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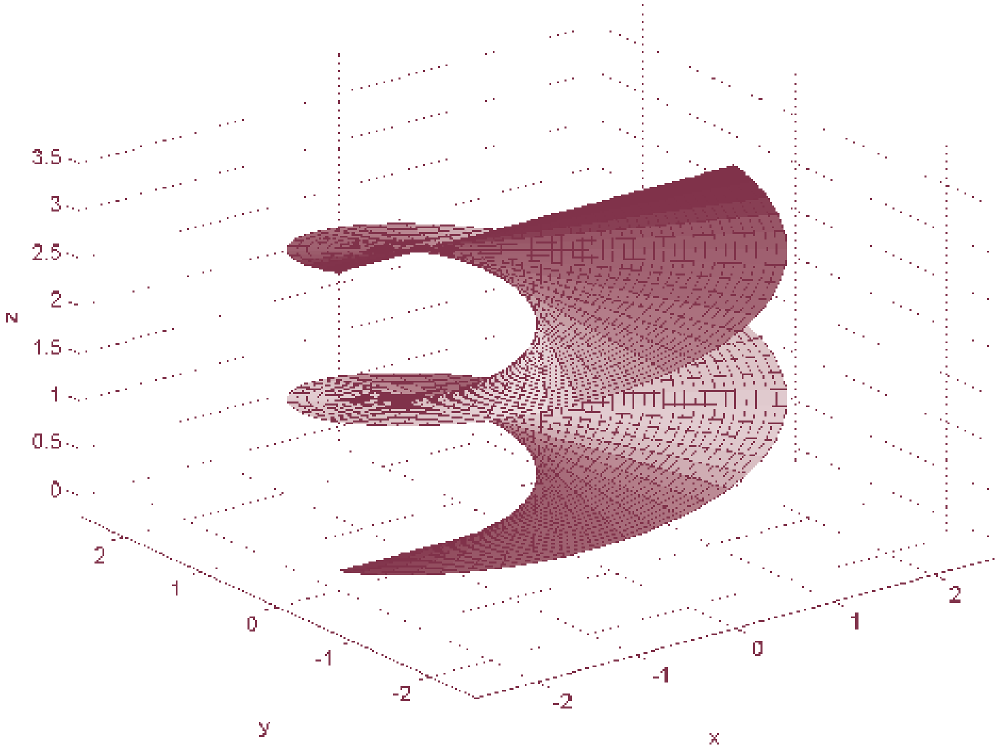

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CONTENTS

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A simple proof of the fundamental theorem of calculus for the Lebesgue integral

Rodrigo López Pouso

Abstract. This paper contains a new elementary proof of the Fundamental Theorem of Calculus for the Lebesgue integral. The hardest part of our proof simply concerns the convergence in L^1 of a certain sequence of step functions, and we prove it using only basic elements from Lebesgue integration theory.

Mathematics Subject Classification (2010): 26A46, 26A36.

Keywords: Fundamental theorem of calculus, Lebesgue integral, absolute continuity.

1. Introduction

Let $f : [a,b] \longrightarrow \mathbb{R}$ be absolutely continuous on $[a,b]$, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\{(a_j, b_j)\}_{j=1}^n$ is a family of pairwise disjoint subintervals of $[a,b]$ satisfying

$$\sum_{j=1}^{n} (b_j - a_j) < \delta,$$

then

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)| < \varepsilon.$$

Classical results ensure that f has a finite derivative almost everywhere in $I = [a,b]$, and that $f' \in \mathrm{L}^1(I)$, see [3] or [9, Corollary 6.83]. These results, which we shall use in this paper, are the first steps in the proof of the main connection between absolute continuity and Lebesgue integration: the Fundamental Theorem of Calculus for the Lebesgue integral.

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Theorem 1.1. *If $f : I = [a,b] \longrightarrow \mathbb{R}$ is absolutely continuous on I, then*

$$f(b) - f(a) = \int_a^b f'(x)\,dx \quad \textit{in Lebesgue's sense.}$$

In this note we present a new elementary proof to Theorem 1.1 which seems more natural and easy than the existing ones. Indeed, our proof can be sketched simply as follows:

1. We consider a well–known sequence of step functions $\{h_n\}_{n\in\mathbb{N}}$ which tends to f' almost everywhere in I and, moreover,

$$\int_a^b h_n(x)\,dx = f(b) - f(a) \quad \text{for all } n \in \mathbb{N}.$$

2. We prove, by means of elementary arguments, that

$$\lim_{n\to\infty} \int_a^b h_n(x)\,dx = \int_a^b f'(x)\,dx.$$

More precise comparison with the literature on Theorem 1.1 and its several proofs will be given in Section 3.

In the sequel m stands for the Lebesgue measure in $\mathbb{R}$.

2. Proof of Theorem 1.1

For each $n \in \mathbb{N}$ we consider the partition of the interval $I = [a,b]$ which divides it into 2^n subintervals of length $(b-a)2^{-n}$, namely

$$x_{n,0} < x_{n,1} < x_{n,2} < \cdots < x_{n,2^n},$$

where $x_{n,i} = a + i(b-a)2^{-n}$ for $i = 0, 1, 2, \ldots, 2^n$.

Now we construct a step function $h_n : [a,b) \longrightarrow \mathbb{R}$ as follows: for each $x \in [a,b)$ there is a unique $i \in \{0, 1, 2, \ldots, 2^n - 1\}$ such that

$$x \in [x_{n,i}, x_{n,i+1}),$$

and we define

$$h_n(x) = \frac{f(x_{n,i+1}) - f(x_{n,i})}{x_{n,i+1} - x_{n,i}} = \frac{2^n}{b-a}[f(x_{n,i+1}) - f(x_{n,i})].$$

On the one hand, the construction of $\{h_n\}_{n\in\mathbb{N}}$ implies that

$$\lim_{n\to\infty} h_n(x) = f'(x) \quad \text{for a.a. } x \in [a,b]. \tag{2.1}$$

To prove (2.1), we fix $x \in (a,b)$ such that $f'(x)$ exists and $x \neq x_{n,i}$ for all $n \in \mathbb{N}$ and all $i \in \{1, 2, \ldots, 2^n - 1\}$. Now for each $n \in \mathbb{N}$ we consider the unique index

$i \in \{0, 1, 2, \ldots, 2^n - 1\}$ such that $x \in (x_{n,i}, x_{n,i+1})$ and we have

$$\begin{aligned}|f'(x) - h_n(x)| &= \left| f'(x) - \frac{f(x_{n,i+1}) - f(x) + f(x) - f(x_{n,i})}{x_{n,i+1} - x_{n,i}} \right| \\ &= \left| f'(x) \frac{x_{n,i+1} - x + x - x_{n,i}}{x_{n,i+1} - x_{n,i}} \right. \\ &\quad - \frac{f(x_{n,i+1}) - f(x)}{x_{n,i+1} - x} \frac{x_{n,i+1} - x}{x_{n,i+1} - x_{n,i}} \\ &\quad \left. - \frac{f(x) - f(x_{n,i})}{x - x_{n,i}} \frac{x - x_{n,i}}{x_{n,i+1} - x_{n,i}} \right| \\ &\leq \left| f'(x) - \frac{f(x_{n,i+1}) - f(x)}{x_{n,i+1} - x} \right| + \left| f'(x) - \frac{f(x) - f(x_{n,i})}{x - x_{n,i}} \right|,\end{aligned}$$

which yields (2.1).

On the other hand, for each $n \in \mathbb{N}$ we compute

$$\int_a^b h_n(x)dx = \sum_{i=0}^{2^n-1} \int_{x_{n,i}}^{x_{n,i+1}} h_n(x)dx = \sum_{i=0}^{2^n-1} [f(x_{n,i+1}) - f(x_{n,i})] = f(b) - f(a),$$

and therefore it only remains to prove that

$$\lim_{n\to\infty} \int_a^b h_n(x)\, dx = \int_a^b f'(x)\, dx.$$

Let us prove that, in fact, we have convergence in $\mathrm{L}^1(I)$, i.e.,

$$\lim_{n\to\infty} \int_a^b |h_n(x) - f'(x)|\, dx = 0. \tag{2.2}$$

Let $\varepsilon > 0$ be fixed and let $\delta > 0$ be one of the values corresponding to $\varepsilon/4$ in the definition of absolute continuity of f.

Since $f' \in \mathrm{L}^1(I)$ we can find $\rho > 0$ such that for any measurable set $E \subset I$ we have

$$\int_E |f'(x)|\, dx < \frac{\varepsilon}{4} \quad \text{whenever } m(E) < \rho. \tag{2.3}$$

The following lemma will give us fine estimates for the integrals when many of the $|h_n|$ are "large". We postpone its proof for better readability.

Lemma 2.1. *For each $\varepsilon > 0$ there exist $k, n_k \in \mathbb{N}$ such that*

$$k \cdot m\left(\left\{x \in I \,:\, \sup_{n \geq n_k} |h_n(x)| > k\right\}\right) < \varepsilon.$$

Lemma 2.1 guarantees that there exist $k, n_k \in \mathbb{N}$ such that

$$k \cdot m\left(\left\{x \in I \,:\, \sup_{n \geq n_k} |h_n(x)| > k\right\}\right) < \min\left\{\delta, \frac{\varepsilon}{4}, \rho\right\}. \tag{2.4}$$

Let us denote

$$A = \left\{x \in I \,:\, \sup_{n \geq n_k} |h_n(x)| > k\right\},$$

which, by virtue of (2.4) and (2.3), satisfies the following properties:

$$m(A) < \delta, \tag{2.5}$$

$$k \cdot m(A) < \frac{\varepsilon}{4}, \tag{2.6}$$

$$\int_A |f'(x)|\,dx < \frac{\varepsilon}{4}. \tag{2.7}$$

We are now in a position to prove that the integrals in (2.2) are smaller than ε for all sufficiently large values of $n \in \mathbb{N}$. We start by noticing that (2.7) guarantees that for all $n \in \mathbb{N}$ we have

$$\begin{aligned}\int_I |h_n(x) - f'(x)|\,dx &= \int_{I\setminus A} |h_n(x) - f'(x)|\,dx + \int_A |h_n(x) - f'(x)|\,dx \\ &< \int_{I\setminus A} |h_n(x) - f'(x)|\,dx + \int_A |h_n(x)|\,dx + \frac{\varepsilon}{4}. \end{aligned} \tag{2.8}$$

The definition of the set A implies that for all $n \in \mathbb{N}$, $n \geq n_k$, we have

$$|h_n(x) - f'(x)| \leq k + |f'(x)| \quad \text{for almost all } x \in I \setminus A,$$

so the Dominated Convergence Theorem yields

$$\lim_{n\to\infty} \int_{I\setminus A} |h_n(x) - f'(x)|\,dx = 0. \tag{2.9}$$

From (2.8) and (2.9) we deduce that there exists $n_\varepsilon \in \mathbb{N}$, $n_\varepsilon \geq n_k$, such that for all $n \in \mathbb{N}$, $n \geq n_\varepsilon$, we have

$$\int_I |h_n(x) - f'(x)|\,dx < \frac{\varepsilon}{2} + \int_A |h_n(x)|\,dx. \tag{2.10}$$

Finally, we estimate $\int_A |h_n|$ for each fixed $n \in \mathbb{N}$, $n \geq n_\varepsilon$. First, we decompose $A = B \cup C$, where

$$B = \{x \in A \,:\, |h_n(x)| \leq k\} \quad \text{and} \quad C = A \setminus B.$$

We immediately have

$$\int_B |h_n(x)|\,dx \leq k \cdot m(B) \leq k \cdot m(A) < \frac{\varepsilon}{4} \tag{2.11}$$

by (2.6).

Obviously, $\int_C |h_n| < \varepsilon/4$ when $C = \varnothing$. Let us see that this inequality holds true when $C \neq \varnothing$. For every $x \in C = \{x \in A \,:\, |h_n(x)| > k\}$ there is a unique index $i \in \{0, 1, 2, \ldots, 2^n - 1\}$ such that $x \in [x_{n,i}, x_{n,i+1})$. Since $|h_n|$ is constant on $[x_{n,i}, x_{n,i+1})$ we deduce that $[x_{n,i}, x_{n,i+1}) \subset C$. Thus there exist indices $i_l \in \{0, 1, 2, \ldots, 2^n - 1\}$, with $l = 1, 2, \ldots, p$ and $i_l \neq i_{\tilde{l}}$ if $l \neq \tilde{l}$, such that

$$C = \bigcup_{l=1}^{p} [x_{n,i_l}, x_{n,i_l+1}).$$

Therefore

$$\sum_{l=1}^{p}(x_{n,i_l+1} - x_{n,i_l}) = m(C) \leq m(A) < \delta \quad \text{by (2.5)},$$

and then the absolute continuity of f finally comes into action:

$$\int_C |h_n(x)|\, dx = \sum_{l=1}^{p} \int_{x_{n,i_l}}^{x_{n,i_l+1}} |h_n(x)|\, dx$$
$$= \sum_{l=1}^{p} |f(x_{n,i_l+1}) - f(x_{n,i_l})| < \frac{\varepsilon}{4}.$$

This inequality, along with (2.10) and (2.11), guarantee that for all $n \in \mathbb{N}$, $n \geq n_\varepsilon$, we have

$$\int_I |h_n(x) - f'(x)|\, dx < \varepsilon,$$

thus proving (2.2) because ε was arbitrary. The proof of Theorem 1.1 is complete.

Now we go back to Lemma 2.1. A more general version, which seems interesting in its own right, will be established instead. We split it into two parts for better readability.

A first result, elementary but absent from many textbooks, complements Tchebyshev's inequality for integrable functions. It is however an old result which we can already find in Hobson's book [6, page 526]. A proof is included for the convenience of readers.

Proposition 2.2. *Let $A \subset \mathbb{R}$ be a measurable set. If $g \in \mathrm{L}^1(A)$ then*

$$\lim_{k\to\infty} k \cdot m\left(\{x \in A \,:\, |g(x)| \geq k\}\right) = 0. \tag{2.12}$$

Proof. Standard results guarantee that

$$\sum_{n=0}^{\infty} n \cdot m(\{x \in A \,:\, n \leq |g(x)| < n+1\}) \leq \sum_{n=0}^{\infty} \int_{\{x\,:\,n\leq|g(x)|<n+1\}} |g(x)|\, dx$$
$$= \int_{\cup_{n=0}^{\infty}\{x\,:\,n\leq|g(x)|<n+1\}} |g(x)|\, dx$$
$$= \int_A |g(x)|\, dx,$$

hence the series on the left-hand side is convergent.

For all $k \in \mathbb{N}$ we have

$$k \cdot m\left(\{x \in A \,:\, |g(x)| \geq k\}\right) = k \cdot \sum_{n=k}^{\infty} m\left(\{x \in A \,:\, n \leq |g(x)| < n+1\}\right)$$
$$\leq \sum_{n=k}^{\infty} n \cdot m(\{x \in A \,:\, n \leq |g(x)| < n+1\}),$$

and then (2.12) obtains. □

Now we proceed with our extended version of Lemma 2.1.

Proposition 2.3. *Let $A \subset \mathbb{R}$ be a measurable set with $m(A) < \infty$. Assume that $g_n : A \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is measurable for each $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} g_n(x) = g(x) \in \mathbb{R} \quad \text{for almost all } x \in A. \tag{2.13}$$

Then for every $k \in \mathbb{N}$ we have

$$\lim_{j\to\infty} m\left(\left\{x \in A \,:\, \sup_{n\geq j} |g_n(x)| \geq k\right\}\right) = m\left(\{x \in A \,:\, |g(x)| \geq k\}\right), \tag{2.14}$$

and, if $g \in \mathrm{L}^1(A)$ then

$$\lim_{k\to\infty} \lim_{j\to\infty} k \cdot m\left(\left\{x \in A \,:\, \sup_{n\geq j} |g_n(x)| \geq k\right\}\right) = 0, \tag{2.15}$$

which implies the result in Lemma 2.1 *for $g_n = h_n$ and $g = f'$.*

Proof. Let $N \subset A$ be a null–measure set such that (2.13) holds for all $x \in A \setminus N$ and let $k \in \mathbb{N}$ be fixed.

We define a family of measurable sets

$$E_j = \left\{x \in A \setminus N \,:\, \sup_{n\geq j} |g_n(x)| \geq k\right\} \quad (j \in \mathbb{N}).$$

Notice that $E_{j+1} \subset E_j$ for every $j \in \mathbb{N}$, and $m(E_1) \leq m(A) < \infty$, hence

$$\lim_{j\to\infty} m(E_j) = m\left(\bigcap_{j=1}^{\infty} E_j\right) = m\left(\{x \in A \setminus N \,:\, |g(x)| \geq k\}\right),$$

so (2.14) is proven.

Now (2.14) and (2.12) yield (2.15). □

3. Final remarks

The sequence $\{h_n\}_{n\in\mathbb{N}}$ is used in other proofs of Theorem 1.1, see [1] or [11]. The novelty in this paper is our elementary and self–contained proof of (2.2).

Our proof avoids somewhat technical results often invoked to prove Theorem 1.1. For instance, we do not use any sophisticated estimate for the measure of image sets such as [4, Theorem 7.20], [9, Lemma 6.88] or [11, Proposition 1.2], see also [7]. We do not use the following standard lemma either (usually proven by means of Vitali's Covering Theorem): an absolutely continuous function having zero derivative almost everywhere is constant, see [4, Theorem 7.16] or [9, Lemma 6.89]. It is worth having a look at [5] for a proof of that lemma using tagged partitions; see also [2] for a proof based on full covers [10].

Concise proofs of Theorem 1.1 follow from the Radon–Nikodym Theorem, see [1], [4] or [8], but this is far from being elementary.

Finally, it is interesting to note that (2.2) is an almost trivial consequence of Lebesgue's Dominated Convergence Theorem when f is Lipschitz continuous on I.

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Certain subclasses of uniformly harmonic β-starlike functions of complex order

Basem Aref Frasin and Nanjundan Magesh

Abstract. In this paper, we introduce a family of harmonic parabolic starlike functions of complex order in the unit disc and coefficient bounds, distortion bounds, extreme points, convolution conditions and convex combination are determined for functions in this family. Further results on integral transforms are discussed. Consequently, many of our results are either extensions or new approaches to those corresponding previously known results.

Mathematics Subject Classification (2010): 30C45.

Keywords: Harmonic functions, univalent functions, starlike functions, convolution (or Hadamard) product.

1. Introduction and definitions

A continuous complex valued function $f = u + iv$ defined in a simply connected complex domain $\mathcal{D}$ is said to be harmonic in $\mathcal{D}$ if both u and v are real harmonic in $\mathcal{D}$. In any simply connected domain we can write $f = h + \bar{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient for f to be locally univalent and sense preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$.

Let $\mathcal{S_H}$ denote the family of functions $f = h + \bar{g}$ that are harmonic univalent and sense preserving in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in \mathcal{S_H}$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

The harmonic function $f = h + \bar{g}$ for $g \equiv 0$ reduces to an analytic function $f = h$. Also let $\mathcal{S}_{\overline{\mathcal{H}}}$ be the subclass of $\mathcal{S_H}$ consisting of functions of the form $f = h + \overline{g}$, where the analytic functions h and g as

$$h(z) = z - \sum_{k=2}^{\infty} |a_k|\, z^k, \qquad g(z) = \sum_{k=1}^{\infty} |b_k|\, z^k. \tag{1.2}$$

In 1984 Clunie and Sheil-Small [3] investigated the class $\mathcal{S}_{\mathcal{H}}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several papers related on $\mathcal{S}_{\mathcal{H}}$ and its subclasses. Jahangiri [7], Silverman[9], Silverman and Silvia [10] studied the harmonic starlike functions. Frasin [5], Frasin and Murugusundaramoorthy [6] and Dixit et al. [4] extended the results by defining the subclasses by means convolution (or Hadamard product) also, see [2, 13].

Recently, Yalçin and Öztürk [12] defined the class $S_{\overline{\mathcal{H}}}(b)$ consisting of functions $f = h + \bar{g} \in S_{\overline{\mathcal{H}}}$ that satisfy the condition

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - 1\right)\right\} > 0,\ b \in \mathbb{C}\setminus\{0\},\ z \in \mathcal{U}. \tag{1.3}$$

Also, they gave necessary and sufficient conditions for the functions to be in $S_{\overline{\mathcal{H}}}(b)$.

Furthermore, Stephen et al. [11] studied a harmonic parabolic starlike functions of complex order denoted by $S_{\overline{\mathcal{H}}}(b,\alpha)$ consisting of functions $f = h + \bar{g} \in S_{\overline{\mathcal{H}}}$ that satisfy the condition

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left((1+e^{i\gamma})\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - (1+e^{i\gamma}) - 1\right)\right\} > \alpha,\ z \in \mathcal{U}, \tag{1.4}$$

where $0 \le \alpha < 1$, $\gamma \in \mathbb{R}$ and $b \in \mathbb{C}\setminus\{0\}$ with $|b| \le 1$. Also, they gave necessary and sufficient conditions for the functions to be in $S_{\overline{\mathcal{H}}}(b,\alpha)$.

Motivated by Frasin and Murugusundaramoorthy [6], Yalçin and Öztürk [12] and Stephen et al. [11], we define a new class of uniformly harmonic $\beta-$ starlike functions of complex order $\mathcal{S}_{\mathcal{H}}(\Phi,\Psi;\alpha,\beta,b;t)$, the subclass of $\mathcal{S}_{\mathcal{H}}$ consisting of functions $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ that satisfy the condition

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left((1+\beta e^{i\gamma})\frac{h(z)*\Phi(z) - \overline{g(z)*\Psi(z)}}{h_t(z) + \overline{g_t(z)}} - \beta e^{i\gamma} - 1\right)\right\} > \alpha,\ z \in \mathcal{U}, \tag{1.5}$$

where $b \in \mathbb{C}\setminus\{0\}$, $\alpha(0 \le \alpha < 1)$, $h_t(z) = (1-t)z + th(z)$, $g_t(z) = tg(z)$, $0 \le t \le 1$, $\Phi(z) = z + \sum\limits_{k=2}^{\infty} \lambda_k z^k$ and $\Psi(z) = z + \sum\limits_{k=2}^{\infty} \mu_k z^k$ are analytic in $\mathcal{U}$ with the conditions $\lambda_k \ge 0$, $\mu_k \ge 0, \beta \ge 0$ and $\gamma \in \mathbb{R}$. The operator “ $*$ ” stands for the convolution of two power series. We further let $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$ denote the subclass of $\mathcal{S}_{\mathcal{H}}(\Phi,\Psi;\alpha,\beta,b;t)$ consisting of functions $f = h + \overline{g} \in \mathcal{S}_{\overline{\mathcal{H}}}$.

We note that by specializing the functions Φ, Ψ and parameters β, γ and t we obtain the well-known harmonic univalent functions as well as many new ones. For example,

1. $\mathcal{S}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \alpha, 1, 1; 1) = G_{\overline{\mathcal{H}}}(\alpha)$ (Rosy et al. [8])
2. $\mathcal{S}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \alpha, \beta, 1; t) = \mathcal{G}_{\overline{\mathcal{H}}}(\alpha,\beta;t)$ (Ahuja et al. [1])
3. $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,1;1) = \overline{\mathcal{HS}}(\Phi,\Psi;\alpha,\beta)$ (Frasin and Murugusundaramoorthy [6])

4. $\mathcal{S}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \alpha, 1, b; 1) = \mathcal{S}_{\overline{\mathcal{H}}}(b, \alpha)$ (Stephen et al [11])
5. $\mathcal{S}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 0, 0, b; 1) = \mathcal{S}_{\overline{\mathcal{H}}}(b)$ (Yalçin and Öztürk [12])
6. $\mathcal{S}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \alpha, 0, 1; 1) = S^*_{\overline{\mathcal{H}}}(\alpha)$ (Jahangiri [7].)
   For $\alpha = 0$ the class $S^*_{\overline{\mathcal{H}}}(\alpha)$ was studied by Silverman and Silvia [10], for $\alpha = 0$ and $b_1 = 0$ see [9, 10].

In this paper, we obtain coefficient bounds, distortion bounds, extreme points, convolution conditions and convex combination for functions in $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$. Further results on integral transforms are also discussed.

2. Coefficient inequalities

Our first theorem gives a sufficient condition for functions in $\mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha, \beta, b; t)$.

Theorem 2.1. *Let $f = h + \bar{g}$ be so that h and g are given by (1.1). If*

$$\sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\alpha)t|b|]}{(1-\alpha)|b|}|a_k| + \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\alpha)t|b|]}{(1-\alpha)|b|}|b_k| \leq 1, \tag{2.1}$$

where $\beta \geq 0$, $0 \leq \alpha < 1$, $0 \leq t \leq 1$, $k(1-\alpha)|b| \leq [(\lambda_k - t)(1+\beta) + (1-\alpha)t|b|]$ and $k(1-\alpha)|b| \leq [(\mu_k + t)(1+\beta) - (1-\alpha)t|b|]$ for $k \geq 2$ then $f \in \mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha, \beta, b; t)$.

Proof. To prove that $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha, \beta, b; t)$, we only need to show that if (2.1) holds, then the required condition (1.5) is satisfied. For (1.5), we can write

$$Re\left\{1 + \frac{1}{b}\left((1+\beta e^{i\gamma})\left(\frac{h(z)*\Phi(z) - \overline{g(z)*\Psi(z)}}{h_t(z) + \overline{g_t(z)}}\right) - \beta e^{i\gamma} - 1\right)\right\} \geq \alpha.$$

Using the fact that $Re\omega \geq \alpha$ if and only if $|1-\alpha+\omega| \geq |1+\alpha-\omega|$, it suffices to show that

$$\begin{aligned}&\Big|\left[(2-\alpha)b - (\beta e^{i\gamma}+1)\right]\left(h_t(z) + \overline{g_t(z)}\right)\\ &\quad + \left(\beta e^{i\gamma}+1\right)\left(h(z)*\Phi(z) - \overline{g(z)*\Psi(z)}\right)\Big|\\ &\quad - \Big|\left[\alpha b + (\beta e^{i\gamma}+1)\right]\left(h_t(z) + \overline{g_t(z)}\right)\\ &\quad - \left(\beta e^{i\gamma}+1\right)\left(h(z)*\Phi(z) - \overline{g(z)*\Psi(z)}\right)\Big|\end{aligned}$$

$$\geq 2(1-\alpha)|b||z| - \sum_{k=2}^{\infty} 2\left[(\lambda_k - t)(1+\beta) + (1-\alpha)t|b|\right]|a_k||z|^k$$

$$- \sum_{k=1}^{\infty} 2\left[(\mu_k + t)(1+\beta) - (1-\alpha)t|b|\right]|b_k||z|^k$$

$$\geq 2(1-\alpha)|b||z| \left\{ 1 - \sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\alpha)t|b|]}{(1-\alpha)|b|}|a_k||z|^{k-1} \right.$$

$$\left. - \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\alpha)t|b|]}{(1-\alpha)|b|}|b_k||z|^{k-1} \right\}$$

$$\geq 2(1-\alpha)|b| \left\{ 1 - \sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\alpha)t|b|]}{(1-\alpha)|b|}|a_k| \right.$$

$$\left. - \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\alpha)t|b|]}{(1-\alpha)|b|}|b_k| \right\} \geq 0,$$

which implies that $f \in \mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha, \beta, b; t)$. The harmonic function

$$f(z) = z + \sum_{k=2}^{\infty} \frac{(1-\alpha)|b|}{[(\lambda_k - t)(1+\beta) + (1-\alpha)t|b|\ ]} x_k z^k + \sum_{k=1}^{\infty} \frac{(1-\alpha)|b|}{[(\mu_k + t)(1+\beta) - (1-\alpha)t|b|\ ]} \overline{y_k z^k},$$

where $\sum\limits_{k=2}^{\infty} |x_k| + \sum\limits_{k=1}^{\infty} |y_k| = 1$,shows that the coefficient bound given by (2.1) is sharp. □

In the following theorem, it is shown that the bound (2.1) is also necessary for functions $f = h + \bar{g}$, where h and g are of the form (1.2).

Theorem 2.2. *Let $f = h + \bar{g}$ be so that h and g are given by (1.2). Then $f \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ if and only if*

$$\sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\alpha)|b|t]}{(1-\alpha)|b|}|a_k| + \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\alpha)|b|t]}{(1-\alpha)|b|}|b_k| \leq 1, \tag{2.2}$$

where $\beta \geq 0$, $0 \leq \alpha < 1$, $0 \leq t \leq 1$, $k(1-\alpha)|b| \leq [(\lambda_k - t)(1+\beta) + (1-\alpha)t|b|]$ and $k(1-\alpha)|b| \leq [(\mu_k + t)(1+\beta) - (1-\alpha)t|b|]$ for $k \geq 2$.

Proof. The 'if part' follows from Theorem 2.1 upon noting that the functions h and g in $f \in \mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha, \beta, b; t)$ are of the form (1.2), then $f \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$. For the 'only if' part, we wish to show that $f \notin \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ if the condition (2.2) does not hold. Note that a necessary and sufficient condition for $f = h + \overline{g}$ given by (1.2) be in $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ is that

$$
\begin{aligned}
& Re\left\{\frac{[b-(1+\beta e^{i\gamma})][h_t(z)+\overline{g_t(z)}] + (1+\beta e^{i\gamma})[h(z)*\Phi(z) - \overline{g(z)*\Psi(z)}]}{b[h_t(z)+\overline{g_t(z)}]} - \alpha\right\} \\
= \ & Re\left\{\frac{(1-\alpha)bz - \sum\limits_{k=2}^{\infty}[(1-\alpha)bt + (1+\beta e^{i\gamma})(\lambda_k - t)]|a_k|z^k - \sum\limits_{k=1}^{\infty}[(1+\beta e^{i\gamma})(\mu_k + t) - (1-\alpha)bt]|b_k|\bar{z}^k}{b(z - \sum\limits_{k=2}^{\infty} t|a_k|z^k + \sum\limits_{k=1}^{\infty} t|b_k|\bar{z}^k)}\right\} \\
= \ & Re\left\{\frac{(1-\alpha)|b|^2 - \sum\limits_{k=2}^{\infty}[(1-\alpha)bt + (1+\beta e^{i\gamma})(\lambda_k - t)]\bar{b}|a_k|z^{k-1} - \frac{\bar{z}}{z}\sum\limits_{k=1}^{\infty}[(1+\beta e^{i\gamma})(\mu_k + t) - (1-\alpha)bt]\bar{b}|b_k|\bar{z}^{k-1}}{|b|^2(1 - \sum\limits_{k=2}^{\infty} t|a_k|z^{k-1} + \sum\limits_{k=1}^{\infty} t|b_k|\bar{z}^{k-1})}\right\} \\
\geq \ & 0.
\end{aligned}
$$

If we choose z to be real $z \to 1^-$ and since $Re(-e^{i\gamma}) \geq -|e^{i\gamma}| = -1$, the above inequality reduces to

$$
\frac{(1-\alpha)|b|^2 - \sum\limits_{k=2}^{\infty}[(1-\alpha)bt + (1+\beta)(\lambda_k - t)]\bar{b}|a_k|r^{k-1} - \sum\limits_{k=1}^{\infty}[(1+\beta)(\mu_k + t) - (1-\alpha)bt]\bar{b}|b_k|r^{k-1}}{|b|^2(1 - \sum\limits_{k=2}^{\infty} t|a_k|r^{k-1} + \sum\limits_{k=1}^{\infty} t|b_k r^{k-1})} \geq 0. \qquad (2.3)
$$

If the condition (2.2) does not hold then the numerator in (2.3) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0,1)$ for which the quotient in (2.3) is negative. This contradicts the condition for $f \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$. Hence the proof is complete. □

3. Extreme points and distortion bounds

In this section, our first theorem gives the extreme points of the closed convex hulls of $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$.

Theorem 3.1. *Let f be given by (1.2). Then $f\in\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$ if and only if*

$$f(z)=\sum_{k=1}^{\infty}(X_kh_k(z)+Y_kg_k(z)), \tag{3.1}$$

where $h_1(z)=z$, $h_k(z)=z-\frac{(1-\alpha)|b|}{[(\lambda_k-t)(1+\beta)+(1-\alpha)|b|t]}z^k$ $(k=2,3,...)$, $g_k(z)=z+\frac{(1-\alpha)|b|}{[(\mu_k+t)(1+\beta)-(1-\alpha)|b|t]}\overline{z}^k$ $(k=1,2,3,...)$, $\sum\limits_{k=1}^{\infty}(X_k+Y_k)=1$, $X_k\geq 0$, $Y_k\geq 0$. In particular, the extreme points of $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For functions f of the form (3.1), we have

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}(X_kh_k(z)+Y_kg_k(z)) \\ &= \sum_{k=1}^{\infty}(X_k+Y_k)\,z-\sum_{k=2}^{\infty}\frac{(1-\alpha)|b|}{[(\lambda_k-t)(1+\beta)+(1-\alpha)|b|t]}X_kz^k \\ &\qquad +\sum_{k=1}^{\infty}\frac{(1-\alpha)|b|}{[(\mu_k+t)(1+\beta)-(1-\alpha)|b|t]}Y_k\overline{z}^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{[(\lambda_k-t)(1+\beta)+(1-\alpha)|b|t]}{(1-\alpha)|b|}\left(\frac{(1-\alpha)|b|}{[(\lambda_k-t)(1+\beta)+(1-\alpha)|b|t]}\right)X_k \\ &\quad+\sum_{k=1}^{\infty}\frac{[(\mu_k+t)(1+\beta)-(1-\alpha)|b|t]}{(1-\alpha)|b|}\left(\frac{(1-\alpha)|b|}{[(\mu_k+t)(1+\beta)-(1-\alpha)|b|t]}\right)Y_k \\ &=\sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k=1-X_1\leq 1 \end{aligned}$$

and so f is in closed convex hulls of $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$. Conversely, suppose that f is in closed convex hulls of $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$. Setting

$$X_k=\frac{[(\lambda_k-t)(1+\beta)+(1-\alpha)|b|t]}{(1-\alpha)|b|}|a_k|,\ \ k=2,3,\dots,$$

and

$$Y_k=\frac{[(\mu_k+t)(1+\beta)-(1-\alpha)|b|t]}{(1-\alpha)|b|}|b_k|,\ \ k=1,2,\dots,$$

where $\sum\limits_{k=1}^{\infty}(X_k+Y_k)=1$. Then note that by Theorem 2.2, $0\le X_k\le 1$ $(k=2,3,\dots)$ and $0\le Y_k\le 1$ $(k=1,2,3,\dots)$. We define $X_1=1-\sum\limits_{k=2}^{\infty}X_k-\sum\limits_{k=1}^{\infty}Y_k$ and by Theorem 2.2 , $X_1\ge 0$. Consequently, we obtain $f(z)=\sum\limits_{k=1}^{\infty}\left(X_kh_k(z)+Y_kg_k(z)\right)$.

Using Theorem 2.2, it is easily seen that $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$ is convex and closed, so closed convex hulls of $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$ is $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$. In other words, the statement of Theorem 3.1 is really for $f\in\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$. □

The following theorem gives the distortion bounds for functions in $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$ which yields a covering result for this class.

Theorem 3.2. *Letf $\in\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$ and $\lambda_2\le\lambda_k$, $\lambda_2\le\mu_k$ for $k\ge 2$, then*

$$|f(z)|\le(1+|b_1|)r+\left(\frac{(1-\alpha)|b|}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}-\frac{[(\mu_1+t)(1+\beta)-(1-\alpha)t|b|]}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}|b_1|\right)r^2$$

and

$$|f(z)|\ge(1-|b_1|)r-\left(\frac{(1-\alpha)|b|}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}-\frac{[(\mu_1+t)(1+\beta)-(1-\alpha)t|b|]}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}|b_1|\right)r^2.$$

Proof. Let $f\in\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$. Taking the absolute value of f, we obtain

$$\begin{aligned}
|f(z)| &\le (1+|b_1|)r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k\\
&\le (1+|b_1|)r+\frac{(1-\alpha)|b|}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}\\
&\quad \left(\sum_{k=2}^{\infty}\frac{[(\lambda_k-t)(1+\beta)+(1-\alpha)t|b|]}{(1-\alpha)|b|}|a_k|\right.\\
&\qquad\left.+\frac{[(\mu_k+t)(1+\beta)-(1-\alpha)t|b|]}{(1-\alpha)|b|}|b_k|\right)r^2\\
&= (1+|b_1|)r\\
&\quad +\left(\frac{(1-\alpha)|b|}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}\right.\\
&\qquad\left.-\frac{[(\mu_1+t)(1+\beta)-(1-\alpha)t|b|]}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}|b_1|\right)r^2.
\end{aligned}$$

Similarly

$$|f(z)| \geq (1-|b_1|)r - \left(\frac{(1-\alpha)|b|}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]} - \frac{[(\mu_1+t)(1+\beta)-(1-\alpha)t|b|]}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}|b_1|\right) r^2.$$

The upper and lower bounds given in Theorem 3.2 are respectively attained for the following functions.

$$f(z) = z + |b_1|\bar{z} + \left(\frac{(1-\alpha)|b|}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]} - \frac{[(\mu_1+t)(1+\beta)-(1-\alpha)t|b|]}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}|b_1|\right) \bar{z}^2$$

and

$$f(z) = (1-|b_1|)z - \left(\frac{(1-\alpha)|b|}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]} - \frac{[(\mu_1+t)(1+\beta)-(1-\alpha)t|b|]}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}|b_1|\right) z^2.$$

□

The following covering result follows from the left hand inequality in Theorem 3.2.

Corollary 3.3. *Let f of the form (1.2) be so that $f \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ and $\lambda_2 \leq \lambda_k$, $\lambda_2 \leq \mu_k$ for $k \geq 2$. Then*

$$\left\{\omega : |\omega| < 1 - \frac{(1-\alpha)|b|}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]} - \left[1 - \frac{[(\mu_1+t)(1+\beta)-(1-\alpha)t|b|]}{[(\lambda_2-t)(1+\beta)+(1-\alpha)t|b|]}\right]|b_1|\right\}.$$

4. Convolution and convex combinations

In this section, we show that the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ is closed under convolution and convex combinations. Now we need the following definition of convolution of two harmonic functions. For $f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \bar{z}^k$ and

$F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| \overline{z}^k$, we define the convolution of two harmonic functions f and F as

$$(f * F)(z) = f(z) * F(z) = z - \sum_{k=2}^{\infty} |a_k||A_k| z^k + \sum_{k=1}^{\infty} |b_k||B_k| \overline{z}^k. \tag{4.1}$$

Using the definition, we show that the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ is closed under convolution.

Theorem 4.1. *For $0 \leq \delta < \alpha < 1$, let $f \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ and $F \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \delta, \beta, b; t)$. Then $f * F \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t) \subset \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \delta, \beta, b; t)$.*

Proof. Let $f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \overline{z}^k$ and $F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| \overline{z}^k$ be in $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \delta, \beta, b; t)$. Then the convolution $f * F$ is given by (4.1), from the assertion that $f * F \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \delta, \beta, b; t)$. We note that $|A_k| \leq 1$ and $|B_k| \leq 1$. In view of Theorem 2.2 and the inequality $0 \leq \delta \leq \alpha < 1$, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\delta)|b|t]}{(1-\delta)|b|} |a_k||A_k| \\
&\quad + \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\delta)|b|t]}{(1-\delta)|b|} |b_k||B_k| \\
&\leq \sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\delta)|b|t]}{(1-\delta)|b|} |a_k| \\
&\quad + \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\delta)|b|t]}{(1-\delta)|b|} |b_k| \\
&\leq \sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\alpha)|b|t]}{(1-\alpha)|b|} |a_k| \\
&\quad + \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\alpha)|b|t]}{(1-\alpha)|b|} |b_k| \\
&\leq 1,
\end{aligned}$$

by Theorem 2.2, $f \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$. By the same token, we then conclude that $f * F \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t) \subset \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \delta, \beta, b; t)$. □

Next, we show that the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ is closed under convex combination of its members.

Theorem 4.2. *The class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \ldots$, let $f_i \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$, where f_i is given by

$$f_i(z) = z - \sum_{k=2}^{\infty} |a_{ik}| z^k + \sum_{k=1}^{\infty} |b_{ik}| \overline{z}^k.$$

For $\sum\limits_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{ik}| \right) z^k + \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{ik}| \right) \overline{z}^k.$$

Since,

$$\sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\alpha)|b|t]}{(1-\alpha)|b|} |a_{ik}| + \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\alpha)|b|t]}{(1-\alpha)|b|} |b_{ik}| \leq 1,$$

from the above equation we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\alpha)|b|t]}{(1-\alpha)|b|} \sum_{i=1}^{\infty} t_i |a_{ik}| \\
&\qquad + \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\alpha)|b|t]}{(1-\alpha)|b|} \sum_{i=1}^{\infty} t_i |b_{ik}| \\
&= \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=2}^{\infty} \frac{[(\lambda_k - t)(1+\beta) + (1-\alpha)|b|t]}{(1-\alpha)|b|} |a_{ik}| \right. \\
&\qquad \left. + \sum_{k=1}^{\infty} \frac{[(\mu_k + t)(1+\beta) - (1-\alpha)|b|t]}{(1-\alpha)|b|} |b_{ik}| \right\} \\
&\leq \sum_{i=1}^{\infty} t_i = 1.
\end{aligned}$$

This is the condition required by (2.2) and so $\sum\limits_{i=1}^{\infty} t_i f_i \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$. □

5. Class preserving integral operator

Finally, we consider the closure property of the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$ under the generalized Bernardi-Libera -Livingston integral operator$\mathcal{L}_c[f]$ which is defined by

$$\mathcal{L}_c[f(z)] = \frac{c+1}{z^c} \int_0^z \xi^{c-1} f(\xi) d\xi \ (c > -1).$$

Theorem 5.1. *Let $f \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$. Then $\mathcal{L}_c[f] \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \alpha, \beta, b; t)$.*

Proof. From the representation of $\mathcal{L}_c[f]$, it follows that

$$\begin{aligned}
\mathcal{L}_c[f(z)] &= \frac{c+1}{z^c}\int_0^z \xi^{c-1}h(\xi)d\xi + \overline{\frac{c+1}{z^c}\int_0^z \xi^{c-1}g(\xi)d\xi} \\
&= \frac{c+1}{z^c}\int_0^z \xi^{c-1}\left(\xi - \sum_{k=2}^{\infty}|a_k|\xi^k\right)d\xi \\
&\quad + \overline{\frac{c+1}{z^c}\int_0^z \xi^{c-1}\left(\sum_{k=1}^{\infty}|b_k|\xi^k\right)d\xi} \\
&= z - \sum_{k=2}^{\infty}A_k z^k + \sum_{k=1}^{\infty}B_k z^k,
\end{aligned}$$

where $A_k = \frac{c+1}{c+k}\ |a_k|$ and $B_k = \frac{c+1}{c+k}\ |b_k|$. Hence

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{[(\lambda_k - t)(1+\beta)+(1-\alpha)t|b|]}{(1-\alpha)|b|}\left(\frac{c+1}{c+k}|a_k|\right) \\
&\quad + \sum_{k=1}^{\infty}\frac{[(\mu_k + t)(1+\beta)-(1-\alpha)t|b|]}{(1-\alpha)|b|}\left(\frac{c+1}{c+n}|b_k|\right) \\
&\qquad \leq \sum_{k=2}^{\infty}\frac{[(\lambda_k - t)(1+\beta)+(1-\alpha)t|b|]}{(1-\alpha)|b|}|a_k| \\
&\qquad\quad + \sum_{k=1}^{\infty}\frac{[(\mu_k + t)(1+\beta)-(1-\alpha)t|b|]}{(1-\alpha)|b|}|b_k| \\
&\qquad \leq 1,
\end{aligned}$$

since $f \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$, therefore by Theorem 2.2, $\mathcal{L}_c[f] \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\alpha,\beta,b;t)$. □

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On a certain class of analytic functions

Saurabh Porwal and Kaushal Kishore Dixit

Abstract. In this paper, authors introduce a new class $R(\beta,\alpha,n)$ of Salagean-type analytic functions. We obtain extreme points of $R(\beta,\alpha,n)$ and some sharp bounds for $Re\left\{\frac{D^n f(z)}{z}\right\}$ and $Re\left\{\frac{D^{n-1} f(z)}{z}\right\}$. Relevant connections of the results presented here with various known results are briefly indicated.

Mathematics Subject Classification (2010): 30C45.

Keywords: Analytic, Salagean derivative, extreme points.

1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$ and normalized by the condition $f(0) = f'(0) - 1 = 0$.

Further, let S be the class of functions in A which are univalent in U. For $0 \leq \beta < 1$, $\alpha > 0$ and $n \in N_0 = N \cup 0$, we let

$$R(\beta,\alpha,n) = \left\{ f(z) \in A : Re\left\{ \frac{D^n f(z) + \alpha(D^{n+1} f(z) - D^n f(z))}{z} \right\} > \beta,\ z \in U \right\},$$

where D^n stands for Salagean derivative operator introduced by Salagean [9].

By specializing the parameters in the subclass $R(\beta,\alpha,n)$, we obtain the following known subclasses of S studied earlier by various researchers.

(i) $R(\beta,\alpha,1) \equiv R(\beta,\alpha)$ studied by Gao and Zhou [4].

(ii) $R(\beta,1,1) \equiv R(\beta)$ studied by various authors ([2], [3] and [8]), see also ([1], [6], [11]).

(iii) $R(\beta,0,1) \equiv R_\beta$ studied by Hallenbeck [5].

Now, we introduce Alexander operator $I^n f(z) : A \to A,\ n \in N_0$ by

$$I^0 f(z) = f(z)$$

$$I^1 f(z) = \int_0^z \frac{f(t)}{t} dt$$

....

$$I^n f(z) = I^1(I^{n-1} f(z)),\ n \in N.$$

Thus

$$I^n f(z) = z + \sum_{k=2}^{\infty} \frac{1}{k^n} a_k z^k.$$

It can be easily seen that

$$D^n(I^n f(z)) = f(z) = I^n(D^n f(z)).$$

In the present paper, we determine extreme points of $R(\beta, \alpha, n)$ and also to obtain some sharp bounds for $Re\left\{\frac{D^n f(z)}{z}\right\}$ and $Re\left\{\frac{D^{n-1} f(z)}{z}\right\}$.

2. Main results

Theorem 2.1. *A function $f(z)$ is in $R(\beta, \alpha, n)$, if and only if $f(z)$ can be expressed as,*

$$f(z) = \int_{|x|=1} \left[(2\beta - 1)z + 2(1-\beta)\overline{x} \sum_{k=0}^{\infty} \frac{(xz)^{k+1}}{(k+1)^n (k\alpha + 1)}\right] d\mu(x), \tag{2.1}$$

where $\mu(x)$ is the probability measure defined on the $X = \{x : |x| = 1\}$. For fixed α, β, n and $R(\beta, \alpha, n)$ the probability measure μ defined on X are one-to-one by the expression (2.1).

Proof. By the definition of $R(\beta, \alpha, n)$, $f(z) \in R(\beta, \alpha, n)$, if and only if

$$\frac{\frac{D^n f(z) + \alpha(D^{n+1} f(z) - D^n f(z))}{z} - \beta}{1 - \beta} \in P,$$

where P denotes the normalized well-known class of analytic functions which have positive real part. By the aid of Herglotz expression of functions in P, we have

$$\frac{\frac{D^n f(z) + \alpha(D^{n+1} f(z) - D^n f(z))}{z} - \beta}{1 - \beta} = \int_{|x|=1} \frac{1 + xz}{1 - xz} d\mu(x),$$

which is equivalent to

$$\frac{D^n f(z) + \alpha(D^{n+1} f(z) - D^n f(z))}{z} = \int_{|x|=1} \frac{1 + (1-2\beta)xz}{1 - xz} d\mu(x).$$

So we have

$$I^n \left[z \left\{ \frac{D^n f(z) + \alpha(D^{n+1} f(z) - D^n f(z))}{z} \right\} \right] = \int_{|x|=1} I^n z \left\{ \frac{1 + (1-2\beta)xz}{1 - xz} \right\} d\mu(x),$$

or

$$f(z)+\alpha(zf'(z)-f(z))=\int_{|x|=1}\left\{z+\sum_{k=2}^{\infty}\frac{2(1-\beta)x^{k-1}z^k}{k^n}\right\}d\mu(x),$$

that is,

$$z^{1-\frac{1}{\alpha}}\int_0^z\left\{\frac{1}{\alpha}f(\zeta)+(\zeta f'(\zeta)-f(\zeta))\right\}\zeta^{\frac{1}{\alpha}-2}d\zeta$$

$$=\frac{1}{\alpha}\int_{|x|=1}\left\{z^{1-\frac{1}{\alpha}}\int_0^z\left\{\zeta+2(1-\beta)\sum_{k=2}^{\infty}\frac{x^{k-1}\zeta^k}{k^n}\right\}\zeta^{\frac{1}{\alpha}-2}\right\}d\mu(x).$$

We obtain

$$f(z)=\int_{|x|=1}\left\{z+2(1-\beta)\sum_{k=2}^{\infty}\frac{x^{k-1}z^k}{k^n(\alpha k+1-\alpha)}\right\}d\mu(x),$$

or equivalently

$$f(z)=\int_{|x|=1}\left\{(2\beta-1)z+2(1-\beta)\overline{x}\sum_{k=0}^{\infty}\frac{(xz)^{k+1}}{(k+1)^n(\alpha k+1)}\right\}d\mu(x).$$

This deductive process can be converse, so we have proved the first part of the theorem. we know that both probability measure μ and class P, class P and $R(\beta,\alpha,n)$ are one-to-one, so the second part of the theorem is true. Thus the proof of Theorem 2.1 is established. □

Corollary 2.2. *The extreme points of the class $R(\beta,\alpha,n)$ are*

$$f_x(z)=(2\beta-1)z+2(1-\beta)\overline{x}\sum_{k=0}^{\infty}\frac{(xz)^{k+1}}{(k+1)^n(\alpha k+1)},\ |x|=1. \tag{2.2}$$

Proof. Using the notation $f_x(z)$ equation (2.1) can be written as

$$f_\mu(z)=\int_{|x|=1}f_x(z)d\mu(x).$$

By Theorem 2.1, the map $\mu\to f_\mu$ is one-to-one so the assertion follows (see [5]). □

Corollary 2.3. *If $f(z)=z+\sum_{k=2}^{\infty}a_kz^k\in R(\beta,\alpha,n)$, then*

$$|a_k(z)|\le\frac{2(1-\beta)}{k^n(\alpha k+1-\alpha)},\ (k\ge 2).$$

The results are sharp.

Proof. The coefficient bounds are maximized at an extreme point. Now from (2.2), $f_x(z)$ can be expressed as

$$f_x(z)=z+2(1-\beta)\sum_{k=2}^{\infty}\frac{x^{k-1}z^k}{k^n(\alpha k+1-\alpha)},\ |x|=1, \tag{2.3}$$

and the result follows. □

Corollary 2.4. *If $f(z) \in R(\beta,\alpha,n)$, then for $|z| = r < 1$*

$$|f(z)| \leq r + 2(1-\beta)\sum_{k=2}^{\infty}\frac{r^k}{k^n(\alpha k + 1 - \alpha)}.$$

The result follows from (2.3).

Next, we determine the sharp lower bound of $Re\left\{\frac{D^n f(z)}{z}\right\}$ and $Re\left\{\frac{D^{n-1} f(z)}{z}\right\}$ for $f(z) \in R(\beta,\alpha,n)$. Since $R(\beta,\alpha,n)$ is rotationally invariant, we may restrict our attention to the extreme point of

$$g(z) = z + 2(1-\beta)\sum_{k=2}^{\infty}\frac{z^k}{k^n(\alpha k + 1 - \alpha)}. \tag{2.4}$$

Theorem 2.5. *If $f(z) \in R(\beta,\alpha,n)$, then for $|z| \leq r < 1$ we have*

$$Re\left\{\frac{D^n f(z)}{z}\right\} \geq 1 + 2(1-\beta)\sum_{k=2}^{\infty}\frac{(-r)^{k-1}}{\alpha(k-1)+1} > 1 + 2(1-\beta)\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{\alpha(k-1)+1}, \tag{2.5}$$

and

$$Re\left\{\frac{D^n f(z)}{z}\right\} \leq 1 + 2(1-\beta)\sum_{k=2}^{\infty}\frac{(-r)^{k-1}}{\alpha(k-1)+1} < 1 + 2(1-\beta)\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{\alpha(k-1)+1}. \tag{2.6}$$

These inequalities are both sharp.

Proof. We need only consider $g(z)$ defined by (2.4). We have

$$\frac{D^n g(z)}{z} = 1 + 2(1-\beta)\sum_{k=2}^{\infty}\frac{z^{k-1}}{\alpha(k-1)+1}. \tag{2.7}$$

It can be written as

$$\frac{D^n g(z)}{z} = 1 + 2\frac{(1-\beta)}{\alpha}\int_0^1 t^{\frac{1}{\alpha}}\frac{z}{1-tz}dt. \tag{2.8}$$

So we have

$$Re\left\{\frac{D^n g(z)}{z}\right\} = 1 + 2\frac{(1-\beta)}{\alpha}\int_0^1 t^{\frac{1}{\alpha}}Re\left\{\frac{z}{1-tz}\right\}dt. \tag{2.9}$$

Since $k(z) = \frac{z}{1-tz}$ is convex in U, $k(\overline{z}) = \overline{k(z)}$ and $k(z)$ maps real axis to real axis, we have

$$-\frac{r}{1+tr} \leq Re\left\{\frac{z}{1-tz}\right\} \leq \frac{r}{1-tr}, \ (|z| \leq r).$$

Substituting the last inequalities in (2.9) and expanding the integrand into the power series of t and integrating it, we can obtain the inequalities (2.5) and (2.6).

The sharpness can be seen from (2.7). □

Theorem 2.6. *$D^{n-1}\left[R(\beta,\alpha,n)\right] \subset S$ for $\beta \geq \beta_0$ and this result can not be extended to $\beta < \beta_0$, where*

$$\beta_0 = 1 + \frac{1}{2}\left(\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{\alpha(k-1)+1}\right)^{-1}.$$

Proof. Let $f(z) \in R(\beta, \alpha, n)$.

Now using (2.5)

$$\left(D^{n-1} f(z)\right)' = 1 + 2(1-\beta) \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{\alpha(k-1)+1} \geq 0. \tag{2.10}$$

$D^{n-1} f(z) \in S$, that is, if $\beta \geq \beta_0$, we have $D^{n-1}\left[R(\beta, \alpha, n)\right] \subset S$. The result can not be extended to $\beta < \beta_0$ because $\left(D^{n-1} f(-1)\right)' = 0$ at $\beta = \beta_0$. Thus $\left(D^{n-1} f(-r)\right)' = 0$ for some $r = r(\beta) < 1$ when $\beta < \beta_0$. □

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Theorem 2.7. *If $f(z) \in R(\beta, \alpha, n)$, then for $|z| \leq r < 1$*

$$Re\left\{\frac{D^{n-1} f(z)}{z}\right\} \geq 1 + 2(1-\beta) \sum_{k=2}^{\infty} \frac{(-r)^{k-1}}{k[\alpha(k-1)+1]} \tag{2.11}$$

$$> 1 + 2(1-\beta) \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k[\alpha(k-1)+1]}.$$

The result is sharp.

Proof. According to the same reasoning as in Theorem 2.5, we need only consider $g(z)$ defined by (2.4). We have

$$\frac{D^{n-1} g(z)}{z} = 1 + 2(1-\beta) \sum_{k=2}^{\infty} \frac{z^{k-1}}{k[\alpha(k-1)+1]}$$

$$= 1 + 2\frac{(1-\beta)}{\alpha} \int_0^1 t^{\frac{1}{\alpha}} \left(\int_0^1 \frac{vz}{1-tvz} dv\right) dt.$$

Thus

$$Re\left\{\frac{D^{n-1} g(z)}{z}\right\} = 1 + 2\frac{(1-\beta)}{\alpha} \int_0^1 t^{\frac{1}{\alpha}} \left(\int_0^1 vRe\left\{\frac{z}{1-tvz}\right\} dv\right) dt$$

$$> 1 - 2\frac{(1-\beta)}{\alpha} \int_0^1 t^{\frac{1}{\alpha}} \left(\int_0^1 \frac{vr}{1+tvr} dv\right) dt$$

$$= 1 + 2(1-\beta) \sum_{k=2}^{\infty} \frac{(-r)^{k-1}}{k[\alpha(k-1)+1]}$$

$$> 1 + 2(1-\beta) \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k[\alpha(k-1)+1]}.$$

The sharpness can be seen from (2.4). □

Remark 2.8. If we put $n = 1$ in Theorem 2.1, 2.5 and 2.7 then we obtain the corresponding results due to Gao and Zhou [4].

Remark 2.9. If we put $n = 1, \alpha = 1$ in Theorem 2.1, 2.5, 2.7 then we obtain the corresponding results due to Silverman [10].

Remark 2.10. If we put $n = 1, \alpha = 0$ in Theorem 2.1, 2.5, 2.7 then we obtain the corresponding results due to Hallenbeck [5].

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The order of convexity for a new integral operator

Laura Stanciu and Daniel Breaz

Abstract. In this paper we consider a new integral operator $I\left(f_1,...,f_n;g_1,...,g_n\right)(z)$ for analytic functions $f_i(z)$, $g_i(z)$ in the open unit disk $\mathcal{U}$. The main object of the present paper is to study the order of convexity for this integral operator.

Mathematics Subject Classification (2010): 30C45.

Keywords: Analytic functions, integral operator, starlike functions, convex functions, general Schwarz lemma.

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f\left(z\right) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$$

and satisfy the following usual normalization condition

$$f\left(0\right) = f'\left(0\right) - 1 = 0.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathcal{U}$.

Definition 1.1. *A function f belonging to $\mathcal{S}$ is a starlike function by the order α, $0 \leq \alpha < 1$ if f satisfies the inequality*

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in \mathcal{U}.$$

We denote this class by $\mathcal{S}^*\left(\alpha\right)$.

Definition 1.2. *A function f belonging to $\mathcal{S}$ is a convex function by the order $\alpha, 0 \leq \alpha < 1$ if f satisfies the inequality*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) > \alpha,\ z \in \mathcal{U}.$$

We denote this class by $\mathcal{K}(\alpha)$.
A function $f \in \mathcal{S}$ is in the class $\mathcal{P}(\alpha)$ if and only if

$$\operatorname{Re}(f'(z)) > \alpha,\ z \in \mathcal{U}.$$

In [1], Frasin and Jahangiri introduced the class $\mathcal{B}(\mu, \alpha)$ defined as follows.

Definition 1.3. *A function $f(z) \in \mathcal{A}$ is said to be a member of the class $\mathcal{B}(\mu, \alpha)$ if and only if*

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < 1-\alpha, \tag{1.1}$$

$z \in \mathcal{U}; 0 \leq \alpha < 1; \mu \geq 0.$

Note that the condition (1.1) is equivalent to

$$\operatorname{Re}\left(f'(z)\left(\frac{z}{f(z)}\right)^{\mu}\right) > \alpha,$$

for $z \in \mathcal{U}; 0 \leq \alpha < 1; \mu \geq 0$.
Clearly, $\mathcal{B}(1,\alpha) = \mathcal{S}^*(\alpha)$, $\mathcal{B}(0,\alpha) = \mathcal{P}(\alpha)$ and $\mathcal{B}(2,\alpha) = \mathcal{B}(\alpha)$ the class which has been introduced and studied by Frasin and Darus [2] (see also [3]).
Let $\mathcal{S}^*_\beta$ be the subclass of $\mathcal{A}$ consisting of the functions f which satisfy the inequality

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \beta,\ 0 < \beta \leq 1; z \in \mathcal{U} \tag{1.2}$$

and let $\mathcal{S}_\beta$ be the subclass of $\mathcal{A}$ consisting of the functions f which satisfy the inequality

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < \beta,\ 0 < \beta \leq 1; z \in \mathcal{U}. \tag{1.3}$$

For $f_i(z),\ g_i(z) \in \mathcal{A}$ and $\delta_i, \gamma_i \in \mathbb{C}$, we define the integral operator $I_\beta(f_1, ..., f_n; g_1, ..., g_n)(z)$ given by

$$I_\beta(f_1, ..., f_n; g_1, ..., g_n)(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\delta_i}\left(e^{g_i(t)}\right)^{\gamma_i} dt. \tag{1.4}$$

In order to prove our main results, we recall the following lemma.

Lemma 1.4. (**General Schwarz Lemma**) (see [4]). *Let the function f be regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m,\ z \in \mathcal{U}_R.$$

The equality can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,\ \text{where } \theta \text{ is constant.}$$

2. The order of convexity for the integral operator $I(f_1,...,f_n;g_1,...,g_n)$

Theorem 2.1. *Let the functions $f_i, g_i \in \mathcal{A}$ and suppose that $|g_i(z)| \le M_i$, $M_i \ge 1$ for all $i \in \{1,2,...,n\}$. If $f_i \in \mathcal{S}^*_{\beta_i}$, $0 < \beta_i \le 1$ and $g_i \in \mathcal{B}(\mu_i, \alpha_i)$, $\mu_i \ge 0, 0 \le \alpha_i < 1$ then the integral operator $I(f_1,...,f_n;g_1,...,g_n)(z)$ defined by (1.4) is in $\mathcal{K}(\lambda)$, where*

$$\lambda = 1 - \sum_{i=1}^{n} [|\delta_i| \beta_i + |\gamma_i| (2-\alpha_i) M_i^{\mu_i}]$$

and

$$\sum_{i=1}^{n} [|\delta_i| \beta_i + |\gamma_i| (2-\alpha_i) M_i^{\mu_i}] < 1, \ \delta_i, \gamma_i \in \mathbb{C}$$

for all $i \in \{1,2,...,n\}$.

Proof. From (1.4) we obtain

$$I'(f_1,...,f_n;g_1,...,g_n)(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z}\right)^{\delta_i} \left(e^{g_i(z)}\right)^{\gamma_i}$$

and

$$I''(f_1,...,f_n;g_1,...,g_n)(z) =$$

$$\sum_{i=1}^{n} \left[\delta_i \left(\frac{f_i(z)}{z}\right)^{\delta_i - 1} \left(\frac{zf_i'(z) - f_i(z)}{z^2}\right) \left(e^{g_i(z)}\right)^{\gamma_i}\right] \prod_{\substack{k=1 \\ k\neq i}}^{n} \left(\frac{f_k(z)}{z}\right)^{\delta_k} \left(e^{g_k(z)}\right)^{\gamma_k}$$

$$+ \sum_{i=1}^{n} \left[\left(\frac{f_i(z)}{z}\right)^{\delta_i} \gamma_i \left(e^{g_i(z)}\right)^{\gamma_i - 1} g_i'(z) e^{g_i(z)}\right] \prod_{\substack{k=1 \\ k\neq i}}^{n} \left(\frac{f_k(z)}{z}\right)^{\delta_k} \left(e^{g_k(z)}\right)^{\gamma_k}.$$

After the calculus we obtain that

$$\frac{zI''(f_1,...,f_n;g_1,...,g_n)(z)}{I'(f_1,...,f_n;g_1,...,g_n)(z)} = \sum_{i=1}^{n} \left[\delta_i \left(\frac{zf_i'(z)}{f_i(z)} - 1\right) + \gamma_i z g_i'(z)\right]. \qquad (2.1)$$

It follows from (2.1) that

$$\left|\frac{zI''(f_1,...,f_n;g_1,...,g_n)(z)}{I'(f_1,...,f_n;g_1,...,g_n)(z)}\right| \le \sum_{i=1}^{n} \left[|\delta_i| \left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + |\gamma_i| \, |zg_i'(z)|\right]$$

$$\le \sum_{i=1}^{n} \left[|\delta_i| \left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + |\gamma_i| \left|g_i'(z) \left(\frac{z}{g_i(z)}\right)^{\mu_i}\right| \left|\frac{g_i(z)}{z}\right|^{\mu_i} |z|\right]. \qquad (2.2)$$

Since $|g_i(z)| \le M_i$, $z \in \mathcal{U}$ applying the General Schwarz Lemma for the functions g_i, we have

$$|g_i(z)| \le M_i |z|, \ z \in \mathcal{U}$$

for all $i \in \{1, 2, ..., n\}$.
Because $f_i \in \mathcal{S}^*_{\beta_i}$, $0 < \beta_i \leq 1$, $i \in \{1, 2, ..., n\}$, we apply in the relation (2.2) the inequalities (1.2) and we obtain

$$\left| \frac{zI''(f_1, ..., f_n; g_1, ..., g_n)(z)}{I'(f_1, ..., f_n; g_1, ..., g_n)(z)} \right| \leq \sum_{i=1}^{n} \left[|\delta_i| \beta_i + |\gamma_i| \left| g_i'(z) \left(\frac{z}{g_i(z)} \right)^{\mu_i} \right| M_i^{\mu_i} \right]. \quad (2.3)$$

From (2.3) and (1.1), we see that

$$\left| \frac{zI''(f_1, ..., f_n; g_1, ..., g_n)(z)}{I'(f_1, ..., f_n; g_1, ..., g_n)(z)} \right|$$

$$\leq \sum_{i=1}^{n} \left[|\delta_i| \beta_i + |\gamma_i| \left(\left| g_i'(z) \left(\frac{z}{g_i(z)} \right)^{\mu_i} - 1 \right| + 1 \right) M_i^{\mu_i} \right]$$

$$\leq \sum_{i=1}^{n} \left[|\delta_i| \beta_i + |\gamma_i| (2 - \alpha_i) M_i^{\mu_i} \right]$$

$$= 1 - \lambda.$$

So, the integral operator $I(f_1, ..., f_n; g_1, ..., g_n)(z)$ defined by (1.4) is in $\mathcal{K}(\lambda)$. This completes the proof. □

Setting $n = 1$ in Theorem 2.1 we obtain

Corollary 2.2. *Let the functions $f, g \in \mathcal{A}$ and suppose that $|g(z)| \leq M$, $M \geq 1$. If $f \in \mathcal{S}^*_\beta$, $0 < \beta \leq 1$ and $g \in \mathcal{B}(\mu, \alpha)$, $\mu \geq 0, 0 \leq \alpha < 1$ then the integral operator*

$$I(f; g)(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\delta \left(e^{g(t)} \right)^\gamma dt$$

is in $\mathcal{K}(\lambda)$, where

$$\lambda = 1 - [|\delta| \beta + |\gamma| (2 - \alpha) M^\mu]$$

and

$$[|\delta| \beta + |\gamma| (2 - \alpha) M^\mu] < 1, \ \ \delta, \gamma \in \mathbb{C}.$$

Theorem 2.3. *Let the functions $f_i, g_i \in \mathcal{A}$ and suppose that $|f_i(z)| \leq M_i$, $|g_i(z)| \leq N_i$, $M_i \geq 1$, $N_i \geq 1$ for all $i \in \{1, 2, ..., n\}$. If $f_i \in \mathcal{S}_{\beta_i}$, $0 < \beta_i \leq 1$ and $g_i \in \mathcal{B}(\mu_i, \alpha_i)$, $\mu_i \geq 0, 0 < \alpha_i < 1$ then the integral operator $I(f_1, ..., f_n; g_1, ..., g_n)(z)$ defined by (1.4) is in $\mathcal{K}(\lambda)$ where*

$$\lambda = 1 - \sum_{i=1}^{n} [|\delta_i| ((\beta_i + 1) M_i + 1) + |\gamma_i| (2 - \alpha_i) N_i^{\mu_i}]$$

and

$$\sum_{i=1}^{n} [|\delta_i| ((\beta_i + 1) M_i + 1) + |\gamma_i| (2 - \alpha_i) N_i^{\mu_i}] < 1, \ \ \delta_i, \gamma_i \in \mathbb{C}$$

for all $i \in \{1, 2, ..., n\}$.

Proof. If we make the similar operations to the proof of Theorem 2.1, we have

$$\frac{zI''(f_1,...,f_n;g_1,...,g_n)(z)}{I(f_1,...,f_n;g_1,...,g_n)(z)}=\sum_{i=1}^{n}\left[\delta_i\left(\frac{zf_i'(z)}{f_i(z)}-1\right)+\gamma_i z g_i'(z)\right]. \tag{2.4}$$

From the relation (2.4), we obtain that

$$\left|\frac{zI''(f_1,...,f_n;g_1,...,g_n)(z)}{I'(f_1,...,f_n;g_1,...,g_n)(z)}\right|$$

$$\leq\sum_{i=1}^{n}\left[|\delta_i|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)+|\gamma_i|\,|zg_i'(z)|\right]$$
$$\leq\sum_{i=1}^{n}\left[|\delta_i|\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)}\right|\left|\frac{f_i(z)}{z}\right|+1\right)+|\gamma_i|\left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i}\right|\left|\frac{g_i(z)}{z}\right|^{\mu_i}|z|\right]. \tag{2.5}$$

Since $|f_i(z)|\leq M_i$, $|g_i(z)|\leq N_i$, $z\in\mathcal{U}$ applying the General Schwarz Lemma for the functions f_i, g_i, we obtain

$$|f_i(z)|\leq M_i|z|,\ z\in\mathcal{U} \text{ and } |g_i(z)|\leq N_i|z|,\ z\in\mathcal{U}$$

for all $i\in\{1,2,...,n\}$.
Because $f_i\in\mathcal{S}_{\beta_i}$, $0<\beta_i\leq 1$ $i\in\{1,2,...,n\}$, we apply in the relation (2.5) the inequality (1.3) and we obtain

$$\left|\frac{zI''(f_1,...,f_n;g_1,...,g_n)(z)}{I'(f_1,...,f_n;g_1,...,g_n)(z)}\right|$$

$$\leq\sum_{i=1}^{n}\left[|\delta_i|\left(\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)}-1\right|+1\right)M_i+1\right)+|\gamma_i|\left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i}\right|N_i^{\mu_i}\right]$$
$$\leq\sum_{i=1}^{n}\left[|\delta_i|\left((\beta_i+1)M_i+1\right)+|\gamma_i|\left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i}\right|N_i^{\mu_i}\right] \tag{2.6}$$

From (2.6) and (1.1) we obtain

$$\left|\frac{zI''(f_1,...,f_n;g_1,...,g_n)(z)}{I'(f_1,...,f_n;g_1,...,g_n)(z)}\right|$$

$$\leq\sum_{i=1}^{n}\left[|\delta_i|\left((\beta_i+1)M_i+1\right)+|\gamma_i|\left(\left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i}-1\right|+1\right)N_i^{\mu_i}\right]$$
$$\leq\sum_{i=1}^{n}\left[|\delta_i|\left((\beta_i+1)M_i+1\right)+|\gamma_i|(2-\alpha_i)N_i^{\mu_i}\right]$$
$$=1-\lambda.$$

So, the integral operator $I(f_1,...,f_n;g_1,...,g_n)(z)$ defined by (1.4) is in $\mathcal{K}(\lambda)$. This completes the proof. □

Setting $n=1$ in Theorem 2.3 we obtain

Corollary 2.4. *Let the functions* $f, g \in \mathcal{A}$ *and suppose that* $|f(z)| \leq M$, $|g(z)| \leq N$, $M \geq 1$, $N \geq 1$. *If* $f \in \mathcal{S}_\beta$, $0 < \beta \leq 1$ *and* $g \in \mathcal{B}(\mu, \alpha)$, $\mu \geq 0, 0 < \alpha < 1$ *then the integral operator*

$$I(f;g)(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\delta \left(e^{g(t)}\right)^\gamma dt$$

is in $\mathcal{K}(\lambda)$ *where*

$$\lambda = 1 - [|\delta|((\beta+1)M+1) + |\gamma|(2-\alpha)N^\mu]$$

and

$$[|\delta|((\beta+1)M+1) + |\gamma|(2-\alpha)N^\mu] < 1, \ \delta, \gamma \in \mathbb{C}.$$

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Neighborhood and partial sums results on the class of starlike functions involving Dziok-Srivastava operator

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Abstract. In this paper, we introduce a new subclasses of univalent functions defined in the open unit disc involving Dziok-Srivastava Operator. The results on partial sums, integral means and neighborhood results are discussed.

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1. Introduction

Denote by $\mathcal{A}$ the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are normalized by $f(0) = 0 = f'(0) - 1$ and univalent in $\mathcal{U}$. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order $\alpha(0 \le \alpha < 1)$ if $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha$ and the class $\mathcal{K}(\alpha)$ of convex functions of order $\alpha(0 \le \alpha < 1)$ if $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$ in $\mathcal{U}$. It readily follows that $f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha)$.

Denote by $\mathcal{T}$ the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \;\; a_n \ge 0, \;\; z \in \mathcal{U} \tag{1.2}$$

studied extensively by Silverman [11].

For positive real values of $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ ($\beta_j \neq 0, -1, \ldots;\ j = 1, 2, \ldots, m$) the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$
\begin{aligned}
{}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \quad &:= \quad \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \qquad (1.3)\\
(l \leq m+1;\ l, m \in N_0 \quad &:= \quad N \cup \{0\}; z \in U)
\end{aligned}
$$

where N denotes the set of all positive integers and $(\lambda)_k$ is the Pochhammer symbol defined by

$$
(\lambda)_n = \begin{cases} 1, & n = 0 \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1), & n \in N. \end{cases} \qquad (1.4)
$$

The notation ${}_lF_m$ is quite useful for representing many well-known functions such as the exponential, the Binomial, the Bessel, the Laguerre polynomial and others; for example see [3].

Let $\mathcal{H}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m) : \mathcal{A} \to \mathcal{A}$ be a linear operator defined by

$$
\begin{aligned}
\mathcal{H}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m) f(z) \quad &:= \quad z\ {}_lF_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2 \ldots, \beta_m; z) * f(z)\\
&= \quad z + \sum_{n=2}^{\infty} \Gamma_n\ a_n\ z^n \qquad (1.5)
\end{aligned}
$$

where

$$
\Gamma_n = \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \frac{1}{(n-1)!}, \qquad (1.6)
$$

unless otherwise stated and $*$ the Hadamard product (or convolution) of two functions $f, g \in \mathcal{A}$ where $f(z)$ of the form (1.1) and $g(z)$ be given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ then $f(z) * g(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathcal{U}$.

For simplicity, we can use a shorter notation $\mathcal{H}_m^l[\alpha_1]$ for $\mathcal{H}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m)$ in the sequel. The linear operator $\mathcal{H}_m^l[\alpha_1]$ is called Dziok-Srivastava operator [3] (see [6, 8]), includes (as its special cases) various other linear operators introduced and studied by Carlson and Shaffer [2], Ruscheweyh [9] and Owa-Srivastava [7]. Motivated by earlier works of Aouf et al.,[1] and Dziok and Raina[4] we define the following new subclass of $\mathcal{T}$ involving hypergeometric functions.

For $0 \leq \lambda \leq 1, 0 < \beta \leq 1, -1 \leq B < A \leq 1, 0 \leq \gamma \leq 1$,we let $\mathcal{HF}_\gamma^\lambda(\alpha, \beta, A, B)$ denote the subclass of $\mathcal{T}$ consisting of functions $f(z)$ of the form (1.2) satisfying the analytic condition

$$
\left| \frac{\frac{zF_\lambda'(z)}{F_\lambda(z)} - 1}{(B-A)\gamma\left[\frac{zF_\lambda'(z)}{F_\lambda(z)} - \alpha\right] - B\left[\frac{zF_\lambda'(z)}{F_\lambda(z)} - 1\right]} \right| < \beta, \qquad z \in U \qquad (1.7)
$$

where

$$
\frac{zF_\lambda'(z)}{F_\lambda(z)} = \frac{z\mathcal{H}f'(z) + \lambda z^2 \mathcal{H}f''(z)}{(1-\lambda)\mathcal{H}f(z) + \lambda z \mathcal{H}f'(z)}, \qquad 0 \leq \lambda \leq 1 \qquad (1.8)
$$

and

$$\mathcal{H}f(z) = z + \sum_{n=2}^{\infty} a_n \Gamma_n z^n \tag{1.9}$$

where Γ_n is given by (1.6)

The main object of the present paper is to investigate (n, δ)- neighborhoods of functions $f(z) \in \mathcal{HF}_{\gamma}^{\lambda}(\alpha, \beta, A, B)$. Furthermore, we obtain Partial sums $f_k(z)$ and Integral means inequality of functions $f(z)$ in the class $\mathcal{HF}_{\gamma}^{\lambda}(\alpha, \beta, A, B)$.

We state the following Lemma , due to Vijaya and Deppa [15] which provide the necessary and sufficient conditions for functions $f(z) \in \mathcal{HF}_{\gamma}^{\lambda}(\alpha, \beta, A, B)$.

Lemma 1.1. *A function $f(z) \in \mathcal{T}$ is in the class $\mathcal{HF}_{\gamma}^{\lambda}(\alpha, \beta, A, B)$ if and only if*

$$\sum_{n=2}^{\infty} \phi_B^A(n, \lambda, \alpha, \beta, \gamma) a_k \leq 1 \tag{1.10}$$

where

$$\phi_B^A(n, \lambda, \alpha, \beta, \gamma) = \frac{(1 + n\lambda - \lambda)[(n-1)(1 - \beta B) + \beta\gamma(B - A)(n - \alpha)]\Gamma_n}{\beta\gamma(B - A)(1 - \alpha)}. \tag{1.11}$$

2. Neighborhood results

In this section we discuss neighborhood results of the class $\mathcal{HF}_{\gamma}^{\lambda}(\alpha, \beta, A, B)$ due to Goodman [5] and Ruscheweyh [10] . We define the $\delta-$ neighborhood of function $f(z) \in \mathcal{T}$.

Definition 2.1. *For $f \in \mathcal{T}$ of the form (1.2) and $\delta \geq 0$. We define a $\delta-$neighbourhood of a function $f(z)$ by*

$$\mathcal{N}_\delta(f) = \left\{ g : g \in \mathcal{T} : g(z) = z - \sum_{n=2}^{\infty} c_n z^n \text{ and } \sum_{n=2}^{\infty} n|a_n - c_n| \leq \delta \right\}. \tag{2.1}$$

In particular, for the identity function $e(z) = z$, we immediately have

$$\mathcal{N}_\delta(e) = \left\{ g : g \in \mathcal{T} : g(z) = z - \sum_{n=2}^{\infty} c_n z^n \text{ and } \sum_{n=2}^{\infty} n|c_n| \leq \delta \right\}. \tag{2.2}$$

Theorem 2.2. *If $\delta = \frac{2}{\phi_B^A(2,\lambda,\alpha,\beta,\gamma)}$ then $\mathcal{HF}_{\gamma}^{\lambda}(\alpha, \beta, A, B) \subset \mathcal{N}_\delta(e)$, where*

$$\phi_B^A(2, \lambda, \alpha, \beta, \gamma) = \frac{(1 + \lambda)[(1 - \beta B) + \beta\gamma(B - A)(2 - \alpha)]\Gamma_2}{\beta\gamma(B - A)(1 - \alpha)}. \tag{2.3}$$

Proof. For a function $f(z) \in \mathcal{HF}_{\gamma}^{\lambda}(\alpha, \beta, A, B)$ of the form (1.2), Lemma 1.1 immediately yields,

$$\sum_{n=2}^{\infty} (1 + n\lambda - \lambda)[(n-1)(1 - \beta B) + \beta\gamma(B - A)(n - \alpha)]\Gamma_n a_n \leq \beta\gamma(B - A)(1 - \alpha)$$

$$(1+\lambda)[1-\beta B+\beta\gamma(B-A)(2-\alpha)]\Gamma_2\sum_{n=2}^{\infty}a_n \le \beta\gamma(B-A)(1-\alpha)$$

$$\sum_{n=2}^{\infty}a_n \le \frac{\beta\gamma(B-A)(1-\alpha)}{(1+\lambda)[1-\beta B+\beta\gamma(B-A)(2-\alpha)]\Gamma_2} = \frac{1}{\phi_B^A(2,\lambda,\alpha,\beta,\gamma)}. \tag{2.4}$$

On the other hand, we find from (1.10) and (2.4) that

$$\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\alpha)]\Gamma_n a_n \le \beta\gamma(B-A)(1-\alpha).$$

That is

$$\sum_{n=2}^{\infty}na_n \le \frac{\beta\gamma(B-A)(1-\alpha)[1-\beta B(1+\lambda-\lambda+1)]}{(1+\lambda)[1-\beta B+\beta\gamma(B-A)(2-\alpha)](1-\beta B)}.$$

Hence

$$\sum_{n=2}^{\infty}na_n \le \frac{2}{\phi_B^A(2,\lambda,\alpha,\beta,\gamma)} = \delta.$$

□

A function $f\in\mathcal{T}$ is said to be in the class $\mathcal{HF}_\gamma^\lambda(\rho,\alpha,\beta,A,B)$ if there exists a function $h\in\mathcal{HF}_\gamma^\lambda(\rho,\alpha,\beta,A,B)$ such that

$$\left|\frac{f(z)}{h(z)}-1\right| < 1-\rho,\ \ (z\in U,\ 0\le\rho<1). \tag{2.5}$$

Now we determine the neighborhood for the class $\mathcal{HF}_\gamma^\lambda(\rho,\alpha,\beta,A,B)$.

Theorem 2.3. *If $h\in\mathcal{HF}_\gamma^\lambda(\rho,\alpha,\beta,A,B)$ and*

$$\rho = 1-\frac{\delta\phi_B^A(2,\lambda,\alpha,\beta,\gamma)}{2[\phi_B^A(2,\lambda,\alpha,\beta,\gamma)-1]} \tag{2.6}$$

then $N_\delta(h)\subset\mathcal{HF}_\gamma^\lambda(\rho,\alpha,\beta,A,B)$ where $\phi_B^A(2,\lambda,\alpha,\beta,\gamma)$ is defined in (2.3).

Proof. Suppose that $f\in N_\delta(h)$ we then find from (2.1) that

$$\sum_{n=2}^{\infty}n|a_n-d_n| \le \delta$$

which implies that the coefficient inequality

$$\sum_{n=2}^{\infty}|a_n-d_n| \le \frac{\delta}{2}.$$

Next, since $h\in\mathcal{HF}_\gamma^\lambda(\alpha,\beta,A,B)$, we have

$$\sum_{n=2}^{\infty}d_n = \frac{1}{\phi_B^A(2,\lambda,\alpha,\beta,\gamma)}$$

so that

$$\begin{aligned}\left|\frac{f(z)}{h(z)}-1\right| &< \frac{\sum\limits_{n=2}^{\infty}|a_n-d_n|}{1-\sum\limits_{n=2}^{\infty}d_n}\\ &\leq \frac{\delta}{2}\times\frac{\phi_B^A(2,\lambda,\alpha\beta,\gamma)}{\phi_B^A(2,\lambda,\alpha,\beta,\gamma)-1}\\ &\leq \frac{\delta\phi_B^A(2,\lambda,\alpha\beta,\gamma)}{2(\phi_B^A(2,\lambda,\alpha\beta,\gamma)-1)}\\ &= 1-\rho\end{aligned}$$

provided that ρ is given precisely by (2.6). Thus by definition, $f \in \mathcal{HF}_\gamma^\lambda(\rho,\alpha,\beta,A,B)$ for ρ given by (2.6), which completes the proof. □

3. Partial sums

Silverman [14] determined the sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions , viz., $\Re\{f(z)/f_k(z)\}$, $\Re\{f_k(z)/f(z)\}$, $\Re\{f'(z)/f_k'(z)\}$ and $Re\left\{f_k'(z)/f'(z)\right\}$ for their sequences of partial sums $f_k(z)=z+\sum\limits_{n=2}^{k}a_nz^n$ of the analytic function $f(z)=z+\sum\limits_{n=2}^{\infty}a_nz^n$. In the following theorems we discuss results on partial sums for functions $f(z)\in\mathcal{HF}_\gamma^\lambda(\alpha,\beta,A,B)$.

Theorem 3.1. *If f of the form (1.2) satisfies the condition (1.10), then*

$$\Re\left\{\frac{f(z)}{f_k(z)}\right\}\geq\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)-1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)} \tag{3.1}$$

and

$$\Re\left\{\frac{f_k(z)}{f(z)}\right\}\geq\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)+1} \tag{3.2}$$

where $\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)$ is given by (1.11). The results are sharp for every k, with the extremal function given by

$$f(z)=z-\frac{1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}z^{n+1}. \tag{3.3}$$

Proof. In order to prove (1.10), it is sufficient to show that

$$\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\left[\frac{f(z)}{f_k(z)}-\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)-1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}\right]\prec\frac{1+z}{1-z}\quad(z\in\mathcal{U})$$

we can write

$$\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\left[\frac{1-\sum_{n=2}^{\infty}a_nz^{n-1}}{1-\sum_{n=2}^{k}a_nz^{n-1}}-\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)-1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}\right]$$

$$=\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\left[\frac{1-\sum_{n=2}^{k}a_nz^{n-1}-\sum_{n=k+1}^{\infty}a_nz^{n-1}}{1-\sum_{n=2}^{k}a_nz^{n-1}}-\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)-1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}\right]$$

$$=\frac{1+w(z)}{1-w(z)}.$$

Then

$$w(z)=\frac{-\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\sum_{n=k+1}^{\infty}a_nz^{n-1}}{2-2\sum_{n=2}^{k}a_nz^{n-1}-\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\sum_{n=k+1}^{\infty}a_nz^{n-1}}.$$

Obviously $w(0)=0$ and $|w(z)|\le\frac{-\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\sum_{n=k+1}^{\infty}a_n}{2-2\sum_{n=2}^{k}a_n-\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\sum_{n=k+1}^{\infty}a_n}$. Now,$|w(z)|\le 1$ if and only if

$$2\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\sum_{n=k+1}^{\infty}a_n\le 2-2\sum_{n=2}^{k}a_n$$

which is equivalent to

$$\sum_{n=2}^{k}a_n+\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\sum_{n=k+1}^{\infty}a_n\le 1. \tag{3.4}$$

In view of (1.10), this is equivalent to showing that

$$\sum_{n=2}^{k}(\phi_B^A(n,\lambda,\alpha,\beta,\gamma)-1)a_n+\sum_{n=k+1}^{\infty}(\phi_B^A(n,\lambda,\alpha,\beta,\gamma)-\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma))a_n\ge 0.$$

To see that the function f given by (3.3) gives the sharp results, we observe for $z=re^{\frac{2\pi i}{n}}$ that

$$\frac{f(z)}{f_k(z)}=1-\frac{1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}z^n\to 1-\frac{1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}$$

where $r\to 1^-$. Thus, we have completed the proof of (3.1).
The proof of (3.2) is similar to (3.1) and will be omitted. □

Theorem 3.2. *If $f(z)$ of the from (1.2) satisfies (1.10) then*

$$\Re\left\{\frac{f'(z)}{f_k'(z)}\right\}\ge\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)-k-1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)} \tag{3.5}$$

and

$$\Re\left\{\frac{f_k'(z)}{f'(z)}\right\} \geq \frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)+k+1} \tag{3.6}$$

where $\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)$ *is given by* (1.11)*. The results are sharp for every* k*, with the extremal function given by* (3.3)*.*

Proof. In order to prove (3.5) it is sufficient to show that

$$\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}{k+1}\left[\frac{f'(z)}{f_k'(z)} - \frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)-n-1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}\right] \prec \frac{1+z}{1-z} \quad (z \in \mathcal{U})$$

we can write

$$\begin{aligned}
&\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}{k+1}\left[\frac{f'(z)}{f_k'(z)} - \frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)-n-1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}\right]\\
&= \frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}{k+1}\left[\frac{1-\sum_{n=2}^{\infty} na_n z^{n-1}}{1-\sum_{n=2}^{k} na_n z^{n-1}} - \frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)-k-1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}\right]\\
&= \frac{1+w(z)}{1-w(z)}.
\end{aligned}$$

Then

$$w(z) = \frac{-\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)(k+1)^{-1}\sum_{n=k+1}^{\infty} na_n z^{n-1}}{2-2\sum_{n=2}^{k} na_n z^{n-1} - \phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)(k+1)^{-1}\sum_{n=k+1}^{\infty} na_n z^{n-1}}.$$

Obviously $w(0)=0$ and

$$|w(z)| \leq \frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)(n+1)^{-1}\sum_{n=k+1}^{\infty} na_n}{2-2\sum_{n=2}^{k} na_n - \phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)(n+1)^{-1}\sum_{n=k+1}^{\infty} na_n}.$$

Now,$|w(z)| \leq 1$ if and only if

$$2\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}{(k+1)}\sum_{n=k+1}^{\infty} na_n \leq 2-2\sum_{n=2}^{k} na_n$$

which is equivalent to

$$\sum_{n=2}^{k} na_n + \frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}{(k+1)}\sum_{n=k+1}^{\infty} na_n \leq 1. \tag{3.7}$$

In view of (1.10), this is equivalent to showing that

$$\sum_{n=2}^{k}(\phi_B^A(n,\lambda,\alpha,\beta,\gamma)-n)a_n + \sum_{n=k+1}^{\infty}\left(\phi_B^A(n,\lambda,\alpha,\beta,\gamma) - \frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}{(k+1)n}\right)a_n \geq 0$$

which completes the proof of (3.5).

The proof of (3.6) is similar to (3.5) and hence we omit proof . □

4. Integral means inequality

In 1975, Silverman [13] found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family $\mathcal{T}$ and applied this function to resolve his integral means inequality, conjectured in Silverman [12] that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\eta d\theta,$$

for all $f \in \mathcal{T}$, $\eta > 0$ and $0 < r < 1$. and settled in Silverman (1997), also proved his conjecture for the subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of $\mathcal{T}$.

Lemma 4.1. *If $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$ with $f(z) \prec g(z)$, then*

$$\int_0^{2\pi} |f(re^{i\theta})|^\mu d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^\mu d\theta$$

where $\mu \geq 0, z = re^{i\theta}$ and $0 < r < 1$.

Application of Lemma 4.1 to function of $f(z)$ in the class $\mathcal{HF}_\gamma^\lambda(\alpha, \beta, A, B)$ gives the following result.

Theorem 4.2. *Let $\mu > 0$. If $f(z) \in \mathcal{HF}_\gamma^\lambda(\alpha, \beta, A, B)$ is given by (1.2) and $f_2(z)$ is defined by*

$$f_2(z) = z - \frac{1}{\phi_B^A(2, \lambda, \alpha, \beta, \gamma)} z^2 \tag{4.1}$$

where $\phi_B^A(2, \lambda, \alpha, \beta, \gamma)$ is defined by (2.3). Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\eta d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta d\theta. \tag{4.2}$$

Proof. For functions f of the form (1.2) is equivalent to proving that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^\infty a_n z^{n-1}\right|^\eta d\theta \leq \int_0^{2\pi} \left|1 - \frac{1}{\phi_B^A(2, \lambda, \alpha, \beta, \gamma)} z\right|^\eta d\theta.$$

By Lemma 4.1, it suffices to show that

$$1 - \sum_{n=2}^\infty a_n z^{n-1} \prec 1 - \frac{1}{\phi_B^A(2, \lambda, \alpha, \beta, \gamma)} z.$$

Setting

$$1 - \sum_{n=2}^\infty a_n z^{n-1} = 1 - \frac{1}{\phi_B^A(2, \lambda, \alpha, \beta, \gamma)} w(z), \tag{4.3}$$

and using (1.10), we obtain

$$\begin{aligned}|w(z)| &= \left|\sum_{n=2}^{\infty} \phi_B^A(n,\lambda,\alpha,\beta,\gamma) a_n z^{n-1}\right| \\ &\le |z| \sum_{n=2}^{\infty} \phi_B^A(n,\lambda,\alpha,\beta,\gamma) a_n \\ &\le |z|,\end{aligned}$$

where $\phi_B^A(n,\lambda,\alpha,\beta,\gamma)$ is given by (1.11) which completes the proof. □

Remark 4.3. We observe that for $\lambda = 0$, if $\mu = 0$ the various results presented in this chapter would provide interesting extensions and generalizations of those considered earlier for simpler and familiar function classes studied in the literature .The details involved in the derivations of such specializations of the results presented in this chapter are fairly straight- forward.

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Fekete-Szegö problem for a class of analytic functions

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Abstract. In the present investigation, by taking $\phi(z)$ as an analytic function, sharp upper bounds of the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for functions belonging to the class $\mathcal{M}^{\alpha}_{g,h}(\phi)$ are obtained. A few applications of our main result are also discussed.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f defined on the unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. For two functions f and g analytic in Δ we say that f is *subordinate* to g or g is *superordinate* to f, denoted by $f \prec g$, if there is an analytic function w with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$.

A function $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is said to be in the class $\mathcal{P}$ if $\operatorname{Re} p(z) > 0$. Let ϕ be an analytic univalent function in Δ with positive real part and $\phi(\Delta)$ be symmetric with respect to the real axis, starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$. Ma and Minda [6] gave a unified presentation of various subclasses of starlike and convex functions by introducing the classes $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ satisfying $zf'(z)/f(z) \prec \phi(z)$ and $1 + zf''(z)/f'(z) \prec \phi(z)$ respectively, which includes several well-known classes as special case. For example, when $\phi(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$ the class $\mathcal{S}^*(\phi)$ reduces to the class $\mathcal{S}^*[A, B]$ introduced by Janowski [3].

Ali *et al.*[1] introduced the class $\mathcal{M}(\alpha,\phi)$ of α-convex functions with respect to ϕ consisting of functions f in $\mathcal{A}$, satisfying

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\phi(z).$$

The class $\mathcal{M}(\alpha,\phi)$ includes several known classes namely $\mathcal{S}^*(\phi)$, $\mathcal{C}(\phi)$ and $\mathcal{M}(\alpha,(1+(1-2\alpha)z)/(1-z)) =: \mathcal{M}(\alpha)$. The class $\mathcal{M}(\alpha)$ is the class of α-convex functions, introduced and studied by Miller and Mocanu [7]. Several coefficient problems for p- valent analytic functions were considered by Ali *et al.* [2].

In 1933, Fekete and Szegö proved that

$$|a_2^2-\mu a_3|\leq\begin{cases}4\mu-3 & (\mu\geq 1),\\ 1+\exp\left(-\frac{2\mu}{1-\mu}\right) & (0\leq\mu\leq 1),\\ 3-4\mu, & (\mu\leq 0)\end{cases}$$

holds for the functions $f\in\mathcal{S}$ and the result is sharp. The problem of finding the sharp bounds for the non-linear functional $|a_3-\mu a_2^2|$ of any compact family of functions is popularly known as the Fekete-Szegö problem. Keogh and Merkes [4], in 1969, obtained the sharp upper bound of the Fekete-Szegö functional $|a_2^2-\mu a_3|$ for functions in some subclasses of $\mathcal{S}$. For many results on Fekete-Szegö problems see [1, 2, 9, 10, 12, 13, 14].

The Hadamard product (or convolution) of $f(z)$, given by (1.1) and $g(z) = z+\sum_{n=2}^{\infty}g_nz^n$ is defined by

$$(f*g)(z) := z+\sum_{n=2}^{\infty}a_ng_nz^n =: (g*f)(z).$$

Recently, using the Hadamard product Murugusundaramoorthy *et al.* [8] introduced a new class $M_{g,h}(\phi)$ of functions $f\in\mathcal{A}$ satisfying

$$\frac{(f*g)(z)}{(f*h)(z)}\prec\phi(z)\quad(g_n>0,h_n>0,g_n-h_n>0),$$

where $g,h\in\mathcal{A}$ and are given by

$$g(z)=z+\sum_{n=2}^{\infty}g_nz^n\ \text{ and }\ h(z)=z+\sum_{n=2}^{\infty}h_nz^n. \tag{1.2}$$

Motivated by the work of Ma and Minda [6] and others [1, 2, 4, 8], in the present paper, we introduce a more general class $M_{g,h}^{\alpha}(\phi)$ defined using convolution and subordination and deduce Fekete-Szegö inequality for this class. Certain applications of our results are also discussed. In fact our results extend several earlier known works in [4, 6, 8].

Definition 1.1. *Let g and h are given by (1.2) with $g_n>0, h_n>0$ and $g_n-h_n>0$. A function $f\in\mathcal{A}$ given by (1.1) is said to be in the class $M_{g,h}^{\alpha}(\phi)$, if it satisfies*

$$(1-\alpha)\frac{(f*g)(z)}{(f*h)(z)}+\alpha\frac{(f*g)'(z)}{(f*h)'(z)}\prec\phi(z)\quad(\alpha\geq 0), \tag{1.3}$$

where ϕ is an analytic function with $\phi(0)=1$ and $\phi'(0)>0$.

Note that in Definition 1.1, we are not assuming $\phi(\Delta)$ to be symmetric with respect to the real axis and starlike with respect to $\phi(0) = 1$. In order to prove the class $M^{\alpha}_{g,h}(\phi)$ is non empty, consider the function $f(z) = z/(1-z)$. Assuming

$$\Phi(z) = (1-\alpha)\frac{(f*g)(z)}{(f*h)(z)} + \alpha\frac{(f*g)'(z)}{(f*h)'(z)},$$

we have $\Phi(z) = 1 + (1+\alpha)(g_2 - h_2)z + \cdots$. Clearly $\Phi(0) = 1$ and $\Phi'(0) = (1+\alpha)(g_2 - h_2) > 0$, thus $f(z) = z/(1-z) \in M^{\alpha}_{g,h}(\phi)$.

Remark 1.2. For various choices of the functions g, h, ϕ and the real number α, the class $M^{\alpha}_{g,h}(\phi)$ reduces to several known classes, we enlist a few of them below:

1. The class $M^{0}_{g,h}(\phi) =: M_{g,h}(\phi)$, introduced and studied by Murugusundaramoorthy *et al.* [8].
2. If we set
$$g(z) = \frac{z}{(1-z)^2}, \quad h(z) = \frac{z}{(1-z)} \tag{1.4}$$
and $\phi(z) = (1+z)/(1-z)$, then the class $M^{\alpha}_{g,h}(\phi)$ reduces to the class of α-convex functions.
3. $M^{\alpha}_{\frac{z}{(1-z)^2},\frac{z}{(1-z)}}(\phi) =: \mathcal{M}(\alpha,\phi)$.
4. For the functions g and h given by (1.4), $M^{\alpha}_{g,h}((1+z)/(1-z)) =: \mathcal{M}(\alpha)$ is the class of α-convex functions.
5. $M^{0}_{\frac{z}{(1-z)^2},\frac{z}{(1-z)}}(\phi) =: \mathcal{S}^*(\phi)$ and $M^{1}_{\frac{z}{(1-z)^2},\frac{z}{(1-z)}}(\phi) =: \mathcal{C}(\phi)$ are the well known classes of ϕ-starlike and ϕ-convex functions respectively.

The following lemmas are required in order to prove our main results. Lemma 1.3 of Ali *et al.* [2], is a reformulation of the corresponding result for functions with positive real part due to Ma and Minda [6].

Let Ω be the class of analytic functions w, normalized by the condition $w(0) = 0$, satisfying $|w(z)| < 1$.

Lemma 1.3. [2] *If $w \in \Omega$ and $w(z) := w_1 z + w_2 z^2 + \cdots (z \in \Delta)$, then*

$$|w_2 - t w_1^2| \le \begin{cases} -t & (t \le -1), \\ 1 & (-1 \le t \le 1), \\ t & (t \ge 1). \end{cases}$$

For $t < -1$ or $t > 1$, equality holds if and only if $w(z) = z$ or one of its rotations. For $-1 < t < 1$, equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if $w(z) = z(\lambda + z)/(1 + \lambda z)$ $(0 \le \lambda \le 1)$ or one of its rotations, while for $t = 1$, equality holds if and only if $w(z) = -z(\lambda + z)/(1 + \lambda z)$ $(0 \le \lambda \le 1)$ or one of its rotations.

Lemma 1.4. [4] *(see also* [11]*) If $w \in \Omega$, then, for any complex number t,*

$$|w_2 - t w_1^2| \le \max\{1; |t|\}$$

and the result is sharp for the functions given by $w(z) = z^2$ or $w(z) = z$.

2. Fekete-Szegö problem

We begin with the following result:

Theorem 2.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If $f(z)$ given by (1.1) belongs to the class $M^{\alpha}_{g,h}(\phi)$, then, for any real number μ,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1 A}{(1+2\alpha)(g_3-h_3)} & (\mu \leq \sigma_1), \\ \frac{B_1}{(1+2\alpha)(g_3-h_3)} & (\sigma_1 \leq \mu \leq \sigma_2), \\ \frac{B_1 A}{(1+2\alpha)(h_3-g_3)} & (\mu \geq \sigma_2), \end{cases} \tag{2.1}$$

where

$$A = \frac{B_2}{B_1} - \frac{[(1+3\alpha)(h_2^2 - h_2 g_2) + \mu(1+2\alpha)(g_3 - h_3)]B_1}{(1+\alpha)^2(g_2-h_2)^2},$$

$$\sigma_1 := \frac{(B_2 - B_1)(1+\alpha)^2(g_2-h_2)^2 - (1+3\alpha)(h_2^2 - h_2 g_2)B_1^2}{(1+2\alpha)(g_3-h_3)B_1^2}$$

and

$$\sigma_2 := \frac{(B_2 + B_1)(1+\alpha)^2(g_2-h_2)^2 - (1+3\alpha)(h_2^2 - h_2 g_2)B_1^2}{(1+2\alpha)(g_3-h_3)B_1^2},$$

and for any complex number μ

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2(1+2\alpha)(g_3-h_3)} \max\{1; |t|\}, \tag{2.2}$$

where

$$t := \frac{[(1+3\alpha)(h_2^2 - h_2 g_2) + \mu(1+2\alpha)(g_3 - h_3)]B_1^2 - B_2(1+\alpha)^2(g_2-h_2)^2}{(1+\alpha)^2(g_2-h_2)^2 B_1}.$$

Proof. If $f \in \mathcal{M}^{\alpha}_{g,h}(\phi)$, then there exits an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$ such that

$$(1-\alpha)\frac{(f*g)(z)}{(f*h)(z)} + \alpha\frac{(f*g)'(z)}{(f*h)'(z)} = \phi(w(z)). \tag{2.3}$$

A computation shows that

$$\frac{(f*g)(z)}{(f*h)(z)} = 1 + a_2(g_2-h_2)z + [a_3(g_3-h_3) + a_2^2(h_2^2 - h_2 g_2)]z^2 + \cdots, \tag{2.4}$$

$$\frac{(f*g)'(z)}{(f*h)'(z)} = 1 + 2a_2(g_2-h_2)z + [3a_3(g_3-h_3) + 4a_2^2(h_2^2 - h_2 g_2)]z^2 + \cdots \tag{2.5}$$

and

$$\phi(w(z)) = 1 + B_1 w_1 z + (B_1 w_2 + B_2 w_1^2)z^2. \tag{2.6}$$

From (2.3), (2.4), (2.5) and (2.6), we have

$$(1+\alpha)(g_2-h_2)a_2 = B_1 w_1 \tag{2.7}$$

and

$$(1+2\alpha)(g_3-h_3)a_3 + (1+3\alpha)(h_2^2 - h_2 g_2)a_2^2 = B_1 w_2 + B_2 w_1^2. \tag{2.8}$$

A computation using (2.7) and (2.8) give

$$|a_3 - \mu a_2^2| = \frac{B_1}{(1+2\alpha)(g_2-h_2)}[w_2 - t w_1^2], \tag{2.9}$$

where

$$t := -\frac{B_2}{B_1} + \frac{[(1+3\alpha)(h_2^2 - h_2 g_2) + \mu(1+2\alpha)(g_3 - h_3)]B_1}{(1+\alpha)^2(g_2 - h_2)^2}. \tag{2.10}$$

Now the first inequality (1.3) is established as follows by an application of Lemma 1.3.

If

$$-\frac{B_2}{B_1} + \frac{[(1+3\alpha)(h_2^2 - h_2 g_2) + \mu(1+2\alpha)(g_3 - h_3)]B_1}{(1+\alpha)^2(g_2 - h_2)^2} \leq -1,$$

then

$$\mu \leq \frac{(B_2 - B_1)(1+\alpha)^2(g_2 - h_2)^2 - (1+3\alpha)(h_2^2 - h_2 g_2)B_1^2}{(1+2\alpha)(g_3 - h_3)B_1^2} := \sigma_1$$

and Lemma 1.3, gives

$$|a_3 - \mu a_2^2| \leq \frac{B_1 A}{(1+2\alpha)(g_3 - h_3)}.$$

For

$$-1 \leq -\frac{B_2}{B_1} + \frac{[(1+3\alpha)(h_2^2 - h_2 g_2) + \mu(1+2\alpha)(g_3 - h_3)]B_1}{(1+\alpha)^2(g_2 - h_2)^2} \leq 1,$$

we have $\sigma_1 \leq \mu \leq \sigma_2$, where σ_1 and σ_2 are as given in the statement of theorem. Now an application of Lemma 1.3 yields

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{(1+2\alpha)(g_3 - h_3)}.$$

For

$$-\frac{B_2}{B_1} + \frac{[(1+3\alpha)(h_2^2 - h_2 g_2) + \mu(1+2\alpha)(g_3 - h_3)]B_1}{(1+\alpha)^2(g_2 - h_2)^2} \geq 1,$$

we have $\mu \geq \sigma_2$ and it follows from Lemma 1.3 that

$$|a_3 - \mu a_2^2| \leq \frac{B_1 A}{(1+2\alpha)(h_3 - g_3)}.$$

Now the second inequality (2.2) fallows by an application of Lemma 1.4 as follows:

$$\begin{aligned} |a_3 - \mu a_2^2| &= \frac{B_1}{(1+2\alpha)(g_2 - h_2)}[w_2 - t w_1^2] \\ &\leq \frac{B_1}{(1+2\alpha)(g_3 - h_3)} \max\{1; |t|\}, \end{aligned}$$

where t is given by (2.10). □

Remark 2.2. If we set $\alpha = 1$, g and h are as given by (1.4), then Theorem 2.1 reduces to the result [6, Theorem 3] of Ma and Minda. When $\alpha = 0$, Theorem 2.1 reduces to the result [8, Theorem 2.1], proved by Murugusundaramoorthy *et al.* Note that there were few typographical errors in the assertion of the result [8, Theorem 2.1] and it is rectified in the following corollary:

Corollary 2.3. [8, Theorem 2.1] *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. *If* $f(z)$ *given by* (1.1) *belongs to the class* $M_{g,h}(\phi)$*, then for any real number* μ,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1}{g_3-h_3}\left(\frac{B_2}{B_1} - \frac{[(h_2^2-h_2g_2)+\mu(g_3-h_3)]B_1}{(g_2-h_2)^2}\right) & (\mu \leq \sigma_1), \\ \frac{B_1}{g_3-h_3} & (\sigma_1 \leq \mu \leq \sigma_2), \\ \frac{B_1}{g_3-h_3}\left(\frac{[(h_2^2-h_2g_2)+\mu(g_3-h_3)]B_1}{(g_2-h_2)^2} - \frac{B_2}{B_1}\right) & (\mu \geq \sigma_2), \end{cases}$$

where

$$\sigma_1 := \frac{(B_2 - B_1)(g_2 - h_2)^2 - (h_2^2 - h_2 g_2)B_1^2}{(g_3 - h_3)B_1^2}$$

and

$$\sigma_2 := \frac{(B_2 + B_1)(g_2 - h_2)^2 - (h_2^2 - h_2 g_2)B_1^2}{(g_3 - h_3)B_1^2}.$$

Here below, we discuss some applications of Theorem 2.1:

Corollary 2.4. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. *Assume that*

$$g(z) = z + \sum_{n=2}^{\infty} \frac{n\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n-\delta+1)} z^n \quad \text{and} \quad h(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n-\delta+1)} z^n.$$

If $f(z)$ *given by* (1.1) *belongs to the class* $M_{g,h}^{\alpha}(\phi)$*, then for any real number* μ,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(2-\delta)(3-\delta)B_1}{12(1+2\alpha)}\left(\frac{B_2}{B_1} - \frac{[12\mu(1+2\alpha)(2-\delta)-4(3-\delta)(1+3\alpha)]B_1}{4(3-\delta)(1+\alpha)^2}\right) & (\mu \leq \sigma_1), \\ \frac{(2-\delta)(3-\delta)B_1}{12(1+2\alpha)} & (\sigma_1 \leq \mu \leq \sigma_2), \\ \frac{(2-\delta)(3-\delta)B_1}{12(1+2\alpha)}\left(\frac{[12\mu(1+2\alpha)(2-\delta)-4(3-\delta)(1+3\alpha)]B_1}{4(3-\delta)(1+\alpha)^2} - \frac{B_2}{B_1}\right) & (\mu \geq \sigma_2), \end{cases}$$

where

$$\sigma_1 := \frac{(3-\delta)[(B_1 - B_2)(1+\alpha)^2 + (1+3\alpha)B_1^2]}{3(2-\delta)(1+2\alpha)B_1^2}$$

and

$$\sigma_2 := \frac{(3-\delta)[(B_1 + B_2)(1+\alpha)^2 + (1+3\alpha)B_1^2]}{3(2-\delta)(1+2\alpha)B_1^2}.$$

Remark 2.5. Taking $\alpha = 8/\pi^2, B_2 = 16/3\pi^2$ and $\delta = 1$ in Corollary 2.4, we have the result of Ma and Minda [5, Theorem 2]. When $\alpha = 0$, the above Corollary 2.4 reduces to [8, Corollary 3.2] of Murugusundaramoorthy *et al.* Note that there were few typographical errors in the assertion of [8, Corollary 3.2] and the following result is the corrected one:

Corollary 2.6. [8, Corollary 3.2] *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. *If* $f(z)$ *given by* (1.1) *belongs to the class* $M_{g,h}(\phi)$*, then, for any real number* μ,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(2-\delta)(3-\delta)B_1}{12}\left(\frac{B_2}{B_1} - \frac{[12\mu(2-\delta)-4(3-\delta)]B_1}{4(3-\delta)}\right) & (\mu \leq \sigma_1), \\ \frac{(2-\delta)(3-\delta)B_1}{12} & (\sigma_1 \leq \mu \leq \sigma_2), \\ \frac{(2-\delta)(3-\delta)B_1}{12}\left(\frac{[12\mu(2-\delta)-4(3-\delta)]B_1}{4(3-\delta)} - \frac{B_2}{B_1}\right) & (\mu \geq \sigma_2), \end{cases}$$

where

$$\sigma_1 := \frac{(3-\delta)[B_1 - B_2 + B_1^2]}{3(2-\delta)B_1^2}$$

and

$$\sigma_2 := \frac{(3-\delta)[B_1 + B_2 + B_1^2]}{3(2-\delta)B_1^2}.$$

Putting $\phi(z) = (1+z)/(1-z)$, g and h are as given by (1.4) in Theorem 2.1, we deduce the following result:

Corollary 2.7. *Let $f(z)$ is given by (1.1) belongs to the class $\mathcal{M}(\alpha)$, then, for any real number μ,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(\alpha^2+8\alpha+3)-4\mu(1+2\alpha)}{(1+\alpha)^2(1+2\alpha)} & (\mu \leq \sigma_1), \\ \frac{1}{(1+2\alpha)} & (\sigma_1 \leq \mu \leq \sigma_2), \\ \frac{4\mu(1+2\alpha)-(\alpha^2+8\alpha+3)}{(1+\alpha)^2(1+2\alpha)} & (\mu \geq \sigma_2), \end{cases}$$

where $\sigma_1 := \frac{1+3\alpha}{2(1+2\alpha)}$ and $\sigma_2 := \frac{\alpha^2+5\alpha+2}{2(1+2\alpha)}$.

Note that for $\alpha = 0$, Corollary 2.7 reduces to a result in [4] (see also [14]). By taking $\phi(z) = (1+z)/(1-z)$, g and h, given by (1.4) in second result of Theorem 1.3, we have the following result:

Corollary 2.8. *Let $f(z)$ is given by (1.1) belongs to the class $\mathcal{M}(\alpha)$, then for any complex number μ*

$$|a_3 - \mu a_2^2| \leq \frac{1}{1+2\alpha} \max\left\{1; \left|\frac{4\mu(1+2\alpha) - (\alpha^2+8\alpha+3)}{(1+\alpha)^2}\right|\right\}.$$

Remark 2.9. For $\alpha = 1$, the above Corollary 2.8 reduces to the result [4, Corollary 1] of Keogh and Merkes.

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Compact composition operators on spaces of Laguerre polynomials kernels

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Abstract. We study the action of the composition operator on the analytic function spaces whose kernels are special cases of Laguerre polynomials. These function spaces become Banach spaces when the kernels are integrated with respect to the complex Borel measures of the unit circle. Necessary and sufficient conditions for the composition operator to be compact are found.

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1. Introduction

Let $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ be the open unit disc in the complex plane $\mathbf{C}$. For $z \in \mathbf{D}$, $t \in \mathbf{R}$ and $a > -1$ the generating function of the associated Laguerre polynomials [7, Formula 5.1.9] is given by

$$G(a,t,z) = (1-z)^{-a-1} \exp\left(\frac{-tz}{1-z}\right) = \sum_{n=0}^{\infty} L_n^{(a)}(t) z^n \tag{1.1}$$

where $L_n^{(a)}(t)$ is the generalized Laguerre polynomial of degree n given by

$$L_n^{(a)}(t) = \sum_{k=0}^{n} \binom{n+a}{n-k} \frac{(-t)^k}{k!}. \tag{1.2}$$

Formula (1.2) can extended for $a \leq -1$ by using the identity in [7, p. 102 , eq. 5.2.1],

$$L_n^{(-a)}(t) = (-t)^a \frac{\Gamma(n-a+1)}{n!} L_{n-a}^{(a)}(t) \text{ for } a \geq 1.$$

The first few terms of $L_n^{(a)}(t)$ (see [3, p. 114]) are given by,

$$\begin{array}{ll} L_0^{(a)}(t) & = 1 \\ L_1^{(a)}(t) & = -t + a + 1 \\ L_2^{(a)}(t) & = \frac{1}{2}t^2 - (a+2)t + \frac{(a+2)(a+1)}{2} \\ L_3^{(a)}(t) & = -\frac{1}{6}t^3 + \frac{(a+3)}{2}t^2 - \frac{(a+2)(a+3)}{2}t + \frac{(a+1)(a+2)(a+3)}{6} \end{array} \tag{1.3}$$

and the recurrence relation for the coefficients $L_n^{(a)}(x)$ in [3, p. 114, Eq. 4.5.5] is given by

$$(n+1)\, L_{n+1}^{(a)}(t) = (2n + a + 1 - t)\, L_n^{(a)}(t) - (n + a)\, L_{n-1}^{(a)}(t). \tag{1.4}$$

The Laguerre generating function in (1.1) can be written in terms of the classical Cauchy kernel $K(z) = (1-z)^{-1}$ as follows

$$G(a,t,z) = [K(z)]^{a+1} \exp[t - tK(z)] \tag{1.5}$$

and special cases of the generating function $G(a,t,z)$ give interesting kernels of analytic functions spaces. For instance we have:

$$\begin{array}{rl} G(0,0,z) & = K(z) = (1-z)^{-1} \\ G(\alpha - 1, 0, z) & = K^{\alpha}(z) = (1-z)^{-\alpha}, \alpha > 0 \\ eG(-1,-1,z) & = K_e(z) = \exp\left[(1-z)^{-1}\right] \end{array} \tag{1.6}$$

where it is known in the litirature that

$$\begin{array}{ll} K(z) = (1-z)^{-1} & \text{is the classical Cauchy kernel,} \\ K^{\alpha}(z) = (1-z)^{-\alpha}, \alpha > 0 & \text{is the fractional Cauchy kernel and} \\ K_e(z) = \exp\left[(1-z)^{-1}\right] & \text{is the exponential Cauchy kernel.} \end{array} \tag{1.7}$$

Using (1.1) the corresponding Taylor series of these kernels are:

$$\begin{aligned} K(z) &= \sum_{n=0}^{\infty} L_n^{(0)}(0) z^n = \sum_{n=0}^{\infty} z^n, \\ K^{\alpha}(z) &= \sum_{n=0}^{\infty} L_n^{(\alpha-1)}(0) z^n = \sum_{n=0}^{\infty} A_n(\alpha) z^n, \\ K_e(z) &= e\sum_{n=0}^{\infty} L_n^{(-1)}(-1) z^n = e\sum_{n=0}^{\infty} A_n z^n. \end{aligned} \tag{1.8}$$

The coefficients above have the following properties:

1. $A_n(\alpha) = (-1)^n \binom{-\alpha}{n} = \binom{n+\alpha-1}{n}$
2. $A_n = L_n^{(-1)}(-1) = \sum_{i=0}^{n} \frac{1}{i!}\binom{n-1}{n-i}$ where $A_0 = A_1 = 1$.
3. $(n+1)\, A_{n+1} = (2n+1)\, A_n - (n-1)\, A_{n-1}$.
4. $\dfrac{A_{n+1}}{A_n} > 1$, and $\dfrac{A_{n+1}}{A_n} \to 1$, as $n \to \infty$.
5. The sequence $\left\{\frac{1}{A_n}\right\}_{n=0}^{\infty}$ is convex.

Results (1) is from [3] while (2)-(5) are in [8].

2. Cauchy type analytic function spaces

Let $\mathbf{T} = \partial\mathbf{D}$ be the boundary of $\mathbf{D}$ and let $H(\mathbf{D})$ denotes the class of holomorphic functions on $\mathbf{D}$. $H(\mathbf{D})$ is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of $\mathbf{D}$. We denote by $\mathbf{M}$ the set of all complex–valued Borel measures on $\mathbf{T}$ and $\mathbf{M}^*$ the subset of $\mathbf{M}$ consisting of probability measures. An analytic function f is subordinate to g in $\mathbf{D}$, written $f(z) \prec g(z)$, if there exists an analytic self-map φ in $\mathbf{D}$ with $\varphi(0) = 0$ and $|\varphi(z)| < 1$, satisfying $f(z) = g[\varphi(z)]$. If in particular g is also univalent in $\mathbf{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbf{D}) \subset g(\mathbf{D})$.

Let $z \in \mathbf{D}$ and let $k \in H(\mathbf{D})$ be one of the kernels in (1.6). We define X to be the subspace of $H(\mathbf{D})$ consisting of functions for which there exists a measure $\mu \in \mathbf{M}$ such that

$$f_\mu(z) = \int_{\mathbf{T}} k(xz)d\mu(x). \tag{2.1}$$

where $x = e^{it} \in \mathbf{T}$. The norm on X defined by

$$\|f_\mu\|_X = \inf_{\mu\in\mathbf{M}} \left\{ \|\mu\| : f_\mu(z) = \int_{\mathbf{T}} k(xz)\, d\mu(x) \right\} \tag{2.2}$$

makes X into a Banach space. If the series expansion of the kernel function k is given by,

$$k(z) = \sum_{n=0}^{\infty} a_n z^n$$

then the series of the function is given by

$$f_\mu(z) = \int_{\mathbf{T}} k(xz)d\mu(x) = \sum_{n=0}^{\infty} a_n \mu_n z^n \tag{2.3}$$

where

$$\mu_n = \int_{\mathbf{T}} x^n d\mu(x) = \int_{-\pi}^{+\pi} e^{int} d\mu\left(e^{it}\right).$$

According to the Lebesgue decomposition theorem $\mathbf{M} = \mathbf{M}_a \oplus \mathbf{M}_s$, where $\mathbf{M}_a := \{\mu_a \in \mathbf{M} : \mu_a \ll m\}$ where m is the normalized Lebesgue measure on the unit circle, and $\mathbf{M}_s := \{\mu_s \in \mathbf{M} : \mu_s \perp m\}$. Thus any $\mu \in \mathbf{M}$ can be written as $\mu = \mu_a + \mu_s$ where $\mu_a \in \mathbf{M}_a$, $\mu_s \in \mathbf{M}_s$ and $\|\mu\| = \|\mu_a\| + \|\mu_s\|$. Consequently the Banach space X may be written as $X = (X)_a \oplus (X)_s$, where $(X)_a$ is isomorphic to $L^1/\overline{H_0^1}$ the closed subspace of $\mathbf{M}$ of absolutely continuous measures, and $(X)_s$ is isomorphic to $\mathbf{M}_s$ the subspace of $\mathbf{M}$ of singular measures. If $f \in X_a$, then the singular part is null and the measure μ for which the integral in (2.1) holds reduces to $d\mu(e^{it}) = g(e^{it})dt$ where $g(e^{it}) \in L^1$ and dt is the Lebesgue measure on $\mathbf{T}$. In this case the functions in $(X)_a$ may be then written as,

$$f_\mu(z) = \int_{-\pi}^{\pi} k\left(e^{it}z\right) g(e^{it})dt$$

where if $g(e^{it})$ is nonnegative then $\|f\|_X = \left\|g(e^{it})\right\|_{L^1}$.

If the kernel function in (2.1) is replaced by $K(z) = (1-z)^{-1}$, $K^{\alpha}(z) = (1-z)^{-\alpha}$ or $K_e(z) = \exp[K(z)]$ respectively then the corresponding analytic function spaces are the classical Cauchy transform space $\mathbf{K}$ [5], the fractional Cauchy transform spaces F_{α} [6] and the exponential Cauchy transform space $\mathbf{K}_e$ introduced in [8], thus using (2.1) and replacing a_n by the appropriate coefficients from (1.8) in (2.3) we get the following:

$$\begin{aligned} \mathbf{K} &= \left\{ f_{\mu} \in H(\mathbf{D}) : f_{\mu}(z) = \int_{\mathbf{T}} K(xz) d\mu(x) = \sum_{n=0}^{\infty} \mu_n z^n \right\} \\ \mathbf{K}_{\alpha} &= \left\{ f_{\mu} \in H(\mathbf{D}) : f_{\mu}(z) = \int_{\mathbf{T}} K^{\alpha}(xz) d\mu(x) = \sum_{n=0}^{\infty} A_n(\alpha) \mu_n z^n \right\} \\ \mathbf{K}_e &= \left\{ f_{\mu} \in H(\mathbf{D}) : f_{\mu}(z) = \int_{\mathbf{T}} K_e(xz) d\mu(x) = \sum_{n=0}^{\infty} e A_n \mu_n z^n \right\} \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \mu_n &= \int_{\mathbf{T}} x^n d\mu(x) = \int_{\mathbf{T}} e^{int} d\mu\left(e^{it}\right) \\ A_n(\alpha) &= (-1)^n \binom{-\alpha}{n} = \binom{n+\alpha-1}{n} \\ A_n &= L_n^{(-1)}(-1) = \sum_{i=0}^{n} \frac{1}{i!} \binom{n-1}{n-i}. \end{aligned} \tag{2.5}$$

Clearly $\mathbf{K}$ is a special case of $\mathbf{K}_{\alpha}$ when $\alpha = 1$. It is also known that $\mathbf{K}_{\alpha} \subset \mathbf{K}_{\beta}$ for $0 < \alpha < \beta$. It was also shown in [8] that $\mathbf{K} \subset (\mathbf{K}_e)_a$ and if $f \in \mathbf{K}$ then $\|f\|_{\mathbf{K}_e} < \|h\|_{L^1} \|f\|_{\mathbf{K}}$ where $h \in L^1$.

The next result gives us examples of elements of $\mathbf{K}_e$.

Lemma 2.1. *Suppose that* $|w| \leq 1$ *and let* $f_w(z) = K_e(wz) = \exp\left[(1-wz)^{-1}\right]$ *for* $|z| < 1$. *Then* $f_w(z) \in \mathbf{K}_e$ *and there exists a probability measure* $\mu \in \mathbf{M}^*$ *such that*

$$f_w(z) = \int_{\mathbf{T}} K_e(xz) d\mu(x) \quad \textit{and} \quad \|f_w\|_{\mathbf{K}_e} = \|\mu\| = 1.$$

Proof. For $|w| \leq 1$ and $|z| < 1$ we have $\operatorname{Re}\{K(wz)\} = \operatorname{Re}\left\{(1-wz)^{-1}\right\} > \frac{1}{2}$. The Riesz-Herglotz formula implies that there exists a probability measure $\mu = \mu_w \in \mathbf{M}^*$ such that

$$K(wz) = (1-wz)^{-1} = \int_{\mathbf{T}} K(xz) d\mu(x) = \int_{\mathbf{T}} (1-xz)^{-1} d\mu(x). \tag{2.6}$$

The left hand side of the above equation is $(1-wz)^{-1} = \sum_{n=0}^{\infty} w^n z^n$ and right hand is $\sum_{n=0}^{\infty} \mu_n z^n$. Equating coefficients of the power series of both sides of (2.6) we get that

$w^n = \mu_n = \int_{\mathbf{T}} x^n d\mu(x)$ for $n = 0, 1, 2, \ldots$ and thus

$$\begin{aligned} f_w(z) = K_e(wz) &= e\sum_{n=0}^{\infty} A_n w^n z^n \\ &= e\sum_{n=0}^{\infty} A_n \left(\int_{\mathbf{T}} x^n d\mu(x)\right) z^n \\ &= \int_{\mathbf{T}} \left(e\sum_{n=0}^{\infty} A_n x^n z^n\right) d\mu(x) \\ &= \int_{\mathbf{T}} K_e(xz)\, d\mu(x) \end{aligned}$$

Hence $f_w \in \mathbf{K}_e$ and since $\mu \in \mathbf{M}^*$, we have $\|f_w\|_{\mathbf{K}_e} = \|\mu\| = 1$. □

Corollary 2.2. *A special case of the previous result is*

$$K_e(xz) \in \mathbf{K}_e \text{ for all } x \in \mathbf{T} \text{ and } \|K_e(xz)\|_{\mathbf{K}_e} = 1.$$

Lemma 2.3. *Suppose* $\{f_{\mu_n}\}_{n=1}^{\infty}$ *is a sequence of functions in* $\mathbf{K}_e$ *such that there is a constant* A *for which which* $\|f_{\mu_n}\|_{\mathbf{K}_e} \leq A$ *for* $n = 1, 2, \ldots..$ *If* $f_\mu(z) = \lim_{n\to\infty} f_{\mu_n}(z)$ *exists for* $|z| < 1$*, then* $f \in \mathbf{K}_e$ *and* $\|f\|_{\mathbf{K}_e} \leq A$.

Proof. Let $z \in \mathbf{D}$ suppose $f_{\mu_n} \in \mathbf{K}_e$ for $n = 1, 2, \ldots$ then by definition we have,

$$f_{\mu_n}(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu_n(x) \text{ and } \mu_n \in \mathbf{M}, \|f_{\mu_n}(z)\|_{\mathbf{K}_e} = \|\mu_n\| \leq A$$

The Banach-Alaoglu theorem yields a subsequence $\{\mu_{n_k}\}$ for $k = 1, 2, \ldots, \|\mu_{n_k}\| \leq A$ and $\mu \in \mathbf{M}, \|\mu\| \leq A$ such that $\mu_{n_k} \to \mu \in \mathbf{M}$ as $k \to \infty$ in the weak* topology. Hence we get,

$$\int_{\mathbf{T}} K_e(xz)\, d\mu_{n_k}(x) \longrightarrow \int_{\mathbf{T}} K_e(xz)\, d\mu(x) \text{ as } k \to \infty.$$

Since we also have that $f_\mu(z) = \lim_{k\to\infty} f_{\mu_{n_k}}(z)$ then

$$f_\mu(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu(x) \in \mathbf{K}_e \text{ and } \|f_\mu\| \leq A.$$ □

3. The composition operator on $\mathbf{K}_e$

If φ is an analytic self map of the unit disc $\mathbf{D}$, we say that φ induces a bounded composition operator C_φ on X if there exists a positive constant A such that for all $f \in X$, $\|C_\varphi(f)\|_X = \|(f \circ \varphi)\|_X \leq A\|f\|_X$. A bounded operator C_φ will be a compact operator if the image of every bounded set of X is relatively compact (i.e. has compact closure) in X. Equivalently C_φ is a compact operator on X if and only if for every bounded sequence $\{f_n\}$ of X, $\{C_\varphi(f_n)\}$ has a convergent subsequence in X.

The composition operator C_φ has been thoroughly studied on the Cauchy space $\mathbf{K}$ such as in [4, 5] and on the fractional Cauchy spaces $\mathbf{K}_\alpha$ such as in [2, 6]. In particular it is known that;

1. If $\alpha > 0$ and φ is conformal automorphism of $\mathbf{D}$, then $C_\varphi(f) = f \circ \varphi \in \mathbf{K}_\alpha$ for every $f \in \mathbf{K}_\alpha$.
2. If $\alpha \geq 1$ and φ is an analytic self map of the unit disc $\mathbf{D}$, then $C_\varphi(f) = f \circ \varphi \in \mathbf{K}_\alpha$ for every $f \in \mathbf{K}_\alpha$.
3. Let G_α denote the set of functions that are subordinate to $K^\alpha(z) = (1-z)^{-\alpha}$ in $\mathbf{D}$. If $\alpha \geq 1$ then a function f belongs to the closed convex hull of G_α if and only if there is a probability measure $\mu \in \mathbf{M}^*$ such that $f(z) = \int_{\mathbf{T}} K^\alpha(xz)d\mu(x)$.
4. C_φ is compact on $\mathbf{K}$ if and only if $C_\varphi(\mathbf{K}) \subset (\mathbf{K})_a$.
5. If $\alpha \geq 1$ then C_φ is compact on $\mathbf{K}_\alpha$ if and only if $C_\varphi[K^\alpha(xz)] \in (\mathbf{K}_\alpha)_a$ for all $|x| = 1$.

Results (1)-(3) are in [6], result (4) it is known from [4] and was extended to result (5) in [2]. The operator C_φ is also bounded and Möbius invariant on $\mathbf{K}_e$. There is no loss of generality in assuming that $\varphi(0) = 0$, and we will assume so throughout the article. Our focus then is only on when the composition operator is compact on $\mathbf{K}_e$.

We need the following interesting two results due to Y. Abu Muhanna and D. Hallenbeck in [1] .

Theorem 3.1. *Let Δ be a bounded convex body, with $0 \in \Delta$ and let H be a covering function mapping the unit disk onto the exterior of the bounded convex body $\Omega = c\Delta$. Suppose that $\log H$ is univalent and also maps the unit disk onto the compliment of a convex set. Then any analytic function f subordinate to H can be expressed as*

$$f(z) = \int_{\mathbf{T}} H(xz)d\mu(x), \tag{3.1}$$

for some positive Borel measure μ on the unit circle with $||\mu|| = 1$.

The previous theorem includes the following special case.

Theorem 3.2. *If φ is analytic self map of the unit disc $\mathbf{D}$, with $\varphi(0) = 0$ then there exist probability measures $\mu, \nu \in \mathbf{M}^*$ such that*

$$C_\varphi[K_e(z)] = K_e(\varphi(z)) = \int \exp(K(xz))\, d\mu(x) = \exp\left(\int_{\mathbf{T}} K(xz)d\nu(x)\right).$$

Then λK_e, with $|\lambda| = 1$ are all of the universal coverings of $c\mathbf{D}$.

Lemma 3.3. *Suppose $g_x(e^{it})$ is a nonnegative L^1-continuous function of x and let $\{\mu_n\}$ be a sequence of nonnegative Borel measures that are weak* convergent to μ. Define*

$$w_n(t) = \int_{\mathbf{T}} g_x(e^{it})\, d\mu_n(x) \;\; \textit{and} \;\; w(t) = \int_{\mathbf{T}} g_x(e^{it})\, d\mu(x)$$

then $\|w_n - w\|_{L^1} \longrightarrow 0$.

Proof. Suppose $g_x\left(e^{it}\right)$ is a nonnegative L^1continuous function of x and for $z \in \mathbf{D}$ let,

$$g_x(z) = \int \operatorname{Re}\left(\frac{1+e^{it}z}{1-e^{it}z}\right) g_x\left(e^{it}\right) d(t) \ ,$$
$$w_n(z) = \int g_x(z)\, d\mu_n(x) \ \text{and}$$
$$w(z) = \int g_x(z)\, d\mu(x)\,.$$

Notice that all functions are positive and harmonic in $\mathbf{D}$ and that the radial limits of $w_n(z)$ and $w(z)$ are $w_n(t)$ and $w(t)$ respectively. Then, for $|z| \leq \rho < 1$,

$$|g_x(z) - g_y(z)| \leq \frac{1}{1-\rho} \left\| g_x\left(e^{it}\right) - g_y\left(e^{it}\right) \right\|_{L^1}$$

The continuity condition implies that $g_x(z)$ is uniformly continuous in x for all $|z| \leq \rho < 1$. Weak star convergence, implies that $w_n(z) \to w(z)$ uniformly on $|z| \leq \rho < 1$ and consequently the convergence is locally uniformly on $\mathbf{D}$. In addition, we have

$$\left\| w_n(\rho e^{it}) \right\|_{L^1} \to \left\| w(\rho e^{it}) \right\|_{L^1}.$$

Hence we conclude that

$$\left\| w_n(\rho e^{it}) - w(\rho e^{it}) \right\|_{L^1} \longrightarrow 0 \text{ as } n \to \infty.$$

Now using Fatou's Lemma we conclude that

$$\left\| w_n(e^{it}) - w(e^{it}) \right\|_{L^1} \longrightarrow 0.$$

□

Lemma 3.4. *Let $g_x\left(e^{it}\right)$ be a nonnegative L^1 continuous function of x such that $\|g_x\|_{L^1} \leq a < \infty$ and $g_x(e^{it})$ defines a bounded operator on $\overline{H_0^1}$.*
Let $f(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu(x)$, and let L be the operator given by

$$L[f(z)] = \iint g_x\left(e^{it}\right) K_e\left(e^{it}z\right) dt d\mu(x)$$

then L is compact operator on $\mathbf{K}_e$.

Proof. First note that the condition that $g_x(e^{it})$ defines a bounded operator on $\overline{H_0^1}$ implies that the L operator is a well defined function on $\mathbf{K}_\alpha$. Let $\{f_n(z)\}$ be a bounded sequence in $\mathbf{K}_\alpha$ and let $\{\mu_n\}$ be the corresponding norm bounded sequence of measures in $\mathbf{M}$. Since every norm bounded sequence of measures in $\mathbf{M}$ has a weak star convergent subsequence, let $\{\mu_n\}$ be such subsequence that is convergent to $\mu \in \mathbf{M}$. We want to show that $\{L(f_n)\}$ has a convergent subsequence in $\mathbf{K}_\alpha$. First, let us assume that $d\mu_n(x) >> 0$ for all n, and let $w_n(t) = \int g_x\left(e^{it}\right) d\mu_n(x)$ and $w(t) = \int g_x\left(e^{it}\right) d\mu(x)$,then we know from the Lemma 3.3 that $w_n(t), w(t) \in L^1$ for all n, and $w_n(t) \to w(t)$ in L^1. Now since $g_x\left(e^{it}\right)$ is a nonnegative continuous function in x and $\{\mu_n\}$ is weak star convergent to μ, then

$$L(f_n(z)) = \iint K_e\left(e^{it}z\right) g_x\left(e^{it}\right) d(t) d\mu_n(x) = \int K_e\left(e^{it}z\right) w_n(t) dt$$
$$L(f(z)) = \iint K_e\left(e^{it}z\right) g_x\left(e^{it}\right) d(t) d\mu(x) = \int K_e\left(e^{it}z\right) w(t)\, dt$$

Furthermore because $w_n(t)$ is nonnegative then

$$\|L(f_n)\|_{\mathbf{K}_e} = \|w_n\|_{L^1}$$
$$\|L(f)\|_{\mathbf{K}_e} = \|w\|_{L^1}$$

Now since $\|w_n - w\|_{L^1} \to 0$ then $\|L(f_n) - L(f)\|_{\mathbf{K}_e} \to 0$ which shows that $\{L(f_n)\}$ has a convergent subsequence in $\mathbf{K}_e$ and thus L is a compact operator for the case where μ is a positive measure.
In the case where μ is complex measure we write

$$d\mu_n(x) = (d\mu_n^1(x) - d\mu_n^2(x)) + i(d\mu_n^3(x) - d\mu_n^4(x))$$

where each $d\mu_n^j(x) >> 0$ and define $w_n^j(t) = \int g_x\left(e^{it}\right) d\mu_n^j(x)$ then

$$w_n(t) = \int g_x\left(e^{it}\right) d\mu_n(x) = \left(w_n^1(t) - w_n^2(t)\right) + i\left(w_n^3(t) - w_n^4(t)\right).$$

Using an argument similar to the one above we get that

$$w_n^j(t), w^j(t) \in L^1, \quad \text{and} \quad \left\|w_n^j - w^j\right\|_{L^1} \longrightarrow 0.$$

Consequently, $\|w_n - w\|_{L^1} \longrightarrow 0$, where

$$w(t) = \left(w^1(t) - w^2(t)\right) + i\left(w^3(t) - w^4(t)\right) = \int g_x\left(e^{it}\right) d\mu(x).$$

Hence,

$$\|L(f_n) - L(f)\|_{F_\alpha} \leq \|w_n - w\|_{L^1} \longrightarrow 0.$$

Finally, we conclude that the operator L is compact. □

Now we are ready to prove our main theorem which characterizes compact composition operators on $\mathbf{K}_e$.

Theorem 3.5. *If φ is analytic self map of the unit disc $\mathbf{D}$, with $\varphi(0) = 0$ then the operator C_φ is compact in $\mathbf{K}_e$ if and only if $C_\varphi[K_e(xz)] \in (\mathbf{K}_e)_a$ for all x such that $|x| = 1$.*

Proof. Assume that C_φ is compact on $\mathbf{K}_e$ and let $\{f_j(z)\}_{j=1}^{\infty}$ be the bounded sequence of functions defined as

$$f_j(z) = K_e(\rho_j xz) = \exp\left(\frac{1}{1-\rho_j xz}\right) = \exp\left[K(\rho_j xz)\right],$$

where $0 < \rho_j < 1$ and $\lim_{j\to\infty} \rho_j = 1$. Clearly, $f_j \in H^\infty \cap \mathbf{K}_e$ and there exist probability measures $\mu_j \in \mathbf{M}^*$ such that

$$f_j(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu_j(x)$$

where $\|f_j\|_{\mathbf{K}_e} = \|\mu_j\| = 1$. Since C_φ is compact on $\mathbf{K}_e$, then $C_\varphi(f_j) \in \mathbf{K}_e$ and $\|C_\varphi(f_j)\| \le \|C_\varphi\| \, \|f_j\|_{\mathbf{K}_e} = \|C_\varphi\|$ for all j. Furthermore $C_\varphi(f_j) \in H^\infty \cap \mathbf{K}_e \subset (\mathbf{K}_e)_a$ for every j and thus there exists a nonnegative L^1 function $g_j(x)$ such that $d\mu_j(x) = g_j(x)\, dt$ and

$$C_\varphi[f_j(z)] = \int_{\mathbf{T}} K_e(xz)\, g_j(x)\, dt.$$

Since the operator C_φ is compact then the sequence $\{C_\varphi(f_j)\}_{j=1}^\infty$ has a convergent subsequence that converges to $C_\varphi[K_e(z)] \in (\mathbf{K}_e)_a$ because of Lemma 2.3 and the fact that $(\mathbf{K}_e)_a$ is a closed subspace of $\mathbf{K}_e$.

For the converse let $f \in \mathbf{K}_e$ then there exists a measure in $\mathbf{M}$ such that

$$f(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu(x).$$

Then

$$(f \circ \varphi)(z) = C_\varphi[f(z)] = \int_{\mathbf{T}} K_e[x\varphi(z)]\, d\mu(x) = \int_{\mathbf{T}} C_\varphi[K_e(xz)]\, d\mu(x)$$

where by assumption $C_\varphi[K_e(xz)] \in (\mathbf{K}_e)_a$ and thus can be written as

$$C_\varphi[K_e(xz)] = \int_{\mathbf{T}} g_x\left(e^{it}\right) K_e\left(e^{it}z\right) dt$$

where $g_x\left(e^{it}\right)$ is a positive L^1-continuous function of x. Hence

$$\begin{aligned} C_\varphi(f)(z) &= \int_{\mathbf{T}} C_\varphi[K_e(xz)]\, d\mu(x) \\ &= \int_{\mathbf{T}} \int_{\mathbf{T}} g_x\left(e^{it}\right) K_e\left(e^{it}z\right) dt d\mu(x) \end{aligned}$$

which was proven to be compact in $\mathbf{K}_e$ in the the previous Lemma 3.4. □

Corollary 3.6. *We have the following.*

1. *The operator C_φ is compact in $\mathbf{K}_e$ if and only if $C_\varphi(\mathbf{K}_e) \subset (\mathbf{K}_e)_a$.*
2. *Let $\varphi \in H(\mathbf{D})$, with $\|\varphi\|_\infty < 1$. Then C_φ is compact on $\mathbf{K}_e$.*

Proof. $C_\varphi[K_e(xz)] = K_e[x\varphi(z)] \in H^\infty \cap \mathbf{K}_e \subset (\mathbf{K}_e)_a$ and is subordinate to $K_e(z)$ hence

$$C_\varphi[K_e(xz)] = \int_{\mathbf{T}} K_e(e^{it}z) g_x\left(e^{it}\right) dt \in (\mathbf{K}_e)_a$$

where $g_x\left(e^{it}\right)$ is a nonnegative L^1 function. □

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Maximum principles for elliptic systems and the problem of the minimum matrix norm of a characteristic matrix, revisited

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Abstract. In 1968, the existence of a maximum principle for some systems of partial differential equations led us to the following problem (see I.A. Rus, Studia Univ. Babeş-Bolyai, 15(1968), No. 1, 19-26 and Glasnik Matematički, 5(1970), No. 2, 356): Let $A \in \mathbb{R}^{n\times n}$ be a matrix and $\|\cdot\|_2$ the spectral norm on $\mathbb{R}^{n\times n}$. The problem is to determine, $\min\limits_{x\in\mathbb{R}}\|A - xI\|_2$. In this paper we study the evolution of this interesting relation between the theory of partial differential equations and the matrix theory. An application of an elliptic partial differential equation with complex valued coefficients is presented. New maximum principles are given and the case of infinite systems is also studied. Some open problems are formulated.

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Keywords: strongly elliptic system, maximum principle, matrix norm, spectral matrix norm, equation with complex valued coefficients, infinite matrix, infinite system.

1. Introduction

Some time ago, studying maximum principle for elliptic systems of second order we was conducted to the following problem (see [37]-[41]):

Let $\|\cdot\|_2$ be the spectral norm on $\mathbb{R}^{n\times n}$ and $A \in \mathbb{R}^{n\times n}$ be a matrix. The problem is to determine, $\min\limits_{x\in\mathbb{R}}\|A - xI\|_2$.

In 1971, E. Deutsch informed me that H. Heinrich (see [15] and [16]) studied a similar problem in the case of Frobenius norm, $\|\cdot\|_F$, column sum norm, $\|\cdot\|_1$, and row sum norm, $\|\cdot\|_\infty$. The problem corresponding to the spectral norm was studied by A.S. Mureşan ([29]) and by I.C. Chifu ([5] and [6]). In 1975, S. Friedland studied the following problem (see [11]):

Let $A, B \in \mathbb{C}^{n\times n}$ be nonzero matrices such that $A \neq xB$ for any $x \in \mathbb{R}$. Let $d := \min_{x\in\mathbb{R}} \|A - xB\|_2$. The problem is to study the solution set of the equation:

$$\|A - xB\|_2 = d.$$

The aim of the present paper is to revisit the "abstract model" in [37], to give new maximum principles in terms of spectral norms and to consider the case of elliptic equations with complex valued coefficients and the case of an infinite system of elliptic equations. Some open problems are also formulated.

2. Preliminaries

2.1. Vector norms and matrix norms

Let us denote by K, $\mathbb{R}$ or $\mathbb{C}$. If $x \in K^n$, then, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $x^* = (x_1, \dots, x_n)$.

We consider on K^n the following norms:

$$\|x\|_\infty := \max_{1\leq k\leq n} |x_k|$$

and

$$\|x\|_p := \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}, \text{ for } p \geq 1.$$

If, $\|\cdot\|$, is a norm on K^n then we denote by the same symbol the operatorial norm (subordonate norm, or natural norm) on $\mathbb{K}^{n\times n}$ corresponding to the norm, $\|\cdot\|$, on K^n.

So, we have $\|A\|_\infty = \max_{1\leq k\leq n}\left(\sum_{j=1}^{n} |a_{kj}|\right)$, $\|A\|_1 = \max_{1\leq j\leq n}\left(\sum_{k=1}^{n} |a_{kj}|\right)$ and $\|A\|_2 = \left(\rho(A^*A)\right)^{\frac{1}{2}}$ - the spectral norm of A.

We also consider on $\mathbb{K}^{n\times n}$ the Frobenius (or Euclidean) norm defined by

$$\|A\|_F := \left(\sum_{k,j=1}^{n} |a_{kj}|^2\right)^{\frac{1}{2}}.$$

This norm is not induced by any norm on K^n, but is a matrix norm, i.e.,

$$\|A \cdot B\|_F \leq \|A\|_F \cdot \|B\|_F, \ \forall\, A, B \in \mathbb{K}^{n\times n}.$$

For an operator norm on $\mathbb{K}^{n\times n}$, $\|\cdot\|$, we have that

$$\|Ax\| \leq \|A\|\|x\|, \ \forall\, A \in \mathbb{K}^{n\times n} \text{ and } x \in K^n.$$

We also have that, $\|A\|_2 \leq \|A\|_F$, $\forall\, A \in \mathbb{K}^{n\times n}$. For the minimum norm problem we mention the following result

Heinrich's Theorem. *Let $A \in \mathbb{C}^{n\times n}$. Then,*

$$\min_{z\in\mathbb{C}}\|A - zI\|_F = \left(\|A\|_F^2 - \frac{1}{n}|trA|^2\right)^{\frac{1}{2}}.$$

From this theorem we have

Theorem 2.1. *Let $A \in \mathbb{R}^{n\times n}$. Then,*

$$\min_{x\in\mathbb{R}}\|A - xI\|_F = \left(\|A\|_F^2 - \frac{1}{n}|trA|^2\right)^{\frac{1}{2}}.$$

For more considerations of the above notions and results see: [17] (especially Chapter 37 by R. Byers and B.N. Dalta), [1], [35], [2], [15], [16], [36], [42], [3], ...

2.2. Elliptic systems of second order

Let $\Omega \subset \mathbb{R}^n$ be an open subset. Let us consider the following second order system of partial differential equations:

$$\sum_{k=1}^{n}\sum_{j=1}^{n} A_{kj}\frac{\partial^2 u}{\partial x_k \partial x_j} + F\left(x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = 0 \tag{2.1}$$

where $A_{kj} : \Omega \to \mathbb{R}^{m\times m}$, $F : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}^m$.

There are many points of view in classifying systems of partial differential equations (see for example, [27], [10], [8], [20], [4], [13], [23], [31], [34], ...).

In this paper we need the following notions.

Definition 2.2. *The system* (2.1) *is called elliptic on Ω if*

$$\det\left(\sum_{k=1}^{n}\sum_{j=1}^{n} A_{kj}(x)\lambda_k\lambda_j\right) \neq 0$$

for all $x \in \Omega$ and all $\lambda \in \mathbb{R}^n \setminus \{0\}$.

Definition 2.3. *The system* (2.1) *is called strongly elliptic on Ω, if*

$$\sum_{k=1}^{n}\sum_{j=1}^{n} \left(\tau^* A_{kj}(x)\tau\right)\lambda_k\lambda_j > 0, \text{ for all } x \in \Omega,$$

for all $\tau \in \mathbb{R}^m \setminus \{0\}$ and all $\lambda \in \mathbb{R}^n \setminus \{0\}$.

Definition 2.4. *The system* (2.1) *satisfies Somigliana's condition on Ω, if*

$$\sum_{k=1}^{n}\sum_{j=1}^{n} \tau_k^* A_{kj}(x)\tau_j > 0, \text{ for all } x \in \Omega,$$

for all $\tau_k \in \mathbb{R}^m$, $k = \overline{1,n}$ with $\sum_{k=1}^{n}\|\tau_k\| \neq 0$.

3. Basic idea and examples

The basic idea of the paper [37] may be presented as follows.

For a subset $\Omega \subset \mathbb{R}^n$ we denote

$$\mathcal{F}(\Omega, \mathbb{R}^m) := \{u \mid u : \Omega \to \mathbb{R}^m\}.$$

Let $D \subset \mathbb{R}^p$, $1 \leq p \leq n$ and $X \subset \mathcal{F}(\Omega, \mathbb{R}^m)$ be a linear subspace. By definition

$$\langle \cdot, \cdot \rangle : X \times X \to \mathcal{F}(D, \mathbb{R})$$

is a generalized inner product on X if the following axioms are satisfied:

(*i*) $\langle u, v\rangle = \langle v, u\rangle$, $\forall\ u, v \in X$;
(*ii*) $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$, $\forall\ u, v \in X$, $\forall\ \lambda \in \mathbb{R}$;
(*iii*) $\langle u_1 + u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle$, $\forall\ u_1, u_2, v \in X$;
(*iv*) $\langle u, u\rangle \geq 0$, $\forall\ u \in X$ and $\langle u, u\rangle = 0 \Leftrightarrow u = 0$.

Let $\langle \cdot, \cdot \rangle$ be a generalized inner product on X, $L : X \to \mathcal{F}(\Omega, \mathbb{R}^m)$ be a linear operator and $Y \subset \mathcal{F}(D, \mathbb{R})$ be a linear subspace. In which conditions for each $u \in X$, there exists a linear operator $T_u : Y \to \mathcal{F}(D, \mathbb{R})$ such that

$$\langle u, L(u)\rangle(x) = \|u\|(x)T_u(\|u\|)(x),$$

for all $x \in D$, with $\|u\|(x) \neq 0$.

Let us consider the equations

$$L(u) = 0 \tag{3.1}$$

and

$$T_u(v) = 0. \tag{3.2}$$

If the pair L, T_u is a solution of the above problem and u is a solution of (3.1) and iff all solution of (3.2) has a property (p), then the norm, $\|u\|$, of u has the property (p).

Example 3.1. Let $\Omega \subset \mathbb{R}^n$ be an open set, $X := C^2(\Omega, \mathbb{R}^m)$, $D := \Omega$, $Y := C^2(\Omega, \mathbb{R})$ and

$$\langle u, v\rangle := \sum_{k=1}^{m} u_k v_k.$$

As L, let us take the following operator

$$L(u) := \Delta u + \sum_{k=1}^{n} B_k \frac{\partial u}{\partial x_k} + Cu$$

where $B_k, C \in \mathcal{F}(\Omega, \mathbb{R}^{m\times m})$.

Let $u \in C^2(\Omega, \mathbb{R}^m)$ and $x \in \Omega$ be such that $\|u\|(x) \neq 0$. Then

$$u(x) = \|u\|(x)e(x),\ \text{with}\ \langle e, e\rangle(x) = 1.$$

So, in all point $x \in \Omega$, where $\|u\|(x) \neq 0$, we have

$$\langle e, \frac{\partial e}{\partial x_k}\rangle = 0,\ k = \overline{1, m}, \quad \text{and} \quad \langle \frac{\partial e}{\partial x_k}, \frac{\partial e}{\partial x_k}\rangle + \langle e, \frac{\partial^2 e}{\partial x_k^2}\rangle = 0.$$

This relations imply that

$$\langle u, L(u)\rangle = \|u\|\langle e, L(\|u\|e)\rangle = \|u\|T_u(\|u\|)$$

where

$$T_u(v) = \Delta v + \sum_{k=1}^{m} \langle e, B_k e\rangle \frac{\partial v}{\partial x_k} + \langle e, Le\rangle v.$$

From a well known maximum principle for an elliptic differential equation we have (see [12], [34], [33])

Theorem 3.2. *Let L be such that*

$$\langle e, Le\rangle(x) < 0 \tag{3.3}$$

for all $e \in C^2(\Omega, \mathbb{R}^m)$, with $\|e\| = 1$ and all $x \in \Omega$.

Then the norm of each solution $u \in C^2(\Omega, \mathbb{R})$ of (3.1) *has no positive local maximums in Ω.*

Remark 3.3. For the case when Ω is open and bounded and $u \in C^2(\Omega, \mathbb{R}^m) \cap C(\overline{\Omega}, \mathbb{R}^m)$ see [37] and [38].

Remark 3.4. The problem is in which conditions on B_k and C we have the condition (3.3) ?

First of all we have that

$$\langle e, Le\rangle = -\sum_{k=1}^{n} \Big\langle \frac{\partial e}{\partial x_k}, \frac{\partial e}{\partial x_k}\Big\rangle + \sum_{k=1}^{n} \Big\langle e, B_k \frac{\partial e}{\partial x_k}\Big\rangle + \langle e, Ce\rangle \prec 0 \tag{3.4}$$

(a function $u \prec 0 \Leftrightarrow u(x) < 0,\ \forall\ x \in \Omega$).

On the other hand we remark that

$$\Big\langle e, B_k \frac{\partial e}{\partial x_k}\Big\rangle = \Big\langle e, (B_k - b_k I)\frac{\partial e}{\partial x_k}\Big\rangle,$$

for all $b_k \in \mathcal{F}(\Omega, \mathbb{R})$.

So, we have the condition

$$\Big\langle e, Le - \sum_{k=1}^{n} b_k I \frac{\partial e}{\partial x_k}\Big\rangle \prec 0 \tag{3.5}$$

and we have that, (3.4) $\Leftrightarrow$ (3.5).

Now, let us suppose that

$$\sum_{k=1}^{m}\sum_{j=1}^{m} C_{kj}(x)\lambda_k\lambda_j \le -c(x)\sum_{k=1}^{m}|\lambda_k|^2 \tag{3.6}$$

for all $\lambda \in \mathbb{R}^m \setminus \{0\}$, with $c(x) \in \mathbb{R}_+^*$, $\forall\ x \in \Omega$.

Since

$$\Big|\Big\langle e, (B_k - b_k I)\frac{\partial e}{\partial x_k}\Big\rangle(x)\Big| \le \|B_k - b_k I\|_2(x)\Big\|\frac{\partial e}{\partial x_k}\Big\|,\ \forall\ x \in \Omega$$

from Theorem 3.2 it follows

Theorem 3.5. *We suppose that in Theorem* 3.2 *we put instead the condition* (3.3) *the following:*

(*i*) *the matrix* C *satisfies condition* (3.6) *with* $c = \sum_{k=1}^{n} c_k^2$;

(*ii*) *there exist* $b_k \in \mathcal{F}(\Omega, \mathbb{R})$ *such that*

$$\|B_k - b_k\|_2 \leq 2c_k, \ \ k = \overline{1, n}.$$

Then we have the conclusions in Theorem 3.2.

Remark 3.6. Since $\|\cdot\|_2 \leq \|\cdot\|_F$, by Theorem 2.1 we can take in Theorem 3.5,

$$c_k := \frac{1}{2}\big(\|B_k\|_F^2 - \frac{1}{m}|trB_k|^2\big)^{\frac{1}{2}}.$$

Remark 3.7. In a similar way we have

Theorem 3.8 (see [37]**,** [38]**).** *Let us consider the following second order system*

$$L(u) := \sum_{k=1}^{n}\sum_{j=1}^{n} A_{kj}\frac{\partial^2 u}{\partial x_k \partial x_j} + \sum_{k=1}^{n} B_k \frac{\partial u}{\partial x_k} + Cu = 0. \tag{3.7}$$

We suppose that:

(*i*) $\Omega \subset \mathbb{R}^n$ *is an open subset and* $A_{kj}, B_k, C : \Omega \to \mathbb{R}^{m\times m}$ *are arbitrary matriceal functions;*

(*ii*) *the system* (3.7) *is strongly elliptic;*

(*iii*) $\langle e, Le\rangle \prec 0$, *for all* $e \in C^2(\Omega, \mathbb{R}^m)$ *such that* $\|e\| = 1$.

In these conditions the norm of each solution, $u \in C^2(\Omega, \mathbb{R}^m)$, *of* (3.7) *has no positive local maximums.*

Remark 3.9. For the maximum principles for elliptic equations and systems see [27], [13], [34], [12], [21], [23], [33], [43], [44], [5], [6], [29], [30], ...

Example 3.10. Let $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^n$ be an open subset, where $\Omega_1 \subset \mathbb{R}^p$, $\Omega_2 \subset \mathbb{R}^{n-p}$, $1 \leq p < n$, are domains with smooth boundary. If $x \in \Omega$, then $x = (x', x'')$ with $x' = (x_1, \ldots, x_p) \in \Omega_1$, $x'' = (x_{p+1}, \ldots, x_n) \in \Omega_2$.

Let $X := C^2(\overline{\Omega}, \mathbb{R}^m)$, $D := \Omega_1$, $Y := C^2(\overline{\Omega}, \mathbb{R})$ and

$$\langle u, v\rangle := \int\limits_{\Omega_2} \left(\sum_{k=1}^{m} u_k v_k\right) dx''.$$

For $x \in \Omega$ such that $\|u\|(x) \neq 0$ we have

$$u(x) = \|u\|(x_1, \ldots, x_p)e(x), \ \text{with } \langle e, e\rangle = 1;$$

and

$$\langle e, \frac{\partial e}{\partial x_k}\rangle = 0, \ k = \overline{1,p}, \ \langle \frac{\partial e}{\partial x_j}, \frac{\partial e}{\partial x_k}\rangle + \langle e, \frac{\partial^2 e}{\partial x_k \partial x_j}\rangle = 0$$

for $k, j = \overline{1,p}$.

Let us take

$$L(u) := \Delta u + \sum_{k=1}^{n} B_k(x_1, \ldots, x_p) \frac{\partial u}{\partial x_k} + C(x_1, \ldots, x_p)u = 0 \tag{3.8}$$

$$L(\|u\|e) = \sum_{k=1}^{p} \frac{\partial^2 \|u\|}{\partial x_k^2} e + 2 \sum_{k=1}^{p} \frac{\partial \|u\|}{\partial x_k} \frac{\partial e}{\partial x_k} + \sum_{k=1}^{p} B_k \frac{\partial \|u\|}{\partial x_k} e + \|u\| Le.$$

So,

$$T_u(v) = \sum_{k=1}^{p} \frac{\partial^2 v}{\partial x_k^2} + \sum_{k=1}^{p} \langle e, B_k e \rangle \frac{\partial v}{\partial x_k} + \langle e, Le \rangle v.$$

From the above considerations, we have

Theorem 3.11. *Let L be such that*

$$\langle e, Le \rangle(x') < 0, \ \forall \ x' \in \Omega_1, \ \text{and}$$

for all $e \in C^2(\overline{\Omega}, \mathbb{R}^m)$ with $\|e\| = 1$.

Then the norm of each solution $u \in C^2(\overline{\Omega}, \mathbb{R}^m)$ of, $L(u) = 0$, has no positive local maximums in Ω.

Remark 3.12. As in the case of condition (3.3), the problem is in which conditions we have

$$\langle e, Le \rangle(x') < 0 \tag{3.9}$$

for all $e \in C^2(\overline{\Omega}, \mathbb{R}^m)$ with $\|e\| = 1$ and all $x' \in \Omega_1$.

First of all we have that

$$\begin{aligned} \langle e, Le \rangle = & \int_{\Omega_2} \sum_{k=1}^{p} \Big(- \sum_{j=1}^{m} \big(\frac{\partial e_j}{\partial x_k}\big)^2 \Big) d\xi'' + \int_{\Omega_2} \sum_{k=p}^{n} \sum_{j=1}^{m} e_j \frac{\partial^2 e_j}{\partial x_k} d\xi'' \\ & + \sum_{k=1}^{p} \int_{\Omega_2} \sum_{j=1}^{m} e_j \big(B_k \frac{\partial e}{\partial x_k}\big)_j d\xi'' + \sum_{k=p+1}^{n} \int_{\Omega_2} \sum_{j=1}^{m} e_j \big(B_k \frac{\partial e}{\partial x_k}\big)_j d\xi'' \\ & + \int_{\Omega_2} \sum c_{kj} e_k e_j d\xi''. \end{aligned}$$

From this relation and for a well known maximum principle for an elliptic equation, we have

Theorem 3.13. *Let us suppose that*

(i) $-\sum_{j=1}^{p} \langle \tau_j, \tau_j \rangle_E + \sum_{k=1}^{p} \langle \eta, (B_k - b_k I)\tau_k \rangle_E + \sum_{k=p+1}^{n} \langle \eta, B_k \tau_k \rangle_E + \langle \eta, c\eta \rangle_e \prec 0$ *for all* $\eta, \tau_k \in \mathbb{R}^m \setminus \{0\}$, *and for some* $b_k \in \mathcal{F}(\Omega_1, \mathbb{R})$, $k = \overline{1,p}$;

(ii) $u \in C^2(\overline{\Omega}, \mathbb{R}^m)$ *is a solution of* (3.9) *such that,* $u\big|_{\Omega_1 \times \partial\Omega_2} = 0$.

Then the norm of u has no positive local maximums in Ω.

Here, $\langle \cdot, \cdot \rangle_E$ denotes the Euclidean inner product.

Remark 3.14. For the case of $p = 1$ see [21].

Remark 3.15. For the case of a class of systems which satisfy Somiglian's condition see [41].

Remark 3.16. It is clear that, in all of above cases, a solution for the minimum norm problem is very important.

4. An application to an elliptic equation with complex valued coefficients

Let us consider the following elliptic equation

$$\Delta u + \sum_{k=1} p_k \frac{\partial u}{\partial x_k} + qu = 0 \tag{4.1}$$

where $p_k, q : \Omega \to \mathbb{C}$ with $\Omega \subset \mathbb{R}^n$ an open subset.

By a solution of (4.1) we understand a function $u \in C^2(\Omega, \mathbb{C})$ which satisfies the equation (4.1).

The equation (4.1) is equivalent with the following system of elliptic equations

$$\Delta \begin{pmatrix} \mathrm{Re}u \\ \mathrm{Im}u \end{pmatrix} + \sum_{k=1}^{m} \begin{pmatrix} \mathrm{Re}p_k & -\mathrm{Im}p_k \\ \mathrm{Im}p_k & \mathrm{Re}p_k \end{pmatrix} \frac{\partial}{\partial x_k} \begin{pmatrix} \mathrm{Re}u \\ \mathrm{Im}u \end{pmatrix}$$
$$+ \begin{pmatrix} \mathrm{Re}q & -\mathrm{Im}q \\ \mathrm{Im}q & \mathrm{Re}q \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Re}u \\ \mathrm{Im}u \end{pmatrix} = 0.$$

If in Theorem 3.5 we take $b_k := \mathrm{Re}p_k$, we have from this theorem the following result.

Theorem 4.1. *Let us consider the equation* (4.1). *We suppose that*

$$\mathrm{Re}q(x) < -\frac{1}{4} \sum_{k=1}^{n} \left(\mathrm{Im}p_k(x)\right)^2, \ \forall \ x \in \Omega.$$

If $u \in C^2(\Omega, \mathbb{C})$ *is a solution of* (4.1), *then,* $|u|$ *has no positive local maximums in* Ω.

Remark 4.2. For a similar results, see [28].

5. Infinite elliptic systems of partial differential equations

We start this section with some words on infinite matrices.

Let $A \in \mathbb{K}^{\mathbb{N}^* \times \mathbb{N}^*}$ be an infinite matrix which is row-column-finite. This matrix induces the linear operator

$$\tilde{A} : l_2(\mathbb{K}) \to \mathbb{K}^{\mathbb{N}^*}.$$

By definition the matrix A is 2-bounded if:

(*i*) $\tilde{A}(l_2(\mathbb{K})) \subset l_2(\mathbb{K})$;

(*ii*) the operator $\tilde{A} : l_2(\mathbb{K}) \to l_2(\mathbb{K})$ is a bounded operator.

By definition, the 2-norm of A is the norm of $\tilde{A}$, i.e.,

$$\|A\|_2 := \sup\left\{\|\tilde{A}x\|_2 \mid x \in l_2(\mathbb{K}) \text{ with } \|x\|_2 = 1\right\}.$$

It is clear that we have

$$\|Ax\|_2 \leq \|A\|_2\|x\|_2,$$

for all $x \in l_2(\mathbb{K})$ and all 2-bounded matrices A.

Example 5.1. Let $\Omega \subset \mathbb{R}^n$ be an open subset. We consider

$$X := \Big\{u : \Omega \to l^2(\mathbb{R}) \;\Big|\; u \in C^2(\Omega, l^2(\mathbb{R})),\ \frac{\partial u}{\partial x_k} \in C^1(\Omega, l^2(\mathbb{R}))$$
$$\text{and } \frac{\partial^2 u}{\partial x_k^2} \in C(\Omega, l^2(\mathbb{R})\Big\},$$

and

$$\langle u, v\rangle := \sum_{k=1}^{\infty} u_k v_k.$$

Now, let us consider the following infinite system

$$L(u) := \Delta u + \sum_{k=1}^{n} B_k \frac{\partial u}{\partial x_k} + Cu = 0 \tag{5.1}$$

where $B_k, C : \Omega \to \mathbb{R}^{\mathbb{N}^* \times \mathbb{N}^*}$ are row-column-finite matrices. We have

Theorem 5.2. *We suppose that:*

(*i*) *the matrices* $B_k(x)$, $C(x)$ *are* 2*-bounded for all* $x \in \Omega$*;*

(*ii*)
$$\langle e, Le\rangle \prec 0,\ \forall\ e \in X \text{ with } \langle e, e\rangle = 1. \tag{5.2}$$

If $u \in X$ *is a solution of* (5.1)*, then* $\|u\|_2$ *has no positive local maximums in* Ω*.*

The proof is similar with that of Theorem 3.2.

Remark 5.3. As in the case of Theorem 3.2, the problem is to study in which conditions on B_k and C we have (5.2). We have

Theorem 5.4. *We suppose that:*

(*i*) *the matrices* $B_k(x)$, $C(x)$ *are* 2*-bounded for all* $x \in \Omega$*;*

(*ii*) *there exist* $c_k, b_k \in \mathcal{F}(\Omega, \mathbb{R})$, $k = \overline{1, m}$ *such that:*

  (*a*) $\|B_k - b_k\|_2 \leq 2c_k$, $k = \overline{1, m}$*;*

  (*b*) $\langle \xi, C(x)\xi\rangle < -\sum_{k=1}^{m} c_k^2,\ \forall\ \xi \in l_2(\mathbb{R}),$ *with* $\langle \xi, \xi\rangle = 1$.

If $u \in X$*, with* $\|u\|_2$*,* $\|\frac{\partial u}{\partial x_k}\|_2$ *and* $\|\frac{\partial^2 u}{\partial x_k^2}\|_2$*,* $k = \overline{1, m}$*, uniformly convergent on each compact in* Ω*, is a solution of* (5.1)*, then* $\|u\|_2$ *has no positive local maximums in* Ω*.*

Remark 5.5. For more informations on infinite matrices see: [35], [7], [14], [25], [18], [19], [22], [24], [26], ...

6. Research directions and open problems

The above considerations give rise to the following questions.

Problem 6.1. Use the above technique to study some maximum principles for the following elliptic system in an open subset $\Omega \subset \mathbb{R}^m$:

$$\sum_{k=1}^{n}\sum_{j=1}^{n} A_{kj}\frac{\partial^2 u}{\partial x_k \partial x_j} + \sum_{k=1}^{n} B_k \frac{\partial u}{\partial x_k} + Cu = 0$$

where $A_{kj}, B_k, C : \Omega \to \mathbb{C}^{m\times m}$ and $u \in C^2(\Omega, \mathbb{C}^n)$.

References: [28], [27], [31], [10], ...

Problem 6.2. Let $\Omega \subset \mathbb{R}^m$ be an open subset. Use the above technique to study maximum principles for the following parabolic system:

$$\sum_{k=1}^{n}\sum_{j=1}^{n} A_{kj}(x,t)\frac{\partial^2 u}{\partial x_k \partial x_j} + \sum_{k=1}^{n} B_k(x,t) \frac{\partial u}{\partial x_k} + C(x,t)u - \frac{\partial u}{\partial t} = 0$$

for $(x,t) \in \Omega\times]0,T[$. Here $A_{kj}, B_k, C : \Omega\times]0,T[\to \mathbb{R}^{m\times m}$.

A similar problem holds for the case of complex valued matrices A_{kj}, B_k and C.

References: [37], [38], [6], ...

Problem 6.3. Use the maximum principles in this paper to study the uniqueness of the solution of Dirichlet problem for elliptic systems.

For example, let us consider the following uniformly elliptic operator in an open and bounded $\Omega \subset \mathbb{R}^n$

$$L = L_0 + c(x) := -\sum_{k=1}^{n}\sum_{j=1}^{n} a_{kj}\frac{\partial^2}{\partial x_k \partial x_j} + \sum_{k=1}^{n} b_k(x) + c(x)$$

with smooth coefficients and smooth boundary Γ of Ω $(u \in C^2(\Omega) \cap C(\overline{\Omega}))$

$$L(u) = f \tag{6.1}$$

$$u\big|_{\Gamma} = g \tag{6.2}$$

The following result is given in [32]:

Theorem of equivalent statements. *We suppose that we have uniqueness for the problem* (6.1) + (6.2). *Then the following statements are equivalent:*

(*i*) *there exists $v \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $v(x) > 0$ for $x \in \overline{\Omega}$ and $L(v) \geq 0$ in Ω;*
(*ii*) *for all smooth $c_1 \geq c$ we have uniqueness for $L_1 = L_0 + c_1$ and Ω;*
(*iii*) *for all smooth open $\Omega_1 \subset \Omega$ we have uniqueness for L and Ω_1;*
(*iv*) *$f \geq 0$ in Ω, $g = 0$ on Γ imply $u \geq 0$ in Ω;*
(*iv′*) *the corresponding Green function for L and Ω, $G(x,y) \geq 0$ for all $x, y \in \Omega$;*
(*v*) *$f = 0$ on Ω, $g \geq 0$ on Γ imply $u \geq 0$ in Ω;*
(*v′*) *$\frac{\partial G(x,y)}{\partial \nu_y} \geq 0$ for all $x \in \Omega$ and $y \in \Gamma$, where ν_y is the inner conormal at $y \in \Gamma$.*

The problem is to give a similar result for a strongly elliptic system of second order.

Problem 6.4. Let $A \in \mathbb{R}^{n\times n}$. Determine some upper estimations for

$$\min_{x\in\mathbb{R}} \|A - xI\|_2.$$

A similar problem for $A \in \mathbb{R}^{\mathbb{N}^*\times\mathbb{N}^*}$.

References: [1], [5], [6], [17], [7], [14], [22], [24], [26], ...

Problem 6.5. Let $A \in \mathbb{C}^{n\times n}$. Determine some upper estimations for

$$\min_{z\in\mathbb{C}} \|A - zI\|_2.$$

A similar problem for $A \in \mathbb{C}^{\mathbb{N}^*\times\mathbb{N}^*}$.

References: [1], [7], [14], [15], [16], [24], ...

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Strong and A-statistical comparisons for double sequences and multipliers

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Abstract. In this work, we obtain strong and A-statistical comparisons for double sequences. Also, we study multipliers for bounded A-statistically convergent and bounded A-statistically null double sequences. Finally, we prove a Steinhaus type result.

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Keywords: A-statistical convergence of double sequences, multipliers.

1. Introduction

Strong and A-statistical comparisons for sequences have been studied in [3]. Demirci, Khan and Orhan [4] have studied multipliers for bounded A-statistically convergent and bounded A-statistically null sequences. Also, Connor, Demirci and Orhan [1] have studied multipliers and factorizations for bounded statistically convergent sequences. Yardımcı [16] has extended the results in [1] using the concept of ideal convergence. Dündar and Altay [6] have obtained analogous results in [16] for bounded ideal convergent double sequences.

In this paper we show that the double sequence $\chi_{\mathbb{N}^2}$, which is the characteristic function of $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, is a multiplier from $W(T, p, q) \cap l_2^\infty$, the space of all bounded strongly T-summable double sequences with index $p, q > 0$, into the bounded summability domain $c_A^2(b)$, when T and A two nonnegative RH-regular summability matrices. Also A-statistical comparisons for both bounded as well as arbitrary double sequences have been characterized.

We first recall the concept of A-statistical convergence for double sequences.

A double sequence $x = (x_{m,n})$ is said to be convergent in the Pringsheim's sense if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$, the set of all natural numbers, such that $|x_{m,n} - L| < \varepsilon$ whenever $m, n > N$. L is called the Pringsheim limit of x and denoted by $P - \lim x = L$ (see [14]). We shall such an x more briefly as "$P-$convergent". By a bounded double sequence we mean there exists a positive number K such that

$|x_{m,n}| < K$ for all $(m,n) \in \mathbb{N}^2$, two-dimensional set of all positive integers. For bounded double sequences, we use the notation

$$||x||_{2,\infty} = \sup_{m,n} |x_{m,n}| < \infty.$$

Note that in contrast to the case for single sequences, a convergent double sequence is not necessarily bounded. Let $A = (a_{j,k,m,n})$ be a four-dimensional summability method. For a given double sequence $x = (x_{m,n})$, the $A-$transform of x, denoted by $Ax := ((Ax)_{j,k})$, is given by

$$(Ax)_{j,k} = \sum_{m,n=1,1}^{\infty,\infty} a_{j,k,m,n} x_{m,n}$$

provided the double series converges in the Pringsheim's sense for $(m,n) \in \mathbb{N}^2$.

A two dimensional matrix transformation is said to be regular if it maps every convergent sequence in to a convergent sequence with the same limit. The well-known characterization for two dimensional matrix transformations is known as Silverman-Toeplitz conditions ([8]). In 1926 Robison [15] presented a four dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is $P-$convergent is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices is known as Robison-Hamilton conditions, or briefly, $RH-$regularity ([7], [15]).

Recall that a four dimensional matrix $A = (a_{j,k,m,n})$ is said to be $RH-$regular if it maps every bounded $P-$convergent sequence into a $P-$convergent sequence with the same $P-$limit. The Robison- Hamilton conditions state that a four dimensional matrix $A = (a_{j,k,m,n})$ is $RH-$regular if and only if

(i) $P - \lim_{j,k} a_{j,k,m,n} = 0$ for each $(m,n) \in \mathbb{N}^2$,

(ii) $P - \lim_{j,k} \sum\limits_{m,n=1,1}^{\infty,\infty} a_{j,k,m,n} = 1$,

(iii) $P - \lim_{j,k} \sum\limits_{m=1}^{\infty} |a_{j,k,m,n}| = 0$ for each $n \in \mathbb{N}$,

(iv) $P - \lim_{j,k} \sum\limits_{n=1}^{\infty} |a_{j,k,m,n}| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum\limits_{m,n=1,1}^{\infty,\infty} |a_{j,k,m,n}|$ is $P-$convergent for every $(j,k) \in \mathbb{N}^2$,

(vi) There exits finite positive integers A and B such that $\sum\limits_{m,n>B} |a_{j,k,m,n}| < A$ holds for every $(j,k) \in \mathbb{N}^2$.

Now let $A = (a_{j,k,m,n})$ be a nonnegative $RH-$regular summability matrix, and let $K \subset \mathbb{N}^2$. Then $A-$density of K is given by

$$\delta_A^2(K) := P - \lim_{j,k} \sum_{(m,n) \in K} a_{j,k,m,n}$$

provided that the limit on the right-hand side exists in the Pringsheim sense. A real double sequence $x = (x_{m,n})$ is said to be $A-$statistically convergent to L if, for every

$\varepsilon > 0,$

$$\delta_A^2(\{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\}) = 0.$$

In this case, we write $st^2_{(A)} - \lim x = L$. Clearly, a $P-$convergent double sequence is $A-$statistically convergent to the same value but its converse it is not always true. Also, note that an $A-$statistically convergent double sequence need not be bounded. For example, consider the double sequence $x = (x_{m,n})$ given by

$$x_{m,n} = \begin{cases} mn, & \text{if } m \text{ and } n \text{ are squares,} \\ 1, & \text{otherwise.} \end{cases}$$

We should note that if we take $A = C(1,1)$,which is double Cesáro matrix, then $C(1,1)$-statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in ([12], [13]).

By st^2_A, $st^{2,0}_A$, $st^2_A(b)$, $st^{2,0}_A(b)$, c^2, $c^2(b), l_2^\infty$ we denote the set of all A-statistically convergent double sequences, the set of all A-statistically null double sequences, the set of all bounded A-statistically convergent double sequences, the set of all bounded A-statistically null double sequences, the set of all convergent double sequences, the set of all bounded convergent double sequences and the set of all bounded double sequences, respectively. From now on the summability field of matrix A will be denoted by c^2_A, i.e.,

$$c^2_A = \left\{ x : P - \lim_{j,k} (Ax)_{j,k} \text{ exists} \right\},$$

and $c^2_A(b) := c^2_A \cap l_2^\infty$.

Let p, q positive real numbers and let $A = (a_{j,k,m,n})$ be a nonnegative RH-regular infinite matrix. Write

$$W(A,p,q) := \left\{ x = (x_{m,n}) : P - \lim_{j,k} \sum_{m,n} a_{j,k,m,n} |x_{m,n} - L|^{pq} = 0 \text{ for some } L \right\};$$

we say that x is strongly A-summable with $p, q > 0$.

Definition 1.1. *Let E and F be two double sequence spaces. A multiplier from E into F is a sequence $u = (u_{m,n})$ such that*

$$ux = (u_{m,n} x_{m,n}) \in F$$

whenever $x = (x_{m,n}) \in E$. The linear space of all such multipliers will be denoted by $m(E,F)$. Bounded multipliers will be denoted by $M(E,F)$. Hence

$$M(E,F) = l_2^\infty \cap m(E,F).$$

If $E = F$, then we write $m(E)$ instead of $m(E,E)$. Hence the inclusion $X \subset Y$ may be interpreted as saying that the sequence $\chi_{\mathbb{N}^2}$ is a multiplier from X to Y.

2. Strong and A-statistical comparisons for double sequences

In this section, we demonstrate equivalent forms of $\chi_{\mathbb{N}^2} \in m\left(W(T,p,q) \cap l_2^{\infty}, c_A^2(b)\right)$ that compares bounded strong summability field of the nonnegative RH-regular summability matrices A and T. Also we will show that these characterize the A-statistical comparisons for both bounded as well as arbitrary double sequences.

Theorem 2.1. *Let $A = (a_{j,k,m,n})$ and $T = (t_{j,k,m,n})$ be nonnegative RH-regular summability matrices. Then the followings are equivalent:*
(i) $\chi_{\mathbb{N}^2} \in m\left(W(T,p,q) \cap l_2^{\infty}, c_A^2(b)\right)$,
(ii) $W(T,p,q) \cap l_2^{\infty} \subseteq c_A^2(b)$,
(iii) $A \in \left(W(T,p,q) \cap l_2^{\infty}, c^2\right)$,
(iv) *For any subset $K \subseteq \mathbb{N}^2$, $\delta_T^2(K) = 0$ implies that $\delta_A^2(K) = 0$,*
(v) $A \in \left(W(T,p,q) \cap l_2^{\infty}, c^2\right)$ *and A preserves the strong limits of T.*

Proof. It is obvious that the first three parts are equivalent. To show that (iii) implies (iv), suppose that (iii) holds. Assume the contrary and let K be a subset of nonnegative integers with $\delta_T^2(K) = 0$ but

$$\limsup_{j,k} \sum_{(m,n)\in K} a_{j,k,m,n} > 0. \tag{2.1}$$

So, K must be an infinitive set since A is RH-regular and $P - \lim_{j,k} a_{j,k,m,n} = 0$ for each $(m,n) \in \mathbb{N}^2$. (Since $\delta_T^2(K) = 0$, and T is RH-regular, it must be that $\mathbb{N} \times \mathbb{N} - K$ must also be infinitive). Now take a sequence x which is the indicator of the set K . Note that for any $p, q > 0$, we have

$$\begin{aligned} P - \lim_{j,k} \sum_{m,n} |t_{j,k,m,n}| \, |x_{m,n} - 0|^{pq} &= P - \lim_{j,k} \sum_{m,n} t_{j,k,m,n} x_{m,n} \\ &= P - \lim_{j,k} \sum_{(m,n)\in K} t_{j,k,m,n} \\ &= \delta_T^2(K) = 0. \end{aligned}$$

Hence, $x \in W(T,p,q) \cap l_2^{\infty}$. By $A \in \left(W(T,p,q) \cap l_2^{\infty}, c^2\right)$, it must be that $(Ax)_{j,k}$ is convergent. Combining this with (2.1) we obtain that the density $\delta_A^2(K)$ exists and so $P - \lim_{j,k} (Ax)_{j,k} = \delta_A^2(K) > 0$. Consider the matrix D that keeps all the columns of A whose positions correspond with the set K and fills the rest of the columns with zero matrices. Because of $P - \lim_{j,k} (Dx)_{j,k} = P - \lim_{j,k} (Ax)_{j,k} > 0$, a straight forward extension of an argument of Maddox provides a contradiction. Suppose now (iv) holds, and let $x \in W(T,p,q) \cap l_2^{\infty}$, so that

$$P - \lim_{j,k} \sum_{m,n} t_{j,k,m,n} \, |x_{m,n} - L|^{pq} = 0,$$

for some number L. So x is T-statistically convergent. Then for any $\varepsilon > 0$, define the set $K = \{(m,n) : |x_{m,n} - L| > \varepsilon\}$. And we have $\delta_T^2(K) = 0$. Then by assumption, it must be that $\delta_A^2(K) = 0$. Since x is bounded, let $|x_{m,n}| \leq C$ for all m, n. So, for any

$p,q>0$, we have

$$\begin{aligned}\sum_{m,n} a_{j,k,m,n}\left|x_{m,n}-L\right|^{pq} &= \sum_{(m,n)\in K} a_{j,k,m,n}\left|x_{m,n}-L\right|^{pq}+\\ &\quad \sum_{(m,n)\in K^c} a_{j,k,m,n}\left|x_{m,n}-L\right|^{pq}\\ &\leq (2C)^{pq}\sum_{(m,n)\in K} a_{j,k,m,n}+\varepsilon^{pq}\sum_{(m,n)\in K^c} a_{j,k,m,n}\\ &\leq (2C)^{pq}\sum_{(m,n)\in K} a_{j,k,m,n}+\varepsilon^{pq}\sum_{(m,n)\in K^c} a_{j,k,m,n}.\end{aligned}$$

Letting $j,k\to\infty$, we obtain

$$P-\lim_{j,k}\sum_{(m,n)} a_{j,k,m,n}\left|x_{m,n}-L\right|^{pq}=0.$$

So that, $(Ax)_{j,k}\to L$ and A preserves the strong limit of T , which gives (v). Observe that (v) trivially implies(iii) . □

The following proposition collects the last result's various equivalent forms. For this purpose we introduce the notation

$$W^L(T,p,q):=\left\{x:P-\lim_{j,k}\sum_{m,n} t_{j,k,m,n}\left|x_{m,n}-L\right|^{pq}=0\right\}.$$

Proposition 2.2. *Let $A=(a_{j,k,m,n})$ and $T=(t_{j,k,m,n})$ be nonnegative RH-regular summability matrices. The following statements are equivalent:*

(i) $st_T^2(b)\subseteq st_A^2(b)$,

(ii) $W(T,p,q)\cap l_2^\infty\subseteq W(A,s,t)\cap l_2^\infty$ for some $p,q,s,t>0$,

(iii) $A\in\left(W(T,p,q)\cap l_2^\infty,c^2\right)$ and A preserves the strong limits of T. That is, $W^L(T,p,q)\cap l_2^\infty\subseteq W^L(A,s,t)\cap l_2^\infty$ for every L,

(iv) For any subset $K\subseteq\mathbb{N}^2$, $\delta_T^2(K)=0$ implies that $\delta_A^2(K)=0$,

(v) $st_T^{2,0}(b)\subseteq st_A^{2,0}(b)$,

(vi) $st_T^2(b)\subseteq st_A^2(b)$ and A preserves the T-statistical limits,

(vii) $W^L(T,p,q)\cap l_2^\infty\subseteq W^L(A,s,t)\cap l_2^\infty$ for some $p,q,s,t>0$ and some real number L,

$(viii)$ $W(T,p,q)\cap l_2^\infty\subseteq c_A^2(b)$ for some $p,q>0$,

(ix) $st_T^2\subseteq st_A^2$ and A preserves the T-statistical limits,

(x) $st_T^2\subseteq st_A^2$.

Proof. At fist we give the following notation:

$$st_T^L(b):=\left\{x\in l_2^\infty: x \text{ is } T-\text{statistically convergent to } L\right\}.$$

Note that

$$st_T^L(b)=W^L(T,p,q)\cap l_2^\infty$$

for any $p,q>0$. Because of this, taking union over all L gives that (i) and (ii) are equivalent. By theorem, we know that (iii) and (iv) are equivalent. Taking union over

L shows that (iii) implies (ii). To show that (ii) implies (iii), clearly (ii) implies that $W(T,p,q) \cap l_2^\infty \subseteq c_A^2(b)$. Hence, $A \in \left(W(T,p,q) \cap l_2^\infty, c^2\right)$. Therefore, by theorem, (iv) holds. Therefore, (iii) holds. Also theorem implies that (iv) and $(viii)$ are equivalent. While (vii) holds some L, if $x \in W^M(T,p,q) \cap l_2^\infty$ then define a new squence $y_{m,n} = x_{m,n} - M + L$. Since $y \in W^L(T,p,q) \cap l_2^\infty$, we have $y \in W^L(A,s,t) \cap l_2^\infty$. This implies that

$$\sum_{m,n} a_{j,k,m,n} \left|x_{m,n} - M\right|^{st} = \sum_{m,n} a_{j,k,m,n} \left|y_{m,n} - L\right|^{pq} \to 0.$$

So that, $y \in W^M(A,s,t) \cap l_2^\infty$. That is,

$$W^M(T,p,q) \cap l_2^\infty \subseteq W^L(A,s,t) \cap l_2^\infty$$

for every M. If supremum over all M takes then (ii) holds. Now (ii) implies (iii) and clearly (iii) implies (vii). Hence (i) and (iii) together imply (vi). Trivially (vi) implies (i) .Also, (vi) implies (v). Conversely (v) implies (vii) with $L = 0$. Hence, (i) through $(viii)$ are all equivalent. So far all arguments were for bounded sequences. Now (ix) implies (x), and (x) implies (i). To show that (i) implies (ix), let $x \in st_T^2$ with T-statistical limit L. For $\varepsilon > 0$, define $h_{m,n} = 0$ if $|x_{m,n} - L| < \varepsilon$ and $h_{m,n} = 1$ otherwise. Hence, any such $h \in st_T^{2,0}(b) \subseteq st_A^{2,0}(b)$ by (v). This implies that $x \in st_A^2$ with L being the A-statistical limit, the proof is complete. □

3. Multipliers

In this section, we introduce multipliers on above some different spaces. Firstly, we give some notations.

Definition 3.1. ([5]) *Let $A = (a_{j,k,m,n})$ be a non-negative RH-regular summability matrix and let $(\alpha_{m,n})$ be a positive non-increasing double sequence. A double sequence $x = (x_{m,n})$ is A-statistically convergent to a number L with the rate of $o(\alpha_{m,n})$ if for every $\varepsilon > 0$,*

$$P - \lim_{j,k\to\infty} \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \left\{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\right\}.$$

In this case, we write

$$x_{m,n} - L = st_A^2 - o(\alpha_{m,n}) \quad as \quad m,n \to \infty.$$

Definition 3.2. ([5]) *Let $A = (a_{j,k,m,n})$ and $(\alpha_{m,n})$ be the same as in Definition* 3.1. *Then, a double sequence $x = (x_{m,n})$ is A-statistically bounded with the rate of $O(\alpha_{m,n})$ if for every $\varepsilon > 0$,*

$$\sup_{j,k} \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in L(\varepsilon)} a_{j,k,m,n} < \infty,$$

where

$$L(\varepsilon) := \left\{(m,n) \in \mathbb{N}^2 : |x_{m,n}| \geq \varepsilon \right\}.$$

In this case, we write

$$x_{m,n} = st_A^2 - O(\alpha_{m,n}) \quad as \quad m,n \to \infty.$$

Now, we define the subspaces of A-statistically convergent double sequences as follows:

$$\begin{aligned}
st_{A,a}^2 &: = \left\{x : x_{m,n} - L = st_A^2 - o\left(\alpha_{m,n}\right),\ \text{as } m,n \to \infty,\ \text{for some } L\right\},\\
st_{A,O(a)}^2 &: = \left\{x : x_{m,n} - L = st_A^2 - O\left(\alpha_{m,n}\right),\ \text{as } m,n \to \infty,\ \text{for some } L\right\},\\
st_{A,a}^{2,0} &: = \left\{x : x_{m,n} = st_A^2 - o\left(\alpha_{m,n}\right),\ \text{as } m,n \to \infty\right\},\\
st_{A,O(a)}^{2,0} &: = \left\{x : x_{m,n} = st_A^2 - O\left(\alpha_{m,n}\right),\ \text{as } m,n \to \infty\right\},\\
st_{A,a}^2(b) &: = st_{A,a}^2 \cap l_2^\infty,\\
st_{A,O(a)}^2(b) &: = st_{A,O(a)}^2 \cap l_2^\infty,\\
st_{A,a}^{2,0}(b) &: = st_{A,a}^{2,0} \cap l_2^\infty,\\
st_{A,O(a)}^{2,0}(b) &: = st_{A,O(a)}^{2,0} \cap l_2^\infty.
\end{aligned}$$

For each $Z \subset \mathbb{N}^2$, we let c_Z^2 denote the set of double sequences which convergence along Z and $c_Z^2\,(b)$ bounded members of c_Z^2. Note that c_Z^2 is the convergence domain of a nonnegative $RH-$regular summability method. It is also easy to verify that $m\left(c_Z^2\right) = c_Z^2$; $M\left(c_Z^2\right) = c_Z^2\,(b)$, and $st_A^2\,(b) = \cup\left\{c_Z^2\,(b) : \delta_A^2\,(Z) = 1\right\}$.

Theorem 3.3. $m\left(st_{A,a}^2\,(b)\right) = st_{A,a}^2\,(b)$, *and* $m\left(st_{A,O(a)}^2\,(b)\right) = st_{A,O(a)}^2\,(b)$.

Proof. Let $u \in m\left(st_{A,a}^2\,(b)\right)$. Then $ux \in st_{A,a}^2\,(b)$ for all $x \in st_{A,a}^2\,(b)$. Especially, $x = \chi_{\mathbb{N}^2} \in st_{A,a}^2\,(b)$, hence $u \in st_{A,a}^2\,(b)$, which shows $m\left(st_{A,a}^2\,(b)\right) \subset st_{A,a}^2\,(b)$. Conversely, suppose that $u \in st_{A,a}^2\,(b)$ and take $x \in st_{A,a}^2\,(b)$. Then, by the discussion preceding Section 2 we get $ux \in st_{A,a}^2\,(b)$, by this $u \in m\left(st_{A,a}^2\,(b)\right)$, i.e., $st_{A,a}^2\,(b) \subset m\left(st_{A,a}^2\,(b)\right)$. The same argument works for the second part of the theorem. □

One may now expect that $m\left(st_{A,a}^{2,0}\,(b)\right) = st_{A,a}^{2,0}\,(b)$. However , as the next example shows, it is not the case.

Example 3.4. Take $\alpha = \chi_{\mathbb{N}^2}$ and $A = C\,(1,1)$. Then $st_{A,a}^{2,0}\,(b) = st^{2,0}\,(b)$, the set of all bounded statistically null double sequences. Now define a double bounded sequence $u = (u_{m,n})$ by

$$u_{m,n} = \begin{cases} 1 & , \quad m,n \text{ are odds,} \\ -1 & , \quad m,n \text{ are evens,} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Then $ux \in st^{2,0}\,(b)$ for every $x \in st^{2,0}\,(b)$. Hence $u \in m\left(st^{2,0}\,(b)\right)$, but $u \notin st^{2,0}\,(b)$. So, the next result characterizes the multipliers from $st_{A,a}^{2,0}\,(b)$ into itself.

Theorem 3.5. $m\left(st_{A,a}^{2,0}\,(b)\right) = l_2^\infty$.

Proof. If $u \in m\left(st_{A,a}^{2,0}(b)\right)$, then $ux \in st_{A,a}^{2,0}(b) \subset l_2^{\infty}$ for all $x \in st_{A,a}^{2,0}(b)$. To show that this implies that $u \in l_2^{\infty}$, first observe that $c_0^2 \subseteqq st_{A,a}^{2,0}(b)$; and from this case $u \in m\left(st_{A,a}^{2,0}(b)\right)$ if and only if the matrix $Tu = (t_{j,k,m,n}) = \left(u_{j,k}\delta_{(m,n)}^{(j,k)}\right)$ maps $st_{A,a}^{2,0}(b)$ into itself, where $\delta_{(m,n)}^{(j,k)}$ is the Kronecker delta. Hence, it also maps c_0^2 into l_2^{∞}, which implies that $\sup_{j,k}\sum_{m,n}|t_{j,k,m,n}| = \sup_{j,k}\sum_{m,n}\left|u_{j,k}\delta_{(m,n)}^{(j,k)}\right| = \sup_{j,k}|u_{j,k}| < \infty$. Conversely, suppose $u \in l_2^{\infty}$ and let $z \in st_{A,a}^{2,0}(b)$, then

$$\{(m,n) : |u_{m,n}z_{m,n}| \geq \varepsilon\} \subseteqq \left\{(m,n) : |z_{m,n}| \geq \frac{\varepsilon}{1+\|u\|_{2,\infty}}\right\}.$$

Thus, since $z_{m,n} = st_A^2{-}o\,(a_{m,n})$, we obtain $u_{m,n}x_{m,n} = st_A^2{-}o\,(a_{m,n})$. Also it is clear that uz is bounded, and hence $l_2^{\infty} \subseteqq m\left(st_{A,a}^{2,0}(b)\right)$, and the proof is complete. □

Theorem 3.6. $m\left(st_A^2(b)\right) = \cup\left\{M\left(c_Z^2\right) : \delta_A^2(Z) = 1\right\}$.

Proof. $m\left(st_A^2(b)\right) = st_A^2(b) = \cup\left\{c_Z^2(b) : \delta_A^2(Z) = 1\right\} = \cup\left\{M\left(c_Z^2\right) : \delta_A^2(Z) = 1\right\}$.

Before proving the following theorem, we observe that, in general,

$$c_0^2 \subseteq m\left(st_A^2(b), c^2\right) \subseteq c^2.$$

The first inclusion follows from noting $ux \in c_0^2 \subseteq st_A^2(b)$ for any $u \in c_0^2$ and $x \in l_2^{\infty}$. The second inclusion follows from $\chi_{\mathbb{N}^2} \in st_A^2(b)$. Note that if $st_A^2(b) = c^2$, then $m\left(st_A^2(b), c^2\right) = c^2$. The next theorem shows that this the only situation for which $m\left(st_A^2(b), c^2\right) = c^2$. □

Theorem 3.7. $m\left(st_A^2(b), c^2\right) = c_0^2$ *and* $m\left(c^2, st_A^2(b)\right) = st_A^2(b)$.

Proof. First we show that $m\left(st_A^2(b), c^2\right) = c_0^2$. All we need to establish is that if $u \in c^2$ and $\lim u = l \neq 0$, then $u \notin m\left(st_A^2(b), c^2\right)$. Let $z \in st_A^2(b)$, $z \notin c^2$, and, without loss of generality, suppose z is $A-$statistically convergent to 1. Then there is an $\varepsilon > 0$ such that $K = \{(m,n) : |z_{m,n} - 1| \geq \varepsilon\}$ is an infinite set. Note that $\delta_A^2(K) = 0$.

Define x by $x_{m,n} = \chi_{K^c}(m,n)$ and observe that x is convergent in $A-$density to 1, hence $x \in st_A^2(b)$. Also note xu converges to $l \neq 0$ along K^c and to 0 along K, hence $xu \notin c^2$ and thus $u \notin m\left(st_A^2(b), c^2\right)$.

Now we show that $m\left(c^2, st_A^2(b)\right) = st_A^2(b)$. As $\chi_{\mathbb{N}^2} \in c^2$, $m\left(c^2, st_A^2(b)\right) \subseteq st_A^2(b)$. The reserve inclusion follows from noting that if $u \in st_A^2(b)$ and $x \in c^2 \subseteq st_A^2(b)$, then ux is $A-$statistically convergent. □

Theorem 3.8. (i) $m\left(c_0^2, st_A^{2,0}(b)\right) = l_2^{\infty}$,

(ii) $m\left(st_A^{2,0}(b), c_0^2\right) = \left\{u \in l_2^{\infty} : u\chi_E \in c_0^2 \text{ for all } E \text{ such that } \delta_A^2(E) = 0\right\}$.

Proof. The proof of (i) follows from noting

$$l_2^{\infty} = m\left(c_0^2, c_0^2\right) \subseteq m\left(c_0^2, st_A^{2,0}(b)\right) \subseteq l_2^{\infty}.$$

Next we prove (ii). First note that if $\delta_A^2(E)=0$, then $\chi_E \in st_A^{2,0}(b)$ and thus, if $u \in m\left(st_A^{2,0}(b), c_0^2\right)$, $u\chi_E \in c_0^2$, or u goes 0 along E.

Hence,
$m\left(st_A^{2,0}(b), c_0^2\right) \subseteq \left\{u \in l_2^\infty : u\chi_E \in c_0^2 \text{ for all } E \text{ such that } \delta_A^2(E)=0\right\}$.

Now suppose that u is a bounded sequence such that u tends to 0 along every $A-$null set and suppose x is bounded and convergent to 0 in $A-$density. Then there is an $K \subseteq \mathbb{N}^2$ such that, $x\chi_{K^c} \in c_0^2$, $\delta_A^2(K)=0$. As $ux = ux\chi_{K^c} + ux\chi_K$ and both terms of the right hand side are null double sequences , $ux \in c_0^2$.

Now suppose $x \in st_A^{2,0}(b)$. Then there is a sequence $\left(x^{j,k}\right)$, each $x^{j,k}$ convergent in $A-$density to 0, such that $x^{j,k}$ converges to x in l_2^∞. Now $ux^{j,k} \to ux$ in l_2^∞, and as $ux^{j,k} \in c_0^2$ for all j,k and c_0^2 is closed, $ux \in c_0^2$. Thus

$$\left\{u \in l_2^\infty : u\chi_E \in c_0^2 \text{ for all } E \text{ such that } \delta_A^2(E)=0\right\} \subseteq m\left(st_A^{2,0}(b), c_0^2\right)$$

and hence the theorem.

Note that $m\left(st_A^{2,0}(b), c_0^2\right)$ can be a variety of spaces. In particular $m\left(c_0^2, c_0^2\right) = l_2^\infty$ and, if $c_{0,Z}^2$ denotes the sequences that converge to 0 along Z, then

$$m\left(c_{0,Z}^2(b), c_0^2\right) = c_{0,Z}^2(b). \qquad \square$$

4. A Steinhaus-type result

The well known Theorem of Steinhaus knows that if T is a regular matrix then $\chi_{\mathbb{N}}$ is not a multipler from l^∞ into $c_T := \{x : Tx \in c\}$. It may be true if regularity condition on A is replaced by coregularity. Maddox [10] proved that $\chi_{\mathbb{N}}$ is not a multipler from l^∞ into $f_T := \{x : Tx \in f\}$ either, where f denotes the space of all almost convergent sequences [9]. It is known that almost convergence and statistical convergence are not compatible summability methods [11]. So there seems some hope that $\chi_{\mathbb{N}}$ might be a multiplier from l^∞ into $(st_A)_T := \{x : Tx \in st_A\}$. However, it has been shown in [1] that it is not the case. Of course $\chi_{\mathbb{N}}$ is not a multipler from l^∞ into the space $(st_{A,a})_T := \{x : Tx \in st_{A,a}\}$ either. Furthermore Demirci, Khan and Orhan gave an alternate proof of it. What we offer in this study is to prove the theorem which is characterized $\chi_{\mathbb{N}^2}$ is not a multiplier from l_2^∞ into $\left(st_{A,a}^2\right)_T$.

Definition 4.1. *Let $A=(a_{j,k,m,n})$ be a non-negative RH-regular summability matrix. The characteristic χ defined by*

$$\chi(A) = \lim_{j,k} \sum_{m,n} a_{j,k,m,n} - \sum_{m,n} \lim_{j,k} a_{j,k,m,n}.$$

If $\chi(A)=0$ then we say A is co-null, if $\chi(A)\neq 0$ then we say A is co-regular.

$$K_0^2 = \{A : \chi(A)=0\},$$
$$K^2 = \{A : \chi(A)\neq 0\}.$$

Now, we give the following lemma before the proof of theorem;

Lemma 4.2. ([2]) $A \in \left(l_2^\infty, c^2(b)\right)$ *if and only if the condition* $\sum_{j,k} |a_{j,k,m,n}| \le C < \infty$ *holds and*

(i) $\lim_{j,k} a_{j,k,m,n} = \alpha_{m,n}$ *for each* $(m,n) \in \mathbb{N}^2$,

(ii) $\lim_{j,k} \sum_{n=1}^{k} |a_{j,k,m,n}|$ *exists for each* $m \in \mathbb{N}$ *and*

(iii) $\lim_{j,k} \sum_{m=1}^{j} |a_{j,k,m,n}|$ *exists for each* $n \in \mathbb{N}$,

(iv) $\sum_{j,k} |a_{j,k,m,n}|$ *converges,*

(v) $\lim_{j,k} \sum_m \sum_n |a_{j,k,m,n} - \alpha_{m,n}| = 0$.

Theorem 4.3. *Let* A *and* B *be conservative matrices and suppose that* $A \in (l_2^\infty, c_B^2(b))$. *Then*

(i) $BA \in K_0^2$,

(ii) *If* $B \in K^2$ *then* $A \in K_0^2$.

Proof. (i) Because of $A \in \left(l_2^\infty, c_B^2(b)\right)$ we have $B(Ax) \in c^2(b)$ for all $x \in l_2^\infty$. Now A and B conservative implies $B(Ax) = (BA)x$ for all $x \in l_2^\infty$, therefore $(BA)x \in c^2(b)$ for all $x \in l_2^\infty$, so that $BA \in \left(l_2^\infty, c^2(b)\right) \subset K_0^2$ from Lemma 4.2.

(ii) By (i) and the fact that χ is a scalar homomorphism we have $\chi(B)\chi(A) = 0$, whence the result. □

Theorem 4.4. *Let* A *be a nonnegative RH-regular summability method. If* T *is a co-regular summabilty matrix, then* $\chi_{\mathbb{N}^2}$ *is not a multiplier from* l_2^∞ *into* $\left(st_{A,a}^2\right)_T := \left\{x : Tx \in st_{A,a}^2\right\}$.

Proof. Suppose $\chi_{\mathbb{N}^2} \in m\left(l_2^\infty, \left(st_{A,a}^2\right)_T\right)$, then $l_2^\infty \subset \left(st_{A,a}^2\right)_T$. Hence $Tx \in l_2^\infty$ and $Tx \in st_{A,a}^2 \subset st_A^2$ for all $x \in l_2^\infty$. Then we have $Tx \in c_A^2$. So $T : l_2^\infty \to c_A^2$. Since A is RH-regular, it follows from Theorem 4.3 that T is co-null double matrix which is a contradiction. □

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On a generalization of Szasz-Durrmeyer operators with some orthogonal polynomials

Serhan Varma and Fatma Taşdelen

Abstract. In this paper, we construct a form of linear positive operators with Brenke type polynomials as a generalization of Szasz-Durrmeyer operators. We obtain convergence properties of our operators with the help of the universal Korovkin-type property and calculate the order of approximation by using classical modulus of continuity. Explicit examples of our operators involving some orthonogal and d-orthogonal polynomials such as the Hermite polynomials $H_k^{(\nu)}(x)$ of variance ν and Gould-Hopper polynomials are given.

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Keywords: Szasz-Durrmeyer operator, modulus of continuity, Brenke type polynomials, Gould-Hopper polynomials, Hermite polynomials.

1. Introduction

Several integral modifications of Szasz operators [10] take part in approximation theory. One of them is the Durrmeyer type integral modification i.e. Szasz-Durrmeyer operators discovered by Mazhar and Totik [8]

$$(S_n^* f)(x) = n \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \int_0^{\infty} e^{-nt} \frac{(nt)^k}{k!} f(t)\, dt \tag{1.1}$$

where $x \geq 0$ and $f \in C[0,\infty)$. Note that the operators (1.1) are linear positive operators.

On the other hand, Jakimovski and Leviatan [6] gave a generalization for Szasz operators by using Appell polynomials. Later, Ciupa [3] investigated the properties of the following operators as a Durrmeyer type integral modification of the operators given in [6]

$$(P_n f)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)\, dt \tag{1.2}$$

where $\lambda \geq 0$, $g(1) \neq 0$ and $p_k(x)$ are the Appell polynomials. The Appell polynomials are defined with the help of the following generating relation

$$g(u)\, e^{ux} = \sum_{k=0}^{\infty} p_k(x)\, u^k \tag{1.3}$$

where $g(u) = \sum\limits_{k=0}^{\infty} a_k u^k$ $(a_0 \neq 0)$ is an analytic function in the disc $|u| < R$ $(R > 1)$. For ensuring the positivity of the operators (1.2), Ciupa considered the assumptions $\frac{a_k}{g(1)} \geq 0$, $k = 0, 1, \ldots$. Notice that for the special case $\lambda = 0$ and $g(u) = 1$, the operators (1.2) return to the Szasz-Durrmeyer operators given by (1.1).

Recently, Varma et al. [12] constructed linear positive operators including Brenke type polynomials. Brenke type polynomials [2] have generating relation of the form

$$A(t)\, B(xt) = \sum_{k=0}^{\infty} p_k(x)\, t^k \tag{1.4}$$

where A and B are analytic functions

$$A(t) = \sum_{r=0}^{\infty} a_r t^r, \qquad a_0 \neq 0, \tag{1.5}$$

$$B(t) = \sum_{r=0}^{\infty} b_r t^r, \qquad b_r \neq 0 \ (r \geq 0) \tag{1.6}$$

and have the following explicit expression

$$p_k(x) = \sum_{r=0}^{k} a_{k-r} b_r x^r, \qquad k = 0, 1, 2, \ldots \ . \tag{1.7}$$

Using the following restrictions

$$\begin{array}{ll} (i) & A(1) \neq 0,\ \frac{a_{k-r} b_r}{A(1)} \geq 0,\ 0 \leq r \leq k,\ k = 0, 1, 2, \ldots, \\ (ii) & B : [0, \infty) \longrightarrow (0, \infty), \\ (iii) & (1.4) \text{ and the power series } (1.5) \text{ and } (1.6) \text{ converge for } \\ & |t| < R \ (R > 1), \end{array} \tag{1.8}$$

Varma et al. introduced the following linear positive operators involving the Brenke type polynomials

$$L_n(f; x) = \frac{1}{A(1)\, B(nx)} \sum_{k=0}^{\infty} p_k(nx)\, f\left(\frac{k}{n}\right) \tag{1.9}$$

where $x \geq 0$ and $n \in \mathbb{N}$.

In this paper, by using the same restrictions given by (1.8), our aim is to construct the Durrmeyer type integral modification of the operators (1.9) as a generalization of Szasz-Durrmeyer operators (1.1) with

$$L_n^*(f; x) = \frac{1}{A(1)\, B(nx)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)\, dt \tag{1.10}$$

where $x \geq 0$, $\lambda \geq 0$ and $n \in \mathbb{N}$.

Remark 1.1. Let $B(t) = e^t$. The operators (1.10) (resp. (1.4)) return to the operators given by (1.2) (resp. (1.3)).

Remark 1.2. Let $\lambda = 0$, $A(t) = 1$ and $B(t) = e^t$. The operators (1.10) reduce to the well-known Szasz-Durrmeyer operators given by (1.1).

The paper is divided into three sections. In the next section, convergence of the operators (1.10) is investigated with the help of the universal Korovkin-type property [1]. The order of approximation is calculated by means of classical modulus of continuity. In section 3, we design the bridge with the notion of approximation theory and orthogonal polynomials. Namely, we give some illustrations with the help of the Hermite polynomials $H_k^{(\nu)}(x)$ of variance ν and Gould-Hopper polynomials for the operators (1.10).

2. Approximation properties of L_n^* operators

In this section, we state our main theorem with the help of the universal Korovkin-type property [1] and calculate the order of approximation by classical modulus of continuity. First of all, we give some definitions and lemmas used in the sequel.

Definition 2.1. *Let $f \in \tilde{C}[0,\infty)$ and $\delta > 0$. The modulus of continuity $\omega(f;\delta)$ of the function f is defined by*

$$\omega(f;\delta) := \sup_{\substack{x,y \in [0,\infty) \\ |x-y| \leq \delta}} |f(x) - f(y)|$$

where $\tilde{C}[0,\infty)$ is the space of uniformly continuous functions on $[0,\infty)$.

Lemma 2.2. *(Varma et al. [12]) For the operators L_n given by the equality (1.9), it holds*

$$L_n(1;x) = 1 \tag{2.1}$$

$$L_n(t;x) = \frac{B'(nx)}{B(nx)}x + \frac{A'(1)}{nA(1)} \tag{2.2}$$

$$L_n(t^2;x) = \frac{B''(nx)}{B(nx)}x^2 + \frac{\left[A(1) + 2A'(1)\right]B'(nx)}{nA(1)B(nx)}x + \frac{A''(1) + A'(1)}{n^2 A(1)} \tag{2.3}$$

for $x \in [0,\infty)$.

Lemma 2.3. *For the operators L_n^*, we have*

$$L_n^*(1;x) = 1 \tag{2.4}$$

$$L_n^*(t;x) = \frac{B'(nx)}{B(nx)}x + \frac{1}{n}\left(\lambda+1+\frac{A'(1)}{A(1)}\right) \tag{2.5}$$

$$L_n^*(t^2;x) = \frac{B''(nx)}{B(nx)}x^2 + \frac{2\left[(\lambda+2)A(1)+A'(1)\right]B'(nx)}{nA(1)B(nx)}x + \frac{A''(1)+2(\lambda+2)A'(1)+(\lambda+1)(\lambda+2)A(1)}{n^2A(1)} \tag{2.6}$$

for $x\in[0,\infty)$.

Proof. For $f(t)=1$, by using the definition of gamma function, we get from (1.10)

$$\begin{aligned} L_n^*(1;x) &= \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}dt \\ &= \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx) = L_n(1;x) \quad . \end{aligned}$$

In view of the equality (2.1), we easily get the equality (2.4).

For $f(t)=t$, we obtain from (1.10)

$$\begin{aligned} L_n^*(t;x) &= \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k+1}dt \\ &= \frac{\lambda+1}{n}\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)+\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\left(\frac{k}{n}\right) \\ &= L_n(t;x)+\frac{\lambda+1}{n}L_n(1;x). \end{aligned}$$

Taking into account the equalities (2.1)-(2.2), we have the equality (2.5).

For $f(t)=t^2$, by virtue of the equalities $(2.1)-(2.3)$, using similar technique leads us to the equality (2.6). □

Let us define the class of E as follows

$$E := \left\{ f : x\in[0,\infty),\ \frac{f(x)}{1+x^2} \text{ is convergent as } x\to\infty \right\}.$$

Theorem 2.4. *Let $f\in C[0,\infty)\cap E$ and assume that*

$$\lim_{y\to\infty}\frac{B'(y)}{B(y)} = 1 \qquad \textit{and} \qquad \lim_{y\to\infty}\frac{B''(y)}{B(y)} = 1. \tag{2.7}$$

Then,

$$\lim_{n\to\infty}L_n^*(f;x) = f(x)$$

uniformly on each compact subset of $[0,\infty)$.

Proof. According to the Lemma 2.3 and taking into account the assumptions (2.7), we find

$$\lim_{n\to\infty} L_n^*\left(t^i;x\right)=x^i, \qquad i=0,1,2.$$

Above mentioned convergences are satisfied uniformly in each compact subset of $[0,\infty)$. By applying the universal Korovkin-type property (vi) of Theorem 4.1.4 [1], we get the desired result. □

Theorem 2.5. *Let $f\in\tilde{C}[0,\infty)\cap E$. L_n^* operators satisfy the following inequality*

$$\left|L_n^*(f;x)-f(x)\right|\le 2\omega\left(f;\sqrt{\gamma_n(x)}\right)$$

where

$$\begin{aligned}\gamma_n(x) &= L_n^*\left((t-x)^2;x\right)=\frac{B''(nx)-2B'(nx)+B(nx)}{B(nx)}x^2\\ &+\frac{2\left[\left[(\lambda+2)A(1)+A'(1)\right]B'(nx)-\left[A'(1)+(\lambda+1)A(1)\right]B(nx)\right]}{nA(1)B(nx)}x\\ &+\frac{A''(1)+2(\lambda+2)A'(1)+(\lambda+1)(\lambda+2)A(1)}{n^2A(1)}.\end{aligned}$$

Proof. From (2.4) and the property of modulus of continuity, we deduce

$$\begin{aligned}&\left|L_n^*(f;x)-f(x)\right|\\ \le\ & \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^\infty e^{-nt}t^{\lambda+k}\left|f(t)-f(x)\right|dt\\ \le\ & \left\{1+\frac{1}{\delta}\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^\infty e^{-nt}t^{\lambda+k}\left|t-x\right|dt\right\}\omega(f;\delta).\end{aligned}$$

By using the Cauchy-Schwarz inequality for the integral, we have

$$\begin{aligned}&\left|L_n^*(f;x)-f(x)\right|\\ \le\ & \left\{1+\frac{1}{\delta}\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\left(\int_0^\infty e^{-nt}t^{\lambda+k}dt\right)^{1/2}\right.\\ &\left.\times\left(\int_0^\infty e^{-nt}t^{\lambda+k}(t-x)^2dt\right)^{1/2}\right\}\omega(f;\delta). \qquad (2.8)\end{aligned}$$

By applying the Cauchy-Schwarz inequality for the sum, (2.8) leads to

$$\begin{aligned}
&\left|L_n^*(f;x)-f(x)\right| \\
\leq\ & \left\{1+\frac{1}{\delta}\left(\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}dt\right)^{1/2}\right. \\
&\left.\times\left(\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}(t-x)^2dt\right)^{1/2}\right\}\omega(f;\delta) \\
=\ & \left\{1+\frac{1}{\delta}\left(L_n^*(1;x)\right)^{1/2}\left(L_n^*\left((t-x)^2;x\right)\right)^{1/2}\right\}\omega(f;\delta) \quad .
\end{aligned}$$

In view of Lemma 2.3, we get the desired result for $\delta=\delta_n=\sqrt{\gamma_n(x)}$. □

Remark 2.6. Note that in Theorem 2.5, when $n\to\infty$, $\gamma_n(x)$ tends to zero under the assumptions (2.7).

3. Examples

Example 3.1. The Hermite polynomials $H_k^{(\nu)}(x)$ of variance ν [9] have the following generating functions of the form

$$e^{-\frac{\nu t^2}{2}+xt}=\sum_{k=0}^{\infty}\frac{H_k^{(\nu)}(x)}{k!}t^k \tag{3.1}$$

and the explicit representations

$$H_k^{(\nu)}(x)=\sum_{r=0}^{\left[\frac{k}{2}\right]}\left(\frac{-\nu}{2}\right)^r\frac{k!}{r!(k-2r)!}x^{k-2r}$$

where, as usual, [.] denotes the integer part. It is obvious that the Hermite polynomials $H_k^{(\nu)}(x)$ of variance ν are Brenke type polynomials for

$$A(t)=e^{-\frac{\nu t^2}{2}} \quad \text{and} \quad B(t)=e^t \quad .$$

Under the assumption $\nu\leq 0$; the restrictions (1.8) and assumptions (2.7) for the operators L_n^* given by (1.10) are satisfied. With the help of generating functions (3.1), we get the explicit form of L_n^* operators involving the Hermite polynomials $H_k^{(\nu)}(x)$ of variance ν by

$$H_n^*(f;x)=e^{-nx+\frac{\nu}{2}}\sum_{k=0}^{\infty}\frac{H_k^{(\nu)}(nx)}{k!}\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}f(t)\,dt \tag{3.2}$$

where $x\in[0,\infty)$.

Example 3.2. Gould-Hopper polynomials [5] have the generating functions of the type

$$e^{ht^{d+1}} \exp(xt) = \sum_{k=0}^{\infty} g_k^{d+1}(x,h) \frac{t^k}{k!} \tag{3.3}$$

and the explicit representations

$$g_k^{d+1}(x,h) = \sum_{s=0}^{\left[\frac{k}{d+1}\right]} \frac{k!}{s!\,(k-(d+1)\,s)!} h^s x^{k-(d+1)s} \quad .$$

Gould-Hopper polynomials $g_k^{d+1}(x,h)$ are d-orthogonal polynomial set of Hermite type [4]. Van Iseghem [11] and Maroni [7] discovered the notion of d-orthogonality. Gould-Hopper polynomials are Brenke type polynomials with

$$A(t) = e^{ht^{d+1}} \quad \text{and} \quad B(t) = e^t \quad .$$

Under the assumption $h \geq 0$; the restrictions (1.8) and assumptions (2.7) for the operators L_n^* given by (1.10) are satisfied. With the help of generating functions (3.3), we obtain the explicit form of L_n^* operators including Gould-Hopper polynomials by

$$G_n^*(f;x) = e^{-nx-h} \sum_{k=0}^{\infty} \frac{g_k^{d+1}(nx,h)}{k!} \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)\,dt \tag{3.4}$$

where $x \in [0,\infty)$.

Remark 3.3. It is worthy to note that for $h = 0$ and $\nu = 0$, respectively, we obtain that

$$g_k^{d+1}(nx,0) = (nx)^k \quad \text{and} \quad H_k^{(0)}(nx) = (nx)^k .$$

Substituting $H_k^{(0)}(nx) = (nx)^k$ for $\nu = 0$ in the operators (3.2) and similarly $g_k^{d+1}(nx,0) = (nx)^k$ for $h = 0$ in the operators (3.4), with the special case $\lambda = 0$, we get the well-known Szasz-Durrmeyer operators given by (1.1). By the help of H_n^* and G_n^* operators, we introduce an interesting generalization of Szasz-Durrmeyer operators with the Hermite polynomials $H_k^{(\nu)}(x)$ of variance ν and Gould-Hopper polynomials.

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On Cheney and Sharma type operators reproducing linear functions

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Abstract. With the help of generating functions, we present general conditions to construct positive linear operators which reproduce linear functions. The results are used to present a modification of the Cheney and Sharma operators and the rate of convergence is studied.

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Keywords: Positive linear operators, generating functions, rate of convergence.

1. Introduction

For an interval I, let $C(I)$ ($C_B(I)$) be the space of the real (bounded) continuous functions defined on I. As usual, we denote $e_k(x) = x^k$, for $k \in \mathbb{N}_0$.

In [5] Cheney and Sharma introduced a modification of Meyer-König and Zeller operators by defining, for a fixed $t \leq 0$, $f \in C[0,1]$ and $x \in [0,1)$

$$L_{n,t}(f,x) = (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{k=0}^{\infty} f(x_{n,k})\, L_k^{(n)}(t) x^k, \tag{1.1}$$

where

$$x_{n,k} = \frac{k}{n+k},$$

and the functions $L_k^{(n)}(t)$ are the Laguerre polynomials. It is known that (see [12], p. 101, eq. 5.1.6))

$$L_k^{(n)}(t) = \sum_{j=0}^{k} \binom{n+k}{k-j} \frac{(-t)^j}{j!}. \tag{1.2}$$

Hence $L_k^{(n)}(t) \geq 0$ (for $t \leq 0$) and the operators (1.1) are positive. On the other hand, it follows from the properties of Laguerre polynomials that $L_{n,t}(e_0) = e_0$ (see [12], p. 101, eq. 5.1.9)). It can be proved that $L_{n,t}(e_1) = e_1$ if and only if $t = 0$ (see [1]). This property was asserted in [5], but the proof given there is not correct. When $t = 0$, we obtain what are usually called the (slight modification of the) Meyer-König and Zeller operators (see [11]).

The Meyer-König and Zeller operators have been intensively studied and several modifications have been proposed (for instance, see [1], [6], [9], [10], [13] and the references therein).

In Section 3 of this paper we show that the nodes $x_{n,k}$ in (1.1) can be selected in such a way that the new operators reproduce linear functions, and we also give an estimate of the rate of convergence (in terms of the so called Ditzian-Totik moduli). First, in Section 2, we analyze the problem for general positive linear operators constructed by means of generating functions. Finally, in the last section we provide another example to show that the general approach of Section 2 can be used to modify other known operators.

2. Generating functions

Let us begin with a general approach to construct positive linear operators.

Theorem 2.1. *Fix $a > 0$ and sequence $\{a_k\}$ of positive real numbers such that*

$$\limsup_{k\to\infty} \left(\frac{a_k}{k!}\right)^{1/k} = \frac{1}{a} \tag{2.1}$$

and set

$$g(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k, \qquad | z |< a. \tag{2.2}$$

Let $\{y_k\}_{k=0}^{\infty}$ be any increasing sequence of points satisfying $y_k \in [0, a)$.

(i) *If $f : [0, a) \to \mathbb{R}$ is a bounded function and $x \in [0, a)$, then the series*

$$L(f, x) = \frac{1}{g(x)} \sum_{k=0}^{\infty} \frac{a_k}{k!} f(y_k)\, x^k, \tag{2.3}$$

defines a function that is continuous on $[0, a)$.

(ii) *The map L defines a positive linear operator in $C_B[0, a)$ which reproduces the constant functions.*

(iii) *One has $L(e_1) = e_1$ if (and only if) $y_0 = 0$ and*

$$y_{k+1} = \frac{(k + 1)a_k}{a_{k+1}}, \quad \text{for all} \quad k \geq 0. \tag{2.4}$$

(iv) *Suppose that $y_k \longrightarrow a$ and $g(x) \longrightarrow \infty$ as $x \longrightarrow a$. If $f \in C[0, a]$ and we set $L(f, a) = f(a)$, then $L(f) \in C[0, a]$.*

Proof. (i) It follows from (2.1) that g is an analytic function in the domain $| z |< a$. If $| f(x) |\leq C(f)$ for $x \in [0, a)$, then $| L(f, x) |\leq C(f)g(x)$ and the series converges uniformly on the compact subsets of $[0, a)$.

(ii) It is clear that $L(f)$ is well defined for each $f \in C_B[0, 1]$ and L is a positive linear operator on this space. The assertion $L(e_0) = e_0$ follows from (2.2).

(iii) If $L(e_1) = e_1$, then $L(e_1, 0) = a_0 e_1(y_0) = e_1(0) = 0$. Since $a_0 > 0$, we obtain $y_0 = 0$. On the other hand, if $y_0 = 0$, then $L(e_1, 0) = 0 = e_1(0)$.

For $0 < x < a$,

$$L(e_1, x) = \frac{x}{g(x)} \sum_{k=1}^{\infty} \frac{a_k}{k!}\, y_k\, x^{k-1} = \frac{x}{g(x)} \sum_{k=0}^{\infty} \frac{a_k}{k!}\, \frac{a_{k+1} y_{k+1}}{a_k(k+1)}\, x^k$$

$$= x + \frac{x}{g(x)} \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{a_{k+1}\, y_{k+1}}{a_k(k+1)} - 1 \right) x^k .$$

Thus, $L(e_1, x) = x$ if and only if

$$\frac{a_{k+1}\, y_{k+1}}{a_k(k+1)} = 1, \quad \text{for all} \quad k \geq 0,$$

and this is equivalent to (2.4).

(iv) Fix $\varepsilon > 0$ and $t > 0$ such that $\mid f(x) - f(a) \mid < \varepsilon/2$, whenever $\mid x - a \mid < t$. Since $y_k \longrightarrow a$, there exists a natural m such that $\mid y_k - a \mid < t$, for all $k > m$. Set

$$C = \sup_{x \in [0,a]} \left| \sum_{k=0}^{m} \frac{a_k}{k!} x^k \right| .$$

On the other hand, there exists $\delta > 0$ such that, if $0 < a - x < \delta$, then

$$\frac{1}{g(x)} < \frac{\varepsilon}{4C(1 + \|f\|)},$$

where we consider the sup norm on $[0, a]$. Therefore, if $0 < a - x < \delta$, then

$$\mid L(f, x) - f(x) \mid = \left| \frac{1}{g(x)} \sum_{k=0}^{\infty} \frac{a_k}{k!} \Big(f(y_k) - f(a) \Big) x^k \right|$$

$$\leq \frac{2\|f\|}{g(x)} \sum_{k=0}^{m} \frac{a_k}{k!} x^k + \frac{1}{g(x)} \sum_{k=m+1}^{\infty} \frac{a_k}{k!} \mid f(y_k) - f(a) \mid x^k$$

$$\leq \frac{2\|f\| C}{g(x)} + \frac{\varepsilon}{2} \frac{1}{g(x)} \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k < \varepsilon .$$

This proves the assertion. □

Notice that, in order to use condition (2.4), we also need the inequality

$$a_k < a\, \frac{a_{k+1}}{k+1}, \qquad (k \geq 0),$$

which follows from the conditions $y_k \in [0, a)$.

In the next result we consider the case $a = 1$.

Theorem 2.2. *Suppose that the analytic function h has the expansion*

$$h(z) = \sum_{k=0}^{\infty} b_k z^k, \qquad \mid z \mid < 1,$$

with $0 < b_{k-1} < b_k$, for all $k \in \mathbb{N}$. Then the equation

$$L(f, x) = \frac{1}{h(x)} \sum_{k=0}^{\infty} b_k\, f\left(\frac{b_{k-1}}{b_k} \right) x^k, \qquad x \in [0, 1), \tag{2.5}$$

where $b_{-1} = 0$*, defines a positive linear operator,* $L : C_B[0,1) \to C_B[0,1)$ *such that* $L(e_0) = e_0$ *and* $L(e_1) = e_1$*.*

Moreover, if

$$\lim_{x\to 1-} h(x) = \infty, \qquad \lim_{k\to\infty} \frac{b_{k-1}}{b_k} = 1 \tag{2.6}$$

and we set $L(f,1) = f(1)$*, then* $L : C[0,1] \to C[0,1]$*.*

Proof. With the notation given above, one has $k!b_k = a_k$ and $a \geq 1$. Thus, in this case, equation (2.4) can be written as

$$y_{k+1} = \frac{(k+1)a_k}{a_{k+1}} = \frac{b_k}{b_{k+1}} < 1, \quad \text{for all} \quad k \geq 0. \qquad \square$$

3. A variation of Cheney and Sharma operators

Theorem 3.1. *Fix* $t \leq 0$ *and let the numbers* $L_k^{(n)}(t)$ $(n \in \mathbb{N},\ k \geq 0)$ *be defined by* (1.2) *and set* $L_{-1}^{(n)} = 0$*. Then the equation*

$$S_{n,t}(f,x) \;=\; (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{k=0}^{\infty} L_k^{(n)}(t)\; f\left(\frac{L_{k-1}^{(n)}(t)}{L_k^{(n)}(t)}\right) x^k, \tag{3.1}$$

where $L_{-1}^{(n)}(t) = 0$ *and* $x \in [0,1)$*, defines a positive linear operator on* $C_B[0,1)$ *such that*

$$S_{n,t}(e_0,x) = 1, \qquad S_{n,t}(e_1,x) = x \tag{3.2}$$

and

$$0 \;\leq\; S_{n,t}(e_2,x) - x^2 \;\leq\; x^2 + \frac{2x(1-x)(1-tx)^2}{n}\,. \tag{3.3}$$

Moreover, if for $f \in C[0,1]$ *we set* $S_{n,t}(f,1) = f(1)$*, then* $S_{n,t} : C[0,1] \to C[0,1]$*.*

Proof. Since t will be fixed, in order to simplify, we write $L_k^{(n)}$ instead of $L_k^{(n)}(t)$.

It is known that (see [12], p. 102, Eq. (5.1.14) and (5.1.13))

$$\frac{L_k^{(n)}}{L_{k+1}^{(n)}} = \frac{k+1}{n+k+1} + \frac{t}{n+k+1}\,\frac{L_k^{(n+1)}}{L_{k+1}^{(n)}} \tag{3.4}$$

and

$$\frac{L_k^{(n)}}{L_{k+1}^{(n)}} = 1 - \frac{L_{k+1}^{(n-1)}}{L_{k+1}^{(n)}}. \tag{3.5}$$

From the last equation we know that $L_k^{(n)} < L_{k+1}^{(n)}$. Thus the operators (3.1) are well defined and it follows from the first part of Theorem 2.2 that (3.2) holds.

Let us verify (3.3). Since S_{n-t} is a positive linear operator, from (3.2) we know that $0 \leq S_{n,t}((e_1-x)^2,x) = S_{n,t}(e_2,x) - x^2$.

From (1.2) we know that

$$L_k^{(n+1)} = \sum_{j=0}^{k} \frac{k+1-j}{n+1+j} \binom{n+k+1}{k+1-j} \frac{(-t)^j}{j!} < \frac{k+1}{n+1} L_{k+1}^{(n)}. \tag{3.6}$$

and from (3.4) and (3.5) we obtain (recall that $t \le 0$)

$$\begin{aligned} L_k^{(n)} \frac{L_k^{(n)}}{L_{k+1}^{(n)}} &= L_k^{(n)} - L_{k+1}^{(n-1)} \frac{L_k^{(n)}}{L_{k+1}^{(n)}} \\ &= L_k^{(n)} - L_{k+1}^{(n-1)} \left(\frac{k+1}{n+k+1} + \frac{t}{n+k+1} \frac{L_k^{(n+1)}}{L_{k+1}^{(n)}} \right) \\ &\le L_k^{(n)} - L_{k+1}^{(n-1)} \frac{k+1}{n+k+1} - L_{k+1}^{(n-1)} \frac{t(k+1)}{(n+1)(n+k+1)} . \end{aligned}$$

Let us set $g_n(x) = (1-x)^{n+1} \exp(tx/(1-x))$. From the last estimate, taking into account that $L_{-1}^{(n)}(t) = 0$, for $n > 2$ and $x \in (0,1)$ we obtain

$$\begin{aligned} S_{n,t}(e_2, x) \;&=\; g_n(x) \sum_{k=1}^{\infty} L_{k-1}^{(n)} \left(\frac{L_{k-1}^{(n)}}{L_k^{(n)}} \right) x^k \;=\; x\, g_n(x) \sum_{k=0}^{\infty} L_k^{(n)} \left(\frac{L_k^{(n)}}{L_{k+1}^{(n)}} \right) x^k \\ &\le\; x\, g_n(x) \sum_{k=0}^{\infty} \left(L_k^{(n)} - \frac{n+1+t}{n+1} \frac{k+1}{n+k+1} L_{k+1}^{(n-1)} \right) x^k \\ &= x - \frac{(n+1+t)}{n+1} g_n(x) \sum_{k=0}^{\infty} \left(\frac{k+1}{n+k+1} L_{k+1}^{(n-1)} \right) x^{k+1} \\ &= x - \frac{(n+1+t)}{n+1} g_n(x) \sum_{k=0}^{\infty} \left(L_k^{(n-1)} \frac{k}{n+k} \right) x^k \\ &= x - \frac{(n+1+t)}{n+1} g_n(x) \sum_{k=0}^{\infty} \left(L_k^{(n-1)} \frac{k}{n-1+k} \left(1 - \frac{1}{n+k} \right) \right) x^k \\ &= x - \frac{(n+1+t)(1-x)}{n+1} L_{n-1,t}(e_1, x) \\ &\qquad + \frac{n+1+t}{n+1} g_n(x) \sum_{k=0}^{\infty} \left(L_k^{(n-1)} \frac{k}{(n-1+k)(n+k)} \right) x^k \\ &= x^2 - \frac{tx(1-x)}{n+1} \Big(L_{n-1,t}(e_1, x) - x \Big) - \frac{tx(1-x)}{n+1} \\ &\quad + g_n(x) \sum_{k=1}^{\infty} \left(L_k^{(n-1)} \frac{k}{(n-1+k)(n+k)} \right) x^k \\ &\le x^2 - \frac{tx(1-x)}{n+1} \Big(L_{n-1,t}(e_1, x) - x \Big) - \frac{tx(1-x)}{n+1} \\ &\qquad + \frac{1}{n} g_n(x) \sum_{k=1}^{\infty} \left(L_k^{(n-1)} \frac{k}{n-2+k} \right) x^k \\ &= x^2 - \frac{tx(1-x)}{n+1} \Big(L_{n-1,t}(e_1, x) - x \Big) - \frac{tx(1-x)}{n+1} + \frac{1-x}{n} L_{n-1,t}(e_1, x) . \end{aligned}$$

Since (see Theorem 1 of [1])

$$L_{n,t}(e_1,x) \leq x - \frac{tx}{n+1},$$

one has

$$S_{n,t}(e_2,x) \leq x^2 + \frac{(tx)^2(1-x)}{(n+1)^2} - \frac{tx(1-x)}{n+1} + \frac{1-x}{n}\left(x - \frac{tx}{n}\right)$$

$$\leq x^2 + \frac{x(1-x)}{n} - \frac{2tx(1-x)(2-tx)}{n}.$$

For the last assertion, first notice that, since $t \leq 0$,

$$\lim_{x\to 1-} \frac{1}{(1-x)^{n+1}} \exp\left(\frac{-tx}{1-x}\right) = \infty.$$

In order to finish, we only need to verify the second equality in (2.6). But it is a known result (for instance, see [3]). □

Theorem 3.2. *Fix* $\alpha \in (0,1/2]$ *and set* $\varphi(x) = (x(1-x))^\alpha$. *For* $t \leq 0$, *let the operators* $S_{n,t}$ *be defined as in Theorem* 3.1. *For* $f \in C[0,1]$, $x \in [0,1]$, *and* $n > 2$ *one has*

$$|f(x) - S_{n,t}(f,x)| \leq \Big(\frac{3}{2} + 3(1-tx)^2\Big)\omega_2^\varphi\left(f, \sqrt{\frac{(x(1-x))^{1-2\alpha}}{n}}\right),$$

where

$$\omega_2^\varphi(f,h) = \sup_{0\leq s\leq h} \sup_{x\pm s\varphi(x)\in[0,1]} \left|\Delta^2_{h\varphi(x)} f(x)\right|$$

and $\Delta^2_{h\varphi(x)} f(x) = f(x - s\varphi(x))) - 2f(x) + f(x + s\varphi(x))$.

Proof. The result follows from (3.3) and Theorem 11 of [4]. □

4. Another example

Fix $r \in \mathbb{Z}$ and, for $n \in \mathbb{N}$, consider the identity

$$\frac{1}{(1-z)^{n+r}} = \sum_{k=0}^{\infty} b_{n,k} z^k \qquad |z| < 1,$$

where

$$b_{n,k} = \binom{n+r+k-1}{k}.$$

We also set $b_{n,-1} = 0$. Notice that, for $k \geq 1$,

$$\frac{b_{n,k-1}}{b_{n,k}} = \frac{k}{n+r+k-1}.$$

Thus we have all the conditions of Theorem 2.2.

Theorem 4.1. *Fix $r \in \mathbb{N}$. For $n \in \mathbb{N}$, $f \in C[0,1]$ and $x \in [0,1)$ set*

$$M_{n,r}(f,x) = (1-x)^{n+r} \sum_{k=0}^{\infty} \binom{n+r+k-1}{k} x^k f\left(\frac{k}{n+r+k-1}\right),$$

and

$$M_{n,r}(f,1) \;=\; f(1)\,.$$

(i) *For each $n \in \mathbb{N}$, $M_{n,r} : C[0,1] \to C[0,1]$ is a positive linear operator such that*

$$M_{n,r}(e_0,x) = 1, \qquad M_{n,r}(e_1,x) = x$$

and, for $n > 2$

$$\frac{x(1-x)^2}{2(n+r-2)} \;\le\; M_{n,r}(e_2,x) - x^2 \;\le\; \frac{x(1-x)^2}{n+r-2}\,. \tag{4.1}$$

(ii) *Fix $\alpha \in (0,1/2]$ and set $\varphi(x) = (x(1-x)^2)^\alpha$. For $f \in C[0,1]$, $x \in [0,1]$, and $n > 2$ one has*

$$|f(x) - M_{n,r}(f,x)| \le 3\omega_2^{\varphi}\left(f, \sqrt{\frac{\varphi^{1-2\alpha}(x)}{n+r-2}}\right).$$

Proof. We only need to verify (4.1). With the notation given above, one has

$$\begin{aligned}
M_{n,r}(e_2,x) - x^2 &= (1-x)^{n+r} \sum_{k=1}^{\infty} b_{n,k-1} \frac{b_{n,k-1}}{b_{n,k}} x^k - x^2 \\
&= x\left((1-x)^{n+r} \sum_{k=1}^{\infty} b_{n,k-1} \frac{b_{n,k-1}}{b_{n,k}} x^{k-1} - x\right) \\
&= x(1-x)^{n+r} \sum_{k=0}^{\infty} b_{n,k}\left(\frac{b_{n,k}}{b_{n,k+1}} - \frac{b_{n,k-1}}{b_{n,k}}\right) x^k \\
&= x(1-x)^{n+r} \sum_{k=0}^{\infty} b_{n,k}\left(\frac{k+1}{n+r+k} - \frac{k}{r+n+k-1}\right) x^k \\
&= x(1-x)^{n+r} \sum_{k=0}^{\infty} b_{n,k}\left(\frac{n+r-1}{(n+r+k)(r+n+k-1)}\right) x^k \\
&= x(1-x)^{n+r} \sum_{k=0}^{\infty} b_{n-1,k}\left(\frac{1}{n+r+k}\right) x^k \\
&= x(1-x)^{n+r} \sum_{k=0}^{\infty} \frac{(n+r+k-2)(n+r+k-3)!}{(n+r-2)k!(n+r-3)!}\left(\frac{1}{n+r+k}\right) x^k \\
&= \frac{x(1-x)^{n+r}}{n+r-2} \sum_{k=0}^{\infty} b_{n-2,k}\left(\frac{n+r+k-2}{n+r+k}\right) x^k.
\end{aligned}$$

Therefore, for $n > 2$,

$$\frac{x(1-x)^2}{2(n+r-2)} = x(1-x)^{n+r} \sum_{k=0}^{\infty} b_{n-2,k} \frac{1}{2(n+r-2)} x^k$$

$$\leq \frac{x(1-x)^{n+r}}{n+r-2}\sum_{k=0}^{\infty} b_{n-2,k}\left(\frac{n+r+k-2}{n+r+k}\right)x^k$$

$$= M_{n,r}(e_2,x) - x^2$$

$$\leq \frac{x(1-x)^2}{n+r-2}(1-x)^{n-2+r}\sum_{k=0}^{\infty} b_{n-2,k}\, x^k \;=\frac{x(1-x)^2}{n+r-2}. \qquad \square$$

Remark 4.2. In [8] (p.17), Götz introduced the operators

$$M^*_{n,r}(f,x) = (1-x)^{n+r}\sum_{k=0}^{\infty}\binom{n+r+k-1}{k}x^k f\left(\frac{k}{n+k}\right).$$

They are similar to the Meyer-König and Zeller operators, but they do not reproduce linear functions (see also [2], p. 126).

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Estimates for the operator of r-th order on simplex

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Abstract. We present general estimates with optimal constants of the degree of approximation by Kirov - Popova operators using weighted K-functionals of first order.

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Keywords: Positive linear operators, Peetre's K -functional, degree of approximation.

1. Introduction

In [2] Kirov and Popova associated to each positive linear operator $L : \mathbf{C}[a,b] \longrightarrow \mathbf{C}[a,b]$ a new operator $L_r(f,x) = L\left(T_{r,f,\cdot}(x), x\right), r \in \mathbb{N}$ (a generalization of r-th order), where $T_{r,f,y}(x)$ is the Taylor polynomial of degree r for the function f at y. This operator is linear but not necessarily positive. In [7], using same ideea as in [1] we gave a quantitative estimate for the remainder in Taylor's formula using K-functional K_1^∞ and weighted K-functional $K_{1,\varphi}^\infty$ and we obtained estimates for the operator of r-th order. In this paper we extend the results for the generalization of r-th order of a positive linear operator $L : \mathbf{C}(S) \longrightarrow \mathbf{C}(S)$ defined by $L_r(f,\mathbf{x}) = L\left(T_{r,f,\cdot}(\mathbf{x}), \mathbf{x}\right), r \in \mathbb{N}$, with $T_{r,f,\mathbf{y}}(\mathbf{x}) = \sum\limits_{j=0}^{r} \frac{1}{j!} d^{(j)} f(\mathbf{y})(\mathbf{x}-\mathbf{y})^j$, where $S = \left\{\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d | x_1, \cdots, x_d \geq 0,\, x_1 + \cdots + x_d \leq 1\right\}$ is the simplex in $\mathbb{R}^d$, $d \in \mathbb{N}$.

Starting from the weight function used in [5], we consider the function

$$\varphi(\mathbf{x}) = \left[(x_1 + \cdots + x_d)(1-x_1)\cdots(1-x_d)\right]^\alpha, \ \alpha \in (0,1).$$

We denote by

$$\mathbf{C}_\varphi(S) = \left\{ f \in \mathbf{C}(S \setminus \left\{\mathbf{v}_i,\, i = \overline{0,d}\right\}) \,|\, (\exists) \lim_{\mathbf{x} \to \mathbf{v_i}} f(\mathbf{x})\varphi(\mathbf{x}) \in \mathbb{R},\, i = \overline{0,d} \right\}$$

and

$$\mathbf{W}^1_{\mathbf{C}_\varphi}(S) = \left\{ f \in \mathbf{C}(\mathbf{S}) \mid \frac{\partial f}{\partial x_i} \in \mathbf{C}_\varphi(S),\, i = \overline{1,d} \right\}$$

where $\mathbf{v_i}$, $i = \overline{0,d}$ are simplex vertices. We consider the K-functional

$$K^\infty_{1,\varphi}(f,t) = K^\infty\left(f, t; \mathbf{C}(S), \mathbf{W}^1_{\mathbf{C}_\varphi}(S)\right),\ t > 0,$$

defined for the Banach space $(\mathbf{C}(S), \|\cdot\|)$ and the semi-Banach subspace $\left(\mathbf{W}^1_{\mathbf{C}_\varphi}(S), |\cdot|_{W^1_{C_\varphi}}\right)$, $|f|_{W^1_{C_\varphi}} = \|\varphi\nabla f\|_\infty = \max\limits_{i=\overline{1,d}} \left\| \varphi \frac{\partial f}{\partial x_i} \right\|$ by

$$K^\infty_{1,\varphi}(f,t) = \inf_{g \in \mathbf{W}^1_{\mathbf{C}_\varphi}(S)} \max \left\{ \|f-g\|, t\, \|\varphi\nabla g\|_\infty \right\}.$$

We use the notation e_0 for the function $e_0 : S \subset \mathbb{R}^d \to \mathbb{R}$, $e_0(\mathbf{x}) = 1$ and e_1 for the function $e_1 : S \subset \mathbb{R}^d \to \mathbb{R}^d$, $e_1(\mathbf{x}) = \mathbf{x}$.

2. Estimates with weighted K - functionals $K^\infty_{1,\varphi}$

Lemma 2.1. *If $f \in \mathbf{C^r}(S)$, $r \in \mathbb{N}$, $\mathbf{x} \in S \setminus \left\{\mathbf{v}_i,\, i = \overline{0,d}\right\}$ and $\mathbf{y} \in S$ then $(\forall) t > 0$ for the remainder in Taylor's formula of order r we have the following estimate*

$$|R_{r,f,\mathbf{y}}(\mathbf{x})| \le \left(2\frac{\|\mathbf{y}-\mathbf{x}\|_1^r}{r!} + \frac{\|\mathbf{y}-\mathbf{x}\|_1^{r+1}}{\prod\limits_{k=1}^{r+1}(k-\alpha)t\varphi(\mathbf{x})} \right) \max_{r_1+\cdots+r_d=r} K^\infty_{1,\varphi}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}, t \right). \tag{2.1}$$

Proof. Let $f \in \mathbf{C^r}(S)$ and $\mathbf{x} \in S \setminus \left\{\mathbf{v}_i,\, i = \overline{0,d}\right\}$, $\mathbf{y} \in S$, $\mathbf{x} \neq \mathbf{y}$. We consider $\psi(u) = (1-u)\mathbf{x} + u\mathbf{y}$, $u \in [0,1]$ and $h(u) = f(\psi(u))$.

Step 1. We prove that

$$|R_{r,f,\mathbf{y}}(\mathbf{x})| = |R_{r,h,1}(0)| \le \left(\frac{2}{r!} + \frac{1}{\prod\limits_{k=1}^{r+1}(k-\alpha)t\varphi(\mathbf{x})} \right) K^\infty_{1,\varphi\circ\psi}\left(h^{(r)}, t\right). \tag{2.2}$$

Let $g \in \mathbf{W}^{\mathbf{r+1}}_{\mathbf{C}_{\varphi\circ\psi}}[0,1]$. Let us now make use of the fact that the function

$$u \mapsto \frac{1-u}{\varphi(\psi(u))^{\frac{1}{\alpha}}},\ u \in (0,1)$$

is decreasing [5]. Using the integral form of the remainder we have

$$\begin{aligned}
|R_{r,g,1}(0)| &= \left| \frac{1}{r!} \int_1^0 g^{(r+1)}(u)(0-u)^r du \right| \\
&\leq \frac{1}{r!} \int_0^1 \left| (\varphi \circ \psi)(u) g^{(r+1)}(u) \right| \frac{u^r}{(\varphi \circ \psi)(u)} du \\
&\leq \frac{\left\| (\varphi \circ \psi) g^{(r+1)} \right\|}{r!} \int_0^1 u^r \frac{1}{(\varphi \circ \psi)(0)\,(1-u)^\alpha} du \\
&= \frac{\left\| (\varphi \circ \psi) g^{(r+1)} \right\|}{r! \varphi(\mathbf{x})} \int_0^1 \frac{u^r}{(1-u)^\alpha} du \\
&= \frac{\left\| (\varphi \circ \psi) g^{(r+1)} \right\|}{r! \varphi(\mathbf{x})} B\,(r+1, 1-\alpha) = \frac{\left\| (\varphi \circ \psi) g^{(r+1)} \right\|}{\prod_{k=1}^{r+1} (k-\alpha) \cdot \varphi(\mathbf{x})}
\end{aligned}$$

where B is Euler beta function.

We have

$$\begin{aligned}
|R_{r,h-g,1}(0)| &= \left| (h-g)(0) - \sum_{k=0}^{r} \frac{(h-g)^{(k)}(1)}{k!} (-1)^k \right| \\
&= \left| R_{r-1,h-g,1}(0) - \frac{(h-g)^{(r)}(1)}{r!} (-1)^r \right| \\
&\leq |R_{r-1,h-g,1}(0)| + \frac{\left\| (h-g)^{(r)} \right\|}{r!} \leq 2 \frac{\left\| h^{(r)} - g^{(r)} \right\|}{r!}.
\end{aligned}$$

Then

$$\begin{aligned}
&|R_{r,h,1}(0)| \leq |R_{r,h-g,1}(0)| + |R_{r,g,1}(0)| \\
&\leq \left(2 \frac{1}{r!} \left\| h^{(r)} - g^{(r)} \right\| + \frac{\left\| (\varphi \circ \psi) g^{(r+1)} \right\|}{\prod_{k=1}^{r+1} (k-\alpha) \cdot \varphi(\mathbf{x})} \right) \\
&\leq \left(\frac{2}{r!} + \frac{1}{\prod_{k=1}^{r+1} (k-\alpha) \cdot t\varphi(\mathbf{x})} \right) \cdot \max \left\{ \left\| h^{(r)} - g^{(r)} \right\|, t \left\| (\varphi \circ \psi) g^{(r+1)} \right\| \right\}.
\end{aligned}$$

Since g is arbitrary this implies (2.2).

Step 2. From (2.2) it results

$$|R_{r,f,\mathbf{y}}(\mathbf{x})| \leq \left(\frac{2}{r!} + \frac{\|\mathbf{y}-\mathbf{x}\|_1}{\prod_{k=1}^{r+1}(k-\alpha)\cdot t\varphi(\mathbf{x})} \right) K_{1,\varphi\circ\psi}^{\infty}\left(h^{(r)}, \frac{t}{\|\mathbf{y}-\mathbf{x}\|_1}\right). \tag{2.3}$$

Let $\varepsilon > 0$. We choose $g_{r_1,\cdots,r_d} \in \mathbf{C^1}(S)$, $r_i \in \mathbb{N}\cup\{0\}$, $i=\overline{1,d} : \, r_1+\cdots+r_d = r$, such that

$$\begin{aligned} & K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}, t\right) + \varepsilon \\ & \quad \geq \max\left\{ \left\| \frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} - g_{r_1,\cdots,r_d} \right\|, t\left\|\varphi\nabla g_{r_1,\cdots,r_d}\right\|_{\infty} \right\}. \end{aligned}$$

We consider the function

$$h_0(u) = \sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!} g_{r_1,\cdots,r_d}(\psi(u)) \prod_{i=1}^{d}(y_i-x_i)^{r_i}.$$

We have

$$K_{1,\varphi\circ\psi}^{\infty}\left(h^{(r)}, \frac{t}{\|\mathbf{y}-\mathbf{x}\|_1}\right) \leq \max\left\{ \left\|h^{(r)}-h_0\right\|, \frac{t}{\|\mathbf{y}-\mathbf{x}\|_1}\left\|(\varphi\circ\psi)h_o'\right\| \right\}.$$

Since

$$\begin{aligned} & \left|h^{(r)}(u)-h_0(u)\right| = \left|d^r f(\psi(u))(\mathbf{y}-\mathbf{x})^r - h_0(u)\right| \\ & = \left| \sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!} \left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} - g_{r_1,\cdots,r_d} \right)(\psi(u)) \prod_{i=1}^{d}(y_i-x_i)^{r_i} \right| \\ & \leq \sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!} \left\| \frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} - g_{r_1,\cdots,r_d} \right\| \cdot \prod_{i=1}^{d}|y_i-x_i|^{r_i}. \end{aligned}$$

hold for u arbitrary this implies

$$\left\|h^{(r)}-h_0\right\| \leq \sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!} \left\| \frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} - g_{r_1,\cdots,r_d} \right\| \cdot \prod_{i=1}^{d}|y_i-x_i|^{r_i}.$$

Also since

$$\begin{aligned} & |\varphi(\psi(u))h_0'(u)| \\ & \quad = \left| \sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!} \varphi(\psi(u)) d g_{r_1,\cdots,r_d}(\psi(u))(\mathbf{y}-\mathbf{x}) \prod_{i=1}^{d}(y_i-x_i)^{r_i} \right| \\ & \quad \leq \sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!} \left\|\varphi\nabla g_{r_1,\cdots,r_d}\right\|_{\infty} \cdot \|\mathbf{y}-\mathbf{x}\|_1 \cdot \prod_{i=1}^{d}|y_i-x_i|^{r_i}. \end{aligned}$$

hold for u arbitrary this implies

$$\|(\varphi\circ\psi)h_0'\| \le \sum_{r_1+\cdots+r_d=r}\frac{r!}{r_1!\cdots r_d!}\left\|\varphi\nabla g_{r_1,\cdots,r_d}\right\|_\infty\cdot\|\mathbf{y}-\mathbf{x}\|_1\cdot\prod_{i=1}^d|y_i-x_i|^{r_i}.$$

Then

$$K^\infty_{1,\varphi\circ\psi}\left(h^{(r)},\frac{t}{\|\mathbf{y}-\mathbf{x}\|_1}\right)\le\max\left\{\left\|h^{(r)}-h_0\right\|,\frac{t}{\|\mathbf{y}-\mathbf{x}\|_1}\left\|(\varphi\circ\psi)h_o'\right\|\right\}$$
$$\le\sum_{r_1+\cdots+r_d=r}\frac{r!}{r_1!\cdots r_d!}\prod_{i=1}^d|y_i-x_i|^{r_i}\cdot$$
$$\cdot\max\left\{\left\|\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}-g_{r_1,\cdots,r_d}\right\|,t\left\|\varphi\nabla g_{r_1,\cdots,r_d}\right\|_\infty\right\}$$
$$\le\sum_{r_1+\cdots+r_d=r}\frac{r!}{r_1!\cdots r_d!}\prod_{i=1}^d|y_i-x_i|^{r_i}\left(K^\infty_{1,\varphi}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right)+\varepsilon\right).$$

Since ε is arbitrary this implies

$$K^\infty_{1,\varphi\circ\psi}\left(h^{(r)},\frac{t}{\|\mathbf{y}-\mathbf{x}\|_1}\right)$$
$$\le\sum_{r_1+\cdots+r_d=r}\frac{r!}{r_1!\cdots r_d!}\prod_{i=1}^d|y_i-x_i|^{r_i}K^\infty_{1,\varphi}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right)$$
$$\le\|\mathbf{y}-\mathbf{x}\|_1^r\max_{r_1+\cdots+r_d=r}K^\infty_{1,\varphi}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right).$$

Finally, with (2.3) result (2.1). □

Theorem 2.2. *Let $r\in\mathbb{N}$, $L:\mathbf{C}(S)\longrightarrow\mathbf{C}(S)$ a positive linear operator and $f\in\mathbf{C^r}(S)$. Then $(\forall)\mathbf{x}\in S\setminus\left\{v_i,\,i=\overline{0,d}\right\}$, $(\forall)t>0$ we have*

$$|L_r(f,\mathbf{x})-f(\mathbf{x})|\le|f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1| \tag{2.4}$$
$$+\left(\frac{2}{r!}L\left(\|e_1-\mathbf{x}e_0\|_1^r,\mathbf{x}\right)+\frac{L\left(\|e_1-\mathbf{x}e_0\|_1^{r+1},\mathbf{x}\right)}{\prod_{k=1}^{r+1}(k-\alpha)t\varphi(\mathbf{x})}\right)\cdot$$
$$\cdot\max_{r_1+\cdots+r_d=r}K^\infty_{1,\varphi}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right)$$

Conversely, if $(\exists)A$, B, $C \geq 0$ *such that*

$$\begin{aligned}|L_r(f,\mathbf{x})-f(\mathbf{x})| &\leq A\cdot|f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1| \qquad (2.5)\\ &+\left(B\cdot L\left(\|e_1-\mathbf{x}e_0\|_1^r,\mathbf{x}\right)+C\cdot\frac{L\left(\|e_1-\mathbf{x}e_0\|_1^{r+1},\mathbf{x}\right)}{t\varphi(\mathbf{x})}\right)\cdot\\ &\cdot\max_{r_1+\cdots+r_d=r}K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right)\end{aligned}$$

holds for all positive linear operator $L:\mathbf{C}(S)\longrightarrow\mathbf{C}(S)$, *any* $f\in\mathbf{C^r}(S)$, *any* $\mathbf{x}\in S$ *and any* $t>0$ *then* $A\geq 1$, $B\geq\frac{2}{r!}$ *and* $C\geq\frac{1}{\prod_{k=1}^{r+1}(k-\alpha)}$.

Proof. From Lemma 2.1 we have

$$\begin{aligned}|L_r(f,\mathbf{x})-f(\mathbf{x})| &= |L(T_{r,f,\cdot}(\mathbf{x}),\mathbf{x})-f(\mathbf{x})|\\ &= |L(f(\mathbf{x})e_0-R_{r,f,\cdot}(\mathbf{x}),\mathbf{x})-f(\mathbf{x})|\\ &\leq |f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1|+|L(R_{r,f,\cdot}(\mathbf{x}),\mathbf{x})|\\ &\leq |f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1|\\ &+\left(\frac{2}{r!}L\left(\|e_1-\mathbf{x}e_0\|_1^r,\mathbf{x}\right)+\frac{L\left(\|e_1-\mathbf{x}e_0\|_1^{r+1},\mathbf{x}\right)}{\prod_{k=1}^{r+1}(k-\alpha)t\varphi(\mathbf{x})}\right)\cdot\\ &\cdot\max_{r_1+\cdots+r_d=r}K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right).\end{aligned}$$

which is (2.4).

We prove now the converse part. If we choose $L(h,\mathbf{x})=0$ and $f=e_0$ and replace in (2.5) we obtain $A\geq 1$.

To show that $B\geq\frac{2}{r!}$ we choose $L(h,\mathbf{x})=h(0)$ and $f(\mathbf{x})=2(x_1+\cdots+x_d)^{r+a}$ with $a>0$. For $g=(r+a)\cdot(r+a-1)\cdots(a+1)\,e_0$ we have

$$\begin{aligned}K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right) &\leq \max\left\{\left\|\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}-g\right\|,t\,\|\varphi\nabla g\|\right\}\\ =\left\|\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}-g\right\| &= (r+a)\cdot(r+a-1)\cdots(a+1).\end{aligned}$$

From (2.5) we obtain

$$\begin{aligned}&2(x_1+\cdots+x_d)^{r+a}\\ &\leq\left(B(x_1+\cdots+x_d)^r+C\frac{(x_1+\cdots+x_d)^{r+1}}{t\varphi(\mathbf{x})}\right)(r+a)\cdot(r+a-1)\cdots(a+1).\end{aligned}$$

Passing to the limit $t\to\infty$, $a\to 0$ we obtain $B\geq\frac{2}{r!}$.

To show that $C \geq \frac{1}{\prod_{k=1}^{r+1}(k-\alpha)}$ we choose

$$L(h,\mathbf{x}) = h(0) \text{ and } f(\mathbf{x}) = (x_1 + \cdots + x_d)^{r-\alpha+1}.$$

We have

$$K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}, t\right) \leq t\left\|\varphi\frac{\partial^{r+1} f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}\right\| = t\prod_{k=1}^{r+1}(k-\alpha).$$

From (2.5) we obtain

$$(x_1 + \cdots + x_d)^{r-\alpha+1} \leq B(x_1 + \cdots + x_d)^r t\prod_{k=1}^{r+1}(k-\alpha)$$

$$+C\frac{(x_1 + \cdots + x_d)^{r+1}}{\varphi(\mathbf{x})}\cdot\prod_{k=1}^{r+1}(k-\alpha).$$

Passing to the limit $t \to 0$ we obtain

$$C \geq [(1-x_1)\cdots(1-x_d)]^{\alpha}\cdot\frac{1}{\prod_{k=1}^{r+1}(k-\alpha)}$$

and passing to the limit $\mathbf{x} \to \mathbf{0}$ we obtain $C \geq \frac{1}{\prod_{k=1}^{r+1}(k-\alpha)}$. □

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On some vertical cohomologies of complex Finsler manifolds

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Abstract. In this paper we study some vertical cohomologies of complex Finsler manifolds as vertical cohomology attached to a function and vertical Lichnerowicz cohomology. We also study a relative vertical cohomology attached to a function associated to a holomorphic Finsler subspace.

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Introduction

The study of vertical cohomology of complex Finsler manifolds was initiated by Pitiş and Munteanu in [13]. The main goal of this paper is to study some other vertical cohomologies for forms of type (p,q,r,s) on complex Finsler manifolds as cohomology attached to a function defined in [12] and Lichnerowicz cohomology studied by many authors, e.g. [3, 8, 16]. In this sense, in the first section following [1, 2, 9] and [13], we briefly recall some preliminaries notions about complex Finsler manifolds and $\overline{v}$-cohomology groups. In the second section, we define a vertical cohomology attached to a function for forms of type (p,q,r,s) on a compex Finsler manifold (M,F) and we explain how this cohomology depends on the function. In particular, we show that if the function does not vanish, then our cohomology is isomorphic with the vertical cohomology of (M,F). In the third section we define and we study a vertical Lichnerowicz cohomology for forms of type (p,q,r,s) on a complex Finsler manifold (M,F) and in the last section, we construct a relative vertical cohomology attached to a function associated to a holomorphic Finsler subspace. The methods used here are closely related to those used by [4], [12] and [16].

1. Preliminaries

1.1. Complex Finsler manifolds

Let $\pi : T^{1,0}M \to M$ be the holomorphic tangent bundle of a n-dimensional complex manifold M. Denote by $(\pi^{-1}(U), (z^k, \eta^k))$, $k = 1, \ldots, n$ the induced complex coordinates on $T^{1,0}M$, where $(U, (z^k))$ is a local chart domain of M. At local change charts on $T^{1,0}M$, the transformation rules of these coordinates are given by

$$z'^k = z'^k(z) \ , \ \eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j, \tag{1.1}$$

where z'^k are holomorphic functions and $\det(\frac{\partial z'^k}{\partial z^j}) \neq 0$.

It is well known that $T^{1,0}M$ has a natural structure of $2n$-dimensional complex manifold, because the transition functions $\frac{\partial z'^k}{\partial z^j}$ are holomorphic.

Denote by $\widetilde{M} = T^{1,0}M - \{o\}$, where o is the zero section of $T^{1,0}M$, and we consider $T_{\mathbb{C}}\widetilde{M} = T^{1,0}\widetilde{M} \oplus T^{0,1}\widetilde{M}$ the complexified tangent bundle of the real tangent bundle $T_{\mathbb{R}}\widetilde{M}$, where $T^{1,0}\widetilde{M}$ and $T^{0,1}\widetilde{M} = \overline{T^{1,0}\widetilde{M}}$ are the holomorphic and antiholomorphic tangent bundles of $\widetilde{M}$, respectively.

Let $V^{1,0}\widetilde{M} = \ker \pi_*$ be the holomorphic vertical bundle over $\widetilde{M}$ and $\mathcal{V}^{1,0}(\widetilde{M})$ the module of its sections, called *vector fields of v-type.*

A given supplementary subbundle $H^{1,0}\widetilde{M}$ of $V^{1,0}\widetilde{M}$ in $T^{1,0}\widetilde{M}$, i.e.

$$T^{1,0}\widetilde{M} = H^{1,0}\widetilde{M} \oplus V^{1,0}\widetilde{M} \tag{1.2}$$

defines a *complex nonlinear connection* on $\widetilde{M}$, briefly c.n.c. and we denote by $\mathcal{H}^{1,0}(\widetilde{M})$ the module of its sections, called *vector fields of h-type.*

By conjugation over all, we get a decomposition of the complexified tangent bundle, namely $T_{\mathbb{C}}\widetilde{M} = H^{1,0}\widetilde{M} \oplus V^{1,0}\widetilde{M} \oplus H^{0,1}\widetilde{M} \oplus V^{0,1}\widetilde{M}$.

The elements of the conjugates are called *vector fields of* $\overline{h}$*-type* and $\overline{v}$*-type*, respectively.

If $N_k^j(z, \eta)$ are the local coefficients of the c.n.c. then the following set of complex vector fields

$$\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j\frac{\partial}{\partial \eta^j}\}\ ,\ \{\frac{\partial}{\partial \eta^k}\}\ ,\ \{\frac{\delta}{\delta \overline{z}^k} = \frac{\partial}{\partial \overline{z}^k} - N_{\overline{k}}^{\overline{j}}\frac{\partial}{\partial \overline{\eta}^j}\}\ ,\ \{\frac{\partial}{\partial \overline{\eta}^k}\} \tag{1.3}$$

are called the local adapted bases of $\mathcal{H}^{1,0}(\widetilde{M})$, $\mathcal{V}^{1,0}(\widetilde{M})$, $\mathcal{H}^{0,1}(\widetilde{M})$ and $\mathcal{V}^{0,1}(\widetilde{M})$, respectively. The dual adapted bases are given by

$$\{dz^k\}\ ,\ \{\delta\eta^k = d\eta^k + N_j^k dz^j\}\ ,\ \{d\overline{z}^k\}\ ,\ \{\delta\overline{\eta}^k = d\overline{\eta}^k + N_{\overline{j}}^{\overline{k}} d\overline{z}^j\}. \tag{1.4}$$

Throughout this paper, we consider the abreviate notations $\partial_k = \frac{\partial}{\partial z^k}$, $\dot{\partial}_k = \frac{\partial}{\partial \eta^k}$, $\delta_k = \frac{\delta}{\delta z^k}$ and its conjugates $\partial_{\overline{k}} = \frac{\partial}{\partial \overline{z}^k}$, $\dot{\partial}_{\overline{k}} = \frac{\partial}{\partial \overline{\eta}^k}$, $\delta_{\overline{k}} = \frac{\delta}{\delta \overline{z}^k}$.

Let us consider M be a strongly pseudoconvex complex Finsler manifold [1], i.e. M is endowed with a complex Finsler metric $F : T^{1,0}M \to \mathbb{R}_+ \cup \{0\}$ satisfying:

(1) F^2 is smooth on $\widetilde{M}$;

(2) $F(z, \eta) > 0$ for all $(z, \eta) \in \widetilde{M}$ and $F(z, \eta) = 0$ if and only if $\eta = 0$;

(3) $F(z, \lambda\eta) = |\lambda| F(z, \eta)$ for all $(z, \eta) \in T^{1,0}M$ and $\lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$;
(4) the complex hessian $(G_{j\overline{k}}) = (\dot{\partial}_j \dot{\partial}_{\overline{k}} (F^2))$ is positive definite on $\widetilde{M}$.

Let $(G^{\overline{m}j})$ be the inverse of $(G_{j\overline{m}})$. According to [1] and [9], a c.n.c. on (M, F) depending only on the complex Finsler metric F is the Chern-Finsler c.n.c., locally given by

$$\overset{CF}{N_k^j} = G^{\overline{m}j} \partial_k \, \dot{\partial}_{\overline{m}} \, (F^2) \tag{1.5}$$

and it has an important property, namely

$$\overset{CF}{R^i_{kj}} = \delta_k \overset{CF}{N^i_j} - \delta_j \overset{CF}{N^i_k} = 0. \tag{1.6}$$

In the sequel we will consider the adapted frames and coframes with respect to the Chern-Finsler c.n.c. and the hermitian metric structure G on $\widetilde{M}$ given by the Sasaki type lift of the fundamental tensor $G_{j\overline{k}}$, locally given by

$$G = G_{j\overline{k}} dz^j \otimes d\overline{z}^k + G_{j\overline{k}} \delta\eta^j \otimes \delta\overline{\eta}^k. \tag{1.7}$$

1.2. Vertical cohomology

According to [13], the set $\mathcal{A}(\widetilde{M})$ of complex valued differential forms on $\widetilde{M}$ is given by the direct sum

$$\mathcal{A}(\widetilde{M}) = \bigoplus_{p,q,r,s=0}^{n} \mathcal{A}^{p,q,r,s}(\widetilde{M}), \tag{1.8}$$

where $\mathcal{A}^{p,q,r,s}(\widetilde{M})$ or simply $\mathcal{A}^{p,q,r,s}$ is the set of all $(p+q+r+s)$-forms which can be non zero only when these act on p vector fields of h-type, on q vector fields of $\overline{h}$-type, on r vector fields of v-type, and on s vector fields of $\overline{v}$-type. The elements of $\mathcal{A}^{p,q,r,s}$ are called (p, q, r, s)*-forms* on $\widetilde{M}$.

With respect to the adapted coframes $\{dz^k, d\overline{z}^k, \delta\eta^k, \delta\overline{\eta}^k\}$ of $T^*_{\mathbb{C}}\widetilde{M}$ a form $\varphi \in \mathcal{A}^{p,q,r,s}$ is locally given by

$$\varphi = \frac{1}{p!q!r!s!} \varphi_{I_p \overline{J_q} K_r \overline{H_s}} dz^{I_p} \wedge d\overline{z}^{J_q} \wedge \delta\eta^{K_r} \wedge \delta\overline{\eta}^{H_s}, \tag{1.9}$$

where I_p denotes the ordered p-tuple $(i_1 \ldots i_p)$, J_q the ordered q-tuple $(j_1 \ldots j_q)$, K_r the ordered r-tuple $(k_1 \ldots k_r)$, H_s the ordered s-tuple $(h_1 \ldots h_s)$ and $dz^{I_p} = dz^{i_1} \wedge \ldots \wedge dz^{i_p}$, $d\overline{z}^{J_q} = d\overline{z}^{j_1} \wedge \ldots \wedge d\overline{z}^{j_q}$, $\delta\eta^{K_r} = \delta\eta^{k_1} \wedge \ldots \wedge \delta\eta^{k_r}$ and $\delta\overline{\eta}^{H_s} = \delta\overline{\eta}^{h_1} \wedge \ldots \wedge \delta\overline{\eta}^{h_s}$, respectively.

We notice that these forms are the $(p + r, q + s)$ *complex type* and according to [13] if (M, F) is a complex Finsler manifold endowed with the Chern-Finsler c.n.c., then by (1.6) the exterior differential d admits the decomposition

$$d\mathcal{A}^{p,q,r,s} \subset \mathcal{A}^{p+1,q,r,s} \oplus \mathcal{A}^{p,q+1,r,s} \oplus \mathcal{A}^{p,q,r+1,s} \oplus \mathcal{A}^{p,q,r,s+1} \oplus$$

$$\oplus \mathcal{A}^{p+1,q+1,r-1,s} \oplus \mathcal{A}^{p+1,q,r-1,s+1} \oplus \mathcal{A}^{p+1,q+1,r,s-1} \oplus \mathcal{A}^{p,q+1,r+1,s-1} \tag{1.10}$$

which allows us to define eight morphisms of complex vector spaces if we consider the different components, namely

$$\begin{array}{lll} \partial_h & : & \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p+1,q,r,s} \quad ; \quad \partial_v : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p,q,r+1,s} \\ \partial_{\overline{h}} & : & \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p,q+1,r,s} \quad ; \quad \partial_{\overline{v}} : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p,q,r,s+1} \\ \partial_1 & : & \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p+1,q+1,r-1,s} \, ; \, \partial_2 : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p+1,q,r-1,s+1} \\ \partial_3 & : & \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p+1,q+1,r,s-1} \, ; \, \partial_4 : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p,q+1,r+1,s-1} \end{array}$$

We remark that these operators and the classical operators ∂ and $\overline{\partial}$ that appear in the decomposition $d = \partial + \overline{\partial}$ of the differential on a complex manifold are related by $\partial = \partial_h + \partial_v + \partial_3 + \partial_4$ and $\overline{\partial} = \partial_{\overline{h}} + \partial_{\overline{v}} + \partial_1 + \partial_2$.

The conjugated vertical differential operator $\partial_{\overline{v}}$ is locally given by

$$\partial_{\overline{v}}\varphi = \sum_k \dot{\partial}_{\overline{k}} \, (\varphi_{I_p \overline{J_q} K_r \overline{H_s}}) \delta\overline{\eta}^k \wedge dz^{I_p} \wedge d\overline{z}^{J_q} \wedge \delta\eta^{K_r} \wedge \delta\overline{\eta}^{H_s}, \tag{1.11}$$

where the sum is after the indices $i_1 \leq \ldots \leq i_p$, $j_1 \leq \ldots \leq j_q$, $k_1 \leq \ldots \leq k_r$ and $h_1 \leq \ldots \leq h_s$, respectively. Also it satisfies

$$\partial_{\overline{v}}(\varphi \wedge \psi) = \partial_{\overline{v}}\varphi \wedge \psi + (-1)^{\deg \varphi} \varphi \wedge \partial_{\overline{v}}\psi$$

for any $\varphi \in \mathcal{A}^{p,q,r,s}$ and $\psi \in \mathcal{A}^{p',q',r',s'}$.

This operator has the property $\partial_{\overline{v}}^2 = 0$ and in [13], a classical theory of de Rham cohomology is developed for the conjugated vertical differential $\partial_{\overline{v}}$, see also [9] pag. 89. Namely, the sequence

$$0 \longrightarrow \Phi^{p,q,r} \xrightarrow{i} \mathcal{F}^{p,q,r0} \xrightarrow{\partial_{\overline{v}}} \mathcal{F}^{p,q,r,1} \xrightarrow{\partial_{\overline{v}}} \mathcal{F}^{p,q,r,2} \xrightarrow{\partial_{\overline{v}}} \ldots \xrightarrow{\partial_{\overline{v}}} \ldots,$$

is a fine resolution for the sheaf $\Phi^{p,q,r}$ of germs of $\partial_{\overline{v}}$-closed $(p,q,r,0)$-forms on $\widetilde{M}$, where $\mathcal{F}^{p,q,r,s}$ are the sheaves of germs of (p,q,r,s)-forms. It is also given a de Rham type theorem for the $\overline{v}$-cohomology groups $H^{p,q,r,s}(\widetilde{M}) = Z^{p,q,r,s}(\widetilde{M})/B^{p,q,r,s}(\widetilde{M})$ of the complex Finsler manifold:

$$H^s(\widetilde{M}, \Phi^{p,q,r}) \approx H^{p,q,r,s}(\widetilde{M}),$$

where $Z^{p,q,r,s}(\widetilde{M})$ is the space of $\partial_{\overline{v}}$-closed (p,q,r,s)-forms and $B^{p,q,r,s}(\widetilde{M})$ is the space of $\partial_{\overline{v}}$-exact (p,q,r,s)-forms globally defined on $\widetilde{M}$.

2. Vertical cohomology attached to a function

In this section, we consider a new $\overline{v}$-cohomology attached to a function on the complex Finsler manifold (M,F). This new cohomology is also defined in terms of forms of type (p,q,r,s) on $\widetilde{M}$. More precisely, if (M,F) is a complex Finsler manifold and f is a function on $\widetilde{M}$, we define the coboundary operator

$$\partial_{\overline{v},f} : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p,q,r,s+1} \, , \; \partial_{\overline{v},f}\varphi = f\partial_{\overline{v}}\varphi - (p+q+r+s)\partial_{\overline{v}}f \wedge \varphi. \tag{2.1}$$

It is easy to check that $\partial_{\overline{v},f}^2 = 0$ and denote by $H_f^{p,q,r,s}(\widetilde{M})$ the cohomology groups of the differential complex $(\mathcal{A}^{p,q,r,\bullet}(\widetilde{M}), \partial_{\overline{v},f})$, called the *vertical de Rham cohomology groups attached to the function f* of complex Finsler manifold (M,F).

More generally, for any integer k, we define the coboundary operator

$$\partial_{\overline{v},f,k} : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p,q,r,s+1}\ ,\ \partial_{\overline{v},f,k}\varphi = f\partial_{\overline{v}}\varphi - (p+q+r+s-k)\partial_{\overline{v}}f \wedge \varphi. \quad (2.2)$$

We still have $\partial^2_{\overline{v},f,k} = 0$ and we denote by $H^{p,q,r,s}_{f,k}(\widetilde{M})$ the cohomology of this complex. We shall restrict our attention to the cohomology $H^{p,q,r,s}_{f}(\widetilde{M})$ but most results readily generalize to the cohomology $H^{p,q,r,s}_{f,k}(\widetilde{M})$.

Using (2.1), by direct calculus we obtain

Proposition 2.1. *If $f,g \in \mathcal{F}(\widetilde{M})$ then*

(i) $\partial_{\overline{v},f+g} = \partial_{\overline{v},f} + \partial_{\overline{v},g}$, $\partial_{\overline{v},0} = 0$, $\partial_{\overline{v},-f} = -\partial_{\overline{v},f}$;
(ii) $\partial_{\overline{v},fg} = f\partial_{\overline{v},g} + g\partial_{\overline{v},f} - fg\partial_{\overline{v}}$, $\partial_{\overline{v},1} = \partial_{\overline{v}}$, $\partial_{\overline{v}} = \frac{1}{2}(f\partial_{\overline{v},\frac{1}{f}} + \frac{1}{f}\partial_{\overline{v},f})$, *and*
(iii) $\partial_{\overline{v},f}(\varphi \wedge \psi) = \partial_{\overline{v},f}\varphi \wedge \psi + (-1)^{\deg \varphi}\varphi \wedge \partial_{\overline{v},f}\psi$.

Dependence on the function

A natural question to ask about the cohomology $H^{p,q,r,s}_{f}(\widetilde{M})$ is how it depends on the function f. Similar with the Proposition 3.2. from [12], we explain this fact for our vertical cohomology. We have

Proposition 2.2. *If $h \in \mathcal{F}(\widetilde{M})$ does not vanish, then the cohomology groups $H^{p,q,r,s}_{f}(\widetilde{M})$ and $H^{p,q,r,s}_{fh}(\widetilde{M})$ are isomorphic.*

Proof. For each $p,q,r,s \in \mathbb{N}$, consider the linear isomorphism

$$\phi^{p,q,r,s} : \mathcal{A}^{p,q,r,s}(\widetilde{M}) \to \mathcal{A}^{p,q,r,s}(\widetilde{M})\ ,\ \phi^{p,q,r,s}(\varphi) = \frac{\varphi}{h^{p+q+r+s}}. \quad (2.3)$$

If $\varphi \in \mathcal{A}^{p,q,r,s}(\widetilde{M})$, one checks easily that

$$\phi^{p,q,r,s+1}(\partial_{\overline{v},fh}\varphi) = \partial_{\overline{v},f}(\phi^{p,q,r,s}(\varphi)), \quad (2.4)$$

so $\phi^{p,q,r,s}$ induces an isomorphism between the cohomologies $H^{p,q,r,s}_{f}(\widetilde{M})$ and $H^{p,q,r,s}_{fh}(\widetilde{M})$. □

Corollary 2.3. *If the function f does not vanish, then $H^{p,q,r,s}_{f}(\widetilde{M})$ is isomorphic to the vertical de Rham cohomology $H^{p,q,r,s}(\widetilde{M})$.*

Proof. We take $h = \frac{1}{f}$ in the above proposition. □

3. Vertical Lichnerowicz cohomology

In this section we define a vertical Lichnerowicz cohomology for (p,q,r,s)-forms on a complex Finsler manifold (M,F) following the classical definition, e.g. [3, 8, 16].

Let $\omega \in \mathcal{A}^{0,0,0,1}(\widetilde{M})$ be a $\partial_{\overline{v}}$-closed $(0,0,0,1)$-form on $\widetilde{M}$ and the map

$$\partial_{\overline{v},\omega} : \mathcal{A}^{p,q,r,s}(\widetilde{M}) \to \mathcal{A}^{p,q,r,s+1}(\widetilde{M})\ ,\ \partial_{\overline{v},\omega} = \partial_{\overline{v}} - \omega \wedge . \quad (3.1)$$

Since $\partial_{\overline{v}}\omega = 0$, we easily obtain that $\partial^2_{\overline{v},\omega} = 0$. The differential complex

$$0 \longrightarrow \mathcal{A}^{p,q,r,0}(\widetilde{M}) \xrightarrow{\partial_{\overline{v},\omega}} \mathcal{A}^{p,q,r,1}(\widetilde{M}) \xrightarrow{\partial_{\overline{v},\omega}} \dots \xrightarrow{\partial_{\overline{v},\omega}} \mathcal{A}^{p,q,r,n}(\widetilde{M}) \longrightarrow 0 \quad (3.2)$$

is called $\overline{v}$-*Lichnerowicz complex* of complex Finsler manifold (M,F); its cohomology groups $H^{p,q,r,s}_{\omega}(\widetilde{M})$ are called $\overline{v}$-*Lichnerowicz cohomology groups* of the complex Finsler manifold (M,F).

This is a version adapted to our study of the classical Lichnerowicz cohomology, motivated by Lichnerowicz's work [8] or Lichnerowicz-Jacobi cohomology on Jacobi and locally conformal symplectic geometry manifolds, see [3, 7]. We also notice that Vaisman in [16] studied it under the name of "adapted cohomology" on locally conformal Kähler (LCK) manifolds.

We notice that, locally, the $\overline{v}$-Lichnerowicz complex becames the $\overline{v}$-complex after a change $\varphi\mapsto e^f\varphi$ with f a function which satisfies $\partial_{\overline{v}}f=\omega$, namely $\partial_{\overline{v},\omega}$ is the unique differential in $\mathcal{A}^{p,q,r,s}(\widetilde{M})$ which makes the multiplication by the smooth function e^f an isomorphism of cochain $\overline{v}$-complexes $e^f:(\mathcal{A}^{p,q,r,\bullet}(\widetilde{M}),\partial_{\overline{v},\omega})\to(\mathcal{A}^{p,q,r,\bullet}(\widetilde{M}),\partial_{\overline{v}})$.

Proposition 3.1. *The $\overline{v}$-Lichnerowicz cohomology depends only on the $\overline{v}$-class of ω. In fact, we have $H^{p,q,r,s}_{\omega-\partial_{\overline{v}}f}(\widetilde{M})\approx H^{p,q,r,s}_{\omega}(\widetilde{M})$.*

Proof. Since $\partial_{\overline{v},\omega}(e^f\varphi)=e^f\partial_{\overline{v},\omega-\partial_{\overline{v}}f}\varphi$ it results that the map $[\varphi]\mapsto[e^f\varphi]$ is an isomorphism between $H^{p,q,r,s}_{\omega-\partial_{\overline{v}}f}(\widetilde{M})$ and $H^{p,q,r,s}_{\omega}(\widetilde{M})$. □

Example 3.2. Let us consider $\omega:=\overline{\gamma}$ to be the conjugated vertical Liouville 1-form (the dual of the conjugated vertical Liouville vector field $\overline{\Gamma}=\overline{\eta}^k\frac{\partial}{\partial\overline{\eta}^k}$). Then, by the homogeneity conditions of complex Finsler metric, it is locally given by

$$\overline{\gamma}=\frac{G_{j\overline{k}}\eta^j}{F^2}\delta\overline{\eta}^k=\frac{1}{F^2}\frac{\partial F^2}{\partial\overline{\eta}^k}\delta\overline{\eta}^k=\partial_{\overline{v}}(\log F^2). \tag{3.3}$$

Then $\overline{\gamma}$ is a $\partial_{\overline{v}}$-closed $(0,0,0,1)$-form on $\widetilde{M}$ and we can consider the associated $\overline{v}$-Lichnerowicz cohomology groups $H^{p,q,r,s}_{\overline{\gamma}}(\widetilde{M})$.

As in the classical case, using the definition of $\partial_{\overline{v},\omega}$ we easily obtain

$$\partial_{\overline{v},\omega}(\varphi\wedge\psi)=\partial_{\overline{v}}\varphi\wedge\psi+(-1)^{\deg\varphi}\varphi\wedge\partial_{\overline{v},\omega}\psi.$$

Also, if ω_1 and ω_2 are two $\partial_{\overline{v}}$-closed $(0,0,0,1)$-forms on $\widetilde{M}$ then

$$\partial_{\overline{v},\omega_1+\omega_2}(\varphi\wedge\psi)=\partial_{\overline{v},\omega_1}\varphi\wedge\psi+(-1)^{\deg\varphi}\varphi\wedge\partial_{\overline{v},\omega_2}\psi,$$

which says that the wedge product induces the map

$$\wedge:H^{p,q,r,s_1}_{\omega_1}(\widetilde{M})\times H^{p,q,r,s_2}_{\omega_2}(\widetilde{M})\to H^{p,q,r,s_1+s_2}_{\omega_1+\omega_2}(\widetilde{M}).$$

Corollary 3.3. *The wedge product induces the following homomorphism*

$$\wedge:H^{p,q,r,s}_{\omega}(\widetilde{M})\times H^{p,q,r,s}_{-\omega}(\widetilde{M})\to H^{p,q,r,2s}(\widetilde{M}).$$

Now, using an argument inspired from [16], we prove that the $\overline{v}$-Lichnerowicz cohomology spaces $H^{p,q,r,s}_{\omega}(\widetilde{M})$ can also be obtained as the $\overline{v}$-cohomology spaces of $\widetilde{M}$ with the coefficients in the sheaf $\Phi^{p,q,r}_{\omega}$ of germs of $\partial_{\overline{v},\omega}$-closed $(p,q,r,0)$-forms.

Firstly, we notice that $\partial_{\overline{v},\omega}$ satisfies a Dolbeault type Lemma for (p,q,r,s)-forms on $\widetilde{M}$. Indeed, let φ be a local (p,q,r,s)-form such that $\partial_{\overline{v},\omega}\varphi=0$. Since $\partial_{\overline{v}}\omega=0$ and the lemma has to be local, we may suppose $\omega=-(\partial_{\overline{v}}\alpha)/\alpha$, where α is a nonzero smooth function on $\widetilde{M}$. Then, $\partial_{\overline{v},\omega}\varphi = 0$ means $\partial_{\overline{v}}(\alpha\varphi) = 0$, whence by Dolbeault type lemma for the operator $\partial_{\overline{v}}$, see [13], we have $\varphi = \partial_{\overline{v},\omega}(\psi/\alpha)$ for some local $(p,q,r,s-1)$-form ψ. This is exactly the requested result.

Then, we see that

$$0 \longrightarrow \Phi_{\omega}^{p,q,r} \xrightarrow{i} \mathcal{A}^{p,r,r,0}(\widetilde{M}) \xrightarrow{\partial_{\overline{v},\omega}} \mathcal{A}^{p,q,r,1}(\widetilde{M}) \xrightarrow{\partial_{\overline{v},\omega}} \dots \tag{3.4}$$

is a fine resolution of $\Phi_{\omega}^{p,q,r}$, which leads to

Proposition 3.4. *For every $\partial_{\overline{v}}$-closed $(0,0,0,1)$-form ω, one has the isomorphisms*

$$H^s(\widetilde{M}, \Phi_{\omega}^{p,q,r}) \approx H_{\omega}^{p,q,r,s}(\widetilde{M}).$$

For every ω as above, let us consider now the auxiliary vertical operator

$$\widetilde{\partial}_{\overline{v}} = \partial_{\overline{v}} - \frac{p+q+r+s}{2}\omega\wedge, \tag{3.5}$$

where (p,q,r,s) is the type of the form acted on. We notice that $\widetilde{\partial}_{\overline{v}}$ is an antiderivation of differential forms and it is easy to see that $\widetilde{\partial}_{\overline{v}}^2 = -\frac{1}{2}\omega\wedge\partial_{\overline{v}}$. Then $\widetilde{\partial}_{\overline{v}}$ defines a *twisted $\overline{v}$-cohomology*, [17], of (p,q,r,s)-forms on $\widetilde{M}$, which is given by

$$H_{\widetilde{\partial}_{\overline{v}}}^{p,q,r,\bullet}(\widetilde{M}) = \frac{\operatorname{Ker}\widetilde{\partial}_{\overline{v}}}{\operatorname{Im}\widetilde{\partial}_{\overline{v}} \cap \operatorname{Ker}\widetilde{\partial}_{\overline{v}}} \tag{3.6}$$

and is isomorphic to the cohomology of the $\overline{v}$-complex $(\widetilde{\mathcal{A}}^{p,q,r,\bullet}(\widetilde{M}), \widetilde{\partial}_{\overline{v}})$ consisting of the (p,q,r,s)-forms $\varphi \in \mathcal{A}^{p,q,r,s}(\widetilde{M})$ satisfying $\widetilde{\partial}_{\overline{v}}^2\varphi = -\omega\wedge\partial_{\overline{v}}\varphi = 0$.

The $\overline{v}$-complex $\widetilde{\mathcal{A}}^{p,q,r,\bullet}(\widetilde{M})$ admits a $\overline{v}$-subcomplex $\mathcal{A}_{\omega}^{p,q,r,\bullet}(\widetilde{M})$, namely, the ideal generated by ω. On this subcomplex, $\widetilde{\partial}_{\overline{v}} = \partial_{\overline{v}}$, which means that it is a $\overline{v}$-subcomplex of the usual $\overline{v}$-de Rham complex of $\widetilde{M}$. Hence, one has the homomorphisms

$$a : H^s(\mathcal{A}_{\omega}^{p,q,r,\bullet}(\widetilde{M})) \to H_{\widetilde{\partial}_{\overline{v}}}^{p,q,r,s}(\widetilde{M}),\ b : H^s(\mathcal{A}_{\omega}^{p,q,r,\bullet}(\widetilde{M})) \to H^{p,q,r,s}(\widetilde{M},\mathbb{C}). \tag{3.7}$$

Now, we can easily construct a homomorphism

$$c : H_{\widetilde{\partial}_{\overline{v}}}^{p,q,r,s}(\widetilde{M}) \to H^{p,q,r,s+1}(\widetilde{M},\mathbb{C}). \tag{3.8}$$

Indeed, if $[\varphi] \in H_{\widetilde{\partial}_{\overline{v}}}^{p,q,r,s}(\widetilde{M})$, where φ is $\widetilde{\partial}_{\overline{v}}$-closed (p,q,r,s)-form, then we put $c([\varphi]) = [\omega\wedge\varphi]$, and this produces the homomorphism from (3.8). We notice that the existence of c gives some relation between $\widetilde{\partial}_{\overline{v}}$ and the $\overline{v}$-cohomology of $\widetilde{M}$ with values in $\mathbb{C}$.

Remark 3.5. From (2.1) and (3.5) one gets

$$\frac{1}{f}\partial_{\overline{v},f} = \partial_{\overline{v}} - \frac{p+q+r+s}{2}\partial_{\overline{v}}(\log f^2)\wedge = \widetilde{\partial}_{\overline{v}},\ \text{with } \omega = \partial_{\overline{v}}(\log f^2). \tag{3.9}$$

Then, if f does not vanish, we have the homomorphisms

$$\widetilde{a} : H_f^{p,q,r,s}(\widetilde{M}) \to H_{\widetilde{\partial}_{\overline{v}}}^{p,q,r,s}(\widetilde{M}). \tag{3.10}$$

In particular, we can choose $f = F$ to be the complex Finsler function, and so $\frac{1}{F}\partial_{\overline{v},F} = \widetilde{\partial}_{\overline{v}}$ with $\omega = \overline{\gamma}$.

4. A relative vertical cohomology attached to a function

The relative de Rham cohomology was first defined in [4] p. 78. In this subsection we construct a similar version for our vertical cohomology of complex Finsler manifolds.

For the begining we need some basic notions about holomorphic Finsler subspaces. For more details see [9, 10, 11].

4.1. Holomorphic Finsler subspaces

Let (M,F) be a complex Finsler space, (z^k,η^k), $k=1,\ldots,n$ complex coordinates in a local chart, and $i:\mathcal{M}\hookrightarrow M$ a holomorphic immersion of an m-dimensional complex manifold $\mathcal{M}$ into M, locally given by $z^k=z^k(\xi^1,\ldots,\xi^m)$. Everywhere the indices $i,j,k,\ldots$ run from 1 to n and $\alpha,\beta,\gamma,\ldots$ run from 1 to $m\leq n$. Let $T^{1,0}\mathcal{M}$ and $T^{1,0}M$ be the corresponding holomorphic tangent bundles. By $i_{*,C}:T^{1,0}\mathcal{M}\to T^{1,0}M$ we denote the inclusion map between the manifolds $T^{1,0}\mathcal{M}$ and $T^{1,0}M$ (the complexified tangent inclusion map), that is $i_{*,C}(\xi,\theta)=(z(\xi),\eta(\xi,\theta))$, where $\xi=(\xi^\alpha)$, $\theta=\theta^\alpha\frac{\partial}{\partial\xi^\alpha}$, $\eta=\eta^k\frac{\partial}{\partial z^k}$. Then $i_{*,C}$ has the following local representation [9]:

$$z^k=z^k(\xi^1,\ldots,\xi^m),\ \eta^k=\theta^\alpha B^k_\alpha(\xi)\text{ where }B^k_\alpha(\xi)=\frac{\partial z^k}{\partial\xi^\alpha}. \tag{4.1}$$

The holomorphic immersion assumption implies that $B^k_{\overline{\alpha}}=\frac{\partial z^k}{\partial\overline{\xi}^\alpha}=0$ and $B^{\overline{k}}_\alpha=\frac{\partial\overline{z}^k}{\partial\xi^\alpha}=0$. In a point of the complexified tangent space $T_{\mathbb{C}}(T^{1,0}\mathcal{M})$, the local frame $\{\frac{\partial}{\partial\xi^\alpha},\frac{\partial}{\partial\theta^\alpha}\}$ is coupled to $\{\frac{\partial}{\partial z^k},\frac{\partial}{\partial\eta^k}\}$ as follows:

$$\frac{\partial}{\partial\xi^\alpha}=B^k_\alpha\frac{\partial}{\partial z^k}+B^k_{0\alpha}\frac{\partial}{\partial\eta^k},\ \frac{\partial}{\partial\theta^\alpha}=B^k_\alpha\frac{\partial}{\partial\eta^k}, \tag{4.2}$$

where $B^k_{0\alpha}=\frac{\partial B^k_\alpha}{\partial\xi^\beta}\theta^\beta$. Its dual basis satisfies the conditions

$$dz^k=B^k_\alpha d\xi^\alpha,\ d\eta^k=B^k_{0\alpha}d\xi^\alpha+B^k_\alpha d\theta^\alpha \tag{4.3}$$

and their conjugates.

In view of (4.1) the complex Finsler function F, with the metric tensor $G_{j\overline{k}}=\dot{\partial}_j\dot{\partial}_{\overline{k}}(F^2)$, induces a complex Finsler function $\mathcal{F}:T^{1,0}\mathcal{M}\to\mathbb{R}_+\cup\{0\}$ given by $\mathcal{F}(\xi,\theta)=F(z(\xi),\eta(\xi,\theta))=F(z^k(\xi),\theta^\alpha B^k_\alpha(\xi))$ with the metric tensor $\mathcal{G}_{\alpha\overline{\beta}}=B^j_\alpha B^{\overline{k}}_{\overline{\beta}}G_{j\overline{k}}$. Here $\mathcal{G}_{\alpha\overline{\beta}}=\dot{\partial}_\alpha\dot{\partial}_{\overline{\beta}}(\mathcal{F}^2)$ and $\dot{\partial}_\alpha=\frac{\partial}{\partial\theta^\alpha}$, $\dot{\partial}_{\overline{\beta}}=\frac{\partial}{\partial\overline{\theta}^\beta}$. By these considerations, the pair $(\mathcal{M},\mathcal{F})$ is said to be a holomorphic subspace of the complex Finsler space (M,F).

From (4.2) it is deduced that the distribution $V^{1,0}\widetilde{\mathcal{M}}$, spanned locally by $\{\frac{\partial}{\partial\theta^\alpha}\}$, $\alpha=1,\ldots,m$, is a subdistribution of the vertical distribution $V^{1,0}\widetilde{M}$ spanned in any point $(z(\xi),\eta(\xi,\theta))$ by $\{\frac{\partial}{\partial\eta^k}\}$, $k=1,\ldots,n$. We consider $V^{1,0\perp}\mathcal{M}$ an orthogonal complement, namely $V^{1,0}\widetilde{M}=V^{1,0}\mathcal{M}\oplus V^{1,0\perp}\mathcal{M}$, spanned in any point by the set of normal vectors $\{N_a=B^k_a\frac{\partial}{\partial\eta^k}\}$, $a=1,\ldots,n-m$, which we may assume orthonormal. Therefore, the functions $B^k_a(\xi,\theta)$ (and their conjugates) will satisfy the conditions $G_{j\overline{k}}(z(\xi),\eta(\xi,\theta))B^j_\alpha B^{\overline{k}}_{\overline{a}}=0$ and $G_{j\overline{k}}(z(\xi),\eta(\xi,\theta))B^j_a B^{\overline{k}}_{\overline{b}}=\delta_{a\overline{b}}$.

Let us now consider the moving frame $\mathcal{R} = \{B^k_\alpha(\xi) B^k_a(\xi,\theta)\}$ along the complex Finsler subspace $(\mathcal{M},\mathcal{F})$ and let $\mathcal{R}^{-1} = (B^\alpha_k B^a_k)^t$ be the inverse matrix associated to the moving frame $\mathcal{R}$. Evidently, B^α_k and B^a_k are functions of ξ,θ and

$$B^\alpha_k B^k_\beta = \delta^\alpha_\beta,\ B^\alpha_k B^k_a = 0,\ B^a_k B^k_b = \delta^a_b,\ B^k_\alpha B^\alpha_j + B^k_a B^a_j = \delta^k_j. \tag{4.4}$$

Let $\mathcal{N} = (\mathcal{N}^\alpha_\beta(\xi,\theta))$ be a c.n.c. on $T^{1,0}\mathcal{M}$ and consider its adapted basis $\{\delta_\alpha := \frac{\delta}{\delta\xi^\alpha} = \frac{\partial}{\partial\xi^\alpha} - \mathcal{N}^\beta_\alpha \frac{\partial}{\partial\theta^\beta}, \dot{\partial}_\alpha := \frac{\partial}{\partial\theta^\alpha}\}$ and $\{\delta_{\overline{\alpha}}, \dot{\partial}_{\overline{\alpha}}\}$ as well as its dual $\{d\xi^\alpha, \delta\theta^\alpha = d\theta^\alpha - \mathcal{N}^\alpha_\beta d\xi^\beta\}$ and $\{d\overline{\xi}^\alpha, \delta\overline{\theta}^\alpha = d\overline{\theta}^\alpha - \mathcal{N}^{\overline{\alpha}}_{\overline{\beta}} d\overline{\xi}^\beta\}$.

The c.n.c. $\mathcal{N}$ on $T^{1,0}\mathcal{M}$ is said to be *induced* by the c.n.c. N on $T^{1,0}M$ if $\delta\theta^\alpha = B^\alpha_k \delta\eta^k$. This condition implies [9], $\mathcal{N}^\alpha_\beta = B^\alpha_k(B^k_{0\beta} + N^k_j B^j_\beta)$.

Proposition 4.1. *([9, 10]). The adapted bases are tied by*

$$\delta_\alpha = B^k_\alpha \delta_k + B^k_a H^a_\alpha\ \dot{\partial}_k,\ \dot{\partial}_\alpha = B^k_\alpha\ \dot{\partial}_k,$$
$$dz^k = B^k_\alpha d\xi^\alpha,\ \delta\eta^k = B^k_\alpha \delta\theta^\alpha + B^k_a H^a_\alpha d\xi^\alpha$$

with $H^a_\alpha = B^a_j(B^j_{0\alpha} + N^j_k B^k_\alpha)$.

By above proposition we easily deduce that $\delta_\alpha = B^k_\alpha \delta_k + H^a_\alpha N_a$ and $\dot{\partial}_k = B^\alpha_k\ \dot{\partial}_\alpha + B^a_k N_a$. A notable result in [9] asserts that the induced c.n.c. by the Chern-Finsler c.n.c. coincides with the intrinsic Chern-Finsler c.n.c. of the holomorphic subspace $(\mathcal{M},\mathcal{F})$.

4.2. Relative vertical cohomology

Now we return to the construction of a relative vertical cohomology attached to a function of complex Finsler manifolds. Let us denote by

$$J_C i = i_{*,C}.$$

By Proposition 4.1. we easily deduce that if $\varphi \in \mathcal{A}^{p,q,r,s}(\widetilde{M})$ is locally given by (1.9) then

$$(J_C i)^*\varphi \in \mathcal{A}^{p,q,r,s}(\widetilde{\mathcal{M}}) \oplus \bigoplus_{h=\overline{1,r};k=\overline{1,s}} \mathcal{A}^{p+h,q+k,r-h,s-k}(\widetilde{\mathcal{M}}). \tag{4.5}$$

Thus, $(J_C i)^*$ does not preserves the (p,q,r,s) type components of a form $\varphi \in \mathcal{A}^{p,q,r,s}(\widetilde{M})$, but we can eliminate this inconvenient if we take $p = q = m = \dim_{\mathbb{C}} \mathcal{M}$. Then, we easily obtain

Proposition 4.2. *If* $\varphi \in \mathcal{A}^{m,m,r,s}(\widetilde{M})$ *then* $(J_C i)^*\varphi \in \mathcal{A}^{m,m,r,s}(\widetilde{\mathcal{M}})$.

Proposition 4.3. *If* $\varphi \in \mathcal{A}^{m,m,r,s}(\widetilde{M})$ *then*

$$\partial_{\overline{v}}(J_C i)^*\varphi = (J_C i)^*\partial_{\overline{v}}\varphi. \tag{4.6}$$

Proof. Let $\varphi = \varphi_{I_m\overline{J_m}K_r\overline{H_s}} dz^{I_m} \wedge d\overline{z}^{J_m} \wedge \delta\eta^{K_r} \wedge \delta\overline{\eta}^{H_s} \in \mathcal{A}^{m,m,r,s}(\widetilde{M})$. By Proposition 4.1. we have

$$(J_C i)^*\varphi = \varphi_{A_m\overline{B_m}C_r\overline{D_s}}(\xi,\theta) d\xi^{A_m} \wedge d\overline{\xi}^{B_m} \wedge \delta\theta^{C_r} \wedge \delta\overline{\theta}^{D_s},$$

where

$$\varphi_{A_m\overline{B_m}C_r\overline{D_s}}(\xi,\theta) = B^{I_m}_{A_m} B^{\overline{J_m}}_{\overline{B_m}} B^{K_r}_{C_r} B^{\overline{H_s}}_{\overline{D_s}} \varphi_{I_m\overline{J_m}K_r\overline{H_s}}(z(\xi),\eta(\xi,\theta)) \tag{4.7}$$

and $B^{I_m}_{A_m} = B^{i_1}_{\alpha_1}(z(\xi)) \cdot \ldots \cdot B^{i_m}_{\alpha_m}(z(\xi))$ etc. Applying $\partial_{\overline{v}}$ from (1.11) it results

$$\partial_{\overline{v}}(J_C i)^*\varphi = \sum_\alpha \dot{\partial}_{\overline{\alpha}}\,(\varphi_{A_m\overline{B_m}C_r\overline{D_s}})\delta\overline{\theta}^\alpha \wedge d\xi^{A_m} \wedge d\overline{\xi}^{B_m} \wedge \delta\theta^{C_r} \wedge \delta\overline{\theta}^{D_s}. \tag{4.8}$$

Similarly, we have

$$(J_C i)^*\partial_{\overline{v}}\varphi =$$
$$\dot{\partial}_{\overline{k}}\,(\varphi_{I_m\overline{J_m}K_r\overline{H_s}})B^{\overline{k}}_{\overline{\alpha}}\delta\overline{\theta}^\alpha \wedge B^{I_m}_{A_m}d\xi^{A_m} \wedge B^{\overline{J_m}}_{\overline{B_m}}d\overline{\xi}^{B_m} \wedge B^{K_r}_{C_r}\delta\theta^{C_r} \wedge B^{\overline{H_s}}_{\overline{D_s}}\delta\overline{\theta}^{D_s}$$

and by (4.2) and (4.7) one gets

$$(J_C i)^*\partial_{\overline{v}}\varphi = \sum_\alpha \dot{\partial}_{\overline{\alpha}}\,(\varphi_{A_m\overline{B_m}C_r\overline{D_s}})\delta\overline{\theta}^\alpha \wedge d\xi^{A_m} \wedge d\overline{\xi}^{B_m} \wedge \delta\theta^{C_r} \wedge \delta\overline{\theta}^{D_s} \tag{4.9}$$

which completes the proof. □

Now, if $f \in \mathcal{F}(M)$ then by (4.6) one gets

$$\partial_{\overline{v},(J_C i)^* f}(J_C i)^*\varphi = (J_C i)^*\partial_{\overline{v},f}\varphi, \text{ for any } \varphi \in \mathcal{A}^{m,m,r,s}(\widetilde{M}). \tag{4.10}$$

Indeed, for $\varphi \in \mathcal{A}^{m,m,r,s}(\widetilde{M})$ by direct calculus we have

$$\begin{aligned}
&\partial_{\overline{v},(J_C i)^* f}((J_C i)^*\varphi)\\
&= (J_C i)^* f\partial_{\overline{v}}((J_C i)^*\varphi) - (2m+r+s)\partial_{\overline{v}}((J_C i)^* f) \wedge (J_C i)^*\varphi\\
&= (J_C i)^* f (J_C i)^*(\partial_{\overline{v}}\varphi) - (2m+r+s)(J_C i)^*(\partial_{\overline{v}} f) \wedge (J_C i)^*\varphi\\
&= (J_C i)^*(f\partial_{\overline{v}}\varphi) - (J_C i)^*((2m+r+s)\partial_{\overline{v}} f \wedge \varphi)\\
&= (J_C i)^*(\partial_{\overline{v},f}\varphi).
\end{aligned}$$

We define the differential complex

$$\ldots \xrightarrow{\widetilde{\partial}_{\overline{v},f}} \mathcal{A}^{m,m,r,s}(J_C i) \xrightarrow{\widetilde{\partial}_{\overline{v},f}} \mathcal{A}^{m,m,r,s+1}(J_C i) \xrightarrow{\widetilde{\partial}_{\overline{v},f}} \ldots$$

where $\mathcal{A}^{m,m,r,s}(J_C i) = \mathcal{A}^{m,m,r,s}(\widetilde{M}) \oplus \mathcal{A}^{m,m,r,s-1}(\widetilde{\mathcal{M}})$ and

$$\widetilde{\partial}_{\overline{v},f}(\varphi,\psi) = (\partial_{\overline{v},f}\varphi, (J_C i)^*\varphi - \partial_{\overline{v},(J_C i)^* f}\psi). \tag{4.11}$$

Taking into account $\partial^2_{\overline{v},f} = \partial^2_{\overline{v},(J_C i)^* f} = 0$ and (4.10) we easily verify that $\widetilde{\partial}^2_{\overline{v},f} = 0$. Denote the cohomology groups of this complex by $H^{m,m,r,s}_f(J_C i)$.

Now, if we regraduate the complex $\mathcal{A}^{m,m,r,s}(\widetilde{\mathcal{M}})$ as

$$\widetilde{\mathcal{A}}^{m,m,r,s}(\widetilde{\mathcal{M}}) = \mathcal{A}^{m,m,r,s-1}(\widetilde{\mathcal{M}}),$$

then we obtain an exact sequence of differential complexes

$$0 \longrightarrow \widetilde{\mathcal{A}}^{m,m,r,s}(\widetilde{\mathcal{M}}) \xrightarrow{\alpha} \mathcal{A}^{m,m,r,s}(J_C i) \xrightarrow{\beta} \mathcal{A}^{m,m,r,s}(\widetilde{M}) \longrightarrow 0 \tag{4.12}$$

with the obvious mappings α and β given by $\alpha(\psi) = (0,\psi)$ and $\beta(\varphi,\psi) = \varphi$, respectively. From (4.12) we have an exact sequence in cohomologies

$$\ldots \longrightarrow H^{m,m,r,s-1}_{(J_C i)^* f}(\widetilde{\mathcal{M}}) \xrightarrow{\alpha^*} H^{m,m,r,s}_f((J_C i)^*) \xrightarrow{\beta^*} H^{m,m,r,s-1}_f(\widetilde{M}) \xrightarrow{\delta^*}$$
$$\xrightarrow{\delta^*} H^{m,m,r,s}_{(J_C i)^* f}(\widetilde{\mathcal{M}}) \xrightarrow{\alpha^*} \ldots.$$

It is easily seen that $\delta^* = (J_C i)^*$. Here μ^* denotes the corresponding map between cohomology groups. Let $\varphi \in \mathcal{A}^{m,m,r,s}(\widetilde{M})$ be a $\partial_{\overline{v},f}$-closed form, and $(\varphi,\psi) \in \mathcal{A}^{m,m,r,s}(J_C i)$. Then $\widetilde{\partial}_{\overline{v},f}(\varphi,\psi) = (0,(J_C i)^*\varphi - \partial_{\overline{v},(J_C i)^* f}\psi)$ and by the definition of the operator δ^* we have

$$\delta^*[\varphi] = [(J_C i)^*\varphi - \partial_{\overline{v},(J_C i)^* f}\psi] = [(J_C i)^*\varphi] = (J_C i)^*[\varphi].$$

Hence we finally get a long exact sequence

$$\ldots \longrightarrow H^{m,m,r,s-1}_{(J_C i)^* f}(\widetilde{\mathcal{M}}) \xrightarrow{\alpha^*} H^{m,m,r,s}_f((J_C i)^*) \xrightarrow{\beta^*} H^{m,m,r,s-1}_f(\widetilde{M}) \xrightarrow{(J_C i)^*}$$

$$\xrightarrow{(J_C i)^*} H^{m,m,r,s}_{(J_C i)^* f}(\widetilde{\mathcal{M}}) \xrightarrow{\alpha^*} \ldots.$$

Finally, similar to [14], we have

Corollary 4.4. *If $(\mathcal{M},\mathcal{F})$ is an m-dimensional holomorphic Finsler subspace of an n-dimensional complex Finsler space (M,F), then*

(i) $\beta^* : H^{m,m,r,m+1}_f(J_C i) \to H^{m,m,r,m+1}_f(\widetilde{M})$ *is an epimorphism;*

(ii) $\alpha^* : H^{m,m,r,n}_{(J_C i)^* f}(\widetilde{\mathcal{M}}) \to H^{m,m,r,n+1}_f(J_C i)$ *is an epimorphism;*

(iii) $\beta^* : H^{m,m,r,s}_f(J_C i) \to H^{m,m,r,s}_f(\widetilde{M})$ *is an isomorphism for $s > m+1$;*

(iv) $\alpha^* : H^{m,m,r,s}_{(J_C i)^* f}(\widetilde{\mathcal{M}}) \to H^{m,m,r,s+1}_f(J_C i)$ *is an isomorphism for $s > n$;*

(v) $H^{m,m,r,s}_f(J_C i) = 0$ *for $s > \max\{m+1,n\}$.*

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Analysis of a viscoelastic unilateral and frictional contact problem with adhesion

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Abstract. We consider a mathematical model which describes the quasistatic frictional contact between a viscoelastic body with long memory and a foundation. The contact is modelled with a normal compliance condition in such a way that the penetration is limited and restricted to unilateral constraint and associated to the nonlocal friction law with adhesion, where the coefficient of friction is solution-independent. The bonding field is described by a first order differential equation. We derive a variational formulation written as the coupling between a variational inequality and a differential equation. The existence and uniqueness result of the weak solution under a smallness assumption on the coefficient of friction is established. The proof is based on arguments of time-dependent variational inequalities, differential equations and Banach fixed point theorem.

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1. Introduction

Contact problems involving deformable bodies are quite frequent in the industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve complicated surface phenomena, and are modelled by highly nonlinear initial boundary value problems. Taking into account various contact conditions associated with more and more complex behavior laws lead to the introduction of new and non standard models, expressed by the aid of evolution variational inequalities. An early attempt to study frictional contact problems within the framework of variational inequalities was made in [10]. The mathematical, mechanical and numerical state of the art can be found in [23]. In this reference we find a detailed analysis and numerical studies of the adhesive contact problems. Recently a new book [25] introduces to the reader the theory of variational inequalities with emphasis on the study of contact mechanics and, more specifically, on antiplane frictional contact problems. Also, recently existence results were established in [1, 5, 6, 8, 11, 20, 26, 29, 30, 31] in the study of unilateral and frictional contact problems with or without adhesion. In [31] a quasistatic viscoelastic unilateral contact problem with adhesion and friction was studied and an existence and uniqueness result was proved for a coefficient of friction sufficiently small. Also in [7] a dynamic contact problem with nonlocal friction and adhesion between two viscoelastic bodies of Kelvin-Voigt type was studied. An existence result was proved without condition on the coefficient of friction. Here as in [16] we study a mathematical model which describes a frictional and adhesive contact problem between a viscoelastic body with long memory and a foundation. The contact is modelled with a normal compliance condition associated to unilateral constraint and the nonlocal friction law with adhesion. Recall that models for dynamic or quasistatic processes of frictionless adhesive contact between a deformable body and a foundation have been studied in [2, 3, 4, 5, 7, 8, 12, 19, 21, 23, 24, 27, 28]. Following [13, 14] we use the bonding field as an additional state variable β, defined on the contact surface of the boundary. The variable satisfies the restrictions $0 \leq \beta \leq 1$. At a point on the boundary contact surface, when $\beta = 1$ the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. However, according to [17], the method presented here considers a compliance model in which the compliance term does not represent necessarily a compact perturbation of the original problem without contact. This leads us to study such models, where a strictly limited penetration is permitted with the limit procedure to the Signorini contact problem. In this work as in [31] we derive a variational formulation of the mechanical problem written as the coupling between a variational inequality and a differential equation. We prove the existence of a unique weak solution if the coefficient of friction is sufficiently small, and obtain a partial regularity result for the solution.

The paper is structured as follows. In section 2 we present some notations and preliminaries. In section 3 we state the mechanical problem and give a variational formulation. In section 4 we establish the proof of our main existence and uniqueness result, Theorem 4.1.

2. Notations and preliminaries

Everywhere in this paper we denote by S^d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$ while $|.|$ represents the Euclidean norm on $\mathbb{R}^d$ and S^d. Thus, for every $u, v \in \mathbb{R}^d$, $u.v = u_i v_i$, $|v| = (v.v)^{\frac{1}{2}}$, and for every $\sigma, \tau \in S^d$, $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $|\tau| = (\tau.\tau)^{\frac{1}{2}}$. Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let ν denote the unit outer normal on Γ. We shall use the notation:

$$H = \left(L^2(\Omega)\right)^d, \; H_1 = \left(H^1(\Omega)\right)^d, \; Q = \left\{\sigma = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\},$$

$$Q_1 = \{\sigma \in Q : div\sigma \in H\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$(u,v)_H = \int_\Omega u_i v_i dx, \quad \langle \sigma, \tau \rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i,j} + u_{j,i});$$

$div\sigma = (\sigma_{ij,j})$ is the divergence of σ. For every $v \in H_1$ we also use the notation v for the trace of v on Γ and we denote by v_ν and v_τ the normal and tangential components of v on the boundary Γ, given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

We define, similarly, by σ_ν and σ_τ the normal and the tangential traces of a function $\sigma \in Q_1$, and when σ is a regular function then

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu,$$

and the following Green's formula holds:

$$\langle \sigma, \varepsilon(v) \rangle_Q + (div\sigma, v)_H = \int_\Gamma \sigma\nu.v da \quad \forall v \in H_1,$$

where da is the surface measure element. Let $T > 0$. For every real Hilbert space X we employ the usual notation for the spaces $L^p(0,T;X)$, $1 \leq p \leq \infty$, and $W^{1,\infty}(0,T;X)$. Recall that the norm on the space $W^{1,\infty}(0,T;X)$ is given by

$$\|u\|_{W^{1,\infty}(0,T;X)} = \|u\|_{L^\infty(0,T;X)} + \|\dot{u}\|_{L^\infty(0,T;X)},$$

where $\dot{u}$ denotes the first derivative of u with respect to time. Finally, we denote by $C([0,T];X)$ the space of continuous functions from $[0,T]$ to X, with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

Moreover, for a real number r, we use r_+ to represent its positive part, that is $r_+ = \max\{r, 0\}$.

3. Problem statement and variational formulation

We consider the following physical setting. A viscoelastic body with long memory occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with a regular boundary Γ that is partitioned into three disjoint measurable parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that $meas(\Gamma_1) > 0$. The body is acted upon by a volume force of density φ_1 on Ω and a surface traction of density φ_2 on Γ_2. The body is in unilateral contact with adhesion following the nonlocal friction law with a foundation, over the potential contact surface Γ_3.
Thus, the classical formulation of the mechanical problem is written as follows.

Problem P_1. Find a displacement $u : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0,T] \to \mathbf{S}^d$ and a bonding field $\beta : \Gamma_3 \times [0,T] \to [0,1]$ such that for all $t \in [0,T]$,

$$\sigma(t) = F\varepsilon(u(t)) + \int_0^t \mathcal{F}(t-s)\,\varepsilon(u(s))\,ds \quad \text{in } \Omega, \tag{3.1}$$

$$div\sigma(t)+\varphi_1(t)=0 \text{ in } \Omega, \tag{3.2}$$

$$u(t)=0 \quad \text{on } \Gamma_1, \tag{3.3}$$

$$\sigma(t)\nu=\varphi_2(t) \quad \text{on } \Gamma_2, \tag{3.4}$$

$$\left.\begin{array}{l} u_\nu(t)\le g,\ \sigma_\nu(t)+p(u_\nu(t))-c_\nu\beta^2(t)R_\nu(u_\nu(t))\le 0 \\ \\ \big(\sigma_\nu(t)+p(u_\nu(t))-c_\nu\beta^2(t)R_\nu(u_\nu(t))\big)(u_\nu(t)-g)=0 \end{array}\right\} \text{ on } \Gamma_3, \tag{3.5}$$

$$\left.\begin{array}{l} \big|\sigma_\tau(t)+c_\tau\beta^2(t)R_\tau(u_\tau(t))\big|\le\mu\,|R\sigma_\nu(u(t))| \\ \\ \big|\sigma_\tau(t)+c_\tau\beta^2(t)R_\tau(u_\tau(t))\big|<\mu\,|R\sigma_\nu(u(t))|\Rightarrow u_\tau(t)=0 \\ \\ \big|\sigma_\tau(t)+c_\tau\beta^2(t)R_\tau(u_\tau(t))\big|=\mu\,|R\sigma_\nu(u(t))|\Rightarrow \\ \\ \exists\lambda\ge 0 \text{ s.t. } u_\tau(t)=-\lambda\big(\sigma_\tau(t)+c_\tau\beta^2(t)R_\tau(u_\tau(t))\big) \end{array}\right\} \text{ on } \Gamma_3, \tag{3.6}$$

$$\dot\beta(t)=-\Big[\beta(t)\big(c_\nu(R_\nu(u_\nu(t)))^2+c_\tau|R_\tau(u_\tau(t))|^2\big)-\varepsilon_a\Big]_+ \quad \text{on } \Gamma_3, \tag{3.7}$$

$$\beta(0)=\beta_0 \quad \text{on } \Gamma_3. \tag{3.8}$$

Equation (3.1) represents the viscoelastic constitutive law with long memory of the material; F is the elasticity operator and $\int_0^t \mathcal{F}(t-s)\,\varepsilon(u(s))\,ds$ is the memory term in which $\mathcal{F}$ denotes the tensor of relaxation; the stress $\sigma(t)$ at current instant t depends on the whole history of strains up to this moment of time. Equation (3.2) represents the equilibrium equation while (3.3) and (3.4) are the displacement and traction boundary conditions, respectively, in which $\sigma\nu$ represents the Cauchy stress vector. The conditions (3.5) represent the unilateral contact with adhesion in which c_ν is a given adhesion coefficient which may dependent on $x\in\Gamma_3$ and R_ν, R_τ are truncation operators defined by

$$R_\nu(s)=\left\{\begin{array}{l} L \ \text{ if } s<-L \\ -s \text{ if } -L\le s\le 0 \\ 0 \text{ if } s>0 \end{array}\right. ,\ R_\tau(v)=\left\{\begin{array}{l} v \ \text{ if } |v|\le L, \\ L\dfrac{v}{|v|} \text{ if } |v|>L. \end{array}\right.$$

Here $L>0$ is the characteristic length of the bond, beyond which the latter has no additional traction (see [23]) and p is a normal compliance function which satisfies the assumption (3.19); g denotes the maximum value of the penetration which satisfies $g\ge 0$. When $u_\nu<0$ i.e. when there is separation between the body and the foundation then the condition (3.5) combined with hypothese (3.19) and definition of R_ν shows that $\sigma_\nu=c_\nu\beta^2R_\nu(u_\nu)$ and does not exeed the value $L\,\|c_\nu\|_{L^\infty(\Gamma_3)}$. When $g>0$, the body may interpenetrate into the foundation, but the penetration is limited that is $u_\nu\le g$. In this case of penetration (i.e. $u_\nu\ge 0$), when $0\le u_\nu<g$ then $-\sigma_\nu=p(u_\nu)$ which means that the reaction of the foundation is uniquely determined by the normal displacement and $\sigma_\nu\le 0$. Since p is an increasing function then the reaction is increasing with the penetration. When $u_\nu=g$ then $-\sigma_\nu\ge p(g)$ and σ_ν is not uniquely determined. When $g>0$ and $p=0$, conditions (3.5) become the Signorini's contact conditions with a gap and adhesion

$$u_\nu\le g,\ \sigma_\nu-c_\nu\beta^2R_\nu(u_\nu)\le 0,\ (\sigma_\nu-c_\nu\beta^2R_\nu(u_\nu))(u_\nu-g)=0.$$

When $g = 0$, the conditions (3.5) combined with hypothese (3.19) lead to the Signorini contact conditions with adhesion, with zero gap, given by

$$u_\nu \leq 0,\ \sigma_\nu - c_\nu \beta^2 R_\nu (u_\nu) \leq 0,\ (\sigma_\nu - c_\nu \beta^2 R_\nu (u_\nu)) u_\nu = 0.$$

These contact conditions were used in [26, 29]. It follows from (3.5) that there is no penetration between the body and the foundation, since $u_\nu \leq 0$ during the process. Also, note that when the bonding field vanishes, then the contact conditions (3.5) become the classical Signorini contact conditions with zero gap, that is,

$$u_\nu \leq 0,\ \sigma_\nu \leq 0,\ \sigma_\nu u_\nu = 0.$$

Conditions (3.6) represent Coulomb's law of dry friction with adhesion where μ denotes the coefficient of friction. Equation (3.7) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [26]. Since $\dot{\beta} \leq 0$ on $\Gamma_3 \times (0, T)$, once debonding occurs bonding cannot be reestablished and, indeed, the adhesive process is irreversible. Also from [18] it must be pointed out clearly that condition (3.7) does not allow for complete debonding in finite time.

We turn now to the variational formulation of Problem P_1. We denote by V the closed subspace of H_1 defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1\},$$

and let the convex subset of admissible displacements given by

$$K = \{v \in V : v_\nu \leq g \ \ a.e. \text{ on } \Gamma_3\}.$$

Since $meas\,(\Gamma_1) > 0$, the following Korn's inequality holds [10],

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{3.9}$$

where $c_\Omega > 0$ is a constant which depends only on Ω and Γ_1. We equip V with the inner product

$$(u, v)_V = \langle \varepsilon(u), \varepsilon(v) \rangle_Q$$

and $\|.\|_V$ is the associated norm. It follows from Korn's inequality (3.9) that the norms $\|.\|_{H_1}$and $\|.\|_V$ are equivalent on V. Then $(V, \|.\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which only depends on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{3.10}$$

We suppose that the body forces and surface tractions have the regularity

$$\varphi_1 \in C([0, T]; H), \qquad \varphi_2 \in C\left([0, T]; \left(L^2(\Gamma_2)\right)^d\right). \tag{3.11}$$

We define the function $f : [0, T] \to V$ by

$$(f(t), v)_V = \int_\Omega \varphi_1(t) .v dx + \int_{\Gamma_2} \varphi_2(t) .v da \ \ \forall v \in V,\ t \in [0, T], \tag{3.12}$$

and we note that (3.11) and (3.12) imply

$$f \in C([0, T]; V).$$

In the study of the mechanical problem P_1 we assume that the elasticity operator F satisfies

$$\left.\begin{array}{l}(a)\ \ F:\Omega\times S^d\ \to S^d;\\[2ex] (b)\text{ there exists } M>0 \text{ such that}\\ \quad |F(x,\varepsilon_1)-F(x,\varepsilon_2)|\leq M|\varepsilon_1-\varepsilon_2|\quad \forall\varepsilon_1,\varepsilon_2\in S^d,\\ \quad a.e.\ x\ \in\Omega;\\[2ex] (c)\text{ there exists } m>0 \text{ such that}\\ \quad\ (F(x,\varepsilon_1)-F(x,\varepsilon_2)).(\varepsilon_1-\varepsilon_2)\geq m|\varepsilon_1-\varepsilon_2|^2,\\ \quad\ \forall\varepsilon_1,\varepsilon_2\in S^d,\ a.e.\ x\ \in\Omega;\\[2ex] (d)\text{ the mapping } x\to F(x,\varepsilon)\ \text{ is Lebesgue measurable on } \Omega\\ \quad \text{for any } \varepsilon\in S^d;\\[2ex] (e)\text{ the mapping } x\to F(x,0)\in Q.\end{array}\right\}\tag{3.13}$$

We also need to introduce the space of the tensors of fourth order defined by

$$Q_\infty=\{\mathcal{E}=(\mathcal{E}_{ijkl}):\mathcal{E}_{ijkl}=\mathcal{E}_{jikl}=\mathcal{E}_{klij}\in L^\infty(\Omega),\ 1\leq i,j,k,l\leq d\},$$

which is the real Banach space with the norm

$$\|\mathcal{E}\|_{Q_\infty}=\max_{1\leq i,j,k,l\leq d}\|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

We assume that the tensor of relaxation $\mathcal{F}$ satisfies

$$\mathcal{F}\in C([0,T];Q_\infty).\tag{3.14}$$

The adhesion coefficients c_ν, c_τ and ε_a satisfy

$$c_\nu,c_\tau\in L^\infty(\Gamma_3),\ \varepsilon_a\in L^2(\Gamma_3)\ \text{ and } c_\nu,c_\tau,\ \varepsilon_a\geq 0\ a.e. \text{ on } \Gamma_3,\tag{3.15}$$

and we assume that the initial bonding field satisfies

$$\beta_0\in L^2(\Gamma_3);\ 0\leq\beta_0\leq 1\ a.e. \text{ on } \Gamma_3.\tag{3.16}$$

Next, we consider the subset W of H_1 defined as

$$W=\{v\in H_1: div\sigma(v)\in H\}$$

and let $j_c:V\times V\to\mathbb{R}$, $j_f:(V\cap W)\times V\to\mathbb{R}$ be the functionals given by

$$j_c(u,v)=\int_{\Gamma_3}p(u_\nu)v_\nu da\quad\forall(u,v)\in V\times V,$$

$$j_f(u,v)=\int_{\Gamma_3}\mu|R\sigma_\nu(u)||v_\tau|da\quad\forall(u,v)\in(V\cap W)\times V,$$

where

$$R:H^{-\frac{1}{2}}(\Gamma)\to L^2(\Gamma_3)\ \text{ is a linear and continuous mapping (see [9]).}\tag{3.17}$$

The coefficient of friction μ is assumed to satisfy

$$\mu\in L^\infty(\Gamma_3)\ \text{ and } \mu\geq 0\ a.e. \text{ on } \Gamma_3.\tag{3.18}$$

Next we let

$$j = j_c + j_f.$$

We also define the functional

$$h : L^2(\Gamma_3) \times V \times V \to \mathbb{R}$$

by

$$h(\beta, u, v) = \int_{\Gamma_3} (-c_\nu \beta^2 R_\nu(u_\nu) v_\nu + c_\tau \beta^2 R_\tau(u_\tau).v_\tau) da, \ \forall (\beta, u, v) \in L^2(\Gamma_3) \times V \times V,$$

where the normal compliance function p satisfies:

$$\begin{cases} (a)\ \ p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+; \\ \\ (b)\ \ \text{there exists } L_p > 0 \text{ such that} \\ \qquad |p(x, r_1) - p(x, r_2)| \leq L_p |r_1 - r_2| \\ \qquad \forall r_1,\ r_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3; \\ \\ (c)\ \ (p(x, r_1) - p(x, r_2))(r_1 - r_2) \geq 0 \\ \qquad \forall r_1,\ r_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3; \\ \\ (d)\ \ \text{the mapping } x \to p(x, r) \text{ is Lebesgue measurable on } \Gamma_3, \\ \qquad \text{for any } r \in \mathbb{R}; \\ \\ (e)\ \ p(x, r) = 0\ \forall r \leq 0,\ a.e.\ x \in \Gamma_3. \end{cases} \tag{3.19}$$

Finally, we need to introduce the following set of the bonding field:

$$B = \left\{\theta : [0, T] \to L^2(\Gamma_3) : 0 \leq \theta(t) \leq 1,\ \forall t \in [0, T],\ a.e.\ \text{on } \Gamma_3\right\}.$$

By a standard procedure based on Green's formula we derive the following variational formulation of Problem P_1, in terms of displacement and bonding field.

Problem P_2. Find a displacement field $u \in C([0, T]; V)$ and a bonding field $\beta \in W^{1,\infty}\left(0, T; L^2(\Gamma_3)\right) \cap B$ such that

$$\begin{aligned} & u(t) \in K \cap W,\ \ \langle F\varepsilon(u(t)), \varepsilon(v) - \varepsilon(u(t))\rangle_Q \\ & \qquad + \left\langle \int_0^t \mathcal{F}(t-s)\, \varepsilon(u(s))\, ds, \varepsilon(v) - \varepsilon(u(t)) \right\rangle_Q \\ & + h(\beta(t), u(t), v - u(t)) + j(u(t), v) - j(u(t), u(t)) \\ & \geq (f(t), v - u(t))_V \quad \forall\, v \in K,\ t \in [0, T], \end{aligned} \tag{3.20}$$

$$\dot{\beta}(t) = -\left[\beta(t)\left(c_\nu (R_\nu(u_\nu(t)))^2 + c_\tau |R_\tau(u_\tau(t))|^2\right) - \varepsilon_a\right]_+ \ \ a.e.\ t \in (0, T), \tag{3.21}$$

$$\beta(0) = \beta_0. \tag{3.22}$$

4. Existence and uniqueness of solution

Our main result in this section is the following theorem.

Theorem 4.1. *Let* (3.11), (3.13), (3.14), (3.15), (3.16), (3.17), (3.18) *and* (3.19) *hold. Then, there exists a constant* $\mu_0 > 0$ *such that Problem* P_2 *has a unique solution if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0.$$

The proof of Theorem 4.1 is carried out in several steps. In the first step, we consider the closed subset Z of the space $C\left([0,T];L^2(\Gamma_3)\right)$ defined as

$$Z = \left\{\theta \in C\left([0,T];L^2(\Gamma_3)\right) \cap B : \theta(0) = \beta_0\right\},$$

where the Banach space $C\left([0,T];L^2(\Gamma_3)\right)$ is endowed with the norm

$$\|\beta\|_k = \max_{t\in[0,T]} \left[\exp(-kt)\,\|\beta(t)\|_{L^2(\Gamma_3)}\right],\ k > 0.$$

Next for a given $\beta \in Z$, we consider the following variational problem.
Problem $P_{1\beta}$. Find $u_\beta \in C([0,T];V)$ such that

$$\begin{aligned} & u_\beta(t) \in K \cap W,\ \langle F\varepsilon(u_\beta(t)), \varepsilon(v) - \varepsilon(u_\beta(t))\rangle_Q \\ & \qquad + \left\langle \int_0^t \mathcal{F}(t-s)\,\varepsilon(u_\beta(s))\,ds, \varepsilon(v) - \varepsilon(u_\beta(t))\right\rangle_Q \\ & +h(\beta(t), u_\beta(t), v - u_\beta(t)) + j(u_\beta(t), v) - j(u_\beta(t), u_\beta(t)) \\ & \geq (f(t), v - u_\beta(t))_V \quad \forall\, v \in K,\ t \in [0,T]. \end{aligned} \tag{4.1}$$

We have the following result.
Theorem 4.2. *There exists a constant* $\mu_1 > 0$ *such that Problem* $P_{1\beta}$ *has a unique solution if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1.$$

To prove this theorem, for $\eta \in C([0,T];Q)$ we consider the following intermediate problem.

Problem $P_{1\beta\eta}$. Find $u_{\beta\eta} \in C([0,T];V)$ such that

$$\begin{aligned} & u_{\beta\eta}(t) \in K \cap W,\ \langle F\varepsilon(u_{\beta\eta}(t)), \varepsilon(v - u_{\beta\eta}(t))\rangle_Q + \langle \eta(t), \varepsilon(v - u_{\beta\eta}(t))\rangle_Q \\ & +h(\beta(t), u_{\beta\eta}(t), v - u_{\beta\eta}(t)) + j(u_{\beta\eta}(t), v) - j(u_{\beta\eta}(t), u_{\beta\eta}(t)) \\ & \geq (f(t), v - u_{\beta\eta}(t))_V \quad \forall v \in K,\ t \in [0,T]. \end{aligned} \tag{4.2}$$

Since Riesz's representation theorem implies that there exists an element $f_\eta \in C([0,T];V)$ such that

$$(f_\eta(t), v)_V = (f(t), v)_V - \langle \eta(t), \varepsilon(v)\rangle_Q,$$

then Problem $P_{1\beta\eta}$ is equivalent to the following problem.

Problem $P_{2\beta\eta}$. Find $u_{\beta\eta} \in C([0,T];V)$ such that

$$u_{\beta\eta}(t) \in K \cap W,\ \langle F\varepsilon(u_{\beta\eta}(t)), \varepsilon(v - u_{\beta\eta}(t))\rangle_Q + h(\beta(t), u_{\beta\eta}(t), v - u_{\beta\eta}(t))$$

$$+j(u_{\beta\eta}(t), v) - j(u_{\beta\eta}(t), u_{\beta\eta}(t)) \geq (f_\eta(t), v - u_{\beta\eta}(t))_V \quad \forall v \in K,\ t \in [0,T]. \tag{4.3}$$

We now show the proposition below.

Proposition 4.3. *There exists a constant* $\mu_1 > 0$ *such that Problem* $P_{2\beta\eta}$ *has a unique solution if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1.$$

We shall prove Proposition 4.3 in several steps by using arguments on Banach fixed point theorem. Indeed, let $q \in C_+$ where C_+ is a non-empty closed subset of $L^2(\Gamma_3)$ defined as

$$C_+ = \left\{s \in L^2(\Gamma_3);\ s \geq 0 \ a.e. \text{ on } \Gamma_3\right\}$$

and let the functional $j_q : V \to \mathbb{R}$ given by

$$j_q(v) = \int_{\Gamma_3} \mu q |v_\tau| \, da \ \forall v \in V.$$

We consider the following auxiliary problem.

Problem $P_{2\beta\eta q}$. Find $u_{\beta\eta q} \in C([0,T];V)$ such that

$$u_{\beta\eta q}(t) \in K,\ \langle F\varepsilon(u_{\beta\eta q}(t)), \varepsilon(v - u_{\beta\eta q}(t))\rangle_Q + h(\beta(t), u_{\beta\eta q}(t), v - u_{\beta\eta q}(t))$$

$$+j_c(u_{\beta\eta q}(t), v - u_{\beta\eta q}(t)) + j_q(v) - j_q(u_{\beta\eta q}(t)) \geq (f_\eta(t), v - u_{\beta\eta q}(t))_V,$$

$$\forall v \in K,\ t \in [0,T]. \tag{4.4}$$

We have the following lemma.

Lemma 4.4. *Problem* $P_{2\beta\eta q}$ *has a unique solution.*

Proof. Let $t \in [0,T]$ and let $A_{\beta(t)} : V \to V$ be the operator defined by

$$\left(A_{\beta(t)}u, v\right)_V = \langle F\varepsilon(u), \varepsilon(v)\rangle_Q + h(\beta(t), u, v) + j_c(u, v) \quad \forall u, v \in V.$$

As in [28], using (3.13) (b), (3.13) (c), (3.15), (3.19) (b), (3.19) (c) and the properties of R_ν and R_τ, we see that the operator $A_{\beta(t)}$ is Lipschitz continuous and strongly monotone. On the other hand the functional $j_q : V \to \mathbb{R}$ is a continuous seminorm; using standard arguments on elliptic variational inequalties (see [25]), it follows that there exists a unique element $u_{\beta\eta q}(t) \in K$ which satisfies the inequality (4.4).
Now, for each $t \in [0,T]$, we define the map $\Psi_t : C_+ \to C_+$ by

$$\Psi_t(q) = |R\sigma_\nu(u_{\beta\eta q}(t))|.$$

We show the following lemma.

Lemma 4.5. *There exists a constant* $\mu_1 > 0$ *such that the mapping* Ψ_t *has a unique fixed point* q^* *and* $u_{\beta\eta q^*}(t)$ *is a unique solution of the inequality* (4.3) *if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1.$$

Proof. Let $q_1,\, q_2 \in C_+$. Using (3.17), it follows that there exists a constant $c_0 > 0$ such that

$$\|\Psi_t(q_1) - \Psi_t(q_2)\|_{L^2(\Gamma_3)} \le c_0 \|\sigma_\nu(u_{\beta\eta q_1}(t)) - \sigma_\nu(u_{\beta\eta q_2}(t))\|_{H^{-\frac{1}{2}}(\Gamma)}. \tag{4.5}$$

Moreover using (3.13) (b) yields

$$\|\sigma_\nu(u_{\beta\eta q_1}(t)) - \sigma_\nu(u_{\beta\eta q_2}(t))\|_{H^{-\frac{1}{2}}(\Gamma)} \le M \|u_{\beta\eta q_1}(t) - u_{\beta\eta q_2}(t)\|_V. \tag{4.6}$$

We also use (3.10), (3.13) (c), (3.19) (c) and the properties of R_ν and R_τ to find after some calculus algebra that

$$\|u_{\beta\eta q_1}(t) - u_{\beta\eta q_2}(t)\|_V \le \frac{\|\mu\|_{L^\infty(\Gamma_3)} d_\Omega}{m} \|q_1 - q_2\|_{L^2(\Gamma_3)}. \tag{4.7}$$

Hence, taking into account (3.18), we combine (4.5), (4.6) and (4.7) to deduce that

$$\|\Psi_t(q_1) - \Psi_t(q_2)\|_{L^2(\Gamma_3)} \le \|\mu\|_{L^\infty(\Gamma_3)} \frac{c_0 M d_\Omega}{m} \|q_1 - q_2\|_{L^2(\Gamma_3)}.$$

Take $\mu_1 = m/c_0 M d_\Omega$, then this inequality shows that if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1$, Ψ is a contraction; thus it has a unique fixed point q^* and $u_{\beta\eta q^*}(t)$ is a unique solution of (4.3).

Denote $u_{\beta\eta q^*} = u_{\beta\eta}$. We now shall see that $u_{\beta\eta} \in C([0,T]; V)$. Indeed, let $t_1,\, t_2 \in [0,T]$. Take $v = u_{\beta\eta}(t_2)$ in (4.3) written for $t = t_1$ and then $v = u_{\beta\eta}(t_1)$ in the same inequality written for $t = t_2$. Using (3.13) (c), (3.17), (3.19) (c) and the properties of R_ν and R_τ, and adding the resulting inequalities, it follows that there exists a constant $c_1 > 0$ such that

$$\|u_{\beta\eta}(t_2) - u_{\beta\eta}(t_1)\|_V \le$$

$$\frac{c_1}{m - \|\mu\|_{L^\infty(\Gamma_3)} c_0 M d_\Omega} (\|\beta(t_2) - \beta(t_1)\|_{L^2(\Gamma_3)} + \|\eta(t_2) - \eta(t_1)\|_Q + \|f(t_2) - f(t_1)\|_V).$$

Then, as $\beta \in C([0,T]; L^2(\Gamma_3))$, $\eta \in C([0,T]; Q)$ and $f \in C([0,T]; V)$, we immediately conclude. We also have that $u_{\beta\eta}(t) \in W, \forall t \in [0,T]$. Indeed, for each $t \in [0,T]$, denote $\sigma(u_{\beta\eta}(t)) = F\varepsilon(u_{\beta\eta}(t)) + \eta(t)$, take $v = u_{\beta\eta}(t) \pm \varphi$ in inequality (4.3) where $\varphi \in (C_0^\infty(\Omega))^d$ and use Green's formula with the regularity $\varphi_1(t) \in H$ leads to $div\sigma(u_{\beta\eta}(t)) \in H$ and then $u_{\beta\eta}(t) \in W$. □

Now we introduce the operator

$$\Lambda_\beta : C([0,T]; Q) \to C([0,T]; Q)$$

defined by

$$\Lambda_\beta \eta(t) = \int_0^t \mathcal{F}(t-s)\, \varepsilon(u_{\beta\eta}(s))\, ds \quad \forall \eta \in C([0,T]; Q),\, t \in [0,T]. \tag{4.8}$$

We have the lemma below.

Lemma 4.6. *The operator Λ_β has a unique fixed point η_β.*

Proof. Let $\eta_1,\ \eta_2\ \in\ C([0,T];Q)$. Using (4.3), (4.8) and (3.14) we obtain for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1$ that

$$\|\Lambda_\beta\eta_1(t) - \Lambda_\beta\eta_2(t)\|_Q \le c_2 \int_0^t \|\eta_1(s) - \eta_2(s)\|_Q\, ds \quad \forall t \in [0,T],$$

where $c_2 > 0$. Reiterating this inequality n times, yields

$$\left\|\Lambda_\beta^n\eta_1 - \Lambda_\beta^n\eta_2\right\|_{C([0,T];Q)} \le \frac{(c_2T)^n}{n!}\|\eta_1 - \eta_2\|_{C([0,T];Q)}.$$

As $\lim_{n\to+\infty} \frac{(c_2T)^n}{n!} = 0$, it follows that for a positive integer n sufficiently large, Λ_β^n is a contraction; then, by using the Banach fixed point theorem, it has a unique fixed point η_β which is also a unique fixed point of Λ_β i.e.,

$$\Lambda_\beta\eta_\beta(t) = \eta_\beta(t) \quad \forall t \in [0,T]. \tag{4.9}$$

Then by (4.3) and (4.9) we conclude that $u_{\beta\eta_\beta}$ is the unique solution of Problem $P_{1\beta}$. □

Next denote $u_\beta = u_{\beta\eta_\beta}$. In the second step we state the following problem.

Problem P_{ad}. Find $\beta^* : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\beta}^*(t) = -\Big[\beta^*(t)\left(c_\nu\left(R_\nu\left(u_{\beta^*\nu}(t)\right)\right)^2 + c_\tau\left|R_\tau\left(u_{\beta^*\tau}(s)\right)\right|^2\right) - \varepsilon_a\Big]_+ \quad a.e.\ t \in (0,T), \tag{4.10}$$

$$\beta^*(0) = \beta_0. \tag{4.11}$$

We obtain the following result.

Proposition 4.7. *Problem P_{ad} has a unique solution β^* which satisfies*

$$\beta^* \in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right) \cap B.$$

Proof. Let $t \in [0,T]$ and consider the mapping $\Phi : Z \to Z$ defined by

$$\Phi\beta(t) = \beta_0 - \int_0^t [\beta(s)(c_\nu(R_\nu(u_{\beta\nu}(s)))^2 + c_\tau|R_\tau(u_{\beta\tau}(s))|^2) - \varepsilon_a]_+ ds,$$

where u_β is the solution of Problem $P_{1\beta}$. For $\beta_1,\ \beta_2 \in Z$, there exists a constant $c_2 > 0$ such that

$$\begin{aligned}&\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_{L^2(\Gamma_3)}\\ &\le c_2\int_0^t \left\|\beta_1(s)\left(R_\nu\left(u_{\beta_1\nu}(s)\right)\right)^2 - \beta_2(s)\left(R_\nu\left(u_{\beta_2\nu}(s)\right)\right)^2\right\|_{L^2(\Gamma_3)} ds\\ &+c_2\int_0^t \left\|\beta_1(s)\left|R_\tau\left(u_{\beta_1\tau}(s)\right)\right|^2 - \beta_2(s)\left|R_\tau\left(u_{\beta_2\tau}(s)\right)\right|^2\right\|_{L^2(\Gamma_3)} ds.\end{aligned}$$

As in [31] we deduce

$$\begin{aligned}&\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_{L^2(\Gamma_3)} \le\\ &c_3\left(\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} ds + \int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V ds\right),\end{aligned} \tag{4.12}$$

for some constant $c_3 > 0$. Now to continue the proof we have needed to prove the following lemma.

Lemma 4.8. *There exists a constant* $\mu_0 > 0$ *such that*

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V \leq c\,\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \ \ \forall t \in [0,T],$$

if

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0.$$

Proof. Let $t \in [0,T]$. Take $u_{\beta_2}(t)$ in (4.1) satisfied by $u_{\beta_1}(t)$, then take $u_{\beta_1}(t)$ in the same inequality satisfied by $u_{\beta_2}(t)$; by adding the resulting inequalities we obtain

$$\begin{aligned}
&\langle F\varepsilon(u_{\beta_1}(t)) - F\varepsilon(u_{\beta_2}(t)), \varepsilon(u_{\beta_1}(t)) - \varepsilon(u_{\beta_2}(t))\rangle_Q \\
&\leq \left\langle \int_0^t \mathcal{F}(t-s)(\varepsilon(u_{\beta_1}(s)) - \varepsilon(u_{\beta_2}(s))\,ds, \varepsilon(u_{\beta_2}(t)) - \varepsilon(u_{\beta_1}(t))\right\rangle_Q \\
&+ h(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t) - u_{\beta_1}(t)) + h(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)) \\
&+ j(u_{\beta_1}(t), u_{\beta_2}(t)) - j(u_{\beta_1}(t), u_{\beta_1}(t)) + j(u_{\beta_2}(t), u_{\beta_1}(t)) - j(u_{\beta_2}(t), u_{\beta_2}(t)).
\end{aligned}$$

Using $(3.13)\,(b)$ this inequality implies

$$\begin{aligned}
&m\,\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 \leq \\
&\left\langle \int_0^t \mathcal{F}(t-s)(\varepsilon(u_{\beta_1}(s)) - \varepsilon(u_{\beta_2}(s))\,ds, \varepsilon(u_{\beta_2}(t)) - \varepsilon(u_{\beta_1}(t))\right\rangle_Q \\
&+ h(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t) - u_{\beta_1}(t)) + h(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)) \\
&+ j(u_{\beta_1}(t), u_{\beta_2}(t)) - j(u_{\beta_1}(t), u_{\beta_1}(t)) + j(u_{\beta_2}(t), u_{\beta_1}(t)) - j(u_{\beta_2}(t), u_{\beta_2}(t)).
\end{aligned} \tag{4.13}$$

We have

$$\begin{aligned}
&\left\langle \int_0^t \mathcal{F}(t-s)(\varepsilon(u_{\beta_1}(s)) - \varepsilon(u_{\beta_2}(s)))\,ds, \varepsilon(u_{\beta_2}(t) - u_{\beta_1}(t))\right\rangle_Q \\
&\leq \left(\int_0^t \|\mathcal{F}(t-s)\|_{Q_\infty} \|u_{\beta_2}(s) - u_{\beta_1}(s)\|_V\,ds\right) \|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V \\
&\leq c_4 \left(\int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V\,ds\right) \|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V,
\end{aligned}$$

for some positive constant c_4. Using Young's inequality, we find

$$\begin{aligned}
&\left\langle \int_0^t \mathcal{F}(t-s)(\varepsilon(u_{\beta_1}(s)) - \varepsilon(u_{\beta_2}(s)))\,ds, \varepsilon(u_{\beta_2}(t) - u_{\beta_1}(t))\right\rangle_Q \\
&\leq \frac{c_4^2}{2m}\left(\int_0^t \|u_{\beta_2}(s) - u_{\beta_1}(s)\|_V\,ds\right)^2 + \frac{m}{2}\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2.
\end{aligned} \tag{4.14}$$

Using the properties of R_ν and R_τ (see [28]), we have

$$h\left(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t) - u_{\beta_1}(t)\right) + h\left(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)\right)$$

$$\leq c_5 \left\|\beta_1(t) - \beta_2(t)\right\|_{L^2(\Gamma_3)} \left\|u_{\beta_1}(t) - u_{\beta_2}(t)\right\|_V,$$

where $c_5 > 0$. Using also (3.10), (3.17) and (3.19) (c) yields

$$j\left(u_{\beta_1}(t), u_{\beta_2}(t)\right) - j\left(u_{\beta_1}(t), u_{\beta_1}(t)\right) + j\left(u_{\beta_2}(t), u_{\beta_1}(t)\right) - j\left(u_{\beta_2}(t), u_{\beta_2}(t)\right)$$

$$\leq c_0 M d_\Omega \left\|\mu\right\|_{L^\infty(\Gamma_3)} \left\|u_{\beta_1}(t) - u_{\beta_2}(t)\right\|_V^2. \tag{4.15}$$

We now combine inequalities (4.13), (4.14) and (4.15) to deduce

$$m\left\|u_{\beta_1}(t) - u_{\beta_2}(t)\right\|_V^2 \leq c_0 M d_\Omega \left\|\mu\right\|_{L^\infty(\Gamma_3)} \left\|u_{\beta_1}(t) - u_{\beta_2}(t)\right\|_V^2$$

$$+\frac{c_4^2}{2m}\left(\int_0^t \left\|u_{\beta_1}(s) - u_{\beta_2}(s)\right\|_V ds\right)^2 + \frac{m}{2}\left\|u_{\beta_1}(t) - u_{\beta_2}(t)\right\|_V^2 \tag{4.16}$$

$$+c_5 \left\|\beta_1(t) - \beta_2(t)\right\|_{L^2(\Gamma_3)} \left\|u_{\beta_1}(t) - u_{\beta_2}(t)\right\|_V.$$

Using Young's inequality we get

$$c_5 \left\|\beta_1(t) - \beta_2(t)\right\|_{L^2(\Gamma_3)} \left\|u_{\beta_1}(t) - u_{\beta_2}(t)\right\|_V$$

$$\leq c_6 \left\|\beta_1(t) - \beta_2(t)\right\|_{L^2(\Gamma_3)}^2 + \frac{m}{4}\left\|u_{\beta_1}(t) - u_{\beta_2}(t)\right\|_V^2 \tag{4.17}$$

for some constant $c_6 > 0$. Then (4.16) and (4.17) imply that

$$\frac{m}{4}\left\|u_{\beta_1}(t) - u_{\beta_2}(t)\right\|_V^2 \leq$$

$$c_0 M d_\Omega \left\|\mu\right\|_{L^\infty(\Gamma_3)} \left\|u_{\beta_1}(t) - u_{\beta_2}(t)\right\|_V^2 + \frac{c_4^2}{2m}\left(\int_0^t \left\|u_{\beta_1}(s) - u_{\beta_2}(s)\right\|_V ds\right)^2$$

$$+c_6 \left\|\beta_1(t) - \beta_2(t)\right\|_{L^2(\Gamma_3)}^2.$$

Let now

$$\mu_0 = \mu_1/4.$$

Then if

$$\left\|\mu\right\|_{L^\infty(\Gamma_3)} < \mu_0,$$

we deduce that there exists a constant $c_7 > 0$ such that

$$\left\|u_{\beta_1}(t) - u_{\beta_2}(t)\right\|_V^2 \leq c_7\left(\int_0^t \left\|u_{\beta_1}(s) - u_{\beta_2}(s)\right\|_V^2 ds + \left\|\beta_1(t) - \beta_2(t)\right\|_{L^2(\Gamma_3)}^2\right).$$

Then using Gronwall's argument, it follows that there exists a constant $c > 0$ such that

$$\left\|u_{\beta_1}(t) - u_{\beta_2}(t)\right\|_V \leq c\left\|\beta_1(t) - \beta_2(t)\right\|_{L^2(\Gamma_3)}. \tag{4.18}$$

□

Now to end the proof of Proposition 4.7 we use (4.12) and (4.18) to deduce

$$\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_{L^2(\Gamma_3)} \leq c_8 \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} ds \quad \forall t \in [0,T],$$

where $c_8 > 0$, and then we obtain

$$\|\Phi\beta_1 - \Phi\beta_2\|_k \leq \frac{c_8}{k} \|\beta_1 - \beta_2\|_k .$$

This inequality shows that for $k > c_8$, Φ is a contraction. Then it has a unique fixed point β^* which satisfies (4.10) and (4.11). We now have all ingredients to prove Theorem 4.1.

Proof of Theorem 4.1. *Existence.* Let $\beta = \beta^*$ and let u_{β^*} the solution of Problem $P_{1\beta}$. We conclude by (4.1), (4.10) and (4.11) that (u_{β^*}, β^*) is a solution to Problem P_2.

Uniqueness. Suppose that (u, β) is a solution of Problem P_2 which satisfies (3.20), (3.21) and (3.22). It follows from (3.20) that u is a solution to Problem $P_{1\beta}$, and from Theorem 4.2 that $u = u_\beta$. Take $u = u_\beta$ in (3.20) and use the initial condition (3.22), we deduce that β is a solution to Problem P_{ad}. Therefore, we obtain from Proposition 4.7 that $\beta = \beta^*$ and then we conclude that (u_{β^*}, β^*) is a unique solution to Problem P_2. □

Let now σ^* be the function defined by (3.1) which corresponds to the function u_{β^*}. Then, it results from (3.13) and (3.14) that $\sigma^* \in C([0,T]; Q)$. Using also a standard argument, it follows from the inequality (3.20) that

$$div\sigma^*(t) + \varphi_1(t) = 0 \text{ in } \Omega, \text{ for all } t \in [0,T].$$

Therefore, using the regularity $\varphi_1 \in C([0,T]; H)$, we deduce that $div\sigma^* \in C([0,T]; H)$ which implies that $\sigma^* \in C([0,T]; Q_1)$. The triple $(u_{\beta^*}, \sigma^*, \beta^*)$ which satisfies (3.1) and (3.20) − (3.22) is called a weak solution of Problem P_1. We conclude that under the stated assumptions, the problem P_1 has a unique weak solution $(u_{\beta^*}, \sigma^*, \beta^*)$. Moreover, the regularity of the weak solution is $u_{\beta^*} \in C([0,T]; V)$, $\sigma^* \in C([0,T]; Q_1)$, $\beta^* \in W^{1,\infty}(0,T; L^2(\Gamma_3)) \cap B$.

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Book reviews

Francis Clarke, Functional Analysis, Calculus of Variations and Optimal Control, Graduate Texts in Mathematics 264, Springer, London - Heidelberg - New York - Dordrecht, 2013, ISBN 978-1-4471-4819-7; ISBN 978-1-4471-4820-3 (eBook); DOI 10.1007/978-1-4471-4820-3, xiv+591 pp.

The book consists of four parts: I. *Functional Analysis*, II. *Optimization and Nonsmooth Analysis*, III. *Calculus of Variations*, and IV. *Optimal Control.* Treating apparently disparate topics, these parts are, in fact, strongly interconnected, each topic being treated in a way that makes it of use for the remaining, and making use of the preceding ones.

The first part can be considered as an introduction to some topics in functional analysis (although some familiarity with basic functional analysis and real analysis is assumed), mainly those useful in the next three parts. To this end, besides some standard material on normed spaces, continuous linear functionals and operators, some topics useful in optimization and calculus of variations, as derivatives, tangents and normals, convex functions and convex analysis, perturbed minimization principles (Ekeland's variational principle) and regularity, are included. Besides this, some proofs are given using optimization methods. For instance, the open mapping theorem is proven via a Decrease Principle, which, in its turn, is a consequence of Ekeland Variational Principle. Some regularity problems are discussed as well, including Graves-Lyusternik theorem. As application, the direct method of the calculus of variations is illustrated by a result on the existence of minima of convex lsc functions defined on convex subsets of reflexive Banach spaces. The direct method will be systematically exploited in the next chapters to prove the existence of solutions in calculus of variations and optimal control problems. Lebesgue spaces $L^p(\Omega)$, $\Omega \subset \mathbb{R}^n$, are treated in detail - duality and reflexivity (via uniform rotundity in the case $1 < p < \infty$ and directly for $p = 1$), weak compactness. Measurable multifunctions, measurable selections and semicontinuity properties of integral functionals, and some weak closure results in L^1 related to differential inclusions are considered as well. This part ends with a short introduction to Hilbert spaces, including a smooth minimization principle and proximal subdifferentials, with applications to dense Fréchet differentiability of convex functions and to Moreau-Yosida approximation.

The second part starts with a discussion, motivated by examples, on the main problems in optimization theory and on various techniques used for their solving ((mainly in the convex case) - deductive and inductive methods, multipliers and Lagrange type results, Kuhn-Tucker theorem. All this discussion leads to the conclusion of the need for a nonsmooth calculus whose development starts in Chapter 10 with the study of generalized gradients for locally Lipschitz functions defined on Banach spaces, the corresponding subdifferentials and the calculus rules. The notion of generalized gradient was introduced by the author and exposed in the book, F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, 1983 (republished as vol. 5 of Classics in Applied Mathematics, SIAM, 1990), a breakthrough in the study of optimization problems. For further developments, see F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R.Wolenski, *Nonsmooth Analysis and Control Theory*, GMT vol. 178, Springer-Verlag, New York, 1998. This study continues in the next chapter with the consideration of proximal subgradients and proximal calculus, Dini and viscosity subdifferentials. For both of these two types of subdifferentials one proves multiplier rules. The second part ends with a chapter, Ch. 12, *Invariance and monotonicity,* dealing with flow invariance for differential inclusions of the form $x'(t) \in F(x(t)),$ a.e. $t \in [a,b]$, where $x : [a,b] \to \mathbb{R}^n$ is absolutely continuous and F is a multifunction from $\mathbb{R}^n$ to $\mathbb{R}^n$.

The third part is devoted to the calculus of variations, modeled on the study of the minimization of the functional $J(x) = \int_a^b \Lambda(t, x(t), v(t))dt$ over a convenable class of functions. In the first chapter of this part, Ch. 14, *The classical theory*, one supposes that $\Lambda : \mathbb{R}^3 \to \mathbb{R}$ is twice differentiable and $x \in C^2[a,b]$. The first chapter of this part (Ch. 14, *The classical theory*) is concerned mainly with necessary conditions, a subject that is over than three hundred years old which is presented in historical perspective, emphasizing the contributions of some great mathematicians - Euler, Lagrange, Legendre, Jacobi, Tonelli, and others. The theory is extended in the next chapter to the class of Lipschitz functions, and to absolutely continuous functions (the natural framework to study the problem) in Chapter 16. A general multiplier rule is proved in Chapter 17 with application to the isoperimetric problem. One considers also nonsmooth Lagrangians and Hamilton-Jacobi methods. In the last chapter of this part, Ch. 20, *Multiple integrals*, the interval $[a,b]$ is replaced by a nonempty open bounded subset Ω of $\mathbb{R}^n$, and the solutions are discussed first in the classical context (that is for $x \in C^2(\bar{\Omega})$), and then one considers Lipschitz solutions and solutions in Sobolev space.

The last part of the book is devoted to optimal control, a natural generalization of the calculus of variations, having as a central topic Pontryagin's maximum principle. This very important result is presented gradually and from different angles, starting with the classical Maximum Principle proved by Pontryagin a.o. around 1960. The author presents several variants and extensions of this principle, culminating with a very general one involving differential inclusions, from which one deduces all previous principles, but which has an intrinsic interest too. No easy proof of this principle is known. Although, not easy too, this new approach gives a full, more streamlined, more unified and self contained treatment of this difficult matter. Existence and regularity and the use of inductive methods to check presumably solutions of control problems are discussed as well.

There are a lot of exercises spread throughout the book, completing the main text with further results and examples. Besides these, each part ends with a set of additional, more demanding exercises (most of them original). Full, partial, or hints only solutions for some of them are given in the endnotes.

Each of the four parts the book, or selections of chapters, can be used for courses on different topics. Some variants, experienced by the author himself, are suggested in the Preface.

Written by an expert in the field, with outstanding contributions to nonsmooth analysis, calculus of variations and optimal control, the present book, written in a live but rigorous style, will help the interested people to a smooth approach and a better understanding of this difficult subject in mathematics, both pure and applied, which is optimal control.

S. Cobzaş

Mikhail Popov and Beata Randrianantoanina, Narrow Operators on Function Spaces and Vector Lattices, Studies in Mathematics, Vol. 45, xiii + 319 pp, Walter de Gruyter, Berlin - New York, 2013, ISBN: 978-3-11-026303-9, e-ISBN: 978-3-11-026334-3, ISSN: 0179-0986.

By an F-space one understands a complete metric linear space. For an atomless finite measure space (Ω, Σ, μ) one denotes by $L_0(\mu)$ the space of equivalence classes of real- or complex-valued μ-measurable functions and let $\Sigma^+ = \{A \in \Sigma : \mu(A) > 0\}$. A Köthe F-space is a subspace E of $L_0(\mu)$ such that (K_i) $y \in E$ and $|x| \le |y| \;\Rightarrow\; y \in E$ and $\|x\| \le \|y\|$, and (K_{ii}) $1_\Omega \in E$. If further E is Banach and (K_{iii}) $E \subset L_1(\mu)$, then E is called a Köthe-Banach space. A *sign* is a measurable function on Ω taking only the values $0, \pm 1$. A sign x is called mean zero if $\int_\Omega x d\mu = 0$. If supp $(x) = A \in \Sigma$, then x is called a sign on A. A continuous linear operator T form a Köthe F-space E to an F-space X is called narrow if for every $A \in \Sigma^+$ and every $\varepsilon > 0$ there exists a sign x of mean 0 on A such that $\|Tx\| < \varepsilon$. Although narrow operators were considered and used under different names by J. Bourgain (1981) and H. P. Rosenthal (1981-1984), they were formally defined and systematically studied by A. Plichko and M. Popov in Dissertationes Mathematicae vol. 306 (1990). Narrow operators turned to be a very important class of continuous linear operators with applications to the study of Banach and quasi-Banach spaces and non-locally convex spaces. The aim of the present book is to give a comprehensive presentation of the modern theory of narrow operators, including very recent results (some available only as preprints, or published for the first time), defined on function spaces and vector lattices. The authors with their collaborators have important contributions to the domain which are included in the book.

The book is divided into twelve chapters. The first one contains preliminary results on F-spaces, Köthe function spaces, vector lattices, measure theory (Maharam theorem), the definition of narrow operators and their initial properties. The term "small" used in Chapter 2, *Each "small" operator is narrow*, refers to compact or AM-operators (send order bounded sets to compact ones), Dunford-Pettis operators, strictly singular operators (an operator $T \in \mathcal{L}(E, X)$ is called Z-strictly singular if it does not preserve an isomorphic copy of any subspace X_0 of E isomorphic to Z). Every narrow operator is strictly singular, but a still open problem, posed by Plichko and Popov in the mentioned paper, is whether or not is every strictly singular operator narrow. Chapter 3 contains some applications of narrow operators to operators on non-locally convex spaces (as e.g., the isomorphic classification of a class of Köthe F-spaces, quotients of $L_p(\mu)$, $0 < p < 1$). In Chapter 4, *Noncompact narrow operators*, it is shown that there exists spaces with non compact narrow operators and one gives a characterization on narrow expectation operators. In Chapter 5 spectral properties and numerical radii for narrow operators are studied, as well as some ideal properties of them - ST is narrow if T is but it could be not narrow if only S is narrow, so the narrow operators form only a left ideal. Chapter 6 is concerned with Daugavet-type properties of narrow operators T (i.e. the equality $\|I + T\| = 1 + \|T\|$) acting on Lebesgue or Lorentz spaces. Plichko and Popov, *loc cit*, have shown that any narrow operator has the Daugavet property. Here one gives an extension of this result.

Chpater 7, *Strict singularity versus narrowness*, is concerned with the problem mentioned above of narrowness of strictly singular operators. This chapter contains some very deep results on the narrowness of some classes of strictly singular operators: ℓ_1- as well L_1-strictly singular operators on L_1 (Bourgain and Rosenthal, (1983) and Rosenthal (1984)), L_p-strictly singular operators on L_p, $1 < p < 2$, (Johnson, Maurey, Schechtman, and Tzafriri, Memoirs AMS, 1979).

In Chapter 8, *Weak embeddings of* L_1, one discusses several types of embeddings – semi-embeddings, G_δ-embeddings and sign-embeddings. The important result of Talagrand, giving a negative answer to the three-space problem for isomorphic embeddings of L_1, which asserts that there exists a subspace Z of L_1 such that neither Z nor L_1/Z contains an isomorph of L_1, is included.

Chapter 9, *Spaces X for which every operator* $T \in \mathcal{L}(L_p, X)$ *is narrow,* contains characterizations of these spaces (for instance, in terms of ranges of vector measures) and some particular spaces for which this is true are emphasized as, for instance, $\mathcal{L}(E, c_0(\Gamma))$, $\mathcal{L}(L_p, L_r)$ for $1 \le p < 2$ and $p < r < \infty$. In Section 9.5 one gives a partial answer to the problem concerning strictly singular operators – every ℓ_2-strictly singular operator from $\mathcal{L}(L_p, X)$ with $1 < p < \infty$ and X Banach with an unconditional basis is narrow.

Chapter 10, *Narrow operators on vector lattices,* is concerned with an extension of narrowness to vector lattices as given in a paper by Maslyuchenko, Mykhaylyuk and Popov, Positivity (2009). In Chapter 11, some variants of the notion of narrowness are briefly discussed. Among them, one used by V. Kadets, R. Shvidkoy and D. Werner in the study of Daugavet property.

There are 28 research problems spread through the book. For the convenience of the reader these are collected in the last chapter, Chapter 12, with references to the places where they occurred and bibliographical references. The book ends with a complete bibliography of 142 titles, a Name Index and a Subject Index.

The book is clearly written with full proofs, most of which, in spite of the fact that some have been simplified by the authors or by colleagues of them, still remain long and involved. The book includes topics which are of great interest for researchers in functional analysis, mainly in Banach and quasi-Banach spaces and operator theory, making available for the first time in book form many results scattered through various publications. Undoubtedly it will become a standard reference in the area.

S. Cobzaş

Glen Van Brummelen, The Mathematics of the Heavens and the Earth - The Early History of Trigonometry, Princeton University Press, Princeton and Oxford, 2009, xvii+329 pp, Hardbound, ISBN 978-0-691-12973-0.

The book under review is written by a well known expert in the field of history of mathematics. It is - as its subtitle emphasizes - a concise history of the early plane and spherical trigonometry from the dawn of civilization to the Middle Ages (1550).

In the first chapter *Precursors*, the author enumerates the earliest trigonometric results from Egypt, Babylon and Ancient Greece, like finding the slope of a pyramid, arc measurement and the 360° circle or Aristarchus and Archimedes reasoning for determining the shape of Earth, Moon and Sun and the ratio of the distances to the Moon and to the Sun.

In the second chapter *Alexandrian Greece*, the author follows the development of trigonometry in the major works of Hipparchus, Theodosius from Bithynia, Menelaus and Claudius Ptolemy. Their results were meant to explain the observed motion of the Sun, to solve other problems from Astronomy - like motion of the planets, timekeeping, or from Geography - like construction of the latitude arcs on a map.

In the following chapter *India*, the author describes how the first trigonometric results arose in this part of the world. The *chord* function used by the Greek scholars is transformed in a function proportional to the today *sine* function. The tables of this function were computed with increased accuracy using higher degree interpolation schemes for its approximation. This tables were used to solve spherical astronomy problems - like finding the right ascension of a point on the ecliptic, or other astronomical issues - like planetary equation.

The next chapter *Islam* trails the development of spherical and plane trigonometry in the area occupied by the Islamic world stretching from the borders of India through middle East across northern Africa and Spain. In this new world born in early 7^{th} century we find traces of Indian learning, but the Muslin scholars developed their own techniques to solve astronomical problems. For their religious practice they needed *qibla* - the direction in which they have to be oriented to face Mecca during their daily prayers. Moreover, they used trigonometry to design astronomical instruments like *horary quadrants*, which let them find the time of the day by the altitude of the Sun.

The last chapter *The West to 1550* is devoted to the application of trigonometry in navigation over seas. During Middle Ages and Renaissance in Europe flourished this new activity which benefited from trigonometry results. In the mean time, plane trigonometry was developed by scholars like Regiomontanus, Werner, Copernic, Rheticus, Otho and Pitiscus. The author emphasized their struggle to build precise tables of trigonometric functions (mainly sine and cosine), from 0° to 90°, with increments smaller than one degree.

The book includes a number of excerpts of translation from the original texts written by the scholars mentioned above. These are meant to give the reader of today the opportunity to experience directly what the ancient authors wrote and judge their reasoning. These old writings could be obscure for the today reader, the reason why the author provides after each text an explanation which allows a better understanding of the content.

After few *Concluding Remarks*, the book ends with an extensive bibliography which contains almost all important works for the history of trigonometry. The reference sources used to write this book are cited in *Preface*. Beside the excerpts from the old texts, the book contains reproductions from old printed works.

Being devoted to the history of mathematics, the book could be used by the teachers who want to make their lessons more attractive. Most of the excerpts from the ancient texts stand on their own and are ready to be used to illustrate trigonometric and astronomical concepts by providing historical context.

I highly recommend the book to all those interested in the way in which the ancient people solve their practical problems and hope that the next volume of this interesting history of spherical and plane trigonometry will appear soon.

Cristina Blaga

Bernard Dacorogna and Chiara Tanteri, Mathematical Analysis for Engineers, x+359 pp, Imperial College Press, World Scientific, Singapore, 2012, ISBN: 13 978-1-84816-912-8, ISBN: 10 1-84816-912-4.

This is the translation of the third French edition of a successful book destined to engineering students at l'Ecole Polythechnique Fédérale de Lausanne, but it can be also profitably used by students in mathematics and physics. The prerequisites are a basic course in analysis - differential and integral calculus.

The book is concerned with three main topics: I. *Vector analysis*, II. *Complex analysis*, an III. *Fourier analysis*. The first part contains exercises on the differential operators of mathematical physics (divergence, gradient, curl, Laplace operator), line integrals and gradient vector fields, surface integrals and Stokes theorem.

The second part is devoted to basic results in complex analysis: holomorphic functions and Cauchy-Riemann equations, complex integration, residues and their applications, conformal mappings.

The third part is concerned with Fourier series and Fourier transform, Laplace transform, and applications to ordinary and to partial differential equations.

What makes the book particularly useful are the theoretical results (definitions, basic theorems and formulae) and the examples preceding each section. These results are stated with mathematical rigor, without comments or proofs, but with references to precise pages of some books from the bibliography. Also an appendix to the first part contains some complementary results on topology, function spaces, curves and surfaces (with examples of some important curves and surfaces). The majority of the exercises help the students to master the concepts and techniques of the field, but there are several, marked by *, which present some further theoretical developments. The detailed solutions to exercises are given in the fourth part of the book.

This very well organized book will be useful to students in engineering, but also to those in mathematics and physics, as a complementary material to courses in analysis (real and complex), Fourier analysis and differential equations (both ordinary and partial).

Damian Trif