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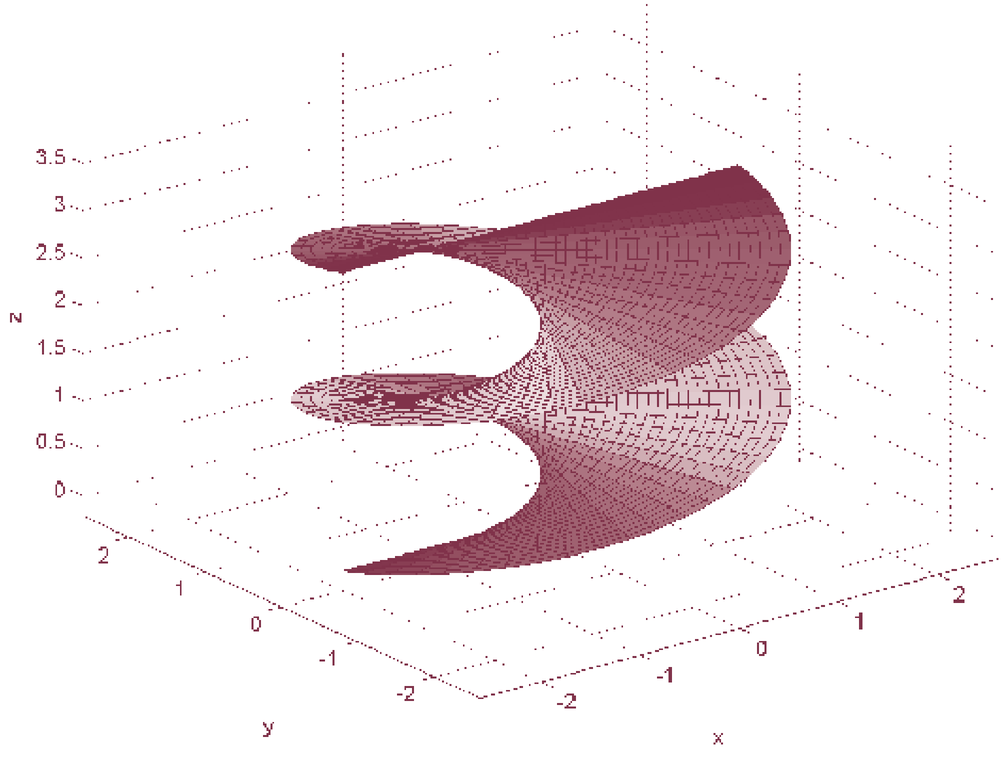

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CONTENTS

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On saturated triples associated to some block algebras of finite groups

Constantin-Cosmin Todea

Abstract. Saturated triples are recently defined in [3]. Block algebras of finite groups give an important example of such saturated triples. In this short article we prove that the principal block and the group algebra viewed as an algebra acted by the group of automorphisms of the finite group provides a new example of saturated triples.

Mathematics Subject Classification (2010): 20C20.

Keywords: Block algebra, finite group, permutation algebra.

1. Preliminaries

We follow [4] to recall definitions and basic properties of block algebras of finite groups. Let G be a finite group and let k be an algebraically closed field of characteristic p such that p divides the order of G. A *block algebra* of kG is an indecomposable factor B of kG as an algebra. The block algebras are in bijection with the primitive idempotents of the center $Z(kG)$. We denote by $\mathrm{Bl}(kG)$ the finite set of block algebras (i.e. primitive idempotents in $Z(kG)$). If B is a block algebra we have $B = bkG$, where b is the corresponding primitive idempotent from $Z(kG)$.

kG is a p-permutation G-algebra, where G acts by conjugation and then $\mathrm{Bl}(kG)$ are actually the primitive idempotents of $(kG)^G = Z(kG)$. If G acts on a set X we denote by $Orb_G(X)$ its orbits and if $C \in Orb_G(X)$ we denote by C^+ the sum of all elements in the orbit C. If we use a set of indices I for the orbits we usually mean an arbitrary family of orbits, if not, it means that we consider all the orbits. We follow [2, 2.1] for results and notations regarding permutation algebras. By [2, Lemma 2.2] the set $\{C^+ : C \in Orb_G(G)\}$ is a k-basis of $(kG)^G$, where G acts by conjugation on G. We denote by N the element $\sum_{g\in G} g$.

The augmentation map $\varepsilon : kG \to k$ is the surjective homomorphism of k-algebras defined by $\varepsilon(g) = 1_k$ for any $g \in G$, that is, for an element $\sum_{x\in G} \alpha_x x \in kG$ we have

$$\varepsilon(\sum_{x\in G} \alpha_x x) = \sum_{x\in G} \alpha_x.$$

$ker(\varepsilon)$ is a maximal ideal of kG, the *augmentation ideal* and is generated as a k-vector space by $\{g-1 : g \in G\}$. Let ε^G be the restriction of ε to $(kG)^G$ (as a map $\varepsilon^G : Z(kG) \to k$) and let $\{C_i\}_{i\in I} \subseteq Orb_G(G)$. Then it is easy to see that

$$\varepsilon^G(\sum_{i\in I} \alpha_i C_i^+) = \sum_{i\in I} \mid C_i \mid \alpha_i.$$

Proposition 1.1. *With the above notations we have:*
(1) ε^G *is a surjective homomorphism.*
(2) $ker(\varepsilon^G) \neq 0$ *and* $ker(\varepsilon^G)$ *is a maximal ideal of* $Z(kG)$.

Proof. (1) Obviously ε^G is a k-algebra homomorphism. We only prove that is surjective. Let $\alpha \in k$. Then is easy to see that

$$a = (\alpha + 1_k)1_G + \sum_{C\in Orb_G(G)\setminus\{1_G\}} C^+ \in (kG)^G.$$

We have that

$$\varepsilon^G(a) = (\alpha + 1_k)1_k + \sum_{C\in Orb_G(G)\setminus\{1_G\}} \mid C \mid 1_k = \alpha + 1_k + \mid G \mid 1_k - 1_k = \alpha.$$

(2) Since $\varepsilon^G(N) = \mid G \mid = 0$ then $N \in ker(\varepsilon^G)$. □

2. The action of $\mathrm{Aut}(G)$ on kG

We consider in this section an action of $\mathrm{Aut}(G)$ on kG, which we describe in the following lines. If $f \in \mathrm{Aut}(G)$ then there is $\overline{f} \in \mathrm{Aut}_k(kG)$, where $\mathrm{Aut}_k(kG)$ represents the group of all k-algebra automorphisms of kG. For $\sum_{x\in G} \alpha_x x \in kG$ we have that $\overline{f}$ is defined by

$$\overline{f}(\sum_{x\in G} \alpha_x x) = \sum_{x\in G} \alpha_x f(x).$$

Now kG becomes an $\mathrm{Aut}(G)$-algebra where $f \in \mathrm{Aut}(G)$ acts on $a \in kG$ by ${}^f a = \overline{f}(a)$. By [2] we have that $(kG)^{Aut(G)}$ has as k-basis the set

$$\{C^+ \mid C \in Orb_{\mathrm{Aut}(G)} G\}.$$

As above, let $\varepsilon^{Aut(G)}$ be the restriction of ε to $(kG)^{\mathrm{Aut}(G)}$, that is the map

$$\varepsilon^{\mathrm{Aut}(G)} : (kG)^{\mathrm{Aut}(G)} \to k.$$

Proposition 2.1. *With the above notations we have:*
(1) $\varepsilon^{\mathrm{Aut}(G)}$ *is a surjective homomorphism.*
(2) $ker(\varepsilon^{\mathrm{Aut}(G)}) \neq 0$ *and* $ker(\varepsilon^{\mathrm{Aut}(G)})$ *is a maximal ideal of* $(kG)^{\mathrm{Aut}(G)}$.

Proof. (1) We have the same proof as in Proposition 1.1, using the element

$$a' = (\alpha + 1_k)1_G + \sum_{C\in Orb_{\mathrm{Aut}(G)}(G)\setminus\{1_G\}} C^+ \in (kG)^{\mathrm{Aut}(G)}.$$

(2) Similarly we have that $N \in ker(\varepsilon^{\mathrm{Aut}(G)})$. □

We denote by $b_0 \in \mathrm{Bl}(kG)$ the unique block such that $b_0N = N$, equivalently b_0 is the unique block such that $\varepsilon(b_0) \neq 0$. We call b_0 the *principal* block of kG, see [4, Section 40].

Proposition 2.2. *Let $f \in \mathrm{Aut}(G)$. If $b \in \mathrm{Bl}(kG)$ then $\overline{f}(b) \in \mathrm{Bl}(kG)$. Moreover for the principal block we have $\overline{f}(b_0) = b_0$.*

Proof. It is easy to verify that $\overline{f}(b)$ is an idempotent. To prove that it is central let $g \in G$. Then

$$\overline{f}(b)g = \overline{f}(b)\overline{f}(f^{-1}(g)) = \overline{f}(bf^{-1}(g)) = \overline{f}(f^{-1}(g)b) = g\overline{f}(b).$$

It is easy to check, by contradiction, that $\overline{f}(b)$ is primitive in $Z(kG)$.

For the second part, since $N \in (kG)^{Aut(G)}$ we have that

$$\overline{f}(b_0)N = \overline{f}(b_0)\overline{f}(N) = \overline{f}(b_0N) = \overline{f}(N) = N.$$

□

3. Saturated triples

From [2] we know that (A, b, G) is a *saturated triple* if b is a central idempotent, primitive in A^G such that for any (A, b, G)-Brauer pair (Q, e) we have that e is primitive in $A(Q)^{C_G(Q,e)}$, where A is a p-permutation algebra. See [2, IV, Section 2] for more details.

Theorem 3.1. *With the above notation we have that the triple $(kG, \mathrm{Aut}(G), b_0)$ is a saturated triple.*

Proof. We have that $(kG)^G = (kG)^{\mathrm{Inn}(G)}$, where $\mathrm{Inn}(G)$ is the normal subgroup in $\mathrm{Aut}(G)$ of inner automorphisms. Then $(kG)^{\mathrm{Aut}(G)} \subseteq (kG)^G$. By Proposition 2.2 we have that b_0 remains primitive in $(kG)^{\mathrm{Aut}(G)}$.

Let Q be a p-subgroup of $\mathrm{Aut}(G)$ and e a primitive idempotent of $Z(kG(Q))$. Since $\mathrm{Aut}(G)$ acts on G (the action given in Section 2), by [1, 2.5] we have that $kG(Q) \cong kC_G(Q)$, where

$$C_G(Q) = \{g \in G \mid f(g) = g, \forall f \in Q\}.$$

We prove next that e remains primitive in $kC_G(Q)^{C_{\mathrm{Aut}(G)}(Q,e)}$, where

$$C_{\mathrm{Aut}(G)}(Q, e) = \{f \mid f \in \mathrm{Aut}(G), \overline{f}(e) = e,\ f \circ q = q \circ f, \forall q \in Q\}.$$

We consider $\mathrm{Inn}_{C_G(Q)}(G)$ as the following subset of $\mathrm{Aut}(G)$, given by

$$\mathrm{Inn}_{C_G(Q)}(G) = \{c_x \mid x \in C_G(Q),\ c_x : G \to G,\ c_x(g) = xgx^{-1}, \forall g \in G\}.$$

It is easy to check that $\mathrm{Inn}_{C_G(Q)}(G)$ is a subgroup of $\mathrm{Aut}(G)$. If we restrict an element $c_x \in \mathrm{Inn}_{C_G(Q)}(G)$ to $C_G(Q)$ we have that $\mathrm{Im}(c_x \mid_{C_G(Q)}) = C_G(Q)$, then it follows that

$$kC_G(Q)^{C_G(Q)} = kC_G(Q)^{\mathrm{Inn}_{C_G(Q)}(G)} \tag{3.1}$$

Next we prove that $\mathrm{Inn}_{C_G(Q)}(G)$ is a subset of $C_{\mathrm{Aut}(G)}(Q, e)$ (in particular it is a subgroup). Let $c_x \in \mathrm{Inn}_{C_G(Q)}(G), q \in Q, g \in G$. We have that

$$\overline{c_x}(e) = xex^{-1} = e,$$

since $e \in kC_G(Q)^{C_G(Q)}$. The following statements holds

$$(c_x \circ q)(g) = xq(g)x^{-1};$$
$$(q \circ c_x)(g) = q(xgx^{-1}) = q(x)q(g)q(x)^{-1} = xq(g)x^{-1},$$

since $q \in Q$ and $x \in C_G(Q)$.

We now obtain that $\mathrm{Inn}_{C_G(Q)}(G) \leq C_{\mathrm{Aut}(G)}(Q, e)$, hence e remains primitive in

$$kC_G(Q)^{C_{\mathrm{Aut}(G)}(Q,e)} \subseteq kC_G(Q)^{\mathrm{Inn}_{C_G(Q)}(G)} = Z(kC_G(Q)),$$

where the last equality is proved in (3.1). □

Remark 3.2. Theorem 3.1 remains valid if we replace b_0 with any block b such that $b \in (kG)^{\mathrm{Aut}(G)}$, which remains primitive in this smaller algebra $(kG)^{\mathrm{Aut}(G)}$.

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On certain subclasses of analytic functions

Imran Faisal, Zahid Shareef and Maslina Darus

Abstract. In the present paper, we introduce and study certain new subclasses of analytic functions in the open unit disk U. Some inclusion relationships and integral preserving properties have also discussed in particular with reference to a new integral operator.

Mathematics Subject Classification (2010): 30C45.

Keywords: Analytic functions, starlike functions, inclusion relationship.

1. Introduction

Let A be the class of functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic and normalized in the open unit disk $U = \{z : |z| < 1\}$.

Next we define some well known subclasses such as starlike, convex, close-to-convex and quasi-convex functions of A, denoted by $S^*(\xi)$, $C(\xi)$, $K(\rho,\xi)$ and $K^*(\rho,\xi)$ respectively as follow(cf.[1]-[3]):

$$\begin{aligned}
S^*(\xi) &= \left\{ f \in A : \Re\left(\frac{zf'(z)}{f(z)}\right) > \xi, z \in \mathbb{U} \right\}, 0 \le \xi < 1. \\
C(\xi) &= \left\{ f \in A : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \xi, z \in \mathbb{U} \right\}, 0 \le \xi < 1. \\
K(\rho,\xi) &= \left\{ f \in A : \exists g(z) \in S^*(\xi) \wedge \Re\left(\frac{zf'(z)}{g(z)}\right) > \rho, z \in \mathbb{U} \right\}, 0 \le \rho < 1. \\
K^*(\rho,\xi) &= \left\{ f \in A : \exists g(z) \in C(\xi) \wedge \Re\left(\frac{(zf'(z))'}{g'(z)}\right) > \rho, z \in \mathbb{U} \right\}.
\end{aligned}$$

Note that

$$f(z) \in C(\xi) \Leftrightarrow zf'(z) \in S^*(\xi) \ \wedge \ f(z) \in K^*(\rho,\xi) \Leftrightarrow zf'(z) \in K(\rho,\xi).$$

For $f \in A$ and $\beta, \gamma \geq 0$, we define a new differential operator as follows:

$$\begin{aligned}
\Theta^0(\beta,\gamma)f(z) &= f(z) \\
(\gamma+\beta+1)\Theta^1(\beta,\gamma)f(z) &= \beta f(z) + (\gamma+1)(zf'(z)) \\
&\vdots \\
\Theta^n(\beta,\gamma)f(z) &= z + \sum_{k=2}^{\infty}\left(\frac{\beta+k(\gamma+1)}{\gamma+\beta+1}\right)^n a_k z^k. \qquad (1.1)
\end{aligned}$$

This operator is closely related to the following operators:

1. $\Theta^n(\lambda,0)f(z) = \Theta^n(\lambda)f(z) = z + \sum\limits_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n a_k z^k$,(see [4, 5]);
2. $\Theta^n(1,0)f(z) = \Theta^n f(z) = z + \sum\limits_{k=2}^{\infty}\left(\frac{k+1}{2}\right)^n a_k z^k$,(see[6]);
3. $\Theta^n(0,0)f(z) = \Theta^n f(z) = z + \sum\limits_{k=2}^{\infty}(k)^n a_k z^k$,(see[7]).

(1.1) $\Rightarrow (\gamma+1)z(\Theta^n(\beta,\gamma)f(z))' = (\gamma+1+\beta)\Theta^{n+1}(\beta,\gamma)f(z) - \beta\Theta^n(\beta,\gamma)f(z)$.

Now for linear operator $\Theta^n(\beta,\gamma)$ we define the following subclasses of A:

$$\begin{aligned}
S_n^*(\xi,\beta,\gamma) &= \{f \in A : \Theta^n(\beta,\gamma)f \in S^*(\xi)\}; \\
C_n(\xi,\beta,\gamma) &= \{f \in A : \Theta^n(\beta,\gamma)f \in C(\xi)\}; \\
K_n(\rho,\xi,\beta,\gamma) &= \{f \in A : \Theta^n(\beta,\gamma)f \in K(\rho,\xi)\}; \\
K_n^*(\rho,\xi,\beta,\gamma) &= \{f \in A : \Theta^n(\beta,\gamma)f \in K^*(\rho,\xi)\}.
\end{aligned}$$

2. Inclusion relationships

Lemma 2.1. [8, 9] *Let $\varphi(\mu, \upsilon)$ be a complex function such that $\varphi : D \to \mathbb{C}$, $D \subseteq \mathbb{C}\times\mathbb{C}$, and let $\mu = \mu_1 + i\mu_2$, $\upsilon = \upsilon_1 + i\upsilon_2$. Suppose that $\varphi(\mu,\upsilon)$ satisfies the following conditions:*

1. *$\varphi(\mu,\upsilon)$ is continuous in D;*
2. *$(1,0) \in D$ and $\Re\varphi(1,0) > 0$;*
3. *$\Re\varphi(i\mu_2,\upsilon_1) \leq 0$ for all $(i\mu_2,\upsilon_1) \in D$ such that $\upsilon_1 \leq -\frac{1}{2}(1+\mu_2^2)$.*

Let $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be analytic in $\mathbb{U}$, such that $(h(z), zh'(z)) \in D$ for all $z \in \mathbb{U}$. If $\Re\{\varphi(h(z), zh'(z))\} > 0 (z \in \mathbb{U})$, then $\Re\{h(z)\} > 0$.

Theorem 2.2. *Let $f \in A$, $0 \leq \xi < 1$, $\beta, \gamma \geq 0, n \in \mathbb{N}$ then*

$$S_{n+1}^*(\xi,\beta,\gamma) \subseteq S_n^*(\xi,\beta,\gamma) \subseteq S_{n-1}^*(\xi,\beta,\gamma).$$

Proof. Let $f \in S_{n+1}^*(\xi,\beta,\gamma)$, and suppose that

$$\frac{z(\Theta^n(\beta,\gamma)f(z))'}{\Theta^n(\beta,\gamma)f(z)} = \xi + (1-\xi)h(z).$$

Since

$$(1+\frac{\beta}{\gamma+1})\frac{\Theta^{n+1}(\beta,\gamma)f(z)}{\Theta^n(\beta,\gamma)f(z)} = \xi + (1-\xi)h(z) + \frac{\beta}{\gamma+1},$$

therefore

$$\frac{z(\Theta^{n+1}(\beta,\gamma)f(z))'}{\Theta^{n+1}(\beta,\gamma)f(z)}-\xi = (1-\xi)h(z)+\frac{(\gamma+1)(1-\xi)zh'(z)}{\beta+(\gamma+1)\xi+(1-\xi)h(z)}.$$

Taking $h(z)=\mu=\mu_1+i\mu_2$ and $zh'(z)=\upsilon=\upsilon_1+i\upsilon_2$, we define $\varphi(\mu,\upsilon)$ by:

$$\begin{aligned}
\varphi(\mu,\upsilon) &= (1-\xi)\mu+\frac{(\gamma+1)(1-\xi)\upsilon}{\beta+(\gamma+1)\xi+(1-\xi)\mu}.\\
\Rightarrow \Re\{\varphi(i\mu_2,\upsilon_1)\} &= \frac{[1+(\gamma+1)\xi](\gamma+1)(1-\xi)\upsilon_1}{(1+(\gamma+1)\xi)^2+(1-\xi)^2\mu_2^2},\\
\Re\{\varphi(i\mu_2,\upsilon_1)\} &\leq -\frac{[1+(\gamma+1)\xi](\gamma+1)(1-\xi)(1+\mu_2^2)}{(1+(\gamma+1)\xi)^2+(1-\xi)^2\mu_2^2}<0.
\end{aligned}$$

Clearly $\varphi(\mu,\upsilon)$ satisfies the conditions of Lemma 2.1. Hence $\Re\{h(z)\}>0(z\in\mathbb{U})$, implies $f\in S_n^*(\xi,\beta,\gamma)$. □

Theorem 2.3. *Let $f\in A$, $0\leq\xi<1$, $\beta,\gamma\geq 0, n\in\mathbb{N}_0$ then*

$$C_{n+1}(\xi,\beta,\gamma)\subseteq C_n(\xi,\beta,\gamma)\subseteq C_{n-1}(\xi,\beta,\gamma).$$

Proof. Let $f\in C_{n+1}(\xi,\beta,\gamma)\Rightarrow\Theta^{n+1}(\beta,\gamma)f\in C(\xi)\Leftrightarrow z(\Theta^{n+1}(\beta,\gamma)f)'\in S^*(\xi)\Rightarrow \Theta^{n+1}(\beta,\gamma)(zf')\in S^*(\xi)\Rightarrow zf'\in S_{n+1}^*(\xi,\beta,\gamma)\subseteq S_n^*(\xi,\beta,\gamma)\Rightarrow zf'\in S_n^*(\xi,\beta,\gamma)\Rightarrow \Theta^n(\beta,\gamma)(zf')\in S^*(\xi)\Rightarrow z(\Theta^n(\beta,\gamma)f)'\in S^*(\xi)\Leftrightarrow\Theta^n(\beta,\gamma)f\in C(\xi)\Rightarrow f\in C_n(\xi,\beta,\gamma)$. □

Theorem 2.4. *Let $f\in A$, $0\leq\xi<1$, $\beta,\gamma\geq 0, 0\leq\rho<1, n\in\mathbb{N}_0$ then*

$$K_{n+1}(\rho,\xi,\beta,\gamma)\subseteq K_n(\rho,\xi,\beta,\gamma)\subseteq K_{n-1}(\rho,\xi,\beta,\gamma).$$

Proof. Let $f\in K_{n+1}(\rho,\xi,\beta,\gamma)$ and suppose that

$$\left(\frac{z(\Theta^n(\beta,\gamma)f(z))'}{\Theta^n(\beta,\gamma)g(z)}\right) = \rho+(1-\rho)h(z), z\in\mathbb{U}.$$

Using (1.1) we have

$$\frac{z(\Theta^{n+1}(\beta,\gamma)f(z))'}{\Theta^{n+1}(\beta,\gamma)g(z)} = \frac{\frac{(\gamma+1)z(\Theta^n(\beta,\gamma)f'(z))'}{\Theta^n(\beta,\gamma)g(z)}+\frac{\beta z(\Theta^n(\beta,\gamma)zf'(z))}{\Theta^n(\beta,\gamma)g(z)}}{\frac{(\gamma+1)z(\Theta^n(\beta,\gamma)g(z))'}{\Theta^n(\beta,\gamma)g(z)}+\beta}.$$

Since $\frac{(\Theta^n(\beta,\gamma)zf'(z))}{\Theta^n(\beta,\gamma)g(z)}=\rho+(1-\rho)h(z)$ and $g(z)\in S_{n+1}^*(\xi,\beta,\gamma)\subseteq S_n^*(\xi,\beta,\gamma)$. Therefore

$$\frac{z(\Theta^{n+1}(\beta,\gamma)f(z))'}{\Theta^{n+1}(\beta,\gamma)g(z)}-\rho = (1-\rho)h(z)+\frac{(\gamma+1)(1-\rho)zh'(z)}{(\gamma+1)(\xi+(1-\xi)H(z))+\beta}.$$

Taking $h(z)=\mu=\mu_1+i\mu_2$ and $zh'(z)=\upsilon=\upsilon_1+i\upsilon_2$, we define $\varphi(\mu,\upsilon)$ by

$$\varphi(\mu,\upsilon) = (1-\rho)\mu+\frac{(\gamma+1)(1-\rho)\upsilon}{(\gamma+1)(\xi+(1-\xi)H(z))+\beta}.$$

This implies

$$\Re[\varphi(i\mu_2,\upsilon_1)]=-\frac{(\gamma+1)(1+\mu_2^2)(1-\rho)[\beta+\xi(\gamma+1)+(\gamma+1)(1-\xi)h_1(x_1,y_1)]}{[\beta+\xi(\gamma+1)+(\gamma+1)(1-\xi)h_1(x_1,y_1)]^2+[(\gamma+1)(1-\xi)h_2(x_1,y_1)]^2}<0.$$

Hence, the function $\varphi(\mu, \upsilon)$ satisfies the conditions of Lemma 2.1. Implies $\Re\{h(z)\} > 0 (z \in \mathbb{U})$ gives $f \in K_n(\rho, \xi, \beta, \gamma)$. □

Similarly we proved the following theorem.

Theorem 2.5. *Let $f \in A$, $0 \le \xi < 1$, $0 \le \rho < 1$, $\beta, \gamma \ge 0, n \in \mathbb{N}_0$ then*

$$K^*_{n+1}(\rho, \xi, \beta, \gamma) \subseteq K^*_n(\rho, \xi, \beta, \gamma) \subseteq K^*_{n-1}(\rho, \xi, \beta, \gamma).$$

3. Integral operator

For $c > -1$ and $f \in A$, the integral operator $L_c(f) : A \to A$ is defined by

$$L_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt. \tag{3.1}$$

The operator $L_c(f)$ was introduced by Bernardi [10].

Theorem 3.1. *Let $c > -1$, $0 \le \xi < 1$. If $f \in S^*_n(\xi, \beta, \gamma)$, then $L_c(f) \in S^*_n(\xi, \beta, \gamma)$.*

Proof. By using (3.1) we get

$$\begin{aligned}
\frac{z(\Theta^n(\beta,\gamma)L_cf(z))'}{\Theta^n(\beta,\gamma)L_cf(z)} &= (c+1)\frac{\Theta^n(\beta,\gamma)f(z)}{\Theta^n(\beta,\gamma)L_cf(z)} - c. \\
\text{Let } \frac{z(\Theta^n(\beta,\gamma)L_cf(z))'}{\Theta^n(\beta,\gamma)L_cf(z)} &= \xi + (1-\xi)h(z),\ h(z) = 1 + c_1z + c_2z^2 + \cdots . \\
\frac{z(\Theta^n(\beta,\gamma)L_cf(z))'}{\Theta^n(\beta,\gamma)L_cf(z)} - \xi &= (1-\xi)h(z) + \frac{(1-\xi)zh'(z)}{\xi + (1-\xi)h(z) + c}.
\end{aligned}$$

This implies

$$\varphi(\mu, \upsilon) = (1-\xi)\mu + \frac{(1-\xi)\upsilon}{\xi + c + (1-\xi)\mu}, \text{ (same as Theorem 2.2)},$$

and

$$\Re[\varphi(i\mu_2, \upsilon_1)] = \frac{(\xi+c)(1-\xi)\upsilon_1}{[\xi+c]^2 + [(1-\xi)\mu_2]^2} \le \frac{-(\xi+c)(1-\xi)(1+\mu_2^2)}{2[\xi+c]^2 + 2[(1-\xi)\mu_2]^2} < 0.$$

After using Theorem 2.1 and Lemma 2.1., we have $L_c(f) \in S^*_n(\xi, \lambda, \alpha, \beta, \mu)$. □

Theorem 3.2. *Let $c > -1$, $0 \le \xi < 1$. If $f(z) \in C_n(\xi, \beta, \gamma)$, then $L_c(f) \in C_n(\xi, \beta, \gamma)$.*

Proof. Proof is same as that of Theorem 2.3. □

Theorem 3.3. *Let $c > -1$, $0 \le \xi < 1$, $0 \le \rho < 1$. If $f(z) \in K_n(\rho, \xi, \beta, \gamma)$, then $L_c(f) \in K_n(\rho, \xi, \beta, \gamma)$.*

Proof. Since $f \in K_n(\xi, \beta, \gamma) \Rightarrow \Theta^n(\beta, \gamma)f \in K(\rho, \xi)$. Let

$$\frac{z(\Theta^n(\beta,\gamma)L_cf(z))'{}'}{\Theta^n(\beta,\gamma)L_cg(z)} = \rho + (1-\rho).$$

Using (3.1) we have

$$\left(\frac{z(\Theta^n(\beta,\gamma)f(z))'}{\Theta^n(\beta,\gamma)g(z)}\right) = \frac{\frac{z(\Theta^n(\beta,\gamma)L_c(zf'(z))'}{\Theta^n(\beta,\gamma)L_cg(z)} + c\frac{(\Theta^n(\beta,\gamma)L_c(zf'(z))}{\Theta^n(\beta,\gamma)L_cg(z)}}{\frac{z(\Theta^n(\beta,\gamma)L_c(g(z))'}{\Theta^n(\beta,\gamma)L_cg(z)} + c}.$$

$$\begin{aligned}\text{Since}\, g(z) \in S_n^*(\xi,\beta,\gamma) &\Rightarrow L_c(g(z)) \in S_n^*(\xi,\beta,\gamma). \text{Let}\\ \frac{z(\Theta^n(\beta,\gamma)L_c(g(z))'}{\Theta^n(\beta,\gamma)L_cg(z)} &= \xi + (1-\xi)H(z),\ \Re(H(z)) > 0.\\ \Rightarrow \left(\frac{z(\Theta^n(\beta,\gamma)f(z))'}{\Theta^n(\beta,\gamma)g(z)}\right) - \rho &= (1-\rho)h(z) + \frac{(1-\rho)zh'(z)}{\xi + (1-\xi)H(z) + c}.\end{aligned}$$

Using method of Theorem 2.3, we get

$$\varphi(\mu,\upsilon) = (1-\rho)\mu + \frac{(1-\rho)\upsilon}{\xi + (1-\xi)H(z) + c}.$$

Taking $h(z) = \mu = \mu_1 + i\mu_2$ and $zh'(z) = \upsilon = \upsilon_1 + i\upsilon_2$, we define the function $\varphi(\mu,\upsilon)$ by:

$$\Re[\varphi(i\mu_2,\upsilon_1)] = -\frac{1}{2}\frac{(1+\mu_2^2)(1-\rho)[(\xi+c)+(1-\xi)h_1(x_1,y_1)]}{[(\xi+c)+(1-\xi)h_1(x_1,y_1)]^2 + [(1-\xi)h_2(x_2,y_2)]^2} < 0.$$

Hence, by using Lemma 2.1. we have $L_c(f) \in K_n(\rho,\xi,\beta,\gamma)$. □

Similarly we proved the following theorem.

Theorem 3.4. *Let $c > -1$, $0 \le \xi < 1$, and $0 \le \rho < 1$. If $f(z) \in K_n^*(\rho,\xi,\beta,\gamma)$, then $L_c(f) \in K_n^*(\rho,\xi,\beta,\gamma)$.*

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Partial sums of harmonic univalent functions

Saurabh Porwal and Kaushal Kishore Dixit

Abstract. In this paper, authors obtain conditions under which the partial sums of the Libera integral operator of functions in the class $HP(\alpha)$, $(0 \leq \alpha < 1)$, consisting of harmonic univalent functions $f = h+\overline{g}$ for which $Re\{h'(z)+g'(z)\} > \alpha$, belong to the similar class $HP(\beta)$, $(0 \leq \beta < 1)$. Further, we improve a recent result on partial sums of functions of bounded turning in [6].

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Keywords: Harmonic, univalent, partial Sums.

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h+\overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|, z \in D$ (see Clunie and Sheil-Small [2]).

Denote by S_H the class of functions $f = h+\overline{g}$ which are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ g(z) = \sum_{k=1}^{\infty} b_k z^k,\ |b_1| < 1. \tag{1.1}$$

For basic results on harmonic functions one may refer to the following standard introductory text book by Duren [3].

Note that S_H reduces to the class S of normalized analytic univalent functions if the co-analytic part of its member is zero. For this class $f(z)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ z \in U. \tag{1.2}$$

For $0 \leq \alpha < 1$, $B(\alpha)$ denote the class of functions of the form (1.2) such that $Re\{f'(z)\} > \alpha$ in U. The functions in $B(\alpha)$ are called functions of bounded turning (cf. [5]).

Recently, Yalcin et al.[13] introduced the subclass $HP(\alpha)$ of S_H consisting of functions f of the form (1.1) satisfying the condition

$$Re\{h'(z) + g'(z)\} > \alpha. \tag{1.3}$$

In [13], $HP^*(\alpha)$ denote the subclass of $HP(\alpha)$ consisting of functions $f = h + \overline{g}$ such that h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \; g(z) = -\sum_{k=1}^{\infty} |b_k| z^k. \tag{1.4}$$

We note that for f of the form (1.2), $HP(\alpha)$ reduces to the class $B(\alpha)$ satisfying the condition $Re\{f'(z)\} > \alpha$ in U.

For f of the form (1.2), the Libera integral operator F is given by

$$F(z) = \frac{2}{z}\int_0^z f(\varsigma)\, d\varsigma = z + \sum_{k=2}^{\infty} \frac{2}{k+1} a_k z^k. \tag{1.5}$$

For $f = h + \overline{g}$ in S_H, where h and g are given by (1.1), the Libera integral operator led us to define integral operator given by

$$F(z) = \frac{2}{z}\int_0^z h(\varsigma)\, d\varsigma + \overline{\frac{2}{z}\int_0^z g(\varsigma)\, d\varsigma} = z + \sum_{k=2}^{\infty} \frac{2}{k+1} a_k z^k + \sum_{k=1}^{\infty} \overline{\frac{2}{k+1} b_k z^k}. \tag{1.6}$$

The nth partial sums $F_n(z)$ of the integral operator $F(z)$ for functions f of the form (1.1) are given by

$$\begin{aligned} F_n(z) &= z + \sum_{k=2}^{n} \frac{2}{k+1} a_k z^k + \sum_{k=1}^{n} \overline{\frac{2}{k+1} b_k z^k}. \\ &= H_n(z) + \overline{G_n(z)}. \end{aligned} \tag{1.7}$$

The nth partial sums $F_n(z)$ of the Libera integral operator $F(z)$ for analytic univalent functions of the form (1.2) have been studied by various authors in ([6], [8]) (See also [1], [7], [9], [10], [11], [12]), yet analogous results on harmonic univalent functions have not been so far explored. Motivated with the work of Jahangiri and Farahmand [6], an attempt has been made to systematically study the partial sums of harmonic univalent functions.

2. Main results

To derive our first main result, we need the following three lemmas. The first lemma is due to Gasper [4], the second is due to Jahangiri and Farahmand [6] and the third is a well-known and celebrated result (cf. [5]) that can be derived from the Herglotz representation for positive real part functions.

Lemma 2.1. *Let θ be a real number and let m and k be natural numbers. Then*

$$\frac{1}{3}+\sum_{k=1}^{m}\frac{\cos(k\theta)}{k+2}\geq 0. \tag{2.1}$$

Lemma 2.2. *For $z\in U$,*

$$Re\left(\sum_{k=1}^{m}\frac{z^k}{k+2}\right)>-\frac{1}{3}. \tag{2.2}$$

Lemma 2.3. *Let $P(z)$ be analytic in U, $P(0)=1$ and $Re(P(z))>\frac{1}{2}$ in U. For functions Q analytic in U, the convolution function $P*Q$ takes values in the convex hull of the image on U under Q.*

The operator "$*$" stands for the Hadamard product or convolution of two power series $f(z)=\sum_{k=0}^{\infty}a_kz^k$ and $g(z)=\sum_{k=0}^{\infty}b_kz^k$ is given by

$$(f*g)(z)=f(z)*g(z)=\sum_{k=0}^{\infty}a_kb_kz^k.$$

Theorem 2.4. *If f of the form (1.1) with $b_1=0$ and $f\in HP(\alpha)$, then $F_n\in HP\left(\frac{4\alpha-1}{3}\right)$, for $\frac{1}{4}\leq\alpha<1$.*

Proof. Let f be of the form (1.1) and belong to $HP(\alpha)$ for $\frac{1}{4}\leq\alpha<1$.

Since

$$Re\{h'(z)+g'(z)\}>\alpha,$$

we have

$$Re\left\{1+\frac{1}{2(1-\alpha)}\left(\sum_{k=2}^{\infty}ka_kz^{k-1}+\sum_{k=2}^{\infty}kb_kz^{k-1}\right)\right\}>\frac{1}{2}. \tag{2.3}$$

Applying the convolution properties of power series to $H_n'(z)+G_n'(z)$, we may write

$$H_n'(z)+G_n'(z)=1+\sum_{k=2}^{n}\frac{2k}{k+1}a_kz^{k-1}+\sum_{k=2}^{n}\frac{2k}{k+1}b_kz^{k-1}$$

$$=\left(1+\frac{1}{2(1-\alpha)}\left(\sum_{k=2}^{\infty}k(a_k+b_k)z^{k-1}\right)\right)*\left(1+(1-\alpha)\sum_{k=2}^{n}\frac{4}{k+1}z^{k-1}\right)$$

$$=P(z)*Q(z). \tag{2.4}$$

From Lemma 2.2 for $m=n-1$, we obtain

$$Re\left(\sum_{k=2}^{n}\frac{z^{k-1}}{k+1}\right)>-\frac{1}{3}. \tag{2.5}$$

By applying a simple algebra to inequality (2.5) and $Q(z)$ in (2.4)), one may obtain

$$Re(Q(z))=Re\left\{1+(1-\alpha)\sum_{k=2}^{n}\frac{4}{k+1}z^{k-1}\right\}>\frac{4\alpha-1}{3}.$$

On the other hand, the power series $P(z)$ in (2.4) in conjunction with the condition (2.3) yields

$$Re\left(P\left(z\right)\right) > \frac{1}{2}.$$

Therefore, by Lemma 2.3, $Re\left\{H_n'\left(z\right) + G_n'\left(z\right)\right\} > \frac{4\alpha-1}{3}$.

This completes the proof of Theorem 2.4. □

If f of the form (1.2) in Theorem 2.4, we obtain the following result of Jahangiri and Farahmand in [6].

Corollary 2.5. *If f of the form (1.2) and $f \in B\left(\alpha\right)$, then $F_n \in B\left(\frac{4\alpha-1}{3}\right)$, for $\frac{1}{4} \leq \alpha < 1$.*

To prove our next theorem, we need the following Lemma due to Yalcin et al. [13].

Lemma 2.6. *Let $f = h + \overline{g}$ be given by (1.4). Then $f \in HP^*\left(\alpha\right)$ if and only if*

$$\sum_{k=2}^{\infty} k\left|a_k\right| + \sum_{k=1}^{\infty} k\left|b_k\right| \leq 1-\alpha,\ 0 \leq \alpha < 1.$$

Theorem 2.7. *Let f be of the form (1.4) with $b_1 = 0$ and $f \in HP^*\left(\alpha\right)$, then the functions $F\left(z\right)$ defined by (1.6) belongs to $HP^*\left(\rho\right)$, where $\rho = \frac{1+2\alpha}{3}$. The result is sharp. Further, the converse need not to be true.*

Proof. Since $f \in HP^*\left(\alpha\right)$, Lemma 2.6 ensures that

$$\sum_{k=2}^{\infty} \frac{k}{1-\alpha}\left(\left|a_k\right| + \left|b_k\right|\right) \leq 1. \tag{2.6}$$

Also, from (1.6) we have

$$F\left(z\right) = z - \sum_{k=2}^{\infty} \frac{2}{k+1}\left|a_k\right| z^k - \sum_{k=2}^{\infty} \frac{2}{k+1}\left|b_k\right| \bar{z}^k.$$

Let $F\left(z\right) \in HP^*\left(\sigma\right)$, then, by Lemma 2.6, we have

$$\sum_{k=2}^{\infty}\left(\frac{k}{1-\sigma}\right)\left(\frac{2}{k+1}\left|a_k\right| + \frac{2}{k+1}\left|b_k\right|\right) \leq 1.$$

Thus we have to find largest value of σ so that the above inequality holds. Now this inequality holds if

$$\sum_{k=2}^{\infty}\left(\frac{k}{1-\sigma}\right)\left(\frac{2}{k+1}\left|a_k\right| + \frac{2}{k+1}\left|b_k\right|\right) \leq \sum_{k=2}^{\infty}\frac{k}{1-\alpha}\left(\left|a_k\right| + \left|b_k\right|\right).$$

or, if

$$\left(\frac{k}{1-\sigma}\right)\frac{2}{k+1} \leq \frac{k}{1-\alpha},\ \text{for each}\quad k = 2, 3, 4......$$

which is equivalent to

$$\sigma \leq \frac{k-1+2\alpha}{k+1} = \rho_k,\ k = 2, 3, 4.......$$

It is easy to verify that ρ_k is an increasing function of k. Therefore, $\rho = \inf_{k\geq 2} \rho_k = \rho_2$ and, hence

$$\rho = \frac{1+2\alpha}{3}.$$

To show the sharpness, we take the function $f(z)$ given by

$$f(z) = z - \frac{(1-\alpha)}{2}|x|z^2 - \frac{(1-\alpha)}{2}|y|\bar{z}^2, \text{ where } |x|+|y| = 1.$$

Then

$$\begin{aligned} F(z) &= z - \frac{(1-\alpha)}{3}|x|z^2 - \frac{(1-\alpha)}{3}|y|\bar{z}^2 \\ &= H(z) + \overline{G(z)} \end{aligned}$$

and therefore

$$\begin{aligned} H'(z) + G'(z) &= 1 - \frac{2(1-\alpha)}{3}|x|z - \frac{2(1-\alpha)}{3}|y|z \\ &= \frac{3 - 2(1-\alpha)(|x|+|y|)z}{3} \\ &= \frac{1+2\alpha}{3}, \text{ for } z \to 1. \end{aligned}$$

Hence, the result is sharp.

We now show that the converse of above theorem need not to be true. To this end, we consider the function

$$F(z) = z - \frac{(1-\sigma)}{3}|x|z^3 - \frac{(1-\sigma)}{3}|y|\bar{z}^3,$$

where

$$|x|+|y| = 1, \ \sigma = \frac{2\alpha+1}{3}.$$

Lemma 2.6 guarantees that $F(z) \in HP^*(\sigma)$.

But the corresponding function

$$f(z) = z - \frac{2(1-\sigma)}{3}|x|z^3 - \frac{2(1-\alpha)}{3}|y|\bar{z}^3,$$

does not belong to $HP^*(\alpha)$, since, for this $f(z)$ the coefficient inequality of Lemma 2.6 is not satisfied. □

In next theorem, we improve the result of Theorem 2.4 for functions f of the form (1.4) for this we need the following Lemma due to Yalcin et al. [13].

Lemma 2.8. *If* $0 \leq \alpha_1 \leq \alpha_2 < 1$, *then*

$$HP^*(\alpha_2) \subseteq HP^*(\alpha_1).$$

Theorem 2.9. *Let f of the form* (1.4) *with* $b_1 = 0$ *and* $f \in HP^*(\alpha)$. *Then the function* $F_n(z)$ *defined by* (1.7) *belong to* $HP^*\left(\frac{2\alpha+1}{3}\right)$.

Proof. Since

$$f\left(z\right)=z-\sum_{k=2}^{\infty}\left|a_k\right|z^k-\sum_{k=2}^{\infty}\left|b_k\right|\bar{z}^k.$$

Then

$$F\left(z\right)=z-\sum_{k=2}^{\infty}\frac{2}{k+1}\left|a_k\right|z^k-\sum_{k=2}^{\infty}\frac{2}{k+1}\left|b_k\right|\bar{z}^k.$$

By using Theorem 2.7, we have

$$F(z)\in HP^{*}(\sigma),\ \text{where } \sigma=\frac{2\alpha+1}{3}.$$

Now

$$F_n\left(z\right)=z-\sum_{k=2}^{n}\frac{2}{k+1}\left|a_k\right|z^k-\sum_{k=2}^{n}\frac{2}{k+1}\left|b_k\right|\bar{z}^k.$$

To show that $F_n\left(z\right)\in HP^{*}\left(\sigma\right)$, we have

$$\begin{aligned}&\sum_{k=2}^{n}\left(\frac{k}{1-\sigma}\right)\left(\frac{2}{k+1}\left|a_k\right|+\frac{2}{k+1}\left|b_k\right|\right)\\ \leq&\sum_{k=2}^{\infty}\left(\frac{k}{1-\sigma}\right)\left(\frac{2}{k+1}\left|a_k\right|+\frac{2}{k+1}\left|b_k\right|\right)\\ \leq&1.\end{aligned}$$

Thus $F_n\left(z\right)\in HP^{*}\left(\sigma\right)$.

In next theorem, we improve a result of Jahangiri and Farahmand in [6] when f has form $f(z)=z-\sum_{k=2}^{\infty}\left|a_k\right|z^k$, for this we need the following Lemma. □

Lemma 2.10. *If* $0\leq\alpha_1\leq\alpha_2<1$, *then*

$$B\left(\alpha_2\right)\subseteq B\left(\alpha_1\right).$$

Proof. The proof of the above lemma is straightforward, so we omit the details. □

Theorem 2.11. *Let* $f(z)=z-\sum_{k=2}^{\infty}\left|a_k\right|z^k$. *If* $f(z)\in B\left(\alpha\right)$, *then*

$$F_n(z)=z-\sum_{k=2}^{n}\frac{2}{k+1}\left|a_k\right|z^k$$

belongs to $B\left(\frac{2\alpha+1}{3}\right)$.

Proof. The proof of this theorem is much akin to that of Theorem 2.9 and therefore we omit the details. □

Remark 2.12. For $\frac{1}{4}\leq\alpha<1$, $f(z)\in B\left(\alpha\right)$ Jahangiri and Farahmand [6] shows that $F_n(z)\in B\left(\frac{4\alpha-1}{3}\right)$ and our result states that $F_n\left(z\right)\in B\left(\frac{2\alpha+1}{3}\right)$.
Since $\frac{2\alpha+1}{3}>\frac{4\alpha-1}{3}$, for $\frac{1}{4}\leq\alpha<1$, and using Lemma 2.10, we have

$$B\left(\frac{2\alpha+1}{3}\right)\subset B\left(\frac{4\alpha-1}{3}\right).$$

Hence our result provides a smaller class in comparison to the class given by Jahangiri and Farahmand [6].

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Subordination results for a class of Bazilević functions with respect to symmetric points

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Abstract. In this paper, using the principle of subordination we introduce the class of Bazilević functions with respect to k-symmetric points. Several subordination results are obtained for this classes of functions involving a certain family of linear operators.

Mathematics Subject Classification (2010): 30C45, 30C50.

Keywords: Analytic functions, Bazilević functions, k-symmetric points, differential operators, differential subordination.

1. Introduction, definitions and preliminaries

Let $\mathcal{H}$ be the class of functions analytic in the open unit disc $\mathcal{U} = \{z : | z |< 1\}$. Let $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ \ldots .$$

Let

$$\mathcal{A}_n = \{f \in \mathcal{H},\ f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots\}$$

and let $\mathcal{A} = \mathcal{A}_1$.

Let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. Also let $\mathcal{P}$ to denote the class of functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (z \in \mathcal{U}),$$

which satisfy the condition $Re(p(z)) > 0$.

We denote by $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{C}^*$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$. One of our favorite reference of the field is [4] which covers most of the topics in a lucid and economical style.

Let the functions $f(z)$ and $g(z)$ be members of $\mathcal{A}$. we say that the function g is subordinate to f (or f is superordinate to g), written $g \prec f$, if there exists a function

w analytic in $\mathcal{U}$, with $w(0) = 0$ and $|w(z)| < 1$ and such that $g(z) = f(w(z))$. In particular, if g is univalent, then $f \prec g$ if $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. Using the concept of subordination of analytic functions, Ma and Minda[6] introduced the class $\mathcal{S}^*(\phi)$ of functions in $\mathcal{A}$ satisfying $\frac{zf'(z)}{f(z)} \prec \phi$ where $\phi \in \mathcal{P}$ with $\phi'(0) > 0$ maps $\mathcal{U}$ onto a region starlike with respect to 1 and symmetric with respect to real axis.

For a fixed non zero positive integer k and $f_k(z)$ defined by the following equality

$$f_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon_k^{-\nu} f(\varepsilon_k^{\nu} z) \qquad \left(\varepsilon_k = \exp\left(\frac{2\pi i}{k}\right)\right), \tag{1.1}$$

a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}_s^{(k)}(\phi)$ if and only if it satisfies the condition

$$\frac{zf'(z)}{f_k(z)} \prec \phi(z) \quad (z \in \mathcal{U}), \tag{1.2}$$

where $\phi \in \mathcal{P}$, the class of functions with positive real part.

Similarly, a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{C}_s^{(k)}(\phi)$ if and only if it satisfies the condition

$$\frac{(zf'(z))'}{f_k'(z)} \prec \phi(z) \quad (z \in \mathcal{U}), \tag{1.3}$$

where $\phi \in \mathcal{P}$, $k \geq 1$ is a fixed positive integer and $f_k(z)$ is defined by equality (1.1). The classes $\mathcal{S}_s^{(k)}(\phi)$ and $\mathcal{C}_s^{(k)}(\phi)$ were introduced and studied by Wang et. al. [11]. Motivated by the class of univalent Bazilević functions, we introduce the following: For $0 \leq \gamma < \infty$, a function $f(z) \in \mathcal{A}$ is said to be in $\mathcal{B}_k(\gamma; \phi)$ if and only if it satisfies the condition

$$\frac{zf'(z)}{[f_k(z)]^{1-\gamma}\,[g_k(z)]^{\gamma}} \prec \phi(z), \quad (z \in \mathcal{U};\, g \in \mathcal{S}_s^{(k)}(\phi)) \tag{1.4}$$

where $\phi \in \mathcal{P}$ and $g_k(z) \neq 0$ for all $z \in \mathcal{U}$ is defined as in (1.1).

For complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ ($\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\, \mathbb{Z}_0^- = 0, -1, -2, \ldots;\, j = 1, \ldots, s$), we define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by

$${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_q)_n}{(\beta_1)_n \ldots (\beta_s)_n} \frac{z^n}{n!}$$

$$(q \leq s+1;\, q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\, z \in \mathcal{U}),$$

where $\mathbb{N}$ denotes the set of positive integers and $(x)_k$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & if\, k = 0 \\ x(x+1)(x+2) \ldots (x+k-1) & if\, k \in \mathbb{N} = \{1, 2, , \ldots\}. \end{cases}$$

Corresponding to a function $\mathcal{G}_{q,s}(\alpha_1, \beta_1; z)$ defined by

$$\mathcal{G}_{q,s}(\alpha_1, \beta_1; z) := z\, {}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z), \tag{1.5}$$

Selvaraj and Karthikeyan in [9] recently introduced the following operator $D^{m,\,q}_{\lambda,\,s}(\alpha_1,\,\beta_1)f\,:\,\mathcal{A}\longrightarrow\mathcal{A}$ by

$$D^{0,\,q}_{\lambda,\,s}(\alpha_1,\,\beta_1)f(z)=f(z)*\mathcal{G}_{q,\,s}(\alpha_1,\,\beta_1;\,z)$$

$$D^{1,\,q}_{\lambda,\,s}(\alpha_1,\,\beta_1)f(z)=(1-\lambda)(f(z)*\mathcal{G}_{q,\,s}(\alpha_1,\,\beta_1;\,z))+\lambda\,z(f(z)*\mathcal{G}_{q,\,s}(\alpha_1,\,\beta_1;\,z))^{'} \quad (1.6)$$

$$D^{m,\,q}_{\lambda,\,s}(\alpha_1,\,\beta_1)f(z)=D^{1,\,q}_{\lambda,\,s}(D^{m-1,\,q}_{\lambda,\,s}(\alpha_1,\,\beta_1)f(z)) \quad (1.7)$$

If f of the form $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, then from (1.6) and (1.7) we may easily deduce that

$$D^{m,\,q}_{\lambda,\,s}(\alpha_1,\,\beta_1)f(z)=z+\sum_{n=2}^{\infty}\big[1+(n-1)\lambda\big]^m\frac{(\alpha_1)_{n-1}\,\ldots\,(\alpha_q)_{n-1}}{(\beta_1)_{n-1}\,\ldots\,(\beta_s)_{n-1}}\,\frac{a_nz^n}{(n-1)!} \quad (1.8)$$

where $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $\lambda\geq 0$. We remark that, for choice of the parameter $m=0$, the operator $D^{m,\,q}_{\lambda,\,s}(\alpha_1,\,\beta_1)f(z)$ reduces to the well-known Dziok- Srivastava operator [1] and for $q=2$, $s=1$; $\alpha_1=\beta_1$, $\alpha_2=1$ and $\lambda=1$, we get the operator introduced by G. Ş. Sălăgean [8]. Also many (well known and new) integral and differential operators can be obtained by specializing the parameters.

Throughout this paper we assume that

$$m,\,q,\,s\in N_0,\quad \varepsilon_k=\exp\left(\frac{2\pi i}{k}\right)$$

and

$$f^{q,\,s}_{k,\,\lambda}(\alpha_1,\,\beta_1;\,m;\,z)=\frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon_k^{-\nu}D^{m,\,q}_{\lambda,\,s}(\alpha_1,\,\beta_1)f(\varepsilon_k^{\nu}z). \quad (1.9)$$

Clearly, for $k=1$, we have

$$f^{q,\,s}_{1,\,\lambda}(\alpha_1,\,\beta_1;\,m;\,z)=D^{m,\,q}_{\lambda,\,s}(\alpha_1,\,\beta_1)f(z).$$

Lemma 1.1. [3]*Let h be convex in $\mathcal{U}$, with $h(0)=a$, $\delta\neq 0$ and $Re\,\delta\geq 0$. If $p\in\mathcal{H}(a,\,n)$ and*

$$p(z)+\frac{zp^{'}(z)}{\delta}\prec h(z),$$

then

$$p(z)\prec q(z)\prec h(z),$$

where

$$q(z)=\frac{\delta}{n\,z^{\delta/n}}\int_0^z h(t)\,t^{(\delta/n)-1}dt.$$

The function q is convex and is the best $(a,\,n)$-dominant.

Lemma 1.2. [7]*Let h be starlike in $\mathcal{U}$, with $h(0)=0$. If $p\in\mathcal{H}(a,\,n)$ satisfies*

$$zp^{'}(z)\prec h(z),$$

then

$$p(z)\prec q(z)=a+n^{-1}\int_0^z h(t)\,t^{-1}\,dt.$$

The function q is convex and is the best $(a,\,n)$-dominant.

Remark 1.3. The Lemma 1.1 for the case of $n=1$ was earlier given by Suffridge [10].

2. Main results

We begin with the following

Theorem 2.1. *Let $f,\, g \in \mathcal{A}$ with $f(z),\ f^{'}(z),\ f_k(z) \neq 0$ and $g_k(z) \neq 0$ for all $z \in \mathcal{U} \setminus \{0\}$. Also let h be convex in $\mathcal{U}$ with $h(0) = 1$ and $Re\, h(z) > 0$. Further suppose that $g \in \mathcal{S}_s^{(k)}(\phi)$ and*

$$\left(\frac{z(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))^{'}}{[f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)]^{1-\gamma}\,[g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)]^{\gamma}}\right)^2\left[3+2\left\{\frac{z(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))^{''}}{(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))^{'}}-\right.\right.$$
$$\left.\left.(1-\gamma)\frac{z\left(f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)\right)^{'}}{f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)}-\gamma\frac{z\left(g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)\right)^{'}}{g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)}\right\}\right]\prec h(z). \tag{2.1}$$

Then

$$\frac{z(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))^{'}}{[f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)]^{1-\gamma}\,[g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)]^{\gamma}}\prec\phi(z)=\sqrt{Q(z)} \tag{2.2}$$

where

$$Q(z)=\frac{1}{z}\int_0^z h(t)\,dt$$

and ϕ is convex and is the best dominant.

Proof. Let

$$p(z)=\frac{z(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))^{'}}{[f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)]^{1-\gamma}\,[g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)]^{\gamma}}\quad(z\in\mathcal{U};\ \gamma\geq 0),$$

then $p(z) \in \mathcal{H}(1,\, 1)$ with $p(z) \neq 0$.

Since h is convex, it can be easily seen that Q is convex and univalent in $\mathcal{U}$. If we make the change of the variables $P(z) = p^2(z)$, then $P(z) \in \mathcal{H}(1,\, 1)$ with $P(z) \neq 0$ in $\mathcal{U}$.

By a straight forward computation, we have

$$\frac{zP^{'}(z)}{P(z)}=2\left[1+\frac{z(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))^{''}}{(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))^{'}}-(1-\gamma)\frac{z\left(f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)\right)^{'}}{f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)}-\right.$$
$$\left.\gamma\frac{z\left(g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)\right)^{'}}{g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)}\right].$$

Thus by (2.1), we have

$$P(z)+zP^{'}(z)\prec h(z)\quad(z\in\mathcal{U}). \tag{2.3}$$

Now by Lemma 1.1, we deduce that

$$P(z)\prec Q(z)\prec h(z).$$

Since $Re\,h(z) > 0$ and $Q(z) \prec h(z)$ we also have $Re\,Q(z) > 0$. Hence the univalence of Q implies the univalence of $\sqrt{Q(z)}$, $p^2(z) \prec Q(z)$ implies that $p(z) \prec \sqrt{Q(z)}$ and the proof is complete. $\square$

Corollary 2.2. *Let $f, g \in \mathcal{A}$ with $f'(z)$, $f_k(z)$ and $g_k(z) \neq 0$ for all $z \in \mathcal{U} \setminus \{0\}$. If $g \in \mathcal{S}_s^{(k)}$ and $Re\,[\Omega(z)] > \eta \quad (0 \leq \eta < 1)$, where*

$$\Omega(z) = \left(\frac{z(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))'}{[f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)]^{1-\gamma}\,[g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)]^{\gamma}} \right)^2$$

$$\left[3 + 2\left\{ \frac{z(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))''}{(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))'} - (1-\gamma)\frac{z\left(f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)\right)'}{f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)} \right.\right.$$
$$\left.\left. -\gamma\frac{z\left(g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)\right)'}{g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)} \right\} \right],$$

then

$$Re\left[\frac{z(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))'}{[f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)]^{1-\gamma}\,[g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)]^{\gamma}} \right] > \lambda(\eta),$$

where $\lambda(\eta) = [2(1-\eta)\cdot\log 2 + (2\eta-1)]^{\frac{1}{2}}$. This result is sharp.

Proof. If we let $h(z) = \dfrac{1+(2\eta-1)z}{1+z}$ $0 \leq \eta < 1$ in Theorem 2.1.
It follows that $Q(z)$ is convex and $Re\,Q(z) > 0$. Therefore

$$\min_{|z|\leq 1} Re\,\sqrt{Q(z)} = \sqrt{Q(1)} = [2(1-\eta)\cdot\log 2 + (2\eta-1)]^{\frac{1}{2}}\,.$$

Hence the proof of the Corollary.

If we let $m = \gamma = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$ in the Corollary 2.2, then we have the following

Corollary 2.3. *Let $f \in \mathcal{A}$ with $f'(z)$ and $f_k(z) \neq 0$ for all $z \in \mathcal{U} \setminus \{0\}$. If*

$$Re\left\{ \left(\frac{zf'(z)}{f_k(z)} \right)^2 \left[3 + \frac{2\,zf''(z)}{f'(z)} - \frac{2\,zf_k'(z)}{f_k(z)} \right] \right\} > \eta,$$

then

$$Re\,\frac{zf'(z)}{f_k(z)} > \lambda(\eta),$$

where $\lambda(\eta) = [2(1-\eta)\cdot\log 2 + (2\eta-1)]^{\frac{1}{2}}$. This result is sharp.

If we let $\gamma = 1$, $m = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$ in the Corollary 2.2, then we have the following

Corollary 2.4. *Let $f,\, g \in \mathcal{A}$ with $f^{'}(z)$ and $g_k(z) \neq 0$ for all $z \in \mathcal{U} \setminus \{0\}$. If $g \in \mathcal{S}_s^{(k)}$ and*

$$Re \left\{ \left(\frac{zf^{'}(z)}{g_k(z)} \right)^2 \left[3 + \frac{2\,zf^{''}(z)}{f^{'}(z)} - \frac{2\,zg_k^{'}(z)}{g_k(z)} \right] \right\} > \eta,$$

then

$$Re\, \frac{zf^{'}(z)}{g_k(z)} > \lambda(\eta),$$

where $\lambda(\eta) = [2(1-\eta)\cdot \log 2 + (2\eta - 1)]^{\frac{1}{2}}$. This result is sharp.

Remark 2.5. If we let $k = 1$ in Corollary 2.4 and in Corollary 2.3, then we have the condition for usual starlikeness and close-to-convex respectively.

Theorem 2.6. *Let $f,\, g \in \mathcal{A}$ with $f(z)$, $f^{'}(z)$ and $g_k(z) \neq 0$ for all $z \in \mathcal{U} \setminus \{0\}$. Further suppose h is starlike with $h(0) = 0$ in the unit disk $\mathcal{U}$, $g \in \mathcal{S}_s^{(k)}(\phi)$ and*

$$\begin{aligned} 1 + \frac{z(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))^{''}}{(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))^{'}} - (1-\gamma)\frac{z\left(f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)\right)^{'}}{f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)} - \\ \gamma\frac{z\left(g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)\right)^{'}}{g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)} \prec h(z) \quad (z \in \mathcal{U};\, \gamma \geq 0). \end{aligned} \tag{2.4}$$

Then

$$\frac{z(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))^{'}}{[f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)]^{1-\gamma}\,[g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)]^{\gamma}} \prec \phi(z) = \exp\left(\int_0^z \frac{h(t)}{t}\,dt\right) \tag{2.5}$$

where ϕ is convex and is the best dominant.

Proof. Let

$$\begin{aligned} \Psi(z) = 1 + \frac{z(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))^{''}}{(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))^{'}} - \\ (1-\gamma)\frac{z\left(f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)\right)^{'}}{f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)} - \gamma\frac{z\left(g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)\right)^{'}}{g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)}. \end{aligned} \tag{2.6}$$

Since $f,\, g \in \mathcal{A}$ with $f^{'}(z)$, $f_k(z)$ and $g_k(z) \neq 0$ for all $z \in \mathcal{U} \setminus \{0\}$, therefore

$$\Psi(z) = z + b_1 z + b_2 z^2 + \ldots.$$

Obviously Ψ is analytic in $\mathcal{U}$. Thus we have

$$\Psi(z) = h(z) \quad (z \in \mathcal{U}).$$

Now by Lemma, we deduce that

$$\int_0^z \frac{\Psi(t)}{t}\,dt \prec \int_0^z \frac{h(t)}{t}\,dt. \tag{2.7}$$

Hence using

$$\frac{\Psi(z)}{z} = \frac{d}{dz}\left[\log\left\{\frac{z(D^{m,\,q}_{\lambda,\,s}(\alpha_1,\,\beta_1)f(z))'}{[f^{q,\,s}_{k,\,\lambda}(\alpha_1,\,\beta_1;\,m;\,z)]^{1-\gamma}\,[g^{q,\,s}_{k,\,\lambda}(\alpha_1,\,\beta_1;\,m;\,z)]^{\gamma}}\right\}\right]$$

in (2.7), we arrive at the desired result. □

If we let $m = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$ in the Theorem 2.6, then we have the following

Corollary 2.7. *Let $f,\, g \in \mathcal{A}$ with $f(z)$, $f'(z)$ and $g_k(z) \neq 0$ for all $z \in \mathcal{U}\setminus\{0\}$. Further suppose h is starlike with $h(0) = 0$ in the unit disk $\mathcal{U}$ and*

$$1 + \frac{zf''(z)}{f'(z)} - (1-\gamma)\frac{zf_k'(z)}{f_k(z)} - \gamma\frac{zg_k'(z)}{g_k(z)} \prec h(z) \quad (z \in \mathcal{U};\, \gamma \geq 0).$$

Then

$$\frac{zf'(z)}{[f_k(z)]^{1-\gamma}\,[g_k(z)]^{\gamma}} \prec \phi(z) = \exp\left(\int_0^z \frac{h(t)}{t}\,dt\right)$$

where ϕ is convex and is the best dominant.

For $k = 1$ in the Corollary 2.7, we get result obtained by Goyal and Goswami in [2].

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On certain subclasses of meromorphic functions with positive coefficients

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Abstract. In this paper we introduce and study a new subclass of meromorphic univalent functions defined by convolution structure. We investigate various important properties and characteristics properties for this class. Further we obtain partial sums for the same.

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Keywords: Meromorphic functions, Hadamard product(or convolution), meromorphic starlike functions, meromorphic convex functions.

1. Introduction

Let $\mathcal{S}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in $\mathbb{U} = \{z : z \in \mathbb{C}, \ \ |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$. Denote by $S^*(\gamma)$ and $K(\gamma)$, $(0 \le \gamma < 1)$ the subclasses of function in $\mathcal{S}$ that are starlike and convex functions of order γ respectively. Analytically, $f \in S^*(\gamma)$ if and only if, f is of the form (1.1) and satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \quad z \in \mathbb{U},$$

similarly, $f \in K(\gamma)$, if and only if, f is of the form (1.1) and satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma, \quad z \in \mathbb{U},$$

Also denote by T the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ \ a_n \ge 0, \tag{1.2}$$

introduced and studied by Silverman [18], let $T^*(\gamma) = T \cap \mathcal{S}^*(\gamma)$, $CV(\gamma) = T \cap K^*(\gamma)$. The classes $T^*(\gamma)$ and $K^*(\gamma)$ possess some interesting properties and have been extensively studied by Silverman [18] and others.

In 1991, Goodman [7, 8] introduced an interesting subclass uniformly convex (uniformly starlike) of the class CV of convex functions (ST starlike functions) denoted by UCV (UST). A function $f(z)$ is uniformly convex (uniformly starlike) in $\mathbb{U}$ if $f(z)$ in $CV(ST)$ has the property that for every circular arc γ contained in $\mathbb{U}$, with center ξ also in $\mathbb{U}$, the arc $f(\gamma)$ is a convex arc (starlike arc) with respect to $f(\xi)$. Motivated by Gooodman [7, 8], Rønning [16, 17] introduced and studied the following subclasses of $\mathcal{S}$. A function $f \in \mathcal{S}$ is said to be in the class $S_p(\gamma, k)$ uniformly $k-$starlike functions if it satisfies the condition

$$\text{Re}\,\left(\frac{zf'(z)}{f(z)} - \gamma\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|,\ (0 \leq \gamma < 1; k \geq 0)\ z \in \mathbb{U} \tag{1.3}$$

and is said to be in the class $UCV(\gamma, k)$, uniformly $k-$convex functions if it satisfies the condition

$$\text{Re}\,\left(1 + \frac{zf''(z)}{f'(z)} - \gamma\right) > k\left|\frac{zf''(z)}{f'(z)}\right|,\ (0 \leq \gamma < 1; k \geq 0)\ z \in \mathbb{U}. \tag{1.4}$$

Indeed it follows from (1.3) and (1.4) that

$$f \in UCV(\gamma, k) \Leftrightarrow zf' \in S_p(\gamma, k). \tag{1.5}$$

Further Ahuja et al.[1], Bharathi et al. [6], Murugusundaramoorthy and Magesh [11] and others have studied and investigated interesting properties for the classes $S_p(\gamma, k)$ and $UCV(\gamma, k)$.

Let Σ denote the class of functions of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n,\ \ a_n \geq 0, \tag{1.6}$$

which are analytic in the punctured open unit disk $\mathbb{U}^* := \{z : z \in \mathbb{C}, 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}$.

Let $\Sigma_{\mathcal{S}}$, $\Sigma^*(\gamma)$ and $\Sigma_K(\gamma)$, $(0 \leq \gamma < 1)$ denote the subclasses of Σ that are meromorphic univalent, meromorphically starlike functions of order γ and meromophically convex functions of order γ respectively. Analytically, $f \in \Sigma^*(\gamma)$ if and only if, f is of the form (1.6) and satisfies

$$-\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma,\quad z \in \mathbb{U},$$

similarly, $f \in \Sigma_K(\gamma)$, if and only if, f is of the form (1.6) and satisfies

$$-\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma,\quad z \in \mathbb{U},$$

and similar other classes of meromorphically univalent functions have been extensively studied by (for example) Altintas et al [2], Aouf [3], Mogra et al. [12], Uralegadi et al. [21, 22, 23] and others(see[10, 13, 14]).

Let $f, g \in \Sigma$, where f is given by (1.6) and g is defined by

$$g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n. \tag{1.7}$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{1.8}$$

Motivated by Ravichandaran et al [15] and Atshan et al [5], now, we define a new subclass $\Sigma^*(g, \gamma, k, \lambda)$ of Σ.

Definition 1.1. *For* $0 \leq \gamma < 1$, $k \geq 0$ *and* $0 \leq \lambda < \frac{1}{2}$, *we let* $\Sigma^*(g, \gamma, k, \lambda)$ *be the subclass of* $\Sigma_{\mathcal{S}}$ *consisting of functions of the form (*1.6*) and satisfying the analytic criterion*

$$\begin{aligned} -\ Re \left(\frac{z(f * g)'(z)}{(f * g)(z)} + \lambda \frac{z^2(f * g)''(z)}{(f * g)(z)} + \gamma \right) \\ > k \left| \frac{z(f * g)'(z)}{(f * g)(z)} + \lambda \frac{z^2(f * g)''(z)}{(f * g)(z)} + 1 \right|. \end{aligned} \tag{1.9}$$

Also by suitably choosing $g(z)$ involved in the class, the class $\Sigma^*(g, \gamma, k, \lambda)$ reduces to various new subclasses. These considerations can fruitfully be worked out and we skip the details in this regard.

The main object of this paper is to study some usual properties of the geometric function theory such as the coefficient bounds, extreme points, radii of meromorphic starlikeness and meromorphic convexity for the class $\Sigma^*(g, \gamma, k, \lambda)$. Further, we obtain partial sums for aforementioned class.

2. Coefficients inequalities

In this section we obtain necessary and sufficient condition for a function f to be in the class $\Sigma^*(g, \gamma, k, \lambda)$. In this connection we state the following lemmas without proof.

Lemma 2.1. *If* γ *is a real number and* $w = -(u + iv)$ *is a complex number, then* $Re\ (w) \geq \gamma \Leftrightarrow |w + (1 - \gamma)| - |w - (1 + \gamma)| \geq 0$.

Lemma 2.2. *If* $w = u + iv$ *is a complex number and* γ, k *are real numbers, then*

$$-\ Re\ (w) \geq k|w + 1| + \gamma \Leftrightarrow\ -\ Re(w(1 + ke^{i\theta}) + ke^{i\theta}) \geq \gamma, \quad -\pi \leq \theta \leq \pi.$$

Theorem 2.3. *Let* $f \in \Sigma$ *be given by (*1.6*). Then* $f \in \Sigma^*(g, \gamma, k, \lambda)$ *if and only if*

$$\sum_{n=1}^{\infty} [n(k+1)(1 + (n-1)\lambda) + (k + \gamma)]\ a_n\ b_n \leq (1 - \gamma) - 2\lambda(k + 1). \tag{2.1}$$

Proof. Let $f \in \Sigma^*(g,\gamma,k,\lambda)$. Then by definition and using Lemma 2.2, it is enough to show that

$$-\operatorname{Re}\left\{\left(\frac{z(f*g)'(z)}{(f*g)(z)}+\lambda\frac{z^2(f*g)''(z)}{(f*g)(z)}\right)(1+ke^{i\theta})+ke^{i\theta}\right\}>\gamma,\quad -\pi\le\theta\le\pi. \tag{2.2}$$

For convenience, we let

$$A(z) := -[z(f*g)'(z)+\lambda z^2(f*g)''(z)](1+ke^{i\theta})-ke^{i\theta}(f*g)(z)$$
$$B(z) := (f*g)(z).$$

That is, the equation (2.2) is equivalent to

$$-\operatorname{Re}\left(\frac{A(z)}{B(z)}\right)\ge\gamma.$$

In view of Lemma 2.1, we only need to prove that

$$|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|\ge 0.$$

Therefore

$$|A(z)+(1-\gamma)B(z)| \ge (2-\gamma-2\lambda(k+1))\frac{1}{|z|} -\sum_{n=1}^{\infty}[n(1+(n-1)\lambda)-(k+1)+k+\gamma-1)]b_na_n|z|^n$$

and

$$|A(z)-(1+\gamma)B(z)|\le(\gamma+2\lambda(k+1))\frac{1}{|z|}+\sum_{n=1}^{\infty}[n(1+(n-1)\lambda)(k+1)+k+\gamma+1]b_na_n|z|^n$$

It is now easy to show that

$$\begin{aligned}&|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|\\ &\ge(2(1-\gamma)-4\lambda(k+1))\frac{1}{|z|}-2\sum_{n=1}^{\infty}[n(1+(n-1)\lambda)(k+1)+(\gamma+k)]b_na_n|z|^n\\ &\ge 0,\end{aligned}$$

by the given condition (2.1). Conversely, suppose $f \in \Sigma^*(g,\gamma,k,\lambda)$. Then by Lemma 2.2, we have (2.2).

Choosing the values of z on the positive real axis the inequality (2.2) reduces to

$$\operatorname{Re}\left\{\frac{(1-\gamma-2\lambda(ke^{i\theta}+1))\frac{1}{z^2}+\sum\limits_{n=1}^{\infty}[n(1+(n-1)\lambda)(1+ke^{i\theta})+(\gamma+ke^{i\theta})]b_na_nz^{n-1}}{\frac{1}{z^2}+\sum\limits_{n=1}^{\infty}b_na_nz^{n-1}}\right\}\ge 0.$$

Since $\operatorname{Re}(-e^{i\theta})\ge-|e^{i\theta}|=-1$, the above inequality reduces to

$$\operatorname{Re}\left\{\frac{(1-\gamma-2\lambda(k+1))\frac{1}{r^2}+\sum\limits_{n=1}^{\infty}[n(1+k)(1+(n-1)\lambda)+(\gamma+k)b_na_nr^{n-1}}{\frac{1}{r^2}+\sum\limits_{n=1}^{\infty}b_na_nr^{n-1}}\right\}\ge 0.$$

Letting $r \to 1^-$ and by the mean value theorem we have desired inequality (2.1). $\square$

Corollary 2.4. *If $f \in \Sigma^*(g, \gamma, k, \lambda)$ then*

$$a_n \leq \frac{(1-\gamma) - 2\lambda(k+1)}{[n(k+1)(1+(n-1)\lambda) + (k+\gamma)]\ b_n}. \tag{2.3}$$

By taking $\lambda = 0$, in Theorem 2.3, we get the following corollary.

Corollary 2.5. *Let $f(z) \in \Sigma$ be given by (1.6). Then $f \in \Sigma(\gamma, k)$ if and only if*

$$\sum_{n=1}^{\infty}[n(k+1) + (k+\gamma)]\ b_n\ a_n \leq (1-\gamma).$$

Next we obtain the growth theorem for the class $\Sigma^*(g, \gamma, k, \lambda)$.

Theorem 2.6. *If $f \in \Sigma^*(g, \gamma, k, \lambda)$ and $b_n \geq b_1 (n \geq 1)$, then*

$$\frac{1}{r} - \frac{(1-\gamma) - 2\lambda(k+1)}{(2k+\gamma+1)\ b_1} r \leq |f(z)| \leq \frac{1}{r} + \frac{(1-\gamma) - 2\lambda(k+1)}{(2k+\gamma+1)\ b_1} r \quad (|z| = r)$$

and

$$\frac{1}{r^2} - \frac{(1-\gamma) - 2\lambda(k+1)}{(2k+\gamma+1)\ b_1} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{(1-\gamma) - 2\lambda(k+1)}{(2k+\gamma+1)\ b_1} \quad (|z| = r).$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{(1-\gamma) - 2\lambda(k+1)}{(2k+\gamma+1)\ b_1} z. \tag{2.4}$$

Proof. Since $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, we have

$$|f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \leq \frac{1}{r} + r\sum_{n=1}^{\infty} a_n. \tag{2.5}$$

Since for $n \geq 1$, $(2k+\gamma+1)\ b_1 \leq [n(k+1) + (k+\gamma)]b_n$, using Theorem 2.3, we have

$$\begin{aligned} (2k+\gamma+1)\ b_1 \sum_{n=1}^{\infty}\ a_n &\leq \sum_{n=1}^{\infty}[n(k+1)(1+(n-1)\lambda) + (k+\gamma)]\ a_n\ b_n \\ &\leq (1-\gamma) - 2\lambda(k+1). \end{aligned}$$

That is,

$$\sum_{n=1}^{\infty} a_n \leq \frac{(1-\gamma) - 2\lambda(k+1)}{(2k+\gamma+1)\ b_1}.$$

Using the above equation in (2.5), we have

$$|f(z)| \leq \frac{1}{r} + \frac{(1-\gamma) - 2\lambda(k+1)}{(2k+\gamma+1)\ b_1} r$$

and

$$|f(z)| \geq \frac{1}{r} - \frac{(1-\gamma) - 2\lambda(k+1)}{(2k+\gamma+1)\ b_1} r.$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)\ b_1} z$. Similarly we have,

$$|f'(z)| \geq \frac{1}{r^2} - \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)\ b_1}$$

and

$$|f'(z)| \leq \frac{1}{r^2} + \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)\ b_1}.$$

□

Let the functions $f_j(z)$ $(j = 1, 2, ..., m)$ be given by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0,\ n \in \mathbb{N}, n \geq 1. \tag{2.6}$$

We state the following closure theorem for the class $\Sigma^*(g, \gamma, k, \lambda)$ without proof.

Theorem 2.7. *Let the function $f_j(z)$ defined by (2.6) be in the class $\Sigma^*(g, \gamma, k, \lambda)$ for every $j = 1, 2, ..., m$. Then the function $f(z)$ defined by*

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,$$

belongs to the class $\Sigma^*(g, \gamma, k, \lambda)$, where $a_n = \frac{1}{m} \sum_{j=1}^{m} a_{n,j}$, $(n = 1, 2, ..)$.

Theorem 2.8. *(Extreme Points) Let*

$$f_0(z) = \frac{1}{z} \text{ and } f_n(z) = \frac{1}{z} + \frac{(1-\gamma)-2\lambda(k+1)}{[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]\ b_n} z^n, \quad (n \geq 1). \tag{2.7}$$

Then $f \in \Sigma^*(g, \gamma, k, \lambda)$, if and only if it can be represented in the form

$$f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z), \ \ (\mu_n \geq 0, \sum_{n=0}^{\infty} \mu_n = 1). \tag{2.8}$$

Proof. Suppose $f(z)$ can be expressed as in (2.8). Then

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \mu_n f_n(z) \\ &= \mu_0 f_0(z) + \sum_{n=1}^{\infty} \mu_n f_n(z) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \mu_n \frac{(1-\gamma)-2\lambda(k+1)}{[n(1+k)(1+(n-1)\lambda)+(k+\gamma)]\ b_n} z^n. \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} \mu_n \frac{(1-\gamma)-2\lambda(k+1)}{[n(1+k)(1+(n-1)\lambda)+(k+\gamma)]\ b_n} \times \frac{[n(1+k)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}{(1-\gamma)-2\lambda(k+1)} z^n$$

$$= \sum_{n=1}^{\infty} \mu_n - 1 = 1 - \mu_0 \leq 1.$$

So by Theorem 2.3, $f \in \Sigma^*(g, \gamma, k, \lambda)$.

Conversely, we suppose $f \in \Sigma^*(g, \gamma, k, \lambda)$. Since

$$a_n \leq \frac{(1-\gamma)-2\lambda(k+1)}{[n(1+k)(1+(n-1)\lambda)+(\gamma+k)]\ b_n}, \quad n \geq 1.$$

We set,

$$\mu_n = \frac{[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}{(1-\gamma)-2\lambda(k+1)} a_n, \quad n \geq 1$$

and $\mu_0 = 1 - \sum\limits_{n=1}^{\infty} \mu_n$. Then we have,

$$f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z) = \mu_0 f_0(z) + \sum_{n=1}^{\infty} \mu_n f_n(z).$$

Hence the results follows. □

3. Radii of meromorphically starlikeness and meromorphically convexity

Theorem 3.1. *Let $f \in \Sigma^*(g, \gamma, k, \lambda)$. Then f is meromorphically starlike of order $\delta (0 \leq \delta < 1)$ in the disc $|z| < r_1$, where*

$$r_1 = \inf_n \left[\left(\frac{1-\delta}{n+2-\delta} \right) \frac{[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}{(1-\gamma)-2\lambda(k+1)} \right]^{\frac{1}{n+1}} \qquad (n \geq 1),$$

The result is sharp for the extremal function $f(z)$ given by (2.7).

Proof. The function $f \in \Sigma^*(g, \gamma, k, \lambda)$ of the form (1.6) is meromorphically starlike of order δ in the disc $|z| < r_1$, if and only if it satisfies the condition

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| < 1 - \delta. \tag{3.1}$$

Since

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq \left| \frac{\sum_{n=1}^{\infty}(n+1)a_n z^{n+1}}{1+\sum_{n=1}^{\infty} a_n z^{n+1}} \right| \leq \frac{\sum_{n=1}^{\infty}(n+1)|a_n||z|^{n+1}}{1-\sum_{n=1}^{\infty}|a_n||z|^{n+1}}.$$

The above expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty} \frac{n+2-\delta}{1-\delta} |a_n|\ |z|^{n-1} < 1.$$

Using the fact, that $f \in \Sigma^*(g,\gamma,k,\lambda)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}{(1-\gamma)-2\lambda(k+1)} a_n \ < 1.$$

We say (3.1) is true if

$$\frac{n+2-\delta}{1-\delta}|z|^{n+1} < \frac{[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}{(1-\gamma)-2\lambda(k+1)}.$$

Or, equivalently,

$$|z|^{n+1} < \frac{(1-\delta)}{(n+2-\delta)} \frac{[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}{(1-\gamma)-2\lambda(k+1)}$$

which yields the starlikeness of the family. □

Theorem 3.2. *Let $f \in \Sigma^*(g,\gamma,k,\lambda)$. Then f is meromorphically convex of order δ $(0 \le \delta < 1)$ in the unit disc $|z| < r_2$, where*

$$r_2 = \inf_n \left[\left(\frac{1-\delta}{n(n+2-\delta)}\right) \frac{[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}{(1-\gamma)-2\lambda(k+1)}\right]^{\frac{1}{n+1}} \qquad (n \ge 1),$$

The result is sharp for the extremal function $f(z)$ given by (2.4).

Proof. The proof is analogous to that of Theorem 3.1, and we omit the details. □

4. Partial sums

Let $f \in \Sigma$ be a function of the form (1.6). Motivated by Silverman [19] and Silvia [20] see also [4], we define the partial sums f_m defined by

$$f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} a_n z^n \quad (m \in \mathbb{N}). \tag{4.1}$$

In this section, we consider partial sums of functions from the class $\Sigma^*(g,\gamma,k,\lambda)$ and obtain sharp lower bounds for the real part of the ratios of f to f_m and f' to f'_m.

Theorem 4.1. *Let $f \in \Sigma^*(g,\gamma,k,\lambda)$ be given by (1.6) and define the partial sums $f_1(z)$ and $f_m(z)$, by*

$$f_1(z) = \frac{1}{z} \ \text{ and } \ f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} |a_n| z^n, \ (m \in \mathbb{N}/\{1\}). \tag{4.2}$$

Suppose also that

$$\sum_{n=1}^{\infty} d_n |a_n| \le 1,$$

where

$$d_n \geq \begin{cases} 1 & \text{for } n = 1, 2, 3, \dots, m \\ \frac{[n(1+k)(1+(n-1)\lambda)+(\gamma+k)]\ b_n}{(1-\gamma)-2\lambda(k+1)} & \text{for } n = m+1, m+2, m+3 \cdots \end{cases}. \tag{4.3}$$

Then $f \in \Sigma^*(g, \gamma, k, \lambda)$. *Furthermore,*

$$Re\left(\frac{f(z)}{f_m(z)}\right) > 1 - \frac{1}{d_{m+1}} \tag{4.4}$$

and

$$Re\left(\frac{f_m(z)}{f(z)}\right) > \frac{d_{m+1}}{1 + d_{m+1}}. \tag{4.5}$$

Proof. For the coefficients d_n given by (4.3) it is not difficult to verify that

$$d_{n+1} > d_n > 1. \tag{4.6}$$

Therefore we have

$$\sum_{n=1}^{m} |a_n| + d_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} d_n |a_n| \leq 1 \tag{4.7}$$

by using the hypothesis (4.3). By setting

$$\begin{aligned} g_1(z) &= d_{m+1}\left(\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right)\right) \\ &= 1 + \frac{d_{m+1} \sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=1}^{m} a_n z^{n-1}}, \end{aligned}$$

then it suffices to show that

$$\text{Re}\ (g_1(z)) \geq 0 \ \ (z \in \mathbb{U}^*)$$

or,

$$\left|\frac{g_1(z) - 1}{g_1(z) + 1}\right| \leq 1 \ \ (z \in \mathbb{U}^*)$$

and applying (4.7), we find that

$$\begin{aligned} \left|\frac{g_1(z) - 1}{g_1(z) + 1}\right| &\leq \frac{d_{m+1} \sum\limits_{n=m+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=1}^{m} |a_n| - d_{m+1} \sum\limits_{n=m+1}^{\infty} |a_n|} \\ &\leq 1, \ \ z \in \mathbb{U}^*, \end{aligned}$$

which readily yields the assertion (4.4) of Theorem 4.1. In order to see that

$$f(z) = \frac{1}{z} + \frac{z^{m+1}}{d_{m+1}} \tag{4.8}$$

gives sharp result, we observe that for $z = re^{i\pi/m}$ that $\frac{f(z)}{f_m(z)} = 1 - \frac{r^{m+2}}{d_{m+1}} \to 1 - \frac{1}{d_{m+1}}$ as $r \to 1^-$.

Similarly, if we take

$$g_2(z) \quad = \quad (1+d_{m+1})\left(\frac{f_m(z)}{f(z)} - \frac{d_{m+1}}{1+d_{m+1}}\right)$$

and making use of (4.7), we deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1+d_{m+1})\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{m}|a_n| - (1-d_{m+1})\sum\limits_{n=m+1}^{\infty}|a_n|}$$

which leads us immediately to the assertion (4.5) of Theorem 4.1. The bound in (4.5) is sharp for each $m \in \mathbb{N}$ with the extremal function $f(z)$ given by (4.8). $\square$

Theorem 4.2. *If $f(z)$ of the form (1.6) satisfies the condition (2.1). Then*

$$Re\left(\frac{f'(z)}{f'_m(z)}\right) \geq 1 - \frac{m+1}{d_{m+1}}$$

and

$$Re\left(\frac{f'_m(z)}{f'(z)}\right) \geq \frac{d_{m+1}}{m+1+d_{m+1}},$$

where

$$d_n \geq \begin{cases} n & \textit{for } n = 2,3,\ldots,m \\ \frac{n[n(1+k)(1+(n-1)\lambda)+(\gamma+k)]\ b_n}{(1-\gamma)-2\lambda(k+1)} & \textit{for } n = m+1, m+2, m+3\cdots \end{cases}.$$

The bounds are sharp, with the extremal function $f(z)$ of the form (2.4).

Proof. The proof is analogous to that of Theorem 4.1, and we omit the details. $\square$

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Inclusion results for four dimensional Cesàro submethods

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Abstract. We define submethods of four dimensional Cesàro matrix. Comparisons between these submethods are established.

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Keywords: Double sequence, P-convergence, four dimensional Cesàro matrix.

1. Introduction

Some equivalance results for Cesàro submethods have been studied by Goffman and Petersen [2], Armitage and Maddox [1] and Osikiewicz [5]. In this paper we consider the same concept for four dimensional Cesàro method $C_1 := (C, 1, 1)$. First we recall some definitions.

A double sequence $[x] = (x_{jk})$ is said to be $P-$*convergent* (i.e., it is convergent in Pringsheim sense) to L if for all $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon)$ such that $|x_{nm} - L| < \varepsilon$ for all $n, m \geq n_0$ [7]. In this case we write $P - \lim_{j,k} x_{jk} = L$. Recall that $[x]$ is bounded if and only if

$$\|x\|_{(\infty,2)} := \sup_{j,k} |x_{jk}| < \infty.$$

By $l_{(\infty,2)}$ we denote the set of all bounded double sequences.
Note that a $P-$*convergent* double sequence need not be in $l_{(\infty,2)}$. Let

$$P - l_{(\infty,2)} := \left\{ [x] = (x_{jk}) : \sup_{n \geq h_1, m \geq h_2} |x_{jk}| < \infty,\ for\ some\ h_1,\ h_2 \in \mathbb{N} \right\}$$

and call it the space of all $P-$*bounded double sequences* where $\mathbb{N}$ denotes the set of all positive integers. If a double sequence is $P-$*convergent* then it is $P-$*bounded* and it is easy to see that $P - \lim [x] [y] = 0$ whenever $P - \lim [x] = 0$ and $[y]$ is $P-$*bounded.*

Let $A = (a_{jk}^{nm})$ be a four dimensional summability matrix and $[x] = (x_{jk})$ be a double sequence. If $[Ax] := \{(Ax)_{nm}\}$ is $P-convergent$ to L then we say $[x]$ is $A-summable$ to L where

$$(Ax)_{nm} := \sum_{j,k} a_{jk}^{nm} x_{jk}, \; for \; all \; n, m \in \mathbb{N}.$$

A is said to be $RH-regular$ if it maps every bounded $P-convergent$ sequence into a $P-convergent$ sequence with the same $P-$limit [3]. Some recent developments concerning the summability by four dimensional matrices may be found in [6].

Recall that four dimensional Cesàro matrix $C_1 = (c_{jk}^{nm})$ is defined by

$$c_{jk}^{nm} = \begin{cases} \frac{1}{nm}, & j \le n \; and \; k \le m \\ 0, & otherwise. \end{cases}$$

The double index sequence $\beta = \beta(n, m)$ is defined as $\beta(n, m) = (\lambda(n), \mu(m))$ where $\lambda(n)$ and $\mu(m)$ are strictly increasing single sequences of positive integers. Let $[x] = (x_{jk})$ be a double sequence. We say $[y] = (y_{jk})$ is a subsequence of $[x]$ if $y_{jk} = x_{\beta(j,k)}$ for all $j, k \in \mathbb{N}$.

Let $\beta(n, m) = (\lambda(n), \mu(m))$ be a double index sequence and $[x] = (x_{jk})$ be a double sequence. Then the Cesàro submethod $C_\beta := (C_\beta, 1, 1)$ is defined to be

$$(C_\beta x)_{nm} = \frac{1}{\lambda(n)\mu(m)} \sum_{(j,k)=(1,1)}^{(\lambda(n),\mu(m))} x_{jk}$$

where $\sum_{(j,k)=(1,1)}^{(\lambda(n),\mu(m))} x_{jk} = \sum_{j=1}^{\lambda(n)} \sum_{k=1}^{\mu(m)} x_{jk}$. Since $\{(C_\beta x)_{nm}\}$ is a subsequence of $\{(Cx)_{nm}\}$, the method C_β is $RH-regular$ for any β.

Let $x = (x_k)$ be a single sequence and $[x^c] = (x_{jk}^c)$, $[x^r] = (x_{jk}^r)$ be two double sequences such that

$$x_{jk}^c = x_j, \text{ for all } k \in \mathbb{N}$$
$$x_{jk}^r = x_k, \text{ for all } j \in \mathbb{N}.$$

It easy to see that the following statements are equivalent:

(a) $\lim x = L$; (b) $P-\lim\,[x^c] = L$; (c) $P-\lim\,[x^r] = L$.

The next result follows easily.

Proposition 1.1. *Let $[x] = (x_{jk})$ be a double sequence such that $x_{jk} = y_j z_k$ for all $j, k \in \mathbb{N}$ where $y = (y_j)$ and $z = (z_k)$ are single sequences (we call such a double sequence as a factorable double sequence). If y, z are convergent to L_1, L_2 respectively then $[x]$ is $P-$convergent to $L_1 L_2$.*

2. Inclusion results

Let A and B two four dimensional summability matrix methods. If every double sequence which is A summable is also B summable to the same limit, then we say B includes A and we write $A \subseteq B$.

In [1] Armitage and Maddox have given an inclusion theorem for submethods of ordinary Cesàro method. Now, we give an analog of that result for four dimensional Cesàro submethods.

Theorem 2.1. *Let* $\beta_1(n,m) = \left(\lambda^{(1)}(n), \mu^{(1)}(m)\right)$ *and* $\beta_2(n,m) = \left(\lambda^{(2)}(n), \mu^{(2)}(m)\right)$ *be two double index sequences.*

i) If $E(\lambda^{(2)})\backslash E\left(\lambda^{(1)}\right)$ *and* $E(\mu^{(2)})\backslash E(\mu^{(1)})$ *are finite sets then* $C_{\beta_1} \subseteq C_{\beta_2}$.

ii) If $C_{\beta_1} \subseteq C_{\beta_2}$ *then* $E(\lambda^{(2)})\backslash E\left(\lambda^{(1)}\right)$ *or* $E(\mu^{(2)})\backslash E(\mu^{(1)})$ *is finite set, where*

$$E\left(\lambda^{(i)}\right) := \left\{\lambda^{(i)}(n) : n \in \mathbb{N}\right\} \text{ and } E\left(\mu^{(i)}\right) := \left\{\mu^{(i)}(m) : m \in \mathbb{N}\right\};\ i=1,2.$$

Proof. i) If $E(\lambda^{(2)})\backslash E\left(\lambda^{(1)}\right)$ and $E(\mu^{(2)})\backslash E(\mu^{(1)})$ are finite then there exists n_0 such that $\left\{\lambda^{(2)}(n) : n \geq n_0\right\} \subset E\left(\lambda^{(1)}\right)$ and $\left\{\mu^{(2)}(m) : m \geq n_0\right\} \subset E(\mu^{(1)})$. Let $n(j)$ and $m(k)$ be two increasing index sequences such that for all $n, m \geq n_0$

$$\lambda^{(2)}(n) = \lambda^{(1)}(n(j)) \text{ and } \mu^{(2)}(m) = \mu^{(1)}(m(k)).$$

Then $P - \lim(C_{\beta_1}x)_{nm} = L$ implies $P - \lim(C_{\beta_1}x)_{n(j),m(k)} = L$. Hence this implies $P - \lim(C_{\beta_2}x)_{nm} = L$.

ii) Suppose that C_{β_1} implies C_{β_2} but that $E(\lambda^{(2)})\backslash E\left(\lambda^{(1)}\right)$ and $E(\mu^{(2)})\backslash E(\mu^{(1)})$ are infinite sets. Then there are strictly increasing sequences $\lambda^{(2)}(n(j))$ and $\mu^{(2)}(m(k))$ such that for all j, $k \in \mathbb{N}$ $\lambda^{(2)}(n(j)) \notin E\left(\lambda^{(1)}\right)$ and $\mu^{(2)}(m(k)) \notin E(\mu^{(1)})$. Define $[t] = (t_{nm})$ by

$$t_{nm=}\begin{cases} jk, & \text{if } n = \lambda^{(2)}(n(j)) \text{ and } m = \mu^{(2)}(m(k)) \\ 0, & \text{otherwise} \end{cases}.$$

Let $(Cs)_{nm} = t_{nm}$, i.e. $\dfrac{1}{nm}\sum\limits_{(j,k)=(1,1)}^{(n,m)} s_{jk} = t_{nm}$. If $n \in E\left(\lambda^{(1)}\right)$ and $m \in E(\mu^{(1)})$ then $t_{nm} = 0$ which implies the sequence $[s]$ is $C_{\beta_1} -$ *summable* to zero. Now we define a double index sequence β_3 as

$$\beta_3 = (\lambda^{(2)}(n(j)), \mu^{(2)}(m(k))).$$

Since

$$\frac{1}{\lambda^{(2)}(n(j))\mu^{(2)}(m(k))} \sum_{(p,q)=(1,1)}^{\left((\lambda^{(2)}(n(j))\mu^{(2)}(m(k)))\right)} s_{pq} = C_{\lambda^{(2)}(n(j)),\mu^{(2)}(m(k))}$$

and $t_{nm} = jk$ for $n \in \left\{\lambda^{(2)}(n(j))\right\}$ and $m \in \left\{\mu^{(2)}(m(k))\right\}$ we have $[s] \notin C_{\beta_3}$ which implies $[s] \notin C_{\beta_2}$. □

Osikiewicz [5] has given a characterization for equivalence of Cesàro method and its submethods. The following theorem is an analog for four dimensional Cesàro method and its submethods.

Theorem 2.2. *Let $\beta = (\lambda(n), \mu(m))$ be a double index sequence.*

i) If

$$\lim_n \frac{\lambda(n+1)}{\lambda(n)} = \lim_m \frac{\mu(m+1)}{\mu(m)} = 1 \tag{2.1}$$

then C_1 and C_β are equivalent for bounded double sequences.

ii) If C_1 and C_β are equivalent for bounded double sequences then

$$\lim_n \frac{\lambda(n+1)}{\lambda(n)} = 1 \text{ or } \lim_m \frac{\mu(m+1)}{\mu(m)} = 1.$$

Proof. i) By Theorem 2.1 we have $C_1 \subseteq C_\beta$. Let $[x] = (x_{jk})$ be a bounded double sequence that is C_β summable to L and assume

$$\lim_n \frac{\lambda(n+1)}{\lambda(n)} = \lim_m \frac{\mu(m+1)}{\mu(m)} = 1.$$

Consider the sets $F_1 = \mathbb{N} \backslash E(\lambda) =: \{\alpha_1(n)\}$ and $F_2 = \mathbb{N} \backslash E(\mu) =: \{\alpha_2(m)\}$.

Case I. If the sets F_1 and F_2 are finite, then Theorem 2.1 implies that $C_\beta \subseteq C_1$.

Case II. Assume F_1 and F_2 are both infinite sets. Then there exists an n_0 such that for $n, m \geq n_0$, $\alpha_1(n) > \lambda(1)$ and $\alpha_2(m) > \mu(1)$. Since $E(\lambda) \cap F_1 = \varnothing$ and $E(\mu) \cap F_2 = \varnothing$, for all $n, m \geq n_0$, there exist $p, q \in \mathbb{N}$ such that $\lambda(p) < \alpha_1(n) < \lambda(p+1)$ and $\mu(q) < \alpha_2(m) < \mu(q+1)$. It can be written that $\alpha_1(n) = \lambda(p) + a$ and $\alpha_2(m) = \mu(q) + b$, where

$$0 < a < \lambda(p+1) - \lambda(p) \text{ and } 0 < b < \mu(q+1) - \mu(q). \tag{2.2}$$

Now define a double index sequence β' as

$$\beta'(n,m) = (\alpha_1(n), \alpha_2(m)).$$

Then for $n, m \geq n_0$,

$$\begin{aligned}
\left|(C_{\beta'}x)_{nm} - (C_\beta x)_{pq}\right| &= \left| \frac{1}{\alpha_1(n)\alpha_2(m)} \sum_{(j,k)=(1,1)}^{(\alpha_1(n),\alpha_2(m))} x_{jk} - \frac{1}{\lambda(p)\mu(q)} \sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(q))} x_{jk} \right| \\
&= \left| \frac{1}{(\lambda(p)+a)(\mu(q)+b)} \sum_{(j,k)=(1,1)}^{(\lambda(p)+a,\mu(q)+b)} x_{jk} - \frac{1}{\lambda(p)\mu(q)} \sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(q))} x_{jk} \right| \\
&= \left| \frac{1}{(\lambda(p)+a)(\mu(q)+b)} \sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(q))} x_{jk} - \frac{1}{\lambda(p)\mu(q)} \sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(q))} x_{jk} \right. \\
&\quad + \frac{1}{(\lambda(p)+a)(\mu(q)+b)} \left\{ \sum_{(j,k)=(1,\mu(q)+1)}^{(\lambda(p),\mu(q)+b)} x_{jk} + \sum_{(j,k)=(\lambda(p)+1,1)}^{(\lambda(p)+a,\mu(q))} x_{jk} \right. \\
&\quad \left. \left. + \sum_{(j,k)=(\lambda(p)+1,\mu(q)+1)}^{(\lambda(p)+a,\mu(q)+b)} x_{jk} \right\} \right| \\
&\leq \|x\|_{(\infty,2)} \sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(q))} \left| \frac{1}{(\lambda(p)+a)(\mu(q)+b)} - \frac{1}{\lambda(p)\mu(q)} \right|
\end{aligned}$$

$$+\|x\|_{(\infty,2)}\frac{b\lambda(p)+a\mu(q)+ab}{(\lambda(p)+a)\,(\mu(q)+b)}$$
$$\leq 2\|x\|_{(\infty,2)}\frac{b\lambda(p)+a\mu(q)+ab}{\lambda(p)\mu(q)}.$$

By 2.2 we have

$$\begin{aligned}\left|(C_{\beta'}x)_{nm}-(C_{\beta}x)_{pq}\right| &\leq 2\|x\|_{(\infty,2)}\frac{b\lambda(p)+a\mu(q)+ab}{\lambda(p)\mu(q)}\\ &\leq 2\|x\|_{(\infty,2)}\left(\frac{\lambda(p+1)\mu(q+1)}{\lambda(p)\mu(q)}-1\right).\end{aligned}\tag{2.3}$$

Since

$$\left|(C_{\beta'}x)_{nm}-L\right|\leq\left|(C_{\beta'}x)_{nm}-(C_{\beta}x)_{pq}\right|+\left|(C_{\beta}x)_{pq}-L\right|$$

it follows from 2.1, 2.3 and Proposition 1.1 that $P-\lim_{n,m}(C_{\beta'}x)_{nm}=L$.

As the double sequence $\{(C_1x)_{nm}\}$ may be partitioned into two subsequences $\left\{(C_{\beta'}x)_{nm}\right\}$ and $\left\{(C_{\beta}x)_{nm}\right\}$, each having the common P-limit L, $[x]$ must be C_1- *summable* to L. Hence $C_\beta\subseteq C_1$.

Case III. Assume F_1 is infinite set and F_2 is finite set and define a double index sequence β' as

$$\beta'(n,m)=(\alpha_1(n),\mu(m))\,.$$

Now using the same argument in Case II with taking $b=0$ we have

$$\left|(C_{\beta'}x)_{nm}-(C_{\beta}x)_{pq}\right|\leq 2\|x\|_{(\infty,2)}\left(\frac{\lambda(p+1)}{\lambda(p)}-1\right).\tag{2.4}$$

Since

$$\left|(C_{\beta'}x)_{nm}-L\right|\leq\left|(C_{\beta'}x)_{nm}-(C_{\beta}x)_{pq}\right|+\left|(C_{\beta}x)_{pq}-L\right|$$

it follows from 2.1, 2.4 and Proposition 1.1 that $P-\lim_{n,m}(C_{\beta'}x)_{nm}=L$.

As the double sequence $\{(C_1x)_{nm}\}$ may be partitioned into two subsequences $\left\{(C_{\beta'}x)_{nm}\right\}$ and $\left\{(C_{\beta}x)_{nm}\right\}$, each having the common P-limit L, $[x]$ must be C_1- *summable* to L. Hence $C_\beta\subseteq C_1$.

Case IV. If F_1 is finite set and F_2 is infinite set, then we can get the proof as in Case III by changing the roles of F_1 and F_2.

Hence for all cases we get $C_\beta\subseteq C_1$.

ii) Assume that $\limsup_n\dfrac{\lambda(n+1)}{\lambda(n)}>1$ and $\limsup_m\dfrac{\mu(m+1)}{\mu(m)}>1$. Then, we choose two strictly increasing sequences of positive integers $n(j)$ and $m(k)$ such that

$$\lim_j\frac{\lambda(n(j)+1)}{\lambda(n(j))}=L_1>1\text{ and }\lim_k\frac{\mu(m(k)+1)}{\mu(m(k))}=L_2>1\tag{2.5}$$

with $\lambda(n(j)+1)-\lambda(n(j))$ and $\mu(m(k)+1)-\mu(m(k))$ are odd. Let I_j and S_k be the intervals $[\lambda(n(j))+1,\lambda(n(j)+1)-1]$ and $[\mu(m(k))+1,\mu(m(k)+1)-1]$, respectively. $|I_j|$ and $|S_k|$ will always be even by the choice of $n(j)$ and $m(k)$, where $|E|$ is the number of the integers in E. If we define a double sequence $[x]$ by $x_{pq}=0$

if $p \in \left[\lambda(n(j))+1, \lambda(n(j))+\frac{|I_j|}{2}\right]$ or $q \in \left[\mu(m(k))+1, \mu(m(k))+\frac{|S_k|}{2}\right]$, $x_{pq}=1$ if $p \in \left(\lambda(n(j))+\frac{|I_j|}{2}, \lambda(n(j)+1)-1\right]$ and $q \in \left(\mu(m(k))+\frac{|S_k|}{2}, \mu(m(k)+1)-1\right]$, $x_{pq}=0$ if $p \notin |I_j|$ or $q \notin |S_k|$ and p or q is odd, $x_{pq}=1$ if $p \notin |I_j|$ or $q \notin |S_k|$ and p and q are even, for $j,k=1,2,....$ Then for given j,k we have $\sum_{(p,q)\in I_j\times S_k} x_{pq} = \frac{|I_j|\,|S_k|}{4}$ and for given n,m we have

$$\left(C_\beta x\right)_{nm} = \frac{1}{\lambda(n)\mu(m)} \sum_{(p,q)=(1,1)}^{(\lambda(n),\mu(m))} x_{pq} = \frac{1}{\lambda(n)\mu(m)} \left[\left|\frac{\lambda(n)}{2}\right|\right] \left[\left|\frac{\mu(m)}{2}\right|\right]$$

where $[|K|]$ denotes the greatest integer that is not greater than K. Hence, we have $P-\lim_{n,m} \left(C_\beta x\right)_{nm} = \frac{1}{4}$. Now define a double index sequence $\sigma(j,k)$ by

$$\sigma(j,k) = (a(j), b\,(k))$$

where $a(j) = \lambda(n(j)) + \frac{|I_j|}{2}$ and $b(k) = \mu(m(k)) + \frac{|S_k|}{2}$. For all j we get

$$\left(C_\sigma x\right)_{jk} = \frac{1}{a(j)b\,(k)} \sum_{(p,q)=(1,1)}^{(a(k),b(k))} x_{pq}$$

$$= \frac{1}{\left(\lambda(n(j))+\frac{|I_j|}{2}\right)} \frac{1}{\left(\mu(m(k))+\frac{|S_k|}{2}\right)} \sum_{(p,q)=(1,1)}^{\left(\lambda(n(j))+\frac{|I_j|}{2}, \mu(m(k))+\frac{|S_k|}{2}\right)} x_{pq}$$

$$= \frac{1}{\left(\lambda(n(j))+\frac{|I_j|}{2}\right)} \frac{1}{\left(\mu(m(k))+\frac{|S_k|}{2}\right)} \sum_{(p,q)=(1,1)}^{(\lambda(n(j)),\mu(m(k)))} x_{pq}$$

$$\approx \frac{1}{\left(\lambda(n(j))+\frac{|I_j|}{2}\right)} \frac{1}{\left(\mu(m(k))+\frac{|S_k|}{2}\right)} \frac{\lambda(n(j))}{2} \frac{\mu(m(k))}{2}$$

$$= \frac{\lambda(n(j))}{2\lambda(n(j))+|I_j|} \frac{\mu(m(k))}{2\mu(m(k))+|S_k|}$$

$$= \frac{\lambda(n(j))}{2\lambda(n(j))+\lambda(n(j)+1)-\lambda(n(j))-1} \frac{\mu(m(k))}{2\mu(m(k))+\mu(m(k)+1)-\mu(m(k))-1}$$

$$= \frac{1}{\frac{\lambda(n(j)+1)}{\lambda(n(j))}+1-\frac{1}{\lambda(n(j))}} \frac{1}{\frac{\mu(m(k)+1)}{\mu(m(k))}+1-\frac{1}{\mu(m(k))}}.$$

From 2.5 and Proposition 1.1 we have

$$P-\lim_{j,k} \left(C_\sigma x\right)_{jk} = \frac{1}{L_1+1} \frac{1}{L_2+1} < \frac{1}{4}.$$

Since $\left\{(C_\sigma x)_{jk}\right\}$ and $\left\{(C_\beta x)_{nm}\right\}$ are two subsequences of $\left\{(C_1 x)_{nm}\right\}$ with $P-\lim_{j,k}(C_\sigma x)_{jk} < \frac{1}{4}$ and $P-\lim_{n,m}(C_\beta x)_{nm} = \frac{1}{4}$, $[x]$ cannot be $C_1 - summable$.On the other hand, we may choose $n(j)$ and $m(k)$ such that

$$\lambda(n(j)+1)-\lambda(n(j)) \text{ and } \mu(m(k)+1)-\mu(m(k)) \text{ are even}$$

or

$$\lambda(n(j)+1)-\lambda(n(j)) \text{ is odd and } \mu(m(k)+1)-\mu(m(k)) \text{ is even}$$

or

$$\lambda(n(j)+1)-\lambda(n(j)) \text{ is even and } \mu(m(k)+1)-\mu(m(k)) \text{ is odd}$$

and we will continue the proof in the same way. Hence, we have C_1 and C_β are not equivalent for bounded sequences. $\square$

Osikiewicz [5] has given an inclusion result between submethods of the ordinary Cesàro method. The following theorem gives similar results for four dimensional Cesàro submethods.

Theorem 2.3. *Let $\beta_1(n,m) = (\lambda^{(1)}(n), \mu^{(1)}(m))$ and $\beta_2(n,m) = (\lambda^{(2)}(n), \mu^{(2)}(m))$ be two double index sequences such that*

$$P-\lim_{nm}\frac{\lambda^{(1)}(n)\mu^{(1)}(m)}{\lambda^{(2)}(n)\mu^{(2)}(m)} = 1$$

then C_{β_1} and C_{β_2} are equivalent for bounded double sequences.

Proof. Let $[x]$ be a bounded double sequence, and define two double sequences $T(n,m)$ and $t(n,m)$ by

$$T(n,m) = \max\left\{\lambda^{(1)}(n)\mu^{(1)}(m), \lambda^{(2)}(n)\mu^{(2)}(m)\right\}$$

and

$$t(n,m) = \min\left\{\lambda^{(1)}(n)\mu^{(1)}(m), \lambda^{(2)}(n)\mu^{(2)}(m)\right\}.$$

It is easy to see that $P-\lim_{nm}\frac{t(n,m)}{T(n,m)} = 1$. Now define two double index sequences $T^*(n,m) = (T_1(n), T_2(m))$ and $t^*(n,m) = (t_1(n), t_2(m))$ by

$$T^*(n,m) = \begin{cases} \left(\lambda^{(1)}(n), \mu^{(1)}(m)\right), & \lambda^{(1)}(n)\mu^{(1)}(m) = T(n,m) \\ \left(\lambda^{(2)}(n), \mu^{(2)}(m)\right), & \lambda^{(2)}(n)\mu^{(2)}(m) = T(n,m) \end{cases}$$

and

$$t^*(n,m) = \begin{cases} \left(\lambda^{(1)}(n), \mu^{(1)}(m)\right), & \lambda^{(1)}(n)\mu^{(1)}(m) = t(n,m) \\ \left(\lambda^{(2)}(n), \mu^{(2)}(m)\right), & \lambda^{(2)}(n)\mu^{(2)}(m) = t(n,m). \end{cases}$$

Note that $T(n,m) = T_1(n)T_2(m)$ and $t(n,m) = t_1(n)t_2(m)$. Then for fixed n,m we get

$$
\begin{aligned}
\left|(C_{\beta_1}x)_{nm} - (C_{\beta_2}x)_{nm}\right| &= \left| \frac{1}{\lambda^{(1)}(n)\mu^{(1)}(m)} \sum_{(j,k)=(1,1)}^{\left(\lambda^{(1)}(n),\mu^{(1)}(m)\right)} x_{jk} - \right. \\
&\quad \left. \frac{1}{\lambda^{(2)}(n)\mu^{(2)}(m)} \sum_{(j,k)=(1,1)}^{\left(\lambda^{(2)}(n),\mu^{(2)}(m)\right)} x_{jk} \right| \\
&= \left| \frac{1}{T(n,m)} \sum_{(j,k)=(1,1)}^{T^*(n,m)} x_{jk} - \frac{1}{t(n,m)} \sum_{(j,k)=(1,1)}^{t^*(n,m)} x_{jk} \right| \\
&= \left| \sum_{(j,k)=(1,1)}^{t^*(n,m)} \left(\frac{1}{T(n,m)} - \frac{1}{t(n,m)} \right) x_{jk} \right. \\
&\quad + \frac{1}{T(n,m)} \left\{ \sum_{(j,k)=(t_1(n)+1,1)}^{(T_1(n),t_2(m))} x_{jk} + \sum_{(j,k)=(1,t_2(m)+1)}^{(t_1(n),T_2(m))} x_{jk} \right. \\
&\quad \left. \left. + \sum_{(j,k)=(t_1(n)+1,t_2(m)+1)}^{(T_1(n),T_2(m))} x_{jk} \right\} \right| \\
&\le \|x\|_{(\infty,2)} \sum_{(j,k)=(1,1)}^{t^*(n,m)} \frac{T_1(n)T_2(m) - t_1(n)t_2(m)}{T_1(n)T_2(m)t_1(n)t_2(m)} \\
&\quad + \|x\|_{(\infty,2)} \frac{1}{T_1(n)T_2(m)} \left\{ \left(T_1(n) - t_1(n)\right) t_2(m) \right. \\
&\quad \left. + t_1(n) \left(T_2(m) - t_2(m)\right) + \left(T_1(n) - t_1(n)\right)\left(T_2(m) - t_2(m)\right) \right\} \\
&= 2\|x\|_{(\infty,2)} \frac{T_1(n)T_2(m) - t_1(n)t_2(m)}{T_1(n)T_2(m)} \\
&= 2\|x\|_{(\infty,2)} \left(1 - \frac{t_1(n)t_2(m)}{T_1(n)T_2(m)}\right) \\
&= 2\|x\|_{(\infty,2)} \left(1 - \frac{t(n,m)}{T(n,m)}\right). \qquad (2.6)
\end{aligned}
$$

Since

$$\left|(C_{\beta_1}x)_{nm} - L\right| \le \left|(C_{\beta_1}x)_{nm} - (C_{\beta_2}x)_{nm}\right| + \left|(C_{\beta_2}x)_{nm} - L\right|,$$

2.6 implies that $[x]$ is C_{β_1} summable to L provided that $[x]$ is C_{β_2} summable to L. Hence, C_{β_1} is equivalent to C_{β_2} for bounded double sequences. □

We have compared C_β and C_1 for bounded double sequences in Theorem 2.2. Next, replacing the convergence condition in 2.1 by $P-$*boundedness*, we show that

C_β is equivalent to C_1 for nonnegative double sequences that are $C_\beta - summable$ to 0.

Theorem 2.4. *Let $\beta = (\lambda(n), \mu(m))$ be a double index sequence. Then the following statements are equivalent:*

i) The double sequence $[y] = (y_{nm})$ defined by

$$y_{nm} = \left(\frac{\lambda(n+1)\mu(m+1)}{\lambda(n)\mu(m)}\right), \ for\ all\ n, m \in \mathbb{N} \tag{2.7}$$

is $P - bounded$.

ii) $[x]$ is C_1 summable to 0 where $[x]$ is a nonnegative double sequence that is C_β summable to 0.

Proof. Let $[x]$ is a nonnegative double sequence that is C_β summable to 0 and assume that the double sequence $[y]$ defined by 2.7 is $P - bounded$. Consider the sets

$$F_1 = \mathbb{N}\backslash E(\lambda) =: \{\alpha_1(n)\} \text{ and } F_2 = \mathbb{N}\backslash E(\mu) =: \{\alpha_2(m)\}.$$

Case I. If the sets F_1 and F_2 are finite then Theorem 2.1 implies that $C_\beta \subseteq C_1$.
Case II. Assume F_1 and F_2 are both infinite sets. Then there exists an n_0 such that for $n, m \geq n_0$, $\alpha_1(n) > \lambda(1)$ and $\alpha_2(m) > \mu(1)$. Since $E(\lambda) \cap F_1 = \varnothing$ and $E(\mu) \cap F_2 = \varnothing$, for all $n, m \geq n_0$, there exist $p, q \in \mathbb{N}$ such that $\lambda(p) < \alpha_1(n) < \lambda(p+1)$ and $\mu(q) < \alpha_2(m) < \mu(q+1)$. It can be written that $\alpha_1(n) = \lambda(p)+a$ and $\alpha_2(m) = \mu(q)+b$, where

$$0 < a < \lambda(p+1) - \lambda(p) \ and\ 0 < b < \mu(q+1) - \mu(q). \tag{2.8}$$

Now define a double index sequence β' as

$$\beta'(n, m) = (\alpha_1(n), \alpha_2(m)).$$

Then for $n, m \geq n_0$ we have,

$$\begin{aligned}
(C_{\beta'}x)_{nm} &= \frac{1}{\alpha_1(n)\alpha_2(m)} \sum_{(j,k)=(1,1)}^{(\alpha_1(n),\alpha_2(m))} x_{jk} = \frac{1}{(\lambda(p)+a)(\mu(q)+b)} \sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(q))} x_{jk} \\
&+ \frac{1}{(\lambda(p)+a)(\mu(q)+b)} \left\{ \sum_{(j,k)=(1,\mu(q)+1)}^{(\lambda(p),\mu(q)+b)} x_{jk} + \sum_{(j,k)=(\lambda(p)+1,1)}^{(\lambda(p)+a,\mu(q))} x_{jk} + \right. \\
&\left. \sum_{(j,k)=(\lambda(p)+1,\mu(q)+1)}^{(\lambda(p)+a,\mu(q)+b)} x_{jk} \right\} \\
&\leq \frac{1}{\lambda(p)\mu(q)} \sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(q))} x_{jk} + \frac{3}{(\lambda(p)+a)(\mu(q)+b)} \sum_{(j,k)=(1,1)}^{(\lambda(p+1),\mu(q+1))} x_{jk} \\
&\leq \frac{1}{\lambda(p)\mu(q)} \sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(q))} x_{jk}
\end{aligned}$$

$$+3\frac{\lambda(p+1)\mu(q+1)}{(\lambda(p)+a)\,(\mu(q)+b)}\frac{1}{\lambda(p+1)\mu(q+1)}\sum_{(j,k)=(1,1)}^{(\lambda(p+1),\mu(q+1))}x_{jk}$$

$$\leq (C_\beta x)_{pq}+3\frac{\lambda(p+1)\mu(q+1)}{\lambda(p)\mu(q)}(C_\beta x)_{p+1,q+1}. \tag{2.9}$$

Since $P-\lim[x]=0$ and $[y]$ is $P-\textit{bounded}$, from 2.9 we get

$$P-\lim_{n,m}(C_{\beta'}x)_{nm}=0.$$

As the double sequence $\{(C_1x)_{nm}\}$ may be partitioned into two subsequences $\{(C_{\beta'}x)_{nm}\}$ and $\{(C_\beta x)_{nm}\}$, each having the common P-limit 0, $[x]$ must be $C_1-\textit{summable}$ to 0.

Case III. Assume F_1 is infinite set and F_2 is finite set and define a double index sequence β' as

$$\beta'(n,m)=(\alpha_1(n),\mu(m)).$$

Then for all $n\geq n_0$ and for all $m\in\mathbb{N}$

$$\begin{aligned}
(C_{\beta'}x)_{nm} &= \frac{1}{\alpha_1(n)\mu(m)}\sum_{(j,k)=(1,1)}^{(\alpha_1(n),\mu(m))}x_{jk}\\
&= \frac{1}{(\lambda(p)+a)\,\mu(m)}\left\{\sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(m))}x_{jk}+\sum_{(j,k)=(\lambda(p)+1,1)}^{(\lambda(p)+a,\mu(m))}x_{jk}\right\}\\
&\leq \frac{1}{\lambda(p)\mu(m)}\sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(m))}x_{jk}+\frac{1}{(\lambda(p)+a)\,\mu(m)}\sum_{(j,k)=(1,1)}^{(\lambda(p+1),\mu(m+1))}x_{jk}\\
&\leq \frac{1}{\lambda(p)\mu(m)}\sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(m))}x_{jk}\\
&\quad+\frac{\lambda(p+1)\mu(m+1)}{(\lambda(p)+a)\,\mu(m)}\frac{1}{\lambda(p+1)\mu(m+1)}\sum_{(j,k)=(1,1)}^{(\lambda(p+1),\mu(m+1))}x_{jk}\\
&\leq (C_\beta x)_{pq}+\frac{\lambda(p+1)\mu(m+1)}{\lambda(p)\mu(m)}(C_\beta x)_{p+1,m+1}.
\end{aligned}$$

Then as in Case II we have $P-\lim_{n,m}(C_1x)_{nm}=0$.

Case IV. If F_1 is finite set and F_2 is infinite set, then we can get the proof as in Case III by interchanging the roles of F_1 and F_2.

Conversely assume that $[y]$ is not $P-\textit{bounded}$. Then there exist two index sequences $n(j)$ and $m(k)$ such that

$$P-\lim\frac{\lambda(n(j)+1)\mu(m(k)+1)}{\lambda(n(j))\mu(m(k))}=\infty, \tag{2.10}$$

$$\lambda(n(j)+1)>2\lambda(n(j)) \text{ and } \mu(m(k)+1)>2\mu(m(k)), \tag{2.11}$$

for all a, $b \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$

$$\lim_j \frac{\lambda(n(j))}{\lambda(n(j+a))} = 1 \text{ and } \lim_k \frac{\mu(m(k))}{\mu(m(k+b))} = 1 \tag{2.12}$$

and for all $s \in \mathbb{N}$

$$\lambda(n(1))\mu(m(1)) + ... + \lambda(n(s-1))\mu(m(s-1)) < \lambda(n(s))\mu(m(s)).$$

Now define the double sequence $[x] = (x_{pq})$ by

$$x_{pq} = \begin{cases} 1, & \begin{cases} p \in (\lambda(n(t)), 2\lambda(n(t))] \\ \quad and \\ q \in (\mu(m(t)), 2\mu(m(t))] \end{cases} \quad t = 1, 2, ... \\ 0, & \quad otherwise \end{cases}.$$

For fixed n, m such that $\lambda(n(1)+1) \leq \lambda(n)$ and $\mu(m(1)+1) \leq \mu(m)$, there exist j, k such that $\lambda(n(j)+1) \leq \lambda(n) \leq \lambda(n(j+1))$ and $\mu(m(k)+1) \leq \mu(m) \leq \mu(m(k+1))$. Then we have,

$$(C_\beta x)_{nm} = \frac{1}{\lambda(n)\mu(m)} \sum_{(p,q)=(1,1)}^{(\lambda(n),\mu(m))} x_{pq}$$

$$= \frac{1}{\lambda(n)\mu(m)} \left\{ \sum_{(p,q)=(\lambda(n(1))+1,\mu(m(k))+1)}^{(2\lambda(n(1)),2\mu(m(1)))} 1 + ... + \sum_{(p,q)=(\lambda(n(i))+1,\mu(m(i))+1)}^{(2\lambda(n(i)),2\mu(m(i)))} \right\}$$

$$= \frac{\lambda(n(i)+1)\mu(m(i)+1)}{\lambda(n)\mu(m)} \frac{\lambda(n(1))\mu(m(1)) + ... + \lambda(n(i))\mu(m(i))}{\lambda(n(i)+1)\mu(m(i)+1)}$$

$$\leq \frac{\lambda(n(i)+1)\mu(m(i)+1)}{\lambda(n)\mu(m)} \frac{2\lambda(n(i))\mu(m(i))}{\lambda(n(i)+1)\mu(m(i)+1)}$$

$$\leq \frac{2\lambda(n(i))\mu(m(i))}{\lambda(n(i)+1)\mu(m(i)+1)} \tag{2.13}$$

where $i = \min\{j, k\}$. Hence, from 2.10 and 2.13 we get

$$P - \lim_{n,m} (C_\beta x)_{nm} = 0. \tag{2.14}$$

Now let $\beta'(j,k) = (\alpha(j), \gamma(k))$ be a double index sequences where

$$\alpha(j) = 2\lambda(n(j)) \text{ and } \gamma(k) = 2\mu(m(k)).$$

Then we get

$$
\begin{aligned}
(C_{\beta'}x)_{jk} &= \frac{1}{\alpha(j)\gamma(k)} \sum_{(j,k)=(1,1)}^{(\alpha(j),\gamma(k))} x_{jk} \\
&= \frac{1}{4\lambda(n(j))\mu(m(k))} \sum_{(j,k)=(1,1)}^{(2\lambda(n(j)),2\mu(m(k)))} x_{jk} \\
&= \frac{1}{4\lambda(n(j))\mu(m(k))} \left\{ \sum_{(j,k)=(1,1)}^{(\lambda(n(j)),\mu(m(k)))} x_{jk} + \sum_{(j,k)=(\lambda(n(j))+1,1)}^{(2\lambda(n(j)),\mu(m(k)))} x_{jk} \right. \\
&\quad \left. + \sum_{(j,k)=(1,\mu(m(k))+1)}^{(\lambda(n(j)),2\mu(m(k)))} x_{jk} + \sum_{(j,k)=(\lambda(n(j))+1,\mu(m(k))+1)}^{(2\lambda(n(j)),2\mu(m(k)))} x_{jk} \right\} \\
&= \frac{1}{4}\left(C_{\beta}x\right)_{jk} + \frac{3}{4}\frac{\lambda(n(i))\mu(m(i))}{\lambda(n(j))\mu(m(k))}.
\end{aligned}
$$

Since $i = \min\{j,k\}$, there exist nonnegative integers a, b such that $i = j + a$ and $i = k + b$. Then 2.12, 2.14 and Proposition 1.1 imply that $P - \lim_{j,k} \left(C_{\beta'}x\right)_{jk} = \frac{3}{4}$.

Hence $[x]$ is not C_1 summable. □

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Rate of convergence for Szász type operators including Sheffer polynomials

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Abstract. In the present paper, we study the rate of convergence of Szász type and Kantorovich-Szász type operators involving Sheffer polynomials with the help of modulus of continuity and examine these type operators including reverse Bessel polynomials which are Sheffer type. Furthermore, we compute error estimation for a function f by operators containing reverse Bessel polynomials.

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1. Introduction

One of the fundamental problems of approximation theory is to approximate function f by functions which have better properties than f. In 1953, Korovkin [4] discovered the most powerful and simplest criterion for positive approximation processes. This theory has widely affected not only classical approximation theory but also such other areas of mathematics as partial differential equations, harmonic analysis, orthogonal polynomials and wavelet analysis.

In 1950, Szász [7] introduced and exhaustively investigated the operator

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right) \tag{1.1}$$

defined on the set of real valued function on $[0,\infty)$.

Jakimovski and Leviatan [3] presented a new type of operators which involves Appell polynomials as follows

$$P_n(f;x) := \frac{e^{-nx}}{A(1)}\sum_{k=0}^{\infty}p_k(nx)\,f\left(\frac{k}{n}\right)\ . \tag{1.2}$$

In the above relation p_k are Appell polynomials defined by the generating functions

$$A(u)\, e^{ux} = \sum_{k=0}^{\infty} p_k(x)\, u^k$$

where $A(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ $(a_0 \neq 0)$ be an analytic function in the disc $|z| < R$ $(R > 1)$ and suppose $A(1) \neq 0$. They obtained important results analogue to Szász [7]. If we take $A(z) = 1$, we get Szász operators (1.1) by using above generating functions.

Ismail [2] generalized Jakimovski and Leviatan's work by dealing with the approximation of Szász operators with the help of Sheffer polynomials as follows

$$T_n(f;x) := \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx)\, f\left(\frac{k}{n}\right) \tag{1.3}$$

whenever the right hand side of (1.3) exists. In the relation (1.3) p_k are Sheffer polynomials given by the generating functions

$$A(u)\, e^{xH(u)} = \sum_{k=0}^{\infty} p_k(x)\, u^k \tag{1.4}$$

where

$$\begin{aligned} A(z) &= \sum_{k=0}^{\infty} a_k z^k \qquad (a_0 \neq 0) \\ H(z) &= \sum_{k=1}^{\infty} h_k z^k \qquad (h_1 \neq 0) \end{aligned} \tag{1.5}$$

be analytic functions in the disc $|z| < R$ $(R > 1)$. Under the restrictions:

$$\begin{array}{ll} \text{(i)} & \text{For } x \in [0,\infty) \text{ and } k \in \mathbb{N} \cup \{0\}, p_k(x) \geq 0, \\ \text{(ii)} & A(1) \neq 0 \text{ and } H'(1) = 1, \\ \text{(iii)} & \text{(1.4) relation is valid for } |u| < R \text{ and} \\ & \text{the power series given by (1.5) converges for } |z| < R\,(R > 1), \end{array} \tag{1.6}$$

Ismail showed that the same type of results which are obtained by Jakimovski and Leviatan are still valid for the operators including Sheffer polynomials known as more general class of polynomials than Appell polynomials. It is clear that the operators (1.3) contain (1.1) and (1.2). Furthermore, Ismail introduced Kantorovich generalization of the operators (1.3) as

$$T_n^*(f;x) := n\frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) \int_{k/n}^{(k+1)/n} f(s)\, ds\,. \tag{1.7}$$

Ismail proved that for $f \in C[0,\infty)$, $|f(x)| \leq ce^{Kx}$ $K \in \mathbb{R}$ and $c \in \mathbb{R}^+$, the operators $T_n(f;.)$ converge to the function f in each compact subset of $[0,\infty)$ by using the methods of Szász [7]. And he also investigated that the operators T_n^* converge for the functions $f \in C[0,\infty)$, $\int\limits_0^t f(s)\, ds = \mathcal{O}\left(e^{Kt}\right)$, $(t \to \infty)$.

As it is known, there are two main problems in approximation theory. One of them is existence of approximation, the other is rate of convergence.

The purpose of the present paper is to study the rate of approximation of the sequences of operators T_n and T_n^* by means of the modulus of continuity. Moreover, since these operators are of general form, we give an example of these type operators T_n and T_n^* including reverse Bessel polynomials [5] and obtain error estimation for operators T_n including reverse Bessel polynomials with the help of Maple13.

2. Approximation properties of T_n and T_n^* operators

We begin by considering the following definition of the class E as follows:

$$E := \left\{ f : \forall x \in [0,\infty)\,,\ \ |f(x)| \le ce^{Kx} \quad K \in \mathbb{R} \text{ and } c \in \mathbb{R}^+ \right\}.$$

In the sequel, we shall need the following auxiliary result.

Lemma 2.1. *For $x \in [0,\infty)$, we have*

$$\begin{aligned} T_n(1;x) &= 1, \\ T_n(\xi;x) &= x + \frac{A'(1)}{nA(1)}, \\ T_n\left(\xi^2;x\right) &= x^2 + \left(\frac{2A'(1)}{A(1)} + H''(1) + 1\right)\frac{x}{n} + \frac{A'(1) + A''(1)}{n^2 A(1)}. \end{aligned}$$

Proof. With the help of generating functions of Sheffer polynomials (1.4), we get the assertion of our lemma. □

Lemma 2.2. *For $x \in [0,\infty)$, the following equality holds:*

$$T_n\left((\xi - x)^2;x\right) = \frac{H''(1)+1}{n}x + \frac{A'(1) + A''(1)}{n^2 A(1)}.$$

Proof. From the linearity property of T_n operators and Lemma 2.1, one can find the above relation. □

Definition 2.3. *The modulus of continuity of a function $f \in \tilde{C}[0,\infty)$ is a function $\omega(f;\delta)$ defined by the relation*

$$\omega(f;\delta) := \sup_{\substack{|x-y|\le\delta \\ x,y\in[0,\infty)}} |f(x) - f(y)|$$

where $\tilde{C}[0,\infty)$ is uniformly continuous functions space.

Ismail proved that for $f \in C[0,\infty) \cap E$ the operators $T_n(f;.)$ converge to the function f in each compact subset of $[0,\infty)$ by using the methods of Szász [7]. On the other hand, if we consider the Lemma 2.1, we obtain the approximation result through the instrument of universal Korovkin-type theorem whenever functions belong to the convenient set. The following theorem is the quantitative version of the result of Ismail only in a particular case. While the result of Ismail holds for functions $f \in C[0,\infty) \cap E$, Theorem 2.4 is valid for restrictive functions $f \in \tilde{C}[0,\infty) \cap E$. Now we are going to state the degree of convergence of the former operator by using the modulus of continuity.

Theorem 2.4. *If $f \in \tilde{C}[0,\infty) \cap E$, then for any $x \in [0,\infty)$ we have*

$$|T_n(f;x) - f(x)| \leq \left(1 + \sqrt{(H''(1)+1)x + \frac{A'(1)+A''(1)}{nA(1)}}\right)\omega\left(f;\frac{1}{\sqrt{n}}\right). \tag{2.1}$$

Proof. By the aid of Lemma 2.1 and property of modulus of continuity, one has the following expression

$$|T_n(f;x) - f(x)| \leq \left\{1 + \frac{1}{\delta}\frac{e^{-nxH(1)}}{A(1)}\sum_{k=0}^{\infty} p_k(nx)\left|\frac{k}{n} - x\right|\right\}\omega(f;\delta). \tag{2.2}$$

On the other hand, making use of the Cauchy-Schwarz inequality we can write

$$\sum_{k=0}^{\infty} p_k(nx)\left|\frac{k}{n} - x\right| \leq A(1)\,e^{nxH(1)}\sqrt{\frac{H''(1)+1}{n}x + \frac{A'(1)+A''(1)}{n^2A(1)}}.$$

Combining the above relation with (2.2), also choosing $\delta = \frac{1}{\sqrt{n}}$, we obtain (2.1). This completes the proof. □

We will need the following lemmas for proving our results about order of convergence for T_n^* operators. Let us consider the class

$$\mathcal{E} := \left\{f : [0,\infty) \to \mathbb{R} \mid \int_0^t f(s)\,ds = \mathcal{O}\left(e^{Kt}\right),\ \ (t \to \infty)\right\}.$$

Lemma 2.5. *There hold the equalities*

$$\begin{aligned}
T_n^*(1;x) &= 1,\\
T_n^*(\xi;x) &= x + \left(\frac{1}{2} + \frac{A'(1)}{A(1)}\right)\frac{1}{n},\\
T_n^*(\xi^2;x) &= x^2 + \left(\frac{2A'(1) + \left(H''(1)+2\right)A(1)}{A(1)}\right)\frac{x}{n}\\
&\quad + \frac{3\left(A''(1)+2A'(1)\right)+A(1)}{3n^2A(1)}.
\end{aligned}$$

Proof. As a consequence of the Lemma 2.1, we immediately get the desired conclusion. □

Therefore by the Lemma 2.5 and by the linearity of T_n^*, we can state the following result.

Lemma 2.6. *For all $x \in [0, \infty)$, we have*

$$T_n^*\left((\xi - x)^2; x\right) = \frac{H''(1) + 1}{n} x + \frac{A(1) + 3\left(A''(1) + 2A'(1)\right)}{3n^2 A(1)}.$$

Through the instrument of Lemma 2.5 and universal Korovkin-type theorem, if function f belongs to appropriate set then the operators given by (1.7) convergence uniformly to the function f in each compact subset of $[0, \infty)$. We obtain quantitative version of the theorem of Ismail only in specific case in the following theorem. Hence, we are going to prove the degree of convergence for T_n^* with the help of modulus of continuity.

Theorem 2.7. *Let f be a function of class $\tilde{C}[0, \infty) \cap \mathcal{E}$. Then for any $x \in [0, \infty)$, we have*

$$|T_n^*(f; x) - f(x)| \leq (1 + \lambda_n(x))\, \omega\left(f; \frac{1}{\sqrt{n}}\right) \tag{2.3}$$

where

$$\lambda_n(x) := \sqrt{(H''(1) + 1)x + \frac{A(1) + 3(A''(1) + 2A'(1))}{3nA(1)}}.$$

Proof. According to Lemma 2.5 and using the property of modulus of continuity, it follows that

$$|T_n^*(f; x) - f(x)| \leq n \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) \left(\frac{1}{n} + \frac{1}{\delta} \int_{k/n}^{(k+1)/n} |s - x|\, ds \right) \omega(f; \delta). \tag{2.4}$$

By a simple application of the Cauchy-Schwarz inequality to the right hand side of (2.4), we obtain

$$|T_n^*(f; x) - f(x)| \leq \left\{ 1 + \frac{1}{\delta}\sqrt{n} \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) \sqrt{\int_{k/n}^{(k+1)/n} (s - x)^2\, ds} \right\} \omega(f; \delta).$$

Once again, by the Cauchy-Schwarz inequality and then with the help of Lemma 2.6 we find that

$$|T_n^*(f; x) - f(x)| \leq$$
$$\left(1 + \frac{1}{\delta} \frac{1}{\sqrt{n}} \sqrt{(H''(1) + 1)x + \frac{A(1) + 3(A''(1) + 2A'(1))}{3nA(1)}} \right) \omega(f; \delta). \tag{2.5}$$

In the inequality (2.5) with choosing $\delta = \frac{1}{\sqrt{n}}$, we get (2.3). □

Remark 2.8. A general estimate in terms of modulus of continuity was given by Shisha and Mond [6]. In the proof of Theorem 2.4 and Theorem 2.7, we follow the method of proof in this mentioned estimate.

3. Example of these type operators

Example 3.1. The Bessel polynomials [5] which are defined by

$$y_k(x) = \sum_{j=0}^{k} \frac{(k+j)!}{(k-j)!j!} \left(\frac{x}{2}\right)^j$$

are an orthogonal sequence of polynomials. Carlitz [1] subsequently constructed a related set of polynomials known as reverse Bessel polynomials as follows

$$\begin{aligned} \theta_k(x) &= x^k y_{k-1}\left(\frac{1}{x}\right) \\ &= \sum_{j=1}^{k} \frac{(2k-j-1)!}{(j-1)!\,(k-j)!2^{k-j}} x^j \ . \end{aligned} \tag{3.1}$$

The generating function for $\theta_k(x)$ is

$$\exp\left[x\left(1-\sqrt{1-2t}\right)\right] = \sum_{k=0}^{\infty} \frac{\theta_k(x)}{k!} t^k \ . \tag{3.2}$$

If we consider the generating function (3.2), then reverse Bessel polynomials are Sheffer type polynomials. Taking into account of explicit formula (3.1), reverse Bessel polynomials $\theta_k(x)$ are positive for $x \geq 0$. Now, let be

$$p_k(x) = \frac{\theta_k\left(2\sqrt{2}x\right)}{4^k k!}.$$

Then by virtue of (3.2), we easily find $A(t) = 1$ and $H(t) = 2\sqrt{2}\left(1-\sqrt{1-\frac{t}{2}}\right)$. From these facts; $A(1) \neq 0$, $H'(1) = 1$ and $p_k(x) \geq 0$ $(x \geq 0)$ are verified. Therefore, we get operators $\widetilde{T_n}$ and $\widetilde{T_n^*}$ including reverse Bessel polynomials as follows:

$$\begin{aligned} \widetilde{T_n}(f;x) \ &: = e^{-2\left(\sqrt{2}-1\right)nx} \sum_{k=0}^{\infty} \frac{\theta_k\left(2\sqrt{2}nx\right)}{4^k k!} f\left(\frac{k}{n}\right) , \\ \widetilde{T_n^*}(f;x) \ &: = ne^{-2\left(\sqrt{2}-1\right)nx} \sum_{k=0}^{\infty} \frac{\theta_k\left(2\sqrt{2}nx\right)}{4^k k!} \int_{k/n}^{(k+1)/n} f(s)\,ds \ . \end{aligned}$$

Remark 3.2. Taking into account Theorem 2.4, we obtain quantitative error estimate for the approximation by $\widetilde{T_n}$ positive linear operators as follows

$$\left|\widetilde{T_n}(f;x) - f(x)\right| \leq \left(1+\sqrt{\frac{3}{2}x}\right)\omega\left(f;\frac{1}{\sqrt{n}}\right) \ ; \quad f \in \tilde{C}[0,\infty) \cap E.$$

Remark 3.3. According to Theorem 2.7, quantitative estimate of the rate of convergence is available for $\widetilde{T_n^*}$ positive linear operators as follows

$$\left|\widetilde{T_n^*}(f;x) - f(x)\right| \leq \left(1+\sqrt{\frac{3}{2}x+\frac{1}{3n}}\right)\omega\left(f;\frac{1}{\sqrt{n}}\right) \ ; \quad f \in \tilde{C}[0,\infty) \cap \mathcal{E}.$$

Remark 3.4. Due to reverse Bessel polynomials are not Appell polynomials, Jakimovski and Leviatan's result [3] does not involve the convergence of $\widetilde{T_n}(f;x)$ to $f(x)$ and of $\widetilde{T_n^*}(f;x)$ to $f(x)$.

Example 3.5. Let us take $f(x) = \dfrac{x}{\sqrt{1+x^4}}$. We compute error estimation by using modulus of continuity for operators $\widetilde{T_n}$ which contain reverse Bessel polynomials in the Table 1 with the help of Maple13 and give its algorithm after the Table 1.

n	Error estimate by $\widetilde{T_n}$ operators including $\{\theta_k(x)\}_{k=1}^{\infty}$ sequence
10	0.7000346345
10^2	0.2224633643
10^3	0.0703525749
10^4	0.0222474486
10^5	0.0070352610
10^6	0.0022247448
10^7	0.0007035261
10^8	0.0002224744
10^9	0.0000703526
10^{10}	0.0000222474

Table 1. The error bound of function f by using modulus of continuity.

Algorithm 3.6. The results with the following algorithm are shown in Table 1.

We derive error estimates for the convergence to the function

$$f(x) = \frac{x}{\sqrt{1+x^4}}$$

with $\widetilde{T_n}$ operators including reverse Bessel polynomials.

```
>restart;
>f:=x->sqrt(x^2/(1+x^4));
>n:=1:
>for i from 1 to 10 do
>n:=10*n;
>delta:=evalf(1/sqrt(n));
>omega(f,delta):=evalf(maximize(expand(abs(f(x+h)-f(x))),
x=0..1-delta,h=0..delta)):
>error:=evalf((1+sqrt(3/2))*omega(f,delta));
>end do;
```

Example 3.7. For $n = 10, 20, 50$; the convergence of $\widetilde{T_n}(f;x)$ to function

$$f(x) = 1 + \sin\left(-2x^2\right)$$

is illustrated in Figure 1 and its algorithm is presented after the Figure 1. Because of our machines have not enough speed and power to compute the complicated infinite series, we have to investigate our approximation result for finite sum.

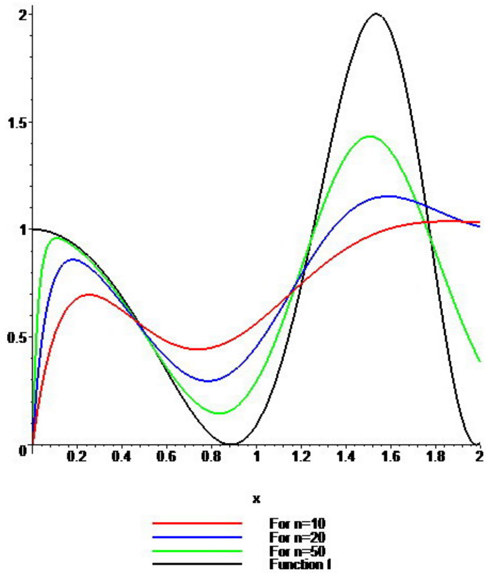

Figure 1. Approximation by $\widetilde{T_n}(f;x)$ operator for the function f.

Algorithm 3.8. >restart;

```
>with(plots):
>f:=x->1+sin(-2*x^2);
>Gt:=0:
>m:=100;
>G:=(k,n,x)->sum(f(k/n)*((2*k-j-1)!)/(((j-1)!)*((k-j)!)*((k)!)
*(2^(3*k-5*j/2)))*(n*x)^j,j=1..k);
>for i from 1 to m do
>Gt:=Gt+simplify(G(i,n,x))
>end do:
>B:=unapply(exp(-2*(sqrt(2)-1)*n*x)*Gt,n):
>p1:=plot(evalf(simplify(B(10))),x=0..2,color=red):
>p2:=plot(evalf(simplify(B(20))),x=0..2,color=blue):
>p3:=plot(evalf(simplify(B(50))),x=0..2,color=green):
>p4:=plot(f(x),x=0..2,color=black):
>display(p1,p2,p3,p4);
```

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Multivariate Voronovskaya asymptotic expansions for general singular operators

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Abstract. In this article we continue with the study of approximation properties of smooth general singular integral operators over R^N, $N \geq 1$. We produce multivariate Voronovskaya asymptotic type results and give quantitative results regarding the rate of convergence of multivariate singular integral operators to unit operator. We list specific multivariate singular integral operators that fulfill our theory.

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Keywords: Multivariate general singular operator, multivariate Picard, Gauss-Weierstrass, Poisson Cauchy and trigonometric singular integrals, multivariate Voronovskaya type asymptotic expansion, rate of convergence.

1. Introduction

The main motivation for this work comes from [2], [3], [4]. We present here multivariate Voronovskaya type asymptotic expansions regarding the multivariate singular integral operators, see Theorem 2.2 and Corolaries 2.3, 2.4. In Theorem 2.6 we give the simultaneous corresponding Voronovskaya asymptotic expansion for our operators. Our expansions give also the rate of convergence of multivariate general singular integral operators to unit operator. In section 3 we list the multivariate singular Picard, Gauss Weierstrass, Poisson-Cauchy and Trigonometric operators that fulfill our results.

2. Main results

Here $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, we define

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, ..., r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 0, \end{cases} \tag{2.1}$$

and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, ..., m \in \mathbb{N}. \tag{2.2}$$

See that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1, \tag{2.3}$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}. \tag{2.4}$$

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$, $\xi_n > 0$, $n \in \mathbb{N}$.

We now define the multiple smooth singular integral operators

$$\theta_{r,n}^{[m]} (f; x_1, ..., x_N) := \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j) \, d\mu_{\xi_n}(s), \tag{2.5}$$

where $s := (s_1, ..., s_N)$, $x := (x_1, ..., x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function, and also $(\xi_n)_{n\in\mathbb{N}}$ is a bounded sequence of positive real numbers.

The above $\theta_{r,n}^{[m]}$ are not in general positive operators and they preserve constants, see [1].

We make

Remark 2.1. Here $f \in C^m(\mathbb{R}^N)$, $m, N \in \mathbb{N}$. Let $l = 0, 1, ..., m$. The lth order partial derivative is denoted by $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$ and $|\alpha| := \sum_{i=1}^{N} \alpha_i = l$.

Consider $g_z(t) := f(x_0 + t(z - x_0))$, $t \geq 0$; $x_0, z \in \mathbb{R}^N$.

Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^{N} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^j f \right] (x_{01} + t(z_1 - x_{01}), ..., x_{0N} + t(z_N - x_{0N})), \tag{2.6}$$

for all $j = 0, 1, ..., m$.

In particular we choose $z = (z_1, ..., z_N) = (x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j) = x + sj$, and $x_0 = (x_{01}, ..., x_{0N}) = (x_1, x_2, ..., x_N) = x$, to get $g_{x+sj}(t) := f(x + t(sj))$.

Notice $g_{x+sj}(0) = f(x)$.

Also for $\widetilde{j} = 0, 1, ..., m-1$ we have

$$g_{x+sj}^{(\widetilde{j})}(0) = \sum_{|\alpha|=\widetilde{j}} \left(\frac{\widetilde{j}!}{\prod_{i=1}^{N} \alpha_i!} \right) \left(\prod_{i=1}^{N} (s_i j)^{\alpha_i} \right) f_\alpha(x). \tag{2.7}$$

Furthermore we get

$$\frac{g_{x+sj}^{(m)}(\theta)}{m!}=\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}(s_ij)^{\alpha_i}\right)f_\alpha(x+\theta(sj)), \tag{2.8}$$

$0\leq\theta\leq1$.

For $\widetilde{j}=1,...,m-1$, and $\alpha:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, $|\alpha|:=\sum_{i=1}^{N}\alpha_i=\widetilde{j}$, we define

$$c_{\alpha,n,\widetilde{j}}:=c_{\alpha,n}:=\int_{\mathbb{R}^N}\prod_{i=1}^{N}s_i^{\alpha_i}d\mu_{\xi_n}(s_1,...,s_N). \tag{2.9}$$

Consequently we obtain

$$\sum_{\widetilde{j}=1}^{m}\frac{\int_{\mathbb{R}^N}g_{x+sj}^{(\widetilde{j})}(0)\,d\mu_{\xi_n}(s)}{\widetilde{j}!}$$

$$=\sum_{\widetilde{j}=1}^{m-1}j^{\widetilde{j}}\left(\sum_{|\alpha|=\widetilde{j}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)c_{\alpha,n}f_\alpha(x)\right). \tag{2.10}$$

Next we observe by multivariate Taylor's formula that

$$f(x+js)=g_{x+js}(1)=\sum_{\widetilde{j}=0}^{m-1}\frac{g_{x+js}^{(\widetilde{j})}(0)}{\widetilde{j}!}+\frac{g_{x+js}^{(m)}(\theta)}{m!}, \tag{2.11}$$

where $\theta\in(0,1)$. Which leads to

$$\int_{\mathbb{R}^N}(f(x+sj)-f(x))\,d\mu_{\xi_n}(s) \tag{2.12}$$

$$=\sum_{\widetilde{j}=1}^{m-1}j^{\widetilde{j}}\left(\sum_{|\alpha|=\widetilde{j}}\frac{1}{\left(\prod_{i=1}^{N}\alpha_i!\right)}c_{\alpha,n,\widetilde{j}}f_\alpha(x)\right)$$

$$+j^m\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\int_{\mathbb{R}^N}\prod_{i=1}^{N}s_i^{\alpha_i}f_\alpha(x+\theta sj)\,d\mu_{\xi_n}(s).$$

Hence

$$\theta_{r,n}^{[m]}(f;x)-f(x)=\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\int_{\mathbb{R}^N}\left(f(x+sj)-f(x)\right)d\mu_{\xi_n}(s)$$

$$=\sum_{\widetilde{j}=1}^{m-1}\sum_{j=1}^{r}\alpha_{j,r}^{[m]}j^{\widetilde{j}}\left(\sum_{|\alpha|=\widetilde{j}}\frac{1}{\left(\prod_{i=1}^{N}\alpha_i!\right)}c_{\alpha,n,\widetilde{j}}f_\alpha(x)\right) \tag{2.13}$$

$$+\sum_{j=1}^{r}\alpha_{j,r}^{[m]}j^m\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\int_{\mathbb{R}^N}\prod_{i=1}^{N}s_i^{\alpha_i}f_\alpha(x+\theta sj)\,d\mu_{\xi_n}(s)$$

$$=\sum_{\widetilde{j}=1}^{m-1}\delta_{\widetilde{j},r}^{[m]}\left(\sum_{|\alpha|=\widetilde{j}}\frac{c_{\alpha,n,\widetilde{j}}f_\alpha(x)}{\left(\prod_{i=1}^{N}\alpha_i!\right)}\right) \tag{2.14}$$

$$+\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}s_i^{\alpha_i}\right)\left(\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}f_\alpha(x+\theta sj)\right)d\mu_{\xi_n}(s).$$

Thus we have

$$\psi:=\theta_{r,n}^{[m]}(f;x)-f(x)-\sum_{\widetilde{j}=1}^{m-1}\delta_{\widetilde{j},r}^{[m]}\left(\sum_{|\alpha|=\widetilde{j}}\frac{c_{\alpha,n,\widetilde{j}}f_\alpha(x)}{\left(\prod_{i=1}^{N}\alpha_i!\right)}\right) \tag{2.15}$$

$$=\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}s_i^{\alpha_i}\right)\left(\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}f_\alpha(x+\theta sj)\right)d\mu_{\xi_n}(s). \tag{2.16}$$

Call

$$\phi_\alpha(x,s):=\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}f_\alpha(x+\theta sj). \tag{2.17}$$

Thus

$$\psi = \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i} \right) \phi_\alpha (x, s)\, d\mu_{\xi_n} (s)\,. \tag{2.18}$$

Consider

$$\Delta_{\xi_n} := \frac{\psi}{\xi_n^m}. \tag{2.19}$$

Assume f_α is bounded for all $\alpha : |\alpha| = m$, by $M > 0$. I.e. $\|f_\alpha\|_\infty \leq M$. Therefore

$$|\phi_\alpha (x, s)| \leq \left(\sum_{j=1}^{r} \binom{r}{j} \right) M = (2^r - 1)\, M. \tag{2.20}$$

Consequently

$$|\Delta_{\xi_n}| \leq \frac{(2^r - 1)\, M}{\xi_n^m} \left(\sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) d\mu_{\xi_n} (s) \right). \tag{2.21}$$

Assume for $|\alpha| = m$ that

$$\xi_n^{-m} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) d\mu_{\xi_n} (s) \leq \rho, \text{ for any } (\xi_n)_{n\in\mathbb{N}}\,. \tag{2.22}$$

Therefore

$$|\Delta_{\xi_n}| \leq (2^r - 1)\, M\rho \left(\sum_{|\alpha|=m} \frac{1}{\left(\prod_{i=1}^{N} \alpha_i! \right)} \right) =: \lambda, \quad \lambda > 0. \tag{2.23}$$

Hence

$$\frac{|\psi|}{\xi_n^m} \leq \lambda \quad \text{and} \quad \frac{|\psi|\, \xi_n^\gamma}{\xi_n^m} \leq \lambda \xi_n^\gamma \to 0, \tag{2.24}$$

where $0 < \gamma \leq 1$, as $\xi_n \to 0+$.

I.e.

$$\frac{|\psi|}{\xi_n^{m-\gamma}} \to 0, \quad \text{as} \ \ \xi_n \to 0+, \tag{2.25}$$

which means $\psi = 0\,(\xi_n^{m-\gamma})$.

We proved

Theorem 2.2. *Let $f \in C^m(\mathbb{R}^N)$, $m, N \in \mathbb{N}$, with all $\|f_\alpha\|_\infty \leq M$, $M > 0$, all $\alpha : |\alpha| = m$. Let $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ bounded sequence, μ_{ξ_n} probability Borel measures on $\mathbb{R}^N$. Call $c_{\alpha,n,\widetilde{j}} = \int_{\mathbb{R}^N} \left(\prod_{i=1}^N s_i^{\alpha_i}\right) d\mu_{\xi_n}(s)$, all $|\alpha| = \widetilde{j} = 1, ..., m-1$. Assume $\xi_n^{-m} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i}\right) d\mu_{\xi_n}(s) \leq \rho$, all $\alpha : |\alpha| = m$, $\rho > 0$, for any such $(\xi_n)_{n\in\mathbb{N}}$. Also $0 < \gamma \leq 1$, $x \in \mathbb{R}^N$. Then*

$$\theta_{r,n}^{[m]}(f; x) - f(x) = \sum_{\widetilde{j}=1}^{m-1} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n,\widetilde{j}} f_\alpha(x)}{\left(\prod_{i=1}^N \alpha_i!\right)} \right) + 0\left(\xi_n^{m-\gamma}\right). \tag{2.26}$$

When $m = 1$ the sum collapses.

Above we assume $\theta_{r,n}^{[m]}(f; x) \in \mathbb{R}$, $\forall\, x \in \mathbb{R}^N$.

Corollary 2.3. *Let $f \in C^1(\mathbb{R}^N)$, $N \geq 1$, with all $\left\|\frac{\partial f}{\partial x_i}\right\|_\infty \leq M$, $M > 0$, $i = 1, ..., N$. Let $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ bounded sequence, μ_{ξ_n} probability Borel measures on $\mathbb{R}^N$. Assume*

$$\xi_n^{-1} \int_{\mathbb{R}^N} |s_i|\, d\mu_{\xi_n}(s) \leq \rho, \quad \text{all} \;\; i = 1, ..., N, \tag{2.27}$$

$\rho > 0$, for any such $(\xi_n)_{n\in\mathbb{N}}$. Also $0 < \gamma \leq 1$, $x \in \mathbb{R}^N$. Then

$$\theta_{r,n}^{[1]}(f; x) - f(x) = 0\left(\xi_n^{1-\gamma}\right). \tag{2.28}$$

Above we assume $\theta_{r,n}^{[1]}(f; x) \in \mathbb{R}$, $\forall\, x \in \mathbb{R}^N$.

Corollary 2.4. *Let $f \in C^2(\mathbb{R}^2)$, with all $\left\|\frac{\partial^2 f}{\partial x_1^2}\right\|_\infty, \left\|\frac{\partial^2 f}{\partial x_2^2}\right\|_\infty, \left\|\frac{\partial^2 f}{\partial x_1 \partial x_2}\right\|_\infty \leq M$, $M > 0$. Let $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ bounded sequence, μ_{ξ_n} probability Borel measures on $\mathbb{R}^2$.*

Call

$$c_1 = \int_{\mathbb{R}^2} s_1 d\mu_{\xi_n}(s), \quad c_2 = \int_{\mathbb{R}^2} s_2 d\mu_{\xi_n}(s). \tag{2.29}$$

Assume

$$\xi_n^{-2} \int_{\mathbb{R}^2} s_1^2 d\mu_{\xi_n}(s), \quad \xi_n^{-2} \int_{\mathbb{R}^2} s_2^2 d\mu_{\xi_n}(s), \quad \xi_n^{-2} \int_{\mathbb{R}^2} |s_1|\,|s_2|\, d\mu_{\xi_n}(s) \leq \rho,$$

$\rho > 0$, for any such $(\xi_n)_{n\in\mathbb{N}}$. Also $0 < \gamma \leq 1$, $x \in \mathbb{R}^2$. Then

$$\theta_{r,n}^{[2]}(f; x) - f(x) = \tag{2.30}$$

$$\left(\sum_{j=1}^r \alpha_{j,r}^{[2]} j\right) \left(c_1 \frac{\partial f}{\partial x_1}(x) + c_2 \frac{\partial f}{\partial x_2}(x)\right) + 0\left(\xi_n^{2-\gamma}\right).$$

We continue with

Theorem 2.5. *Let $f \in C^l\left(\mathbb{R}^N\right)$, $l, N \in \mathbb{N}$. Here μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$, $\xi_n > 0$, $\left(\xi_n\right)_{n\in\mathbb{N}}$ a bounded sequence. Let $\beta := (\beta_1, ..., \beta_N)$, $\beta_i \in \mathbb{Z}^+$, $i = 1, ..., N$; $|\beta| := \sum_{i=1}^{N} \beta_i = l$. Here $f(x + sj)$, $x, s \in \mathbb{R}^N$, is μ_{ξ_n}-integrable wrt s, for $j = 1, ..., r$. There exist μ_{ξ_n}-integrable functions $h_{i_1,j}$, $h_{\beta_1,i_2,j}$, $h_{\beta_1,\beta_2,i_3,j}, ..., h_{\beta_1,\beta_2,...,\beta_{N-1},i_N,j} \geq 0$ $(j = 1, ..., r)$ on $\mathbb{R}^N$ such that*

$$\left|\frac{\partial^{i_1} f(x+sj)}{\partial x_1^{i_1}}\right| \leq h_{i_1,j}(s), \quad i_1 = 1, ..., \beta_1, \tag{2.31}$$

$$\left|\frac{\partial^{\beta_1+i_2} f(x+sj)}{\partial x_2^{i_2}\partial x_1^{\beta_1}}\right| \quad \leq \quad h_{\beta_1,i_2,j}(s), \quad i_2 = 1, ..., \beta_2,$$

$$\vdots$$

$$\left|\frac{\partial^{\beta_1+\beta_2+...+\beta_{N-1}+i_N} f(x+sj)}{\partial x_N^{i_N}\partial x_{N-1}^{\beta_{N-1}}...\partial x_2^{\beta_2}\partial x_1^{\beta_1}}\right| \leq h_{\beta_1,\beta_2,...,\beta_{N-1},i_N,j}(s), \quad i_N = 1, ..., \beta_N,$$

$\forall\ x, s \in \mathbb{R}^N$.

Then, both of the next exist and

$$\left(\theta_{r,n}^{[m]}(f; x)\right)_\beta = \theta_{r,n}^{[m]}(f_\beta; x). \tag{2.32}$$

Proof. By H. Bauer [5], pp. 103-104. □

We finish with

Theorem 2.6. *Let $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $m, l, N \in \mathbb{N}$. Assumptions of Theorem 2.5 are valid. Call $\gamma = 0, \beta$. Assume $\|f_{\gamma+\alpha}\|_\infty \leq M$, $M > 0$, for all $\alpha : |\alpha| = m$. Let $\xi_n > 0$, $\left(\xi_n\right)_{n\in\mathbb{N}}$ bounded sequence, μ_{ξ_n} probability Borel measures on $\mathbb{R}^N$. Call $c_{\alpha,n,\widetilde{j}} = \int_{\mathbb{R}^N}\left(\prod_{i=1}^{N} s_i^{\alpha_i}\right) d\mu_{\xi_n}(s)$, all $|\alpha| = \widetilde{j} = 1, ..., m-1$. Assume $\xi_n^{-m}\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) d\mu_{\xi_n}(s) \leq \rho$, all $\alpha : |\alpha| = m$, $\rho > 0$, for any such $\left(\xi_n\right)_{n\in\mathbb{N}}$. Also $0 < \gamma \leq 1$, $x \in \mathbb{R}^N$. Then*

$$\left(\theta_{r,n}^{[m]}(f; x)\right)_\gamma - f_\gamma(x) = \sum_{\widetilde{j}=1}^{m-1} \delta_{\widetilde{j},r}^{[m]}\left(\sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}(x)}{\left(\prod_{i=1}^{N} \alpha_i!\right)}\right) + 0\left(\xi_n^{m-\gamma}\right). \tag{2.33}$$

When $m = 1$ the sum collapses.

3. Applications

Let all entities as in section 2. We define the following specific operators:
i) The general multivariate Picard singular integral operators:

$$P_{r,n}^{[m]}\left(f;x_1,...,x_N\right) := \frac{1}{\left(2\xi_n\right)^N}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\cdot \tag{3.1}$$

$$\int_{\mathbb{R}^N} f\left(x_1+s_1j,x_2+s_2j,...,x_N+s_Nj\right)e^{-\frac{\left(\sum_{i=1}^{N}\left|s_i\right|\right)}{\xi_n}}ds_1...ds_N.$$

ii) The general multivariate Gauss-Weierstrass singular integral operators:

$$W_{r,n}^{[m]}\left(f;x_1,...,x_N\right) := \frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\cdot \tag{3.2}$$

$$\int_{\mathbb{R}^N} f\left(x_1+s_1j,x_2+s_2j,...,x_N+s_Nj\right)e^{-\frac{\left(\sum_{i=1}^{N}s_i^2\right)}{\xi_n}}ds_1...ds_N.$$

iii) The general multivariate Poisson-Cauchy singular integral operators:

$$U_{r,n}^{[m]}\left(f;x_1,...,x_N\right) := W_n^N\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\cdot \tag{3.3}$$

$$\int_{\mathbb{R}^N} f\left(x_1+s_1j,...,x_N+s_Nj\right)\prod_{i=1}^{N}\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^{\beta}}ds_1...ds_N,$$

with $\alpha \in \mathbb{N}$, $\beta > \frac{1}{2\alpha}$, and

$$W_n := \frac{\Gamma\left(\beta\right)\alpha\xi_n^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}. \tag{3.4}$$

iv) The general multivariate trigonometric singular integral operators:

$$T_{r,n}^{[m]}\left(f;x_1,...,x_N\right) := \lambda_n^{-N}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\cdot \tag{3.5}$$

$$\int_{\mathbb{R}^N} f\left(x_1+s_1j,...,x_N+s_Nj\right)\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}ds_1...ds_N,$$

where $\beta \in \mathbb{N}$, and

$$\lambda_n := 2\xi_n^{1-2\beta}\pi\left(-1\right)^{\beta}\beta\sum_{k=1}^{\beta}\left(-1\right)^k\frac{k^{2\beta-1}}{(\beta-k)!\,(\beta+k)!}. \tag{3.6}$$

One can apply the results of this article to the operators $P_{r,n}^{[m]}$, $W_{r,n}^{[m]}$, $U_{r,n}^{[m]}$, $T_{r,n}^{[m]}$ (special cases of $\theta_{r,n}^{[m]}$) and derive interesting results. We intend to do that in a future article.

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Tripled fixed point theorems in partially ordered metric spaces

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Abstract. The notion of tripled fixed point is introduced by Berinde and Borcut [1]. In this manuscript, some new tripled fixed point theorems are obtained by using a generalization of the results of Luong and Thuang [11].

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Keywords: Tripled fixed point, coupled fixed point, partially ordered set.

1. Introduction

Existence and uniqueness of a fixed point for contraction type mappings in partially ordered metric spaces were discussed first by Ran and Reurings [15] in 2004. Later, so many results were reported on existence and uniqueness of a fixed point and its applications in partially ordered metric spaces (see e.g. [1]-[18]).

In 1987, Guo and Lakshmikantham [6] introduced the notion of the coupled fixed point. The concept of coupled fixed point reconsidered in partially ordered metric spaces by Bhaskar and Lakshmikantham [5] in 2006. In this remarkable paper, by introducing the notion of a mixed monotone mapping the authors proved some coupled fixed point theorems for mixed monotone mapping and considered the existence and uniqueness of solution for periodic boundary value problem.

The triple $(X, d, \leq)$ is called partially ordered metric spaces if $(X, \leq)$ is a partially ordered set and (X, d) is a metric space. Further, if (X, d) is a complete metric space, then the triple $(X, d, \leq)$ is called partially ordered complete metric spaces. Throughout the manuscript, we assume that $X \neq \emptyset$ and

$$X^k = \underbrace{X \times X \times \cdots X}_{k-\text{many}}.$$

Then the mapping $\rho_k : X^k \times X^k \to [0, \infty)$ such that

$$\rho_k(\mathbf{x}, \mathbf{y}) := d(x_1, y_1) + d(x_2, y_2) + \cdots + d(x_k, y_k),$$

forms a metric on X^k where $\mathbf{x} = (x_1, x_2, ..., x_k), \mathbf{y} = (y_1, y_2, \cdots, y_k) \in X^k$, $k \in \mathbb{N}$.

We state the notions of a mixed monotone mapping and a coupled fixed point as follows.

Definition 1.1. ([5]) *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to has the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Definition 1.2. ([5]) *An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if*

$$x = F(x, y) \text{ and } y = F(y, x).$$

In [5] Bhaskar and Lakshmikantham proved the existence of coupled fixed points for an operator $F : X \times X \to X$ having the mixed monotone property on $(X, d, \leq)$ by supposing that there exists a $k \in [0, 1)$ such that

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)], \text{ for all } u \leq x,\ y \leq v. \tag{1.1}$$

under the assumption one of the following condition:

1. Either F is continuous, or
2. (i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x,\ \forall n$;
   (ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n,\ \forall n$.

Very recently, Borcut and Berinde [1] gave the natural extension of Definition 1.1 and Definition 1.2.

Definition 1.3. *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$. The mapping F is said to has the mixed monotone property if for any $x, y, z \in X$*

$$x_1,\ x_2 \in X, \quad x_1 \leq x_2 \Longrightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1,\ y_2 \in X, \quad y_1 \leq y_2 \Longrightarrow F(x, y_1, z) \geq F(x, y_2, z),$$

$$z_1,\ z_2 \in X, \quad z_1 \leq z_2 \Longrightarrow F(x, y, z_1) \leq F(x, y, z_2),$$

Definition 1.4. *Let $F : X^3 \to X$. An element (x, y, z) is called a tripled fixed point of F if*

$$F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z.$$

We recall the main theorem of Borcut and Berinde [1] which is inspired by the main theorem in [5].

Theorem 1.5. *Let $(X, \leq, d)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \times X \times X \to X$ such that F has the mixed monotone property and*

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w), \tag{1.2}$$

for any $x, y, z \in X$ for which $x \leq u$, $v \leq y$ and $z \leq w$. Suppose either F is continuous or X has the following properties:

1. *if a non-decreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,*
2. *if a non-increasing sequence $y_n \to y$, then $y \leq y_n$ for all n,*
3. *if a non-decreasing sequence $z_n \to z$, then $z_n \leq z$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, z_0)$ and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z,$$

that is, F has a tripled fixed point.

In this paper, we prove the existence and uniqueness of a tripled fixed point of $F : X^3 \to X$ satisfying nonlinear contractions in the context of partially ordered metric spaces.

2. Existence of a tripled fixed point

In this section we show the existence of a tripled fixed point. For this purpose, we state the following technical lemma which will be used in the proof of the main theorem efficiently.

Throughout the paper $M = [m_{ij}]$ is a matrix of real numbers and $M^t = [m_{ji}]$ denotes the transpose of M.

Lemma 2.1. *Let* $M = \begin{bmatrix} a & b & c \\ b & a+c & 0 \\ c & b & a \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{12} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$ *with $a + b + c < 1$. Then for* $M^n = \begin{bmatrix} m_{11}^n & m_{12}^n & m_{13}^n \\ m_{21}^n & m_{22}^n & m_{23}^n \\ m_{31}^n & m_{32}^n & m_{33}^n \end{bmatrix}$ *we have*

$$m_{11}^n + m_{12}^n + m_{13}^n = m_{21}^n + m_{22}^n + m_{23}^n = m_{31}^n + m_{32}^n + m_{33}^n = (a + b + c)^n < 1.$$

Proof. We use mathematical induction. For $n = 1$,

$$M = \begin{bmatrix} a & b & c \\ b & a+c & 0 \\ c & b & a \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{12} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix},$$

then by assumption

$$m_{11} + m_{12} + m_{13} = m_{21} + m_{22} + m_{23} = m_{31} + m_{32} + m_{33} = a + b + c < 1.$$

For $n = 2$,

$$
\begin{aligned}
M^2 &= \begin{bmatrix} a & b & c \\ b & a+c & 0 \\ c & b & a \end{bmatrix} \begin{bmatrix} a & b & c \\ b & a+c & 0 \\ c & b & a \end{bmatrix} \\
&= \begin{bmatrix} a^2+b^2+c^2 & b\,(a+c)+ab+bc & 2ac \\ b\,(a+c)+ab & (a+c)^2+b^2 & bc \\ b^2+2ac & b\,(a+c)+ab+bc & a^2+c^2 \end{bmatrix} \\
M^2 &= \begin{bmatrix} m_{11}^2 & m_{12}^2 & m_{13}^2 \\ m_{21}^2 & m_{22}^2 & m_{23}^2 \\ m_{31}^2 & m_{32}^2 & m_{33}^2 \end{bmatrix} = \begin{bmatrix} a^2+b^2+c^2 & b\,(a+c)+ab+bc & 2ac \\ b\,(a+c)+ab & (a+c)^2+b^2 & bc \\ b^2+2ac & b\,(a+c)+ab+bc & a^2+c^2 \end{bmatrix}
\end{aligned}
$$

Since $(a+b+c)^2 = a^2+2ab+2ac+b^2+2bc+c^2$ then

$$
\begin{aligned}
m_{11}^2 + m_{12}^2 + m_{13}^2 &= m_{21}^2 + m_{22}^2 + m_{23}^2 \\
&= m_{31}^2 + m_{32}^2 + m_{33}^2 \\
&= (a+b+c)^2 < 1.
\end{aligned}
$$

Suppose it is true for an arbitrary n, that is, for $M^n = \begin{bmatrix} m_{11}^n & m_{12}^n & m_{13}^n \\ m_{21}^n & m_{22}^n & m_{23}^n \\ m_{31}^n & m_{32}^n & m_{33}^n \end{bmatrix}$ we have

$$
\begin{aligned}
m_{11}^n + m_{12}^n + m_{13}^n &= m_{21}^n + m_{22}^n + m_{23}^n \\
&= m_{31}^n + m_{32}^n + m_{33}^n \\
&= (a+b+c)^n < 1.
\end{aligned}
$$

Then,

$$
\begin{aligned}
M^{n+1} &= M^n M = \begin{bmatrix} m_{11}^n & m_{12}^n & m_{13}^n \\ m_{21}^n & m_{22}^n & m_{23}^n \\ m_{31}^n & m_{32}^n & m_{33}^n \end{bmatrix} \begin{bmatrix} a & b & c \\ b & a+c & 0 \\ c & b & a \end{bmatrix} \\
&= \begin{bmatrix} am_{11}^n + bm_{12}^n + cm_{13}^n & m_{12}^n\,(a+c) + bm_{11}^n + bm_{13}^n & am_{13}^n + cm_{11}^n \\ am_{21}^n + bm_{22}^n + cm_{23}^n & m_{22}^n\,(a+c) + bm_{21}^n + bm_{23}^n & am_{23}^n + cm_{21}^n \\ am_{31}^n + bm_{32}^n + cm_{33}^n & m_{32}^n\,(a+c) + bm_{31}^n + bm_{33}^n & am_{33}^n + cm_{31}^n \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
m_{11}^{n+1} + m_{12}^{n+1} + m_{13}^{n+1} &= am_{11}^n + bm_{12}^n + cm_{13}^n + m_{12}^n\,(a+c) \\
&\qquad + bm_{11}^n + bm_{13}^n + am_{13}^n + cm_{11}^n \\
&= m_{11}^n(a+b+c) + m_{12}^n(a+b+c) + m_{13}^n\,(a+b+c) \\
&= (m_{11}^n + m_{12}^n + m_{13}^n)\,(a+b+c) \\
&= (a+b+c)^n\,(a+b+c) < 1.
\end{aligned}
$$

Analogously we get that

$$
\begin{aligned}
m_{21}^{n+1} + m_{22}^{n+1} + m_{23}^{n+1} &= m_{31}^{n+1} + m_{32}^{n+1} + m_{33}^{n+1} \\
&= (a+b+c)^{n+1} < 1
\end{aligned}
$$

$\square$

Theorem 2.2. *Let $(X, d, \leq)$ be a partially ordered complete metric space. Let $F : X^3 \to X$ be a mapping having the mixed monotone property on X. Assume that there exist* *non-negative numbers a, b, c and L such that $a+b+c<1$ for which*

$$\begin{aligned} d(F(x,y,z),F(u,v,w)) &\leq ad(x,u)+bd(y,v)+cd(z,w) \\ &+L\min\left\{\begin{array}{c} d(F(x,y,z),x),d(F(x,y,z),y),d(F(x,y,z),z), \\ d(F(x,y,z),u),d(F(x,y,z),v),d(F(x,y,z),w), \\ d(F(u,v,w),x),d(F(u,v,w),y),d(F(u,v,w),z), \\ d(F(u,v,w),u),d(F(u,v,w),v),d(F(u,v,w),w) \end{array}\right\}, \end{aligned} \tag{2.1}$$

for all $x \geq u,\ y \leq v,\ z \geq w$. Assume that X has the following properties:

(a) F is continuous, or,
(b) (i) if non-decreasing sequence $x_n \to x$ (respectively, $z_n \to z$), then $x_n \leq x$ (respectively, $z_n \leq z$), for all n,
(ii) if non-increasing sequence $y_n \to y$, then $y_n \geq y$ for all n.

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0,y_0,z_0), \quad y_0 \geq F(y_0,x_0,y_0), \quad z_0 \leq F(x_0,y_0,z_0)$$

then there exist $x, y, z \in X$ such that

$$F(x,y,z)=x \text{ and } F(y,x,y)=y \text{ and } F(z,y,x)=z$$

Proof. We construct a sequence $\{(x_n,y_n,z_n)\}$ in the following way: Set

$$x_1=F(x_0,y_0,z_0)\geq x_0, \quad y_1=F(y_0,x_0,y_0)\leq y_0, z_1=F(z_0,y_0,x_0)\geq z_0,$$

and by the mixed monotone property of F, for $n \geq 1$, inductively we get

$$\begin{aligned} x_n &= F(x_{n-1},y_{n-1},z_{n-1}) \geq x_{n-1} \geq \cdots \geq x_0, \\ y_n &= F(y_{n-1},x_{n-1},y_{n-1}) \leq y_{n-1} \leq \cdots \leq y_0, \\ z_n &= F(z_{n-1},y_{n-1},x_{n-1}) \geq z_{n-1} \geq \cdots \geq z_0. \end{aligned} \tag{2.2}$$

Moreover,

$$\begin{aligned} d(x_n,x_{n+1}) &\leq ad(x_{n-1},x_n)+bd(y_{n-1},y_n)+cd(z_{n-1},z_n) \\ &+L\min\left\{\begin{array}{c} d(x_n,x_{n-1}),d(x_n,y_{n-1}),d(x_n,z_{n-1}), \\ d(x_n,x_n),d(x_n,y_n),d(x_n,z_n), \\ d(x_{n+1},x_{n-1}),d(x_{n+1},y_{n-1}),d(x_{n+1},z_{n-1}), \\ d(x_{n+1},x_n),d(x_{n+1},y_n),d(x_{n+1},z_n), \end{array}\right\} \\ &\leq ad(x_{n-1},x_n)+bd(y_{n-1},y_n)+cd(z_{n-1},z_n) \end{aligned} \tag{2.3}$$

$$\begin{aligned} d(y_n,y_{n+1}) &\leq ad(y_{n-1},y_n)+bd(x_{n-1},x_n)+cd(y_{n-1},y_n) \\ &+L\min\left\{\begin{array}{c} d(y_n,y_{n-1}),d(y_n,x_{n-1}),d(y_n,y_{n-1}), \\ d(y_n,y_n),d(y_n,x_n),d(y_n,y_n), \\ d(y_{n+1},y_{n-1}),d(y_{n+1},x_{n-1}),d(y_{n+1},y_{n-1}), \\ d(y_{n+1},y_n),d(y_{n+1},y_n),d(y_{n+1},y_n), \end{array}\right\} \\ &\leq (a+c)d(y_{n-1},y_n)+bd(x_{n-1},x_n) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} d(z_n, z_{n+1}) &\leq ad(z_{n-1}, z_n) + bd(y_{n-1}, y_n) + cd(x_{n-1}, x_n) \\ &+L \min \left\{ \begin{array}{c} d(z_n, x_{n-1}), d(z_n, y_{n-1}), d(z_n, z_{n-1}), \\ d(z_n, z_n), d(z_n, y_n), d(z_n, x_n), \\ d(z_{n+1}, x_{n-1}), d(z_{n+1}, y_{n-1}), d(z_{n+1}, z_{n-1}), \\ d(z_{n+1}, z_n), d(z_{n+1}, y_n), d(z_{n+1}, x_n), \end{array} \right\} \\ &\leq ad(z_{n-1}, z_n) + bd(y_{n-1}, y_n) + cd(x_{n-1}, x_n) \end{aligned} \tag{2.5}$$

Thus, from by (2.3)-(2.5) and Lemma 2.1, we obtain that

$$D_{n+1} \leq MD_n \leq \cdots \leq M^n D_1 \tag{2.6}$$

where

$$D_{n+1} = \begin{bmatrix} d(x_n, x_{n+1}) \\ d(y_n, y_{n+1}) \\ d(z_n, z_{n+1}) \end{bmatrix} \text{ and } M = \begin{bmatrix} a & b & c \\ b & a+c & 0 \\ c & b & a \end{bmatrix} = [m_{ij}] \text{ as in Lemma 2.1.}$$

We show that $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ are Cauchy sequences.

Due to Lemma 2.1 and (2.6), we have

$$\begin{aligned} d(x_p, x_q) &\leq d(x_p, x_{p-1}) + d(x_{p-1}, x_{p-2}) + \cdots + d(x_{q+1}, x_p) \\ &\leq \begin{bmatrix} m_{11}^p \\ m_{12}^p \\ m_{13}^p \end{bmatrix}^t D_1 + \begin{bmatrix} m_{11}^{p-1} \\ m_{12}^{p-1} \\ m_{13}^{p-1} \end{bmatrix}^t D_1 + \cdots + \begin{bmatrix} m_{11}^{q+1} \\ m_{12}^{q+1} \\ m_{13}^{q+1} \end{bmatrix}^t D_1 \\ &= \begin{bmatrix} m_{11}^{q+1} + \cdots + m_{11}^{p-1} + m_{11}^p \\ m_{12}^{q+1} + \cdots + m_{12}^{p-1} + m_{12}^p \\ m_{13}^{q+1} + \cdots + m_{13}^{p-1} + m_{13}^p \end{bmatrix}^t D_1 \\ &\leq (m_{11}^{q+1} + \cdots + m_{11}^{p-1} + m_{11}^p) d(x_1, x_0) \\ &\quad + (m_{12}^{q+1} + \cdots + m_{12}^{p-1} + m_{12}^p) d(y_1, y_0) \\ &\quad + (m_{13}^{q+1} + \cdots + m_{13}^{p-1} + m_{13}^p) d(z_1, z_0) \\ &\leq (k^q + k^{q+1} + \cdots + k^{p-1}) d(x_1, x_0) \\ &\quad + (k^q + k^{q+1} + \cdots + k^{p-1}) d(y_1, y_0) \\ &\quad + (k^q + k^{q+1} + \cdots + k^{p-1}) d(z_1, z_0) \\ &= (k^q + k^{q+1} + \cdots + k^{p-1})(d(x_1, x_0) + d(y_1, y_0) + d(z_1, z_0)) \\ &\leq k^q \tfrac{1-k^{p-q}}{1-k} (d(x_1, x_0) + d(y_1, y_0) + d(z_1, z_0)) \end{aligned} \tag{2.7}$$

where $k = a+b+c < 1$. Thus (2.7) yields that $\{x_n\}$ is a Cauchy sequence. Analogously, one can show $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences.

Since X is a complete metric space, there exist $x, y, z \in X$ such that

$$\lim_{n\to\infty} x_n = x, \ \lim_{n\to\infty} y_n = y \text{ and } \lim_{n\to\infty} z_n = z. \tag{2.8}$$

Now, suppose that assumption (a) holds. Taking the limit as $n \to \infty$ in (2.2) and by (2.8), we get

$$\begin{aligned} x &= \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_{n-1}, y_{n-1}, z_{n-1}) \\ &= F(\lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} z_{n-1}) \\ &= F(x, y, z) \end{aligned}$$

and

$$\begin{aligned} y &= \lim_{n\to\infty} y_n = lim_{n\to\infty} F(y_{n-1}, x_{n-1}, y_{n-1}) \\ &= F(\lim_{n\to\infty} y_{n-1}, lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}) \\ &= F(y, x, y) \end{aligned}$$

$$\begin{aligned} z &= \lim_{n\to\infty} z_n = \lim_{n\to\infty} F(z_{n-1}, y_{n-1}, x_{n-1}) \\ &= F(\lim_{n\to\infty} z_{n-1}, \lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} x_{n-1}) \\ &= F(z, y, x). \end{aligned}$$

Thus we proved that $x = F(x, y, z), y = F(y, x, y)$ and $z = F(z, y, x)$.

Finally, suppose that (b) holds. Since $\{x_n\}$ (respectively, $\{z_n\}$) is non-decreasing sequence and $x_n \to x$ (respectively, $z_n \to z$) and as $\{y_n\}$ is non-increasing sequence and $y_n \to y$,by assumption (b), we have $x_n \geq x$ (respectively, $z_n \geq z$) and $y_n \leq y$ for all n. We have

$$\begin{aligned} d(F(x_n, y_n, z_n), F(x, y, z)) &\leq ad(x_n, x) + bd(y_n, y) + cd(z_n, z) \\ &+L \min \left\{ \begin{array}{c} d(F(x_n, y_n, z_n), x), d(F(x_n, y_n, z_n), y), \\ d(F(x_n, y_n, z_n), z), d(F(x_n, y_n, z_n), x_n), \\ d(F(x_n, y_n, z_n), y_n), d(F(x_n, y_n, z_n), z_n), \\ d(F(x, y, z), x), d(F(x, y, z), y), \\ d(F(x, y, z), z), d(F(x, y, z), x_n), \\ d(F(x, y, z), y_n), d(F(x, y, z), z_n), \end{array} \right\} \end{aligned} \tag{2.9}$$

Taking $n \to \infty$ in (2.9) we get $d(x, F(x, y, z)) \leq 0$ which implies $F(x, y, z) = x$. Analogously, we can show that $F(y, x, y) = y$ and $F(z, y, x) = z$.
Therefore, we proved that F has a tripled fixed point. □

Corollary 2.3. *(Main Theorem of* [1]*) Let $(X, d, \leq)$ be a partially ordered complete metric space. Suppose $F : X \times X \times X \to X$ such that F has the mixed monotone property and*

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w), \tag{2.10}$$

for any $x, y, z \in X$ for which $x \leq u$, $v \leq y$ and $z \leq w$. Suppose either F is continuous or X has the following properties:

1. *if a non-decreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,*
2. *if a non-increasing sequence $y_n \to y$, then $y \leq y_n$ for all n,*
3. *if a non-decreasing sequence $z_n \to z$, then $z_n \leq z$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, z_0)$ and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z,$$

that is, F has a tripled fixed point.

Proof. Taking $L = 0$ in Theorem (2.2), we obtain Corollary 2.3. □

3. Uniqueness of tripled fixed point

In this section we shall prove the uniqueness of tripled fixed point. For a partially ordered set $(X, \leq)$, we endow the product $X \times X \times X$ with the following partial order relation: for all $(x, y, z), (u, v, w) \in (X \times X)$,

$$(x, y, z) \leq (u, v, w) \Leftrightarrow x \leq u, y \geq v \text{ and } z \leq w.$$

We say that (x, y, z) is equal to (u, v, r) if and only if $x = u,\ y = v,\ z = r$.

Theorem 3.1. *In addition to hypotheses of Theorem 2.2, suppose that for all $(x, y, z), (u, v, r) \in X \times X \times X$, there exists $(a, b, c) \in X \times X \times X$ that is comparable to (x, y, z) and (u, v, r), then F has a unique triple fixed point.*

Proof. The set of triple fixed point of F is not empty due to Theorem 2.2. Assume, now, (x, y, z) and (u, v, r) are the triple fixed points of F, that is,

$$\begin{array}{ll} F(x, y, z) = x, & F(u, v, r) = u, \\ F(y, x, y) = y, & F(v, u, v) = v, \\ F(z, y, x) = z, & F(r, v, u) = r, \end{array}$$

We shall show that (x, y, z) and (u, v, r) are equal. By assumption, there exists $(p, q, s) \in X \times X \times X$ that is comparable to (x, y, z) and (u, v, r). Define sequences $\{p_n\}, \{q_n\}$ and $\{s_n\}$ such that

$$p = p_0, \quad q = q_0, \quad s = s_0, \quad \text{and}$$

$$\begin{aligned} p_n &= F(p_{n-1}, q_{n-1}, s_{n-1}), \\ q_n &= F(q_{n-1}, p_{n-1}, q_{n-1}), \\ s_n &= F(s_{n-1}, q_{n-1}, p_{n-1}), \end{aligned} \tag{3.1}$$

for all n. Since (x, y, z) is comparable with (p, q, s), we may assume that $(x, y, z) \geq (p, q, s) = (p_0, q_0, s_0)$. Recursively, we get that

$$(x, y, z) \geq (p_n, q_n, s_n) \quad \text{for all } n. \tag{3.2}$$

By (3.2) and (2.1), we have

$$\begin{aligned} d(x, p_{n+1}) &= d(F(x, y, z), F(p_n, q_n, s_n)) \\ &\leq ad(x, p_n) + bd(y, q_n) + cd(z, s_n) \\ &\quad + L \min \left\{ \begin{array}{c} d(F(x, y, z), x), d(F(x, y, z), y), \\ d(F(x, y, z), z), d(F(x, y, z), p_n), \\ d(F(x, y, z), q_n), d(F(x, y, z), s_n), \\ d(F(p_n, q_n, s_n), x), d(F(p_n, q_n, s_n), y), \\ d(F(p_n, q_n, s_n), z), d(F(p_n, q_n, s_n), p_n), \\ d(F(p_n, q_n, s_n), q_n), d(F(p_n, q_n, s_n), s_n) \end{array} \right\} \\ &\leq ad(x, p_n) + bd(y, q_n) + cd(z, s_n) \\ &\leq k^n [d(x, p_0) + d(y, q_0) + d(z, s_0)] \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} d(q_{n+1}, y) &= d(F(q_n, p_n, q_n), F(y, x, y)) \\ &\leq ad(y, q_n) + bd(x, p_n) + cd(y, q_n) \\ &+ L \min \left\{ \begin{array}{c} d(F(y,x,y),y), d(F(y,x,y),x), \\ d(F(y,x,y),y), d(F(y,x,y),q_n), \\ d(F(y,x,y),p_n), d(F(y,x,y),q_n), \\ d(F(F(q_n,p_n,q_n),y), d(F(q_n,p_n,q_n),x), \\ d(F(q_n,p_n,q_n),y), d(F(q_n,p_n,q_n),q_n), \\ d(F(q_n,p_n,q_n),p_n), d(F(p_n,q_n,s_n),q_n) \end{array} \right\} \\ &\leq ad(y, q_n) + bd(x, p_n) + cd(y, q_n) \\ &\leq k^n[d(x, p_0) + d(y, q_0) + d(z, s_0)] \end{aligned} \tag{3.4}$$

$$\begin{aligned} d(z, s_{n+1}) &= d(F(x, y, z), F(p_n, q_n, s_n)) \\ &\leq ad(x, p_n) + bd(y, q_n) + cd(z, s_n) \\ &+ L \min \left\{ \begin{array}{c} d(F(z,x,y),z), d(F(z,x,y),y), \\ d(F(z,x,y),x), d(F(z,x,y),s_n), \\ d(F(z,x,y),q_n), d(F(z,x,y),p_n), \\ d(F(s_n,q_n,p_n),z), d(F(s_n,q_n,p_n),y), \\ d(F(s_n,q_n,p_n),x), d(F(s_n,q_n,p_n),s_n), \\ d(F(s_n,q_n,p_n),q_n), d(F(s_n,q_n,p_n),p_n) \end{array} \right\} \\ &\leq ad(x, p_n) + bd(y, q_n) + cd(z, s_n) \\ &\leq k^n[d(x, p_0) + d(y, q_0) + d(z, s_0)] \end{aligned} \tag{3.5}$$

Letting $n \to \infty$ in (3.3) -(3.5), we get

$$\lim_{n\to\infty} d(x, p_{n+1}) = 0, \lim_{n\to\infty} d(y, q_{n+1}) = 0, \lim_{n\to\infty} d(z, s_{n+1}) = 0. \tag{3.6}$$

Analogously, we obtain

$$\lim_{n\to\infty} d(u, p_{n+1}) = 0, \lim_{n\to\infty} d(v, q_{n+1}) = 0, \lim_{n\to\infty} d(r, s_{n+1}) = 0. \tag{3.7}$$

By (3.6) and (3.7) we have $x = u, y = v, z = r$. □

4. Examples

In this sections we give some examples to show that our results are effective.

Example 4.1. Let $X = [0,1]$ with the metric $d(x,y) = |x - y|$, for all $x, y \in X$ and the usual ordering.
Let $F : X^3 \to X$ be given by

$$F(x,y,z) = \frac{4x^2 - 4y^2 + 8z^2 + 8}{65}, \text{ for all } x, y, z, w \in X$$

It is easy to check that all the conditions of Corollary 2.3 are satisfied and $(\frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ is the unique tripled fixed point of F where $a = b = \frac{8}{65}$ and $c = \frac{16}{65}$.

But if we changed the space $X = [0,1]$ with $X = [0,\infty)$, then conditions of Corollary 2.3 are not satisfied anymore. Indeed, $F(8,0,8) = \frac{776}{65}$ and $F(7,0,7) = \frac{596}{65}$. Thus,$d(F(8,0,8), F(7,0,7)) = \frac{36}{13}$. On the other hand, $d(8,7) = 1$. Thus, there exist no non-negative real number a, b, c with $a + b + c < 1$ and satisfies the conditions of Corollary 2.3. But, for $L = 2$ and $a = b = \frac{8}{65}, c = \frac{16}{65}$, the conditions of Theorem 2.2 are satisfied. Moreover, $(\frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ is the unique tripled fixed point of F.

Example 4.2. Let $X = [0, \infty)$ with the metric $d(x, y) = |x - y|$, for all $x, y \in X$ and the following order relation:

$$x, y \in X,\ \ x \preceq y\ \ \Leftrightarrow\ \ x = y = 0\,\text{or}\ \ (x, y \in (0, \infty)\text{and}\ x \leq y),$$

where $\leq$ be the usual ordering.
Let $F : X^3 \to X$ be given by

$$F(x, y, z) = \begin{cases} 1, & if \quad xyz \neq 0 \\ 0, & if \quad xyz = 0 \end{cases}$$

for all $x, y, z \in X$.
It is easy to check that all the conditions of Corollary 2.3 are satisfied. Applying Corollary 2.3 we conclude that F has a tripled fixed point. In fact, F has two tripled fixed points. They are $(0, 0, 0)$ and $(1, 1, 1)$. Therefore, the conditions of Corollary 2.3 are not sufficient for the uniqueness of a tripled fixed point.

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On a bounded critical point theorem of Schechter

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Abstract. A new proof based on Bishop-Phelps' variational principle is given to a critical point theorem of Schechter for extrema in a ball of a Hilbert space. The same technique is used to obtain a similar result in annular domains. Comments on the involved boundary conditions and an application to a two-point boundary value problem are included. An alternative variational approach to the compression-expansion Krasnoselskii's fixed point method is thus provided. In addition, estimations from below are obtained here for the first time, in terms of the energetic norm.

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1. Introduction

Let X be a Hilbert space with inner product $(.,.)$ and norm $|.|$, identified to its dual, let X_R denote the closed ball of X of radius R centered at the origin and let ∂X_R be its boundary. In [13], [14], the following critical point theorem for minima located in X_R, in a slightly different form, was proved by using pseudogradients and deformation arguments.

Theorem 1.1 (Schechter's theorem for minima). *Let $F : X_R \to \mathbf{R}$ be a C^1-functional, bounded from below. There exists a sequence (x_n), $x_n \in X_R$, such that $F(x_n) \to$ $\inf F(X_R)$ and one of the following two situations holds:*

(a) $F'(x_n) \to 0$;

(b) $|x_n| = R$, $(F'(x_n), x_n) \leq 0$ *for all n, and*

$$F'(x_n) - \frac{(F'(x_n), x_n)}{R^2} x_n \to 0. \tag{1.1}$$

If in addition $(F'(x), x) \geq -a > -\infty$ for all $x \in \partial X_R$, F satisfies a Palais-Smale type compactness condition guarantying that any sequence as above has a convergent subsequence, and the boundary condition

$$F'(x) + \mu x \neq 0 \text{ for all } x \in \partial X_R \text{ and } \mu > 0 \tag{1.2}$$

holds, then there exists $x \in X_R$ *with*

$$F(x) = \inf F(X_R), \quad F'(x) = 0.$$

It is easy to see that under the assumption $(F'(x), x) \geq -a > -\infty$ for all $x \in \partial X_R$, Theorem 1.1 yields Schechter's original statement [14, Theorem 5.3.3]: There exists a sequence (x_n), $x_n \in X_R \setminus \{0\}$, such that

$$F(x_n) \to \inf F(X_R), \ (F'(x_n), x_n) \to b \leq 0 \text{ and } F'(x_n) - \frac{(F'(x_n), x_n)}{|x_n|^2} x_n \to 0.$$

The dual result for maxima in X_R is the following theorem, a slightly modified form of Theorem 5.5.5 in [14].

Theorem 1.2 (Schechter's theorem for maxima). *Let* $F : X_R \to \mathbf{R}$ *be a* C^1*-functional, bounded from above. There exists a sequence* (x_n), $x_n \in X_R$, *such that* $F(x_n) \to \sup F(X_R)$ *and one of the following two situations holds:*

(a) $F'(x_n) \to 0$;

(b) $|x_n| = R$, $(F'(x_n), x_n) \geq 0$ *for all* n, *and*

$$F'(x_n) - \frac{(F'(x_n), x_n)}{R^2} x_n \to 0. \tag{1.3}$$

If in addition $(F'(x), x) \leq a < \infty$ *for all* $x \in \partial X_R$, F *satisfies a Palais-Smale type compactness condition guarantying that any sequence as above has a convergent subsequence, and the boundary condition*

$$F'(x) + \mu x \neq 0 \textit{ for all } x \in \partial X_R \textit{ and } \mu < 0$$

holds, then there exists $x \in X_R$ *with*

$$F(x) = \sup F(X_R), \quad F'(x) = 0.$$

In this paper we first present a simple and direct proof of these results using Bishop-Phelps' variational principle. Similar results are then obtain for the localization of critical points of extremum in annular domains. Comments on the involved boundary conditions and an application to a two-point boundary value problem are included. The results are related to those from our previous papers [10] and [11].

We finish this introductory section by the statement of Bishop-Phelps' theorem [3], [6].

Theorem 1.3 (Bishop–Phelps' theorem). *Let* (M, d) *be a complete metric space,* $\varphi : M \to \mathbf{R}$ *lower semicontinuous and bounded from below and* $\varepsilon > 0$. *Then for any* $x_0 \in M$, *there exists* $x \in M$ *such that*

(i) $\varphi(x) \leq \varphi(x_0) - \varepsilon d(x_0, x)$;

(ii) $\varphi(x) < \varphi(y) + \varepsilon d(y, x)$ *for every* $y \neq x$.

Notice the equivalence of Bishop-Phelps' theorem and Ekeland's variational principle (see, e.g. [5], [9]).

2. New proof of Schechter's theorem

Proof of Theorem 1.1. We apply Bishop-Phelps' theorem to $M = X_R$, $\varphi = F$, $\varepsilon = \frac{1}{n}$ $(n \in \mathbf{N} \setminus \{0\})$ and $x_0 \in X_R$ with $F(x_0) \leq \inf F(X_R) + \frac{1}{n}$. It follows that there exists $x_n \in X_R$ such that

$$F(x_n) \leq F(x_0) - \frac{1}{n}|x_n - x_0| \leq F(x_0) \leq \inf F(X_R) + \frac{1}{n} \tag{2.1}$$

and

$$F(x_n) < F(y) + \frac{1}{n}|y - x_n| \quad \text{for every } y \in X_R \text{ with } y \neq x_n. \tag{2.2}$$

Clearly, (2.1) implies $F(x_n) \to \inf F(X_R)$. Two cases are possible: (c1) There is a subsequence of (x_n) with $|x_n| < R$; (c2) The terms of the sequence (x_n), except possibly a finite number, belong to ∂X_R.

In case (c1), we may suppose that $|x_n| < R$ for all n. For a fixed n and any $z \in X$ with $|z| = 1$, we take $y := x_n - tz$ which still belongs to X_R, for $t > 0$ small enough. Then (2.2) gives us

$$-t(F'(x_n), z) + o(t) + \frac{t}{n} > 0.$$

Dividing by t and letting t tend to zero, we obtain $(F'(x_n), z) \leq \frac{1}{n}$, whence $|F'(x_n)| \leq \frac{1}{n}$, that is $F'(x_n) \to 0$. Thus, in case (c1), property (a) holds.

In case (c2) we may assume that $|x_n| = R$ for all n. The key remark is that an element of the form $y = x_n - tz$ with $|z| = 1$, still belongs to the ball for $t > 0$ small enough, whenever $(x_n, z) > 0$. Indeed

$$|x_n - tz|^2 = t^2 - 2t(x_n, z) + R^2 \leq R^2$$

for $0 < t \leq 2(x_n, z)$. Hence, for such a z, we still have de conclusion $(F'(x_n), z) \leq \frac{1}{n}$. By density, the same inequality holds even if $(x_n, z) = 0$. Therefore

$$(F'(x_n), z) \leq \frac{1}{n} \text{ for every } z \in X \text{ with } |z| = 1 \text{ and } (x_n, z) \geq 0. \tag{2.3}$$

Now two subcases are possible: in the first one $(F'(x_n), x_n) > 0$ for a subsequence, when in view of the above remark, $(F'(x_n), F'(x_n)) \leq \frac{1}{n}|F'(x_n)|$, that is $|F'(x_n)| \leq \frac{1}{n}$. Thus, in this case, $F'(x_n) \to 0$ and (a) holds. In the second subcase, $(F'(x_n), x_n) \leq 0$ for all n except possibly a finite number of indices. Then we take in (2.3) $z = \frac{1}{|w_n|}w_n$, where $w_n = F'(x_n) - \frac{(F'(x_n), x_n)}{R^2}x_n$. Clearly $(x_n, w_n) = 0$. Hence

$$(F'(x_n), w_n) \leq \frac{1}{n}|w_n|.$$

Also, since $(x_n, w_n) = 0$, one has

$$(F'(x_n), w_n) = \left(F'(x_n) - \frac{(F'(x_n), x_n)}{R^2}x_n,\ w_n\right) = |w_n|^2.$$

Hence $|w_n|^2 \leq \frac{1}{n}|w_n|$, whence $|w_n| \leq \frac{1}{n}$ and so $w_n \to 0$. Therefore (b) holds.

For the last part of the theorem, it suffices to see that in case (b), under the additional assumption that $(F'(x), x) \geq -a > -\infty$ for all $x \in \partial X_R$, we may suppose that the sequence of real numbers $(F'(x_n), x_n)$ converges to some $b \leq 0$. Then, in view of the Palais-Smale condition, if at least for a subsequence $x_n \to x$, we have $F'(x) + \mu x = 0$, where $x \in \partial X_R$ since $|x_n| = R$, and $\mu = -b/R^2 \geq 0$. The case $\mu > 0$ being excluded by the boundary condition (1.2), it remains that $F'(x) = 0$. □

Proof of Theorem 1.2. Apply Theorem 1.1 to the functional $-F$. □

3. Critical points of extremum in annular domains

In this section, the same technique based on Bishop-Phelps' theorem is used in order to localize critical points of extremum in the annular domain

$$X_{r,R} := \{x \in X : r \leq |x| \leq R\},$$

where $0 < r < R$.

Theorem 3.1. *Let $F : X_{r,R} \to \mathbf{R}$ be a C^1-functional, bounded from below. Then there exists a sequence (x_n), $x_n \in X_{r,R}$ such that $F(x_n) \to \inf F(X_{r,R})$, and one of the following three situations holds:*

(a) $F'(x_n) \to 0$;

(b) $|x_n| = r$, $(F'(x_n), x_n) \geq 0$ *for all* n, *and*

$$F'(x_n) - \frac{(F'(x_n), x_n)}{r^2} x_n \to 0; \tag{3.1}$$

(c) $|x_n| = R$, $(F'(x_n), x_n) \leq 0$ *for all* n, *and*

$$F'(x_n) - \frac{(F'(x_n), x_n)}{R^2} x_n \to 0. \tag{3.2}$$

If in addition

$$(F'(x), x) \leq a < \infty \quad \text{for } |x| = r, \quad (F'(x), x) \geq -a > -\infty \quad \text{for } |x| = R, \tag{3.3}$$

F satisfies a Palais-Smale type compactness condition guaranteeing that any sequence as above has a convergent subsequence, and the boundary conditions

$$\begin{aligned} F'(x) + \mu x &\neq 0 \quad \text{for } |x| = r \text{ and } \mu < 0; \\ F'(x) + \mu x &\neq 0 \quad \text{for } |x| = R \text{ and } \mu > 0, \end{aligned} \tag{3.4}$$

hold, then there exists $x \in X_{r,R}$ with

$$F(x) = \inf F(X_{r,R}), \quad F'(x) = 0.$$

Proof. Applying Bishop-Phelps' theorem on $M = X_{r,R}$ we find a sequence (x_n) of elements of $X_{r,R}$ such that $F(x_n) \leq \inf F(X_{r,R}) + \frac{1}{n}$ and

$$F(x_n) < F(y) + \frac{1}{n}|x_n - y| \quad \text{for } y \neq x_n. \tag{3.5}$$

Then, there must be a subsequence in one of the following cases: (c1) $r < |x_n| \leq R$; (c2) $|x_n| = r$. In case (c1) we may repeat the proof of Theorem 1.1 since in view of the strict inequality $|x_n| > r$, all choices of y of the form $x_n - tz$ from that proof are still possible, that is satisfy $|y| \geq r$ too, provided that $t > 0$ is small enough. Hence in case (c1) one of the situations (a), (c) holds. Assume now that we are in case (c2).

There exist two subcases: (1) $(F'(x_n), x_n) < 0$ for a subsequence. Then we can apply (3.5) to $y = x_n - tF'(x_n)$ since

$$|y|^2 = r^2 - 2t(F'(x_n), x_n) + t^2 |F'(x_n)|^2 \geq r^2 + t^2 |F'(x_n)|^2 \geq r^2$$

for all $t > 0$. In addition $|y| \leq R$ if t is small enough. Thus $y \in X_{r,R}$ for sufficiently small $t > 0$. Now (3.5) implies

$$-t(F'(x_n), F'(x_n)) + o(t) + \frac{1}{n}|F'(x_n)| \geq 0,$$

whence after dividing by t and passing to the limit with $t \to 0$, we deduce $|F'(x_n)| \leq \frac{1}{n}$, that is (a) also holds in this subcase. Assume now subcase: (2) $(F'(x_n), x_n) \geq 0$ for all n, except possibly a finite number of indices. Then in (3.5), we take $y = x_n - tz_n$, where $t > 0$, $z_n = F'(x_n) - \mu_n x_n$ and $\mu_n = \frac{(F'(x_n), x_n)}{r^2}$. Since $(x_n, z_n) = 0$, we have

$$|y|^2 = r^2 - 2t(x_n, z_n) + t^2 |z_n|^2 = r^2 + t^2 |z_n|^2 \geq r^2$$

for every $t > 0$. Hence $|y| \geq r$. In addition, from $|x_n| < R$, we have $|y| \leq R$ for all $t > 0$ small enough. Thus $y \in X_{r,R}$ and (3.5) applies and yields

$$(F'(x_n), z_n) \leq \frac{1}{n}|z_n|.$$

Consequently

$$|z_n|^2 = (F'(x_n) - \mu_n x_n, z_n) = (F'(x_n), z_n) \leq \frac{1}{n}|z_n|,$$

whence $|z_n| \leq \frac{1}{n}$ and so $z_n \to 0$, that is situation (b) holds. The proof of the last part of the theorem is similar to that of Theorem 1.1. □

Obviously, a dual result of Theorem 3.1 for maxima in annular domains can easily be stated.

Notice that *cone versions* of Theorems 1.1, 1.2 and 3.1 can be stated in sets of the form $K_R = \{x \in K : |x| \leq R\}$ and $K_{r,R} = \{x \in K : r \leq |x| \leq R\}$, respectively, where K is a wedge in X, and in particular, a cone (see the forthcoming paper [12]). Their statement is similar, with the additional cone invariance condition

$$x - F'(x) \in K \text{ for every } x \in K. \tag{3.6}$$

4. Comments on the boundary conditions

4.1. Meaning of the boundary conditions

The meaning of boundary conditions (3.4) and the natural way they are related to the minimization problem we are concerning with, can be well understood in one-dimension. Thus, for a C^1-function $F : D = [-R, -r] \cup [r, R] \to \mathbf{R}$, conditions (3.4) reduce to

$$\begin{aligned} F'(-r) &\geq 0, \quad F'(r) \leq 0, \\ F'(-R) &\leq 0, \quad F'(R) \geq 0. \end{aligned} \tag{4.1}$$

Indeed, if for example, $F'(-r) < 0$, then taking $\mu = \frac{F'(-r)}{r}$ we have $F'(-r) + \mu \cdot (-r) = 0$ where $\mu < 0$, which contradicts (3.4). Similar explanations can be done for the other three inequalities. Now remembering that we are interested in critical points of F which minimizes F, that is in points x with $F'(x) = 0$ and $F(x) = \inf F(D)$, we immediately can see that under conditions (4.1), if at some point x from the boundary of D, i.e. $x \in \{-R, -r, r, R\}$, functional F attains its minimum over D, then x must be a critical point of F, i.e. $F'(x) = 0$.

Therefore, conditions (4.1), and in general conditions (3.4), exclude the possibility for F to attain its minimum at a noncritical point of the boundary.

4.2. Connection with fixed point theory

In case that F' has the representation $F'(x) = x - N(x)$, when the critical points of F are the fixed points of N, conditions (3.4) read as follows

$$\begin{aligned} N(x) &\neq \lambda x \;\text{ for } |x| = r \text{ and } \lambda < 1, \\ N(x) &\neq \lambda x \;\text{ for } |x| = R \text{ and } \lambda > 1, \end{aligned}$$

and can be seen as a compression property of N over the annular domain $X_{r,R}$. The corresponding boundary conditions for the dual result for maxima are

$$\begin{aligned} N(x) &\neq \lambda x \;\text{ for } |x| = r \text{ and } \lambda > 1, \\ N(x) &\neq \lambda x \;\text{ for } |x| = R \text{ and } \lambda < 1, \end{aligned}$$

and expresses an expansion property of N over $X_{r,R}$.

Thus we may say that compression is related to minima, while expansion is related to maxima.

5. Application to boundary value problems

In this section we present a variational method for localization in annular domains of the solutions of boundary value problems, as an alternative approach to Krasnoselskii's fixed point method intensively used in the literature (see, e.g. [1], [2], [4], [7], [8], [15]). By our knowledge, this is for the first time that estimations from below are obtained in terms of the energetic norm. We shall discuss the two-point boundary value

$$\begin{cases} -u''(t) = f(u(t)), & t \in [0,1] \\ u(0) = u(1) = 0, \end{cases} \tag{5.1}$$

where f is a continuous function on $\mathbf{R}$, nonnegative and nondecreasing on $\mathbf{R}_+$. We seek positive solutions which are symmetric with respect to the middle of the interval $[0,1]$, that is $u(1-t) = u(t)$ for every $t \in \left[0, \frac{1}{2}\right]$. Thus we shall consider the Hilbert space

$$X = \widehat{H}_0^1(0,1) := \left\{ u \in H_0^1(0,1) : u(1-t) = u(t) \text{ for all } t \in \left[0, \frac{1}{2}\right] \right\},$$

endowed with the inner product

$$(u,v) = \int_0^1 u'(t)\, v'(t)\, dt$$

and norm

$$|u| = \left(\int_0^1 u'(t)^2 \, dt \right)^{1/2} = \left(2 \int_0^{\frac{1}{2}} u'(t)^2 \, dt \right)^{1/2}.$$

We also consider the functional

$$F : \widehat{H}_0^1(0,1) \to \mathbf{R}, \;\; F(u) = \int_0^1 \left(\frac{1}{2} u'(t)^2 - g(u(t)) \right) dt,$$

where $g(\tau) = \int_0^\tau f(s) \, ds$. Clearly F is a C^1-functional and

$$F'(u) = u - N(u), \tag{5.2}$$

where

$$N(u)(t) = \int_0^1 G(t,s) f(u(s)) \, ds,$$

and G is the Green function $G(t,s) = s(1-t)$ for $0 \le s \le t \le 1$, $G(t,s) = t(1-s)$ for $0 \le t < s \le 1$. Hence the solutions of (5.1) are critical points of F. Note that F is bounded from below on each ball of its domain. Indeed, if $|u| \le R$, then

$$|u(t)| = \left| \int_0^t u'(s) \, ds \right| \le \left(\int_0^1 1^2 ds \right)^{1/2} \left(\int_0^1 u'(s)^2 \, ds \right)^{1/2} \le R \tag{5.3}$$

for all $t \in [0,1]$. Hence

$$F(u) \ge - \int_0^1 g(u(t)) \, dt \ge - \max_{|\tau| \le R} g(\tau) > -\infty.$$

Theorem 5.1. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function, nondecreasing on $\mathbf{R}_+$, with $f(\tau) > 0$ for all $\tau > 0$. Assume that there are two numbers $0 < r < R$ such that*

$$f\left(\frac{3r}{25} \right) \ge \frac{125r}{9}, \tag{5.4}$$

$$f(R) \le R. \tag{5.5}$$

Then (5.1) has a nonnegative concave solution u with $\frac{3r}{25} \le u\left(\frac{1}{5}\right) \le R$, which minimizes F on the set of all nonnegative functions $v \in \widehat{H}_0^1(0,1)$ satisfying $r \le |v| \le R$.

Proof. We shall apply the cone version of Theorem 3.1, where the cone is $K = \widehat{H}_0^1(0,1;\mathbf{R}_+)$. First note that the boundedness conditions (3.3) are satisfied in view of (5.2) and of the property of N of sending bounded sets into bounded sets. Also, the Palais-Smale condition holds due to the complete continuity of the operator N, and the invariance condition (3.6) is guaranteed by the positivity of N. Thus it remains to check the boundary conditions (3.4). Assume first that the condition corresponding to the sphere $|u| = R$ does not hold. Then there is $u \in \widehat{H}_0^1(0,1;\mathbf{R}_+)$ with $|u| = R$ and $\mu > 0$ such that $F'(u) + \mu u = 0$. Then $N(u) = (1+\mu) u$, that is

$$-u''(t) = \frac{1}{1+\mu} f(u(t)) \quad \text{on } [0,1] \text{ and } u(0) = u(1) = 0. \tag{5.6}$$

Clearly, since f is nonnegative, u is concave. In addition since u is symmetric with respect to $1/2$, u is increasing on $[0,1/2]$ and decreasing on $[1/2,1]$. If we multiply by $u(t)$, we integrate over $[0,1]$, and we take into account (5.3), we obtain

$$R^2 = |u|^2 = \frac{1}{1+\mu}\int_0^1 f(u(t))\,u(t)\,dt < \int_0^1 f(u(t))\,u(t)\,dt \le f(R)\,R,$$

which contradicts (5.5). Next assume that the boundary condition on the sphere $|u| = r$ does not hold. Then, for some $u \in \widehat{H}_0^1(0,1;\mathbf{R}_+)$ with $|u| = r$ and $\mu < 0$, we have $F'(u) + \mu u = 0$, that is $N(u) = (1+\mu)\,u$, or equivalently (5.6), where this time $\frac{1}{1+\mu} > 1$. Now fix any number $a \in (0,1/2)$ and as above, after multiplication and integration, obtain

$$\begin{aligned} r^2 &= |u|^2 = \frac{1}{1+\mu}\int_0^1 f(u(t))\,u(t)\,dt > \int_0^1 f(u(t))\,u(t)\,dt \qquad (5.7)\\ &= 2\int_0^{\frac{1}{2}} f(u(t))\,u(t)\,dt \ge 2\int_a^{\frac{1}{2}} f(u(t))\,u(t)\,dt \ge 2\left(\frac{1}{2}-a\right) f(u(a))\,u(a). \end{aligned}$$

On the other hand, u' being decreasing, we have

$$u(a) = \int_0^a u'(t)\,dt \ge a u'(a). \qquad (5.8)$$

In addition it is not difficult to prove the inequality

$$u'(a) \ge (1-2a)\,u'(0). \qquad (5.9)$$

Indeed, if we let $\phi(t) = u'(t) - (1-2t)\,u'(0)$ for $t \in \left[0,\frac{1}{2}\right]$, then

$$\phi'(t) = u''(t) + 2u'(0) = -\frac{1}{1+\mu} f(u(t)) + 2u'(0).$$

Since $f(u(t))$ is increasing on $[0,1/2]$, we deduce that ϕ' is decreasing, so ϕ is concave. In addition $\phi(0) = \phi(1/2) = 0$. Hence $\phi(t) \ge 0$ for all $t \in [0,1/2]$. Thus (5.9) is true. An other remark is that

$$r^2 = \int_0^1 u'(t)^2\,dt = 2\int_0^{\frac{1}{2}} u'(t)^2\,dt \le u'(0)^2,$$

whence

$$u'(0) \ge r. \qquad (5.10)$$

Now (5.8), (5.9) and (5.10) give $u(a) \ge a(1-2a)\,r$. This together with (5.7) implies

$$r^2 > a(1-2a)^2 f(a(1-2a)\,r)\,r,$$

that is

$$f(a(1-2a)\,r) < \frac{r}{a(1-2a)^2}.$$

For $a = 1/5$ this contradicts (5.4). Therefore all the assumptions of the cone version of Theorem 3.1 hold. $\square$

Finally we note that Theorem 3.1 in abstract setting and Theorem 5.1 for a concrete application, immediately yield multiplicity results of solutions if their hypotheses are satisfied for several finitely or infinitely many pairs of numbers r, R.

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The variation of curves length reported to cone metric

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Abstract. On a Lorentz manifold (M,g) we consider a timelike, parallel and unitary vector field Z. We define the Z-length of a curve and we obtain their first and second variation.

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Keywords: Lorentz manifold, field of tangent cones, first and second variation.

1. Introduction

In 1988 Dan I. Papuc has started the study of differential manifold endowed with a field of tangent cones. This mathematical structure includes also the Lorentz manifold (M, g) with the cone of future directed, timelike vector fields. The future-directed cone is defined by normalized vector field Z. So, we have in each point $p \in M$ the structure (T_pM, K_p) where $K_p = \{v \in T_pM \mid g(v, v) \leq 0, g(v, Z_p) < 0\}$. This implies a Krein space where the following order relation is defined:

$$v \leq w \text{ if and only if } v - w \in K_p$$

Moreover, this order relation involves the definition of a norm [3], [4] named Z-norm through:

$$|v|_{Z_p} = \inf\{\lambda \geq 0 \mid -\lambda Z_p \leq v \leq \lambda Z_p\}$$

The expression of the Z- norm is by [5]:

$$|v|_{Z_p} = |g(v, Z_p)| + \sqrt{g(v, v) + [g(v, Z_p)]^2}$$

For a smooth curve $\lambda : [a, b] \to M$, we define its Z- length the value:

$$\begin{aligned} L_Z(\lambda) &= \int_a^b |\lambda'(t)|_{Z_p} dt \\ &= \int_a^b \left\{ |g(\lambda'(t), Z_{\lambda(t)})| + \sqrt{g(\lambda'(t), \lambda'(t)) + g^2(\lambda'(t), Z_{\lambda(t)})} \right\} dt \end{aligned}$$

We state that the curve $\lambda : [a,b] \to M$ is Z-global in $x_0 = \lambda(t_0)$ if $\lambda'(t_0)$ and $Z_{\lambda(t)}$ are collinear.
For the calculus of the first variation we need to consider some restrictive hypotheses:
A) The future-directed, normalized, timelike vector field Z is parallel, this meaning

$$\nabla_X Z = 0, \forall X \in \mathcal{X}(M).$$

B) The curve λ, $\lambda : [a,b] \to M$ is not Z-global in any of its points.

For this hypothesis we make the following remarks:

Remark 1.1. The necessary condition for B) hypothesis involves $h(\lambda'(t), \lambda'(t)) \neq 0$ where $h(X,Y) = g(X,Y) + g(X,Z)g(Y,Z)$.

We have $h(X,X) = 0 \Leftrightarrow Gram\{X.Z\} = \begin{vmatrix} g(X,X) & g(X,Z) \\ g(Z,X) & g(Z,Z) \end{vmatrix} = 0 \Leftrightarrow \{X,Z\}$ are collinear.

Remark 1.2. If $\phi : (-\varepsilon, \varepsilon) \times [t_p, t_g] \to M$ is a piecewise smooth variation of a timelike, future directed curve λ which is not Z-global, then it exists $\delta > 0$ with the property that $\phi(u, \ .\ ) : [t_p, t_g] \to M$ is timelike, future directed and not Z-global for every $|u| < \delta$.

Beem [2] (page 253) proved the previous statement for a geodesic segment λ. Without any difficulty, we can give up on the restriction of geodesic segment, considering λ a piecewise smooth timelike future directed curve. It still remains to demonstrate that $\phi(u, \ .\ ) : [t_p, t_g] \to M$ is not Z-global for $|u| < \delta$.

Firstly, the smooth differentiation of ϕ involves the fact that it exists $\varepsilon_1 < \varepsilon$ as $\phi : [-\varepsilon_1, \varepsilon_1] \times [t_p, t_g] \to M$ is differentiable on compact. Consequently, we can extend to an open set which contains $[-\varepsilon_1, \varepsilon_1] \times [t_p, t_g]$. Because λ is timelike and not Z-global, we have $\phi'(u, t_p^+)$, $\phi'(u, t_q^-)$ timelike vectors which are not collinear with $Z_{\phi(u,t_p^+)}$, respectively $Z_{\phi(u,t_q^-)}$ for $\forall$ $|u| < \delta_1$. We assume that for any $\delta > 0$, $\phi(u_0, \ .\ ) : [t_p, t_g] \to M$ is not Z-global, for $\forall$ $|u_0| < \delta_1$. So it exists a sequence $u_n \to 0$ for which $\phi'(u_n, t_n)$ is collinear with $Z_{\phi(u_n,t_n)}$. Hence $(u_n, t_n) \in [-\varepsilon_1, \varepsilon_1] \times [t_p, t_q]$, which is compact, it results that we have an accumulation point $(0, t)$. So, $\phi'(0, t)$ is collinear with $Z_{\phi(0,t)}$, or $\lambda'(t)$ is collinear with $Z_{\lambda(t)}$, that involves λ being Z-global in $x = \lambda(t)$, affirmation excluded by the hypothesis.

Secondly, we consider the case where $\phi : (-\varepsilon, \varepsilon) \times [t_p, t_g] \to M$ is a piecewise smooth variation of piecewise smooth timelike, future directed and not Z-global curve λ. There is a partition $t_p = t_0 < t_1 < ... < t_k = t_q$ so that $\phi|_{(-\varepsilon,\varepsilon)\times[t_{i-1},t_i]}$ is a smooth timelike not Z-global future directed variation of $\lambda|_{[t_{i-1},t_i]}$, $\forall i = \overline{1,k}$. According to the above results, we have δ_i, $i = \overline{1,k}$ so that $\phi(u, \ ) : [t_{i-1}, t_i] \to M$ is not Z-global and $\phi'(u, t_{i-1}^+)$, $\phi'(u, t_i^-)$ are not collinear with $Z_{\phi(u,t_{i-1}^+)}$, respectively $Z_{\phi(u,t_i^-)}$ for $|u| < \delta_i$. Considering $\delta = \min\limits_{i=\overline{1,k}} \delta_i$, we obtain that $\phi(u, \ ) : [t_p, t_g] \to M$ is not a Z-global curve for $|u| < \delta$.

C) We assume that $\lambda : [t_p, t_q] \to M$ is a timelike future directed curve, h-unitary parametrized, this means $h(\lambda', \lambda') = g(\lambda', \lambda') + g(\lambda', Z)^2 = 1$.

Remark 1.3. Obviously, the h-unitary condition implies that the λ curve is not Z-global as stated in Remark 1.1.

Considering $\Phi : \mathcal{A} \to M$, $\mathcal{A} := (-\varepsilon, \varepsilon) \times [t_p, t_q]$ a proper smooth causal variation of λ, the curves $\phi(u, \ ) : [t_p, t_q] \to M$, $|u| < \varepsilon$ are not Z-global in any of their points. So:

1. $\Phi(0, t) = \lambda(t)$, $\forall t \in [t_p, t_q]$
2. $\Phi(u, t_p) = p$, $\Phi(u, t_q) = q$
3. $\Phi \in \mathcal{C}^3(\mathcal{A})$
4. $g(V, V) < 0$, $g(V, Z) < 0$, $g(V, Z)^2 + g(V, V) \neq 0$ where $V = \phi_*(\frac{\partial}{\partial t})$

We note the variation vector field by $X = \phi_*(\frac{\partial}{\partial u})$ where $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial t}\}$ is a base in $T_{(u,t)}\mathcal{A}$. We note $L_Z(u) = L_Z(\phi(u, \ ))$. Considering (A) and (B) hypotheses to be valid, we have to demonstrate:

Lemma 1.4. *The first variation of Z- length of λ is:*

$$\frac{d}{du} L_Z(0) = -\int_{t_p}^{t_q} \frac{1}{\sqrt{h(\lambda'.\lambda')}} h(\lambda'', P^h_{(\lambda')^\perp} X)|_{(0,t)} dt$$

where $P^h_{(Y)^\perp} X = X - \frac{h(X,Y)}{h(Y,Y)} Y$ is the X projection, respecting the bilinear form h on $Y^\perp$.

If, additionally, we assume that λ is a geodesic segment, and all three A, B, C hypotheses are being satisfied, we prove:

Lemma 1.5. *The second variation of Z- length of λ is:*

$$\frac{d^2}{du^2} L_Z(0) = -\int_{t_p}^{t_q} h(R(X, \lambda')\lambda' + N'', N)|_{(0,t)} dt +$$
$$+ \{h(\nabla_X X, \lambda') - g(\nabla_X X, Z) + h(N', N)\}|_{(0,t)}\Big|_{t_p}^{t_q}$$

where $N = X - h(X, V)V$.

2. The first variation

We make the following remark:
$h : TM \times TM \to \mathbb{R}$ is a bilinear metric, positive form, semidefined, degenerated, with its signature $(n-1, 0, 1)$ because:

$$X(h(Y_1, Y_2)) = X[g(Y_1, Y_2) + g(Y_1, Z)g(Y_2, Z)] =$$
$$= g(\nabla_X Y_1, Y_2) + g(Y_1, \nabla_X Y_2) +$$
$$+[g(\nabla_X Y_1, Z) + g(Y_1, \nabla_X Z)]g(Y_2, Z) +$$
$$+g(Y_1, Z)[g(\nabla_X Y_2, Z) + g(Y_2, \nabla_X Z)]$$
$$= g(\nabla_X Y_1, Y_2) + g(Y_1, \nabla_X Y_2) + g(\nabla_X Y_1, Z)g(Y_2, Z) +$$
$$+g(Y_1, Z)g(\nabla_X Y_2, Z) = h(\nabla_X Y_1, Y_2) + h(Y_1, \nabla_X Y_2)$$

where we used the $A)$ hypothesis about Z, namely $\nabla_X Z = 0$.

We have for the Z- length of curve $\phi(u, \ ) : [t_p, t_q] \to M$ the expression:

$$L_Z(u) = \int_{t_p}^{t_q} \left\{ -g(V, Z) + \sqrt{h(V, V)} \right\} dt$$

and:

$$\frac{d}{du} L_Z(u) = \int_{t_p}^{t_q} \left\{ -\frac{\partial}{\partial u}[g(V, Z)] + \frac{\partial}{\partial u} \sqrt{h(V, V)} \right\} dt =$$

$$= \int_{t_p}^{t_q} \left\{ -g(\nabla_X V, Z) - g(V, \nabla_X Z) + \frac{1}{\sqrt{h(V, V)}} h(\nabla_X V, V) \right\} dt$$

Since $[X, V] = 0$, then $\nabla_X V = \nabla_V X$ and so:

$$\frac{d}{du} L_Z(u) = \int_{t_p}^{t_q} \left\{ -g(\nabla_X V, Z) + \frac{h(\nabla_V X, V)}{\sqrt{h(V, V)}} \right\} dt \tag{2.1}$$

We calculate:

$$\frac{\partial}{\partial t}[g(X, Z)] = g(\nabla_V X, Z) + g(X, \nabla_V Z) = g(\nabla_V X, Z) \tag{2.2}$$

$$\frac{\partial}{\partial t} \left[\frac{h(X, V)}{\sqrt{h(V, V)}} \right] =$$

$$= \frac{[h(\nabla_V X, V) + h(X, \nabla_V V)]\sqrt{h(V, V)} - h(X, V) \frac{h(\nabla_V V, V)}{\sqrt{h(V, V)}}}{h(V, V)}$$

$$= \frac{h(\nabla_V X, V)}{\sqrt{h(V, V)}} + \frac{h(X, \nabla_V V)}{\sqrt{h(V, V)}} - \frac{h(X, V) h(\nabla_V V, V)}{\left[\sqrt{h(V, V)}\right]^3} =$$

$$= \frac{h(\nabla_V X, V)}{\sqrt{h(V, V)}} + \frac{1}{\sqrt{h(V, V)}} \left[h(\nabla_V V, X) - \frac{h(X, V)}{h(V, V)} h(\nabla_V V, V) \right] =$$

$$= \frac{h(\nabla_V X, V)}{\sqrt{h(V, V)}} + \frac{1}{\sqrt{h(V, V)}} h\left(\nabla_V V, X - \frac{h(X, V)}{h(V, V)} V \right) \tag{2.3}$$

Replacing the results from (2.2) and (2.3) in (2.1), we obtain the following:

$$\frac{d}{du} L_Z(0) =$$

$$= \int_{t_p}^{t_q} \left\{ -\frac{\partial}{\partial t}[g(X,Z)] + \frac{\partial}{\partial t}\left[\frac{h(X,V)}{\sqrt{h(V,V)}}\right] - \frac{1}{\sqrt{h(V,V)}} h(\nabla_V V, P^h_{V^\perp} X) \right\}\Bigg|_{(0,t)} dt$$

$$= -\int_{t_p}^{t_q} \left\{ \frac{1}{\sqrt{h(V,V)}} h(\nabla_V V, P^h_{V^\perp} X) \right\}\Bigg|_{(0,t)} dt +$$

$$+ \left\{ \frac{h(X,V)}{\sqrt{h(V,V)}} - g(X,Z) \right\}\Bigg|_{(0,t)}\Bigg|_{t_p}^{t_q}$$

$$= -\int_{t_p}^{t_q} \left\{ \frac{1}{\sqrt{h(V,V)}} h(\nabla_V V, P^h_{V^\perp} X) \right\}\Bigg|_{(0,t)} dt$$

$$= -\int_{t_p}^{t_q} \left\{ \frac{1}{\sqrt{h(\lambda',\lambda')}} h(\lambda'', P^h_{(\lambda')^\perp} X) \right\}\Bigg|_{(0,t)} dt$$

We have used the hypothesis of proper variation, namely: $X(t_p^+) = X(t_q^-) = 0$

Remark 2.1. If λ is a geodesic segment, then $\frac{d}{du} L_Z(0) = 0$ because $\nabla_V V|_{(0,t)} = \frac{D\lambda'}{\partial t} = 0$. In consequence, the geodesic segments, that are not Z-global, are stationary points for the Z- length functional.

Remark 2.2. Let it be $\lambda : [t_p, t_q] \to M$ a piecewise smooth timelike and future directed curve, $h-$unitary parametrized which is not Z- global. Noting with $L_Z^i(u)$ the Z-length of the uniparametric variation of the curve $\lambda|_{[t_{i-1},t_i]}$, $i = \overline{1,k}$ we have:

$$\frac{dL_Z^i(0)}{du} = -\int_{t_{i-1}}^{t_i} h(\lambda'', P^h_{(\lambda')^\perp} X)\Big|_{(0,t)} dt + \\ + \{h(\lambda', X) - g(X,Z)\}|_{(0,t)}\Big|_{t_{i-1}}^{t_i}$$

therefore

$$\frac{dL_Z}{du}(0) = \sum_{i=1}^{k} \frac{dL_Z^i(0)}{du} = -\int_{t_p}^{t_q} h\left(\lambda'', P^h_{(\lambda')^\perp} X\right)\Big|_{(0,t)} dt - \\ - \sum_{i=1}^{k-1} h\left(X(t_i), \Delta_{t_i}(\lambda')\right)$$

where $\Delta_{t_i}(\lambda') = \lambda'\left(t_i^+\right) - \lambda'\left(t_i^-\right)$, $\forall i = \overline{1, k-1}$ and we have taken into account the fact that the variation is proper, so: $X(\lambda(t_p)) = X(\lambda(t_q)) = 0$.

Remark 2.3. Considering H a spatial hypersurface and assuming that $\lambda : [t_p, t_q] \to M$ is a timelike future directed geodesical segment, not Z-global with $\lambda(t_p) = p \in H$ and

$\lambda(t_p) = q \notin H$, then:

$$0 = \frac{dL_Z(0)}{du} = g(X_p, Z_p) - g(X_p, \lambda'(t_p)) - g(X_p, Z_p) g(\lambda'(t_p), Z_p) =$$
$$= -g(X_p, \lambda'(t_p) + g(\lambda'(t_p), Z_p) Z_p - Z_p)$$

It results that $\lambda'(t_p) + [g(\lambda'(t_p), Z_p) - 1] Z_p$ has to be g orthogonal on H.

If $\lambda : [t_p, t_q] \to M$ is a geodesic segment, an affine parametrization of λ so that $g(\lambda'(t_p), Z_p) = 1$ can be found. In this case, the above condition becomes: $\lambda'(t_p)$ is g orthogonal on H.

3. The second variation

We calculate the second variation of the Z-length of the geodesical curve λ, h-unitary parametrized. Starting with the formula (2.1), we have:

$$\frac{d}{du}[h(\nabla_V X, V)] = h(\nabla_X \nabla_V X, V) + h(\nabla_V X, \nabla_X V)$$
$$\frac{d}{du}[h(V, V)] = 2h(\nabla_X V, V)$$

Then

$$\begin{aligned} I &\stackrel{def}{=} \frac{d}{du}\left\{-g(\nabla_V X, Z) + h(V, V)^{-\frac{1}{2}} h(\nabla_V X, V)\right\} \qquad (3.1) \\ &= -g(\nabla_X \nabla_V X, Z) - h(V, V)^{-\frac{3}{2}} h(\nabla_X V, V) h(\nabla_V X, V) + \\ &\quad + h(V, V)^{-\frac{1}{2}} [h(\nabla_X \nabla_V X, V) + h(\nabla_V X, \nabla_X V)] \\ &= -g(\nabla_X \nabla_V X, Z) - h(V, V)^{-\frac{3}{2}} [h(\nabla_V X, V)]^2 + \\ &\quad + h(V, V)^{-\frac{1}{2}} [h(\nabla_X \nabla_V X, V) + h(\nabla_V X, \nabla_X V)] \end{aligned}$$

We define

$$N \stackrel{def}{=} X - h(X, V) V \qquad (3.2)$$

the h-normal on V vector field.

Next we calculate:

$$\begin{aligned} \nabla_V [h(X, V) V]|_{(0,t)} &= \left\{\frac{d}{dt}[h(X, V)]\right\} V + h(X, V) \nabla_V V|_{(0,t)} \qquad (3.3) \\ &= \left\{\frac{d}{dt}[h(X, V)]\right\} V \end{aligned}$$

$$h(\nabla_V N, V)|_{(0,t)} = \frac{d}{dt}\{h(N, V)\} - h(N, \nabla_V V)|_{(0,t)} = 0 \qquad (3.4)$$

$$\begin{aligned} h(\nabla_V N, \nabla_V [h(X, V) V])|_{(0,t)} &= h\left(\nabla_V N, \left\{\frac{d}{dt}[h(X, V)]\right\} V\right) \qquad (3.5) \\ &= \frac{d}{dt}[h(X, V)] h(\nabla_V N, V)|_{(0,t)} = 0 \end{aligned}$$

$$h(\nabla_V X, \nabla_X V)|_{(0,t)} = h(\nabla_V X, \nabla_V X)|_{(0,t)}$$
$$\stackrel{(3.2)}{=} h(\nabla_V \{N + h(X,V)V\}, \nabla_V \{N + h(X,V)V\})|_{(0,t)}$$
$$\stackrel{(3.3)}{=} h\left(\nabla_V N + \frac{d}{dt}\{h(X,V)\} V, \nabla_V N + \frac{d}{dt}\{h(X,V)\} V\right)|_{(0,t)}$$
$$= h(\nabla_V N, \nabla_V N) + 2\frac{d}{dt}\{h(X,V)\} h(\nabla_V N, V) + \tag{3.6}$$
$$+ \left[\frac{d}{dt}\{h(X,V)\}\right]^2 h(V,V)|_{(0,t)}$$
$$\stackrel{(3.4)}{=} h(\nabla_V N, \nabla_V N) + \left[\frac{d}{dt}\{h(X,V)\}\right]^2 |_{(0,t)}$$

$$h(\nabla_V N, V)|_{(0,t)} = h(\nabla_V \{N + h(X,V)V\}, V)|_{(0,t)} = h(\nabla_V N, V) +$$
$$+ h(\nabla_V h(X,V)V, V) \stackrel{(3.3)}{=} h(\nabla_V N, V) + \frac{d}{dt}\{h(X,V)\}|_{(0,t)} \tag{3.7}$$

Replacing these results in (3.1) we have:

$$I|_{(0,t)} = -g(\nabla_X \nabla_V X, Z) + h(\nabla_X \nabla_V X, V) + h(\nabla_V N, \nabla_V N)|_{(0,t)}$$

We get:

$$\frac{d^2 L_Z(0)}{du^2} = \int_{t_p}^{t_q} \{-g(\nabla_X \nabla_V X, Z) + h(\nabla_X \nabla_V X, V) + h(\nabla_V N, \nabla_V N)\}|_{(0,t)} dt \tag{3.8}$$

and for this equality, we calculate:

$$\nabla_X \nabla_V X = \nabla_V \nabla_X X - R(V,X)X \tag{3.9}$$

$$g(\nabla_V \nabla_X X, Z)|_{(0,t)} = \frac{d}{dt}\{g(\nabla_X X, Z)\} - g(\nabla_X X, \nabla_V Z)|_{(0,t)} \tag{3.10}$$
$$= \frac{d}{dt}\{g(\nabla_X X, Z)\}|_{(0,t)}$$

$$-g(\nabla_X \nabla_V X, Z)|_{(0,t)} = g(R(V,X)X - \nabla_V \nabla_X X, Z)|_{(0,t)} \tag{3.11}$$
$$= g(R(V,X)X, Z) - \frac{d}{dt}\{g(\nabla_X X, Z)\}|_{(0,t)}$$

$$h(\nabla_X \nabla_V X, V)|_{(0,t)} = h(\nabla_V \nabla_X X - R(V,X)X, V)|_{(0,t)} \tag{3.12}$$
$$= \frac{d}{dt}\{h(\nabla_X X, V)\} - h(\nabla_X X, \nabla_V V) - h(R(V,X)X, V)|_{(0,t)}$$
$$= -h(R(V,X)X, V) + \frac{d}{dt}\{h(\nabla_X X, V)\}|_{(0,t)}$$

$$h(\nabla_V N, \nabla_V V) = \frac{d}{dt}\{h(N, \nabla_V N)\} - h(N, \nabla_V \nabla_V N) \tag{3.13}$$

Replacing the results from (3.11), (3.12) and (3.13), (3.8) becomes:

$$\frac{d^2 L_Z(0)}{du^2} =$$

$$= \int_{t_p}^{t_q} \left\{ g(R(V,X)X,Z) - \frac{d}{dt}\{g(\nabla_X X, Z)\} - h(R(V,X)X,V) \right\} |_{(0,t)} dt+ \quad (3.14)$$
$$+ \int_{t_p}^{t_q} \left\{ -h(N, \nabla_V \nabla_V N) + \frac{d}{dt}\{h(\nabla_X X, V)\} + \frac{d}{dt}\{h(N, \nabla_V N)\} \right\} |_{(0,t)} dt$$
$$= \int_{t_p}^{t_q} \{g(R(V,X)X,Z) - h(R(V,X)X,V) - h(N, \nabla_V \nabla_V N)\} |_{(0,t)} dt+$$
$$+ \{-g(\nabla_X X, Z) + h(\nabla_X X, V) + h(N, \nabla_V N)\} |_{(0,t)} \Big|_{t_p}^{t_q}$$

For the (3.14) expressions we have:

$$g(R(V,X)X,Z) = -g(R(V,X)Z,X) = \quad (3.15)$$
$$= -g(\nabla_V \nabla_X Z - \nabla_X \nabla_V Z, X) = 0$$

$$h(R(V,X)X,V) = g(R(V,X)X,V) + g(R(V,X)X,Z)g(V,Z) \quad (3.16)$$
$$= g(R(V,X)X,V) = g(R(X,V)V,X) = g(R(X,V)V, N + h(X,V)V)$$
$$= g(R(X,V)V,N) + g(R(X,V)V,V)h(X,V) = g(R(X,V)V,N)$$

$$g(R(X,V)V,Z) = -g(R(X,V)Z,V) = 0 \quad (3.17)$$

With the results from (3.15), (3.16) and (3.17) replaced in (3.14) we obtain:

$$\frac{d^2 L_Z(0)}{du^2} =$$

$$= \int_{t_p}^{t_q} \{-g(R(X,V)V,N) - h(N, \nabla_V \nabla_V N)\} |_{(0,t)} dt+$$
$$+ \left\{ \{h(\nabla_X X, V) + h(N, \nabla_V N) - g(\nabla_X X, Z)\} |_{(0,t)} \right\} \Big|_{t_p}^{t_q}$$
$$= \int_{t_p}^{t_q} \{-g(R(X,V)V,N) - g(R(X,V)V,Z)g(N,Z) - h(N, \nabla_V \nabla_V N)\} |_{(0,t)} dt+$$
$$+ \left\{ \{h(\nabla_X X, V) + h(N, \nabla_V N) - g(\nabla_X X, Z)\} |_{(0,t)} \right\} \Big|_{t_p}^{t_q}$$
$$= \int_{t_p}^{t_q} \{-h(R(X,V)V,N) - h(N, \nabla_V \nabla_V N)\} |_{(0,t)} dt+$$
$$+ \left\{ \{h(\nabla_X X, V) + h(N, \nabla_V N) - g(\nabla_X X, Z)\} |_{(0,t)} \right\} \Big|_{t_p}^{t_q}$$

In conclusion the second variation formula is the following:

$$\frac{d^2 L_Z(0)}{du^2} = - \int_{t_p}^{t_q} h(R(X,\lambda')\lambda' + N'', N)|_{(0,t)} dt+$$
$$+ \{h(\nabla_X X, \lambda') + h(N', N) - g(\nabla_X X, Z)\} |_{(0,t)} \Big|_{t_p}^{t_q}$$

Remark 3.1. In the hypothesis according to which $\phi : (-\varepsilon, \varepsilon) \times [t_p, t_q] \to M$ is a proper variation of λ, we have that: $X(t_p^+) = X(t_q^-) = 0$ and therefore $N(t_p^+) = N(t_q^-) = 0$. This simplifies the second variation formula:

$$\frac{d^2 L_Z(0)}{du^2} = -\int_{t_p}^{t_q} h(R(X, \lambda')\lambda' + N'', N)|_{(0,t)} dt +$$
$$+ \{h(\nabla_X X, \lambda') - g(\nabla_X X, Z)\}\,|_{(0,t)}\Big|_{t_p}^{t_q}$$

Remark 3.2. In the hypothesis stating that ϕ is a canonical proper variation of λ, (meaning that $\phi(u, t) = \exp_{\lambda(t)} uY(t)$, with $Y(t)$ a vector field along λ, that respects:

$$Y(t_p) = Y(t_q) = 0, g(Y(t), \lambda'(t)) = 0, \forall t \in [t_p, t_q])$$

we have that the curves: $u \longmapsto \phi(u, t_0)$ are geodesics, namely

$$\nabla_X X = 0 \quad \text{and} \quad X = \phi_*(\frac{\partial}{\partial u})|_{(0,t)} = Y(t).$$

This implies the following expression for the second variation:

$$\frac{d^2 L_Z(0)}{du^2} = -\int_{t_p}^{t_q} h(R(Y, \lambda')\lambda' + N'', N)|_{(0,t)} dt.$$

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A unified theory of weakly contra-(μ, λ)-continuous functions in generalized topological spaces

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Abstract. We introduce a new notion called weakly contra-(μ, λ)-continuous functions as functions on generalized topological spaces [17]. We obtain some characterizations and several properties of such functions. The functions enable us to formulate a unified theory of several modifications of weak contra-continuity due to Baker [9].

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1. Introduction

In [15]-[25], Á. Császár founded the theory of generalized topological spaces, and studied the elementary character of these classes. Especially he introduced the notion of continuous functions on generalized topological spaces and investigated characterizations of generalized continuous functions (= (μ, λ)-continuous functions in [17]). We recall some notions defined in [17]. Let X be a non-empty set and $\exp X$ the power set of X. We call a class $\mu \subseteq \exp X$ a generalized topology [17] (briefly, GT) if $\phi \in \mu$ and the arbitrary union of elements of μ belongs to μ. A set X with a GT μ on it is called a generalized topological space (briefly, GTS) and is denoted by (X, μ).

For a GTS (X, μ), the elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ- closed set containing A; and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A (see [17], [22]). Obviously in a topological space (X, τ), if one takes τ as the GT, then c_μ becomes the usual closure operator.

It is easy to observe that i_μ and c_μ are idempotent and monotonic, where γ : $\exp X \to \exp X$ is said to be idempotent if $A \subseteq B \subseteq X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic if $\gamma(A) \subseteq \gamma(B)$. It is also well known from [20] and [23] that let μ be a

GT on $X, A \subset X$ and $x \in X$, then (1) $x \in c_\mu(A)$ if and only if $M \cap A \neq \phi$ for every $M \in \mu$ containing x and (2) $c_\mu(X - A) = X - i_\mu(A)$.

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *regular closed* (resp. *regular open*) if $\mathrm{Cl}(\mathrm{Int}(A)) = A$ (resp. $\mathrm{Int}(\mathrm{Cl}(A)) = A$).

Definition 2.1. *Let (X, τ) be a topological space. A subset A of X is said to be* semi-open [40] *(resp.* preopen [42]*,* α-open [47]*,* β-open [1] *or* semi-preopen [4]*,* b-open [5]*) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(A)) \cup \mathrm{Int}(\mathrm{Cl}(A))$).*

We note that for any topological space (X, τ), the collection of all open (resp. semi-open, preopen, α-open, β-open, b-open) sets in X is denoted by τ (resp. $\mathrm{SO}(X)$ $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$ or $\mathrm{SPO}(X)$, $\mathrm{BO}(X)$). These collection form a GT.

Definition 2.2. *The complement of a semi-open (resp. preopen, α-open, β-open, b-open) set is said to be* semi-closed [14] *(resp.* preclosed [42]*,* α-closed [43]*,* β-closed [1] *or* semi-preclosed [4]*,* b-closed [5]*).*

Definition 2.3. *The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, b-closed) sets of X containing A is called the* semi-closure [14] *(resp.* preclosure [32]*,* α-closure [43]*,* β-closure [2] *or* semi-preclosure [4]*,* b-closure [5]*) of A and is denoted by sCl(A) (resp. pCl(A), αCl(A), ${}_\beta$Cl(A) or spCl(A), bCl(A)).*

Definition 2.4. The union of all semi-open (resp. preopen, α-open, β-open, b-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior* or *semi-preinterior*, *b-interior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$ or $\mathrm{spInt}(A)$, $\mathrm{bInt}(A)$).

Throughout the present paper, (X, τ) and (Y, σ) denote topological spaces and $f : (X, \tau) \to (Y, \sigma)$ presents a (single valued) function from a topological space (X, τ) into a topological space (Y, σ).

Definition 2.5. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *semi-continuous* [40] (resp. *precontinuous* [42], *α-continuous* [43], *β-continuous* [1], *b-continuous* [31]) if for each point $x \in X$ and each open set V of Y containing $f(x)$, there exists a semi-open (resp. preopen, α-open, β-open , b-open) set U of X containing x such that $f(U) \subset V$.

Definition 2.6. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *weakly continuous* [39] (resp. *weakly quasicontinuous* [59] or *weakly semi-continuous* [7], [13], [38], *almost weakly continuous* [37] or *quasi precontinuous* [55], *weakly α-continuous* [48], *weakly β-continuous* [56], weakly b-continuous [60]) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open (resp. semi-open, preopen, α-open, β-open, b-open) set U of X containing x such that $f(U) \subset \mathrm{Cl}(V)$.

A unified theory of weakly continuous functions is investigated in [58].

Definition 2.7. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *slightly continuous* [34] (resp. *slightly semi-continuous* [53], *slightly precontinuous* or *faintly semi-continuous* [54], *slightly β-continuous* [49], *slightly b-continuous* [31]) if for each point $x \in X$ and each clopen set V of Y containing $f(x)$, there exists an open (resp. semi-open, preopen, β-open , b-open) set U of X containing x such that $f(U) \subset V$.

A unified theory of slightly continuous functions is investigated in [57].

Definition 2.8. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *weakly contra-continuous* [9] (resp. weakly contra-precontinuous [11], weakly contra-β-continuous [10]) if for each open set V of Y and each closed set A of Y such that $A \subset V$, $\mathrm{Cl}(f^{-1}(A)) \subset f^{-1}(V)$ (resp. $\mathrm{pCl}(f^{-1}(A)) \subset f^{-1}(V)$, $\mathrm{spCl}(f^{-1}(A)) \subset f^{-1}(V)$).

Definition 2.9. [51] A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *weakly contra-semi-continuous* (resp. *weakly contra-α-continuous*, *weakly contra-γ-continuous* or weakly contra-b-continuous) if for each open set V of Y and each closed set A of Y such that $A \subset V$, $\mathrm{sCl}(f^{-1}(A)) \subset f^{-1}(V)$ (resp. $\alpha\mathrm{Cl}(f^{-1}(A)) \subset f^{-1}(V)$, $\mathrm{bCl}(f^{-1}(A)) \subset f^{-1}(V)$).

3. Weakly contra-(μ, λ)-continuous functions

Definition 3.1. *Let $f : (X, \mu) \to (Y, \lambda)$ be a function on generalized topological spaces. Then the function f is said to be*

1. *(μ, λ)-continuous* [17] *if $G \in \lambda$ implies that $f^{-1}(G) \in \mu$.*
2. *weakly (μ, λ)-continuous* [44] *if for each $x \in X$ and each λ-open set V containing $f(x)$, there exists a μ-open set U containing x such that $f(U) \subseteq c_\lambda(V)$.*
3. *almost (μ, λ)-continuous* [45] *if for each $x \in X$ and each λ-open set V containing $f(x)$, there exists a μ-open set U containing x such that $f(U) \subseteq i_\lambda(c_\lambda(V))$.*
4. *almost (μ, λ)-open if $f(U) \subseteq i_\lambda(c_\lambda(f(U)))$ for every $U \in \mu$.*
5. *contra-(μ, λ)-continuous* [3] *if $f^{-1}(V)$ is μ-closed in X for each λ-open set of Y.*

Definition 3.2. *Let (X, μ) and (Y, λ) be GTS's. Then a function $f : X \to Y$ is said to be* weakly contra-(μ, λ)-continuous *if for each λ-open set V of Y and each λ-closed set A of Y such that $A \subset V$, $c_\mu(f^{-1}(A)) \subset f^{-1}(V)$.*

Remark 3.3. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, $\mu = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{BO}(X)$) and $\lambda = \sigma$. If $f : (X, \mu) \to (Y, \lambda)$ is weakly contra-(μ, λ)-continuous, then f is weakly contra-continuous [51] (resp. weakly contra-semi-continuous, weakly contra-precontinuous, weakly contra-α-continuous, weakly contra-β-continuous, weakly contra-b-continuous).

Theorem 3.4. *If a function $f : (X, \mu) \to (Y, \lambda)$ is contra-(μ, λ)-continuous, then f is weakly contra-(μ, λ)-continuous.*

Proof. Let V be any λ-open set of Y and A a λ-closed set of Y such that $A \subset V$. Since f is contra-(μ, λ)-continuous, $f^{-1}(V) = c_\mu(f^{-1}(V))$. Therefore, we have $c_\mu(f^{-1}(A)) \subset c_\mu(f^{-1}(V)) = f^{-1}(V)$. This shows that f is weakly contra-(μ, λ)-continuous. $\square$

Remark 3.5. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and μ a GT on X. If $\mu = \tau$ (resp. PO(X), $\beta(X)$), then by Theorem 3.4 we obtain the results established in Theorem 3.1 of [9] (resp. Theorem 3.3 of [11], Theorem 3.3 of [10]).

Theorem 3.6. *If a function $f : (X, \mu) \to (Y, \lambda)$ is (μ, λ)-continuous, then f is weakly contra-(μ, λ)-continuous.*

Proof. Let V be any λ-open set of Y and A a λ-closed set of Y such that $A \subset V$. Since f is (μ, λ)-continuous, then we have $c_\mu(f^{-1}(A)) = f^{-1}(A) \subset f^{-1}(V)$. Therefore f is weakly contra-(μ, λ)-continuous. □

Remark 3.7. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and μ a GT on X. If $\mu = \tau$ (resp. PO(X), $\beta(X)$), then by Theorem 3.6 we obtain the results established in Theorem 3.4 of [9] (resp. Theorem 3.2 of [11], Theorem 3.2 of [10]).

Definition 3.8. *A function $f : (X, \mu) \to (Y, \lambda)$ is said to be* slightly (μ, λ)-continuous *if for each point $x \in X$ and each λ-clopen set V of Y containing $f(x)$, there exists $U \in \mu$ containing x such that $f(U) \subset V$.*

Theorem 3.9. *Let (X, μ) and (Y, λ) be GTSs. For a function $f : (X, \mu) \to (Y, \lambda)$, the following statements are equivalent:*

1. *f is slightly (μ, λ)-continuous;*
2. *for every λ-clopen set $V \subseteq Y$, $f^{-1}(V)$ is μ-open;*
3. *for every λ-clopen set $V \subseteq Y$, $f^{-1}(V)$ is μ-closed;*
4. *for every λ-clopen set $V \subseteq Y$, $f^{-1}(V)$ is μ-clopen.*

Proof. (1) ⇒ (2): Let V be a λ-clopen subset of Y and let $x \in f^{-1}(V)$. Since f is slightly (μ, λ)-continuous, there exists a μ-open set U_x in X containing x such that $f(U_x) \subseteq V$; hence $U_x \subseteq f^{-1}(V)$. We obtain that $f^{-1}(V) = \cup\{U_x | x \in f^{-1}(V)\}$. Thus $f^{-1}(V)$ is μ-open.
(2) ⇒ (3): Let V be a λ-clopen subset of Y. Then $Y \backslash V$ is λ-clopen. By (2) $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V)$ is μ-open. Thus $f^{-1}(V)$ is μ-closed.
(3) ⇒ (4): It can be shown easily.
(4) ⇒ (1): Let V be a λ-clopen subset in Y containing $f(x)$. By (4), $f^{-1}(V)$ is μ-clopen. Take $U = f^{-1}(V)$. Then, $f(U) \subseteq V$. Hence, f is slightly (μ, λ)-continuous. □

Theorem 3.10. *If a function $f : (X, \mu) \to (Y, \lambda)$ is weakly contra-(μ, λ)-continuous, then f is slightly (μ, λ)-continuous.*

Proof. Let V be a λ-clopen set of Y. If we put $A = V$, then by the weak contra-(μ, λ)-continuity we have $c_\mu(f^{-1}(V)) \subset f^{-1}(V)$ and hence $c_\mu(f^{-1}(V)) = f^{-1}(V)$. It follows from Theorem 3.9 that f is slightly (μ, λ)-continuous. □

Remark 3.11. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and μ be a GT on X. If $\mu = \tau$ (resp. PO(X), $\beta(X)$), then by Theorem 3.10 we obtain the results established in Theorem 3.7 of [9] (resp. Theorem 3.6 of [11], Theorem 3.6 of [10]).

Lemma 3.12. *If a function $f : (X, \mu) \to (Y, \lambda)$ is weakly (μ, λ)-continuous, then f is slightly (μ, λ)-continuous.*

Proof. This follows easily from Theorem 3.9. $\square$

The following implications are hold:

DIAGRAM

$$\begin{tikzcd}
& \text{contra-}(\mu,\lambda)\text{-continuity} \arrow[r] \arrow[d] & \text{weak } (\mu,\lambda)\text{-continuity} \arrow[d] \\
(\mu,\lambda)\text{-continuity} \arrow[r] & \text{weak contra-}(\mu,\lambda)\text{-continuity} \arrow[r] & \text{slightly } (\mu,\lambda)\text{-continuity}
\end{tikzcd}$$

Remark 3.13. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and μ be a GT on X. If $\mu = \tau$ (resp. PO(X), $\beta(X)$), then by DIAGRAM we obtain the diagram constructed in [9] (resp. [11], [10]).

Definition 3.14. *A generalized topological space (Y, λ) is said to be* 0-λ-dimensional *if each point of Y has a neighborhood base consisting of λ-clopen sets.*

Theorem 3.15. *Let (Y, λ) be 0-λ-dimensional. Then for a function $f : (X, \mu) \to (Y, \lambda)$, the following properties are equivalent:*

1. *f is (μ, λ)-continuous;*
2. *f is weakly contra-(μ, λ)-continuous;*
3. *f is slightly (μ, λ)-continuous.*

Proof. The proofs of the implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) follow from DIAGRAM.
(3) $\Rightarrow$ (1): Let $x \in X$ and let V be a λ-open subset of Y containing $f(x)$. Since Y is 0-λ-dimensional, there exists a λ-clopen set U containing $f(x)$ such that $U \subseteq V$. Since f is slightly (μ, λ)-continuous, then there exists a μ-open subset G in X containing x such that $f(G) \subseteq U \subseteq V$. Thus, f is (μ, λ)-continuous. $\square$

Definition 3.16. A topological space (Y, σ) is said to be *extremally disconnected* [61] (briefly E.D.) if the closure of each open set of Y is open in Y.

Theorem 3.17. *If $f : (X, \mu) \to (Y, \sigma)$ is weakly contra-(μ, σ)-continuous and (Y, σ) is E.D., then f is weakly (μ, σ)-continuous.*

Proof. Let V be an open set of Y. Since (Y, σ) is E.D., Cl(V) is clopen. Since f is weakly contra-(μ, σ)-continuous, $c_\mu(f^{-1}(V)) \subset c_\mu(f^{-1}(\mathrm{Cl}(V))) \subset f^{-1}(\mathrm{Cl}(V))$. Hence $c_\mu(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ for every open set V of Y. We claim that f is weakly (μ, σ)-continuous if $c_\mu(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ for every open set V of Y. Now let $x \in X$ and V be any open set containing $f(x)$. Since $V \cap (Y - Cl(V)) = \phi$, clearly $f(x) \notin Cl(Y - Cl(V))$ and hence $x \notin f^{-1}(Cl(Y - Cl(V)))$. Since $Y - Cl(V)$ is open, we have $x \notin c_\mu(f^{-1}(Y - Cl(V))$. Therefore, there exists a μ-open set U containing x such that $U \cap f^{-1}(Y - Cl(V)) = \phi$; hence $f(U) \cap (Y - Cl(V)) = \phi$. This shows that $f(U) \subseteq Cl(V)$. Therefore, f is weakly (μ, σ)-continuous. $\square$

Remark 3.18. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and μ a GT on X. If $\mu = \tau$ (resp. PO(X), $\beta(X)$), then by Theorem 3.17 we obtain the results established in Corollary 3.9 of [9] (resp. Corollary 3.10 of [11], Corollary 3.9 of [10]).

4. Weak contra-(μ, λ)-continuity and $(g\mu, \lambda)$-continuity

Definition 4.1. Let (X, τ) be a topological space. A subset A of X is said to be

1. *g-closed* [41] if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,
2. *αg-closed* [27] if $\alpha\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,
3. *gs-closed* [26] if $\mathrm{sCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,
4. *gp-closed* [6] if $\mathrm{pCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,
5. *gsp-closed* [28] if $\mathrm{spCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,
6. *γg-closed* [33] if $\mathrm{bCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.

Definition 4.2. *Let μ be a GT on a topological space (X, τ). Then a subset A of X is said to be generalized μ-closed (briefly $g\mu$-closed)* [52] *if $c_\mu(A) \subset U$ whenever $A \subset U$ and $U \in \tau$. The complement of a $g\mu$-closed set is called a generalized μ-open (or simply $g\mu$-open) set.*

Remark 4.3. [52] Let (X, τ) be a topological space and μ be a GT on X. Then every $g\mu$-closed set reduces to a g-closed (resp. gs-closed, gp-closed, αg-closed, gsp-closed, γg-closed) set if one takes μ to be τ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{BO}(X)$).

Definition 4.4. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be g-continuous* [12] *or weakly g-continuous* [46] *(resp. gs-continuous* [26]*, gp-continuous* [6]*, αg-continuous* [27]*, gsp-continuous* [28]*, γg-continuous* [33]*) if $f^{-1}(K)$ is g-closed (resp. gs-closed, gp-closed, αg-closed, gsp-closed, γg-closed) in X for every closed set K of Y.*

Definition 4.5. *Let (X, τ) be a topological space and μ be a GT on X. A function $f : (X, \mu) \to (Y, \lambda)$ is said to be $(g\mu, \lambda)$-continuous if $f^{-1}(K)$ is $g\mu$-closed for every λ-closed set K of Y.*

Remark 4.6. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, μ, λ be GTS's on X, Y and $\sigma = \lambda$. If $\mu = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{SPO}(X)$, $\mathrm{BO}(X)$) and $f : (X, \mu) \to (Y, \lambda)$ is $(g\mu, \lambda)$-continuous, then f is g-continuous (resp. gs-continuous, gp-continuous, αg-continuous, gsp-continuous, γg-continuous).

Definition 4.7. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *approximately continuous* [8] (resp. *approximately semi-continuous*, *approximately precontinuous* [11], *approximately α-continuous*, *approximately β-continuous* [10], *approximately b-continuous*) if $\mathrm{Cl}(A) \subset f^{-1}(V)$ (resp. $\mathrm{sCl}(A) \subset f^{-1}(V)$, $\mathrm{pCl}(A) \subset f^{-1}(V)$, $\alpha\mathrm{Cl}(A) \subset f^{-1}(V)$, $\mathrm{spCl}(A) \subset f^{-1}(V)$, $\mathrm{bCl}(A) \subset f^{-1}(V)$) whenever V is open in Y and A is g-closed (resp. gs-closed, gp-closed, αg-closed, gsp-closed, γg-closed) in X such that $A \subset f^{-1}(V)$.

Definition 4.8. *Let (X, τ) be a topological space and μ be a GT on X. A function $f : (X, \mu) \to (Y, \lambda)$ is said to be approximately (μ, λ)-continuous if $c_\mu(A) \subset f^{-1}(V)$ whenever V is λ-open in Y and A is $g\mu$-closed in X such that $A \subset f^{-1}(V)$.*

Remark 4.9. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, μ, λ be GTS's on X, Y and $\sigma = \lambda$. If $\mu = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{SPO}(X)$, $\mathrm{BO}(X)$) and $f : (X, \mu) \to (Y, \lambda)$ is approximately (μ, λ)-continuous, then f is approximately continuous (resp. approximately semi-continuous, approximately precontinuous, approximately α-continuous, approximately β-continuous, approximately b-continuous).

Theorem 4.10. *If $f : (X, \mu) \to (Y, \lambda)$ is a $(g\mu, \lambda)$-continuous and approximately (μ, λ)-continuous function, then f is weakly contra-(μ, λ)–continuous.*

Proof. Let V be a λ-open set of Y and A a λ-closed set of Y such that $A \subset V$. Since f is $(g\mu, \lambda)$-continuous, $f^{-1}(A)$ is $g\mu$-closed. Since $f^{-1}(A) \subset f^{-1}(V)$ and f is approximately (μ, λ)-continuous, $c_\mu(f^{-1}(A)) \subset f^{-1}(V)$. This shows that f is weakly contra-(μ, λ)-continuous. $\square$

Remark 4.11. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, μ, λ be GTS's on X, Y and $\sigma = \lambda$. If $\mu = \tau$ (resp. PO(X), $\beta(X)$), then by Theorem 4.10 we obtain the results established in Theorem 3.11 of [9] (resp. Theorem 3.11 of [11], Theorem 3.10 of [10]).

Theorem 4.12. *If $f : (X, \mu) \to (Y, \lambda)$ is weakly contra-(μ, λ)-continuous and $f(A)$ is λ-closed in Y for every $g\mu$-closed set A of X, then f is approximately (μ, λ)-continuous.*

Proof. Let V be any λ-open set of Y and A any $g\mu$-closed set of X such that $A \subset f^{-1}(V)$. Then $f(A)$ is λ-closed and $f(A) \subset V$. Since f is weakly contra-(μ, λ)-continuous, $c_\mu(f^{-1}(f(A))) \subset f^{-1}(V)$ and hence $c_\mu(A) \subset c_\mu(f^{-1}(f(A))) \subset f^{-1}(V)$. This shows that f is approximately (μ, λ)-continuous. □

Remark 4.13. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and μ, λ be GT's on X, Y. If $\mu = \tau$ (resp. PO(X), $\beta(X)$) and $\lambda = \sigma$, then by Theorem 4.12 we obtain the results established in Theorem 3.12 of [9] (resp. Theorem 3.12 of [11], Theorem 3.11 of [10]).

Definition 4.14. *A topological space (X, τ) is said to be* strongly S-closed [29] *(resp.* strongly S-semi-closed, strongly S-preclosed [11]*,* strongly S-α-closed*,* strongly S_β-closed [10]*,* strongly S-b-closed*) if every cover of X by closed (resp. semi-closed, preclosed, α-closed, β-closed, b-closed) sets of (X, τ) has a finite subcover.*

Definition 4.15. *A GTS (X, μ) is said to be* strongly S-μ-closed *if every cover of X by μ-closed sets of (X, μ) has a finite subcover.*

Remark 4.16. Let (X, τ) be a topological space and $\mu = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, SPO(X), BO(X)). If (X, μ) is strongly S-μ-closed, then (X, τ) is strongly S-closed (resp. strongly S-semi-closed, strongly S-preclosed, strongly S-α-closed, strongly S_β-closed, strongly S-b-closed).

Definition 4.17. A topological space (X, τ) is called a P_Σ*-space* [9] or *C-space* [10], [11] if for every open set U and each $x \in U$, there exists a closed set A such that $x \in A \subset U$.

Theorem 4.18. *Let $f : (X, \mu) \to (Y, \sigma)$ be a weakly contra-(μ, σ)-continuous function, μ a GT on X and (Y, σ) a C-space. If (X, μ) is strongly S-μ-closed, then $f(X)$ is compact.*

Proof. Let (X, μ) be strongly S-μ-closed and $\{V_\alpha : \alpha \in \Delta\}$ any cover of $f(X)$ by open sets of (Y, σ). For each $x \in X$, there exists $\alpha(x) \in \Delta$ such that $f(x) \in V_{\alpha(x)}$. Since Y is a C-space, there exists a closed set $A_{\alpha(x)}$ such that $f(x) \in A_{\alpha(x)} \subset V_{\alpha(x)}$. Since f is weakly contra-(μ, σ)-continuous, $c_\mu(f^{-1}(A_{\alpha(x)})) \subset f^{-1}(V_{\alpha(x)})$. The family $\{c_\mu(f^{-1}(A_{\alpha(x)})) : x \in X\}$ is a μ-closed cover of X. Since X is strongly S-μ-closed, there exist a finite number of points, say, $x_1, x_2, ..., x_n$ in X such that $X = \bigcup\{c_\mu(f^{-1}(A_{\alpha(x_k)})) : x_k \in X, 1 \le k \le n\}$. Therefore, we obtain

$$f(X) = \bigcup\{f(c_\mu(f^{-1}(A_{\alpha(x_k)}))) : x_k \in X, 1 \le k \le n\} \subset \bigcup\{V_{\alpha(x_k)} : x_k \in X, 1 \le k \le n\}.$$

This shows that $f(X)$ is compact. □

Remark 4.19. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and μ be a GT on X. If $\mu = \tau$ (resp. PO(X), $\beta(X)$), then by Theorem 4.18 we obtain the results established in Theorem 4.1 of [9] (resp. Theorem 4.1 of [11], Theorem 4.11 of [10]).

Lemma 4.20. [3] *For a function $f : (X, \mu) \to (Y, \lambda)$, the following properties are equivalent:*

1. *f is contra-(μ, λ)-continuous;*
2. *for every λ-closed subset F of Y, $f^{-1}(F)$ is μ-open in X.*

We will denote by $\mathcal{M}_\mu$ the union of all μ-open sets in a GTS (X, μ).

Definition 4.21. *A generalized topological space (X, μ) is said to be μ-compact if every μ-open cover of $\mathcal{M}_\mu$ has a finite subcover.*

Theorem 4.22. *Let $f : (X, \mu) \to (Y, \sigma)$ be a contra-(μ, σ)-continuous surjection and μ a GT on X. If (X, μ) is μ-compact, then (Y, σ) is strongly S-closed.*

Proof. Let (X, μ) be μ-compact and $\{V_\alpha : \alpha \in \Delta\}$ any cover of Y by closed sets of (Y, σ). Since f is contra-(μ, σ)-continuous, by Lemma 4.20 the family $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a μ-open cover of $\mathcal{M}_\mu$. Since (X, μ) is μ-compact, there exists a finite subset Δ_0 of Δ such that $\mathcal{M}_\mu = \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}$. Therefore, $Y = f(\mathcal{M}_\mu) = \cup\{V_\alpha : \alpha \in \Delta_0\}$. This shows that (Y, σ) is strongly S-clsoed. □

Remark 4.23. Let (X, τ) be a topological space. If $\mu = \tau$ (resp. SO(X), PO(X), α(X)), then by Theorem 4.22 we obtain the results established in Theorem 4.2 of [30] (resp. Theorem 4.2 of [30], Corollary 5.1 of [36], Corollary 5.1 of [35]).

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Flow of Herschel-Bulkley fluid through a two dimensional thin layer

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Abstract. The paper is devoted to the study of asymptotic behaviour of the solution of two dimensional steady flow of Herschel-Bulkley fluid through a thin layer. We prove some convergence results when the thicness tends to zero and we give the mechanical interpretation of the results.

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Keywords: asymptotic behaviour, Herschel-Bulkley fluid, thin layer.

1. Introduction

The rigid, viscoplastic and incompressible fluid of Herschel-Bulkley has been studied and used by many mathematicians, physicists and engineers, in order to model the flow of metals, plastic solids and a variety of polymers. Due to existence of yield limit, the model can capture phenomena connected with the development of discontinuous stresses. A particularity of Herschel-Bulkley fluid lies in the presence of rigid zones located in the interior of the flow and as the yield limit increases, the rigid zones become larger and may completely block the flow, this phenomenon is known as the blockage property. The literature concerning this topic is extensive; see e.g. [10, 11, 12, 13].

The purpose of this paper is to study the asymptotic behaviour of steady flow of Herschel-Bulkley fluid in a two dimensional thin layer.

The paper is organized as follows. In Section 2 we present the mechanical problem of the steady flow of Herschel-Bulkley fluid in a two dimensional thin layer. We introduce some notations and preliminaries. Moreover, we define some function spaces and we recall the variational formulation. In Section 3, we are interested in the asymptotic behaviour, to this aim we prove some convergence results concerning the velocity and pressure when the thickness tends to zero. In addition, the uniqueness of limit solution has been also established. Finally, we will discuss in Section 4 the mechanical interpretation of the convergence results.

2. Problem statement

Denoting by I the open interval $I =]0,1[$. Introducing the function $h : I \longrightarrow \mathbb{R}_+^*$ such that $h \in \mathcal{C}^1(I)$.

Considering the following domains

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 \mid x \in I \text{ and } 0 < y < h(x)\right\},$$
$$\Omega^\varepsilon = \left\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \in I \text{ and } 0 < x_2 < \varepsilon h(x_1)\right\},$$

where $\varepsilon > 0$.

Remark that if $(x_1,x_2) \in \Omega^\varepsilon$ then $(x,y) = \left(x_1, \dfrac{x_2}{\varepsilon}\right) \in \Omega$. This permits us to define, for every function $\varphi^\varepsilon : \Omega^\varepsilon \longrightarrow \mathbb{R}$, the function $\widehat{\varphi^\varepsilon} : \Omega \longrightarrow \mathbb{R}$ given by

$$\widehat{\varphi^\varepsilon}(x,y) = \varphi^\varepsilon(x_1,x_2).$$

Let $1 < p \leq 2$, p' the conjugate p, $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ and $\mathbf{f} \in L^{p'}(\Omega)^2$ a given function. We define the function $\mathbf{f}^\varepsilon \in L^{p'}(\Omega^\varepsilon)$ such that $\widehat{\mathbf{f}^\varepsilon} = \mathbf{f}$.

We consider a mathematical problem modelling the steady flow of a rigid viscoplastic and incompressible Herschel-Bulkley fluid in the domain Ω^ε. We suppose that the consistency and yield limit of the fluid are respectively $\mu\varepsilon^p$, $g\varepsilon$ where μ, $g > 0$ and p represents the power law index. The fluid is acted upon by given volume forces of density $\mathbf{f}^\varepsilon$. On $\partial\Omega^\varepsilon$ we suppose that the velocity is known and equal to zero.

We denote by $\mathbb{S}_2$ the space of symmetric tensors on $\mathbb{R}^2$. We define the inner product and the Euclidean norm on $\mathbb{R}^2$ and $\mathbb{S}_2$, respectively, by

$$u \cdot v = u_i v_i \quad \forall\, u,\ v \in \mathbb{R}^2 \text{ and } \sigma \cdot \tau = \sigma_{ij}\tau_{ij} \quad \forall\, \sigma,\ \tau \in \mathbb{S}_2.$$
$$|u| = (u \cdot u)^{\frac{1}{2}} \quad \forall u \in \mathbb{R}^2 \text{ and } |\sigma| = (\sigma \cdot \sigma)^{\frac{1}{2}} \quad \forall \sigma \in \mathbb{S}_2.$$

Here and below, the indices i and j run from 1 to 2 and the summation convention over repeated indices is used. We denote by $\widetilde{\sigma^\varepsilon}$ the deviator of $\sigma^\varepsilon = \left(\sigma_{ij}^\varepsilon\right)$ given by

$$\widetilde{\sigma^\varepsilon} = \left(\widetilde{\sigma_{ij}^\varepsilon}\right), \quad \widetilde{\sigma_{ij}^\varepsilon} = \sigma_{ij}^\varepsilon + p^\varepsilon \delta_{ij},$$

where p^ε represents the hydrostatic pressure and $\delta = (\delta_{ij})$ denotes the identity tensor. We consider the rate of deformation operator defined for every $\mathbf{v}^\varepsilon \in W^{1,p}(\Omega^\varepsilon)^2$ by

$$D(\mathbf{v}^\varepsilon) = (D_{ij}(\mathbf{v}^\varepsilon)), \quad D_{ij}(\mathbf{v}^\varepsilon) = \frac{1}{2}\left(v_{i,j}^\varepsilon + v_{j,i}^\varepsilon\right).$$

The steady flow of Herschel-Bulkley fluid in the domain Ω^ε is given by the following mechanical problem.

Problem P_ε. Find the velocity field $\mathbf{u}^\varepsilon = (u_i^\varepsilon) : \Omega^\varepsilon \longrightarrow \mathbb{R}^2$, the stress field $\sigma^\varepsilon = \left(\sigma_{ij}^\varepsilon\right) : \Omega^\varepsilon \longrightarrow \mathbb{S}_2$ and the pressure $p^\varepsilon : \Omega^\varepsilon \longrightarrow \mathbb{R}$ such that

$$\mathrm{div}\sigma^\varepsilon + \mathbf{f}^\varepsilon = 0 \text{ in } \Omega^\varepsilon. \tag{2.1}$$

$$\left\{\begin{array}{ll} \widetilde{\sigma^\varepsilon} = \mu\varepsilon^p \left|D(\mathbf{u}^\varepsilon)\right|^{p-2} D(\mathbf{u}^\varepsilon) + g\varepsilon \dfrac{D(\mathbf{u}^\varepsilon)}{|D(\mathbf{u}^\varepsilon)|} & \text{if } |D(\mathbf{u}^\varepsilon)| \neq 0 \\ \left|\widetilde{\sigma^\varepsilon}\right| \leq g\varepsilon & \text{if } |D(\mathbf{u}^\varepsilon)| = 0 \end{array}\right. \text{ in } \Omega^\varepsilon. \tag{2.2}$$

$$\mathrm{div}\mathbf{u}^\varepsilon = 0 \text{ in } \Omega^\varepsilon. \tag{2.3}$$

$$\mathbf{u}^{\varepsilon} = 0 \text{ on } \partial\Omega^{\varepsilon}. \tag{2.4}$$

Here, the flow is given by the equation (2.1). Equation (2.2) represents the constitutive law of Herschel-Bulkley fluid. (2.3) represents the incompressibility condition. Equality (2.4) gives the velocity on the boundary $\partial\Omega^{\varepsilon}$.

Let us define now the following Banach spaces

$$W_{\mathrm{div}}^{p,\varepsilon} = \left\{ \mathbf{v} \in W_0^{1,p}(\Omega^{\varepsilon})^2 : div(\mathbf{v}) = 0 \text{ in } \Omega^{\varepsilon} \right\}, \tag{2.5}$$

$$W_{\mathrm{div}}^{p} = \left\{ \mathbf{v} \in W_0^{1,p}(\Omega)^2 : div(\mathbf{v}) = 0 \text{ in } \Omega \right\}, \tag{2.6}$$

$$W_p = \left\{ \varphi \in L^p(\Omega) : \frac{\partial \varphi}{\partial y} \in L^p(\Omega) \right\}. \tag{2.7}$$

$$L_0^p(\Omega^{\varepsilon}) = \left\{ \varphi^{\varepsilon} \in L^p(\Omega^{\varepsilon}) : \int_{\Omega^{\varepsilon}} \varphi^{\varepsilon}(x_1, x_2)\, dx_1 dx_2 = 0 \right\}, \tag{2.8}$$

$$L_0^p(\Omega) = \left\{ \varphi \in L^p(\Omega) : \int_{\Omega} \varphi(x, y)\, dx dy = 0 \right\}, \tag{2.9}$$

For the rest of this article, we will denote by c possibly different positive constants depending only on the data of the problem.

The use of Green's formula permits us to derive the following variational formulation of the mechanical problem (P_{ε}), see [13].

Problem $P\mathrm{V}_{\varepsilon}$. For prescribed data $\mathbf{f}^{\varepsilon} \in L^{p'}(\Omega^{\varepsilon})^2$. Find $(\mathbf{u}^{\varepsilon}, p^{\varepsilon}) \in W_{\mathrm{div}}^{p,\varepsilon} \times L_0^{p'}(\Omega^{\varepsilon})$ satisfying the variational inequality

$$\begin{gathered}
\mu\varepsilon^p \int_{\Omega^{\varepsilon}} |D(\mathbf{u}^{\varepsilon})|^{p-2} D(\mathbf{u}^{\varepsilon}) \cdot D(\mathbf{v} - \mathbf{u}^{\varepsilon})\, dx_1 dx_2 + \\
g\varepsilon \int_{\Omega^{\varepsilon}} |D(\mathbf{v})|\, dx_1 dx_2 - g\varepsilon \int_{\Omega^{\varepsilon}} |D(\mathbf{u}^{\varepsilon})|\, dx_1 dx_2 \\
\geq \int_{\Omega^{\varepsilon}} \mathbf{f}^{\varepsilon} \cdot (\mathbf{v} - \mathbf{u}^{\varepsilon})\, dx_1 dx_2 + \int_{\Omega^{\varepsilon}} p^{\varepsilon} \mathrm{div}(\mathbf{v} - \mathbf{u}^{\varepsilon})\, dx_1 dx_2 \quad \forall \mathbf{v} \in W_0^{1,p}(\Omega^{\varepsilon})^2.
\end{gathered} \tag{2.10}$$

It is known that this variational problem has a unique solution $(\mathbf{u}^{\varepsilon}, p^{\varepsilon}) \in W_{\mathrm{div}}^{p,\varepsilon} \times L_0^{p'}(\Omega^{\varepsilon})$, see for more details [10, 13].

3. Asymptotic behaviour

In this section we establish some results concerning the asymptotic behaviour of the solution when ε tends to zero.

We begin by recalling the following lemmas, see [1, 3, 7].

Lemma 3.1. *1. Poincaré's inequality. For every $\mathbf{v} \in W_0^{1,p}(\Omega^{\varepsilon})^2$ we have*

$$\|\mathbf{v}^{\varepsilon}\|_{L^p(\Omega^{\varepsilon})^2} \leq \varepsilon \left\| \frac{\partial \mathbf{v}^{\varepsilon}}{\partial x_2} \right\|_{L^p(\Omega^{\varepsilon})^2}. \tag{3.1}$$

2. Korn's inequality. For every $\mathbf{v} \in W_0^{1,p}(\Omega^\varepsilon)^2$ *there exists a positive constant* C_0 *independent on* ε, *such that*

$$\|\nabla \mathbf{v}^\varepsilon\|_{L^p(\Omega^\varepsilon)^4} \leq C_0 \|D(\mathbf{v}^\varepsilon)\|_{L^p(\Omega^\varepsilon)^4}. \tag{3.2}$$

Lemma 3.2 (Minty). *Let* E *be a Banach spaces,* $A : E \longrightarrow E'$ *a monotone and hemi-continuous operator,* $J : E \longrightarrow]-\infty, +\infty]$ *a proper and convex functional. Let* $u \in E$ *and* $f \in E'$. *Then the following assertions are equivalent:*

1. $\langle Au; v-u\rangle_{E'\times E} + J(v) - J(u) \geq \langle f; v-u\rangle_{E'\times E} \quad \forall v \in E.$
2. $\langle Av; v-u\rangle_{E'\times E} + J(v) - J(u) \geq \langle f; v-u\rangle_{E'\times E} \quad \forall v \in E.$

The main results of this section are stated by the following proposition.

Proposition 3.3. *Let* $(\mathbf{u}^\varepsilon, p^\varepsilon) \in W_{\mathrm{div}}^{p,\varepsilon} \times L_0^{p'}(\Omega^\varepsilon)$ *be the solution of variational problem* (PV_ε). *Then, there exists* $(\widehat{\mathbf{u}}, \widehat{p}) \in W_p^2 \times L_0^{p'}(\Omega)$ *such that*

$$\widehat{\mathbf{u}^\varepsilon} \longrightarrow \widehat{\mathbf{u}} \text{ in } W_p^2 \text{ weakly,} \tag{3.3}$$

$$\frac{\partial \widehat{u_2^\varepsilon}}{\partial y} \longrightarrow 0 \text{ in } L^p(\Omega) \text{ weakly,} \tag{3.4}$$

$$\widehat{p^\varepsilon} \longrightarrow \widehat{p} \text{ in } L_0^{p'}(\Omega) \text{ weakly.} \tag{3.5}$$

Proof. Choosing $\mathbf{v} = 0$ as test function in inequality (2.10), we deduce that

$$\mu\varepsilon^p \|D(\mathbf{u}^\varepsilon)\|_{L^p(\Omega^\varepsilon)^4}^p \leq \int_{\Omega^\varepsilon} \mathbf{f}^\varepsilon \cdot \mathbf{u}^\varepsilon dx_1 dx_2.$$

This permits us to obtain, making use of Poincaré's and Korn's inequalities and by passage to variables x and y

$$\|\widehat{\mathbf{u}^\varepsilon}\|_{L^p(\Omega)^2} \leq c, \tag{3.6}$$

$$\left\|\frac{\partial \widehat{\mathbf{u}^\varepsilon}}{\partial y}\right\|_{L^p(\Omega)^2} \leq c, \tag{3.7}$$

$$\left\|\frac{\partial \widehat{\mathbf{u}^\varepsilon}}{\partial x}\right\|_{L^p(\Omega)^2} \leq \frac{c}{\varepsilon}. \tag{3.8}$$

Moreover, we get using the incompressibility condition (2.3) and Green's formula, for any function $\varphi^\varepsilon \in W_0^{1,p'}(\Omega^\varepsilon)$

$$\int_\Omega \frac{\partial \widehat{u_2^\varepsilon}}{\partial y} \widehat{\varphi^\varepsilon} dxdy = \varepsilon \int_\Omega u_1^\varepsilon \frac{\partial \widehat{\varphi^\varepsilon}}{\partial x} dxdy.$$

Which gives, making use (2.6)

$$\left\|\frac{\partial \widehat{u_2^\varepsilon}}{\partial y}\right\|_{W^{-1,p}(\Omega)} \leq c\varepsilon. \tag{3.9}$$

We can then extract a subsequence still denoted by $\widehat{\mathbf{u}^\varepsilon}$, such that

$$\widehat{\mathbf{u}^\varepsilon} \longrightarrow \widehat{\mathbf{u}} \text{ in } L^p(\Omega)^2 \text{ weakly,} \tag{3.10}$$

$$\frac{\partial \widehat{\mathbf{u}^\varepsilon}}{\partial y} \longrightarrow \frac{\partial \widehat{\mathbf{u}}}{\partial y} \text{ in } L^p(\Omega)^2 \text{ weakly,} \tag{3.11}$$

$$\frac{\partial \widehat{u_2^\varepsilon}}{\partial y} \longrightarrow 0 \text{ in } L^p(\Omega) \text{ weakly,} \tag{3.12}$$

Let now $\mathbf{v}^\varepsilon \in W_0^{1,p}(\Omega^\varepsilon)^2$, we obtain by setting $\mathbf{u}^\varepsilon - \mathbf{v}^\varepsilon$ as test function in inequality (2.10), using the incompressibility condition (2.3), Green's formula and Hölder's inequality

$$\int_{\Omega^\varepsilon} \nabla p^\varepsilon \cdot \mathbf{v}^\varepsilon dx_1 dx_2 \leq \mu \varepsilon^p \left(\int_{\Omega^\varepsilon} |D(\mathbf{u}^\varepsilon)|^p dx_1 dx_2 \right)^{\frac{1}{p'}} \left(\int_{\Omega^\varepsilon} |D(\mathbf{v}^\varepsilon)|^p dx_1 dx_2 \right)^{\frac{1}{p}}$$
$$+\varepsilon^{\frac{1}{p'}+1} g\,(\operatorname{meas}(\Omega))^{\frac{1}{p'}} \left(\int_{\Omega^\varepsilon} |D(\mathbf{v}^\varepsilon)|^p dx_1 dx_2 \right)^{\frac{1}{p}} + \varepsilon \left\| \widehat{\mathbf{f}^\varepsilon} \right\|_{L^{p'}(\Omega)^2} \left\| \widehat{\mathbf{v}^\varepsilon} \right\|_{W_0^{1,p}(\Omega)^2}. \tag{3.13}$$

On the other hand, it is easy to check that after some algebraic manipulations we find

$$\left(\int_{\Omega^\varepsilon} |D(\mathbf{v}^\varepsilon)|^p dx_1 dx_2 \right)^{\frac{1}{p}} \leq \varepsilon^{\frac{1}{p}-1} \left\| \widehat{\mathbf{v}^\varepsilon} \right\|_{W_0^{1,p}(\Omega)^2}. \tag{3.14}$$

Thus, from (3.7), (3.8), (3.13) and (3.14) we get

$$\int_{\Omega^\varepsilon} \nabla p^\varepsilon \cdot \mathbf{v}^\varepsilon dx_1 dx_2 \leq c\varepsilon \left\| \widehat{\mathbf{v}^\varepsilon} \right\|_{W_0^{1,p}(\Omega)^2}. \tag{3.15}$$

Passing to the variables x and y in (3.15) we find the following estimates

$$\left\| \widehat{p^\varepsilon} \right\|_{L_0^{p'}(\Omega)} \leq c\ , \tag{3.16}$$

$$\left\| \frac{\partial \widehat{p^\varepsilon}}{\partial x} \right\|_{W^{-1,p'}(\Omega)} \leq c\ , \tag{3.17}$$

$$\left\| \frac{\partial \widehat{p^\varepsilon}}{\partial y} \right\|_{W^{-1,p'}(\Omega)} \leq c\varepsilon. \tag{3.18}$$

Consequently, we can extract a subsequence still denoted by $\widehat{p^\varepsilon}$ such that

$$\widehat{p^\varepsilon} \longrightarrow \widehat{p} \text{ in } L_0^{p'}(\Omega) \text{ weakly,} \tag{3.19}$$

which achieves the proof. $\square$

This proof permits also to deduce that the limit pressure verify $\widehat{p}(x,y) = \widehat{p}(x)$.

Proposition 3.4. *The velocity limit given by* (3.3) *verifies*

$$\int_0^{h(x)} \widehat{u_1}(x,y)\, dy = 0 \quad \forall x \in I. \tag{3.20}$$

Proof. We know from the incompressibility condition (2.3) that

$$\int_{\Omega^\varepsilon} \operatorname{div}\mathbf{u}^\varepsilon (x_1, x_2)\, \varphi (x_1)\, dx_1 dx_2 = 0 \text{ for all } \varphi \in \mathcal{D}(I).$$

This implies, using Green's formula

$$\int_{\Omega^\varepsilon} u_1^\varepsilon (x_1, x_2) \frac{d\varphi}{dx_1} (x_1)\, dx_1 dx_2 = \int_{\Omega^\varepsilon} \frac{\partial u_2^\varepsilon}{\partial x_2} (x_1, x_2)\, \varphi (x_1)\, dx_1 dx_2.$$

Hence, by passage to the variables x and y and using Fubini's theorem and Green's formula, we can infer

$$-\int_0^1 \varphi (x) \left(\frac{d}{dx} \int_0^{h(x)} \widehat{u_1^\varepsilon} (x, y)\, dy \right) dx = 0 \quad \forall \varphi \in \mathcal{D}(I).$$

Then

$$\frac{d}{dx} \int_0^{h(x)} \widehat{u_1^\varepsilon} (x, y)\, dy = 0.$$

Moreover, the fact that $\widehat{u_1^\varepsilon} \in L^p (\Omega)$ and $h \in \mathcal{C}^1 (I)$ gives, using the Sobolev embedding $W^{1,p}(I) \subset \mathcal{C}^0 \left(\overline{I}\right)$

$$\int_0^{h(x)} \widehat{u_1^\varepsilon} (x, y)\, dy \in \mathcal{C}^0 \left(\overline{I}\right).$$

Thus, by passage to the limit when ε tends to zero, taking into account the boundary condition (2.4), the assertion (3.20) can be deduced.

We derive in the proposition below the strong equation verified by the limit solution $(\widehat{\mathbf{u}}, \widehat{p}) \in W_p^2 \times L_0^{p'} (\Omega)$. □

Proposition 3.5. *If* $\dfrac{\partial \widehat{u_1}}{\partial y} \neq 0$, *then the limit point* $(\widehat{u_1}, \widehat{p})$ *given by* (3.3) *and* (3.5) *verify the limit problem*

$$-\frac{\partial}{\partial y} \left(\frac{\mu}{2^{\frac{p}{2}}} \left| \frac{\partial \widehat{u_1}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_1}}{\partial y} + \frac{\sqrt{2}}{2} g \operatorname{sign} \left(\frac{\partial \widehat{u_1}}{\partial y} \right) \right) = \widehat{f_1} - \frac{d\widehat{p}}{dx} \quad in \quad W^{-1,p'}(\Omega). \tag{3.21}$$

Proof. Introducing the operator A defined as follows

$$A : W_0^{1,p} (\Omega^\varepsilon)^2 \longrightarrow W^{-1,p'} (\Omega^\varepsilon)^2,$$

$$\langle A\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon \rangle_{W^{-1,p'}(\Omega^\varepsilon)^2 \times W_0^{1,p}(\Omega^\varepsilon)^2} = \mu \varepsilon^p \int_{\Omega^\varepsilon} |D(\mathbf{v}^\varepsilon)|^{p-2} D(\mathbf{v}^\varepsilon) \cdot D(\mathbf{v}^\varepsilon)\, dx_1 dx_2.$$

It is easy to verify that A is monotone and hemi-continuous (see for more details the reference [13]). Moreover, we know that the functional

$$\mathbf{v}^\varepsilon \in W_0^{1,p} (\Omega^\varepsilon)^2 \longrightarrow g\varepsilon \int_{\Omega^\varepsilon} |D(\mathbf{v}^\varepsilon)|\, dx_1 dx_2$$

is proper and convex. Then, the use of Minty's lemma permits us to affirm that (2.10) is equivalent to the following inequality

$$\begin{aligned}
&\mu\varepsilon^{p}\int_{\Omega^{\varepsilon}}\left|D\left(\mathbf{v}^{\varepsilon}\right)\right|^{p-2}D\left(\mathbf{v}^{\varepsilon}\right)\cdot D\left(\mathbf{v}^{\varepsilon}-\mathbf{u}^{\varepsilon}\right)dx_{1}dx_{2}+\\
&g\varepsilon\int_{\Omega^{\varepsilon}}\left|D\left(\mathbf{v}^{\varepsilon}\right)\right|dx_{1}dx_{2}-g\varepsilon\int_{\Omega^{\varepsilon}}\left|D\left(\mathbf{u}^{\varepsilon}\right)\right|dx_{1}dx_{2}\\
&\geq\int_{\Omega^{\varepsilon}}\mathbf{f}^{\varepsilon}\cdot\left(\mathbf{v}^{\varepsilon}-\mathbf{u}^{\varepsilon}\right)dx_{1}dx_{2}+\int_{\Omega^{\varepsilon}}p^{\varepsilon}\mathrm{div}\left(\mathbf{v}^{\varepsilon}-\mathbf{u}^{\varepsilon}\right)dx_{1}dx_{2}\quad\forall\mathbf{v}^{\varepsilon}\in W_{0}^{1,p}\left(\Omega^{\varepsilon}\right)^{2}.
\end{aligned}$$

Our goal now is to pass to the limit when ε tends to zero. To this aim, we use Proposition 3.4 and the weak lower semi-continuity of the convex and continuous functional $\mathbf{v}^{\varepsilon}\in W_{0}^{1,p}\left(\Omega^{\varepsilon}\right)\longrightarrow g\varepsilon\int_{\Omega^{\varepsilon}}\left|D\left(\mathbf{v}^{\varepsilon}\right)\right|dx_{1}dx_{2}$. We find the following limit inequality

$$\begin{aligned}
&\mu\int_{\Omega}\frac{1}{2^{\frac{p-2}{2}}}\left[\left|\frac{\partial\widehat{v_{1}}}{\partial y}\right|^{2}+\left|\frac{\partial\widehat{v_{2}}}{\partial y}\right|^{2}\right]^{\frac{p-2}{2}}\left[\frac{1}{2}\frac{\partial\widehat{v_{1}}}{\partial y}\frac{\partial\left(\widehat{v_{1}}-\widehat{u_{1}}\right)}{\partial y}+\frac{\partial\widehat{v_{2}}}{\partial y}\frac{\partial\left(\widehat{v_{2}}-\widehat{u_{2}}\right)}{\partial y}\right]dxdy\\
&+g\int_{\Omega}\left[\frac{1}{2}\left|\frac{\partial\widehat{v_{1}}}{\partial y}\right|^{2}+\left|\frac{\partial\widehat{v_{2}}}{\partial y}\right|^{2}\right]^{\frac{1}{2}}dxdy-g\int_{\Omega}\left[\frac{1}{2}\left(\frac{\partial\widehat{u_{1}}}{\partial y}\right)^{2}+\left(\frac{\partial\widehat{u_{2}}}{\partial y}\right)^{2}\right]^{\frac{1}{2}}dxdy\\
&\geq\int_{\Omega}\widehat{\mathbf{f}}\cdot\left(\widehat{\mathbf{v}}-\widehat{\mathbf{u}}\right)dxdy+\int_{\Omega}\widehat{p}\mathrm{div}\left(\widehat{\mathbf{v}}-\widehat{\mathbf{u}}\right)dxdy\quad\forall\mathbf{v}^{\varepsilon}\in W_{0}^{1,p}\left(\Omega^{\varepsilon}\right).
\end{aligned}\tag{3.22}$$

Furthermore, from (3.3) and (3.4) we find

$$\frac{\partial\widehat{u_{2}}}{\partial y}=0\text{ in }\Omega.$$

It follows, keeping in mind (3.20), that

$$\widehat{\mathbf{u}}\left(x,y\right)=\left(\widehat{u_{1}}\left(x,y\right),0\right).$$

This permits also to choose $\widehat{\mathbf{v}_{2}}=0$ in (3.22).

Considering now the operator A such that

$$A:W_{p}\longrightarrow W_{p}',$$

$$\langle A\widehat{u_{1}},\widehat{v_{1}}\rangle_{W_{p}'\times W_{p}}=\frac{\mu}{2^{\frac{p}{2}}}\int_{\Omega^{\varepsilon}}\left|\frac{\partial\widehat{u_{1}}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_{1}}}{\partial y}\frac{\partial\widehat{v_{1}}}{\partial y}dxdy.$$

It is clear that the operator A is monotone and hemi-continuous and the functional $\widehat{v_{1}}\in W_{p}\longrightarrow\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial\widehat{v_{1}}}{\partial y}\right|dxdy$ is proper and convex. Hence, we deduce using again Minty's lemma

$$\begin{aligned}
&\frac{\mu}{2^{\frac{p}{2}}}\int_{\Omega}\left|\frac{\partial\widehat{u_{1}}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_{1}}}{\partial y}\frac{\partial\left(\widehat{v_{1}}-\widehat{u_{1}}\right)}{\partial y}dxdy+\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial\widehat{v_{1}}}{\partial y}\right|dxdy-\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial\widehat{u_{1}}}{\partial y}\right|dxdy\\
&\geq\int_{\Omega}\widehat{f_{1}}\left(\widehat{v_{1}}-\widehat{u_{1}}\right)dxdy-\int_{\Omega}\frac{d\widehat{p}}{dx}\left(\widehat{v_{1}}-\widehat{u_{1}}\right)dxdy\quad\forall\widehat{v_{1}}\in W_{p}.
\end{aligned}\tag{3.23}$$

This yields, via Green's formula

$$\begin{aligned}
&-\frac{\mu}{2^{\frac{p}{2}}}\int_{\Omega}\frac{\partial}{\partial y}\left(\left|\frac{\partial\widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_1}}{\partial y}\right)(\widehat{v_1}-\widehat{u_1})\,dxdy+\\
&\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial\widehat{v_1}}{\partial y}\right|dxdy-\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial\widehat{u_1}}{\partial y}\right|dxdy\\
&\geq\int_{\Omega}\widehat{f}_1(\widehat{v_1}-\widehat{u_1})\,dxdy-\int_{\Omega}\frac{d\widehat{p}}{dx}(\widehat{v_1}-\widehat{u_1})\,dxdy\quad\forall\widehat{v_1}\in W_p.
\end{aligned}\tag{3.24}$$

Due to the fact that $W_0^{1,p}(\Omega)$ is dense in W_p, see [1], we can take $\widehat{v_1}=\widehat{u_1}\pm\varphi$ in (3.24), where $\varphi\in W_0^{1,p}(\Omega)$ to obtain the following inequalities

$$-\frac{\mu}{2^{\frac{p}{2}}}\int_{\Omega}\frac{\partial}{\partial y}\left(\left|\frac{\partial\widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_1}}{\partial y}\right)\varphi dxdy+\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial(\widehat{u_1}+\varphi)}{\partial y}\right|dxdy-\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial\widehat{u_1}}{\partial y}\right|dxdy$$

$$\geq\int_{\Omega}\widehat{f}_1\varphi dxdy-\int_{\Omega}\frac{d\widehat{p}}{dx}\varphi dxdy\quad\forall\varphi\in W_0^{1,p}(\Omega).$$

and

$$\begin{aligned}
&\frac{\mu}{2^{\frac{p}{2}}}\int_{\Omega}\frac{\partial}{\partial y}\left(\left|\frac{\partial\widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_1}}{\partial y}\right)\varphi dxdy+\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial(\widehat{u_1}-\varphi)}{\partial y}\right|dxdy-\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial\widehat{u_1}}{\partial y}\right|dxdy\\
&\geq-\int_{\Omega}\widehat{f}_1\varphi dxdy+\int_{\Omega}\frac{d\widehat{p}}{dx}\varphi dxdy\quad\forall\varphi\in W_0^{1,p}(\Omega).
\end{aligned}$$

Replacing in these two inequalities the test function φ by $\lambda\varphi$, $\lambda>0$, dividing the obtained inequalities by λ. The passage to the limit when λ tends to 0 implies, under the hypothesis $\dfrac{\partial\widehat{u_1}}{\partial y}\neq 0$, that

$$\begin{aligned}
&-\frac{\mu}{2^{\frac{p}{2}}}\int_{\Omega}\frac{\partial}{\partial y}\left(\left|\frac{\partial\widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_1}}{\partial y}\right)\varphi dxdy+\frac{\sqrt{2}}{2}g\int_{\Omega}\operatorname{sign}\left(\frac{\partial\widehat{u_1}}{\partial y}\right)\frac{\partial\varphi}{\partial y}dxdy\\
&\geq\int_{\Omega}\widehat{f}_1\varphi dxdy-\int_{\Omega}\frac{d\widehat{p}}{dx}\varphi dxdy\quad\forall\varphi\in W_0^{1,p}(\Omega).
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\mu}{2^{\frac{p}{2}}}\int_{\Omega}\frac{\partial}{\partial y}\left(\left|\frac{\partial\widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_1}}{\partial y}\right)\varphi dxdy-\frac{\sqrt{2}}{2}g\int_{\Omega}\operatorname{sign}\left(\frac{\partial\widehat{u_1}}{\partial y}\right)\frac{\partial\varphi}{\partial y}dxdy\\
&\geq-\int_{\Omega}\widehat{f}_1\varphi dxdy+\int_{\Omega}\frac{d\widehat{p}}{dx}\varphi dxdy\quad\forall\varphi\in W_0^{1,p}(\Omega).
\end{aligned}$$

Consequently, we get combining these two inequalities and using a simple integration by parts

$$-\int_{\Omega}\frac{\partial}{\partial y}\left[\frac{\mu}{2^{\frac{p}{2}}}\left(\left|\frac{\partial\widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_1}}{\partial y}\right)+\frac{\sqrt{2}}{2}g\mathrm{sign}\left(\frac{\partial\widehat{u_1}}{\partial y}\right)\right]\varphi dxdy$$
$$=\int_{\Omega}\left(\widehat{f}_1-\frac{d\widehat{p}}{dx}\varphi\right)dxdy\quad\forall\varphi\in W_0^{1,p}(\Omega).$$

Which eventually gives (3.21). □

From now on we will denote by $(\widehat{u},\widehat{p})\in W_p\times L_0^{p'}(\Omega)$ the solution of the limit problem (3.21).

The following proposition shows the uniqueness of the limit solution $(\widehat{u},\widehat{p})$.

Proposition 3.6. *The limit strong problem (*3.21*) has a unique, solution $(\widehat{u},\widehat{p})$ in $W_p\times L_0^{p'}(\Omega)$ with the condition (*3.20*).*

Proof. Suppose that the limit problem (3.21) has at least two solutions $(\widehat{u_1},\widehat{p_1})$, $(\widehat{u_2},\widehat{p_2})\in W_p\times L_0^{p'}(\Omega)$. In particular, $(\widehat{u_1},\widehat{p_1}),(\widehat{u_2},\widehat{p_2})$ are solutions of the weak formulation (3.23). Then

$$\frac{\mu}{2^{\frac{p}{2}}}\int_{\Omega}\left|\frac{\partial\widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_1}}{\partial y}\frac{\partial(\widehat{v}-\widehat{u_1})}{\partial y}dxdy+\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial\widehat{v}}{\partial y}\right|dxdy-\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial\widehat{u_1}}{\partial y}\right|dxdy$$
$$\geq\int_{\Omega}\widehat{f}_1(\widehat{v}-\widehat{u_1})dxdy-\int_{\Omega}\frac{d\widehat{p_1}}{dx}(\widehat{v}-\widehat{u_1})dxdy\quad\forall\widehat{v}\in W_p,\tag{3.25}$$

and

$$\frac{\mu}{2^{\frac{p}{2}}}\int_{\Omega}\left|\frac{\partial\widehat{u_2}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_2}}{\partial y}\frac{\partial(\widehat{v}-\widehat{u_2})}{\partial y}dxdy+\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial\widehat{v}}{\partial y}\right|dxdy-\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial\widehat{u_2}}{\partial y}\right|dxdy$$
$$\geq\int_{\Omega}\widehat{f}_1(\widehat{v}-\widehat{u_2})dxdy-\int_{\Omega}\frac{d\widehat{p_2}}{dx}(\widehat{v}-\widehat{u_2})dxdy\quad\forall\widehat{v}\in W_p.\tag{3.26}$$

Setting $\widehat{v}=\widehat{u_2}$, $\widehat{v}=\widehat{u_1}$ as test functions in (3.25) and (3.26), respectively. Subtracting the two obtained inequalities, we can infer

$$\frac{\mu}{2^{\frac{p}{2}}}\int_{\Omega}\left[\left|\frac{\partial\widehat{u_2}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_2}}{\partial y}-\left|\frac{\partial\widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_1}}{\partial y}\right]\frac{\partial(\widehat{u_2}-\widehat{u_1})}{\partial y}dxdy$$
$$\leq\int_{\Omega}\frac{d(\widehat{p_1}-\widehat{p_2})}{dx}(\widehat{u_2}-\widehat{u_1})dxdy\quad\forall\widehat{v}\in W_p.\tag{3.27}$$

Observe that for every $x,y\in\mathbb{R}^n$,

$$\left(|x|^{p-2}x-|y|^{p-2}y\right)\cdot(x-y)\geq(p-1)\frac{|x-y|^2}{(|x|+|y|)^{2-p}},\quad 1<p\leq 2.$$

This leads, making use (3.27), to

$$\frac{\mu(p-1)}{2^{\frac{p}{2}}}\int_{\Omega}\frac{\left|\dfrac{\partial(\widehat{u_2}-\widehat{u_1})}{\partial y}\right|^2}{\left(\left|\dfrac{\partial\widehat{u_1}}{\partial y}\right|+\left|\dfrac{\partial\widehat{u_2}}{\partial y}\right|\right)^{2-p}}dxdy\leq\int_{\Omega}\frac{d(\widehat{p_1}-\widehat{p_2})}{dx}(\widehat{u_2}-\widehat{u_1})\,dxdy$$

$$=\int_0^1\left(\frac{d(\widehat{p_1}-\widehat{p_2})}{dx}\int_0^{h(x)}(\widehat{u_2}-\widehat{u_1})\,dy\right)dx.$$

The use of (3.20) gives

$$\int_{\Omega}\frac{\left|\dfrac{\partial(\widehat{u_2}-\widehat{u_1})}{\partial y}\right|^2}{\left(\left|\dfrac{\partial\widehat{u_1}}{\partial y}\right|+\left|\dfrac{\partial\widehat{u_2}}{\partial y}\right|\right)^{2-p}}dxdy=0. \tag{3.28}$$

On the other hand, the application of Hölder's inequality leads to

$$\int_{\Omega}\left|\frac{\partial(\widehat{u_2}-\widehat{u_1})}{\partial y}\right|^p dxdy$$

$$\leq c\left(\int_{\Omega}\frac{\left|\dfrac{\partial\widehat{u_2}}{\partial y}-\dfrac{\partial\widehat{u_1}}{\partial y}\right|^2}{\left(\left|\dfrac{\partial\widehat{u_1}}{\partial y}\right|+\left|\dfrac{\partial\widehat{u_2}}{\partial y}\right|\right)^{2-p}}dxdy\right)^{\frac{p}{2}}\left(\int_{\Omega}\left(\left|\frac{\partial\widehat{u_1}}{\partial y}\right|+\left|\frac{\partial\widehat{u_2}}{\partial y}\right|\right)^p dxdy\right)^{\frac{2-p}{2}}.$$

Which gives, keeping in mind (3.28)

$$\frac{\partial(\widehat{u_2}-\widehat{u_1})}{\partial y}=0.$$

Since $\widehat{u_2}(x,h(x))=\widehat{u_1}(x,h(x))=0$, we deduce that $\widehat{u_2}=\widehat{u_1}$ a.e. in Ω.

Finally, to prove the uniqueness of the pressure, we use equation (3.21) with the two pressures $\widehat{p_1}$ and $\widehat{p_2}$. We find

$$\frac{d(\widehat{p_1}-\widehat{p_2})}{dx}=0.$$

Then, due to fact that $\widehat{p_1},\ \widehat{p_2}\in L_0^{p'}(\Omega)$, the result can be easily deduced. □

4. Mechanical interpretation

Suppose that $\dfrac{\partial\widehat{u}}{\partial y}\neq 0$ and let σ^{ε} be the stress tensor associated to $\mathbf{u}^{\varepsilon}$. Then using the constitutive law of Herschel-Bulkley fluid, we can infer

$$\int_{\Omega^{\varepsilon}}\left|\widetilde{\sigma^{\varepsilon}}\right|^{p'}dx_1dx_2\leq\int_{\Omega^{\varepsilon}}\left(\mu\varepsilon^p\left|D(\mathbf{u}^{\varepsilon})\right|^{p-2}D(\mathbf{u}^{\varepsilon})+g\varepsilon\right)^{p'}dx_1dx_2.$$

We can then easily prove, by passage to the variables x and y, that

$$\frac{1}{\varepsilon}\left\|\widehat{\widetilde{\sigma^{\varepsilon}}}\right\|_{L^{p'}(\Omega)^4} \leq c.$$

Thus, we can extract a subsequence still denoted by σ^{ε} such that

$$\frac{1}{\varepsilon}\widehat{\widetilde{\sigma^{\varepsilon}}} \longrightarrow \widehat{\sigma} \text{ in } L^{p'}(\Omega)^4 \text{ weakly.}$$

On the other hand, we know from the flow equation (2.1) that

$$\sum_{j=1}^{2}\frac{\partial \widetilde{\sigma_{ij}^{\varepsilon}}}{\partial x_j} = \frac{\partial p^{\varepsilon}}{\partial x_i} - \widetilde{f_i^{\varepsilon}}, \quad i = 1,2 \text{ in } \Omega^{\varepsilon}.$$

By passage to the variables x and y, taking into account the fact that $\widehat{p}(x,y) = \widehat{p}(x)$, we obtain the following equations

$$\left\{\begin{array}{l} \dfrac{\partial \widehat{\widetilde{\sigma_{11}^{\varepsilon}}}}{\partial x} + \dfrac{1}{\varepsilon}\dfrac{\partial \widehat{\widetilde{\sigma_{12}^{\varepsilon}}}}{\partial y} = \dfrac{d\widehat{p^{\varepsilon}}}{dx} - \widehat{f_1^{\varepsilon}} \\ \dfrac{\partial \widehat{\widetilde{\sigma_{21}^{\varepsilon}}}}{\partial x} + \dfrac{1}{\varepsilon}\dfrac{\partial \widehat{\widetilde{\sigma_{22}^{\varepsilon}}}}{\partial y} = -\widehat{f_2^{\varepsilon}} \end{array}\right. \text{ in } \Omega. \tag{4.1}$$

The passage to the limit leads

$$\frac{\partial \widehat{\sigma_{21}}}{\partial y} = \frac{d\widehat{p}}{dx} - \widehat{f}_1. \tag{4.2}$$

By comparison with equation (3.21), we find

$$\widehat{\sigma_{21}} = \frac{\mu}{2^{\frac{p}{2}}}\left|\frac{\partial \widehat{u}}{\partial y}\right|^{p-2}\frac{\partial \widehat{u}}{\partial y} + \frac{\sqrt{2}}{2} g \operatorname{sign}\left(\frac{\partial \widehat{u}}{\partial y}\right).$$

Which means that if $\dfrac{\partial \widehat{u}}{\partial y} \neq 0$ then $|\widehat{\sigma_{21}}| > \frac{\sqrt{2}}{2}g$. Hence, if $|\widehat{\sigma_{21}}| \leq \frac{\sqrt{2}}{2}g$ we get

$$\frac{\partial \widehat{u}}{\partial y} = 0.$$

This permits us to deduce that at the limit the flow can be described by the following one dimensional constitutive law

$$\left\{\begin{array}{ll} \widehat{\tau} = \dfrac{\mu}{2^{\frac{p}{2}}}\left|\dfrac{\partial \widehat{u}}{\partial y}\right|^{p-2}\dfrac{\partial \widehat{u}}{\partial y} + \dfrac{\sqrt{2}}{2} g \operatorname{sign}\left(\dfrac{\partial \widehat{u}}{\partial y}\right) & \text{if } \dfrac{\partial \widehat{u}}{\partial y} \neq 0 \\ |\widehat{\tau}| \leq \frac{\sqrt{2}}{2} g & \text{if } \dfrac{\partial \widehat{u}}{\partial y} = 0 \end{array}\right. \text{ in } \Omega, \tag{4.3}$$

where $\widehat{\tau}$ is the stress of the limit model. Such constitutive law has been studied by many engineers for the particular case of Bingham fluid i.e. $p = 2$, see for example [9]. Indeed, the case $p = 2$ corresponds to the Bingham flow. For $\mu = 2\mu^*$, $g = \sqrt{2}g^*$ and $p = 2$ the result in [4] are recovered.

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Book reviews

Jean-Paul Penot, Calculus Without Derivatives, Graduate Texts in Mathematics 266, Springer, New York - Heidelberg - Dordrecht - London, 2013, ISBN: 978-1-4614-4537-1/hbk; 978-1-4614-4538-8/ebook, xx + 524 pp.

Differential calculus offers efficient tools for the study of extrema of differentiable functions. In the case of nondifferentiable functions, defined on subsets of a Banach space X, more refined methods are needed – the differentials, which are elements of the dual space X^*, are replaced by various kinds of subdifferentials, which are subsets of X^*, and various types of cones (tangent, normal, regression, etc) enter the scene. In this new area, called nonsmooth analysis, sets and functions play interchangeable roles, leading to more flexibility in the treatment of optimization problems. As it is expected, the definitions of these new objects and the proofs of their properties use tools from set-valued analysis and differential calculus, topics that are treated in the first two chapters of the book, 1. *Metric and topological tools*, and 2. *Elements of differentiable calculus*. The first chapter contains some results from topology, set-valued analysis (continuity properties of multimaps, limits of sets), the variational principles of Ekeland, Deville-Godefroy-Zizler and Stegall, with applications to fixed point theorems, openness and regularity results (Robinson-Ursescu theorem), and well-posedness of optimization problems. In the second chapter, one studies the fundamental properties of the Fréchet, Hadamard and Gâteaux derivatives. Kantorovich's theorem on the Newton method is used to prove the Lyusternik-Graves theorem, the inverse and the implicit function theorems are proved with a special attention paid to the Lipschitz behavior of the inverse. As applications one studies the relevance of tangent and normal cones in optimization – Fermat's rule, Lagrange multipliers, Lyusternik theorem – and one gives a short introduction to the calculus of variations.

Convex functions, the subject of Chapter 3. *Elements of convex analysis*, have nice continuity and differentiability properties, allowing a good subdifferential calculus which serves as a model for more general subdifferentials studied in the subsequent chapters. This chapter contains the basic results of convex analysis – subdifferentials, the Legendre-Fenchel transform and conjugate functions. Besides the exact rules of the calculus with subdifferentials of convex functions, fuzzy rules (meaning approximate rules) are considered as well, paving the way to similar rules in the non-convex case and showing at the same time that convex analysis is a part of a more general construction.

Containing classical material, as well as some more special topics useful for nonsmooth analysis, each of these first three chapters is of independent interest and can serve as a base for a one semester course on the corresponding topic.

The rest of the book, chapters 4. *Elementary and viscosity subdifferentials*, 5. *Circa-subdifferentials, Clarke subdifferentials*, 6. *Limiting subdifferentials*, and 7. *Graded subdifferentials, Ioffe subdifferentials*, are devoted to nonsmooth analysis. As the author explains, the consideration of various kinds of subdifferentials is motivated by the optimization problem we are studying and by the choice of the space we are working in. Two, somewhat antagonistic, criteria governing the choice of a subdifferential are the accurateness of the information he supplies and the availability of calculus rules. In most cases only fuzzy calculus rules are available, a case already considered for the subdifferentials of convex functions, but this reflects the fact that, very often in real situations, only approximate values of the differentials can be computed, not the exact ones. In author's opinion, the abundance of subdifferentials is not a sign of disorder, but rather reflects the richness of the domain, a unity in diversity. He succeeds to treat in a unitary way various kinds of subdifferentials encountered in nonsmooth analysis, having as primary models convex analysis and classical differential calculus and putting in evidence connections between directional derivatives and tangent cones, on one side, and subdifferentials and normal cones, on the other side. In some cases there is not a complete duality between these two classes of objects, only "one-way routes" being available. These four notions, together with the graphical derivatives and coderivatives for multimaps, are considered by the author as "the six pillars of nonsmooth analysis".

Written by a reputed specialist in the domain, with substantial contributions to convex and nonsmooth analysis (as a significant sample we mention the Michel-Penot subdifferential), and based on a large bibliography (1003 titles from which 173 belong to the author himself, alone or in cooperation), the book presents in a unitary and systematic way a lot of results and tools used in modern optimization theory, some of them still under construction. Each subsection is followed by a set of exercises illustrating the main text by examples, or completing it, some of them with hints and for the more demanding ones a reference to a paper (or book) being given. (Some of these are rather cryptic, as, for instance Exercise

By collecting together a lot of results in nonsmooth analysis and presenting them in a coherent and accessible way, the author rendered a great service to the mathematical community. The book can be considered as an incentive for newcomers to enter this area of research, which, by the variety of tools and methods, may look discouraging at the first sight. The specialists will find also a lot of systematized information, and, as we have already told, the first three chapters can be used for independent graduate courses.

S. Cobzaş

Jean-Baptiste Hiriart-Urruty, Bases, outils et principes pour lanalyse variationnelle, Mathématiques et Applications, Springer-Verlag, Berlin - Heidelberg, 2013, ISSN 1154-483X, ISBN 978-3-642-30734-8, ISBN 978-3-642-30735-5 (eBook), DOI 10.1007/978-3-642-30735-5.

As the author explains in Preface, the book contains topics which can be taught and assimilated by the students attending the course Master 2 Recherche in the first semester (25-30 teaching ours). For this reason he made a rigorous selection, keeping only results which resisted the time and are essential for the area. The first chapter, *Prolégomènes,* contains an introduction to existence results in constrained optimization, emphasizing the roles played by topologies (norm and weak on a normed space E, and norm and weak* on its dual E^*), and by convexity. A lot of supplementary results are contained in the exercises at the end of this chapter.

The second chapter, *Conditions nécessaires d'optimalité approchée*, is concerned mainly with the variational principles of Ekeland and Borwein-Preiss (this one in Hilbert space framework). As application, a detailed study of best approximation in a Hilbert space H by elements of a nonempty closed S is done – continuity and differentiability properties of the distance function d_S and of the metric projection p_S, and dense existence results. In Annexe one discusses Fréchet, Gâteaux and Hadamard differentiability. The results of this chapter are completed and developed in the third chapter *Autour de la projection sur un convexe fermé: la decomposition de Moreau.* The term "opératoire" (operative) in the heading of the fourth chapter, *Analyse convexe opératoire*, means that the presentation is restricted to definitions, essential techniques and tools of convex analysis, destined to be used in situations where the convexity is not available.

The last chapter, Ch. 6, *Sous-différentiel généralisés de fonctions non différentiables*, is concerned with Clarke's directional generalized derivatives and subdifferentials for locally Lipschitz functions defined on an open subset of a Banach space – definitions, properties, calculus rules and applications to necessary conditions in optimization problems. Connections with tangent (contigent) cones and Clarke's normal cone are established, opening the way to nonsmooth geometry. Other types of subdifferentials – Clarke's subdifferential for arbitrary lower semi-continuous functions (not necessarily locally Lipschitz), Fréchet subdifferentials, proximal subdifferentials, viscosity subdifferentials, and back and forth rules with the corresponding tangent and normal cones – are briefly discussed at the end of this chapter.

All the notions are carefully motivated and the results are discussed and illustrated by concrete examples. Each section ends with a set of good exercises completing the main text. The bibliographic references are given at the end of each chapter.

Written in an informal and colloquial style, with witty remarks and quotations of mathematical, but also of general nature, the book is a good introduction to nonsmooth analysis. It can be used as a "très bon appéritif" to more advanced books in this domain as, for instance, the two volume book by J.-B. Hiriart-Urruty and C. Lemarechal, *Convex Analysis and Minimization Algorithms* I and II (Grundlehren des mathematischen Wissenschaften Vol. 305 and 306, Springer 1993, reprinted in 1996, or in abridged form *Fundamentals of convex analysis*, Grundlehren Text Editions

Springer-Verlag 2001, W. Schirotzek, *Nonsmooth Analysis*, Universitext, Springer 2007, and the recent one by J.-P. Penot, *Calculus Without Derivatives*, GTM, Vol. 266, Springer 2013. Or, as the author nicely says at the end of his exposition, "le lecteur-étudiant pourra se faire les dents sur des problèmes variationnels ou d'optimisation non-résolus".

S. Cobzaş

Luboš Pick, Alois Kufner, Oldřich John and Svatopluk Fučik, Function Spaces, 2nd Revised and Extended Edition, Series in Nonlinear Analysis and Applications, Vol. 14, xv+479 pp, Walter de Gruyter, Berlin - New York, 2013, ISBN: 978-3-11-025041-1, e-ISBN: 978-3-11-025042-8, ISSN: 0941-813X.

This is the second edition of the successful book by A. Kufner, O. John and S. Fučik, *Function Spaces*, Noordhoff, Leyden, and Academia, Praha, 1977. This new edition is dedicated to Professor Svatopluk Fučik who passed away not long after the first edition appeared. Over the 35 years passed since then a lot of new results appeared, several books were published, so that three of the authors (L.P., A.K. and O.J.) with the strong support of de Gruyter Publishing House, considered appropriate to write a new revised and updated edition of the book. Because this new edition is based on the 1997 version of the book, Svatopluk Fučik was included between the authors. Also, as the collected material was too long for a single volume, they decided to split in into two volumes. The first volume is dedicated to the study of function spaces, based on intrinsic properties of functions such as size, continuity, smoothness, various forms of control over the mean oscillation, and so on. The second volume will be concerned with function spaces of Sobolev type, in which the key notion is that of weak derivative of functions of several variables.

In the first chapter of the book, *Preliminaries,* for easy reference, the authors collect notions and results (without proofs) from functional analysis and measure theory used throughout the book.

The function spaces treated in the first volume are well illustrated by the headings of the chapters: Ch. 2, *Spaces of smooth functions* (completeness and compactness, separability, extension of functions, Hölder and Lipschitz spaces), Ch. 3, *Lebesgue spaces* (mollifiers, density results, separability, dual spaces, reflexivity, dual spaces, weighted Hardy inequalities, the space L^∞, weak convergence, compactness, Schauder bases), Ch. 4, *Orlicz spaces* (Young functions, Jensen inequality, the condition Δ_2, Hölder inequality, completeness and compactness, separability, isomorphisms, Schauder bases), Ch. 5, *Morrey and Campanato spaces* (these are subspaces of the Lebesgue spaces defined through a mean oscillation property). The spaces studied in Ch. 6, *Banach function spaces*, are Banach spaces of measurable functions defined through a Banach function norm. This abstract scheme covers many examples of scales of function spaces as Lebesgue, Orlicz, and Morrey spaces, as well as rearrangement invariant function spaces (studied in Chapter 7) and Lorentz spaces (studied in Chapters 8, *Lorentz spaces*, and 10, *Classical Lorentz spaces*). Ch. 9, *Generalized Lorentz-Zygmund spaces*, is concerned with a class of functions defined with the help of logarithmic functions raised to different powers near 0 and near infinity (such functions are, in a sense, "broken" in 1, a reason for which they are called also broken logarithmic functions). The last chapter of this volume, Ch. 11, *Variable-exponent Lebesgue spaces*, is concerned with a class of spaces that turned to be of great importance in the study of mathematical models of electrorheological fluids, as it was shown by M. Ružička, LNM, vol. 1748, Springer 2000.

The book contains a lot of results about various classes of function spaces, of great importance in various areas of mathematics, especially for partial differential equations, presented in a clear manner in the elegant typographical layout of de Gruyter Publishres. Undoubtedly, that, together with the second volume, this welcome new edition will become a standard reference in the domain.

Radu Precup