\varphi(a)+t\varphi(b))dt$$
$$\leq (m+1)(f(a)+f(b)).$$

Using the substitution $x = t\varphi(a) + (1-t)\varphi(b)$, we have

$$\int_0^1 f(t\varphi(a)+m(1-t)\varphi(b))dt = \int_0^1 f((1-t)\varphi(a)+mt\varphi(b))dt$$
$$= \frac{1}{m\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{m\varphi(b)} f(x)dx,$$

and using the substitution $x = t\varphi(a) + (1-t)\varphi(b)$, we have

$$f(mt\varphi(a)+(1-t)\varphi(b))dt = f(m(1-t)\varphi(a)+t\varphi(b))dt$$
$$= \frac{1}{\varphi(b)-m\varphi(a)}\int_{m\varphi(a)}^{\varphi(b)} f(x)dx.$$

Using the above equations, we get the required inequality. $\square$

4. Semi-φ and strongly log-φ convexity

Definition 4.1. [3] *A function $f : I \to [0, \infty)$ is a semi log-φ convex if, for all $t \in (0, 1)$ and $x, y \in I$, one has*

$$f(t\varphi(x) + (1 - t)\varphi(y)) \leq f(x)^t f(y)^{1-t}.$$

Polyak [9] introduced strongly convex functions which plays an important role in optimization theory and mathematical economics.

A function $f : I \to \mathbb{R}$ is said to be strongly convex with modulus $c > 0$ on I if

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + ct(1 - t)(x - y)^2,$$

for all $x, y \in I$ and $t \in (0, 1)$.

Sarikaya [11] defined strongly log-convex functions as:

Definition 4.2. *A positive function $f : I \to (0, \infty)$ is said to be strongly log-convex with respect to $c > 0$ if*

$$f(tx + (1 - t)y) \leq f(x)^t f(y)^{1-t} - ct(1 - t)(x - y)^2,$$

for all $x, y \in I$ and $t \in (0, 1)$.

In this section we relate Hermite Hadamard type inequalities to some special means. Firstly, let us recall the following means for positive $a, b \in \mathbb{R}$:
Arithmetic mean:

$$A(a, b) = \frac{a + b}{2},$$

Geometric mean:

$$G(a, b) = \sqrt{ab},$$

Logarithmic mean:

$$L(a, b) = \frac{b - a}{\log(b) - \log(a)}.$$

Theorem 4.3. *If the positive function $f : I \to (0, \infty)$ is semi log-φ convex function and Lebesgue integrable on I, then*

$$\frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} G(f(x), f(\varphi(a) + \varphi(b) - x))dx \leq G(f(a), f(b)),$$

for all $a, b \in I$, $a < b$.

Proof. Since f is semi log-φ convex, we have

$$f(t\varphi(a) + (1 - t)\varphi(b)) \leq f(a)^t f(b)^{1-t}, \ \forall\, t \in (0, 1)$$

and

$$f((1 - t)\varphi(a) + t\varphi(b)) \leq f(a)^{1-t} f(b)^t, \ \forall\, t \in (0, 1).$$

By multiplying both inequalities, we get

$$f(t\varphi(a) + (1 - t)\varphi(b))f((1 - t)\varphi(a) + t\varphi(b)) \leq f(a)f(b).$$

Now, taking square root, we get

$$G(f(t\varphi(a) + (1 - t)\varphi(b)), f((1 - t)\varphi(a) + t\varphi(b))) \leq G(f(a), f(b)).$$

By integrating over the interval $(0,1)$ and replacing $x = t\varphi(a) + (1-t)\varphi(b)$, we get the required inequality. $\square$

Theorem 4.4. *If the positive function $f : I \to (0,\infty)$ is semi log-φ convex function and Lebesgue integrable on I, then*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)dx \le L(f(b), f(a)) \le \frac{f(a)+f(b)}{2},$$

for all $a, b \in I,\ a < b$.

Proof. Since f is semi log-φ convex, we have

$$f(t\varphi(a) + (1-t)\varphi(b)) \le f(a)^t f(b)^{1-t},\ \forall\, t \in (0,1).$$

Integrating over the interval $(0,1)$, we get

$$\int_0^1 f(t\varphi(a) + (1-t)\varphi(b))dt \le \int_0^1 f(a)^t f(b)^{1-t} dt$$
$$= \frac{f(b)-f(a)}{\log f(b) - \log f(a)} = L(f(b), f(a)) \le \frac{f(a)+f(b)}{2}.$$

Substituting $x = t\varphi(a) + (1-t)\varphi(b)$, we get the required result. $\square$

Theorem 4.5. *If the functions $f, g : I \to (0,+\infty)$ are semi log-φ convex and Lebesgue integrable on I, then*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)g(x)dx \le L(f(b)g(b), f(a)g(a))$$
$$\le \frac{1}{4}\{(f(b)+f(a))L(f(b), f(a)) + (g(a)+g(b))L(g(b), g(a))\},$$

for all $a, b \in I,\ a < b$.

Proof. Since f, g are semi log-φ convex, we have

$$f(t\varphi(a) + (1-t)\varphi(b)) \le f(a)^t f(b)^{1-t},\ \forall\, t \in (0,1)$$

and

$$g(t\varphi(a) + (1-t)\varphi(b)) \le g(a)^t g(b)^{1-t},\ \forall\, t \in (0,1).$$

Multiplying both inequalities and integrating over the interval $(0,1)$, we get

$$\int_0^1 f(t\varphi(a) + (1-t)\varphi(b))g(t\varphi(a) + (1-t)\varphi(b))dt$$
$$\le \int_0^1 f(a)^t f(b)^{1-t} g(a)^t g(b)^{1-t} dt$$
$$= \frac{f(b)g(b) - f(a)g(a)}{\log(f(b)g(b)) - \log(f(a)g(a))}$$
$$= L(f(b)g(b), f(a)g(b)). \tag{4.1}$$

By Young's inequality, we have

$$\int_0^1 f(a)^t f(b)^{1-t} g(a)^t g(b)^{1-t} dt$$

$$\leq \frac{1}{2}\int_0^1 \{[f(a)^t f(b)^{1-t}]^2 + [g(a)^t g(b)^{1-t}]^2\}dt$$
$$= \frac{1}{4}\left[\frac{(f(b))^2 - (f(a))^2}{\log(f(b)) - \log(f(a))} + \frac{(g(b))^2 - (g(a))^2}{\log(g(b)) - \log(g(a))}\right]$$
$$= \frac{1}{4}\left\{(f(a) + f(b))L(f(b), f(a)) + (g(a) + g(b))L(g(b), g(a))\right\}. \quad (4.2)$$

Using (4.1) and (4.2) and substituting $x = t\varphi(a)+(1-t)\varphi(b)$, we get the required result. □

Definition 4.6. *Let $f : I \to (0,\infty)$ be a positive function. We say that f is strongly log-φ convex with respect to $c > 0$ if*

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq f(\varphi(x))^t f(\varphi(y))^{1-t} - ct(1-t)(\varphi(x) - \varphi(y))^2,$$

for all $x, y \in I$ and $t \in (0,1)$.

Remark 4.7. From the above inequality, using arithmetic mean- geometric mean, we have

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq f(\varphi(x))^t f(\varphi(y))^{1-t} - ct(1-t)(\varphi(x) - \varphi(y))^2$$

$$\leq tf(\varphi(x)) + (1-t)f(\varphi(y)) - ct(1-t)(\varphi(x) - \varphi(y))^2$$

$$\leq \max\{f(\varphi(x)), f(\varphi(y))\} - ct(1-t)(\varphi(x) - \varphi(y))^2.$$

Example 4.8. Let

$$\varphi(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0. \end{cases}$$

Then for $c = \frac{1}{4}$ the function

$$f(x) = \begin{cases} 0, & -1 < x < 1 \\ 1, & \text{otherwise} \end{cases}$$

is strongly log-φ convex function with respect to c on $\mathbb{R}$.

Theorem 4.9. *Let $\varphi : [a,b] \to [a,b]$ be a continuous function and $f : I \to (0,\infty)$ be a positive strongly log-φ convex function with respect to $c > 0$, where $a, b \in I$. If f is Lebesgue integrable on I then*

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) + \frac{c}{2}(\varphi(a)-\varphi(b))^2 \leq \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} G(f(x), f(\varphi(a)+\varphi(b)-x))dx$$
$$\leq \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)dx$$
$$\leq L(f(\varphi(b)), f(\varphi(a))) - \frac{c}{6}(\varphi(a) - \varphi(b))^2$$
$$\leq \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{c}{6}(\varphi(a) - \varphi(b))^2.$$

Proof. Since f is strongly log-φ convex, we have for $t \in (0,1)$

$$f(\frac{\varphi(a)+\varphi(b)}{2})$$

$$\leq \sqrt{f(t\varphi(a)+(1-t)\varphi(b))f((1-t)\varphi(a)+t\varphi(b))} - \frac{c}{4}(\varphi(a)-\varphi(b))^2(1-2t)^2$$

$$\leq \frac{f(t\varphi(a)+(1-t)\varphi(b))}{2} + \frac{f((1-t)\varphi(a)+t\varphi(b))}{2} - \frac{c}{4}(\varphi(a)-\varphi(b))^2(1-2t)^2.$$

Integrating the above inequality over $(0,1)$ and substituting $x = t\varphi(a)+(1-t)\varphi(b)$ we get

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) + \frac{c}{12}(\varphi(a)-\varphi(b))^2$$

$$\leq \int_{\varphi(a)}^{\varphi(b)} G(f(x), f(\varphi(a)+\varphi(b)-x))dx \tag{4.3}$$

$$\leq \int_{\varphi(a)}^{\varphi(b)} A(f(x), f(\varphi(a)+\varphi(b)-x))dx. \tag{4.4}$$

Using $\int_{\varphi(a)}^{\varphi(b)} f(x)dx = \int_{\varphi(a)}^{\varphi(b)} f(\varphi(a)+\varphi(b)-x)dx$, (4.3) becomes

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) + \frac{c}{12}(\varphi(a)-\varphi(b))^2$$

$$\leq \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} G(f(x), f(\varphi(a)+\varphi(b)-x))dx$$

$$\leq \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx.$$

Again, using strongly log-φ convexity of f, we get

$$\frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx = \int_0^1 f(t\varphi(a)+(1-t)\varphi(b))dt$$

$$\leq \int_0^1 [f(\varphi(a)]^t [f(\varphi(b)]^{1-t}dt - \int_0^1 ct(1-t)(\varphi(a)-\varphi(b))^2 dt$$

$$= \frac{f(\varphi(b))-f(\varphi(a))}{\log(f(\varphi(b)))-\log(f(\varphi(a)))} - \frac{c}{6}(\varphi(a)-\varphi(b))^2$$

$$= L(f(\varphi(b)), f(\varphi(a))) - \frac{c}{6}(\varphi(a)-\varphi(b))^2$$

$$\leq A(f(\varphi(b)), f(\varphi(a))) - \frac{c}{6}(\varphi(a)-\varphi(b))^2$$

$$= \frac{f(\varphi(b))+f(\varphi(a))}{2} - \frac{c}{6}(\varphi(a)-\varphi(b))^2.$$

□

Theorem 4.10. *Let $\varphi : [a, b] \to [a, b]$ be a continuous function, where $a, b \in I$, and let $f : I \to (0, \infty)$ be a positive strongly log-φ convex function with respect to $c > 0$. If f is Lebesgue integrable on I then*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)f(\varphi(b)+\varphi(a)-x)dx$$

$$\leq f(\varphi(a))f(\varphi(b)) + \frac{c^2}{30}(\varphi(b)-\varphi(a))^4$$

$$-4c\frac{(\varphi(b)-\varphi(a))^2}{(\log(f(\varphi(b)))-\log(f(\varphi(a))))^2}[A(f(\varphi(b)), f(\varphi(a))) - L(f(\varphi(b)), f(\varphi(a)))].$$

Proof. Since f is strongly log-φ convex, we have for $t \in (0, 1)$

$$f(t\varphi(a)+(1-t)\varphi(b)) \leq f(\varphi(a))^t f(\varphi(b))^{1-t} - ct(1-t)(\varphi(a)-\varphi(b))^2,$$

and

$$f((1-t)\varphi(a)+t\varphi(b)) \leq f(\varphi(a))^{1-t} f(\varphi(b))^t - ct(1-t)(\varphi(a)-\varphi(b))^2.$$

Multiplying both inequalities and integrating over $(0, 1)$, we get

$$\int_0^1 f(t\varphi(a)+(1-t)\varphi(b))f((1-t)\varphi(a)+t\varphi(b))dt$$

$$\leq f(\varphi(a))f(\varphi(b)) - (\varphi(a)-\varphi(b))^2\int_0^1 ct(1-t)\left\{f(\varphi(b))\left[\frac{f(\varphi(a))}{f(\varphi(b))}\right]^t\right.$$

$$\left. +f(\varphi(a))\left[\frac{f(\varphi(b))}{f(\varphi(a))}\right]^t\right\}dt + c^2(\varphi(a)-\varphi(b))^4\int_0^1 t^2(1-t)^2dt. \tag{4.5}$$

Since

$$\int_0^1 t(1-t)\left[\frac{f(\varphi(a))}{f(\varphi(b))}\right]^t dt$$

$$= \frac{2}{f(\varphi(b))(\log(f(\varphi(a)))-\log(f(\varphi(b))))^2}[A(f(\varphi(b)), f(\varphi(a))) - L(f(\varphi(b)), f(\varphi(a)))]. \tag{4.6}$$

Similarly,

$$\int_0^1 t(1-t)\left[\frac{f(\varphi(a))}{f(\varphi(b))}\right]^t dt$$

$$= \frac{2}{f(\varphi(a))(\log(\varphi(b))-\log(\varphi(a)))^2}\Big[A(f(\varphi(b)), f(\varphi(a))) - L(f(\varphi(b)), f(\varphi(a)))\Big]. \tag{4.7}$$

Substituting (4.6) and (4.7) in (4.5) and replacing $x = t\varphi(a) + (1-t)\varphi(b)$, we get the required inequality. □

Theorem 4.11. *Let $\varphi : [a,b] \to [a,b]$ be a continuous function, where $a,b \in I$, and let $f,g : I \to (0,\infty)$ be a positive strongly log-φ convex functions with respect to $c > 0$. If f and g are Lebesgue integrable, then*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)g(x)dx$$

$$\leq L(f(\varphi(b))g(\varphi(b)), f(\varphi(a))g(\varphi(a))) + \frac{c^2}{30}(\varphi(a)-\varphi(b))^4 - 2c(\varphi(b)-\varphi(a))^2$$

$$\times\left[\frac{A(f(\varphi(b)), f(\varphi(a))) - L(f(\varphi(b)), f(\varphi(a)))}{(\log(f(\varphi(b))) - \log(f(\varphi(a))))^2} + \frac{A(g(\varphi(b)), g(\varphi(a))) - L(g(\varphi(b)), g(\varphi(a)))}{(\log(g(\varphi(b))) - \log(g(\varphi(a))))^2}\right]$$

$$\leq \frac{1}{4}\left[\{f(\varphi(a))+f(\varphi(b))\}L(f(\varphi(b)), f(\varphi(a)))+\{g(\varphi(a))+g(\varphi(b))\}L(g(\varphi(b)), g(\varphi(a)))\right]$$

$$+\frac{c^2}{30}(\varphi(a)-\varphi(b))^4 - 2c(\varphi(b)-\varphi(a))^2$$

$$\times\left[\frac{A(f(\varphi(b)), f(\varphi(a))) - L(f(\varphi(b)), f(\varphi(a)))}{(\log(f(\varphi(b))) - \log(f(\varphi(a))))^2} + \frac{A(g(\varphi(b)), g(\varphi(a))) - L(g(\varphi(b)), g(\varphi(a)))}{(\log(g(\varphi(b))) - \log(g(\varphi(a))))^2}\right].$$

Proof. The proof is similar to Theorem 4.10 □

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On vector variational-like inequalities and vector optimization problems in Asplund spaces

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Abstract. In this paper, we consider different kinds of generalized invexity for vector valued functions and a vector optimization problem. Some relations between some vector variational-like inequalities and a vector optimization problem are established using the properties of Mordukhovich limiting subdifferentials under $C-\eta-$strong pseudomonotonicity.

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1. Introduction

In 1998, Giannessi [9] first used, so called, Minty type vector variational inequality (in short, $MVVI$) to establish the necessary and sufficient conditions for a point to be an efficient solution of a vector optimization problem (in short, (VOP)) for differentiable and convex functions. Since then, several researchers have studied (VOP) by using different kinds of $MVVI$ under different assumptions, see [1, 2, 10, 15, 19] and the references therein. Consequently, vector variational inequalities have been generalized in various directions, in particular, vector variational-like inequality problems, see [1, 13, 14, 20, 23, 28] and the references therein. The vector variational-like inequalities are closely related to the concept of the invex and preinvex functions which generalize the notion of the convexity of functions . The concept of the invexity was first introduced by Hanson [12]. More recently, the characterization and applications for generalized invexity were studied by many authors, see [11, 13, 19, 21, 24, 25, 27] and the references therein.

The relation between the vector variational inequality and the smooth vector optimization problem has been studied by many authors (see, for example, [9, 23, 26] and the references therein). Yang et al. [26] extended the result of Giannessi [9, 10] for differentiable but pseudoconvex functions. Yang and Yang [23] gave some relations between Minty variational-like inequalities and the vector optimization problems for differentiable but pseudo-invex vector-valued functions. Yang et al. [25, 26] and Garzon et al. [6, 7] studied the relations between generalized invexity of a differentiable function and generalized monotonicity of its gradient mapping. Very recently, Rezaie and Zafarani [20] showed some relations between the vector variational-like inequalities and vector optimization problems for nondifferential functions under generalized monotonicity. Al-Homidan and Ansari [1] studied the relation among the generalized Minty vector variational-like inequality, generalized Stampacchia vector variational-like inequality and vector optimization problems for nondifferential and nonconvex functions with Clarke's generalized directional derivative and then, Ansari and Lee [2] showed that for pseudoconvex functions with upper Dini directional derivative, similar results holds. Ansari, Rezaie and Zafarani [3] considered generalized Minty vector variational-like inequality problems, Stampacchia vector variational-like inequality problems and nonsmooth vector optimization problems under nonsmooth pseudo-invexity assumptions. They also considered the weak formulations of generalized Minty vector variational-like inequality problems and generalized Stampacchia vector variational-like inequality problems in a very general setting and established the existence results for their solutions. The main results in [1] and [20] were obtained in the setting of Clarke subdifferential. Since the class of Clarke subdifferential is larger than the class of Mordukhovich subdifferential, it is necessary to study the vector variational-like inequalities and vector optimization problems in the setting of Mordukhovich subdifferential (see [5, 16, 17]). Oveisiha and Zafarani [18] established some properties of pseudo-invex functions and Mordukhovich limiting subdifferential and relations between vector variational-like inequalities and vector optimization problems. Chen and Huang [4] considered the Minty vector variational-like inequality, Stampacchia vector variational-like inequality and the weak formulations of these inequalities, defined by means of Mordukhovich limiting subdifferentials in Asplund spaces. They established some relations between the vector variational-like inequalities and vector optimization problems using the properties of Mordukhovich limiting subdifferential. Farajzadeh et al. [8] considered generalized variational-like inequalities with set-valued mappings in topological spaces, which include as a special case the strong vector variational-like inequalities. Motivated and inspired by the work mentioned above, in this paper we consider the Minty vector variational-like inequality, Stampacchia vector variational-like inequality and the weak formulations of these inequalities, defined by means of Mordukhovich limiting subdifferentials in Asplund spaces. Some relations between vector variational-like inequalities and a vector optimization problem (respectively, between Minty vector variational-like inequality and Stampacchia vector variational-like inequality) are established using the properties of Mordukhovich limiting subdifferentials under different kinds of generalized invexity (respectively, $C-\eta-$strong pseudomonotonicity).

2. Preliminaries

Let X be a Banach space endowed with a norm $\|.\|$ and X^* its dual space with a norm $\|.\|_*$. Denote $\langle .,. \rangle$, $[x,y]$, $]x,y[$ the dual pair between X and X^*, the line segment for $x,y \in X$ and $[x,y]\backslash\{x,y\}$, respectively. Let Ω be a nonempty open subset of X.

When functions are not differentiable, we use the concept of subdifferential: Fréchet subdifferential, Limiting subdifferential and Clarke-Rockafellar subdifferential.

Definition 2.1. *Let X be a Banach space and $f : X \to \mathbf{R} \cup \{\infty\}$ a proper l.s.c. function. We say that f is Fréchet-subdifferentiable and ξ^* is Fréchet-subderivative of f at x ($\xi^* \in \partial_F f(x)$) if $x \in dom\ f$ and*

$$\liminf_{\|h\| \to 0} \frac{f(x+h) - f(x) - \langle \xi^*, h\rangle}{\|h\|} \geq 0.$$

Definition 2.2. [16] *Let $x \in \Omega$ and $\varepsilon \geq 0$. The set of $\varepsilon -$normals to Ω at x is defined by*

$$\widehat{N}_\varepsilon(x,\Omega) = \{x^* \in X^* \mid \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u-x\rangle}{\|u-x\|} \leq \varepsilon\}.$$

If $x \notin \Omega$, we put $\widehat{N}_\varepsilon(x,\Omega) = \varnothing$ for all $\varepsilon \geq 0$.

Definition 2.3. [16] *Let $\overline{x} \in \Omega$. Then $x^* \in X^*$ is a limiting normal to Ω at $\overline{x}$ if there are sequences $\varepsilon_k \searrow 0$, $x_k \xrightarrow{\Omega} \overline{x}$ and $x_k^* \xrightarrow{w^*} \overline{x}^*$ such that $x_k^* \in \widehat{N}_\varepsilon(x_k,\Omega)$, for all $k \in \mathbf{N}$. The set of such normals*

$$N(\overline{x},\Omega) = \limsup_{\substack{x \to \bar{x} \\ \varepsilon \searrow 0}} \widehat{N}_\varepsilon(x,\Omega)$$

is the limiting normal cone to Ω at $\overline{x}$. If $\overline{x} \notin \Omega$, we put $N(\overline{x},\Omega) = \varnothing$.

Remark 2.4. Note that the symbol $u \xrightarrow{\Omega} x$ means that $u \to x$ with $u \in \Omega$. The symbol $\xrightarrow{w^*}$ stands for convergence in weak* topology.

Definition 2.5. [16] *Considering the extended-real-valued function $\varphi : X \to \overline{\mathbf{R}} = [-\infty, +\infty]$ we say that φ is proper if $\varphi(x) > -\infty$ for all $x \in X$ and its domain, $dom\varphi = \{x \in X : \varphi(x) < \infty\}$, is nonempty. The epigraph of φ is defined as*

$$epi\varphi = \{(x,a) \in X \times \mathbf{R} / \varphi(x) \leq a\}.$$

Definition 2.6. [16] *Considering a point $\overline{x} \in X$ with $\mid \varphi(\overline{x}) \mid < \infty$, the set*

$$\partial_L \varphi(\overline{x}) = \{x^* \in X^* \mid (x^*, -1) \in N((\overline{x}, \varphi(\overline{x})), epi\ \varphi)\}$$

is the limiting subdifferential of φ at $\bar{x}$ and its elements are limiting subdifferentials of φ at this point. If $\mid \varphi(\overline{x}) \mid = \infty$, we put $\partial_L \varphi(\overline{x}) = \varnothing$.

Remark 2.7. [16] It is well known that

$$\partial_F f(x) \subseteq \partial_L f(x) \subseteq \partial_C f(x),$$

where $\partial_C f$ is the Clarke subdifferential.

Definition 2.8. *A Banach space X is Asplund, or it has the Asplund property, if every convex continuous function $\varphi : U \to \mathbf{R}$ defined on an open convex subset U of X is Fréchet differentiable on a dense subset of U.*

Remark 2.9. One of the most popular Asplund spaces is any reflexive Banach space [16].

Theorem 2.10. [16] *Let X be an Asplund space and $\varphi : X \to \overline{\mathbf{R}}$ be proper and l.s.c. around $\overline{x} \in dom\varphi$, then*

$$\partial_L \varphi(\overline{x}) = \limsup_{\substack{x \to \overline{x} \\ \varepsilon \searrow 0}} \partial_F \varphi(x).$$

For more details and applications, see [16].

Definition 2.11. *Let $\eta : X \times X \to X$. A subset Ω of X is said to be invex with respect to η if for any $x, y \in \Omega$ and $\lambda \in [0,1]$, we have $y + \lambda\eta(x, y) \in \Omega$.*

Hereafter, unless otherwise specified, we assume that X is an Asplund space and $\Omega \subseteq X$ is a nonempty open invex set with respect to the mapping $\eta : \Omega \times \Omega \to X$.

Definition 2.12. *A mapping $\eta : \Omega \times \Omega \to X$ is said to be skew if for any $x, y \in \Omega$,*

$$\eta(x, y) + \eta(y, x) = 0.$$

Definition 2.13. *Let $x_0 \in \Omega$. A mapping $\eta : \Omega \times \Omega \to X$ is said to be skew at x_0 if for any $x \in \Omega$, $x \neq x_0$,*

$$\eta(x, x_0) + \eta(x_0, x) = 0.$$

Definition 2.14. [21] *Let $f : \Omega \to \mathbf{R}$ be a function. f is said to be*

1. *$weakly - quasi - invex$ with respect to η on Ω if for any $x, y \in \Omega$,*
$$f(x) \le f(y) \Rightarrow \exists\ \xi^* \in \partial_L f(y)\ \langle \xi^*, \eta(x, y)\rangle \le 0;$$
2. *$quasi - invex$ with respect to η on Ω if for any $x, y \in \Omega$,*
$$f(x) \le f(y) \Rightarrow \forall\ \xi^* \in \partial_L f(y)\ \langle \xi^*, \eta(x, y)\rangle \le 0;$$
3. *$pseudo - invex$ with respect to η on Ω if for any $x, y \in \Omega$,*
$$\langle \xi^*, \eta(x, y)\rangle \ge 0,\ \exists\ \xi^* \in \partial_L f(y) \Rightarrow f(x) \ge f(y).$$

In some results of the paper we need to consider some further assumptions on η. These assumptions are known in invexity literature (Jabarootian and Zafarani (2006) [13]).

Condition C. Let $\eta : \Omega \times \Omega \to X$. Then for any $x, y \in \Omega$, $\lambda \in [0,1]$,

$$\begin{cases} C_1 : \eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y); \\ C_2 : \eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y). \end{cases}$$

Remark 2.15. Yang et al. [27] have shown that if $\eta : \Omega \times \Omega \to X$ satisfies **condition C**, then for all $x, y \in \Omega$, $\lambda \in [0,1]$,

$$\eta(y + \lambda\eta(x, y), y) = \lambda\eta(x, y).$$

Definition 2.16. *Let $\eta : \Omega \times \Omega \to X$, $x_0 \in \Omega$. We say that $\eta : \Omega \times \Omega \to X$ satisfies* **condition C** *at x_0 if for all $x \in \Omega$, $\lambda \in [0, 1]$,*

$$\eta(x_0 + \lambda\eta(x, x_0), x_0) = \lambda\eta(x, x_0).$$

Definition 2.17. *Let $f = (f_1, ..., f_n) : \Omega \to \mathbf{R}^n$ be a vector-valued function and $x_0 \in \Omega$. f is said to be*

1. *pseudo − invex with respect to η on Ω if for any $x, y \in \Omega$,*
$$f(x) - f(y) \in -\mathbf{R}^n_+\backslash\{0\} \implies \langle \partial_L f(y), \eta(x, y)\rangle \subseteq -\mathbf{R}^n_+\backslash\{0\};$$
2. *quasi − invex with respect to η on Ω if for any $x, y \in \Omega$,*
$$\langle \xi^*, \eta(x, y)\rangle \in \mathbf{R}^n_+\backslash\{0\},\ \exists\ \xi^* \in\ \partial_L f(y) \implies f(x) - f(y) \in \mathbf{R}^n_+\backslash\{0\};$$
3. *weakly − quasi − invex with respect to η on Ω if for any $x, y \in \Omega$,*
$$\langle \partial_L f(y), \eta(x, y)\rangle \subseteq \mathbf{R}^n_+\backslash\{0\} \implies f(x) - f(y) \in \mathbf{R}^n_+\backslash\{0\};$$
4. *weakly − quasi − invex at x_0 with respect to η if for any $x \in \Omega$,*
$$\langle \partial_L f(x_0), \eta(x, x_0)\rangle \subseteq \mathbf{R}^n_+\backslash\{0\} \implies f(x) - f(x_0) \in \mathbf{R}^n_+\backslash\{0\}.$$

Remark 2.18. Next, we provide an example which shows that a function $f = (f_1, ..., f_n)$ it can be pseudo-invex with respect to η on Ω and there exists $k, 1 \leq k \leq n$, such that f_k is not pseudo-invex with respect to η on Ω.

Example 2.19. Let us consider $X = \mathbf{R}$, $\Omega = [-1, 1]$, $f = (f_1, f_2) : \Omega \to \mathbf{R}^2$ defined as

$$f_1(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ x, & x < 0, \end{cases}$$

$$f_2(x) = x$$

and $\eta : \Omega \times \Omega \to \mathbf{R}$ defined as

$$\eta(x, y) = x - y.$$

We have

$$\partial_L f(x) = \begin{cases} (\frac{1}{2\sqrt{x}}, 1) & x > 0, \\ [0, \infty[\times\{1\}, & x = 0, \\ (1, 1), & x < 0. \end{cases}$$

It is not difficult to see that f is pseudo-invex with respect to η. Function f_1 is not pseudo-invex with respect to η on Ω because for $x = -1, y = 0$ there exists $\xi^* = 0 \in \partial_L f(y)$ such that $\langle \xi^*, \eta(x, y)\rangle = 0$ and $f(x) < f(y)$.

Definition 2.20. [8] *A set valued mapping $F : \Omega \to 2^{X^*}$ is said to be $C - \eta-$strong pseudomonotone if for any $x, y \in \Omega$,*

$$\langle Fx, \eta(x, y)\rangle \nsubseteq -C(x)\backslash\{0\} \Longrightarrow \langle Fy, \eta(y, x)\rangle \subseteq -C(y).$$

Definition 2.21. *A set valued mapping $F : \Omega \to 2^{X^*}$ is said to be $(C, K) - \eta-$strong pseudomonotone if for any $x, y \in \Omega$,*

$$\langle Fx, \eta(x, y)\rangle \nsubseteq C \Longrightarrow \langle Fy, \eta(y, x)\rangle \subseteq K.$$

Definition 2.22. *A set valued mapping* $F : \Omega \to 2^{X^*}$ *is said to be strictly* $(C,K)-\eta-$*strong pseudomonotone if for any* $x, y \in \Omega, x \neq y$,

$$\langle Fx, \eta(x,y)\rangle \nsubseteq C \Longrightarrow \langle Fy, \eta(y,x)\rangle \subseteq K.$$

Let $f = (f_1, ..., f_n) : \Omega \to \mathbf{R}^n$ be a vector-valued function, where $f_i : \Omega \to \mathbf{R}$ $(i = 1, ..., n)$ is non-differentiable locally Lipschitz function.

In this paper, we consider the following vector optimization problem:

$$(VOP) \quad \text{Minimize } f(x) = (f_1(x), ..., f_n(x)) \quad \text{subject to } x \in \Omega.$$

Definition 2.23. *A point* $x_0 \in \Omega$ *is said to be an efficient (or Pareto) solution (respectively, weak efficient solution) of* (VOP) *if for all* $x \in \Omega$,

$$\begin{aligned} f(x) - f(x_0) &= (f_1(x) - f_1(x_0), ..., f_n(x) - f_n(x_0)) \notin -\mathbf{R}^n_+ \setminus \{0\}, \\ (\textit{respectively, } f(x) - f(x_0) &= (f_1(x) - f_1(x_0), ..., f_n(x) - f_n(x_0)) \notin -int\mathbf{R}^n_+), \end{aligned}$$

where $\mathbf{R}^n_+$ *is the nonnegative orthant of* $\mathbf{R}^n$ *and* 0 *is the origin of the nonnegative orthant.*

3. Characterization

We consider the following Minty vector variational-like inequality problems and Stampacchia vector variational-like inequality problems.

$(GGMVVLIP)$ Find $x_0 \in \Omega$ such that, for all $x \in \Omega$ and all $\xi_i \in \partial_L f_i(x)$ $(i = 1, ..., n)$,

$$\langle \xi^*, \eta(x, x_0)\rangle = (\langle \xi^*_1, \eta(x, x_0)\rangle, ..., \langle \xi^*_n, \eta(x, x_0)\rangle) \notin -\mathbf{R}^n_+ \setminus \{0\}.$$

$(GMVVLIP)$ Find $x_0 \in \Omega$ such that, for all $x \in \Omega$ there exists $\xi_i \in \partial_L f_i(x)$ $(i = 1, ..., n)$,

$$\langle \xi^*, \eta(x, x_0)\rangle = (\langle \xi^*_1, \eta(x, x_0)\rangle, ..., \langle \xi^*_n, \eta(x, x_0)\rangle) \notin -\mathbf{R}^n_+ \setminus \{0\}.$$

$(WGGMVVLIP)$ Find $x_0 \in \Omega$ such that, for all $x \in \Omega$ and all $\xi_i \in \partial_L f_i(x)$ $(i = 1, ..., n)$,

$$\langle \xi^*, \eta(x, x_0)\rangle = (\langle \xi^*_1, \eta(x, x_0)\rangle, ..., \langle \xi^*_n, \eta(x, x_0)\rangle) \notin -int\mathbf{R}^n_+.$$

$(WGMVVLIP)$ Find $x_0 \in \Omega$ such that, for all $x \in \Omega$ there exists $\xi_i \in \partial_L f_i(x)$ $(i = 1, ..., n)$,

$$\langle \xi^*, \eta(x, x_0)\rangle = (\langle \xi^*_1, \eta(x, x_0)\rangle, ..., \langle \xi^*_n, \eta(x, x_0)\rangle) \notin -int\mathbf{R}^n_+.$$

$(SVVLIP)$ Find $x_0 \in \Omega$ such that, for all $x \in \Omega$ there exists $\xi_i \in \partial_L f_i(x_0)$ $(i = 1, ..., n)$,

$$\langle \xi^*, \eta(x, x_0)\rangle = (\langle \xi^*_1, \eta(x, x_0)\rangle, ..., \langle \xi^*_n, \eta(x, x_0)\rangle) \notin -\mathbf{R}^n_+ \setminus \{0\}.$$

$(WSVVLIP)$ Find $x_0 \in \Omega$ such that, for all $x \in \Omega$ there exists $\xi_i \in \partial_L f_i(x_0)$ $(i = 1, ..., n)$,

$$\langle \xi^*, \eta(x, x_0)\rangle = (\langle \xi^*_1, \eta(x, x_0)\rangle, ..., \langle \xi^*_n, \eta(x, x_0)\rangle) \notin -int\mathbf{R}^n_+.$$

Theorem 3.1. *If x_0 is a solution of $(SVVLIP)$, $\partial_L f$ is strictly $(int\mathbf{R}^n_+, int\mathbf{R}^n_+) - \eta-$strong pseudomonotone, η is skew at x_0 and $\langle\ \partial_L f(x_0), \eta(x_0, x_0)\rangle \subseteq \mathbf{R}^n_+$, then x_0 is a solution of $(GGMVVLIP)$.*

Proof. Suppose that x_0 is not a solution of $(GGMVVLIP)$.
Since $\langle \partial_L f(x_0), \eta(x_0, x_0)\rangle \subseteq \mathbf{R}^n_+$ it follows that there exist $\overline{x} \in \Omega, \overline{x} \neq x_0$, $\overline{\zeta} \in \partial_L f(\overline{x})$ such that

$$\langle \overline{\zeta}, \eta(\overline{x}, x_0)\rangle \in -\mathbf{R}^n_+ \backslash \{0\}.$$

Therefore,

$$\langle \overline{\zeta}, \eta(\overline{x}, x_0)\rangle \notin int\mathbf{R}^n_+. \tag{3.1}$$

Since $\partial_L f$ is strictly $(int\mathbf{R}^n_+, int\mathbf{R}^n_+) - \eta-$strong pseudomonotone, by (3.1) we obtain

$$\langle \partial_L f(x_0), \eta(x_0, \overline{x})\rangle \subseteq int\mathbf{R}^n_+. \tag{3.2}$$

Since η is skew at x_0, by (3.2) it follows that

$$\langle \partial_L f(x_0), \eta(\overline{x}, x_0)\rangle \subseteq -int\mathbf{R}^n_+,$$

which contradicts the fact that x_0 is a solution of $(SVVLIP)$. Therefore, it follows that x_0 is a solution of $(GGMVVLIP)$. □

Example 3.2. Let us consider $X = \mathbf{R}$, $\Omega = [-1, 1]$, $f : \Omega \to \mathbf{R}$ defined as

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

and $\eta : \Omega \times \Omega \to \mathbf{R}$ defined as

$$\eta(x, y) = x - y.$$

We have

$$\partial_L f(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & x > 0, \\ [0, \infty[\cup \{-1\}, & x = 0, \\ -1, & x < 0. \end{cases}$$

and $\partial_L f$ is strictly $(int\mathbf{R}_+, int\mathbf{R}_+) - \eta-$strong pseudomonotone. It is not difficult to see that $x_0 = 0$ is a solution of $(SVVLIP)$ and η is skew at x_0. Therefore, x_0 is a solution of $(GGMVVLIP)$.

Corollary 3.3. *If x_0 is a solution of $(SVVLIP)$, $\partial_L f$ is strictly $(int\mathbf{R}^n_+, int\mathbf{R}^n_+) - \eta-$strong pseudomonotone and η is skew at x_0, then x_0 is a solution of $(GMVVLIP)$.*

Corollary 3.4. *If x_0 is a solution of $(WSVVLIP)$, $\partial_L f$ is strictly $(int\mathbf{R}^n_+, int\mathbf{R}^n_+) - \eta-$strong pseudomonotone, η is skew at x_0 and $\langle\ \partial_L f(x_0), \eta(x_0, x_0)\rangle \subseteq \mathbf{R}^n_+ \backslash \{0\}$, then x_0 is a solution of $(GGMVVLIP)$.*

Corollary 3.5. *If x_0 is a solution of $(WSVVLIP)$, $\partial_L f$ is strictly $(int\mathbf{R}^n_+, int\mathbf{R}^n_+) - \eta-$strong pseudomonotone and η is skew at x_0, then x_0 is a solution of $(WGMVVLIP)$.*

Theorem 3.6. *If x_0 is a solution of (VOP), f is quasi-invex with respect to η on Ω and η is skew, then x_0 is a solution of $(GGMVVLIP)$.*

Proof. Suppose that x_0 is not a solution of $(GGMVVLIP)$. It follows that there exist $\overline{x} \in \Omega$, $\overline{\zeta} \in \partial_L f(\overline{x})$ such that we have

$$\langle \overline{\zeta}, \eta(\overline{x}, x_0)\rangle \in -\mathbf{R}^n_+ \backslash \{0\}. \tag{3.3}$$

Since η is skew, by (3.3) we obtain

$$\langle \overline{\zeta}, \eta(x_0, \overline{x})\rangle \in \mathbf{R}^n_+ \backslash \{0\}.$$

Since f is quasi-invex, it follows that

$$f(x_0) - f(\overline{x}) \in \mathbf{R}^n_+ \backslash \{0\},$$

which contradicts the fact that x_0 is a solution of (VOP). Therefore, x_0 is a solution of $(GGMVVLIP)$. □

Remark 3.7. In [4] (Theorem 3.1) the authors obtained this result by assuming that $f_i (i = 1, ..., n)$ are invex with respect to η on Ω. Next, we provide an example which shows that a function $f = (f_1, ..., f_n)$ it can be quasi-invex with respect to η on Ω and there exists $k, 1 \leq k \leq n$, such that f_k is not invex with respect to η on Ω.

Example 3.8. Let us consider $X = \mathbf{R}$, $\Omega = [-\frac{1}{5}, \frac{1}{5}]$, $f = (f_1, f_2) : \Omega \to \mathbf{R}^2$ defined as

$$f_1(x) = \begin{cases} x^2 + 2x, & x > 0, \\ -x, & x \leq 0, \end{cases}$$

$$f_2(x) = \begin{cases} x^3 - 2x^2 + x, & x \geq 0, \\ -x, & x < 0, \end{cases}$$

and $\eta : \Omega \times \Omega \to \mathbf{R}$ defined as

$$\eta(x, y) = x - y.$$

We have

$$\partial_L f(x) = \begin{cases} (2x + 2, 3x^2 - 4x + 1), & x > 0, \\ (k, t), & k \in \{2, -1\}, t \in \{1, -1\}, x = 0. \\ (-1, -1), & x < 0. \end{cases}$$

It is easy to observe that $x_0 = 0$ is a solution of (VOP), η is skew and function f is quasi-invex with respect to η on Ω. Function f_2 is not invex with respect to η on Ω because for $x = 1, y = 0$ we obtain

$$f_2(1) - f_2(0) < \langle \xi^*, \eta(1, 0)\rangle,$$

for $\xi^* = 1$.

Corollary 3.9. *If x_0 is a solution of (VOP), f is quasi-invex with respect to η on Ω and η is skew, then x_0 is a solution of $(GMVVLIP)$.*

Theorem 3.10. *If x_0 is a solution of (VOP), f is weakly quasi-invex at x_0 with respect to η on Ω and η is skew at x_0, then x_0 is a solution of $(GMVVLIP)$.*

Proof. Suppose that x_0 is not a solution of $(GMVVLIP)$. Therefore, there exists $\overline{x} \in \Omega$ such that for all $\xi^* \in \partial_L f(\overline{x})$ we have

$$\langle \xi^*, \eta(\overline{x}, x_0)\rangle \in -\mathbf{R}^n_+ \backslash \{0\}. \tag{3.4}$$

Hence,

$$\langle \partial_L f(\overline{x}), \eta(\overline{x}, x_0)\rangle \subseteq -\mathbf{R}^n_+ \backslash \{0\}. \tag{3.5}$$

Since η is skew at x_0 we obtain

$$\langle \partial_L f(\overline{x}), \eta(x_0, \overline{x})\rangle \subseteq \mathbf{R}^n_+ \backslash \{0\}.$$

Since f is weakly quasi-invex at x_0 with respect to η on Ω it follows that

$$f(x_0) - f(\overline{x}) \in \mathbf{R}^n_+ \backslash \{0\},$$

which contradicts the fact that x_0 is a solution of (VOP). Therefore, x_0 is a solution of $(GMVVLIP)$. □

Remark 3.11. In [18] (Theorem 13) the authors obtained this result by assuming that $f_i (i = 1, ..., n)$ are pseudo-invex with respect to η on Ω. Next, we provide an example which shows that a function $f = (f_1, ..., f_n)$ it can be weakly quasi-invex with respect to η on Ω and there exists $k, 1 \leq k \leq n$, such that f_k is not pseudo-invex with respect to η on Ω.

Example 3.12. Let us consider $X = \mathbf{R}$, $\Omega = [-1, 1]$, $f = (f_1, f_2) : \Omega \to \mathbf{R}^2$ defined as

$$f_1(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ x, & x < 0, \end{cases}$$

$$f_2(x) = \begin{cases} \frac{1}{2}\sqrt{x}, & x \geq 0, \\ -x, & x < 0, \end{cases}$$

$x_0 = 0$ and $\eta : \Omega \times \Omega \to \mathbf{R}$ defined as

$$\eta(x, y) = x - y.$$

We obtain that

$$\partial_L f_1(x) = \begin{cases} (\frac{1}{2\sqrt{x}}, \frac{1}{4\sqrt{x}}), & x > 0, \\ [0, \infty[\times([0, \infty[\cup\{-1\}), & x = 0, \\ (1, -1), & x < 0. \end{cases}$$

It is not difficult to verify that f is weakly quasi-invex at x_0 with respect to η, $x_0 = 0$ is solution of (VOP), η is skew at x_0 and f_1 is not pseudo-invex with respect to η on Ω because for $x = -1, y = 0$ there exists $\xi^* = 0 \in \partial_L f(y)$ such that $\langle \xi^*, \eta(x, y)\rangle = 0$ and $f(x) < f(y)$.

Theorem 3.13. *Suppose that x_0 is a solution of $(SVVLIP)$ and f is pseudo-invex with respect to η on Ω. Then, x_0 is a solution of (VOP).*

Proof. Suppose that x_0 is not a solution of (VOP). Therefore, there exists $\overline{x} \in \Omega$ such that

$$f(\overline{x}) - f(x_0) \in -\mathbf{R}^n_+ \backslash \{0\}.$$

Since f is pseudo-invex with respect to η on Ω, it follows that

$$\langle \partial_L f(x_0), \eta(\overline{x}, x_0)\rangle \subseteq -\mathbf{R}^n_+ \backslash \{0\},$$

which contradicts the fact that x_0 is a solution of $(SVVLIP)$. Therefore, x_0 is a solution of (VOP).

Remark 3.14. In [4] (Theorem 3.2) the authors obtained this result by assuming that $f_i (i = 1, ..., n)$ are invex with respect to η on Ω. Next, we provide an example which shows that a function $f = (f_1, ..., f_n)$ it can be pseudo-invex with respect to η on Ω and there exists $k, 1 \leq k \leq n$, such that f_k is not invex with respect to η on Ω.

Example 3.15. Let us consider $X = \mathbf{R}$, $\Omega = [-1, 1]$, $f = (f_1, f_2) : \Omega \to \mathbf{R}^2$ defined as

$$f_1(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ -x, & x < 0, \end{cases}$$

$$f_2(x) = x$$

and $\eta : \Omega \times \Omega \to \mathbf{R}$ defined as

$$\eta(x, y) = x - y.$$

We have

$$\partial_L f_1(x) = \begin{cases} (\frac{1}{2\sqrt{x}}, 1), & x > 0, \\ ([0, \infty[\cup\{-1\}) \times \{1\}, & x = 0, \\ (-1, 1), & x < 0. \end{cases}$$

It is not difficult to see that $x_0 = 0$ is solution of $(SVVLIP)$, f is pseudo-invex with respect to η. Function f_1 is not invex with respect to η on Ω because for $x = 1, y = 0$ we obtain

$$f(1) - f(0) < \langle \xi^*, \eta(1, 0) \rangle,$$

for $\xi^* = 2$.

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New univalence criteria for some integral operators

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Abstract. In this work we consider some integral operators for analytic functions in the open unit disk and we obtain new univalence criteria for these integral operators, using Mocanu's and Şerb's Lemma, Pascu's Lemma.

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Keywords: Integral operator, univalence.

1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

normalized by $f(0) = f'(0) - 1 = 0$, which are analytic in the open unit disk

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

We denote $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathcal{U}$. We consider the integral operators

$$H_n(z) = \left\{\gamma \int_0^z u^{\gamma-1} \left(\frac{f_1(u)}{u}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{u}\right)^{\alpha_n} du\right\}^{\frac{1}{\gamma}}, \tag{1.1}$$

$$T_n(z) = \left\{\gamma \int_0^z u^{\gamma-1} \left(\frac{f_1(u)}{u}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{u}\right)^{\alpha_n} (f_1'(u))^{\beta_1} \cdots (f_n'(u))^{\beta_n} du\right\}^{\frac{1}{\gamma}}, \tag{1.2}$$

for the functions $f_j \in \mathcal{A}$ and the complex numbers $\gamma, \alpha_j, \beta_j, \gamma \neq 0, j = 1, n$.

In this work we define a new general integral operator V_n given by

$$V_n(z) = \left\{\delta \int_0^z u^{\delta-1} \left(\frac{f_1(u)}{g_1(u)}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{g_n(u)}\right)^{\alpha_j} \left(\frac{f_1'(u)}{g_1'(u)}\right)^{\beta_1} \cdots \left(\frac{f_n'(u)}{g_n'(u)}\right)^{\beta_n} du\right\}^{\frac{1}{\delta}} \tag{1.3}$$

for $f_j, g_j \in \mathcal{A}$ and the complex numbers $\alpha_j, \beta_j, \delta, \delta \neq 0, j = \overline{1, n}$.

The integral operator V_n is the most general integral operator.

Remarks. For different particular cases for parameters δ, α_j, β_j, $j = \overline{1,n}$, we obtain the integral operators which have been defined and studied by Kim-Merkes, Pfaltzgraff, Pascu, Pescar, Owa, D. Breaz and N. Breaz, Frasin, Ovesea.

- i1) For $n = 1, \delta = 1, \beta_1 = 0, g_1(z) = z$ we obtain the integral operator which was introduced and studied by Kim-Merkes [4].
- i2) For $n = 1, \delta = 1, \alpha_1 = 0$ we have the integral operator that was introduced and studied by Pfaltzgraff [10].
- i3) For $n = 1, \beta_1 = 0, g_1(z) = z$ we obtain the integral operator which was defined and studied by Pescar and Pascu [8].
- i4) For $n = 1, \alpha_1 = 0, g_1(z) = z$ we have the integral operator, which was defined and studied by Pescar and Owa [9].
- i5) For $g_i(z) = z, i = \overline{1,n}, \delta = \gamma$ and $\beta_1 = \cdots = \beta_n = 0$ we obtain the integral operator H_n, (1.1), which was defined and studied by D. Breaz and N. Breaz [1], and this integral operator is a generalization of the integral operator defined by Pescar and Pascu [8].
- i6) For $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0, \delta = \gamma, g_i(z) = z, i = \overline{1,n}$ we have the integral operator which was defined and studied by D. Breaz, N. Breaz [2] and this integral operator is a generalization of the integral operator defined by Pescar and Owa [9].
- i7) For $n = 1, g_1(z) = z$ we obtain the integral operator which is defined and studied by Ovesea [6].
- i8) For $g_i(z) = z, i = \overline{1,n}, \delta = \gamma$ we obtain the integral operator T_n that was defined and studied by Frasin [3], and this integral operator is a generalization of the integral operator defined by Ovesea [6].

In this paper we derive certain sufficient conditions of univalence for the integral operators H_n, T_n, V_n, using Mocanu's and Şerb's Lemma, Pascu's Lemma.

2. Preliminary results

In order to prove main results we will use the lemmas.

Lemma 2.1. *Mocanu and Şerb* [5]*. Let $M_0 = 1,5936...$ the positive solution of equation*

$$(2 - M)e^M = 2. \tag{2.1}$$

If $f \in \mathcal{A}$ and

$$\left| \frac{f''(z)}{f'(z)} \right| \leq M_0, \quad (z \in \mathcal{U}), \tag{2.2}$$

then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1, \quad (z \in \mathcal{U}). \tag{2.3}$$

The edge M_0 is sharp.

Lemma 2.2. *Pascu* [7]. *Let α be a complex number, Re $\alpha > 0$ and the function $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \tag{2.4}$$

for all $z \in \mathcal{U}$, then for any complex number β, Re $\beta \ge$ Re α, the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)du\right]^{\frac{1}{\beta}} \tag{2.5}$$

is regular and univalent in $\mathcal{U}$.

3. Main results

Theorem 3.1. *Let β, γ, α_j be complex numbers, $j = \overline{1,n}$, Re $\beta > 0, M_0$ the positive solution of the equation (2.1), $M_0 = 1,5936...$ and $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots$, $j = \overline{1,n}$.*

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \le M_0, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{3.1}$$

$$Re\ \beta \ge |\alpha_1| + \cdots + |\alpha_n|, \tag{3.2}$$

then for all γ be complex numbers, Re $\gamma \ge$ Re β, the integral operator H_n given by (1.1) is in the class $\mathcal{S}$.

Proof. Let's consider the function

$$h_n(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{u}\right)^{\alpha_n} du, \quad (z \in \mathcal{U}), \tag{3.3}$$

which is regular in $\mathcal{U}$ and $h_n(0) = h_n'(0) - 1 = 0$.

We have

$$\frac{zh_n''(z)}{h_n'(z)} = \alpha_1\left(\frac{zf_1'(z)}{f_1(z)} - 1\right) + \cdots + \alpha_n\left(\frac{zf_n'(z)}{f_n(z)} - 1\right)$$

and hence, we obtain

$$\frac{1-|z|^{2Re\ \beta}}{Re\ \beta}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le \frac{1-|z|^{2Re\ \beta}}{Re\ \beta}\sum_{j=1}^n \left[|\alpha_j|\left|\frac{zf_j'(z)}{f_j(z)} - 1\right|\right], \tag{3.4}$$

for all $z \in \mathcal{U}$.

From (3.1) and Lemma Mocanu and Şerb we obtain

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| < 1, \quad (z \in \mathcal{U}; j = \overline{1,n}). \tag{3.5}$$

By (3.4) and (3.5) we get

$$\frac{1-|z|^{2Re\ \beta}}{Re\ \beta}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le \frac{|\alpha_1| + \cdots + |\alpha_n|}{Re\ \beta}, \quad (z \in \mathcal{U}). \tag{3.6}$$

From (3.2) and (3.6) we have

$$\frac{1-|z|^{2Re\ \beta}}{Re\ \beta}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le 1, \quad (z\in\mathcal{U}). \tag{3.7}$$

From (3.3) we get $h_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\gamma_1}\cdots\left(\frac{f_n(z)}{z}\right)^{\gamma_n}$ and by Lemma Pascu it results that $H_n \in \mathcal{S}$. □

Corollary 3.2. *Let α, β be complex numbers, $Re\ \beta > 0$, M_0 the positive solution of the equation (2.1), $M_0 = 1,5936...$ and $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \le M_0, \quad (z\in\mathcal{U}; j=\overline{1,n}), \tag{3.8}$$

$$Re\ [n(\alpha-1)+1] \ge Re\ \beta \ge n|\alpha-1|, \quad (n\in\mathbb{N}-\{0\}), \tag{3.9}$$

then the integral operator $G_{\alpha,n}$ defined by

$$G_{\alpha,n} = \left\{[n(\alpha-1)+1]\int_0^z (f_1(u))^{\alpha-1}\cdots(f_n(u))^{\alpha-1}\,du\right\}^{\frac{1}{n(\alpha-1)+1}} \tag{3.10}$$

is in the class $\mathcal{S}$.

Proof. From (3.10) we have

$$G_{\alpha,n}(z) = \left\{[n(\alpha-1)+1]\int_0^z u^{n(\alpha-1)}\left(\frac{f_1(u)}{u}\right)^{\alpha-1}\cdots\left(\frac{f_n(u)}{u}\right)^{\alpha-1}du\right\}^{\frac{1}{n(\alpha-1)+1}} \tag{3.11}$$

We take $\gamma = n(\alpha-1)+1$, $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha - 1$ in Theorem 3.1 and we obtain the Corollary 3.2. □

Theorem 3.3. *Let $\delta, \alpha_j, \beta_j$ be complex numbers, $j = \overline{1,n}$, $Re\ \delta > 0, M_0$ the positive solution of the equation (2.1), $M_0 = 1,5936...$ and $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \le M_0, \quad (z\in\mathcal{U},\ j=\overline{1,n}), \tag{3.12}$$

$$\left[\sum_{j=1}^n |\alpha_j|\right](2Re\ \delta+1)^{\frac{2Re\ \delta+1}{2Re\ \delta}} + 2M_0\left[\sum_{j=1}^n|\beta_j|\right]Re\ \delta \le$$
$$\le (2Re\ \delta+1)^{\frac{2Re\ \delta+1}{2Re\ \delta}}Re\ \delta, \tag{3.13}$$

then for all γ be complex numbers, $Re\ \gamma \ge Re\ \delta > 0$ the integral operator T_n given by (1.2) is in the class $\mathcal{S}$.

Proof. We consider the function

$$g_n(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{u}\right)^{\alpha_n} \cdot (f_1'(u))^{\beta_1} \cdots (f_n'(u))^{\beta_n}\, du. \tag{3.14}$$

The function g_n is regular in $\mathcal{U}$ and we have $g_n(0) = g_n'(0) - 1 = 0$.

From (3.14) we obtain

$$\frac{zg_n''(z)}{g_n'(z)} = \sum_{j=1}^{n} \left[\alpha_j \left(\frac{zf_j'(z)}{f_j(z)} - 1\right)\right] + \sum_{j=1}^{n} \left[\beta_j \frac{zf_j''(z)}{f_j'(z)}\right] \tag{3.15}$$

and hence, we get

$$\frac{1-|z|^{2Re\ \delta}}{Re\ \delta} \left|\frac{zg_n''(z)}{g_n'(z)}\right| \le$$

$$\le \frac{1-|z|^{2Re\ \delta}}{Re\ \delta} \sum_{j=1}^{n} \left[|\alpha_j| \left|\frac{zf_j'(z)}{f_j(z)} - 1\right| + |\beta_j||z| \left|\frac{f_j''(z)}{f_j'(z)}\right|\right], \tag{3.16}$$

for all $z \in \mathcal{U}$.

From (3.12), Lemma Mocanu and Şerb, by (3.16) we have

$$\frac{1-|z|^{2Re\ \delta}}{Re\ \delta} \left|\frac{zg_n''(z)}{g_n'(z)}\right| \le$$

$$\le \frac{1-|z|^{2Re\ \delta}}{Re\ \delta} \sum_{j=1}^{n} |\alpha_j| + \frac{1-|z|^{2Re\ \delta}}{Re\ \delta} |z| M_0 \sum_{j=1}^{n} |\beta_j|. \tag{3.17}$$

Since

$$\max_{|z|\le 1} \frac{1-|z|^{2Re\ \delta}}{Re\ \delta} |z| = \frac{2}{(2Re\ \delta + 1)^{\frac{2Re\ \delta + 1}{2Re\ \delta}}},$$

from (3.17) we obtain

$$\frac{1-|z|^{2Re\ \delta}}{Re\ \delta} \left|\frac{zg_n''(z)}{g_n'(z)}\right| \le$$

$$\le \frac{1}{Re\ \delta} \sum_{j=1}^{n} |\alpha_j| + \frac{2M_0}{(2Re\ \delta + 1)^{\frac{2Re\ \delta + 1}{2Re\ \delta}}} \sum_{j=1}^{n} |\beta_j|, \tag{3.18}$$

for all $z \in \mathcal{U}$.

From (3.13) and (3.18) we get

$$\frac{1-|z|^{2Re\ \delta}}{Re\ \delta} \left|\frac{zg_n''(z)}{g_n'(z)}\right| \le 1, \quad (z \in \mathcal{U}). \tag{3.19}$$

From (3.14) we have $g_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdots \left(\frac{f_n(z)}{z}\right)^{\alpha_n} \cdot (f_1'(z))^{\beta_1} \cdots (f_n'(z))^{\beta_n}$ and by Lemma Pascu we obtain that $T_n \in \mathcal{S}$. □

Remark 3.4. For $\beta_1 = \beta_2 = \cdots = \beta_n = 0, \delta = \beta$, from Theorem 3.3 we obtain the Theorem 3.1.

Corollary 3.5. *Let $\delta, \alpha_j, \beta_j$ be complex numbers, $j = \overline{1,n}$, $0 < Re\ \delta \le 1, M_0$ the positive solution of the equation (2.1), $M_0 = 1,5936...$ and $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \le M_0, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{3.20}$$

$$(2Re\ \delta + 1)^{\frac{2Re\ \delta+1}{2Re\ \delta}} \sum_{j=1}^{n} |\alpha_j| + 2(Re\ \delta)M_0 \left[\sum_{j=1}^{n} |\beta_j|\right] \le$$
$$\le (2Re\ \delta + 1)^{\frac{2Re\ \delta+1}{2Re\ \delta}} Re\ \delta, \tag{3.21}$$

then the integral operator I_n defined by

$$I_n(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{u}\right)^{\alpha_n} \cdot (f_1'(u))^{\beta_1} \cdots (f_n'(u))^{\beta_n}\, du. \tag{3.22}$$

belongs to the class $\mathcal{S}$.

Proof. We take $\gamma = 1$ in the Theorem 3.3. □

Theorem 3.6. *Let $\gamma, \alpha_j, \beta_j$ be complex numbers, $j = \overline{1,n}$, $Re\ \gamma > 0, M_0$ the positive solution of the equation (2.1), $M_0 = 1,5936...$ and $f_j, g_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + \ldots$, $g_j(z) = z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \le M_0, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{3.23}$$

$$\left|\frac{g_j''(z)}{g_j'(z)}\right| \le M_0, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{3.24}$$

$$(2Re\ \gamma + 1)^{\frac{2Re\ \gamma+1}{2Re\ \gamma}} \sum_{j=1}^{n} |\alpha_j| + 2(Re\ \gamma)M_0 \sum_{j=1}^{n} |\beta_j| \le$$
$$\le \frac{(2Re\ \gamma + 1)^{\frac{2Re\ \gamma+1}{2Re\ \gamma}} Re\ \gamma}{2}, \tag{3.25}$$

then for every complex number δ , $Re\ \delta \ge Re\ \gamma$ the integral operator V_n defined by (1.3) is in the class $\mathcal{S}$.

Proof. We consider the function

$$p_n(z) = \int_0^z \left(\frac{f_1(u)}{g_1(u)}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{g_n(u)}\right)^{\alpha_n} \cdot \left(\frac{f_1'(u)}{g_1'(u)}\right)^{\beta_1} \cdots \left(\frac{f_n'(u)}{g_n'(u)}\right)^{\beta_n} du \tag{3.26}$$

The function p_n is regular in $\mathcal{U}$ and $p_n(0) = p_n'(0) - 1 = 0$.

We have

$$\frac{zp_n''(z)}{p_n'(z)} = \sum_{j=1}^{n} \left[\alpha_j \left(\frac{zf_j'(z)}{f_j(z)} - \frac{zg_j'(z)}{g_j(z)} \right) + \beta_j \left(\frac{zf_j''(z)}{f_j'(z)} - \frac{zg_j''(z)}{g_j'(z)} \right) \right],$$

and hence, we get

$$\frac{zp_n''(z)}{p_n'(z)} =$$

$$\sum_{j=1}^{n} \left\{ \alpha_j \left[\left(\frac{zf_j'(z)}{f_j(z)} - 1 \right) - \left(\frac{zg_j'(z)}{g_j(z)} - 1 \right) \right] + \beta_j \left(\frac{zf_j''(z)}{f_j'(z)} - \frac{zg_j''(z)}{g_j'(z)} \right) \right\} \tag{3.27}$$

for all $z \in \mathcal{U}$.

From (3.27) we obtain

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma} \left| \frac{zp_n''(z)}{p_n'(z)} \right| \leq \tag{3.28}$$

$$\leq \frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma} \left\{ \sum_{j=1}^{n} \left[|\alpha_j| \left(\left| \frac{zf_j'(z)}{f_j(z)} - 1 \right| + \left| \frac{zg_j'(z)}{g_j(z)} - 1 \right| \right) + \right. \right.$$

$$\left. \left. + |\beta_j| \left(\left| \frac{zf_j''(z)}{f_j'(z)} \right| + \left| \frac{zg_j''(z)}{g_j'(z)} \right| \right) \right] \right\}$$

for all $z \in \mathcal{U}$.

From (3.23), (3.24) and Lemma Mocanu and Şerb we have

$$\left| \frac{zf_j'(z)}{f_j(z)} - 1 \right| < 1, \tag{3.29}$$

$$\left| \frac{zg_j'(z)}{g_j(z)} - 1 \right| < 1, \tag{3.30}$$

for all $z \in \mathcal{U}$, $j = \overline{1,n}$ and hence, we get

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma} \left| \frac{zp_n''(z)}{p_n'(z)} \right| \leq$$

$$\leq \frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma} \cdot 2 \sum_{j=1}^{n} |\alpha_j| + \frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma} |z| \cdot 2M_0 \sum_{j=1}^{n} |\beta_j|. \tag{3.31}$$

Since

$$\max_{|z|\leq 1} \frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma} |z| = \frac{2}{(2Re\,\gamma+1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}}, \tag{3.32}$$

from (3.31) we obtain

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma} \left| \frac{zp_n''(z)}{p_n'(z)} \right| \leq \frac{2}{Re\,\gamma} \sum_{j=1}^{n} |\alpha_j| + \frac{4M_0}{(2Re\,\gamma+1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}} \sum_{j=1}^{n} |\beta_j|, \tag{3.33}$$

for all $z \in \mathcal{U}$.

From (3.25) and (3.33) we get

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}\left|\frac{zp_n''(z)}{p_n'(z)}\right| \le 1, \quad (z \in \mathcal{U}). \tag{3.34}$$

From (3.26) we obtain

$$p_n'(z) = \left(\frac{f_1(z)}{g_1(z)}\right)^{\alpha_1} \cdots \left(\frac{f_n(z)}{g_n(z)}\right)^{\alpha_n} \cdot \left(\frac{f_1'(z)}{g_1'(z)}\right)^{\beta_1} \cdots \left(\frac{f_n'(z)}{g_n'(z)}\right)^{\beta_n}$$

and by Lemma Pascu it results that $V_n \in \mathcal{S}$. □

Corollary 3.7. *Let γ, α_j be complex numbers, $j = \overline{1,n}$, $Re\ \gamma > 0, M_0$ the positive solution of the equation (2.1), $M_0 = 1,5936...$ and $f_j, g_j \in \mathcal{A}$,*

$$f_j(z) = z + a_{2j}z^2 + \dots, \;\; g_j(z) = z + b_{2j}z^2 + \dots, \;\; j = \overline{1,n}.$$

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \le M_0, \quad (z \in \mathcal{U}, \; j = \overline{1,n}), \tag{3.35}$$

$$\left|\frac{g_j''(z)}{g_j'(z)}\right| \le M_0, \quad (z \in \mathcal{U}, \; j = \overline{1,n}), \tag{3.36}$$

and

$$\sum_{i=1}^{n} |\alpha_j| \le \frac{Re\,\gamma}{2} \tag{3.37}$$

then for all complex numbers δ , $Re\ \delta \ge Re\ \gamma$ the integral operator K_n defined by

$$K_n(z) = \left\{\delta \int_0^z u^{\delta-1}\left(\frac{f_1(u)}{g_1(u)}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{g_n(u)}\right)^{\alpha_n} du\right\}^{\frac{1}{\delta}}, \tag{3.38}$$

is in the class $\mathcal{S}$.

Proof. We take $\beta_1 = \beta_2 = \cdots = \beta_n = 0$ in Theorem 3.6. □

Corollary 3.8. *Let $\gamma, \alpha_j, \beta_j$ be complex numbers, $j = \overline{1,n}$, $0 < Re\ \gamma \le 1, M_0$ the positive solution of the equation (2.1), $M_0 = 1,5936...$ and $f_j, g_j \in \mathcal{A}$,*

$$f_j(z) = z + a_{2j}z^2 + \dots, \;\; g_j(z) = z + b_{2j}z^2 + \dots, \;\; j = \overline{1,n}.$$

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \le M_0, \quad (z \in \mathcal{U}, \; j = \overline{1,n}), \tag{3.39}$$

$$\left|\frac{g_j''(z)}{g_j'(z)}\right| \le M_0, \quad (z \in \mathcal{U}, \; j = \overline{1,n}), \tag{3.40}$$

and

$$(2Re\,\gamma+1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}\sum_{j=1}^{n}|\alpha_j|+2M_0(Re\,\gamma)\sum_{j=1}^{n}|\beta_j|\le$$

$$\le\frac{(2Re\,\gamma+1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}\cdot Re\,\gamma}{2}, \tag{3.41}$$

then the integral operator J_n *defined by*

$$J_n(z)=\int_0^z\left(\frac{f_1(u)}{g_1(u)}\right)^{\alpha_1}\cdots\left(\frac{f_n(u)}{g_n(u)}\right)^{\alpha_n}\cdot\left(\frac{f_1'(u)}{g_1'(u)}\right)^{\beta_1}\cdots\left(\frac{f_n'(u)}{g_n'(u)}\right)^{\beta_n}du \tag{3.42}$$

is in the class $\mathcal{S}$.

Proof. For $\delta=1$ in Theorem 3.6, we obtain Corollary 3.8. □

Corollary 3.9. *Let* γ,β_j *be complex numbers,* $j=\overline{1,n}$, $Re\,\gamma>0, M_0$ *the positive solution of the equation* (2.1)*,* $M_0=1,5936...$ *and* $f_j,g_j\in\mathcal{A}$,
$f_j(z)=z+a_{2j}z^2+\ldots$, $g_j(z)=z+b_{2j}z^2+\ldots$, $j=\overline{1,n}$.

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right|\le M_0,\quad (z\in\mathcal{U},\ j=\overline{1,n}), \tag{3.43}$$

$$\left|\frac{g_j''(z)}{g_j'(z)}\right|\le M_0,\quad (z\in\mathcal{U},\ j=\overline{1,n}), \tag{3.44}$$

and

$$\sum_{j=1}^{n}|\beta_j|\le\frac{(2Re\,\gamma+1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}}{4M_0}, \tag{3.45}$$

then for all complex number δ, $Re\,\delta\ge Re\,\gamma$, *the integral operator* Q_n *defined by*

$$Q_n(z)=\left\{\delta\int_0^z u^{\delta-1}\left(\frac{f_1'(u)}{g_1'(u)}\right)^{\beta_1}\cdots\left(\frac{f_n'(u)}{g_n'(u)}\right)^{\beta_n}du\right\}^{\frac{1}{\delta}} \tag{3.46}$$

is in the class $\mathcal{S}$.

Proof. For $\alpha_1=\alpha_2=\cdots=\alpha_n=0$ in Theorem 3.6, we obtain Corollary 3.9. □

Corollary 3.10. *Let* γ,β_j *be complex numbers,* $j=\overline{1,n}$, $0<Re\,\gamma\le 1, M_0$ *the positive solution of the equation* (2.1)*,* $M_0=1,5936...$ *and* $f_j,g_j\in\mathcal{A}$,
$f_j(z)=z+a_{2j}z^2+\ldots$, $g_j(z)=z+b_{2j}z^2+\ldots$, $j=\overline{1,n}$.

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right|\le M_0,\quad (z\in\mathcal{U},\ j=\overline{1,n}), \tag{3.47}$$

$$\left|\frac{g_j''(z)}{g_j'(z)}\right| \le M_0, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{3.48}$$

and

$$\sum_{j=1}^{n} |\beta_j| \le \frac{(2Re\,\gamma+1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}}{4M_0}, \tag{3.49}$$

then the integral operator L_n *defined by*

$$L_n(z) = \int_0^z \left(\frac{f_1'(u)}{g_1'(u)}\right)^{\beta_1} \cdots \left(\frac{f_n'(u)}{g_n'(u)}\right)^{\beta_n} du \tag{3.50}$$

is in the class $\mathcal{S}$.

Proof. We take $\delta = 1$ and $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ in Theorem 3.6. □

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Harmonic uniformly $\beta-$starlike functions of complex order defined by convolution and integral convolution

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Abstract. In this paper we introduce and study a subclass of harmonic univalent functions defined by convolution and integral convolution. Coefficient bounds, extreme points, distortion bounds, convolution conditions and convex combinations are determined for functions in this family. Consequently, many of our results are either extensions or new approaches to those corresponding to previously known results.

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1. Introduction

A continuous function $f = u + iv$ is a complex- valued harmonic function in a complex domain Ω if both u and v are real and harmonic in Ω. In any simply-connected domain $D \subset \Omega$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. Moreover,

$$h' = f_z = \frac{\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}}{2} \quad \text{and} \quad \bar{g}' = f_{\bar{z}} = \frac{\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}}{2}$$

are always analytic functions in D. A necessary and sufficient condition for f to be locally univalent and orientation-preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [13]).

Denote by $\mathcal{S}_{\mathcal{H}}$ the family of functions $f = h + \overline{g}$ which are harmonic, univalent and orientation-preserving in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$, the functions

h and g analytic in $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \;\; g(z) = \sum_{k=1}^{\infty} b_k z^k \;\; (|b_1| < 1), \tag{1.1}$$

and f is then given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \quad (|b_1| < 1). \tag{1.2}$$

We note that the family $\mathcal{S}_{\mathcal{H}}$ reduces to the well known class $\mathcal{S}$ of normalized univalent functions if the co-analytic part of f is identically zero ($g \equiv 0$).

Also, we denote by $T\mathcal{S}_{\mathcal{H}}$ the subfamily of $\mathcal{S}_{\mathcal{H}}$ consisting of harmonic functions $f = h + \overline{g}$ such that

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) = \sum_{k=1}^{\infty} |b_k| z^k. \tag{1.3}$$

In [3] Clunie and Sheil-Small investigated the class $\mathcal{S}_{\mathcal{H}}$ as well as its geometric subclasses and their properties. Since then, there have been several studies related to the class $\mathcal{S}_{\mathcal{H}}$ and its subclasses. Following Clunie and Sheil-Small [3], Frasin [7], Jahangiri [9, 10], Silverman [17], Silverman and Silvia [18], Dixit and Porwal [4], Dixit et al. [5, 6] and others have investigated various subclasses of $\mathcal{S}_{\mathcal{H}}$ and its properties.

Recently, Yalçin and Öztürk [20] introduced a new class of harmonic starlike functions of complex order $T\mathcal{S}^*_{\mathcal{H}}(b)$ subclass of $T\mathcal{S}_{\mathcal{H}}$ consisting functions of the form (1.3) and satisfying the condition

$$\mathrm{Re}\left(1 + \frac{1}{b}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - 1\right)\right) > 0, \; z \in \mathcal{U}, \quad b \in \mathbb{C}\backslash\{0\}$$

and settled a conjecture. Further, Halim and Janteng [8] extended the study by introducing a new class $\mathcal{S}^*_{\mathcal{H}}(b, \alpha)$, $0 \le \alpha < 1$ of $\mathcal{S}_{\mathcal{H}}$ consisting functions of the form (1.2) and satisfying the condition

$$\mathrm{Re}\left(1 + \frac{1}{b}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - 1\right)\right) > \alpha, \; z \in \mathcal{U}, \quad b \in \mathbb{C}\backslash\{0\}, \quad 0 \le \alpha < 1$$

and obtained following sufficient condition. If $f = h + \bar{g}$ is given by (1.2) and if

$$\sum_{n=2}^{\infty}\left(\frac{n-1+(1-\alpha)|b|}{(1-\alpha)|b|}\right)|a_n| + \sum_{n=1}^{\infty}\left(\frac{n+1-(1+\alpha)|b|}{(1-\alpha)|b|}\right)|b_n| \le 1$$

then $f \in \mathcal{S}^*_{\mathcal{H}}(b, \alpha)$. Also, they proved that the coefficient condition

$$\sum_{n=2}^{\infty}\left(\frac{n-1+(1-\alpha)|b|}{(1-\alpha)|b|}\right)|a_n| + \sum_{n=2}^{\infty}\left(\frac{n+1-(1+\alpha)|b|}{(1-\alpha)|b|}\right)|b_n| \le 1, \; b_1 = 0$$

is necessary for $f = h + \bar{g}$ is given by (1.3) and belongs to $T\mathcal{S}^*_{\mathcal{H}}(b, \alpha)$.

The convolution of two power series

$$\Phi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k, \text{ and } \Psi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k \tag{1.4}$$

is defined by

$$(\Phi * \Psi)(z) = z + \sum_{k=2}^{\infty} \lambda_k \mu_k z^k, \tag{1.5}$$

where $\lambda_k \geq 0$ and $\mu_k \geq 0$. Also the integral convolution is defined by

$$(\Phi \diamond \Psi)(z) = z + \sum_{k=2}^{\infty} \frac{\lambda_k \mu_k}{k} z^k. \tag{1.6}$$

Motivated by the works of Yalçin and Öztürk [20], Halim and Janteng [8], Janteg and Halim [11] and Magesh and Mayilvaganan [14], we consider the subclass $\mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, b; t)$ of functions of the form (1.2) satisfying the condition

$$\mathrm{Re}\left(1 + \frac{1}{b}\left((1+\beta e^{i\alpha})\frac{h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}}{h_t(z) \diamond \Phi(z) + \overline{g_t(z) \diamond \Psi(z)}} - \beta e^{i\alpha} - 1\right)\right) > \gamma, \ z \in \mathcal{U}, \tag{1.7}$$

where $b \in \mathbb{C}\backslash\{0\}$, $\beta \geq 0$, $0 \leq \gamma < 1$, $\alpha \in \mathbb{R}$, $h_t(z) = (1-t)z + th(z)$, $g_t(z) = tg(z)$, $0 \leq t \leq 1$, Φ and Ψ are of the form (1.4). We further let $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$ denote the subclass of $\mathcal{G}_{\mathcal{H}}(\Phi, \Psi, \beta, \gamma, b; t)$ consisting of functions $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ such that h and g are of the form (1.3).

We note that by specializing the functions Φ, Ψ and parameters β, γ and t we obtain well-known harmonic univalent functions as well as many new ones.

For example,

$$\mathcal{G}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 0, 0, b; 1) = T\mathcal{S}^*_{\mathcal{H}}(b)$$

was introduced by Yalçin and Öztürk [20] and studied by Halim and Janteng [8],

$$\mathcal{G}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 1, \gamma, b; 1) = T\mathcal{S}^*_{\mathcal{H}}(\gamma, b)$$

was introduced by Stephen et al. [19]. Furthermore,

$$\mathcal{G}_{\mathcal{H}}(\frac{z+z^2}{(1-z)^3}, \frac{z+z^2}{(1-z)^3}; 1, \gamma, 1; 1) = \mathcal{HC}(\gamma)$$

was studied by Kim et al. [12] and

$$\mathcal{G}_{\mathcal{H}}(z + \sum_{k=2}^{\infty} k^{n+1} z^k, z + \sum_{k=2}^{\infty} k^{n+1} z^k; 1, \gamma, 1; 1) = RS(\gamma)$$

was studied by Yalcin et al. [21]. Also,

$$\mathcal{G}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 1, \gamma, 1; 1) = G_{\mathcal{H}}(\gamma)$$

was studied by Rosy et al. [16],

$$\mathcal{G}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \beta, \gamma, 1; t) = \mathcal{G}_{\overline{\mathcal{H}}}(\beta, \gamma; t)$$

was considered by Ahuja et al. [1]. Also, the class

$$\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi, \beta, \gamma, 1; t) = \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, t)$$

was studied by Magesh and Porwal [15],

$$\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; 0, \gamma, 1; 1) = \overline{\mathcal{HS}}(\Phi, \Psi; \gamma)$$

was studied by Dixit et al. [5],

$$\mathcal{G}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 0, \gamma, 1; 1) = S^*_{H}(\gamma)$$

and

$$\mathcal{G}_{\overline{\mathcal{H}}}(\frac{z+z^2}{(1-z)^3}, \frac{z+z^2}{(1-z)^3}; 0, \gamma, 1; 1) = K(\gamma)$$

were introduced and studied by Jahangiri [10]. For $\gamma = 0$ the classes $S^*_{\mathcal{H}}(\gamma)$ and $K(\gamma)$ were studied by Silverman and Silvia [18], for $\gamma = 0$ and $b_1 = 0$ see [2, 17].

If we set $\beta = 1$ and $\alpha = 0$ in the above definition we define the unified class of harmonic starlike functions of complex order satisfying the following analytic criteria:

$$\operatorname{Re}\left(1 + \frac{2}{b}\left(\frac{h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}}{h_t(z) \diamond \Phi(z) + \overline{g_t(z) \diamond \Psi(z)}} - 1\right)\right) > \gamma, \ z \in \mathcal{U},$$

where $b \in \mathbb{C}\backslash\{0\}$, $0 \leq \gamma < 1$, $h_t(z) = (1-t)z + th(z)$, $g_t(z) = tg(z)$, $0 \leq t \leq 1$, Φ and Ψ are of the form (1.4).

In this paper we give a sufficient condition for $f = h + \bar{g}$ given by (1.2) to be in $\mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, b; t)$ and it is shown that this condition is also necessary for functions to be in $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$. We also obtain extreme points, distortion bounds, convolution and convex combination properties. Further, we obtain the closure property of this class under integral operator. We remark that the results so obtained for these general families can be viewed as extensions and generalizations for various subclasses of $\mathcal{S}_{\mathcal{H}}$ as listed previously in this section.

2. Coefficient bounds

Our first theorem gives a sufficient condition for functions to be in $\mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, b; t)$.

Theorem 2.1. *Let $f = h + \bar{g}$ be so that h and g are given by (1.1). If*

$$\sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta) + (1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|}|a_k| + \sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta) - (1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|}|b_k| \leq 1, \qquad (2.1)$$

where $\beta \geq 0$, $0 \leq \gamma < 1$, $0 \leq t \leq 1$, $k^2(1-\gamma) \leq [(k-t)(1+\beta) + (1-\gamma)t|b|]\lambda_k$ and $k^2(1-\gamma) \leq [(k+t)(1+\beta) - (1-\gamma)t|b|]\mu_k$ for $k \geq 2$. Then $f \in \mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, b; t)$.

Proof. To prove that $f \in \mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, b; t)$, we only need to show that if (2.1) holds, then the required condition (1.7) is satisfied. For (1.7), we can write

$$\operatorname{Re}\left(1+\frac{1}{b}\left((1+\beta e^{i\alpha})\frac{h(z)*\Phi(z)-\overline{g(z)*\Psi(z)}}{h_t(z)\diamond\Phi(z)+\overline{g_t(z)\diamond\Psi(z)}}-\beta e^{i\alpha}-1\right)\right)=\operatorname{Re}\frac{A(z)}{B(z)}>\gamma,\ z\in\mathcal{U}.$$

Using the fact that $\operatorname{Re}\{\omega\} \geq \gamma$ if and only if $|1-\gamma+\omega| \geq |1+\gamma-\omega|$, it suffices to show that

$$|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)| \geq 0,\ z \in \mathcal{U}, \tag{2.2}$$

where

$$\begin{aligned}A(z) &= (1+\beta e^{i\alpha})[h(z)*\Phi(z)-\overline{g(z)*\Psi(z)}]\\ &\quad + [b-(1+\beta e^{i\alpha})][h_t(z)\diamond\Phi(z)+\overline{g_t(z)\diamond\Psi(z)}]\end{aligned}$$

and

$$B(z) = b[h_t(z)\diamond\Phi(z)+\overline{g_t(z)\diamond\Psi(z)}].$$

Substituting A and B in (2.2) and making use of (2.1), we obtain

$$\begin{aligned}
&|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|\\
&= \Big|(1+\beta e^{i\alpha})[h(z)*\Phi(z)-\overline{g(z)*\Psi(z)}]\\
&\quad +[b-(1+\beta e^{i\alpha})][h_t(z)\diamond\Phi(z)+\overline{g_t(z)\diamond\Psi(z)}]\\
&\quad +(1-\gamma)b\left(h_t(z)\diamond\Phi(z)+\overline{g_t(z)\diamond\Psi(z)}\right)\Big|\\
&\quad -\Big|(1+\beta e^{i\alpha})[h(z)*\Phi(z)-\overline{g(z)*\Psi(z)}]\\
&\quad +[b-(1+\beta e^{i\alpha})][h_t(z)\diamond\Phi(z)+\overline{g_t(z)\diamond\Psi(z)}]\\
&\quad -(1+\gamma)b\left(h_t(z)\diamond\Phi(z)+\overline{g_t(z)\diamond\Psi(z)}\right)\Big|\\
&\geq 2(1-\gamma)|b||z|-\sum_{k=2}^{\infty}2\left[\frac{(k-t)(1+\beta)+(1-\gamma)t|b|}{k}\right]\lambda_k|a_k||z|^k\\
&\quad -\sum_{k=1}^{\infty}2\left[\frac{(k+t)(1+\beta)-(1-\gamma)t|b|}{k}\right]\mu_k|b_k||z|^k\\
&= 2(1-\gamma)|b||z|\left\{1-\sum_{k=2}^{\infty}\left[\frac{(k-t)(1+\beta)+(1-\gamma)t|b|}{k(1-\gamma)|b|}\right]\lambda_k|a_k||z|^{k-1}\right.\\
&\quad \left.-\sum_{k=1}^{\infty}\left[\frac{(k+t)(1+\beta)-(1-\gamma)t|b|}{k(1-\gamma)|b|}\right]\mu_k|b_k||z|^{k-1}\right\}\\
&> 2(1-\gamma)|b|\left\{1-\sum_{k=2}^{\infty}\left[\frac{(k-t)(1+\beta)+(1-\gamma)t|b|}{k(1-\gamma)|b|}\right]\lambda_k|a_k|\right.\\
&\quad \left.-\sum_{k=1}^{\infty}\left[\frac{(k+t)(1+\beta)-(1-\gamma)t|b|}{k(1-\gamma)|b|}\right]\mu_k|b_k|\right\} \geq 0
\end{aligned}$$

which implies that $f \in \mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, b; t)$.

The harmonic function

$$f(z) = z + \sum_{k=2}^{\infty} \frac{k(1-\gamma)|b|}{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k} x_k z^k + \sum_{k=1}^{\infty} \frac{k(1-\gamma)|b|}{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k} \overline{y_k z^k},$$

where

$$\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1,$$

shows that the coefficient bound given by (2.1) is sharp. □

Next, we will show that the sufficient condition (2.1) is also necessary for functions to be in the class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$.

Theorem 2.2. *Let $f = h + \bar{g}$ be so that h and g are given by (1.3). Then $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$ if and only if*

$$\sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|} a_k + \sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|} b_k \leq 1, \tag{2.3}$$

where $\beta \geq 0$, $0 \leq \gamma < 1$, $0 \leq t \leq 1$, $k^2(1-\gamma) \leq [(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k$ and $k^2(1-\gamma) \leq [(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k$ for $k \geq 2$.

Proof. Since $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t) \subset \mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, b; t)$, we only need to prove the only if part of the theorem. To this end, for functions f of the form (1.3), we notice that the condition (1.7) is equivalent to

$$\operatorname{Re}\left\{ \frac{(1+\beta e^{i\alpha})[h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}] + [(1-\gamma)b - (1+\beta e^{i\alpha})][h_t(z) \diamond \Phi(z) + \overline{g_t(z) \diamond \Psi(z)}]}{b[h_t(z) \diamond \Phi(z) + \overline{g_t(z) \diamond \Psi(z)}]} \right\} \geq 0, \ z \in \mathcal{U}.$$

Upon choosing the values of z on the positive real axis where $0 \leq |z| = r < 1$, the above inequality reduces to

$$\operatorname{Re}\left\{\frac{\begin{array}{l}(1-\gamma)b-\sum\limits_{k=2}^{\infty}[(k-t)+(1-\gamma)bt]\frac{\lambda_k}{k}|a_k|r^{k-1}\\ -\sum\limits_{k=1}^{\infty}[(k+t)-(1-\gamma)bt]\frac{\mu_k}{k}|b_k|r^{k-1}\end{array}}{b\left[1-\sum\limits_{k=2}^{\infty}\frac{t\lambda_k}{k}|a_k|r^{k-1}+\sum\limits_{k=1}^{\infty}\frac{t\mu_k}{k}|b_k|r^{k-1}\right]}\right\}$$

$$-\operatorname{Re}\left\{\beta e^{i\alpha}\frac{\sum\limits_{k=2}^{\infty}(k-t)\frac{\lambda_k}{k}|a_k|r^{k-1}+\sum\limits_{k=1}^{\infty}(k+t)\frac{\mu_k}{k}|b_k|r^{k-1}}{b\left[1-\sum\limits_{k=2}^{\infty}\frac{t\lambda_k}{k}|a_k|r^{k-1}+\sum\limits_{k=1}^{\infty}\frac{t\mu_k}{k}|b_k|r^{k-1}\right]}\right\}\geq 0.$$

Since $Re(-e^{i\alpha}) \geq -|e^{i\alpha}| = -1$, the above inequality reduces to

$$\left\{\frac{\begin{array}{l}(1-\gamma)|b|-\sum\limits_{k=2}^{\infty}[(k-t)(1+\beta)+(1-\gamma)t|b|]\frac{\lambda_k}{k}|a_k|r^{k-1}\\ -\sum\limits_{k=1}^{\infty}[(k+t)(1+\beta)-(1-\gamma)t|b|]\frac{\mu_k}{k}|b_k|r^{k-1}\end{array}}{|b|\left[1-\sum\limits_{k=2}^{\infty}\frac{t\lambda_k}{k}|a_k|r^{k-1}+\sum\limits_{k=1}^{\infty}\frac{t\mu_k}{k}|b_k|r^{k-1}\right]}\right\}\geq 0. \tag{2.4}$$

If the condition (2.3) does not hold then the numerator in (2.4) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.4) is negative. This contradicts the condition for $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$, hence the proof is complete. □

3. Extreme points and distortion bounds

In this section, our first theorem gives the extreme points of the closed convex hulls of $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$.

Theorem 3.1. *Let f be given by (1.3). Then $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty}(X_k h_k(z) + Y_k g_k(z)), \tag{3.1}$$

where

$$h_1(z) = z,\ h_k(z) = z - \frac{k(1-\gamma)|b|}{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}z^k\ (k = 2, 3, ...),$$

$$g_k(z) = z + \frac{k(1-\gamma)|b|}{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}\overline{z}^k \ (k = 1, 2, 3, ...),$$
$$\sum_{k=1}^{\infty}(X_k + Y_k) = 1, \ X_k \geq 0, \ Y_k \geq 0.$$

In particular, the extreme points of $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$ *are* $\{h_k\}$ *and* $\{g_k\}$.

Proof. For functions f of the form (3.1), we have

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_k(z)) \\ &= \sum_{k=1}^{\infty} (X_k + Y_k)\, z - \sum_{k=2}^{\infty} \frac{k(1-\gamma)|b|}{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k} X_k z^k \\ &\quad + \sum_{k=1}^{\infty} \frac{k(1-\gamma)|b|}{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k} Y_k \overline{z}^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|} \left(\frac{k(1-\gamma)|b|}{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k} \right) X_k \\ &\quad + \sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|} \left(\frac{k(1-\gamma)|b|}{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k} \right) Y_k \\ &= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1 \end{aligned}$$

and so $f \in clco\ \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$.
Conversely, suppose that $f \in clco\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$ and set

$$X_k = \frac{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|}|a_k|, \ k = 2, 3, \ldots,$$

and

$$Y_k = \frac{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|}|b_k|, \ k = 1, 2, \ldots,$$

where

$$\sum_{k=1}^{\infty}(X_k + Y_k) = 1.$$

Then, by Theorem 2.2, we have $0 \leq X_k \leq 1$ $(k = 2, 3, \ldots)$ and $0 \leq Y_k \leq 1$ $(k = 1, 2, 3, \ldots)$. We define

$$X_1 = 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k$$

and use Theorem 2.2 again to get $X_1 \geq 0$. Consequently, we obtain

$$f(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_k(z)).$$

Another application of Theorem 2.2 shows that $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$ is convex and closed, so *clco* $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)=\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$. In other words, the statement of Theorem 3.1 holds. □

The following theorem gives the distortion bounds for functions in $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$ which yields a covering result for this class.

Theorem 3.2. *Let* $f\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$ *and*

$$A\le[(k-t)(1+\beta)+(1-\gamma)t|b|]\frac{\lambda_k}{k},$$
$$A\le[(k+t)(1+\beta)-(1-\gamma)t|b|]\frac{\mu_k}{k}$$

for $k\ge2$, *where*

$$A=\min\left\{[(2-t)(1+\beta)+(1-\gamma)t|b|]\frac{\lambda_2}{2},\ [(2+t)(1+\beta)-(1-\gamma)t|b|]\frac{\mu_2}{2}\right\}$$

then

$$|f(z)|\le(1+|b_1|)r+\left(\frac{1-\gamma}{A}-\frac{(1+t)(1+\beta)-(1-\gamma)t|b|}{A}|b_1|\right)r^2$$

and

$$|f(z)|\ge(1-|b_1|)r-\left(\frac{1-\gamma}{A}-\frac{(1+t)(1+\beta)-(1-\gamma)t|b|}{A}|b_1|\right)r^2.$$

Proof. Let $f\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$. Taking the absolute value of f, we obtain

$$\begin{aligned}
|f(z)| &\le (1+|b_1|)r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k\\
&\le (1+|b_1|)r+r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|)\\
&= (1+|b_1|)r+\frac{1-\gamma}{A}r^2\sum_{k=2}^{\infty}(\frac{A}{1-\gamma}|a_k|+\frac{A}{1-\gamma}|b_k|)\\
&\le (1+|b_1|)r+\frac{1-\gamma}{A}r^2\sum_{k=2}^{\infty}\left(\frac{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|}|a_k|\right.\\
&\qquad\left.+\frac{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|}|b_k|\right)\\
&\le (1+|b_1|)r+\frac{1-\gamma}{A}\left(1-\frac{(1+t)(1+\beta)-(1-\gamma)t|b|}{(1-\gamma)}|b_1|\right)r^2\\
&= (1+|b_1|)r+\left(\frac{1-\gamma}{A}-\frac{(1+t)(1+\beta)-(1-\gamma)t|b|}{A}|b_1|\right)r^2
\end{aligned}$$

and similarly,

$$|f(z)|\ \ge\ (1-|b_1|)r-\left(\frac{1-\gamma}{A}-\frac{(1+t)(1+\beta)-(1-\gamma)t|b|}{A}|b_1|\right)r^2.$$

The upper and lower bounds given in Theorem 3.2 are respectively attained by the following functions:

$$f(z) = z + |b_1|\bar{z} + \left(\frac{1-\gamma}{A} - \frac{(1+t)(1+\beta)-(1-\gamma)t|b|}{A}|b_1|\right)\bar{z}^2$$

and

$$f(z) = (1-|b_1|)z - \left(\frac{1-\gamma}{A} - \frac{(1+t)(1+\beta)-(1-\gamma)t|b|}{A}|b_1|\right)z^2. \qquad \square$$

The next covering result follows from the left hand inequality in Theorem 3.2.

Corollary 3.3. *Let f of the form (1.3) be so that $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$ and*

$$A \leq [(k-t)(1+\beta)+(1-\gamma)t|b|]\frac{\lambda_k}{k},$$

$$A \leq [(k+t)(1+\beta)-(1-\gamma)t|b|]\frac{\mu_k}{k}$$

for $k \geq 2$, where

$$A = \min\left\{[(2-t)(1+\beta)+(1-\gamma)t|b|]\frac{\lambda_2}{2},\ [(2+t)(1+\beta)-(1-\gamma)t|b|]\frac{\mu_2}{2}\right\}.$$

Then

$$\left\{\omega \in \mathbb{C} : |\omega| < \frac{A+1-\gamma}{A} + \frac{A-1+\gamma}{A}|b_1|\right\} \subset f(\mathcal{U}).$$

4. Convolution and convex combinations

In this section we show that the class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$ is closed under convolution and convex combinations. Now we need the following definition of convolution of two harmonic functions. For

$$f(z) = z - \sum_{k=2}^{\infty}|a_k|z^k + \sum_{k=1}^{\infty}|b_k|\bar{z}^k$$

and

$$F(z) = z - \sum_{k=2}^{\infty}|A_k|z^k + \sum_{k=1}^{\infty}|B_k|\bar{z}^k,$$

we define the convolution of two harmonic functions f and F as

$$(f*F)(z) = f(z)*F(z) = z - \sum_{k=2}^{\infty}|a_k||A_k|z^k + \sum_{k=1}^{\infty}|b_k||B_k|\bar{z}^k. \qquad (4.1)$$

Using the definition, we show that the class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$ is closed under convolution.

Theorem 4.1. *For $0 \leq \gamma < 1$, let $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$ and $F \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$. Then $f*F \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$.*

Proof. Let

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \overline{z}^k$$

and

$$F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| \overline{z}^k$$

be in $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$. Then the convolution $f * F$ is given by (4.1). We wish to show that the coefficient of $f * F$ satisfy the required condition given in Theorem 2.2. For $F \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$, we note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Now for the convolution function $f * F$, we obtain

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta) + (1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|} |a_k||A_k| \\ &\qquad + \sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta) - (1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|} |b_k||B_k| \\ &\leq \sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta) + (1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|} |a_k| \\ &\qquad + \sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta) - (1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|} |b_k| \\ &\leq 1, \end{aligned}$$

since $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$. Therefore $f * F \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$. □

Next, we show that the class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$ is closed under convex combination of its members.

Theorem 4.2. *The class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \dots$ let $f_i(z) \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$, where f_i is given by

$$f_i(z) = z - \sum_{k=2}^{\infty} |a_{ik}| z^k + \sum_{k=1}^{\infty} |b_{ik}| \overline{z}^k.$$

For $\sum\limits_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{ik}| \right) z^k + \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{ik}| \right) \overline{z}^k.$$

Since,

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta) + (1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|} |a_{ik}| \\ &\qquad + \sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta) - (1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|} |b_{ik}| \leq 1, \end{aligned}$$

from the above equation we obtain

$$\sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|} \sum_{i=1}^{\infty} t_i|a_{ik}| + \sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|} \sum_{i=1}^{\infty} t_i|b_{ik}|$$

$$= \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta)+k(1-\gamma)|b|t]}{k(1-\gamma)|b|}|a_{ik}| + \sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta)-k(1-\gamma)|b|t]}{k(1-\gamma)|b|}|b_{ik}| \right\}$$

$$\leq \sum_{i=1}^{\infty} t_i = 1.$$

This is the condition required by (2.3) and so $\sum_{i=1}^{\infty} t_i f_i(z) \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$. □

5. Class preserving integral operator

Finally, we consider the closure property of the class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$ under the generalized Bernardi-Libera-Livingston integral operator$\mathcal{L}_c$ which is defined by

$$\mathcal{L}_c[f(z)] = \frac{c+1}{z^c} \int_0^z \xi^{c-1} f(\xi) d\xi \quad (c > -1).$$

Theorem 5.1. *If $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$, then $\mathcal{L}_c[f(z)] \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$.*

Proof. From the representation of $\mathcal{L}_c[f(z)]$, it follows that

$$\begin{aligned} \mathcal{L}_c[f(z)] &= \frac{c+1}{z^c} \int_0^z \xi^{c-1} h(\xi) d\xi + \overline{\frac{c+1}{z^c} \int_0^z \xi^{c-1} g(\xi) d\xi} \\ &= \frac{c+1}{z^c} \int_0^z \xi^{c-1} \left(\xi - \sum_{k=2}^{\infty} |a_k| \xi^k \right) d\xi + \overline{\frac{c+1}{z^c} \int_0^z \xi^{c-1} \left(\sum_{k=1}^{\infty} |b_k| \xi^k \right) d\xi} \\ &= z - \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k z^k, \end{aligned}$$

where $A_k = \frac{c+1}{c+k}|a_k|$ and $B_k = \frac{c+1}{c+k}|b_k|$. Hence

$$\sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|}\left(\frac{c+1}{c+k}|a_k|\right) + \sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|}\left(\frac{c+1}{c+k}|b_k|\right)$$
$$\leq \sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|}|a_k| + \sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|}|b_k|$$
$$\leq 1,$$

since $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$, therefore by Theorem 2.2, $\mathcal{L}_c(f(z)) \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$. □

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Differential inequalities and criteria for starlike and convex functions

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Abstract. We, here, study a differential inequality involving a multiplier transformation. In particular, we obtain certain new criteria for starlikeness and convexity of normalized analytic functions. We also show that our results generalize some known results.

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1. Introduction

Let $\mathcal{A}$ be the class of all functions f which are analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions that $f(0) = f'(0) - 1 = 0$. Thus, $f \in \mathcal{A}$ has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \; p \in \mathbb{N} = \{1, 2, 3, \cdots\},$$

analytic and multivalent in the open unit disk $\mathbb{E}$. Note that $\mathcal{A}_1 = \mathcal{A}$. For $f \in \mathcal{A}_p$, define the multiplier transformation $I_p(n, \lambda)$ as

$$I_p(n, \lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k + \lambda}{p + \lambda}\right)^n a_k z^k, \; (\lambda \geq 0, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

The operator $I_p(n,\lambda)$ has been recently studied by Aghalary et al. [1]. $I_1(n,0)$ is the well-known Sălăgean [6] derivative operator D^n, defined for $f \in \mathcal{A}$ as under:

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

A function $f \in \mathcal{A}_p$ is said to be p-valent starlike of order α $(0 \leq \alpha < p)$ in $\mathbb{E}$, if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in \mathbb{E}.$$

Let $\mathcal{S}_p^*(\alpha)$ denote the class of all such functions. A function $f \in \mathcal{A}_p$ is said to be p-valent convex of order α $(0 \leq \alpha < p)$ in $\mathbb{E}$, if it satisfies the inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ z \in \mathbb{E}.$$

We denote by $\mathcal{K}_p(\alpha)$, the class of all functions $f \in \mathcal{A}_p$ that are p-valent convex of order α $(0 \leq \alpha < p)$ in $\mathbb{E}$. Note that $\mathcal{S}^*(\alpha) = \mathcal{S}_1^*(\alpha)$ and $\mathcal{K}(\alpha) = \mathcal{K}_1(\alpha)$ are the usual classes of univalent starlike functions (w.r.t. the origin) of order α $(0 \leq \alpha < 1)$ and univalent convex functions of order α $(0 \leq \alpha < 1)$.

For two analytic functions f and g in the unit disk $\mathbb{E}$, we say that f is subordinate to g in $\mathbb{E}$ and write as $f \prec g$ if there exists a Schwarz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z)),\ z \in \mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to: $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Liu [3], studied the differential operator $(1-\lambda)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \lambda\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha}$ to make certain estimates on $\left(\frac{f(z)}{z^p}\right)^{\alpha}$ where $\alpha > 0,\ \lambda \geq 0$ are some real numbers and $f \in \mathcal{A}_p$. As special cases of our main results, we also obtain the differential operators of above nature, but our estimations are on $\frac{zf'(z)}{f(z)}$ and $1 + \frac{zf''(z)}{f(z)}$, consequently we get certain new criteria for starlikeness and convexity of $f \in \mathcal{A}_p$.

To prove our main result, we shall make use of following lemma of Hallenbeck and Ruscheweyh [2].

Lemma 1.1. *Let G be a convex function in $\mathbb{E}$, with $G(0) = a$ and let γ be a complex number, with $\Re(\gamma) > 0$. If $F(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$, is analytic in $\mathbb{E}$ and $F \prec G$, then*

$$\frac{1}{z^\gamma}\int_0^z F(w) w^{\gamma-1}\,dw \prec \frac{1}{nz^{\gamma/n}}\int_0^z G(w) w^{\frac{\gamma}{n}-1}\,dw.$$

2. Main results

Theorem 2.1. *Let* $\alpha,\ \beta,\ \delta$ *be real numbers such that* $\alpha > \dfrac{2}{1-\delta},\ 0 \le \delta < 1,\ \beta > 0$ *and let*

$$0 < M \equiv M(\alpha,\beta,\lambda,\delta,p) = \frac{[\alpha+\beta(p+\lambda)][\alpha(1-\delta)-2]}{\alpha[1+\beta(1-\delta)(p+\lambda)]}, \tag{2.1}$$

If $f \in \mathcal{A}_p$ *satisfies the differential inequality*

$$\left|\left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^{\beta}\left[1-\alpha+\alpha\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right]-1\right| < M(\alpha,\beta,\lambda,\delta,p),\ z \in \mathbb{E}, \tag{2.2}$$

then

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \delta,\ z \in \mathbb{E}.$$

Proof. Let us define

$$\left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^{\beta} = u(z),\ z \in \mathbb{E}.$$

Differentiate logarithmically, we obtain

$$\frac{zI_p'(n,\lambda)f(z)}{I_p(n,\lambda)f(z)} - p = \frac{zu'(z)}{\beta u(z)} \tag{2.3}$$

In view of the equality

$$zI_p'(n,\lambda)f(z) = (p+\lambda)I_p(n+1,\lambda)f(z) - \lambda I_p(n,\lambda)f(z),$$

(2.3) reduces to

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} = 1 + \frac{zu'(z)}{\beta(p+\lambda)u(z)}$$

Therefore, in view of (2.2), we have

$$u(z) + \frac{\alpha}{\beta(p+\lambda)}zu'(z) \prec 1 + Mz. \tag{2.4}$$

The use of Lemma 1.1 $\left(\text{taking } \gamma = \dfrac{\beta(p+\lambda)}{\alpha}\right)$ in (2.4) gives

$$u(z) \prec 1 + \frac{\beta(p+\lambda)Mz}{\alpha+\beta(p+\lambda)},$$

or

$$|u(z)-1| < \frac{\beta(p+\lambda)M}{\alpha+\beta(p+\lambda)} < 1,$$

therefore, we obtain

$$|u(z)| > 1 - \frac{\beta(p+\lambda)M}{\alpha+\beta(p+\lambda)} \tag{2.5}$$

Write $\dfrac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} = (1-\delta)w(z)+\delta,\ 0 \le \delta < 1$ and therefore (2.2) reduces to

$$|(1-\alpha)u(z) + \alpha u(z)[(1-\delta)w(z)+\delta] - 1| < M.$$

We need to show that $\Re(w(z)) > 0,\ z \in \mathbb{E}$. If possible, suppose that $\Re(w(z)) \not> 0,\ z \in \mathbb{E}$, then there must exist a point $z_0 \in \mathbb{E}$ such that $w(z_0) = ix, x \in \mathbb{R}$. To prove the required result, it is now sufficient to prove that

$$|(1-\alpha)u(z_0) + \alpha u(z_0)[(1-\delta)ix + \delta] - 1| \geq M. \tag{2.6}$$

By making use of (2.5), we have

$$\begin{aligned}
&|(1-\alpha)u(z_0) + \alpha u(z_0)[(1-\delta)ix + \delta] - 1| \\
\geq\ & |[1-\alpha(1-\delta) + \alpha(1-\delta)ix]\, u(z_0)| - 1 \\
=\ & \sqrt{[1-\alpha(1-\delta)]^2 + \alpha^2(1-\delta)^2x^2}\ \ |u(z_0)| - 1 \\
\geq\ & |1-\alpha(1-\delta)|\, |u(z_0)| - 1 \\
\geq\ & |1-\alpha(1-\delta)| \left(1 - \frac{\beta(p+\lambda)M}{\alpha + \beta(p+\lambda)}\right) - 1 \geq M.
\end{aligned} \tag{2.7}$$

Now (2.7) is true in view of (2.1) and therefore, (2.6) holds. Hence $\Re(w(z)) > 0$ i.e.

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \delta,\ 0 \leq \delta < 1,\ z \in \mathbb{E}.$$

□

Remark 2.2. From Theorem 2.1, it follows, if $\alpha,\ \beta,\ \delta$ are real numbers such that $\alpha > \frac{2}{1-\delta},\ 0 \leq \delta < 1,\ \beta > 0$ and if $f \in \mathcal{A}_p$ satisfies

$$\left|\left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^\beta \left[\frac{1}{\alpha} - 1 + \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right] - \frac{1}{\alpha}\right| < \frac{[\alpha+\beta(p+\lambda)][\alpha(1-\delta)-2]}{\alpha^2[1+\beta(1-\delta)(p+\lambda)]},$$

then

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \delta,\ z \in \mathbb{E}.$$

Letting $\alpha \to \infty$ in above remark, we get the following result.

Theorem 2.3. *Let $\beta,\ \delta$ be real numbers such that $\beta > 0,\ 0 \leq \delta < 1$ and let $f \in \mathcal{A}_p$ satisfy*

$$\left|\left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^\beta \left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} - 1\right)\right| < \frac{1-\delta}{1+\beta(1-\delta)(p+\lambda)},$$

then

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \delta,\ z \in \mathbb{E}.$$

For $p = 1$ and $\lambda = 0$ in Theorem 2.1, we get the following result involving Sălăgean operator.

Theorem 2.4. *If* $\alpha,\ \beta,\ \delta$ *are real numbers such that* $\alpha > \dfrac{2}{1-\delta},\ 0 \le \delta < 1,\ \beta > 0$ *and if* $f \in \mathcal{A}$ *satisfies the differential inequality*

$$\left|\left(\frac{D^n f(z)}{z}\right)^{\beta}\left[1-\alpha+\alpha\frac{D^{n+1}f(z)}{D^n f(z)}\right]-1\right| < \frac{(\alpha+\beta)[\alpha(1-\delta)-2]}{\alpha[1+\beta(1-\delta)]},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) > \delta,\ z \in \mathbb{E}.$$

Select $p = 1$ and $\lambda = 0$ in Theorem 2.3, we obtain:

Theorem 2.5. *If* $\beta,\ \delta$ *are real numbers such that* $\beta > 0,\ 0 \le \delta < 1$ *and* $f \in \mathcal{A}$ *satisfies*

$$\left|\left(\frac{D^n f(z)}{z}\right)^{\beta}\left(\frac{D^{n+1}f(z)}{D^n f(z)}-1\right)\right| < \frac{1-\delta}{1+\beta(1-\delta)},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) > \delta,\ z \in \mathbb{E}.$$

3. Criteria for starlikeness and convexity

Setting $\lambda = n = 0$ in Theorem 2.1, we obtain the following result.

Corollary 3.1. *Let* $\alpha,\ \beta,\ \delta$ *be real numbers such that* $\alpha > \dfrac{2}{1-\delta},\ 0 \le \delta < 1,\ \beta > 0$ *and let* $f \in \mathcal{A}_p$ *satisfy the differential inequality*

$$\left|(1-\alpha)\left(\frac{f(z)}{z^p}\right)^{\beta}+\alpha\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\beta}-1\right| < \frac{(\alpha+p\beta)[\alpha(1-\delta)-2]}{\alpha[1+p\beta(1-\delta)]},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}_p^*(\gamma),\ 0 \le \gamma < p.$

Writing $\beta = 1$ in above corollary, we obtain:

Corollary 3.2. *Suppose that* $\alpha,\ \delta$ *are real numbers such that* $\alpha > \dfrac{2}{1-\delta},\ 0 \le \delta < 1$ *and suppose that* $f \in \mathcal{A}_p$ *satisfies*

$$\left|(1-\alpha)\frac{f(z)}{z^p}+\alpha\frac{f'(z)}{pz^{p-1}}-1\right| < \frac{(\alpha+p)[\alpha(1-\delta)-2]}{\alpha[1+p(1-\delta)]},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}_p^*(\gamma),\ 0 \le \gamma < p.$

Setting $n = 1$ and $\lambda = 0$ in Theorem 2.1, we obtain the following result.

Corollary 3.3. *Let* $\alpha,\ \beta,\ \delta$ *be real numbers such that* $\alpha > \dfrac{2}{1-\delta},\ 0 \le \delta < 1,\ \beta > 0$ *and let* $f \in \mathcal{A}_p$ *satisfy the differential inequality*

$$\left|(1-\alpha)\left(\frac{f'(z)}{pz^{p-1}}\right)^{\beta} + \frac{\alpha}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{f'(z)}{pz^{p-1}}\right)^{\beta} - 1\right| < \frac{(\alpha+p\beta)[\alpha(1-\delta)-2]}{\alpha[1+p\beta(1-\delta)]},$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}_p(\gamma),\ 0 \le \gamma < p.$

Writing $\beta = 1$ in above corollary, we obtain:

Corollary 3.4. *If* $\alpha,\ \delta$ *are real numbers such that* $\alpha > \dfrac{2}{1-\delta},\ 0 \le \delta < 1$ *and if* $f \in \mathcal{A}_p$ *satisfies*

$$\left|(1-\alpha)\frac{f'(z)}{pz^{p-1}} + \frac{\alpha}{p^2}\frac{f'(z)}{z^{p-1}}\left(1+\frac{zf''(z)}{f'(z)}\right) - 1\right| < \frac{(\alpha+p)[\alpha(1-\delta)-2]}{\alpha[1+p(1-\delta)]},\ z \in \mathbb{E},$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}_p(\gamma),\ 0 \le \gamma < p.$

Writing $\lambda = n = 0$ in Theorem 2.3, we get:

Corollary 3.5. *If* $\beta,\ \delta$ *are real numbers such that* $\beta > 0,\ 0 \le \delta < 1$ *and if* $f \in \mathcal{A}_p$ *satisfies*

$$\left|\left(\frac{f(z)}{z^p}\right)^{\beta}\left(\frac{zf'(z)}{pf(z)} - 1\right)\right| < \frac{1-\delta}{1+p\beta(1-\delta)},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}_p^*(\gamma),\ 0 \le \gamma < p.$

Note that for $\beta = p = 1$, the above corollary gives the result of Oros [5].

Setting $\lambda = 0$ and $n = 1$ in Theorem 2.3, we obtain:

Corollary 3.6. *Assume that* $\beta,\ \delta$ *be real numbers such that* $\beta > 0,\ 0 \le \delta < 1$ *and assume that* $f \in \mathcal{A}_p$ *satisfies*

$$\left|\left(\frac{f'(z)}{pz^{p-1}}\right)^{\beta}\left[\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) - 1\right]\right| < \frac{1-\delta}{1+p\beta(1-\delta)},\ z \in \mathbb{E},$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}_p(\gamma),\ 0 \le \gamma < p.$

Note that for $\beta = p = 1$ and $\delta = 0$, the above corollary deduces to the result of Mocanu [4].

Taking $p = 1$ in Corollary 3.1, we get:

Corollary 3.7. *If $\alpha,\ \beta,\ \delta$ are real numbers such that $\alpha > \dfrac{2}{1-\delta},\ 0 \le \delta < 1,\ \beta > 0$ and if $f \in \mathcal{A}$ satisfies*

$$\left|(1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta} + \alpha\frac{(f(z))^{\beta-1}f'(z)}{z^{\beta-1}} - 1\right| < \frac{(\alpha+\beta)[\alpha(1-\delta)-2]}{\alpha[1+\beta(1-\delta)]},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}^*(\delta)$.

Setting $p = 1$ in Corollary 3.3, we get:

Corollary 3.8. *If $\alpha,\ \beta,\ \delta$ are real numbers such that $\alpha > \dfrac{2}{1-\delta},\ 0 \le \delta < 1,\ \beta > 0$ and if $f \in \mathcal{A}$ satisfies*

$$\left|(f'(z))^{\beta}\left[1-\alpha+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right] - 1\right| < \frac{(\alpha+\beta)[\alpha(1-\delta)-2]}{\alpha[1+\beta(1-\delta)]},\ z \in \mathbb{E},$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \delta,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}(\delta)$.

Put $\lambda = p = 1$ and $n = 0$ in Theorem 2.1, we get:

Corollary 3.9. *Suppose that $\alpha,\ \beta,\ \delta$ are real numbers such that $\alpha > \dfrac{2}{1-\delta},\ 0 \le \delta < 1,\ \beta > 0$ and suppose that $f \in \mathcal{A}$ satisfies*

$$\left|\left(1-\frac{\alpha}{2}\right)\left(\frac{f(z)}{z}\right)^{\beta} + \frac{\alpha}{2}\frac{(f(z))^{\beta-1}f'(z)}{z^{\beta-1}} - 1\right| < \frac{(\alpha+2\beta)[\alpha(1-\delta)-2]}{\alpha[1+2\beta(1-\delta)]},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 2\delta - 1,\ z \in \mathbb{E}.$$

Put $\lambda = p = 1$ and $n = 0$ in Theorem 2.3, we obtain the following result.

Corollary 3.10. *If $f \in \mathcal{A}$ satisfies*

$$\left|\left(\frac{f(z)}{z}\right)^{\beta}\left(\frac{zf'(z)}{f(z)} - 1\right)\right| < \frac{2(1-\delta)}{1+2\beta(1-\delta)},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 2\delta - 1,\ z \in \mathbb{E},$$

where $\beta,\ \delta$ are real numbers such that $\beta > 0,\ 0 \le \delta < 1$.

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Semilinear operator equations and systems with potential-type nonlinearities

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Abstract. The recent results of Precup [6] on the variational characterization of the fixed points of contraction-type operators are applied in this paper to semilinear operator equations and systems with linear parts given by positively defined operators, and nonlinearities of potential-type. Mihlin's variational theory is also involved. Applications are given to elliptic semilinear equations and systems.

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1. Introduction

In this paper, firstly we are concerning with semilinear operator equation of the type:

$$Au = J'(u), \tag{1.1}$$

where A is a positively defined linear operator and the nonlinear term is the Fréchet derivative of a functional J. Secondly we discuss the semilinear operator system

$$\begin{cases} A_1 u = J_{11}(u,v) \\ A_2 v = J_{22}(u,v), \end{cases} \tag{1.2}$$

associated to two positively defined linear operators A_1, A_2 and to two functionals J_1, J_2 where by $J_{11}(u,v), J_{22}(u,v)$ we mean the Fréchet derivatives of $J_1(.,v), J_2(u,.)$, respectively. Our special interest in such kind of equations and systems is represented by semilinear elliptic equations of the type

$$-\Delta u = f(x,u), \tag{1.3}$$

and correspondingly, by the following elliptic system

$$\begin{cases} -\Delta u = f(x,u,v) \\ -\Delta v = g(x,u,v). \end{cases} \tag{1.4}$$

Recently in Precup [6], it was shown that the unique fixed point of a contraction T on a Hilbert space, in case that T has the variational form

$$Tu = u - E'(u),$$

minimizes the functional E. Also, the unique fixed point (u^*, v^*) of a Perov contraction $(T_1(u,v), T_2(u,v))$ with

$$\begin{cases} T_1(u,v) = u - E_{11}(u,v) \\ T_2(u,v) = u - E_{22}(u,v), \end{cases}$$

under some conditions, is a Nash-type equilibrium of the pair of functionals (E_1, E_2), that is:

$$\begin{aligned} E_1(u^*, v^*) &= \min_u E_1(u, v^*) \\ E_2(u^*, v^*) &= \min_v E_2(u^*, v). \end{aligned}$$

The goal of this paper is to apply the above results to the semilinear equation (1.1) and to the system (1.2). To this aim, we fully exploit Mihlin's theory [4], on linear operator equations. In particular we shall derive variational characterizations of the weak solutions of the Dirichlet problem for the equation (1.3) and the system (1.4).

The paper is organized as follows: Section 2 is devoted to preliminaires, and Section 3 contains the main results. More exactly, in Section 3.1 we discuss the case of the equation (1.1), while in Section 3.2 we obtain theoretical results for the system (1.2). Furthermore, in Section 3.3 we apply our first result to an elliptic equation of the type (1.3) and in Section 3.4 we apply our second result to the system (1.4).

2. Preliminaries

2.1. Variational theory of linear operator equations

In this section we sketch Mihlin's variational theory [4] (see also [5]) for linear equations associated to positively defined operators. Let H be a Hilbert space with the inner product denoted by $(.,.)_H$ and the norm $\|.\|_H$. Let $A : D(A) \to H$ be a symmetric, linear and densely defined operator. The operator A is said to be *positively defined*, if for some $\gamma > 0$,

$$(Au, u)_H \geq \gamma^2 \|u\|_H^2, \tag{2.1}$$

for every $u \in D(A)$. For such a linear operator, we endow the linear subspace $D(A)$ of H with the bilinear functional:

$$(u, v)_{H_A} = (Au, v)_H,$$

for every $u, v \in D(A)$. One can verify that $(.,.)_{H_A}$ is an inner product. Consequently, $D(A)$ endowed with the inner product $(.,.)_{H_A}$ is a pre-hilbertian space. This space may not be complete. The completion H_A of $(D(A), (.,.)_{H_A})$ is called the *energetic space* of A. By the construction, $D(A) \subset H_A \subset H$ with dense inclusions. We use the same symbol $(.,.)_{H_A}$ to denote the inner product of H_A. The corresponding norm

$$\|u\|_{H_A} = \sqrt{(u,u)_{H_A}},$$

is called the *energetic norm* associated to A.

If $u \in D(A)$, then $\|u\|_{H_A} = \sqrt{(Au,u)_H}$ and in view of (2.1) one has the *Poincaré inequality*

$$\|u\|_H \leq \frac{1}{\gamma}\|u\|_{H_A}, \tag{2.2}$$

for every $u \in D(A)$. By density the above inequality extends to H_A. We use this inequality in order to identify the elements of H_A with elements from H.

Let H_A' be the dual space of H_A. If we identify H with its dual, then from $H_A \subset H$ we have $H \subset H_A'$.

We attach to the operator A the following problem:

$$Au = f,\ u \in H_A, \tag{2.3}$$

where $f \in H_A'$. By a *weak solution* of (2.3) we mean an element $u \in H_A$ with:

$$(u,v)_{H_A} = (f,v) \tag{2.4}$$

for every $v \in H_A$, where the notation (f,v) stands for the value of the functional f on the element v. In case that $f \in H$, then $(f,v) = (f,v)_H$. Notice that if $u \in D(A)$, then (2.4) becomes $(Au,v)_H = (f,v)$.

Theorem 2.1. *For every $f \in H_A'$ there exists a unique weak solution $u \in H_A$ of the problem* (2.3).

Proof. Consider the functional $F : H_A \to \mathbb{R}$ given by $F(v) = (f,v)$, for $v \in H_A$. Obviously, F is linear. Also

$$|F(v)| = |(f,v)| \leq \|f\|_{H_A'}\|v\|_{H_A}.$$

Hence, F is a linear and continuous functional. By Riesz's theorem, there exists a unique $u \in H_A$ such that $F(v) = (u,v)_{H_A}$ for all $v \in H_A$. Clearly, u is the unique weak solution of (2.3). □

This result allows us to define the solution operator A^{-1} associated to operator A. Thus

$$A^{-1} : H_A' \to H_A\ ,$$
$$A^{-1}f = u, \tag{2.5}$$

where u is the unique weak solution of problem (2.3). The operator A^{-1} is well defined by the above theorem and one has

$$(A^{-1}f, v)_{H_A} = (f,v) \tag{2.6}$$

for all $v \in H_A$ and $f \in H_A'$.

The operator A^{-1} is an isometry between H_A' and H_A, i.e:

$$\|A^{-1}f\|_{H_A} = \|f\|_{H_A'} \tag{2.7}$$

for all $f \in H_A'$. Indeed, in order to show that the inequality $\|A^{-1}f\|_{H_A} \leq \|f\|_{H_A'}$ holds, we replace v with $A^{-1}f$ in (2.6), to obtain $(A^{-1}f, A^{-1}f)_{H_A} = (f, A^{-1}f)$. Therefore

$$\|A^{-1}f\|_{H_A}^2 = (f, A^{-1}f) \leq \|f\|_{H_A'}\|A^{-1}f\|_{H_A}.$$

Hence, $\|A^{-1}f\|_{H_A} \leq \|f\|_{H'_A}$. On the other hand, we have that

$$\|f\|_{H'_A} = \sup_{\substack{v \in H_A \\ v \neq 0}} \frac{|(f,v)|}{\|v\|_{H_A}} = \sup_{\substack{v \in H_A \\ v \neq 0}} \frac{|(A^{-1}f,v)_{H_A}|}{\|v\|_{H_A}}$$
$$\leq \sup_{\substack{v \in H_A \\ v \neq 0}} \frac{\|A^{-1}f\|_{H_A}\|v\|_{H_A}}{\|v\|_{H_A}} = \|A^{-1}f\|_{H_A}.$$

From the above inequalities, (2.7) follows.

We also mention Poincaré's inequality for the inclusion $H \subset H'_A$,

$$\|u\|_{H'_A} \leq \frac{1}{\gamma}\|u\|_H, \qquad u \in H. \tag{2.8}$$

This can be proved as follows:

$$\|u\|_{H'_A} = \sup_{\substack{v \in H_A \\ v \neq 0}} \frac{|(u,v)_H|}{\|v\|_{H_A}} \leq \sup_{\substack{v \in H_A \\ v \neq 0}} \frac{\|u\|_H\|v\|_H}{\|v\|_{H_A}}.$$

Now, using (2.2) we have:

$$\sup_{\substack{v \in H_A \\ v \neq 0}} \frac{\|u\|_H\|v\|_H}{\|v\|_{H_A}} \leq \frac{1}{\gamma}\|u\|_H.$$

Therefore (2.8) holds.

Using (2.7) and (2.8) we see that if $f \in H$, then

$$\|A^{-1}f\|_{H_A} = \|f\|_{H'_A} \leq \frac{1}{\gamma}\|f\|_H. \tag{2.9}$$

For a fixed $f \in H'_A$, one considers the functional:

$$E : H_A \to \mathbb{R},$$
$$E(u) = \frac{1}{2}\|u\|^2{}_{H_A} - (f,u).$$

The functional E is Fréchet differentiable and for any $u, v \in H_A$, we have:

$$(E'(u),v) = \lim_{t\to 0} \frac{E(u+tv)-E(u)}{t} = (u,v)_{H_A} - (f,v) = (u - A^{-1}f, v)_{H_A}. \tag{2.10}$$

Now (2.10) shows that $u \in H_A$ is a weak solution of (2.3) if and only if u is a critical point of E, i.e $E'(u) = 0$.

2.2. Variational properties for contraction-type operators

In this section and in the next one, we summarize the abstract results from the paper Precup [6], concerning the variational characterization of the fixed points of contraction-type operators. The first result refers to usual contractions on a Hilbert space.

Theorem 2.2. *([6]) Let X be a Hilbert space and $T : X \to X$ be a contraction with the unique fixed point u^* (guaranteed by Banach contraction theorem). If there exists a C^1 functional E bounded from below such that*

$$E'(u) = u - T(u) \tag{2.11}$$

for all $u \in X$, then u^* minimizes the functional E, i.e

$$E(u^*) = \inf_X E.$$

2.3. Nash-type equilibrium for Perov contractions

The next result from [6] is about systems of the type

$$\begin{cases} u = T_1(u, v) \\ v = T_2(u, v), \end{cases} \tag{2.12}$$

where $u \in X_1$, $v \in X_2$. In this case, instead of Lipschitz constants in the definition of contractions, we use matrices.

A square matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}^n)$ with nonnegative entries is said to be *convergent to zero* if

$$M^k \to 0, \text{ as } k \to \infty.$$

There are known the following characterizations of the convergent to zero matrices (see, e.g [7], [2]).

The following statements are equivalents:
(i) M is a matrix that is convergent to zero;
(ii) $I - M$ is nonsingular and $(I - M)^{-1} = I + M + M^2 + \ldots$ (where I stands for the unit matrix of the same order as M);
(iii) the eigenvalues of M are located inside the unit disc of the complex plane;
(iv) $I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative entries.

Refering to the system (2.12), we assume that $(X_i, |.|_i)$, $i = 1, 2$, are Hilbert spaces identified to their duals and we denote by $X = X_1 \times X_2$. Also, assume that each equation of the system has a variational form, i.e. that there exist the continuous functionals $E_1, E_2 : X \to \mathbf{R}$ such that $E_1(., v)$ is Fréchet differentiable for every $v \in X_2$, $E_2(u, .)$ is Fréchet differentiable for every $u \in X_1$, and

$$\begin{cases} E_{11}(u, v) = u - T_1(u, v) \\ E_{22}(u, v) = v - T_2(u, v). \end{cases} \tag{2.13}$$

Here $E_{11}(., v), E_{22}(u, .)$ are the Fréchet derivatives of $E_1(., v)$ and $E_2(u, .)$, respectively.

We say that the operator $T : X \to X$, $T(u, v) = (T_1(u, v), T_2(u, v))$ is a *Perov contraction* if there exists a matrix $M = [m_{ij}] \in \mathcal{M}_{2,2}(\mathbf{R}_+)$ which is convergent to zero such that the following matricial Lipschitz condition is satisfied

$$\begin{bmatrix} |T_1(u, v) - T_1(\overline{u}, \overline{v})|_1 \\ |T_2(u, v) - T_2(\overline{u}, \overline{v})|_2 \end{bmatrix} \leq M \begin{bmatrix} |u - \overline{u}|_1 \\ |v - \overline{v}|_2 \end{bmatrix} \tag{2.14}$$

for every $u, \overline{u} \in X_1$ and $v, \overline{v} \in X_2$.

The next theorem gives us a variational characterization of the unique fixed point of a Perov contraction.

Theorem 2.3. *([6]) Assume that the above conditions are satisfied. In addition assume that $E_1(.,v)$ and $E_2(u,.)$ are bounded from below for every $u \in X_1$, $v \in X_2$, and that there are R, $c > 0$ such that one of the following conditions holds:*

$$\begin{aligned} E_1(u,v) &\geq \inf_{X_1} E_1(.,v) + c \quad \text{for } |u|_1 \geq R \text{ and } v \in X_2, \\ E_2(u,v) &\geq \inf_{X_2} E_2(u,.) + c \quad \text{for } |v|_2 \geq R \text{ and } u \in X_1. \end{aligned} \tag{2.15}$$

Then the unique fixed point (u^*, v^*) of T (guaranteed by Perov's fixed point theorem) is a Nash-type equilibrium of the pair of functionals (E_1, E_2), i.e.

$$\begin{aligned} E_1(u^*,v^*) &= \inf_{X_1} E_1(.,v^*) \\ E_2(u^*,v^*) &= \inf_{X_2} E_2(u^*,.). \end{aligned}$$

3. Main results

The main results of the paper are concerning with variational properties of the solutions of semilinear equations having the form

$$Au = J'(u),$$

with a positively defined linear operator A, and of semilinear systems of the type:

$$\begin{cases} A_1 u = J_{11}(u,v) \\ A_2 v = J_{22}(u,v). \end{cases}$$

We shall benefit of Mihlin's variational theory for linear operator equations and we shall apply the general results presented in Sections 2.2 and 2.3.

3.1. Semilinear operator equations with potential-type nonlinearities

First we consider the case of semilinear equations.

Let A be a symmetric, linear and densely defined operator as in Section 2.1 and let $J : H \to \mathbb{R}$ be a $C^1(H)$ functional. We look for weak solutions $u \in H_A$ for the semilinear equation

$$Au = J'(u). \tag{3.1}$$

This equation is equivalent to

$$u = A^{-1}J'(u), \tag{3.2}$$

this is, to the fixed point equation:

$$u = T(u), \tag{3.3}$$

where $T := A^{-1}J'$. We associate to the equation (3.1) the functional

$$E : H_A \to \mathbb{R}, \quad E(u) = \frac{1}{2}\|u\|^2{}_{H_A} - J(u). \tag{3.4}$$

The main result of this section is the following theorem.

Theorem 3.1. *Under the above conditions on A and J, if in addition $J' : H \to H$ satisfies the following conditions:*

$$\|J'(u) - J'(v)\|_H \leq \alpha \|u - v\|_H \tag{3.5}$$

for all $u, v \in H$, and

$$J(u) \leq a\|u\|^2{}_{H_A} + b, \tag{3.6}$$

for all $u \in H_A$, some $\alpha < \gamma^2$, $a \leq \frac{1}{2}$ and $b \geq 0$, then there is a unique weak solution $u^* \in H_A$ of equation (3.1) such that

$$E(u^*) = \inf_{H_A} E.$$

Proof. We apply Theorem 2.2 to $X = H_A$, to the operator $T : H_A \to H_A$, $T := A^{-1}J'$ and to the functional given by (3.4). Since J is of class C^1 on H, it follows that E is of class C^1 on H_A. Indeed,

$$\begin{aligned}(E'(u), v) &= \lim_{t \to 0} \frac{E(u + tv) - E(u)}{t} \\ &= (u, v)_{H_A} - (J'(u), v) \\ &= (u - A^{-1}J'(u), v)_{H_A}.\end{aligned}$$

Therefore, if we identify H'_A to H_A, we have

$$E'(u) = u - T(u).$$

Hence the assumption (2.11) holds. Using (3.6) and $a \leq \frac{1}{2}$, we obtain

$$E(u) = \frac{1}{2}\|u\|^2_{H_A} - J(u) \geq \left(\frac{1}{2} - a\right)\|u\|^2{}_{H_A} - b \geq -b > -\infty,$$

for all $u \in H_A$. Thus, E is bounded from below. It remains to show that T is a contraction on H_A. Using the hypothesis (3.5) and the Poincaré's inequality (2.2), for every $u, v \in H_A$, we have

$$\begin{aligned}\|J'(u) - J'(v)\|_H &\leq \alpha\|u - v\|_H \\ &\leq \frac{\alpha}{\gamma}\|u - v\|_{H_A}.\end{aligned}$$

Since A^{-1} is an isometry between H'_A and H_A, we then deduce that

$$\begin{aligned}\|T(u) - T(v)\|_{H_A} &= \|A^{-1}(J'(u) - J'(v))\|_{H_A} \\ &= \|J'(u) - J'(v)\|_{H'_A} \\ &\leq \frac{1}{\gamma}\|J'(u) - J'(v)\|_H \\ &\leq \frac{\alpha}{\gamma^2}\|u - v\|_{H_A}.\end{aligned}$$

This shows that T is a contraction on H_A, since α was assumed less than γ^2. Thus Theorem 2.2 applies and the result follows. □

3.2. Semilinear operator systems with potential-type nonlinearities

This subsection is devoted to the study of systems of the type:

$$\begin{cases} A_1 u = J_{11}(u,v) \\ A_2 v = J_{22}(u,v), \end{cases} \tag{3.7}$$

where A_1, A_2 are symmetric, linear and densely defined operators on two Hilbert spaces H_1, H_2. Denote $H = H_1 \times H_2$. Also, $J_1, J_2 : H \to \mathbb{R}$ are two $C^1(H)$ functionals and by $J_{11}(u,v)$ we mean the partial derivative of J_1 with respect to u and by $J_{22}(u,v)$ the partial derivative of J_2 with respect to v. We express the above system as a fixed point equation of the type

$$w = T(w) \tag{3.8}$$

for the nonlinear operator $T = (T_1, T_2)$, where $w = (u,v)$, $T_1 : H_{A_1} \times H_{A_2} \to H_{A_1}$, $T_1(u,v) = A_1^{-1} J_{11}$ and $T_2 : H_{A_1} \times H_{A_2} \to H_{A_2}$, $T_2(u,v) = A_2^{-1} J_{22}$. Hence (3.8) can be rewritten explicitly as follows

$$\begin{cases} u = T_1(u,v) \\ v = T_2(u,v). \end{cases} \tag{3.9}$$

This vectorial structure of (3.8) allows the two terms T_1 and T_2 to behave differently one from another and also with respect to the two variables. Also, this requires the use of matrices instead of constants, when Lipschitz conditions are imposed to T_1 and T_2. Each component equation of (3.9) has a variational form. We associate to the equations of (3.9) the functionals $E_1, E_2 : H_{A_1} \times H_{A_2} \to \mathbb{R}$ defined by

$$E_1(u,v) = \frac{1}{2}\|u\|^2_{H_{A_1}} - J_1(u,v) \tag{3.10}$$
$$E_2(u,v) = \frac{1}{2}\|v\|^2_{H_{A_2}} - J_2(u,v).$$

One has

$$\begin{aligned} E_{11}(u,v) &= u - T_1(u,v) \\ E_{22}(u,v) &= v - T_2(u,v), \end{aligned} \tag{3.11}$$

where $E_{11}(.,v), E_{22}(u,.)$ are the Fréchet derivatives of $E_1(.,v)$ and $E_2(u,.)$, respectively.

The main result of this subsection is the following theorem.

Theorem 3.2. *Let the above conditions on A_1, A_2 and J_1, J_2 hold. In addition assume that $J_{11} : H_1 \times H_2 \to H_1$ and $J_{22} : H_1 \times H_2 \to H_2$ satisfy the following conditions:*
(i) there exist $m_{ij} \in \mathbb{R}_+$ $(i,j = 1,2)$ such that

$$\|J_{11}(u,v) - J_{11}(\bar{u},\bar{v})\|_{H_1} \le m_{11}\|u-\bar{u}\|_{H_1} + m_{12}\|v-\bar{v}\|_{H_2} \tag{3.12}$$
$$\|J_{22}(u,v) - J_{22}(\bar{u},\bar{v})\|_{H_2} \le m_{21}\|u-\bar{u}\|_{H_1} + m_{22}\|v-\bar{v}\|_{H_2}$$

for all $u, \bar{u} \in H_1$ *and* $v, \bar{v} \in H_2$, *and the matrix*

$$M = \begin{bmatrix} \frac{m_{11}}{\gamma_1^2} & \frac{m_{12}}{\gamma_1^2} \\ \frac{m_{21}}{\gamma_2^2} & \frac{m_{22}}{\gamma_2^2} \end{bmatrix} \tag{3.13}$$

is convergent to zero;
(ii)

$$\begin{aligned} J_1(u,v) &\le a_1 \|u\|^2_{H_{A_1}} + b_1 \\ J_2(u,v) &\le a_2 \|v\|^2_{H_{A_2}} + b_2 \end{aligned} \tag{3.14}$$

for all $u \in H_{A_1}$, $v \in H_{A_2}$ *and some* $a_1,\ a_2 \le \frac{1}{2}$ *and* $b_1,\ b_2 \ge 0$;
(iii) there are $R,\ c > 0$ *such that one of the following conditions holds:*

$$\begin{aligned} E_1(u,v) &\ge \inf_{H_{A_1}} E_1(.,v) + c \quad \text{for } \|u\|_{H_{A_1}} \ge R \text{ and } v \in H_{A_2}, \\ E_2(u,v) &\ge \inf_{H_{A_2}} E_2(u,.) + c \quad \text{for } \|v\|_{H_{A_2}} \ge R \text{ and } u \in H_{A_1}. \end{aligned} \tag{3.15}$$

Then there is a unique solution $(u^*, v^*) \in H_{A_1} \times H_{A_2}$ *of the system (*3.7*) which is a Nash-type equilibrium of the pair of functionals* (E_1, E_2), *i.e:*

$$\begin{aligned} E_1(u^*,v^*) &= \inf_{H_{A_1}} E_1(.,v^*) \\ E_2(u^*,v^*) &= \inf_{H_{A_2}} E_2(u^*,.). \end{aligned} \tag{3.16}$$

Proof. We apply the Theorem 2.3 to $X_1 = H_{A_1}$, and $X_2 = H_{A_2}$. Using (3.12) we show that T is a Perov contraction. Indeed, for $(u,v) \in X$ we have

$$\begin{aligned} \|T_1(u,v) - T_1(\bar{u},\bar{v})\|_{H_{A_1}} &= \|A_1^{-1}(J_{11}(u,v) - J_{11}(\bar{u},\bar{v}))\|_{H_{A_1}} \\ &= \|J_{11}(u,v) - J_{11}(\bar{u},\bar{v})\|_{H'_{A_1}} \\ &\le \frac{1}{\gamma_1}\|J_{11}(u,v) - J_{11}(\bar{u},\bar{v})\|_{H_1} \\ &\le \frac{m_{11}}{\gamma_1}\|u-\bar{u}\|_{H_1} + \frac{m_{12}}{\gamma_1}\|v-\bar{v}\|_{H_2} \\ &\le \frac{m_{11}}{\gamma_1^2}\|u-\bar{u}\|_{H_{A_1}} + \frac{m_{12}}{\gamma_1^2}\|v-\bar{v}\|_{H_{A_2}}. \end{aligned}$$

A similar inequality holds for T_2. Using (3.14) and $a_1, a_2 \le \frac{1}{2}$ we deduce that

$$E_1(u,v) = \frac{1}{2}\|u\|^2_{H_{A_1}} - J_1(u,v) \ge \left(\frac{1}{2} - a_1\right)\|u\|^2{}_{H_{A_1}} - b_1 \ge -b_1 > -\infty.$$

Thus, E_1 is bounded from below. Analogously, E_2 is bounded from below. Thus Theorem 2.3 is applicable and the result yields. □

3.3. Application to elliptic equations

In this subsection we present an application of Theorem 3.1 to elliptic equations. More exactly, we deal with the Dirichlet problem:

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \ . \end{cases} \tag{3.17}$$

Here Ω is a bounded open subset of $\mathbb{R}^n$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and Δ is the Laplacian. In this specific case $H = L^2(\Omega)$ and $A = -\Delta$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Also, $H_A = H_0^1(\Omega)$ with the inner product

$$(u,v)_{H_0^1} = \int_\Omega \nabla u \nabla v \mathrm{d}x$$

and the norm

$$\|u\|_{H_0^1} = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

The functional $J : L^2(\Omega) \to \mathbb{R}$ is given by

$$J(u) = \int_\Omega F(x, u(x))dx,$$

where $F(x,t) = \displaystyle\int_0^t f(x,s)ds$. Also $\gamma = \sqrt{\lambda_1}$, where λ_1 is the first eigenvalue of the Dirichlet problem for $-\Delta$ (see, e.g [8], [1], [3]). Hence the energy functional associated to (3.17) is the following one:

$$E : H_0^1(\Omega) \to \mathbb{R},$$
$$E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - F(x,u(x)) \right) \mathrm{d}x,$$

Problem (3.17) is equivalent to the fixed point equation:

$$u = (-\Delta)^{-1} N_f(u), \tag{3.18}$$

where N_f is the Nemytskii superposition operator assumed to act from $L^2(\Omega)$ to itself, $N_f(u)(x) = f(x, u(x))$ (see, e.g [8], [9]). Notice that the functional J is C^1 on $L^2(\Omega)$, $J' = N_f$ and

$$E'(u) = u - (-\Delta)^{-1} N_f(u).$$

Theorem 3.3. *Assume that the folllowing conditions are satisfied:*
(i) f satisfies the Carathéodory conditions, i.e
$f(.,y) : \Omega \to \mathbb{R}$ is measurable for each $y \in \mathbb{R}$ and $f(x,.) : \mathbb{R} \to \mathbb{R}$ is continuous for a.e $x \in \Omega$;
(ii) $f(.,0) = 0$ on Ω;
(iii) exists $\alpha \in [0, \lambda_1)$ such that

$$|f(x,u) - f(x,\bar{u})| \leq \alpha |u - \bar{u}|$$

for all $u, \bar{u} \in \mathbb{R}$ and a.e $x \in \Omega$.
Then (3.17) has a unique weak solution $u^* \in H_0^1(\Omega)$ and

$$E(u^*) = \inf_{H_0^1(\Omega)} E.$$

Proof. We shall apply Theorem 3.1. From (*iii*) we deduce that

$$\|N_f(u) - N_f(v)\|_{L^2} \leq \alpha \|u - v\|_{L^2}$$

for $u, v \in L^2(\Omega)$. Hence (3.5) holds. Also, since

$$|f(x,t)| = |f(x,t) - f(x,0)| \leq \alpha |t|,$$

for $u \in H_0^1(\Omega)$, one has

$$\begin{aligned} |J(u)| &\leq \int_\Omega |F(x,u(x))| \mathrm{d}x \leq \int_\Omega \left| \int_0^{u(x)} f(x,s)\mathrm{d}s \right| \mathrm{d}x \\ &\leq \int_\Omega \left| \int_0^{u(x)} |f(x,s)|\mathrm{d}s \right| \mathrm{d}x \leq \int_\Omega \left| \int_0^{u(x)} \alpha |s| \mathrm{d}s \right| \mathrm{d}x \\ &= \frac{\alpha}{2} \int_\Omega u(x)^2 \mathrm{d}x = \frac{\alpha}{2} \|u\|_{L^2}^2 \leq \frac{\alpha}{2\lambda_1} \|u\|_{H_0^1}^2. \end{aligned}$$

Therefore (3.6) holds with $a = \dfrac{\alpha}{2\lambda_1} \leq \dfrac{1}{2}$ and $b = 0$. Thus Theorem 3.1 can be applied and the result follows. □

3.4. Application to elliptic systems

Let Ω be a bounded open subset of $\mathbb{R}^n$ and $f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$. We consider the following system:

$$\begin{cases} -\Delta u = f(x,u,v) & \text{in } \Omega \\ -\Delta v = g(x,u,v) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \\ \quad v = 0 & \text{on } \partial\Omega\ . \end{cases} \tag{3.19}$$

This problem is equivalent to the system:

$$\begin{cases} u = (-\Delta)^{-1} f(.,u,v) \\ v = (-\Delta)^{-1} g(.,u,v). \end{cases}$$

Also a pair $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a solution of (3.19) if and only if

$$\begin{cases} E_{11}(u,v) = 0 \\ E_{22}(u,v) = 0, \end{cases} \tag{3.20}$$

where $E_1, E_2 : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ are defined by

$$\begin{aligned} E_1(u,v) \quad &= \frac{1}{2}\|u\|_{H_0^1}^2 - \int_\Omega F(x,u(x),v(x))\mathrm{d}x \qquad (3.21) \\ E_2(u,v) \quad &= \frac{1}{2}\|v\|_{H_0^1}^2 - \int_\Omega G(x,u(x),v(x))\mathrm{d}x, \end{aligned}$$

and

$$F(x,t,s)=\int_0^t f(x,\tau,s)\mathrm{d}\tau,\quad G(x,t,s)=\int_0^s g(x,t,\tau)\mathrm{d}\tau. \tag{3.22}$$

The functionals $E_1(.,v)$ and $E_2(u,.)$ are C^1 for any fixed u and v, respectively, and

$$\begin{aligned} E_{11}(u,v) &= u-(-\Delta)^{-1}f(.,u,v) \\ E_{22}(u,v) &= v-(-\Delta)^{-1}g(.,u,v). \end{aligned} \tag{3.23}$$

Here again $E_{11}(.,v), E_{22}(u,.)$ are the Fréchet derivatives of $E_1(.,v)$ and $E_2(u,.)$, respectively.

We shall say that a function $H:\Omega\times\mathbb{R}\to\mathbb{R}$ is of *coercive type* if the functional $\Phi: H_0^1(\Omega)\to\mathbb{R}$,

$$\Phi(v)=\frac{1}{2}\|v\|_{H_0^1}^2-\int_\Omega H(x,v(x))\mathrm{d}x \tag{3.24}$$

is coercive, i.e $\Phi(v)\to+\infty$ as $\|v\|_{H_0^1}\to\infty$.

The main result of this subsection is the following theorem.

Theorem 3.4. *Let $f,g:\Omega\times\mathbb{R}^2\to\mathbb{R}$, $f=f(x,y,z)$, $g=g(x,y,z)$ satisfy the Carathédory conditions, i.e $f(.,y,z)$, $g(.,y,z)$ are measurable for each $(y,z)\in\mathbb{R}^2$ and $f(x,.)$, $g(x,.)$ are continuous for a.e $x\in\Omega$. Assume that $f(.,0,0)$, $g(.,0,0)\in L^2(\Omega)$ and that the following conditions hold:*
(i) there exist $m_{ij}\in\mathbb{R}_+$ $(i,j=1,2)$ with:

$$\begin{cases} |f(x,u,v)-f(x,\bar u,\bar v)|\le m_{11}|u-\bar u|+m_{12}|v-\bar v| \\ |g(x,u,v)-g(x,\bar u,\bar v)|\le m_{21}|u-\bar u|+m_{22}|v-\bar v|, \end{cases} \tag{3.25}$$

for all $u,\bar u,v,\bar v\in\mathbb{R}$ and a.e $x\in\Omega$;
(ii) there exist $H,H_1:\Omega\times\mathbb{R}\to\mathbb{R}$ with

$$H_1(x,v)\le G(x,u,v)\le H(x,v), \tag{3.26}$$

for all $u,v\in\mathbb{R}$ and a.e. $x\in\Omega$, where H and H_1 are of coercive type.

If the matrix

$$M=\frac{1}{\lambda_1}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{bmatrix} \tag{3.27}$$

is convergent to zero, then (3.19) has a unique solution $(u^*,v^*)\in H_0^1(\Omega)\times H_0^1(\Omega)$ which is a Nash-type equilibrium of the pair of energy functionals (E_1,E_2) associated to the problem (3.19).

Proof. We shall apply Theorem 3.2. Here $H_1=H_2=L^2(\Omega)$, $A_1=A_2=-\Delta$ and $J_1,J_2:H\to\mathbb{R}$ are given by

$$J_1(u,v)=\int_\Omega F(x,u(x),v(x))\mathrm{d}x,\quad J_2(u,v)=\int_\Omega G(x,u(x),v(x))\mathrm{d}x.$$

Also $\gamma_1 = \gamma_2 = \sqrt{\lambda_1}$. Using (3.25), in the same way as for a single equation, we have that $E_1(.,v)$ and $E_2(u,.)$ are bounded from below for any fixed u and v. In addition, we use the second inequality from (3.26) to obtain:

$$\begin{aligned} E_2(u,v) &= \frac{1}{2}\|v\|^2_{H_0^1} - \int\limits_\Omega G(x,u(x),v(x))\mathrm{d}x \\ &\geq \frac{1}{2}\|v\|^2_{H_0^1} - \int_\Omega H(x,v(x))\mathrm{d}x =: \Phi(v). \end{aligned}$$

Since H is of coercive type, Φ is bounded from below. Hence

$$E_2(u,v) \geq \Phi(v) \geq c > -\infty,$$

for all $u, v \in H_0^1(\Omega)$, that is $E_2(.,v)$ is bounded from below uniformly with respect to v. Denote

$$\Phi_1(v) = \frac{1}{2}\|v\|^2_{H_0^1} - \int_\Omega H_1(x,v(x))\mathrm{d}x.$$

Since Φ is coercive, for each $\lambda > 0$, there is R_λ such that $\Phi(v) \geq \lambda$ for $\|v\|_{H_0^1} \geq R_\lambda$. Take $c > 0$ and $\lambda = \inf \Phi_1 + c$. Then for $\|v\|_{H_0^1} \geq R_\lambda$ and any $u \in H_0^1(\Omega)$ we have

$$E_2(u,v) \geq \Phi(v) \geq \inf \Phi_1 + c.$$

From the first inequality of (3.26), we have $\Phi_1(v) \geq E_2(u,v)$. It follows that

$$E_2(u,v) \geq \inf E_2(u,.) + c$$

for $\|v\|_{H_0^1} \geq R_\lambda$ and all $u \in H_0^1(\Omega)$. This shows that E_2 satisfies the condition (3.15). Furthermore,

$$\begin{aligned} \|J_{11}(u,v) - J_{11}(\bar{u},\bar{v})\|_{L^2} &= \|f(.,u,v) - f(.,\bar{u},\bar{v})\|_{L^2} \\ &\leq m_{11}\|u-\bar{u}\|_{L_2} + m_{12}\|v-\bar{v}\|_{L_2}, \end{aligned}$$

and similarly

$$\begin{aligned} \|J_{22}(u,v) - J_{22}(\bar{u},\bar{v})\|_{L^2} &= \|g(.,u,v) - g(.,\bar{u},\bar{v})\|_{L^2} \\ &\leq m_{21}\|u-\bar{u}\|_{L_2} + m_{22}\|v-\bar{v}\|_{L_2}. \end{aligned}$$

Therefore the hypothesis of Theorem 3.2 are fulfilled and the result follows. □

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Greediness of higher rank Haar wavelet bases in $L^p_w(\mathbb{R})$ spaces

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Abstract. We prove that higher rank Haar wavelet systems are greedy in $L^p_w(\mathbb{R})$, $1 < p < \infty$ if and only if $w \in A^N_p$.

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1. Introduction

Let $\mathcal{X} = \{x_n : n \in \mathbb{N}\}$ be a semi-normalized basis in a Banach space X. This means that $\{x_n\}_{n\in\mathbb{N}}$ is a Schauder basis and is semi-normalized i.e. $0 < \inf_{n\in\mathbb{N}} \|x_n\| \leq \sup_{n\in\mathbb{N}} \|x_n\| < \infty$. For an element $x \in X$ we define the error of the best $m-$term approximation as follows

$$\sigma_m(x, \mathcal{X}) = \inf\{\|x - \sum_{n\in A} \alpha_n x_n\|\},$$

where the inf is taken over all subset $A \subset \mathbb{N}$ of cardinality at most m and all possible scalars α_n. The main question in approximation theory concerns the construction of efficient algorithms for m-term approximation. A computationally efficient method to produce m-term approximations, which has been widely investigated in recent years, is the so called greedy algorithm. For $x \in X$ with $x = \sum_{n=1}^{\infty} a_n x_n$ and $m \in \mathbb{N}$, consider a subset $G(m, x) \subset \mathbb{N}$ of cardinality m such that

$$\min_{n\in G(m,x)} |a_n| \geq \max_{n\in\mathbb{N}\setminus G(m,x)} |a_n|.$$

There is some ambiguity in the choice of the set $G(m, x)$, but our considerations do not depend on the particular choice. Then the m-th greedy approximation of x with respect to the basis $\mathcal{X}$ is defined as

$$\mathcal{G}_m(x, \mathcal{X}) = \sum_{n\in G(m,x)} a_n x_n.$$

Clearly, $\sigma_m(x,\mathcal{X}) \le \|x-\mathcal{G}_m(x,\mathcal{X})\|$. The basis $\mathcal{X}$ is called greedy if there is a constant $C>0$, independent of m, such that for each $m\in\mathbb{N}$ and $x\in X$,

$$\|x-\mathcal{G}_m(x,\mathcal{X})\| \le C\sigma_m(x,\mathcal{X}).$$

Wavelet systems are well known examples of greedy bases for many function and distribution spaces. Indeed, V. N. Temlyakov showed in [13] that the classical dyadic Haar system (and any wavelet system L^p-equivalent to it) is greedy in the Lebesgye spaces $L^p(\mathbb{R}^n)$ for $1<p<\infty$.

When wavelets have sufficient smoothness and decay, they are also greedy bases for the more general Sobolev and Tribel-Lizorkin classes (see [3],[5]). Some example of greedy bases are given in [13], [14]. In most cases these bases are greedy simply because they are symmetric (e.g. Riesz bases for a Hilbert space), or because they are equivalent to the dyadic Haar basis or to a subsequence of the Haar basis (see [7]). S. V. Konyagin and V. N. Temlyakov [8] gave a very useful abstract characterization of greedy bases in a Banach spaces X as those which are unconditional and democratic, the last meaning that for some constant $C>0$

$$\left\|\sum_{n\in A}\frac{x_n}{\|x_n\|}\right\| \le C\left\|\sum_{n\in A'}\frac{x_n}{\|x_n\|}\right\|$$

holds for all finite sets of indices A, $A'\subset\mathbb{N}$ with the same cardinality.

The purpose of this paper is to study the efficiency of greedy algorithms with respect higher rank Haar wavelet system in the spaces $L^p_w(\mathbb{R})$. We recall that, as M. Izuki proved in [6], that the dyadic Haar wavelet system ($N=2$) is greedy in $L^p(\mathbb{R}^n)$ for $1<p<\infty$ if and only if $w\in A^2_p$. Characterization of almost greedy uniformly bounded orthonormal bases in rearrangement invariant Banach function spaces are given [2].

By an N-adic ($N\in\mathbb{N}$, $N\ge 2$) lattice $\mathcal{D}$ we mean the collection of all N-adic intervals, i. e. the collection of all intervals of the form $[jN^{-k},(j+1)N^{-k})$, $j,k\in\mathbb{Z}$. If I is an interval, we denote by $|I|$ its length, and by χ_I its characteristic function. If I is an N-adic interval $[jN^{-k},(j+1)N^{-k})$ then we denote by $I^{(l)}$ the "children" intervals of $I: [jN^{-k}+lN^{-(k+1)}, jN^{-k}+(l+1)N^{-(k+1)})$, $l=0,1,\cdots,N-1$.

We denote by $L^2(\mathbb{R})$ the Hilbert space of square integrable (with respect to the Lebesgue measure) complex-valued functions on $\mathbb{R}$. We consider also $L^p_w(\mathbb{R})$ $(1\le p<\infty)$ spaces, where $w\in L^1_{loc}(\mathbb{R})$ is a positive function called a weight. The norm of a function $f:\mathbb{R}\to\mathbb{C}$ from the space $L^p_w(\mathbb{R})$ is

$$\|f\|_{L^p_w} = \left(\int_{\mathbb{R}} |f(x)|^p w(x)dx\right)^{1/p}.$$

Given a function f, we denote by $\langle f\rangle_I$ its average over the interval I,

$$\langle f\rangle_I = \frac{1}{|I|}\int_I f(x)dx.$$

We are concerned with a special class of weights, called A^N_p. We say that $w\in A^N_p$, $1<p<\infty$ if

$$A_w = \sup_{I\in\mathcal{D}}\langle w\rangle_I\langle w^{-1/(p-1)}\rangle_I^{p-1} < \infty.$$

We say that an $N \times N$ matrix is a Haar wavelet matrix of rank N if it is unitary and the elements of the first row are all equal to $1/\sqrt{N}$. Many classical examples of matrices used in mathematics and signal processing are Haar matrices of specific types. These include the discrete Fourier transform matrices, the discrete cosine transform matrices, Hadamard and Walsh matrices, Radmacher matrices, and Chebyshev matrices (see [12]).

Let $H = (g_{ki})_{k,i=0}^{N-1}$ be a $N \times N$ Haar matrix and $\varphi = \chi_{[0,1)}$. Define the functions

$$\psi^{(k)}(x) = \sqrt{N} \sum_{l=0}^{N-1} g_{kl}\varphi(Nx - l) \;\; k = 1, \cdots, N-1. \tag{1.1}$$

The collection of functions

$$\psi_{j,n}^{(k)} = N^{j/2}\psi^{(k)}(N^j x - n), \; j, n \in \mathbb{Z}, \; k = 1, \cdots, N-1$$

form an orthonormal basis of $L^2(\mathbb{R})$ (see [15]. Bellow we adopt the shorter notation $\psi_{j,n}^{(k)} = \psi_I^{(k)}$, where $I = [nN^{-j}, (n+1)N^{-j})$. The system $\mathcal{X} = \{\psi_I^{(k)}, I \in \mathcal{D}, k = 1, \cdots, N-1\}$, where the functions $\psi^{(k)}$, $k = 1, \cdots, N-1$ are defined by (1.1), is called the Haar wavelet system of rank N (corresponding to Haar matrix H). An important example of a higher rank Haar wavelet system is the system obtained by wavelet functions

$$\psi^{(k)}(x) = \sqrt{N} \sum_{l=0}^{N-1} e^{2\pi ikl/N}\varphi(Nx - l), \;\; k = 1, \cdots, N-1, \tag{1.2}$$

where φ is characteristic function of the interval $[0, 1)$.

Note that the wavelet system constructed by functions (1.2) became of interest in connection with some problems of p-adic (non-Archimedean) mathematical physics (see [10-11]).

For a Haar wavelet system $\mathcal{X}$ and $f \in L^1_{loc}(\mathbb{R})$, we define the Littlewood-Paley operator by

$$Pf(x) = \left(\sum_{k=1}^{N-1} \sum_{I \in \mathcal{D}} | < f, \psi_I^{(k)} > |^2 ||I|^{-1}\chi_I(x) \right)^{1/2},$$

where

$$< f, \psi_I^{(k)} >= \int_{\mathbb{R}} f(x)\overline{\psi_I^{(k)}}(x)dx.$$

The characterization of the spaces $L_w^p(\mathbb{R})$ $(w \in A_p, 1 < p < \infty)$ using higher rank Haar wavelet system $\mathcal{X}$ was given in [9].

Theorem 1.1. ([9]) *Let $\mathcal{X}$ be a Haar wavelet system of rank N and $1 < p < \infty$. The following conditions are equivalent: 1) The system $\mathcal{X}$ is unconditional basis of space $L_w^p(\mathbb{R})$; 2) There exist positive constants c and C such that $c\|f\|_{L_w^p} \le \|P(f)\|_{L_w^p} \le C\|f\|_{L_w^p}$ for all $f \in L_w^p(\mathbb{R})$; 3) $w \in A_p^N$.*

The purpose of this paper is to prove following theorem.

Theorem 1.2. *Let $\mathcal{X} = \{\psi_I^{(k)};\ I \in \mathcal{D},\ k = 1,\cdots,N-1\}$ be a Haar wavelet system of rank N. Suppose $w \in A_p^N$ $(1 < p < \infty)$. Then the system $\{\psi_I^{(k)}/\|\psi_I^{(k)}\|_p;\ I \in \mathcal{D},\ k = 1,\cdots,N-1\}$ forms a greedy basis in the space $L_w^p(\mathbb{R})$.*

2. Proof of Theorem 1.2

For simplicity we shall denote the *normalized characteristic function* of a set of indices $\Gamma \subset \{1,2,\cdots,N-1\} \times \mathcal{D}$ by

$$\mathbf{1}_\Gamma = \sum_{(k,I)\in\Gamma} \frac{\psi_I^k}{\|\psi_I^k\|_p}.$$

Observe that $\mathcal{X}$ is democratic in $L_w^p(\mathbb{R})$ if and only if there exists a function $h : \mathbb{N} \to \mathbb{R}^+$ for which

$$\frac{1}{C}\, h(\mathrm{Card}(\Gamma)) \le \|\mathbf{1}_\Gamma\|_{L_w^p} \le C\, h(\mathrm{Card}(\Gamma)),\ \forall\, \Gamma \subset \{1,2,\cdots,N-1\} \times \mathcal{D} \tag{2.1}$$

for some $C \ge 1$.

Observe that from Theorem 1.1 for a single element ψ_I^k

$$\|\psi_I^k\|_{L_w^p} \asymp \frac{w(I)^{1/p}}{|I|^{1/2}},$$

where $w(I) = \int_I w(x)dx$ and the constants involved in $\asymp$ are independent of ψ_I^k. Thus, using again the expression of the norm $\|\cdot\|_{L_w^p}$ it follows that

$$\|\mathbf{1}_\Gamma\|_{L_w^p} \asymp \left\| \left(\sum_{(k,I)\in\Gamma} \frac{\chi_I}{w(I)^{2/p}} \right)^{1/2} \right\|_{L_w^p} \asymp \left\| \left(\sum_{I\in\widetilde{\Gamma}} \frac{\chi_I}{w(I)^{2/p}} \right)^{1/2} \right\|_{L_w^p}, \tag{2.2}$$

where $\widetilde{\Gamma} = \{I :\ (k,I) \in \Gamma\}$. Note that $\mathrm{Card}(\widetilde{\Gamma}) \asymp \mathrm{Card}(\Gamma)$.

Given a finite set of intervals $\Gamma \subset \mathcal{D}$, we shall denote

$$S_\Gamma(x) = \left(\sum_{I\in\Gamma} \frac{\chi_I(x)}{w(I)^{2/p}} \right)^{1/2}.$$

We "linearize" the square function $S_\Gamma(x)$. Note that this lineralization procedure has been used by other authors in the context of $m-$term approximation (see e.g. [3-5]).

For every $x \in \cup_{I\in\Gamma} I$, we define I_x as the smallest (hence unique) interval in Γ containing x. It is clear that

$$S_\Gamma(x) \ge \frac{\chi_{I_x}(x)}{w(I_x)^{1/p}} \quad \forall\, x \in \cup_{I\in\Gamma} I. \tag{2.3}$$

We now show that the reverse inequality holds with some universal constant.

Note that if $w \in A_p^N$, then there exist C_1, $C_2 > 0$ and $\delta > 0$ such that

$$C_1(|A|/|I|)^p \le w(A)/w(I) \le C_2(|A|/|I|)^\delta \tag{2.4}$$

for all $I \in \mathcal{D}$ and all subsets $A \subset I$ (see [1]).

If we enlarge the sum to include all $N-$adic intervals containing I_x we have

$$S_\Gamma(x)^2 = \sum_{I\in\Gamma} \frac{\chi_I(x)}{w(I)^{2/p}} \leq \sum_{I\in\mathcal{D},\, I\supset I_x} \frac{1}{w(I)^{2/p}}.$$

If $I_x \subset I$ and $|I| = N^j|I_x|$, then by (2.4) we have, $w(I) \geq C_2^{-1}w(I_x)N^{-j\delta}$. Thus,

$$S_\Gamma(x)^2 \leq C\frac{\chi_{I_x}(x)}{w(I_x)^{2/p}}.$$

This and (2.3) show that $S_\Gamma(x) \asymp \frac{\chi_{I_x}(x)}{w(I_x)^{1/p}}$.

Observe that $S_\Gamma(x) \asymp S_{\Gamma_{\min}}(x)$, where $\Gamma_{\min}$ denotes the family of minimal intervals in Γ, that is, $\Gamma_{\min} = \{I_x :\ x \in \cup_{I\in\Gamma} I\}$. Note that the intervals in $\Gamma_{\min}$ are not necessarily pairwise disjoint.

Given a fixed $\Gamma \subset \mathcal{D}$, for any $I \in \Gamma$ we define the set $S(I)$ as the union of all intervals from Γ strictly contained in I. We define also the set $L(I) = I\backslash S(I)$. It is clear that $I \in \Gamma_{\min}$ if and only if $L(I) \neq \emptyset$, and moreover $\cup_{I\in\Gamma} I = \cup_{I\in\Gamma_{\min}} L(I)$, where the sets in the last union are pairwise disjoint. Therefore we can write

$$S_\Gamma(x) \asymp \sum_{I\in\Gamma_{\min}} \frac{\chi_{L(I)}(x)}{w(I)^{1/p}} \tag{2.5}$$

where in the last sum there is a most one non-zero term.

Denote by Γ_S the collection of all intervals I from Γ with property: $|S(I)| > (1-1/N)|I|$. Denote by Γ_L the collection of all intervals I from Γ with property: $|L(I)| \geq |I|/N$. Observe that $\Gamma_L \subset \Gamma_{\min}$. We have

$$(1-1/N)\mathrm{Card}(\Gamma) \leq \mathrm{Card}(\Gamma_L) \leq \mathrm{Card}(\Gamma_{\min}) \leq \mathrm{Card}(\Gamma), \ \ \forall\, \Gamma \subset \mathcal{D}. \tag{2.6}$$

Clearly $\mathrm{Card}(\Gamma_L) \leq \mathrm{Card}(\Gamma_{\min}) \leq \mathrm{Card}(\Gamma)$. Thus, we need to prove only the inequality from the left hand side of (2.6). Given $I \in \mathcal{D}$, we write $I^{(k)}$, $k = 1,\cdots,N$ for the N-adic intervals contained in I of size $N^{-1}|I|$. For $I \in \Gamma_S$ and $k = 1,\cdots,N$ let $I_0^{(k)}$ be the biggest interval from Γ with $I_0^{(k)} \subset I^{(k)}$. Note that the intervals $I_0^{(k)}$ exist for every $I \in \Gamma_S$; otherwise, if for some $k_0 \in \{1,\cdots,N\}$ there is no interval from Γ contained in $I^{(k_0)}$ we have $I^{(k_0)} \subset L(I)$ and then

$$|S(I)| \leq |I\backslash I^{(k_0)}| = (1-1/N)|I|,$$

contradicting the definition of Γ_S.

We claim that if $I, R \in \Gamma_S$ and $I \neq R$, then we necessarily have $I_0^{(k)} \neq R_0^{(l)}$ for all $1 \leq k, l \leq N$. This is trivially true if $I \cap R = \emptyset$. Without loss of generality we may assume $I \subset R$ and also $I \subset R^{(1)}$. It follows that $I_0^{(k)} \neq R_0^{(l)}$ for all $k = 1,\cdots,N$ and all $l = 2,\cdots,N$. Moreover, as $R_0^{(1)}$ is the biggest interval in Γ contained in $R^{(1)}$ and $I \subset R^1$ we have that $I \subset R_0^{(1)} \subset R^{(1)}$. Hence, for all $k = 1,\cdots,N$ we have $I_0^{(k)} \subset I \subset R_0^{(1)}$ and thus $I_0^{(k)} \neq R_0^{(1)}$. In short, to each $I \in \Gamma_S$ we have assigned N different intervals in Γ and these are not associated to any other interval in Γ_S. We conclude that $N\mathrm{Card}(\Gamma_S) \leq \mathrm{Card}(\Gamma)$. Consequently we have

$$\mathrm{Card}(\Gamma_L) = \mathrm{Card}(\Gamma_L) - \mathrm{Card}(\Gamma_S) \geq \mathrm{Card}(\Gamma) - \mathrm{Card}(\Gamma)/N = (1-1/N)\mathrm{Card}(\Gamma).$$

Note that $|I|/N \le |L(I)| \le |I|$ and by (2.4) we have $\|\chi_{L(I)}\|_{L^p_w} \asymp \|\chi_I\|_{L^p_w}$. Using this estimate we can write

$$\|S_\Gamma\|_{L^p_w} \asymp \left\| \sum_{I \in \Gamma_{\min}} \frac{\chi_{L(I)}(x)}{w(I)^{1/p}} \right\|_{L^p_w} \asymp (\mathrm{Card}(\Gamma_{\min}))^{1/p} \asymp (\mathrm{Card}(\Gamma))^{1/p}. \tag{2.7}$$

From (2.2), (2.7) one obtains the estimates (2.1), with $h(n) = n^{1/p}$. □

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Existence and localization of positive solutions to first order differential systems with nonlocal conditions

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Abstract. The purpose of the present work is to study the existence and the localization of positive solutions to nonlocal boundary value problems for first order differential systems. The localization is established by the vector version of Krasnosel'skiĭ's fixed point theorem in cones.

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1. Introduction

Nonlocal problems for different classes of differential equations and systems have been intensively studied in the literature (see, for example, [1], [2], [3], [9] for multi-point nonlocal conditions, and [13], [14] for nonlocal conditions given by Stieltjes integrals). One of the most common technique for the existence and localization of positive solutions to integral and differential equations is based on Krasnosel'skiĭ's fixed point theorem in cones (see, e.g. [4], [7], [8], [11] and [12]).

Motivated by the article of Li and Sun [6], in this paper, we study systems of first order equations with integral boundary conditions, using the vector version of Krasnosel'skiĭ's fixed point theorem in cones given by Precup [10]. This vectorial method allows the nonlinear terms of a system to have different behaviors both in components and variables. More exactly, in this paper we consider the following first order differential system with nonlocal boundary conditions given by linear functionals:

$$\begin{cases} u_1' = f_1(t, u_1, u_2) \\ u_2' = f_2(t, u_1, u_2) \\ u_1(0) - a_1 u_1(1) = g_1[u_1] \\ u_2(0) - a_2 u_2(1) = g_2[u_2] \end{cases} \tag{1.1}$$

where $f_1, f_2 \in C([0,1] \times \mathbb{R}_+^2, \mathbb{R}_+)$; $g_1, g_2 : C[0,1] \to \mathbb{R}$ are two linear functionals given by

$$g_i[u] = \int_0^1 u(s)\, d\gamma_i(s), \tag{1.2}$$

with $g_i[1] < 1$; $\gamma_i \in C^1[0,1]$ increasing and $0 < a_i < 1 - g_i[1]$ $(i = 1, 2)$.

We seek nonnegative solutions (u_1, u_2), $u_1 \geq 0$, $u_2 \geq 0$ on $[0,1]$.

1.1. The integral form of the nonlocal problem

In order to put (1.1) in an operator form, let us first consider the scalar problem:

$$\begin{cases} Lu := u' = h(t), & 0 \leq t \leq 1 \\ u(0) - au(1) = g[u] \end{cases} \tag{1.3}$$

where $h \in C[0,1]$; $g : C[0,1] \to \mathbb{R}$ is a linear functional given by

$$g[u] = \int_0^1 u(s)\, d\gamma(s), \tag{1.4}$$

with $g[1] < 1$; $\gamma \in C^1[0,1]$ increasing; $0 < a < 1 - g[1]$. We shall obtain the integral equation equivalent to the problem (1.3). To this end, we start with the differential equation, which by integration gives

$$u(t) = u(0) + \int_0^t h(s)\, ds. \tag{1.5}$$

Apply g to (1.5) and use its linearity to obtain

$$g[u] = u(0)g[1] + g\left[\int_0^t h(s)\, ds\right].$$

Notice that by $g\left[\int_0^t h(s)\, ds\right]$ we mean the value of functional g on the function $t \mapsto \int_0^t h(s)\, ds$. This together with the boundary condition in (1.3) yields

$$u(0) - au(1) = u(0)g[1] + g\left[\int_0^t h(s)\, ds\right],$$

and then

$$u(0) - u(0)g[1] = au(1) + g\left[\int_0^t h(s)\, ds\right].$$

On the other hand,

$$u(1) = u(0) + \int_0^1 h(s)\, ds.$$

Therefore

$$u(0) - u(0)g[1] = au(0) + a\int_0^1 h(s)\, ds + g\left[\int_0^t h(s)\, ds\right].$$

Hence

$$u(0) = \frac{1}{1 - g[1] - a}\left(g\left[\int_0^t h(s)\, ds\right] + a\int_0^1 h(s)\, ds\right).$$

If we denote $c := \dfrac{1}{1-g[1]-a}$ and we substitute into (1.5), we obtain

$$u(t) = cg\left[\int_0^t h(s)\,ds\right] + ca\int_0^1 h(s)\,ds + \int_0^t h(s)\,ds.$$

Next using the expresion (1.4) of g, we find

$$\begin{aligned} u(t) &= c\int_0^1 \left(\int_0^s h(r)\,dr\right) d\gamma(s) + ca\int_0^1 h(s)\,ds + \int_0^t h(s)\,ds \\ &= c\int_0^1 \gamma^{'}(s)\int_0^s h(r)\,dr\,ds + ca\int_0^1 h(s)\,ds + \int_0^t h(s)\,ds \\ &= \int_0^t \left(c\gamma^{'}(s)\int_0^s h(r)\,dr + h(s)\right) ds + c\int_t^1 \gamma^{'}(s)\int_0^s h(r)\,dr\,ds \\ &\quad + ca\int_0^1 h(s)\,ds. \end{aligned}$$

Integration by parts gives

$$\begin{aligned} u(t) &= c\gamma(s)\int_0^s h(r)\,dr\Big|_0^t - \int_0^t c\gamma(s)h(s)\,ds + c\gamma(s)\int_0^s h(r)\,dr\Big|_t^1 \\ &\quad - \int_t^1 c\gamma(s)h(s)\,ds + \int_0^t h(s)\,ds + ca\int_0^1 h(s)\,ds \\ &= c\gamma(1)\int_0^1 h(s)\,ds - \int_0^1 c\gamma(s)h(s)\,ds + \int_0^t h(s)\,ds + ca\int_0^1 h(s)\,ds \\ &= \int_0^1 c\left(\gamma(1)-\gamma(s)+a\right) h(s)\,ds + \int_0^t h(s)\,ds \\ &= \int_0^t [c(\gamma(1)-\gamma(s)+a)+1]h(s)\,ds + \int_t^1 c(\gamma(1)-\gamma(s)+a)h(s)\,ds. \end{aligned} \tag{1.6}$$

If now, to the nonlocal condition $u(0) - au(1) = g[u]$, we associate the Green function

$$G(t,s) = \begin{cases} c[\gamma(1)-\gamma(s)+a]+1 & \text{for } \ 0 \le s \le t \le 1 \\ c[\gamma(1)-\gamma(s)+a] & \text{for } \ 0 \le t < s \le 1, \end{cases} \tag{1.7}$$

then (1.6) can be written as

$$u(t) = \int_0^1 G(t,s)h(s)\,ds.$$

Thus we have obtained the inverse of the operator L, $L^{-1} : C[0,1] \to C[0,1]$,

$$(L^{-1}h)(t) = \int_0^1 G(t,s)h(s)\,ds.$$

Based on this, the problem of nonnegative solutions of (1.1) is equivalent to the integral system:

$$\begin{cases} u_1(t) = \displaystyle\int_0^1 G_1(t,s)f_1(s,u_1(s),u_2(s))\,ds \\ u_2(t) = \displaystyle\int_0^1 G_2(t,s)f_2(s,u_1(s),u_2(s))\,ds, \end{cases} \tag{1.8}$$

where $G_1(t,s)$ and $G_2(t,s)$ are the Green functions corresponding to the two nonlocal conditions,

$$G_i(t,s) = \begin{cases} c_i[\gamma_i(1) - \gamma_i(s) + a_i] + 1 & \text{for } 0 \le s \le t \le 1 \\ c_i[\gamma_i(1) - \gamma_i(s) + a_i] & \text{for } 0 \le t < s \le 1, \end{cases}$$

where $c_i = \dfrac{1}{1 - g_i[1] - a_i}$ $(i = 1, 2)$.

The following properties are essential for the applicability of Krasnosel'skiĭ's technique:

1) $G_i(t,s) \le H_i(s)$, for all $t, s \in [0,1]$, where

$$H_i(s) = c_i[\gamma_i(1) - \gamma_i(s) + a_i] + 1 \ (i = 1, 2)$$

2) $\delta_i H_i(s) \le G_i(t,s)$ for all $t, s \in [0,1]$, where

$$\delta_i = \min_{s \in [0,1]} \frac{c_i[\gamma_i(1) - \gamma_i(s) + a_i]}{c_i[\gamma_i(1) - \gamma_i(s) + a_i] + 1} \ (i = 1, 2).$$

Let $N : C([0,1], \mathbb{R}^2_+) \to C([0,1], \mathbb{R}^2_+)$, $N = (N_1, N_2)$ be defined by

$$N_i(u_1, u_2)(t) = \int_0^1 G_i(t,s) f_i(s, u_1(s), u_2(s))\, ds \ (i = 1, 2).$$

The above properties of the Green functions imply that for any $t, t^* \in [0,1]$, one has:

$$\begin{aligned} N_i(u_1, u_2)(t) &= \int_0^1 G_i(t,s) f_i(s, u_1(s), u_2(s))\, ds \\ &\ge \delta_i \int_0^1 H_i(s) f_i(s, u_1(s), u_2(s))\, ds \\ &\ge \delta_i \int_0^1 G_i(t^*, s) f_i(s, u_1(s), u_2(s))\, ds = \delta_i N_i(u_1, u_2)(t^*). \end{aligned}$$

This yields the estimation from below

$$N_i(u_1, u_2)(t) \ge \delta_i |N_i(u_1, u_2)|_\infty \quad \text{for all } t \in [0,1] \ \ (i = 1, 2) \tag{1.9}$$

and any nonnegative functions $u_1, u_2 \in C[0,1]$.

Based on these estimations we define the cones:

$$K_i = \{u_i \in C[0,1] : u_i(t) \ge \delta_i |u_i|_\infty, \text{for all } t \in [0,1]\} (i = 1, 2), \tag{1.10}$$

and the product cone $K := K_1 \times K_2$ in $C([0,1], \mathbb{R}^2)$. Due to (1.9) we have the invariance property

$$N(K) \subset K.$$

Therefore, the problem of nonnegative solutions of (1.1) is equivalent to the fixed point problem

$$u = Nu, \ u \in K,$$

for the self-mapping N of K. Note that the continuity of f_1, f_2 implies the complete continuity of N.

Notice that (1.9) represents a weak Harnack type inequality for the nonnegative super solutions of the problem (1.1) (see [5]).

1.2. The vector version of Krasnosel'skiĭ's fixed point theorem in cones

The main tool of our paper is the following vector version of Krasnosel'skiĭ's fixed point theorem in cones given by Precup [10].

Theorem 1.1. *Let* $(X, |\cdot|)$ *be a normed linear space;* $K_1, K_2 \subset X$ *two cones;* $K := K_1 \times K_2$*;* $r, R \in \mathbb{R}^2_+$, $r = (r_1, r_2)$, $R = (R_1, R_2)$ *with* $0 < r < R$ *if* $0 < r_1 < R_1$ *and* $0 < r_2 < R_2$*;* $(K_i)_{r_i,R_i} = \{u_i \in K_i : r_i < |u_i| < R_i\}$, *for* $i = 1, 2$*;* $K_{r,R} := (K_1)_{r_1,R_1} \times (K_2)_{r_2,R_2}$ *and* $N : K_{r,R} \to K$, $N = (N_1, N_2)$ *a compact map. Assume that for each* $i \in \{1, 2\}$, *one of the following conditions is satisfied in* $K_{r,R}$*:*

(a) $N_i(u_1, u_2) \nleq u_i$ *if* $|u_i| = r_i$, *and* $N_i(u_1, u_2) \ngeq u_i$ *if* $|u_i| = R_i$*;*
(b) $N_i(u_1, u_2) \ngeq u_i$ *if* $|u_i| = r_i$, *and* $N_i(u_1, u_2) \nleq u_i$ *if* $|u_i| = R_i$.

Then N *has a fixed point* $u := (u_1, u_2)$ *in* K *with* $r_i < |u_i| < R_i$, *for* $i \in \{1, 2\}$.

Notice that the condition (a) means *compression*, while (b) means *expansion*. Therefore, in Theorem 1.1, the operators N_1, N_2 are both compressing, both expanding, or one compressing and the other one expanding.

2. Main results

2.1. Existence and localization

Using the notations from Section 1.1, we can state the main result of this paper.

Theorem 2.1. *Assume that there exist* α_i, $\beta_i > 0$ *with* $\alpha_i \neq \beta_i$, $i = 1, 2$, *such that*

$$\begin{aligned} A_1\lambda_1 > \alpha_1, \quad B_1\Lambda_1 < \beta_1, \\ A_2\lambda_2 > \alpha_2, \quad B_2\Lambda_2 < \beta_2, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
A_i &= \int_0^1 G_i(t^*, s)\, ds, \text{ for a chosen point } t^* \in [0,1], \\
B_i &= \max_{0\le t\le 1} \int_0^1 G_i(t, s)\, ds, \\
\lambda_1 &= \min\{f_1(t, u_1, u_2) : 0 \le t \le 1,\ \delta_1\alpha_1 \le u_1 \le \alpha_1,\ \delta_2 r_2 \le u_2 \le R_2\}, \\
\lambda_2 &= \min\{f_2(t, u_1, u_2) : 0 \le t \le 1,\ \delta_1 r_1 \le u_1 \le R_1,\ \delta_2\alpha_2 \le u_2 \le \alpha_2\}, \\
\Lambda_1 &= \max\{f_1(t, u_1, u_2) : 0 \le t \le 1,\ \delta_1\beta_1 \le u_1 \le \beta_1,\ \delta_2 r_2 \le u_2 \le R_2\}, \\
\Lambda_2 &= \max\{f_2(t, u_1, u_2) : 0 \le t \le 1,\ \delta_1 r_1 \le u_1 \le R_1,\ \delta_2\beta_2 \le u_2 \le \beta_2\},
\end{aligned}$$

and $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$ $(i = 1, 2)$. *Then* (1.1) *has at least one positive solution* $u = (u_1, u_2)$ *with* $r_i < |u_i|_\infty < R_i$ $(i = 1, 2)$.

Proof. We shall apply Theorem 1.1, with $X = C[0,1]$, $|u| = \max_{0\le t\le 1} |u(t)|$ and K_1, K_2 given by (1.10).

If $u_i \in (K_i)_{r_i,R_i}$, then $r_i < |u_i|_\infty < R_i$ $(i = 1, 2)$. It follows from the definitions of K_i that

$$\delta_i r_i \le u_i(t) \le R_i \ (i = 1, 2)$$

for all $t \in [0,1]$. Also, if we know for example that $|u_1|_\infty = \alpha_1$, then

$$\delta_1\alpha_1 \le u_1(t) \le \alpha_1.$$

We claim that for any $u_i \in (K_i)_{r_i,R_i}$ and $i \in \{1,2\}$, the following properties hold:

$$\begin{aligned} |u_i|_\infty = \alpha_i \quad &\text{implies} \quad N_i(u_1,u_2) \not\le u_i, \\ |u_i|_\infty = \beta_i \quad &\text{implies} \quad N_i(u_1,u_2) \not\ge u_i. \end{aligned} \tag{2.2}$$

Indeed, if $|u_1|_\infty = \alpha_1$ and we would have $N_1(u_1,u_2) \le u_1$, then for the chosen point t^* we obtain using (2.1):

$$\begin{aligned} &\alpha_1 \ge u_1(t^*) \ge N_1(u_1,u_2)(t^*) = \int_0^1 G_1(t^*,s) f_1(s,u_1(s),u_2(s))\,ds \\ &\ge A_1\lambda_1 > \alpha_1. \end{aligned}$$

This yields the contradiction $\alpha_1 > \alpha_1$. Now, if $|u_1|_\infty = \beta_1$ and $N_1(u_1,u_2) \ge u_1$, then for some $t' \in [0,1]$ with $|u_1|_\infty = u_1(t')$ we have

$$\begin{aligned} &\beta_1 = u_1(t') \le N_1(u_1,u_2)(t') = \int_0^1 G_1(t',s) f_1(s,u_1(s),u_2(s))\,ds \\ &\le B_1\Lambda_1 < \beta_1 \end{aligned}$$

whence we deduce that $\beta_1 < \beta_1$, a contradiction. Hence (2.2) holds for $i = 1$. Similary, (2.2) is true for $i = 2$. □

In particular, if f_1 and f_2 do not depend on t, i.e., $f_1 = f_1(u_1,u_2)$ and $f_2 = f_2(u_1,u_2)$, and f_1, f_2 have some monotonicity properties in u_1 and u_2, then we can specify the numbers λ_1, λ_2, Λ_1, Λ_2 and the conditions (2.1) are expressed by values of f_1, f_2 on only four points. Here are five cases from all the sixteen possible:

Case 1) If f_1, f_2 are nondecreasing in u_1 and u_2, then

$\lambda_1 = f_1(\delta_1\alpha_1, \delta_2 r_2)$, $\Lambda_1 = f_1(\beta_1, R_2)$, $\lambda_2 = f_2(\delta_1 r_1, \delta_2\alpha_2)$, $\Lambda_2 = f_2(R_1, \beta_2)$,

and (2.1) becomes

$$\frac{f_1(\delta_1\alpha_1,\delta_2 r_2)}{\alpha_1} > \frac{1}{A_1}, \quad \frac{f_1(\beta_1,R_2)}{\beta_1} < \frac{1}{B_1},$$
$$\frac{f_2(\delta_1 r_1,\delta_2\alpha_2)}{\alpha_2} > \frac{1}{A_2}, \quad \frac{f_2(R_1,\beta_2)}{\beta_2} < \frac{1}{B_2}.$$

Case 2) If f_1 is nondecreasing in u_1 and u_2, while f_2 is nondecreasing in u_1 and nonincreasing in u_2, then

$\lambda_1 = f_1(\delta_1\alpha_1, \delta_2 r_2)$, $\Lambda_1 = f_1(\beta_1, R_2)$, $\lambda_2 = f_2(\delta_1 r_1, \alpha_2)$, $\Lambda_2 = f_2(R_1, \delta_2\beta_2)$,

and (2.1) reduces to

$$\frac{f_1(\delta_1\alpha_1,\delta_2 r_2)}{\alpha_1} > \frac{1}{A_1}, \quad \frac{f_1(\beta_1,R_2)}{\beta_1} < \frac{1}{B_1},$$
$$\frac{f_2(\delta_1 r_1,\alpha_2)}{\alpha_2} > \frac{1}{A_2}, \quad \frac{f_2(R_1,\delta_2\beta_2)}{\beta_2} < \frac{1}{B_2}.$$

Case 3) If f_1 is nondecreasing in u_1 and nonincreasing in u_2, while f_2 is nonincreasing in u_1 and nondecreasing in u_2, then

$\lambda_1 = f_1(\delta_1\alpha_1, R_2)$, $\Lambda_1 = f_1(\beta_1, \delta_2 r_2)$, $\lambda_2 = f_2(R_1, \delta_2\alpha_2)$, $\Lambda_2 = f_2(\delta_1 r_1, \beta_2)$,

and (2.1) reads as

$$\frac{f_1(\delta_1\alpha_1, R_2)}{\alpha_1} > \frac{1}{A_1}, \quad \frac{f_1(\beta_1, \delta_2 r_2)}{\beta_1} < \frac{1}{B_1},$$
$$\frac{f_2(R_1, \delta_2\alpha_2)}{\alpha_2} > \frac{1}{A_2}, \quad \frac{f_2(\delta_1 r_1, \beta_2)}{\beta_2} < \frac{1}{B_2}.$$

Case 4) If f_1 and f_2 are nondecreasing in u_1 and nonincreasing in u_2, then

$$\lambda_1 = f_1(\delta_1\alpha_1, R_2),\ \Lambda_1 = f_1(\beta_1, \delta_2 r_2),\ \lambda_2 = f_2(\delta_1 r_1, \alpha_2),\ \Lambda_2 = f_2(R_1, \delta_2\beta_2),$$

and (2.1) becomes

$$\frac{f_1(\delta_1\alpha_1, R_2)}{\alpha_1} > \frac{1}{A_1}, \quad \frac{f_1(\beta_1, \delta_2 r_2)}{\beta_1} < \frac{1}{B_1},$$
$$\frac{f_2(\delta_1 r_1, \alpha_2)}{\alpha_2} > \frac{1}{A_2}, \quad \frac{f_2(R_1, \delta_2\beta_2)}{\beta_2} < \frac{1}{B_2}.$$

Case 5) If f_1 is nondecreasing in u_1 and u_2, while f_2 is nonincreasing in u_1 and u_2, then

$$\lambda_1 = f_1(\delta_1\alpha_1, \delta_2 r_2),\ \Lambda_1 = f_1(\beta_1, R_2),\ \lambda_2 = f_2(R_1, \alpha_2),\ \Lambda_2 = f_2(\delta_1 r_1, \delta_2\beta_2),$$

and (2.1) reduces to

$$\frac{f_1(\delta_1\alpha_1, \delta_2 r_2)}{\alpha_1} > \frac{1}{A_1}, \quad \frac{f_1(\beta_1, R_2)}{\beta_1} < \frac{1}{B_1},$$
$$\frac{f_2(R_1, \alpha_2)}{\alpha_2} > \frac{1}{A_2}, \quad \frac{f_2(\delta_1 r_1, \delta_2\beta_2)}{\beta_2} < \frac{1}{B_2}.$$

2.2. Multiplicity

Theorem 2.1 guarantees the existence of solutions in an annular set. Clearly, if the assumptions of Theorem 2.1 are satisfied for several disjoint annular sets, then multiple solutions are obtained (see [11]).

Theorem 2.2. (A) *Let $(r^j)_{1\le j\le k}$, $(R^j)_{1\le j\le k}$ $(k \le \infty)$ be increasing finite or infinite sequence in $\mathbb{R}^2_+$, with $0 \le r^j < R^j$ and $R^j < r^{j+1}$ for all j. If the assumptions of Theorem* 2.1 *are satisfied for each couple (r^j, R^j), then* (1.1) *has k (respectively, when $k = \infty$, an infinite sequence of) distinct positive solutions.*

(B) *Let $(r^j)_{j\ge 1}$, $(R^j)_{j\ge 1}$ be decreasing infinite sequence with $0 < r^j < R^j$ and $R^j < r^{j+1}$ for all j. If the assumptions of Theorem* 2.1 *are satisfied for each couple (r^j, R^j), then* (1.1) *has an infinite sequence of distinct positive solutions.*

Proof. It is sufficient to see that

$$K_{r^j, R^j} \cap K_{r^{j+1}, R^{j+1}} = \emptyset \text{ for all } j.$$

To prove this, let us consider two cases. First, if we assume that the sequences (r^j), (R^j) are increasing, then $K_{r^j, R^j} \subset \{u \in K : |u| < R^{j+1}\}$. Similary, if the sequences (r^j), (R^j) are decreasing, one has $K_{r^{j+1}, R^{j+1}} \subset \{u \in K : |u| < r^j\}$. □

2.3. Examples

We conclude by two examples illustrating Theorem 2.1 in the Cases 1) and 5).

Example 2.3. Let

$$f_i(u_1,u_2)=\frac{1}{15}\sqrt{u_1+u_2+1}\ \ for\ i=1,2,$$

$\gamma_1(t)=\frac{1}{2}t, \gamma_2(t)=\frac{3}{4}t$, $a_1=\frac{1}{4}$ and $a_2=\frac{1}{8}$. Then (1.1) becomes

$$\begin{cases} u_1'=\frac{1}{15}\sqrt{u_1+u_2+1} \\ u_2'=\frac{1}{15}\sqrt{u_1+u_2+1} \\ u_1(0)-\frac{1}{4}u_1(1)=\frac{1}{2}\int_0^1 u_1(t)\,dt \\ u_2(0)-\frac{1}{8}u_2(1)=\frac{3}{4}\int_0^1 u_2(t)\,dt, \end{cases} \tag{2.3}$$

or equivalently

$$\begin{cases} u_1(t)=\frac{1}{15}\int_0^1 G_1(t,s)\sqrt{u_1(s)+u_2(s)+1}\,ds \\ u_2(t)=\frac{1}{15}\int_0^1 G_2(t,s)\sqrt{u_1(s)+u_2(s)+1}\,ds, \end{cases} \tag{2.4}$$

where $G_1(t,s)$ and $G_2(t,s)$ are the Green functions

$$G_1(t,s)=\begin{cases} 6-4s \text{ for } 0\le s\le t\le 1 \\ 5-4s \text{ for } 0\le t<s\le 1, \end{cases}$$

$$G_2(t,s)=\begin{cases} 10-8s \text{ for } 0\le s\le t\le 1 \\ 9-8s \ \ \text{ for } 0\le t<s\le 1. \end{cases}$$

In this case, the constants $\delta_1,\delta_2>0$ are the following ones:

$$\delta_1=\delta_2=\frac{1}{2}=:\delta.$$

Now we have to determine A_i and B_i for $i\in\{1,2\}$. We have

$$A_1=\int_0^1 G_1(t^*,s)\,ds=\int_0^{t^*}(6-4s)\,ds+\int_{t^*}^1(5-4s)\,ds=t^*+3.$$

If we choose $t^*=0$, then $A_1=3$. Also

$$A_2=\int_0^1 G_2(t^*,s)\,ds=\int_0^{t^*}(10-8s)\,ds+\int_{t^*}^1(9-8s)\,ds=t^*+5,$$

and if we choose $t^*=0$, then $A_2=5$. In addition

$$B_1=\max_{0\le t\le 1}\int_0^1 G_1(t,s)\,ds=4, \quad B_2=\max_{0\le t\le 1}\int_0^1 G_2(t,s)\,ds=6.$$

In this case $f_1(u_1,u_2)$ and $f_2(u_1,u_2)$ are both nondecreasing in u_1 and u_2 for $u_1,u_2\in\mathbb{R}_+$, so we are in Case 1). We choose $\alpha_1=\alpha_2=:\alpha$, $\beta_1=\beta_2=:\beta$, with $\alpha<\beta$, then $r_1=r_2=\alpha$, $R_1=R_2=\beta$ and $\lambda_1=f_1(\delta\alpha,\delta\alpha)$, $\Lambda_1=f_1(\beta,\beta)$, $\lambda_2=f_2(\delta\alpha,\delta\alpha)$, $\Lambda_2=f_2(\beta,\beta)$. The values of α and β will be precised in what follows. Since

$$\lim_{x\to\infty}\frac{f_i(x,x)}{x}=0 \quad\text{and}\quad \lim_{x\to 0}\frac{f_i(x,x)}{x}=\infty,$$

we may find α small enough and β large enough such that the conditions

$$\frac{f_i(\delta\alpha,\delta\alpha)}{\delta\alpha}>\frac{1}{\delta A_i},\quad \frac{f_i(\beta,\beta)}{\beta}<\frac{1}{B_i}\quad (i=1,2)$$

are satisfied. For instance, we can choose $\alpha=0,2$ and $\beta=0,7$.

Hence the following result holds.

Proposition 2.4. *The system (2.3) has at least one positive solution $u=(u_1,u_2)$ with*

$$0,2<|u_i|_\infty<0,7 \ (i=1,2).$$

Example 2.5. Let $f_1(u_1,u_2)=\frac{1}{15}\sqrt{u_1+u_2+1}$, $f_2(u_1,u_2)=\frac{1}{(2+u_1^2)(4+u_2^2)}$, $\gamma_1(t)=\frac{1}{2}t$, $\gamma_2(t)=\frac{3}{4}t$, $a_1=\frac{1}{4}$ and $a_2=\frac{1}{8}$. Then (1.1) becomes

$$\begin{cases} u_1'=\dfrac{1}{15}\sqrt{u_1+u_2+1}\\ u_2'=\dfrac{1}{(2+u_1^2)(4+u_2^2)}\\ u_1(0)-\dfrac{1}{4}u_1(1)=\dfrac{1}{2}\displaystyle\int_0^1 u_1(t)\,dt\\ u_2(0)-\dfrac{1}{8}u_2(1)=\dfrac{3}{4}\displaystyle\int_0^1 u_2(t)\,dt,\end{cases}\tag{2.5}$$

or equivalently

$$\begin{cases} u_1(t)=\dfrac{1}{15}\displaystyle\int_0^1 G_1(t,s)\sqrt{u_1(s)+u_2(s)+1}\,ds\\ u_2(t)=\displaystyle\int_0^1 G_2(t,s)\frac{1}{(2+u_1(s)^2)(4+u_2(s)^2)}\,ds.\end{cases}\tag{2.6}$$

The Green functions $G_i(t,s)$ and the values of δ_i, A_i, B_i $(i=1,2)$ are the same from the Example 2.3. In this case $f_1(u_1,u_2)$ is nondecreasing in u_1 and u_2, while $f_2(u_1,u_2)$ is nonincreasing in u_1 and u_2, for $u_1,u_2\in\mathbb{R}_+$, so now we are in Case 5). We choose $\alpha_1=\alpha_2=:\alpha$, $\beta_1=\beta_2=:\beta$, with $\alpha<\beta$. Then $r_1=r_2=\alpha$, $R_1=R_2=\beta$ and $\lambda_1=f_1(\delta\alpha,\delta\alpha)$, $\Lambda_1=f_1(\beta,\beta)$, $\lambda_2=f_2(\beta,\alpha)$, $\Lambda_2=f_2(\delta\alpha,\delta\beta)$, where α and β will be precised in what follows. Since

$$\lim_{y\to\infty}\frac{f_1(y,y)}{y}=0 \quad\text{and}\quad \lim_{y\to\infty}\frac{f_2(x,y)}{y}=0,$$

uniformly in $x\geq 0$, we may find $\beta>0$ large enough such that

$$\frac{f_1(\beta,\beta)}{\beta}<\frac{1}{B_1},\quad \frac{f_2(\delta\alpha,\delta\beta)}{\delta\beta}<\frac{1}{\delta B_2}.$$

And since

$$\lim_{x\to 0}\frac{f_1(x,x)}{x}=\infty \quad \text{and} \quad \lim_{x\to 0}\frac{f_2(y,x)}{x}=0,$$

with β fixed as above, we choose α small enough such that

$$\frac{f_1(\delta\alpha,\delta\alpha)}{\delta\alpha}>\frac{1}{\delta A_1},\quad \frac{f_2(\beta,\alpha)}{\alpha}>\frac{1}{A_2}.$$

For example, we can choose $\beta = 0,9$ and $\alpha = 0,2$.

Hence the following result holds.

Proposition 2.6. *The system* (2.5) *has at least one positive solution* $u = (u_1, u_2)$ *with*

$$0,2<|u_i|_\infty<0,9\ (i=1,2).$$

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Ćirić type fixed point theorems

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Abstract. The purpose of this paper is to review some fixed point and strict fixed point results for Ćirić type operators. The data dependence of the fixed point set, the well-posedness of the fixed point problem, the limit shadowing property, as well as, the fractal operator theory associated with these operators are also considered.

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1. Preliminaries

Fixed points and strict fixed points (also called end-points) are important tools for the study of equilibrium problems in Mathematical Economics and Game Theory. Fixed points and the strict fixed points represent optimal preferences in some economical models and respectively different Nash type equilibrium points for some abstract games, see, for example, [1] and [23]. As a consequence, it is important aim to obtain fixed and strict fixed point theorems for multivalued operators in different contexts.

We shall begin by presenting some notions and notations that will be used throughout the paper.

Let us consider the following families of subsets of a metric space (X, d):

$$P(X) := \{Y \in \mathcal{P}(X) \,|Y \neq \emptyset\}\,; P_b(X) := \{Y \in P(X)|\ Y \text{ is bounded }\},$$

$$P_{cl}(X) := \{Y \in P(X)|\ Y \text{ is closed}\}\,; P_{cp}(X) := \{Y \in P(X)|\ Y \text{ is compact}\}$$

Let us define the gap functional between the sets A and B in the metric space (X, d) as:

$$D : P(X) \times P(X) \to \mathbb{R}_+,\ D(A, B) = \inf\{d(a, b) \mid a \in A,\ b \in B\}$$

and the (generalized) Pompeiu-Hausdorff functional as:

$$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},$$

$$H(A,B) = \max\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(A,b)\}.$$

The generalized diameter functional is denoted by $\delta : P(X)\times P(X) \to \mathbb{R}_+\cup\{\infty\}$, and defined by

$$\delta(A,B) = \sup\{d(a,b) \mid a \in A,\ b \in B\}.$$

Denote by $diam(A) := \delta(A,A)$ the diameter of the set A.

Let $T : X \to P(X)$ be a multivalued operator and

$$Graphic(T) := \{(x,y)|y \in T(x)\}$$

the graphic of T. An element $x \in X$ is called a *fixed point* for T if and only if $x \in T(x)$ and it is called a *strict fixed point* if and only if $\{x\} = T(x)$.

The set $Fix(T) := \{x \in X|\ x \in T(x)\}$ is called the fixed point set of T, while $SFix(T) = \{x \in X|\ \{x\} = T(x)\}$ is called the strict fixed point set of T. Notice that $SFix(T) \subseteq Fix(T)$.

We will also denote by

$$O(x_0,n) := \{x_0, t(x_0), t^2(x_0), \cdots, t^n(x_0)\}$$

the orbit of order n of the operator t corresponding to $x_0 \in X$, while

$$O(x_0) := \{x_0, t(x_0), t^2(x_0), \cdots, t^n(x_0), \cdots\}$$

is the orbit of f corresponding to $x_0 \in X$.

In this paper we will survey some fixed point and strict fixed point theorems for singlevalued and multivalued operators satisfying contractive conditions of Ćirić type. We will also present some new results for general classes of Ćirić type operators.

2. A survey of known results

The basic metric fixed point theorems for singlevalued, respectively multivalued operators are Banach's contraction principle (1922), respectively Nadler's contraction principle (1969). A lot of efforts were done to extend these results for so called generalized contractions.

Let (X,d) be a metric space and $f : X \to X$ be a singlevalued operator. Then, by definition (see I.A. Rus [22]), f is called a weakly Picard operator if:

(i) $F_f \neq \emptyset$;

(ii) for all $x \in X$, the sequence $(f^n(x))_{n\in\mathbb{N}} \to x^*(x) \in F_f$ as $n \to \infty$.

In particular, if $F_f = \{x^*\}$, then f is called a Picard operator.

Moreover, if f is a weakly Picard operator and there exists $\tilde{c} > 0$ such that

$$d(x, x^*(x)) \leq \tilde{c}d(x, f(x)),\ \text{for all } x \in X,$$

then f is called a $\tilde{c}$-weakly Picard operator. Similarly, a Picard operator for which there exists $\tilde{c} > 0$ such that

$$d(x, x^*) \leq \tilde{c}d(x, f(x)),\ \text{for all } x \in X,$$

is called a $\tilde{c}$-Picard operator.

A nice extension of Banach's contraction principle was given by Ćirić, Reich and Rus (independently one to the others) in 1971-1972.

More precisely, if (X, d) is a complete metric space and $f : X \to X$ is an operator for which there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that

$$d(f(x), f(y)) \leq ad(x, y) + bd(x, f(x)) + cd(y, f(y)), \text{ for all } x, y \in X,$$

then f is a $\tilde{c}$-Picard operator, with $\tilde{c} := \frac{1}{1-\beta}$, where $\beta := \min\{\frac{a+b}{1-c}, \frac{a+c}{1-b}\} < 1$.

An extension of this result was proved in a paper from 1973 by Hardy and Rogers. The result, in Picard operators language, is as follows.

If (X, d) is a complete metric space and $f : X \to X$ is an operator for which there exist $a, b, c, e, f \in \mathbb{R}_+$ with $a + b + c + e + f < 1$ such that

$$d(f(x), f(y)) \leq ad(x, y) + bd(x, f(x)) + cd(y, f(y)) + ed(x, f(y)) + fd(y, f(x)),$$

for all $x, y \in X$, then f is a $\tilde{c}$-Picard operator, with $\tilde{c} := \frac{1}{1-\beta}$, where

$$\beta := \min\{\frac{a+b+e}{1-c-e}, \frac{a+c+f}{1-b-f}\} < 1.$$

The idea of the proof in the above results is to prove that f is a contraction on the graphic of the operator. i.e.,

$$d(f(x), f^2(x)) \leq \beta d(x, f(x)), \text{ for all } x \in X.$$

Then in 1974, Ćirić proved the following very general result.

If (X, d) is a complete metric space and if $f : X \to X$ is an operator for which there exists $q \in (0, 1)$ such that, for all $x, y \in X$, we have

$$d(f(x), f(y)) \leq q \max\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}, \quad (2.1)$$

then f is a $\tilde{c}$-Picard operator, with $\tilde{c} := \frac{1}{1-q}$.

In this last case, the proof is based on some arguments involving the orbit of order n and the orbit of the operator f.

Remark 2.1. Notice that any Ćirić-Reich-Rus type operator is a Hardy-Rogers type operator and any Hardy-Rogers type operator is a Ćirić type operator. The reverse implications do not hold, as we can see from several examples given in [10], [21], [22].

There are also other generalizations of the above theorems. One of the most general one, was proved by I.A. Rus in 1979.

If (X, d) is a complete metric space and $f : X \to X$ is an operator for which there exists a generalized strict comparison function $\varphi : \mathbb{R}^5_+ \to \mathbb{R}_+$ (which means that φ is increasing in each variable and the function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\Phi(t) := \varphi(t, t, t, t, t)$$

satisfy the conditions that $\Phi^n(t) \to 0$ as $n \to +\infty$, for all $t > 0$ and $t - \Phi(t) \to +\infty$ as $t \to +\infty$) such that

$$d(f(x), f(y)) \leq \varphi(d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))), \text{ for all } x, y \in X,$$

then f is a Φ-Picard operator (i.e., f is a Picard operator and $d(x, x^*) \leq \Phi(d(x, f(x))$, for all $x \in X$).

Notice that if, in particular

$$\varphi(t_1, t_2, t_3, t_4, t_5) := at_1 + bt_2 + ct_3 + et_4 + ft_5,$$

(with $a, b, c, e, f \in \mathbb{R}_+$ and $a + b + c + e + f < 1$), then we obtain the Hardy-Rogers condition on f. Also, if we consider

$$\varphi(t_1, t_2, t_3, t_4, t_5) := q \max\{t_1, t_2, t_3, t_4, t_5\},$$

then we get the Ćirić type condition on f.

Finally, let us recall another nice generalization given, for the case of nonself operators, by Ćirić in 2006.

More precisely, if (X, d) is a complete metric space and $f : X \to X$ is an operator for which there exist five strict comparison functions $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ (i.e., φ_i is increasing, $\varphi_i^n(t) \to 0$ as $n \to +\infty$, for all $t > 0$ and $t - \varphi_i(t) \to +\infty$ as $t \to +\infty$ for each $i \in \{1, 2, 3, 4, 5\}$) such that, for all $x, y \in X$ one have that

$$d(f(x), f(y)) \leq$$
$$\leq \max\{\varphi_1(d(x, y)), \varphi_2(d(x, f(x))), \varphi_3(d(y, f(y))), \varphi_4(d(x, f(y))), \varphi_5(d(y, f(x)))\},$$

then f is a Picard operator.

Notice that, in particular if we define $\varphi_i(t) := qt$ (where $q < 1$) for $i \in \{1, 2, 3, 4, 5\}$, then we obtain again the classical condition of Ćirić.

Passing to the multivalued case, let (X, d) be a metric space and let $T : X \to P_b(X)$ be a multivalued operator with nonempty and bounded values. We will be interested in the study of strict fixed points of multivalued operators satisfying some contractive type conditions with respect to the functional δ.

In 1972, S. Reich proved the following very interesting strict fixed point theorem for multivalued operators.

If (X, d) is a complete metric space and if $T : X \to P_b(X)$ is a multivalued operator for which there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that

$$\delta(T(x), T(y)) \leq ad(x, y) + b\delta(x, T(x)) + c\delta(y, T(y)), \text{ for all } x, y \in X,$$

then the following conclusions hold:

(i) $F_T = (SF)_T = \{x^*\}$;

(ii) for each $x_0 \in X$ there exists a sequence of successive approximations for T starting from x_0 (which means that $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$) convergent to x^*;

(iii) $d(x_n, x^*) \leq \frac{\beta^n}{1-\beta} d(x_0, x_1)$, for $n \in \mathbb{N}^*$ (where $\beta := \min\{\frac{a+b}{1-c}, \frac{a+c}{1-b}\} < 1$).

An important extension of the above result is the following theorem of Ćirić, given in 1972.

Let (X, d) be a complete metric space and let $T : X \to P_b(X)$ be a multivalued operator for which there exists $q \in \mathbb{R}_+$ with $q < 1$ such that, for all $x, y \in X$ the following condition holds

$$\delta(T(x), T(y)) \leq q \max\{d(x, y), \delta(x, T(x)), \delta(y, T(y)), D(x, T(y)), D(y, T(x))\}.$$

Then the following conclusions hold:

(i) $F_T = (SF)_T = \{x^*\}$;

(ii) for each $x_0 \in X$ there exists a sequence of successive approximations for T starting from x_0 convergent to x^*;

(iii) $d(x_n, x^*) \leq \frac{q^{(1-a)n}}{1-q^{1-a}} d(x_0, x_1)$, for $n \in \mathbb{N}^*$ (where $a \in (0,1)$ is an arbitrary real number).

In the above two results, the approach is based on the construction of a selection $t : X \to X$ of T which satisfies the corresponding fixed point theorem (given by Reich and respectively by Ćirić) for singlevalued operators.

A relevant generalization of the above theorems was given by I.A. Rus in 1982, as follows.

Let (X, d) be a complete metric space and let $T : X \to P_b(X)$ be a multivalued operator. Suppose that there exists an increasing function $\varphi : \mathbb{R}^5_+ \to \mathbb{R}_+$ for which there exists $p > 1$ such that the function $\psi : \mathbb{R}^5_+ \to \mathbb{R}_+$ defined by

$$\psi(t_1, t_2, t_3, t_4, t_5) := \varphi(t_1, pt_2, pt_3, t_4, t_5)$$

is a generalized strict comparison function. If, for all $x, y \in X$ the following assumption takes place:

$$\delta(T(x), T(y)) \leq \varphi(d(x, y), \delta(x, T(x)), \delta(y, T(y)), D(x, T(y)), D(y, T(x))),$$

then the following conclusions hold:

(i) $F_T = (SF)_T = \{x^*\}$;

(ii) for each $x_0 \in X$ there exists a sequence of successive approximations for T starting from x_0 convergent to x^*.

Remark 2.2. In none of the above cases, we cannot obtain (without additional assumptions) the conclusion that T is a multivalued Picard operator. Recall that, by definition, $T : X \to P(X)$ is called a multivalued Picard operator (see [15]) if and only if:

(i) $(SF)_T = F_T = \{x^*\}$;

(ii) $T^n(x) \xrightarrow{H_d} \{x^*\}$ as $n \to \infty$, for each $x \in X$.

For example, if, in the case of Reich' strict fixed point theorem, we additionally impose the condition that

$$\min \left\{ \frac{a+b}{1-b}, \frac{a+c}{1-c} \right\} < 1,$$

then we can prove that T is a multivalued Picard operator, see [14].

Remark 2.3. It is an open question if we can get similar results if we replace, in Ćirić' result (or more generally in Rus' theorem) the values $D(x, T(y))$ and $D(y, T(x))$ with $\delta(x, T(y))$ and, respectively $\delta(y, T(x))$.

3. Strict fixed point theorems in metric spaces endowed with a graph

A new research direction in fixed point theory was recently considered by J. Jachymski (see [11]) in the context of a metric space endowed with a graph.

Let (X, d) be a metric space and Δ be the diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, $E(G)$ being the set of the edges of the graph. Assuming that G has no parallel edges, we will suppose that G can be identified with the pair $(V(G), E(G))$.

If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $(x_n)_{n\in\{0,1,2,\cdots,k\}}$ of vertices such that $x_0 = x$, $x_k = y$ and $(x_{i-1}, x_i) \in E(G)$, for $i \in \{1, 2, \cdots, k\}$.

Let us denote by $\tilde{G}$ the undirected graph obtained from G by ignoring the direction of edges. Notice that a graph G is connected if there is a path between any two vertices and it is weakly connected if $\tilde{G}$ is connected.

We will write that $E(G) \in I(T \times T)$ if and only if $x, y \in X$ with $(x, y) \in E(G)$ implies $T(x) \times T(y) \subset E(G)$.

For the particular case of a singlevalued operator $t : X \to X$ the above notations should be considered accordingly. In particular, the condition $E(G) \in I(t \times t)$ means that the operator t is edge preserving (in the sense of the Jachymski's definition of a Banach contraction, see [11]), i.e., for each $x, y \in X$ with $(x, y) \in E(G)$ we have that $(t(x), t(y)) \in E(G)$.

One of the main result of the paper [2] is a fixed point theorem for a singlevalued operator of Ćirić type in metric spaces endowed with a graph. An extended version of that theorem is the following.

Theorem 3.1. *Let (X, d) be a complete metric space and G be a directed graph such that the triple (X, d, G) satisfies the following property:*
(P) for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$ $x_n \to x$ as $n \to \infty$, there exists a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(x_{k_n}, x) \in E(G)$.

Let $t : X \to X$ be a singlevalued operator. Suppose the following assertions hold:

(i) *there exists $a \in [0, 1[$ such that*

$$d(t(x), t(y)) \leq a \cdot \max\{d(x, y), d(x, t(x)), d(y, t(y)), d(x, t(y)), d(y, t(x)\},$$

for all $(x, y) \in E(G)$.

(ii) *there exists $x_0 \in X$ such that $(x_0, t(x_0)) \in E(G)$;*

(iii) *$E(G) \in I(t \times t)$;*

(iv) *if $(x, y) \in E(G)$ and $(y, z) \in E(G)$, then $(x, z) \in E(G)$.*

In these conditions we have:

(a) *(existence) $Fix(t) \neq \emptyset$.*

(b) *(uniqueness) If, in addition, the following implication holds*

$$x^*, y^* \in Fix(t) \ \Rightarrow (x^*, y^*) \in E(G),$$

then $Fix(t) = \{x^*\}$.
Moreover, the sequence $(t^n(x_0))_{n\in\mathbb{N}}$ converges to x^* in (X, d)

Recall now two important stability concepts for the case of fixed point inclusions.

Definition 3.2. *Let (X, d) be a metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. By definition, the fixed point problem is well-posed for T with respect to H if:*

(i) $SFixT = \{x^*\}$;

(ii) *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $H(x_n, T(x_n)) \to 0$, as $n \to \infty$, then $x_n \overset{d}{\to} x^*$, as $n \to \infty$.*

Definition 3.3. *Let (X, d) be a metric space and $T : X \to P(X)$ be a multivalued operator. By definition T has the limit shadowing property if for any sequence $(y_n)_{n\in\mathbb{N}}$ from X such that $D(y_{n+1}, T(y_n)) \to 0$, as $n \to \infty$, there exists $(x_n)_{n\in\mathbb{N}}$ a sequence of successive approximation of T, such that $d(x_n, y_n) \to 0$, as $n \to \infty$.*

Another main result in [2] concerns with the case of multivalued operators satisfying a Ćirić type condition with respect to the functional δ. An extended version of that result is the following theorem.

Theorem 3.4. *Let (X, d) be a complete metric space and G be a directed graph such that the triple (X, d, G) satisfies the following property:*

(P) for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$ $x_n \to x$ as $n \to \infty$, there exists a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(x_{k_n}, x) \in E(G)$.

Let $T : X \to P_b(X)$ be a multivalued operator. Suppose the following assertions hold:

(i) *there exists $a \in [0, 1[$ such that*

$$\delta(T(x), T(y)) \leq a \cdot max\{d(x, y), \delta(x, T(x)), \delta(y, T(y)), D(x, T(y)), D(y, T(x))\},$$

for all $(x, y) \in E(G)$.

(ii) *there exists $x_0 \in X$ such that, for all $y \in T(x_0)$ we have $(x_0, y) \in E(G)$;*

(iii) *$E(G) \in I(T \times T)$;*

(iv) *if $(x, y) \in E(G)$ and $(y, z) \in E(G)$, then $(x, z) \in E(G)$.*

In these conditions we have:

(a) *$Fix(T) = SFix(T) \neq \emptyset$.*

(b) *If, in addition, the following implication holds*

$$x^*, y^* \in Fix(T) \Rightarrow (x^*, y^*) \in E(G),$$

then $Fix(T) = SFix(T) = \{x^*\}$. Moreover, there exists a selection $t : X \to X$ of T satisfying the condition (2.1) on $E(G)$, such that the sequence x^* is a fixed point for t and $(t^n(x_0))_{n\in\mathbb{N}}$ converges to x^* as $n \to +\infty$.

(c) *If T has closed graphic and if, for any sequence $(x_n)_{n\in\mathbb{N}}$ in X for which*

$$H(x_n, T(x_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$(x_n, x^*) \in E(G) \text{ for all } n \in \mathbb{N},$$

then the fixed point problem is well-posed for T with respect to H.

(d) *If* $a < \frac{1}{3}$ *and if, for all sequences* $(y_n)_{n\in\mathbb{N}}$ *in* X *for which*

$$D\left(y_{n+1}, T\left(y_n\right)\right) \to 0 \text{ as } n \to \infty,$$

it follows that

$$(y_n, x^*) \in E(G) \text{ for all } n \in \mathbb{N},$$

then T *has the limit shadowing property.*

A data dependence result for the fixed point of a multivalued operator satisfying a Ćirić type condition with respect to the functional δ is the following.

Theorem 3.5. *Let* (X, d) *be a complete metric space and* G *be a directed graph such that the triple* (X, d, G) *satisfies the following property:*
(P) for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$ *with* $x_n \to x$ *as* $n \to \infty$*, there exists a subsequence* $(x_{k_n})_{n\in\mathbb{N}}$ *of* $(x_n)_{n\in\mathbb{N}}$ *such that* $(x_{k_n}, x) \in E(G)$*.*
Let $T, S : X \to P_b(X)$ *be two multivalued operators. Suppose the following assertions hold:*

(i) *there exists* $a \in [0, 1[$ *such that*

$$\delta(T(x), T(y)) \le a \cdot max\{d(x, y), \delta(x, T(x)), \delta(y, T(y)), D(x, T(y)), D(y, T(x))\},$$

for all $(x, y) \in E(G)$*.*

(ii) *there exists* $x_0 \in X$ *such that, for all* $y \in T(x_0)$ *we have* $(x_0, y) \in E(G)$*;*

(iii) $E(G) \in I(T \times T)$*;*

(iv) *if* $(x, y) \in E(G)$ *and* $(y, z) \in E(G)$*, then* $(x, z) \in E(G)$*.*

(v) *if* $x^*, y^* \in Fix(T)$ *then* $(x^*, y^*) \in E(G)$*.*

(vi) $Fix(S) \ne \emptyset$*.*

(vii) *if* $x^* \in Fix(T)$*, then* $(x^*, y) \in E(G)$*, for each* $y \in Fix(S)$*.*

(viii) *there exists* $\eta > 0$ *such that* $\delta(T(x), S(x)) \le \eta$*, for all* $x \in X$*.*

Then

$$\delta(x^*, Fix(S)) \le \frac{\eta}{1-a},$$

where x^* *is the unique fixed point of* T*.*

Proof. By Theorem 3.4 the operator T has a unique fixed point, i.e.,

$$Fix(T) = SFix(T) = \{x^*\}.$$

Let $y \in Fix(S)$ be arbitrary. Denote by t the selection of T which exists as in the text of Theorem 3.4. Then, we have

$$d(x^*, y) \le d(x^*, t(y)) + d(t(y), y)) \le d(t(x^*), t(y)) + \delta(T(y), S(y))$$
$$\le a \cdot \max\{d(x^*, y), d(x^*, t(x^*)), d(y, t(y)), d(x^*, t(y)), d(y, t(x^*))\} + \eta$$
$$= a \cdot \max\{d(x^*, y), d(y, t(y)), d(x^*, t(y))\} + \eta$$

$\le a \cdot \max\{d(x^*, y), \delta(S(y), T(y)), d(x^*, t(y))\} + \eta \le a \cdot \max\{d(x^*, y), \eta, d(x^*, t(y))\} + \eta.$
We have the following cases:

1) If the above maximum is $d(x^*, y)$, then we obtain that

$$d(x^*, y) \le \frac{\eta}{1-a}.$$

2) If the above maximum is η, then we obtain that

$$d(x^*, y) \leq \eta(1 + a).$$

3) If the above maximum is $d(x^*, t(y))$, then

$$d(x^*, y) \leq ad(x^*, t(y)) + \eta = ad(t(x^*), t(y)) + \eta \leq ad(x^*, y) + \eta.$$

Hence

$$d(x^*, y) \leq \frac{\eta}{1 - a}.$$

As a conclusion, from the above cases we get that

$$\delta(x^*, Fix(S)) \leq \frac{\eta}{1 - a}.$$ □

For other results in the context of metric spaces endowed with a graph or the case of ordered metric spaces we refer to [2], [3], [8], [9], [13], [12], [16], [19], etc.

4. A fractal operator theory for Ćirić-type operators

We will present now an existence and uniqueness result for the multivalued fractal operator generated by a multivalued operator of Ćirić type.

Let (X, d) be a metric space and $F : X \to P(X)$ be a multivalued operator. The multi-fractal operator generated by F is denoted by $\hat{F} : P_{cp}(X) \to P_{cp}(X)$ and is defined by $Y \mapsto F(Y)$

$$F(Y) := \bigcup_{x \in Y} F(x), \text{ for each } Y \in P_{cp}(X)$$

A fixed point for $\hat{F}$ is a fixed set for F, i.e., a nonempty compact set A^* with the property $\hat{F}(A^*) = A^*$.

Concerning the above problem, we have the following result.

Theorem 4.1. *Let (X, d) be a complete metric space and let $F : X \to P_{cl}(X)$ be an upper semicontinuous multivalued operator. Suppose that there exists a continuous and increasing (in each variable) function $\varphi : \mathbb{R}_+^3 \to \mathbb{R}_+$ such that the function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$\Psi(t) := \varphi(t, t, t)$$

satisisfies the following properties:

(i) $\Psi^n(t) \to 0$ as $n \to +\infty$, for all $t > 0$;

(ii) $t - \Psi(t) \to +\infty$ as $t \to +\infty$.

Suppose also that

$$H(F(x), F(y)) \leq \varphi(d(x, y), D(x, F(y)), D(y, F(x))), \text{ for all } x, y \in X.$$

Then the multi-fractal $\hat{F} : P_{cp}(X) \to P_{cp}(X)$ generated by F has a unique fixed point, i.e., there exists a unique $A^* \in P_{cp}(X)$ such that

$$\hat{F}(A^*) = A^*.$$

Proof. We will prove that $\hat{F}$ satisfies the assumptions of Rus' Theorem for singlevalued operators, i.e.,

$$H(\hat{F}(A), \hat{F}(B)) \leq \varphi(H(A,B), H(A,F(B)), H(B,F(A))), \text{ for all } A, B \in P_{cp}(X).$$

Indeed we have:

$$\rho(F(A),F(B)) = \sup_{a\in A} \rho(F(a),F(B)) = \sup_{a\in A}(\inf_{b\in B} \rho(F(a),F(b))) \leq$$

$$\leq \sup_{a\in A}(\inf_{b\in B} H(F(a),F(b))) \leq \sup_{a\in A}(\inf_{b\in B}(\varphi(d(a,b),D(a,F(b)),D(b,F(a)))))$$

$$\leq \sup_{a\in A} \varphi(\inf_{b\in B} d(a,b), \inf_{b\in B} D(a,F(b)), \inf_{b\in B} D(b,F(a)))$$

$$= \sup_{a\in A} \varphi(D(a,B), D(a,F(B)), D(F(a),B))$$

$$= \varphi(\sup_{a\in A} D(a,B), \sup_{a\in A} D(a,F(B)), \sup_{a\in A} D(F(a),B))$$

$$= \varphi(\rho(A,B), \rho(A,F(B)), \rho(F(A),B)) \leq \varphi(H(A,B),H(A,F(B)),H(B,F(A))).$$

By the above inequality and the similar one for $\rho(F(A),F(B))$, we obtain that

$$H(F(A),F(B)) \leq \varphi(H(A,B),H(A,F(B)),H(B,F(A))).$$

As a consequence, by Rus' theorem applied for $\hat{F}$, we get that $\hat{F}$ has a unique fixed point in $P_{cp}(X)$, i.e., there exists a unique $A^* \in P_{cp}(X)$ such that $\hat{F}(A^*) = A^*$. □

Moreover, if (X,d) is a metric space and $F_1,...,F_m : X \to P(X)$ are multivalued operators, then the system $F = (F_1,...,F_m)$ is called an iterated multifunction system (IMS).

If the system $F = (F_1,...,F_m)$ is such that, for each $i \in \{1,2,\cdots m\}$, the multivalued operators $F_i : X \to P_{cp}(X)$ are upper semicontinuos, then the operator T_F defined as

$$T_F(Y) = \bigcup_{i=1}^{m} F_i(Y), \text{ for each } Y \in P_{cp}(X)$$

has the property that $T_F : P_{cp}(X) \to P_{cp}(X)$ and it is called the multi-fractal operator generated by the IMS $F = (F_1,...,F_m)$.

A nonempty compact subset $A^* \subset X$ is said to be a multivalued fractals with respect to the iterated multifunction system $F = (F_1,...,F_m)$ if and only if it is a fixed point for the associated multifractal operator, i.e., $T_F(A^*) = A^*$.

In particular, if F_i are singlevalued continuous operators from X to X, then $f = (f_1,...,f_m)$ is called an iterated function system (briefly IFS) and the operator $T_f : P_{cp}(X) \to P_{cp}(X)$ given by

$$T_f(Y) = \bigcup_{i=1}^{m} f_i(Y), \text{ for each } Y \in P_{cp}(X)$$

is called the fractal operator generated by the IFS f. A fixed point of T_f is called a fractal generated by the IFS f.

An existence and uniqueness result for the multivalued fractal is the following.

Theorem 4.2. *Let (X,d) be a complete metric space and let $F_i : X \to P_{cl}(X)$ $(i \in \{1,2,\cdots,m\})$ be upper semicontinuous multivalued operators. Suppose that there exists continuous and increasing (in each variable) functions $\varphi_i : \mathbb{R}_+^3 \to \mathbb{R}_+$ $(i \in \{1,2,\cdots,m\})$ such that the functions $\Psi_i : \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$\Psi_i(t) := \varphi_i(t,t,t), (i \in \{1,2,\cdots,m\})$$

satisfy, for each $i \in \{1,2,\cdots,m\}$, the following properties:

(i) $\Psi_i^n(t) \to 0$ as $n \to +\infty$, for all $t > 0$;

(ii) $t - \Psi_i(t) \to +\infty$ as $t \to +\infty$.

Suppose also that, for each $i \in \{1,2,\cdots,m\}$, we have that

$$H(F_i(x), F_i(y)) \leq \varphi_i(d(x,y), D(x,F(y)), D(y,F(x))), \text{ for all } x,y \in X.$$

Then the multi-fractal $T_F : P_{cp}(X) \to P_{cp}(X)$ generated by IMS $F := (F_1, ..., F_m)$ has a unique fixed point, i.e., there exists a unique $A^* \in P_{cp}(X)$ such that $T_F(A^*) = A^*$.

Proof. For $A, B \in P_{cp}(X)$ and using the proof of the previous theorem, we have

$$\begin{aligned} H(T_F(A), T_F(B)) &= H(\bigcup_{i=1}^m F_i(A), \bigcup_{i=1}^m F_i(B)) \leq \max_{i\in\{1,2,\cdots,m\}} H(F_i(A), F_i(B)) \\ &\leq \max_{i\in\{1,2,\cdots,m\}} \varphi_i(H(A,B), H(A,F_i(B)), H(B,F_i(A))) \\ &\leq \max_{i\in\{1,2,\cdots,m\}} \varphi_i(H(A,B), H(T_F(B),A), H(T_F(A),B)) \\ &= \bar{\varphi}(H(A,B), H(T_F(B),A), H(T_F(A),B)), \end{aligned}$$

where $\bar{\varphi}(t_1,t_2,t_3) := \max_{i\in\{1,2,\cdots,m\}} \varphi_i(t_1,t_2,t_3)$. The conclusion follows again by Rus's Theorem applied for T_F. □

It is an open question to prove a similar result to Theorem 4.1 or Theorem 4.2 for the multifractal operator T_F generated by an IMS $F = (F_1, ..., F_m)$ of multivalued operators of Ćirić type.

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Iterative regularization methods for ill-posed Hammerstein-type operator equations in Hilbert scales

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Abstract. In this paper we report on a method for regularizing a nonlinear Hammerstein type operator equation in Hilbert scales. The proposed method is a combination of Lavrentieve regularization method and a Modified Newton's method in Hilbert scales . Under the assumptions that the operator F is continuously differentiable with a Lipschitz-continuous first derivative and that the solution of (1.1) fulfills a general source condition, we give an optimal order convergence rate result with respect to the general source function.

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1. Introduction

Let X and Y be Hilbert spaces. In this study we are concerned with the problem of approximately solving the operator equation

$$KF(x) = y, \tag{1.1}$$

where $K : X \to Y$ is a bounded linear operator with its range $R(K)$ not closed in Y and $F : D(F) \subseteq X \to X$ is a nonlinear monotone operator (i.e., $\langle F(u)-F(v), u-v\rangle \geq 0,\ \forall u, v \in D$). We shall use the notations $\langle ., .\rangle_X,\ \langle ., .\rangle_Y$ and $\|.\|_X,\ \|.\|_Y$ for the inner product and the corresponding norm in the Hilbert spaces X, Y, respectively. The equation (1.1) is, in general, ill-posed, in the sense that a unique solution that depends continuously on the data does not exist.

A typical example of a Hammerstein type operator is the nonlinear integral operator

$$(KF(x))(t) := \int_0^1 k(s,t)f(s,x(s))ds$$

where $k(s,t) \in L^2([0,1] \times [0,1])$, $x \in L^2[0,1]$ and $t \in [0,1]$. Here $K : L^2[0,1] \to L^2[0,1]$ is a linear integral operator with kernel $k(t,s)$: defined as

$$Kx(t) = \int_0^1 k(t,s)x(s)ds$$

and $F : D(F) \subseteq L^2[0,1] \to L^2[0,1]$ is a nonlinear superposition operator (cf. [16]) defined as

$$Fx(s) = f(s, x(s)). \tag{1.2}$$

In [14], George and Nair studied a Modified NLR method for obtaining an approximation for the x_0-minimum norm solution (x_0-MNS) of the equation (1.1). Recall that a solution $\hat{x} \in D(F)$ of (1.1) is called an x_0-MNS of (1.1), if

$$\|F(\hat{x}) - F(x_0)\|_X = \min\{\|F(x) - F(x_0)\|_X : AF(x) = y,\ x \in D(F)\}. \tag{1.3}$$

In the following, we always assume the existence of an x_0-MNS for exact data y, i.e.,

$$KF(\hat{x}) = y.$$

Note that, due to the nonlinearity of F, the above solution need not be unique. The element $x_0 \in X$ in (1.3) plays the role of a selection criterion.

Further we assume throughout that X is a real Hilbert space, $y^\delta \in Y$ are the available noisy data with

$$\|y - y^\delta\|_Y \leq \delta \tag{1.4}$$

and $\|F'(x)\|_{X \to X} \leq M$ for all $x \in D$.

Since (1.1) is ill-posed, regularization methods are to be employed for obtaining a stable approximate solution for (1.1). See, for example [18], [24], [7], [9], [10] for various regularization methods for ill-posed operator equations.

In [6], we considered the sequence $\{x^\delta_{n,\alpha_k}\}$ defined iteratively by

$$x^\delta_{n+1,\alpha_k} = x^\delta_{n,\alpha_k} - R_\beta(x_0)^{-1}[F(x^\delta_{n,\alpha_k}) - z^\delta_{\alpha_k} + \alpha_k(x^\delta_{n,\alpha_k} - x_0)] \tag{1.5}$$

where $x^\delta_{0,\alpha_k} := x_0$ is an initial guess and $R_\beta(x_0) := F'(x_0) + \beta I$, with $\beta > \alpha_k$ for obtaining an approximation of $\hat{x}$. Here $z^\delta_{\alpha_k} = (K^*K + \alpha_k I)^{-1}K^*(y^\delta - KF(x_0)) + F(x_0)$ and α_k is the regularization parameter chosen appropriately depending on the inexact data y^δ and the error level δ satisfying (1.4). For this we used the adaptive parameter selection procedure suggested by Pereverzev and Schock [20]. In order to improve the error estimate available in [14], in this paper we consider the Hilbert scale variant of (1.5).

Let $L : D(L) \subset X \to X$, be a linear, unbounded, self-adjoint, densely defined and strictly positive operator on X. We consider the Hilbert scale $(X_r)_{r \in \Re}$ (see [12], [13], [17] and [18]) generated by L for our analysis. Recall (c.f.[12])that the space X_t is the completion of $D := \cap_{k=0}^\infty D(L^k)$ with respect to the norm $\|x\|_t$, induced by the inner product

$$\langle u, v \rangle_t := \langle L^t u, L^t v \rangle, \qquad u, v \in D. \tag{1.6}$$

Moreover, if $\beta \leq \gamma$, then the embedding $X_\gamma \hookrightarrow X_\beta$ is continuous, and therefore the norm $\|.\|_\beta$ is also defined in X_γ and there is a constant $c_{\beta,\gamma}$ such that

$$\|x\|_\beta \leq c_{\beta,\gamma}\|x\|_\gamma,\ x \in X_\gamma.$$

In this paper we consider the sequence $\{x^{\delta}_{n,\alpha_k}\}$ in order to obtain stable approximate solution to (1.1), defined iteratively by

$$x^{\delta}_{n+1,\alpha_k,s} = x^{\delta}_{n,\alpha_k,s} - R_{\beta}(x_0)^{-1}[F(x^{\delta}_{n,\alpha_k,s}) - z^{\delta}_{\alpha_k,s} + \alpha_k L^{s/2}(x^{\delta}_{n,\alpha_k,s} - x_0)], \quad (1.7)$$

where $x^{\delta}_{0,\alpha_k,s} := x_0$ is an initial guess and $R_{\beta}(x_0) := F'(x_0) + \beta L^{s/2}$, with $\beta > \alpha_k$ for obtaining an approximation for $\hat{x}$. Here $z^{\delta}_{\alpha_k,s}$ be as in (2.2) with $\alpha = \alpha_k$ and α_k is the regularization parameter chosen appropriately depending on the inexact data y^{δ} and the error level δ satisfying (1.4). For this we use the adaptive parameter selection procedure suggested by Pereverzev and Schock [20].

This paper is organized as follows. Preparatory results are given in section 2 and section 3 comprises the proposed iterative method. Numerical examples are given in section 4. Finally the paper ends with a conclusion in section 5.

2. Preliminaries

We assume that the ill-posed nature of the operator K is related to the Hilbert scale $\{X_t\}_{t\in\mathbb{R}}$ according to the relation

$$c_1\|x\|_{-a} \le \|Kx\|_Y \le c_2\|x\|_{-a}, \qquad x \in X,$$

for some real numbers a, c_1, and c_2.

Observe that from the relation $\langle Kx, y\rangle_Y = \langle x, K^*y\rangle_X = \langle x, L^{-s}K^*y\rangle_s$ for all $x \in X$ and $y \in Y$, we conclude that $L^{-s}K^* : Y \to X$ is the adjoint of the operator K in X. Consequently $L^{-s}K^*K : X \to X$ is self-adjoint. Further we note that

$$(A_s^*A_s + \alpha I)^{-1}L^{s/2} = L^{s/2}(L^{-s}K^*K + \alpha I)^{-1}$$

where $A_s = KL^{-s/2}$.

One of the crucial results for proving the results in this paper is the following proposition, where f and g are defined by

$$f(t) = \min\{c_1^t, c_2^t\}, \qquad g(t) = max\{c_1^t, c_2^t\}, \qquad t \in \mathbb{R}, |t| \le 1.$$

Proposition 2.1. *(See* [23]*, Proposition 2.1) For $s \ge 0$ and $|\nu| \le 1$,*

$$f(\nu)\|x\|_{-\nu(s+a)} \le \|(A_s^*A_s)^{\nu/2}x\|_X \le g(\nu)\|x\|_{-\nu(s+a)}, \quad x \in H.$$

We make use of the relation

$$\|(A_s + \alpha I)^{-1}A_s^p\|_X \le \alpha^{p-1}, \;\; p > 0, \;\; 0 < p \le 1, \qquad (2.1)$$

which follows from the spectral properties of the positive self-adjoint operator A_s, $s > 0$.

In this section we consider Tikhonov regularized solution $z^{\delta}_{\alpha,s}$ defined by

$$z^{\delta}_{\alpha,s} = (L^{-s}K^*K + \alpha I)^{-1}L^{-s}K^*(y^{\delta} - KF(x_0)) + F(x_0) \qquad (2.2)$$

and obtain an a priori and an a posteriori error estimate for $\|F(\hat{x}) - z^{\delta}_{\alpha,s}\|_X$. The following assumption on source condition is based on a source function φ and a property of the source function φ. We will be using this assumption to obtain an error estimate for $\|F(\hat{x}) - z^{\delta}_{\alpha,s}\|_X$.

Assumption 2.2. *There exists a continuous, strictly monotonically increasing function* $\varphi : (0, \|A_s^* A_s\|] \to (0, \infty)$ *such that the following conditions hold:*

- $\lim_{\lambda \to 0} \varphi(\lambda) = 0,$
- $\sup_{\lambda > 0} \frac{\alpha \varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha), \ \forall \lambda \in (0, \|A_s^* A_s\|]$ *and*
- *there exists* $v \in X$ *with* $\|v\| \leq \overline{E}, \ \overline{E} > 0$ *such that*

$$(A_s^* A_s)^{\frac{s}{2(s+a)}} L^{s/2}(F(\hat{x}) - F(x_0)) = \varphi(A_s^* A_s)v.$$

Remark 2.3. Note that if $F(\hat{x}) - F(x_0) \in X_t$ i.e., $\|F(\hat{x}) - F(x_0)\|_t \leq E$, for some $0 < t \leq 2s + a$, then the above assumption is satisfied. This can be seen as follows.

$$\begin{aligned}(A_s^* A_s)^{\frac{s}{2(s+a)}} L^{s/2}(F(\hat{x}) - F(x_0)) &= (A_s^* A_s)^{\frac{t}{2(s+a)}} (A_s^* A_s)^{\frac{(s-t)}{(2s+2a)}} L^{s/2}(F(\hat{x}) - F(x_0)), \\ &= \varphi(A_s^* A_s)v\end{aligned}$$

where $\varphi(\lambda) = \lambda^{\frac{t}{2(s+a)}}$ and $v = (A_s^* A_s)^{\frac{(s-t)}{(2s+2a)}} L^{s/2}(F(\hat{x}) - F(x_0))$.

Further note that

$$\begin{aligned}\|v\|_X &\leq g(\frac{s-t}{s+a})\|L^{s/2}(F(\hat{x}) - F(x_0))\|_{t-s} \\ &\leq g(\frac{s-t}{s+a})\|(F(\hat{x}) - F(x_0))\|_t \\ &\leq \overline{E}\end{aligned}$$

where $\overline{E} = g(\frac{s-t}{s+a})E$.

Theorem 2.4. ([22, Theorem 2.4]) *Suppose that Assumption* 2.2 *holds and let* $z_{\alpha,s} := z_{\alpha,s}^0$. *Then*

1.
$$\|z_{\alpha,s}^\delta - z_{\alpha,s}\|_X \leq \psi(s)\alpha^{\frac{-a}{2(s+a)}}\delta, \tag{2.3}$$

2.
$$\|F(\hat{x}) - z_{\alpha,s}\|_X \leq \phi(s)\varphi(\alpha), \tag{2.4}$$

3.
$$\|F(x_0) - z_{\alpha,s}\|_X \leq \psi_1(s)\|F(\hat{x}) - F(x_0)\|_X, \tag{2.5}$$

where $\psi(s) = \frac{1}{f(\frac{s}{s+a})}, \quad \phi(s) = \frac{\overline{E}}{f(\frac{s}{s+a})}$ *and* $\psi_1(s) = \frac{g(\frac{s}{s+a})}{f(\frac{s}{s+a})}$.

2.1. Error bounds and parameter choice in Hilbert scales

Let $C_s = \max\{\phi(s), \psi(s)\}$, then by (2.3), (2.4) and triangle inequality, we have

$$\|F(\hat{x}) - z_{\alpha,s}^\delta\|_X \leq C_s(\varphi(\alpha) + \alpha^{\frac{-a}{2(s+a)}}\delta). \tag{2.6}$$

The error estimate $\varphi(\alpha) + \alpha^{\frac{-a}{2(s+a)}}\delta$ in (2.6) attains minimum for the choice $\alpha := \alpha(\delta, s, a)$ which satisfies $\varphi(\alpha) = \alpha^{\frac{-a}{2(s+a)}}\delta$. Clearly $\alpha(\delta, s, a) = \varphi^{-1}(\psi_{s,a}^{-1}(\delta))$, where

$$\psi_{s,a}(\lambda) = \lambda[\varphi^{-1}(\lambda)]^{\frac{a}{2(s+a)}}, \qquad 0 < \lambda \leq \|A_s\|^2 \tag{2.7}$$

and in this case

$$\|F(\hat{x}) - z_{\alpha,s}^\delta\|_X \leq 2C_s\psi_{s,a}^{-1}(\delta),$$

which has at least optimal order with respect to δ, s and a (cf. [20]).

2.2. Adaptive scheme and stopping rule

In this paper we consider the adaptive scheme suggested by Pereverzev and Schock in [20] modified suitably, for choosing the parameter α which does not involve even the regularization method in an explicit manner.
Let $i \in \{0, 1, 2, \cdots, N\}$ and $\alpha_i = \mu^i \alpha_0$ where $\mu = \eta^{2(1+s/a)}, \eta > 1$ and $\alpha_0 = \delta^{2(1+s/a)}$.
Let

$$l := max\{i : \varphi(\alpha_i) \leq \alpha_i^{\frac{-a}{2(s+a)}} \delta\} < N \tag{2.8}$$

and

$$k := max\{i : \|z^\delta_{\alpha_{i,s}} - z^\delta_{\alpha_{j,s}}\|_X \leq 4\alpha_j^{\frac{-a}{2(s+a)}} \delta, j = 0, 1, 2, \cdots, i\}. \tag{2.9}$$

Analogous to the proof of Theorem 4.3 in [11], we have the following Theorem.

Theorem 2.5. ([22, Theorem 2.5]) *Let l be as in (2.8), k be as in (2.9), $\psi_{s,a}$ be as in (2.7) and $z^\delta_{\alpha_k,s}$ be as in (2.2) with $\alpha = \alpha_k$. Then $l \leq k$; and*

$$\|F(\hat{x}) - z^\delta_{\alpha_k,s}\|_X \leq C_s(2 + \frac{4\eta}{\eta - 1})\eta\psi^{-1}_{s,a}(\delta)$$

where C_s is as in (2.6).

3. The method and convergence analysis

In the earlier papers [11, 15] the authors used the following Assumption:

Assumption 3.1. (cf. [21], Assumption 3 (A3)) *There exists a constant $K \geq 0$ such that for every $x, u \in D(F)$ and $v \in X$ there exists an element $\Phi(x, u, v) \in X$ such that $[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \|\Phi(x, u, v)\|_X \leq K\|v\|_X\|x - u\|_X$.*

Assumption 3.2. *For each $x \in B_r(x_0)$ there exists a bounded linear operator G such that*

$$F'(x) = F'(x_0)G(x, x_0)$$

with $\|G(x, x_0)\| \leq k$ where k is a constant.

One of the advantages of the proposed method is that we do not need the above assumption.

The hypotheses of Assumption 3.1 may not hold or may be very expensive or impossible to verify in general (see the numerical examples). In particular, as it is the case for well-posed nonlinear equations the computation of the Lipschitz constant K even if this constant exists is very difficult. Moreover, there are classes of operators for which Assumption 3.1 is not satisfied but the iterative method converges.

In the present paper, we expand the applicability of the method in [6] under less computational cost. We achieve this goal by introducing the following weaker Assumption.

Assumption 3.3. *There exists a constant $k_0 \geq 0$ such that for every $x \in D(F)$ and $v \in X$ there exists an element $\Phi(x, x_0, v) \in X$ such that*

$$[F'(x) - F'(x_0)]v = F'(x_0)\Phi(x, x_0, v), \|\Phi(x, x_0, v)\|_X \leq k_0\|v\|_X\|x - x_0\|_X.$$

Note that

$$k_0 \le K$$

holds in general and $\frac{K}{k_0}$ can be arbitrary large (see Example 4.3). The advantages of the new approach are:

(1) Assumption 3.3 is weaker than Assumption 3.1. Notice that there are classes of operators that satisfy Assumption 3.3 but do not satisfy Assumption 3.1 (see the numerical examples);
(2) The computational cost of finding the constant k_0 is less than that of constant K, even when $K = k_0$;
(3) The sufficient convergence criteria are weaker;
(4) The computable error bounds on the distances involved (including k_0) are less costly and more precise than the old ones (including K);
(5) The information on the location of the solution is more precise;

and

(6) The convergence domain of the iterative method is larger.

These advantages are also very important in computational mathematics since they provide under less computational cost a wider choice of initial guesses for iterative method and the computation of fewer iterates to achieve a desired error tolerance. Numerical examples for (1)-(6) are presented in Section 4.

In this section, we consider the method defined as (1.7) with α_k in place of α for approximating the zero $x^\delta_{\alpha_k,s}$ of the equation,

$$F(x) + \alpha_k L^{s/2}(x - x_0) = z^\delta_{\alpha_k,s} \tag{3.1}$$

and then we show that $x^\delta_{\alpha_k,s}$ is an approximation to the solution $\hat{x}$ of (1.1).

Let $F'(x_0) \in L(X)$ be a bounded positive self-adjoint operator on X and $B_s := L^{-s/4}F'(x_0)L^{-s/4}$. Usually, for the analysis of regularization methods in Hilbert scales, an assumption of the form (cf.[8], [19])

$$\|F'(\hat{x})x\|_X \sim \|x\|_{-b}, \;\; x \in X \tag{3.2}$$

on the degree of ill-posedness is used. In this paper instead of (3.2) we require only a weaker assumption;

$$d_1\|x\|_{-b} \le \|F'(x_0)x\|_X \le d_2\|x\|_{-b}, \qquad x \in D(F), \tag{3.3}$$

for some reals b, d_1, and d_2.

Note that (3.3) is simpler than that of (3.2). Next, we define f_1 and g_1 by

$$f_1(t) = \min\{d_1^t, d_2^t\}, \qquad g_1(t) = max\{d_1^t, d_2^t\}, \qquad t \in \mathbb{R}, \; |t| \le 1.$$

One of the crucial result for proving the results in this paper is the following Proposition.

Proposition 3.4. *(See.* [12]*, Proposition 3.1) For* $s > 0$ *and* $|\nu| \le 1$*,*

$$f_1(\nu/2)\|x\|_{\frac{-\nu(s+b)}{2}} \le \|B_s^{\nu/2}x\|_X \le g_1(\nu/2)\|x\|_{\frac{-\nu(s+b)}{2}}, \; x \in H.$$

Let $\psi_2(s) := \frac{g_1(\frac{-s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})}$, $\overline{\psi_2(s)} := \frac{g_1(\frac{s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})}$.

Lemma 3.5. *Let Proposition* 3.4 *hold. Then for all $h \in X$, the following hold:*

(a) $\|(F'(x_0) + \beta L^{s/2})^{-1} F'(x_0) h\|_X \le \overline{\psi_2(s)} \|h\|_X$

(b) $\|(F'(x_0) + \beta L^{s/2})^{-1} L^{s/2} h\|_X \le \frac{\overline{\psi_2(s)}}{\beta} \|h\|_X$

(c) $\|(F'(x_0) + \beta L^{s/2})^{-1} h\|_X \le \psi_2(s) \beta^{\frac{-b}{(s+b)}} \|h\|_X$

Proof. Observe that by Proposition 3.4,

$$\begin{aligned}
\|(F'(x_0) + \beta L^{s/2})^{-1} F'(x_0) h\|_X &= \|L^{-s/4}(L^{-s/4}F'(x_0)L^{-s/4} + \beta I)^{-1} L^{-s/4} \\
&\quad F'(x_0) L^{-s/4} L^{s/4} h\|_X \\
&\le \frac{1}{f_1(\frac{s}{2(s+b)})} \|B_s^{\frac{s}{2(s+b)}} (B_s + \beta I)^{-1} B_s L^{s/4} h\|_X \\
&\le \frac{1}{f_1(\frac{s}{2(s+b)})} \|(B_s + \beta I)^{-1} B_s\| \|B_s^{\frac{s}{2(s+b)}} L^{s/4} h\|_X \\
&\le \frac{g_1(\frac{s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})} \|L^{s/4} h\|_{-s/2} \\
&\le \frac{g_1(\frac{s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})} \|h\|_X.
\end{aligned}$$

This proves (a). To prove (b) and (c) we observe that

$$\begin{aligned}
\|(F'(x_0) + \beta L^{s/2})^{-1} L^{s/2} h\|_X &\le \|L^{-s/4}(L^{-s/4}F'(x_0)L^{-s/4} + \beta I)^{-1} L^{s/4} h\|_X \\
&\le \frac{1}{f_1(\frac{s}{2(s+b)})} \|B_s^{\frac{s}{2(s+b)}} (B_s + \beta I)^{-1} L^{s/4} h\|_X \\
&\le \frac{1}{f_1(\frac{s}{2(s+b)})} \|(B_s + \beta I)^{-1} B_s^{\frac{s}{2(s+b)}} L^{s/4} h\|_X \\
&\le \frac{g_1(\frac{s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})} \beta^{-1} \|h\|_X \\
&\le \overline{\psi_2(s)} \beta^{-1} \|h\|_X \qquad (3.4)
\end{aligned}$$

and

$$\begin{aligned}
\|(F'(x_0) + \beta L^{s/2})^{-1} h\|_X &\le \|L^{-s/4}(L^{-s/4}F'(x_0)L^{-s/4} + \beta I)^{-1} L^{-s/4} h\|_X \\
&\le \frac{1}{f_1(\frac{s}{2(s+b)})} \|B_s^{\frac{s}{2(s+b)}} (B_s + \frac{\alpha_k}{c} I)^{-1} L^{-s/4} h\|_X \\
&\le \frac{1}{f_1(\frac{s}{2(s+b)})} \|(B_s + \beta I)^{-1} B_s^{\frac{s}{(s+b)}} B_s^{\frac{-s}{2(s+b)}} L^{-s/4} h\|_X \\
&\le \frac{g_1(\frac{-s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})} \beta^{\frac{-b}{(s+b)}} \|h\|_X \\
&\le \psi_2(s) \beta^{\frac{-b}{(s+b)}} \|h\|_X. \qquad (3.5)
\end{aligned}$$

□

Let

$$G(x) = x - R_\beta(x_0)^{-1}[F(x) - z^\delta_{\alpha_k,s} + \alpha_k L^{s/2}(x - x_0)]. \tag{3.6}$$

Note that with the above notation $G(x^\delta_{n,\alpha_k,s}) = x^\delta_{n+1,\alpha_k,s}$.

First we prove that $x^\delta_{n,\alpha_k,s}$ converges to the zero $x^\delta_{\alpha_k,s}$ of

$$F(x) + \alpha_k L^{s/2}(x - x_0) = z^\delta_{\alpha_k,s} \tag{3.7}$$

and then we prove that $x^\delta_{\alpha_k,s}$ is an approximation for $\hat{x}$.

Hereafter we assume that $\|\hat{x} - x_0\|_X < \rho$ where

$$\rho < \frac{1}{\psi_1(s)M}\left(\frac{\beta^{\frac{b}{s+b}}[1 - \overline{\psi_2(s)}(\frac{\beta-\alpha_k}{\beta})]^2}{4k_0\overline{\psi_2(s)}^2} - \psi(s)\frac{\delta_0}{\alpha_0^{\frac{a}{2(s+a)}}}\right)$$

with $\delta_0 < \frac{\beta^{\frac{b}{s+b}}[1-\overline{\psi_2(s)}(\frac{\beta-\alpha_k}{\beta})]^2}{4k_0\psi(s)\overline{\psi_2(s)}^2}\alpha_0^{\frac{-a}{2(s+a)}}$. Let

$$\gamma_\rho := \psi_2(s)\beta^{\frac{-b}{(s+b)}}[\psi_1(s)M\rho + \psi(s)\alpha_0^{\frac{-a}{2(s+a)}}\delta_0].$$

and we define

$$q = \overline{\psi_2(s)}[k_0 r + \frac{\beta - \alpha_k}{\beta}], \;\; r \in (r_1, r_2) \tag{3.8}$$

where

$$r_1 = \frac{[1 - \overline{\psi_2(s)}(\frac{\beta-\alpha_k}{\beta})] - \sqrt{[1 - \overline{\psi_2(s)}(\frac{\beta-\alpha_k}{\beta})]^2 - 4k_0\overline{\psi_2(s)}\gamma_\rho}}{2k_0\overline{\psi_2(s)}}$$

and

$$\begin{aligned} r_2 \;\; = \;\; \min\Bigg\{ & \frac{1-(1-c)\psi_2(s)}{k_0\overline{\psi_2(s)}}, \frac{1}{k_0}[\frac{1}{\overline{\psi_2(s)}} - \frac{\beta-\alpha_k}{\beta}], \\ & \frac{[1 - \overline{\psi_2(s)}(\frac{\beta-\alpha_k}{\beta})] + \sqrt{[1 - \overline{\psi_2(s)}(\frac{\beta-\alpha_k}{\beta})]^2 - 4k_0\overline{\psi_2(s)}\gamma_\rho}}{2k_0\overline{\psi_2(s)}}\Bigg\} \end{aligned}$$

where $0 < c < \alpha_k < 1$ is a constant.

Remark 3.6. Note that for $r \in (r_1, r_2)$ we have $q < 1$ and $\gamma_\rho < \frac{\gamma_\rho}{1-q} \leq r$.

Theorem 3.7. *Let $r \in (r_1, r_2)$ and Assumption 3.3 be satisfied. Then the sequence $(x^\delta_{n,\alpha,s})$ defined in (1.7) is well defined and $x^\delta_{n,\alpha,s} \in B_r(x_0)$ for all $n \geq 0$. Further $(x^\delta_{n,\alpha,s})$ is Cauchy sequence in $B_r(x_0)$ and hence converges to $x^\delta_{\alpha_k,s} \in \overline{B_r(x_0)}$ and $F(x^\delta_{\alpha_k,s}) + \alpha_k L^{s/2}(x^\delta_{\alpha_k,s} - x_0) = z^\delta_{\alpha_k}$.*

Moreover, the following estimate holds for all $n \geq 0$,

$$\|x^\delta_{n,\alpha,s} - x^\delta_{\alpha_k,s}\|_X \leq \frac{\gamma_\rho q^n}{1-q}. \tag{3.9}$$

Proof. Let G be as in (3.6). Then for $u, v \in B_r(x_0)$,

$$\begin{aligned} G(u) - G(v) &= u - v - R_\beta(x_0)^{-1}[F(u) - z^\delta_{\alpha_k} + \alpha_k L^{s/2}(u - x_0)] \\ &\quad + R_\beta(x_0)^{-1}[F(v) - z^\delta_{\alpha_k,s} + \alpha_k L^{s/2}(v - x_0)] \\ &= R_\beta(x_0)^{-1}[R_\beta(x_0)(u - v) - (F(u) - F(v))] \\ &\quad + \alpha_k R_\beta(x_0)^{-1} L^{s/2}(v - u) \\ &= R_\beta(x_0)^{-1}[F'(x_0)(u - v) - (F(u) - F(v)) + \beta L^{s/2}(u - v)] \\ &\quad + \alpha_k R_\beta(x_0)^{-1} L^{s/2}(v - u) \\ &= R_\beta(x_0)^{-1}[F'(x_0)(u - v) - (F(u) - F(v)) + (\beta - \alpha_k) L^{s/2}(u - v)] \\ &= R_\beta(x_0)^{-1} \int_0^1 [F'(x_0) - F'(v + t(u - v)] dt (u - v) \\ &\quad + R_\beta(x_0)^{-1}(\beta - \alpha_k) L^{s/2}(u - v)]. \end{aligned}$$

Thus by Assumption 3.3 and Lemma 3.5 we have

$$\|G(u) - G(v)\|_X \le q\|u - v\|_X. \tag{3.10}$$

Now we shall prove that $x^\delta_{n,\alpha_k,s} \in B_r(x_0)$, for all $n \ge 0$. Note that

$$\begin{aligned} \|x^\delta_{1,\alpha_k,s} - x_0\|_X &= \|(F'(x_0) + \beta L^{s/2})^{-1}(F(x_0) - z^\delta_{\alpha_k,s})\|_X \\ &\le \|L^{-s/4}(L^{-s/4}F'(x_0)L^{-s/4} + \beta I)^{-1} L^{-s/4} \\ &\quad (F(x_0) - z^\delta_{\alpha_k,s})\|_X \\ &\le \frac{1}{f_1(\frac{s}{2(s+b)})} \|B_s^{\frac{s}{2(s+b)}}(B_s + \frac{\alpha_k}{c} I)^{-1} L^{-s/4} \\ &\quad (F(x_0) - z^\delta_{\alpha_k,s})\|_X \\ &\le \frac{1}{f_1(\frac{s}{2(s+b)})} \|(B_s + \beta I)^{-1} B_s^{\frac{s}{(s+b)}} B_s^{\frac{-s}{2(s+b)}} \\ &\quad L^{-s/4}(F(x_0) - z^\delta_{\alpha_k,s})\|_X \\ &\le \frac{g_1(\frac{-s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})} \beta^{\frac{-b}{(s+b)}} \|F(x_0) - z^\delta_{\alpha_k,s}\|_X \\ &\le \psi_2(s) \beta^{\frac{-b}{(s+b)}} [\|F(x_0) - z_{\alpha_k,s}\|_X \\ &\quad + \|z_{\alpha_k,s} - z^\delta_{\alpha_k,s}\|_X] \end{aligned} \tag{3.11}$$

Now using (2.3) and (2.5) in (3.5), one can see that

$$\begin{aligned} \|x^\delta_{1,\alpha_k,s} - x_0\|_X &\le \psi_2(s) \beta^{\frac{-b}{(s+b)}} [\psi_1(s)\|F(\hat{x}) - F(x_0)\|_X + \psi(s) \alpha^{\frac{-a}{2(s+a)}} \delta] \\ &\le \psi_2(s) \beta^{\frac{-b}{(s+b)}} [\psi_1(s) M\rho + \psi(s) \alpha_0^{\frac{-a}{2(s+a)}} \delta_0] = \gamma_\rho. \end{aligned}$$

Assume that $x^\delta_{k,\alpha_k,s} \in B_r(x_0)$, for some k. Then

$$\begin{aligned}\|x^\delta_{k+1,\alpha_k,s} - x_0\|_X &= \|x^\delta_{k+1,\alpha_k,s} - x^\delta_{k,\alpha_k,s} + x^\delta_{k,\alpha_k,s} - x^\delta_{k-1,\alpha_k,s} \\ &\quad + \cdots + x^\delta_{1,\alpha_k,s} - x_0\|_X \\ &\leq \|x^\delta_{k+1,\alpha_k,s} - x^\delta_{k,\alpha_k,s}\|_X + \|x^\delta_{k,\alpha_k,s} - x^\delta_{k-1,\alpha_k,s}\|_X \\ &\quad + \cdots + \|x^\delta_{1,\alpha_k,s} - x_0\|_X \\ &\leq (q^k + q^{k-1} + \cdots + 1)\gamma_\rho \\ &\leq \frac{\gamma_\rho}{1-q} \leq r.\end{aligned}$$

So $x^\delta_{k+1,\alpha_k,s} \in B_r(x_0)$ and hence, by induction $x^\delta_{n,\alpha_k,s} \in B_r(x_0), \quad \forall n \geq 0$. Next we shall prove that $(x^\delta_{k+1,\alpha_k,s})$ is a Cauchy sequence in $B_r(x_0)$.

$$\|x^\delta_{n+m,\alpha_k,s} - x^\delta_{n,\alpha_k,s}\|_X \leq \sum_{i=0}^{m} \|x^\delta_{n+i+1,\alpha_k,s} - x^\delta_{n+i,\alpha_k,s}\|_X \tag{3.12}$$

$$\leq \sum_{i=0}^{m} q^{n+i}\gamma_\rho$$

$$\leq \frac{q^n}{1-q}\gamma_\rho. \tag{3.13}$$

Thus $(x^\delta_{n,\alpha_k,s})$ is a Cauchy sequence in $B_r(x_0)$ and hence converges to some $x^\delta_{\alpha_k,s} \in \overline{B_r(x_0)}$. Now by $n \to \infty$ in (1.7) we obtain $F(x^\delta_{\alpha_k,s}) + \alpha_k L^{s/2}(x^\delta_{\alpha_k,s} - x_0) = z^\delta_{\alpha_k,s}$. This completes the proof of the Theorem.

In addition to the Assumption 2.2, we use the following assumption to obtain the error estimate for $\|\hat{x} - x^\delta_{\alpha_k,s}\|$.

Assumption 3.8. *There exists a continuous, strictly monotonically increasing function* $\varphi_1 : (0, \|B_s\|] \to (0, \infty)$ *such that the following conditions hold:*

- $\lim_{\lambda \to 0} \varphi_1(\lambda) = 0$,
- $\sup_{\lambda > 0} \frac{\alpha\varphi_1(\lambda)}{\lambda + \alpha} \leq \varphi_1(\alpha)\ \forall \lambda \in (0, \|B_s\|]$ *and*
- *there exists* $w \in X$ *with* $\|w\|_X \leq E_2$*, such that*

$$B_s^{\frac{s}{2(s+b)}} L^{s/4}(x_0 - \hat{x}) = \varphi_1(B_s)w$$

Remark 3.9. If $x_0 - \hat{x} \in X_{t_1}$ i.e., $\|x_0 - \hat{x}\|_{t_1} \leq E_1$ for some positive constant E_1 and $0 \leq t_1 \leq s+b$. Then as in Remark 2.3, we have $B_s^{\frac{s}{2(s+b)}} L^{s/4}(x_0 - \hat{x}) = \varphi_1(B_s)w$ where $\varphi_1(\lambda) = \lambda^{t_1/(s+b)}$, $w = B_s^{\frac{s-2t_1}{2(s+b)}} L^{s/4}(\hat{x} - x_0)$ and $\|w\| \leq g_1(\frac{s-2t_1}{2(s+b)})E_1 := E_2$.

Hereafter we assume that $\varphi_1(\alpha_k) \leq \varphi(\alpha_k)$.

Theorem 3.10. *Suppose* $x^\delta_{\alpha_k,s}$ *is the solution of* (3.1) *and Assumptions* 3.3 *and* 3.8 *hold. Then*

$$\|\hat{x} - x^\delta_{\alpha_k,s}\|_X = O(\psi^{-1}(\delta)).$$

Proof. Note that $(F(x^\delta_{\alpha_k,s}) - z^\delta_{\alpha_k,s}) + \alpha_k L^{s/2}(x^\delta_{\alpha_k,s} - x_0) = 0$, so

$$(F'(x_0) + \frac{\alpha_k}{c}L^{s/2})(x^\delta_{\alpha_k,s} - \hat{x}) = (F'(x_0) + \frac{\alpha_k}{c}L^{s/2})(x^\delta_{\alpha_k,s} - \hat{x})$$

$$-(F(x^\delta_{\alpha_k,s}) - z^\delta_{\alpha_k,s}) - \alpha_k L^{s/2}(x^\delta_{\alpha_k,s} - x_0)$$

$$= (\frac{\alpha_k}{c} - \alpha_k)L^{s/2}(x^\delta_{\alpha_k,s} - \hat{x}) + \alpha_k L^{s/2}(x_0 - \hat{x})$$

$$+F'(x_0)(x^\delta_{\alpha_k,s} - \hat{x}) - [F(x^\delta_{\alpha_k,s}) - z^\delta_{\alpha_k,s}]$$

$$= (\frac{\alpha_k}{c} - \alpha_k)L^{s/2}(x^\delta_{\alpha_k,s} - \hat{x}) + \alpha_k L^{s/2}(x_0 - \hat{x})$$

$$+F'(x_0)(x^\delta_{\alpha_k,s} - \hat{x}) - [F(x^\delta_{\alpha_k,s}) - F(\hat{x}) + F(\hat{x}) - z^\delta_{\alpha_k,s}]$$

$$= (\frac{\alpha_k}{c} - \alpha_k)L^{s/2}(x^\delta_{\alpha_k,s} - \hat{x}) + \alpha_k L^{s/2}(x_0 - \hat{x}) - (F(\hat{x}) - z^\delta_{\alpha_k,s})$$

$$+F'(x_0)(x^\delta_{\alpha_k,s} - \hat{x}) - [F(x^\delta_{\alpha_k,s}) - F(\hat{x})].$$

Thus

$$\|x^\delta_{\alpha_k,s} - \hat{x}\|_X \le \|(\frac{\alpha_k}{c} - \alpha_k)(F'(x_0 + \frac{\alpha_k}{c}L^{s/2})^{-1}L^{s/2}(x^\delta_{\alpha_k,s} - \hat{x})\|_X$$

$$+\|\alpha_k(F'(x_0 + \frac{\alpha_k}{c}L^{s/2})^{-1}L^{s/2}(x_0 - \hat{x})\|_X + \|(F'(x_0) + \frac{\alpha_k}{c}L^{s/2})^{-1}$$

$$(F(\hat{x}) - z^\delta_{\alpha_k,s})\|_X + \|(F'(x_0) + \frac{\alpha_k}{c}L^{s/2})^{-1}[F'(x_0)(x^\delta_{\alpha_k,s} - \hat{x})$$

$$-(F(x^\delta_{\alpha_k,s}) - F(\hat{x}))]\|_X$$

$$\le \|(\frac{\alpha_k}{c} - \alpha_k)(F'(x_0 + \frac{\alpha_k}{c}L^{s/2})^{-1}L^{s/2}(x^\delta_{\alpha_k,s} - \hat{x})\|_X$$

$$+\|\alpha_k(F'(x_0) + \frac{\alpha_k}{c}L^{s/2})^{-1}L^{s/2}(x_0 - \hat{x})\|_X$$

$$+\psi_2(s)(\frac{\alpha_k}{c})^{-1}\|F(\hat{x}) - z^\delta_{\alpha_k,k}\|_X + \Gamma \tag{3.14}$$

where $\Gamma := \|(F'(x_0) + \frac{\alpha_k}{c}L^{s/2})^{-1}\int_0^1[F'(x_0) - F'(\hat{x} + t(x^\delta_{\alpha_k,s} - \hat{x})](x^\delta_{\alpha_k,s} - \hat{x})dt\|_X$. Note that

$$\begin{aligned}
&\|(\frac{\alpha_k}{c} - \alpha_k)(F'(x_0 + \frac{\alpha_k}{c}L^{s/2})^{-1}L^{s/2}(x^\delta_{\alpha_k,s} - \hat{x})\|_X \\
\le\ & \frac{\frac{\alpha_k}{c} - \alpha_k}{\frac{\alpha_k}{c}}\overline{\psi_2(s)}\|x^\delta_{\alpha_k,s} - \hat{x}\|_X \\
\le\ & (1-c)\overline{\psi_2(s)}\|x^\delta_{\alpha_k,s} - \hat{x}\|_X,
\end{aligned} \tag{3.15}$$

and by Assumption 3.8, we obtain

$$\begin{aligned}
& \|\alpha_k(F'(x_0) + \frac{\alpha_k}{c}L^{s/2})^{-1}L^{s/2}(x_0 - \hat{x})\|_X \\
= & \|\alpha_k L^{-s/4}(B_s + \frac{\alpha_k}{c})^{-1}L^{s/4}(x_0 - \hat{x})\|_X \\
\leq & \frac{1}{f_1(\frac{s}{2(s+b)})}\|\alpha_k(B_s + \frac{\alpha_k}{c})^{-1}B_s^{\frac{s}{2(s+b)}}L^{s/4}(x_0 - \hat{x})\|_X \\
\leq & \frac{1}{f_1(\frac{s}{2(s+b)})} \sup_{\lambda\in\sigma(F'(x_0))} \frac{\alpha_k\varphi_1(\lambda)}{\lambda + \frac{\alpha_k}{c}} \\
\leq & \sup_{\lambda\in\sigma(F'(x_0))} \frac{\alpha_k\varphi_1(\lambda)}{\lambda + \alpha_k} \\
\leq & \varphi_1(\alpha_k)
\end{aligned} \tag{3.16}$$

and by Assumption 3.3, and Lemma 3.5 we obtain

$$\begin{aligned}
\Gamma \leq & \|(F'(x_0) + \frac{\alpha_k}{c}L^{s/2})^{-1}\int_0^1 [F'(x_0) - F'(\hat{x} + t(x^\delta_{\alpha_k,s} - \hat{x})] \\
& (x^\delta_{\alpha_k,s} - \hat{x})dt\|_X \\
\leq & \overline{\psi_2(s)}k_0 r\|x^\delta_{\alpha_k,s} - \hat{x}\|_X
\end{aligned} \tag{3.17}$$

and hence by (3.15), (3.16), (3.17) and (3.14) we have

$$\begin{aligned}
\|x^\delta_{\alpha_k,s} - \hat{x}\|_X \leq & \frac{\varphi_1(\alpha_k) + C_s\psi_2(s)(2 + \frac{4\eta}{\eta-1})\eta\psi^{-1}_{s,a}(\delta)}{1 - (1-c)\psi_2(s) - \overline{\psi_2(s)}k_0 r} \\
= & O(\psi^{-1}_{s,a}(\delta)).
\end{aligned}$$

This completes the proof of the Theorem.

The following Theorem is a consequence of Theorem 3.7 and Theorem 3.10.

Theorem 3.11. *Let $x^\delta_{n,\alpha_k,s}$ be as in (1.7) with $\alpha = \alpha_k$ and $\delta \in (0, \delta_0]$, assumptions in Theorem 3.7 and Theorem 3.10 hold. Then*

$$\|\hat{x} - x^\delta_{n,\alpha_k,s}\|_X \leq \frac{\gamma_\rho}{1-q}q^n + O(\psi^{-1}_{s,a}(\delta)).$$

Theorem 3.12. *Let $x^\delta_{n,\alpha_k,s}$ be as in (1.7) with $\alpha = \alpha_k$ and $\delta \in (0, \delta_0]$, and assumptions in Theorem 3.11 hold. Let*

$$n_k := \min\{n : \tilde{q}^n \leq \alpha_k^{\frac{-a}{2(s+a)}}\delta\}.$$

Then

$$\|\hat{x} - x^\delta_{n_k,\alpha_k,s}\|_X = O(\psi^{-1}_{s,a}(\delta)).$$

4. Numerical examples

In the next two cases, we present examples for nonlinear equations where Assumption 3.3 is satisfied but not Assumption 3.1.

Example 4.1. Let $X = Y = \mathbb{R}$, $D = [0, \infty), x_0 = 1$ and define function F on D by

$$F(x) = \frac{x^{1+\frac{1}{i}}}{1+\frac{1}{i}} + c_1 x + c_2, \tag{4.1}$$

where c_1, c_2 are real parameters and $i > 2$ an integer. Then $F'(x) = x^{1/i} + c_1$ is not Lipschitz on D. Hence, Assumption 3.1 is not satisfied. However central Lipschitz condition Assumption 3.3 holds for $k_0 = 1$.

Indeed, we have

$$\begin{aligned} \|F'(x) - F'(x_0)\| &= |x^{1/i} - x_0^{1/i}| \\ &= \frac{|x - x_0|}{x_0^{\frac{i-1}{i}} + \cdots + x^{\frac{i-1}{i}}} \end{aligned}$$

so

$$\|F'(x) - F'(x_0)\| \le k_0 |x - x_0|.$$

Example 4.2. We consider the integral equations

$$u(s) = f(s) + \lambda \int_a^b G(s,t) u(t)^{1+1/n} dt, \quad n \in \mathbb{N}. \tag{4.2}$$

Here, f is a given continuous function satisfying $f(s) > 0, s \in [a, b], \lambda$ is a real number, and the kernel G is continuous and positive in $[a, b] \times [a, b]$.

For example, when $G(s, t)$ is the Green kernel, the corresponding integral equation is equivalent to the boundary value problem

$$\begin{aligned} u'' &= \lambda u^{1+1/n} \\ u(a) &= f(a), u(b) = f(b). \end{aligned}$$

These type of problems have been considered in [1]- [5].

Equation of the form (4.2) generalize equations of the form

$$u(s) = \int_a^b G(s,t) u(t)^n dt \tag{4.3}$$

studied in [1]-[5]. Instead of (4.2) we can try to solve the equation $F(u) = 0$ where

$$F : \Omega \subseteq C[a, b] \to C[a, b], \Omega = \{u \in C[a, b] : u(s) \ge 0, s \in [a, b]\},$$

and

$$F(u)(s) = u(s) - f(s) - \lambda \int_a^b G(s,t) u(t)^{1+1/n} dt.$$

The norm we consider is the max-norm.

The derivative F' is given by

$$F'(u)v(s) = v(s) - \lambda(1 + \frac{1}{n}) \int_a^b G(s,t) u(t)^{1/n} v(t) dt, \quad v \in \Omega.$$

First of all, we notice that F' does not satisfy a Lipschitz-type condition in Ω. Let us consider, for instance, $[a,b]=[0,1], G(s,t)=1$ and $y(t)=0$. Then $F'(y)v(s)=v(s)$ and

$$\|F'(x)-F'(y)\| = |\lambda|(1+\frac{1}{n})\int_a^b x(t)^{1/n}dt.$$

If F' were a Lipschitz function, then

$$\|F'(x)-F'(y)\| \le L_1\|x-y\|,$$

or, equivalently, the inequality

$$\int_0^1 x(t)^{1/n}dt \le L_2 \max_{x\in[0,1]} x(s), \tag{4.4}$$

would hold for all $x\in\Omega$ and for a constant L_2. But this is not true. Consider, for example, the functions

$$x_j(t)=\frac{t}{j},\quad j\ge 1,\quad t\in[0,1].$$

If these are substituted into (4.4)

$$\frac{1}{j^{1/n}(1+1/n)} \le \frac{L_2}{j} \Leftrightarrow j^{1-1/n} \le L_2(1+1/n),\quad \forall j\ge 1.$$

This inequality is not true when $j\to\infty$.

Therefore, condition (4.4) is not satisfied in this case. Hence Assumption 3.1 is not satisfied. However, condition Assumption 3.3 holds. To show this, let $x_0(t)=f(t)$ and $\gamma=\min_{s\in[a,b]} f(s), \alpha>0$ Then for $v\in\Omega$,

$$\begin{aligned}\|[F'(x)-F'(x_0)]v\| &= |\lambda|(1+\frac{1}{n})\max_{s\in[a,b]}|\int_a^b G(s,t)(x(t)^{1/n}-f(t)^{1/n})v(t)dt|\\ &\le |\lambda|(1+\frac{1}{n})\max_{s\in[a,b]} G_n(s,t)\end{aligned}$$

where $G_n(s,t)=\frac{G(s,t)|x(t)-f(t)|}{x(t)^{(n-1)/n}+x(t)^{(n-2)/n}f(t)^{1/n}+\cdots+f(t)^{(n-1)/n}}\|v\|$.

Hence,

$$\begin{aligned}\|[F'(x)-F'(x_0)]v\| &= \frac{|\lambda|(1+1/n)}{\gamma^{(n-1)/n}}\max_{s\in[a,b]}\int_a^b G(s,t)dt\|x-x_0\|\\ &\le k_0\|x-x_0\|,\end{aligned}$$

where $k_0=\frac{|\lambda|(1+1/n)}{\gamma^{(n-1)/n}}N$ and $N=\max_{s\in[a,b]}\int_a^b G(s,t)dt$. Then Assumption 3.3 holds for sufficiently small λ.

In the last example, we show that $\frac{K}{k_0}$ can be arbitrarily large in certain nonlinear equation.

Example 4.3. Let $X=D(F)=\mathbb{R}, x_0=0$, and define function F on $D(F)$ by

$$F(x)=d_0x+d_1+d_2\sin e^{d_3x}, \tag{4.5}$$

where $d_i, i=0,1,2,3$ are given parameters. Then, it can easily be seen that for d_3 sufficiently large and d_2 sufficiently small, $\frac{K}{k_0}$ can be arbitrarily large.

5. Conclusion

In this paper we present an iterative regularization method for obtaining an approximate solution of an ill-posed Hammerstein type operator equation $KF(x) = y$ in the Hilbert scale setting where K is a bounded linear operator and F is a nonlinear monotone operator. It is assumed that the available data is y^δ in place of exact data y. We considered the Hilbert space $(X_t)_{t\in\mathbb{R}}$ generated by L for the analysis where $L : D(L) \to X$ is a linear, unbounded, self-adjoint, densely defined and strictly positive operator on X. For choosing the regularization parameter α we used the adaptive scheme of Pereverzev and Schock (2005).

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Book reviews

Łukasz Piasecki, Classification of Lipschitz Mappings, CRC Press, Taylor & Francis Group, Boca Raton 2014, x + 224 pp, ISBN: 13: 978-1-4665-9521-7.

The book is concerned with the study of Lipschitz mappings on metric spaces in connection to fixed point theory. One denotes by $L(k)$ the class of Lipschitz mappings with constant $k > 0$ on a metric space (M, ρ), that is mappings $T\colon M \to M$ satisfying the condition $\rho(Tx, Ty) \le k\rho(x, y)$ for all $x, y \in M$. The smallest Lipschitz constant of the mapping T is denoted by $k(T)$ (or by $k_\rho(T)$, if necessary). The mapping T is called uniformly Lipschitz if there exists $k > 0$ such that $\rho(T^n x, T^n y) \le k\rho(x, y)$ for all $x, y \in M$ and all $n \in \mathbb{N}$. This class is characterized by the condition $k_\infty(T) := \limsup_{n\to\infty} \sqrt[n]{k(T^n)} < \infty$.

It follows that $k(T^{m+n}) \le k(T^m)k(T^n)$ so that one can define the characteristic $k_0(T) = \lim_{n\to\infty} \sqrt[n]{k(T^n)} = \inf\{\sqrt[n]{k(T^n)} : n \in \mathbb{N}\}$ – the analog of the spectral radius of a continuous linear operator on a Banach space. It turns up that $k_0(T) = \inf_d k_d(T)$, where the infimum is taken over all metrics d on M that are Lipschitz equivalent to ρ. An important class of Lipschitz mappings is formed by the nonexpansive ones, i.e. Lipschitz mappings with $k = 1$. The fixed point theory for this class of mappings acting on a Banach space X is tightly connected with the geometric properties of the underlying Banach space X (uniform rotundity, superreflexivity, uniform nonsquareness) as well as with those of the convex set $C \subset X$ on which they act (having normal structure, for instance). Some basic results along with some recent ones in this domain are presented in the seventh chapter of the book.

The main class studied by the author is that of mean Lipschitz functions. A multi-index is an n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_1, \alpha_n > 0, \alpha_i \ge 0$ and $\sum_{i=1}^n \alpha_i = 1$. A mapping $T : M \to M$ is called α-Lipschitzian with constant k if $\sum_{i=1}^n \alpha_i \rho({}^i x, T^i y) \le k\rho(x, y)$ for all $x, y \in M$. The class of these mappings is denoted by $L(\alpha, k)$. Any α-Lipschitz mapping is Lipschitz and $k(T) \le k(\alpha, T)/\alpha_1$. Uniformly k-Lipchitzian mappings are (α, k)-Lipschitz for any multi-index α. Another class is that of the mappings satisfying, for $p \ge 1$ and some $k \ge 0$, the condition

$$\left(\sum_{i=1}^n \alpha_i \rho({}^i x, T^i y)^p\right)^{1/p} \le k\rho(x, y), \quad \forall x, y \in M\,,$$

called (α, p)-Lipschitz mappings with constant k.

The bulk of the book is formed by the chapters 4. *On Lipschitz constants for iterates of mean lipschitzian mappings*, 5. *Subclasses determined by p-averages, 6. Mean* *contractions*, 8. *Mean nonexpansive mappings*, and 9. *Mean lipschitzian mappings with* $k > 1$. These chapters are concerned with the behavior of the quantities $k_0(T)$, $k(T^n)$ and $k_\infty(T)$, including some numerical experiments, for mappings T in these classes of mean-Lipschitz mappings, and are essentially based on results obtained by the author alone or in cooperation with Víctor Pérez García.

The first two chapters 1. *The Lipschitz condition* and 2. *Basic facts on Banach spaces*, contain some preliminary notions and results.

The book is well written and contains new interesting results along with some classical ones in metric fixed point theory. The prerequisites are modest – some basic results in topology and functional analysis – so it can be used by advanced undergraduate and graduate students for an introduction to this domain and by researchers as a reference text. Experts in other areas, as differential equations, dynamical systems, will find it useful as well.

S. Cobzaş

Ioannis M. Roussos, Improper Riemann Integrals, CRC Press, Taylor & Francis Group, Boca Raton 2014, xiv + 675 pp, ISBN: 978-1-4665-8807-3.

The book contains a detailed presentation of the main improper Riemann integrals (with or without parameter) at the master level for students in mathematics, statistics, applied sciences and engineering. As it is well known, the improper Riemann integrals are important tools in various areas of mathematics (differential equations, probability theory) as well as in its applications to physics, mechanics, engineering. New classes of functions (e.g. Euler' Beta and Gamma functions) are introduced as improper Riemann integrals depending on a parameter as well as the integral transforms of Fourier and Laplace.

The presentation is restricted to Riemann integral (including double Riemann integral) and in order to make the book self-contained the principal theorems used in the calculations are included, some with proofs other without. In some cases these results are presented under some restricted conditions, accessible to the undergraduate but sufficient for applications.

The book is divided into two main parts 2. *Real analysis techniques*, and 3. *Complex analysis techniques*. An introductory chapter contains the definition of an improper integral, convergence criteria and some motivating examples.

Chapter 2 contains a detailed study of the properties of improper Riemann integrals depending on a parameter – continuity, differentiability, integrability. The treatment is based on a version of Lebesgue dominated convergence theorem for the Riemann integral. Applications are given to Frullani integrals, the functions Beta and Gamma, and to the Laplace transform.

For reader's convenience Chapter 3 contains a quick introduction to complex analysis with emphasis on the elementary holomorphic functions - the exponential, the trigonometric functions, and the multivalued holomorphic functions - the complex logarithm $\log z$, the power function $z^\alpha = e^{\alpha \log z}$. Here the powerful and relatively simple method of residues is applied to the calculation of some improper Riemann integrals, including a relatively complete treatment of the Fourier transform – definition, Riemann-Lebesgue Lemma, calculus rules, the inversion formula – and a reconsideration of the Laplace transform in the complex case.

The last chapter of the book is 4. *List of non-elementary integrals and sums in text*, contains a record of the most important integrals and sum calculated in the text, with exact reference to the places where they appear.

By collecting a lot of important improper integrals and sums used in various domains, and presenting their calculation in an accessible but rigorous way, the book will be of great use to students in mathematics and related areas and for applied scientists (statisticians, engineers, physicists) as well. A small personal objection – the presentation of some examples is too detailed, and so the abundance of these details hide to some extent the ideas behind.

T. Trif

Miroslav Pavlović, Function Classes on the Unit Disc, Studies in Mathematics, Vol. 52, xiii + 449 pp, Walter de Gruyter, Berlin - New York, 2014, ISBN: 978-3-11-028123-1, e-ISBN: 978-3-11-028190-3, ISSN: 0179-0986.

The book is concerned with spaces of harmonic and of analytic functions in the unit disc $\mathbb{D}$ – Hardy, Bergman, Besov, Lipschitz, Bloch, Hardy-Sobolev, BMO, etc. The approach proposed by the author differs from those contained in the classical books of Zygmund, Duren, Koosis, Garnett, allowing him to present new results and to give simpler and clearer proofs to some known facts (e.g. Fefferman-Stein theorem on subharmonic functions, theorems on conjugate harmonic functions, etc).

The first three chapters, 1. *The Poisson integral and Hardy spaces*, 2. *Subharmonic functions and Hardy spaces*, and 3. *Subharmonic behavior and mixed norm spaces*, are devoted to the spaces $h(\mathbb{D})$ and $H(\mathbb{D})$ of harmonic, respectively analytic, functions in the unit disc $\mathbb{D}$.

In Chapter 4. *Taylor coefficients with applications,* the approach to the mixed-norm Bergman spaces is based on a class of functions, called quasi-nearly subharmonic, introduced by the author. Besov spaces are studied in Chapter 5, while the sixth chapter, *The dual of H^1 and some related spaces*, is concerned with the duality between H^1 and BMO spaces. Chapter 7. *Littlewood-Paley theory*, contains some deep characterizations of H^p, $p > 0$, spaces as well as of hyperbolic Hardy classes.

The Lipschitz classes Λ^p_ω of analytic functions are studied in chapters 8. *Lipschitz spaces of first order*, and 9. *Lipschitz spaces of higher order*, defined by ordinary moduli of continuity, respectively by higher order moduli of smoothness. Chapter 10. *One-to-one mappings,* is devoted to the problem of membership of univalent and quasiconformal harmonic mappings in some classical spaces, while Chapter 11. *Coefficients multipliers,* some multiplier results are presented following some ideas of Kalton and of the author, including compact multipliers and multipliers on spaces with non-normal weight.

Chapter 12. *Toward a theory of vector-valued spaces,* presents some results on spaces of harmonic and analytic functions with values in a Banach or a quasi-Banach space X.

Author's booklet, *Introduction to the Function Spaces on the Disk*, Matematički Institut SANU, Special Publications, vol. 20, Belgrade 2004, contains some material included in the present book, although, as the author mentions in the Preface, this new book can not be considered as an expanded version of the former one – they have only nonempty intersection.

The reading of the book assumes familiarity with real, complex and functional analysis (at the level of Rudin's *Real and Complex Analysis*). For reader's convenience, two appendices, A. *Quasi-Banach spaces*, and B. *Interpolation and maximal functions*, are added to the main text. Sixteen research problems are included, and each chapter ends with a section of historical notes and references to further results.

The book is well written and contains a lot of deep and interesting results, including personal contributions of the author. It can be recommended to specialists as a reference text and to post-graduate students for study.

Gabriela Kohr