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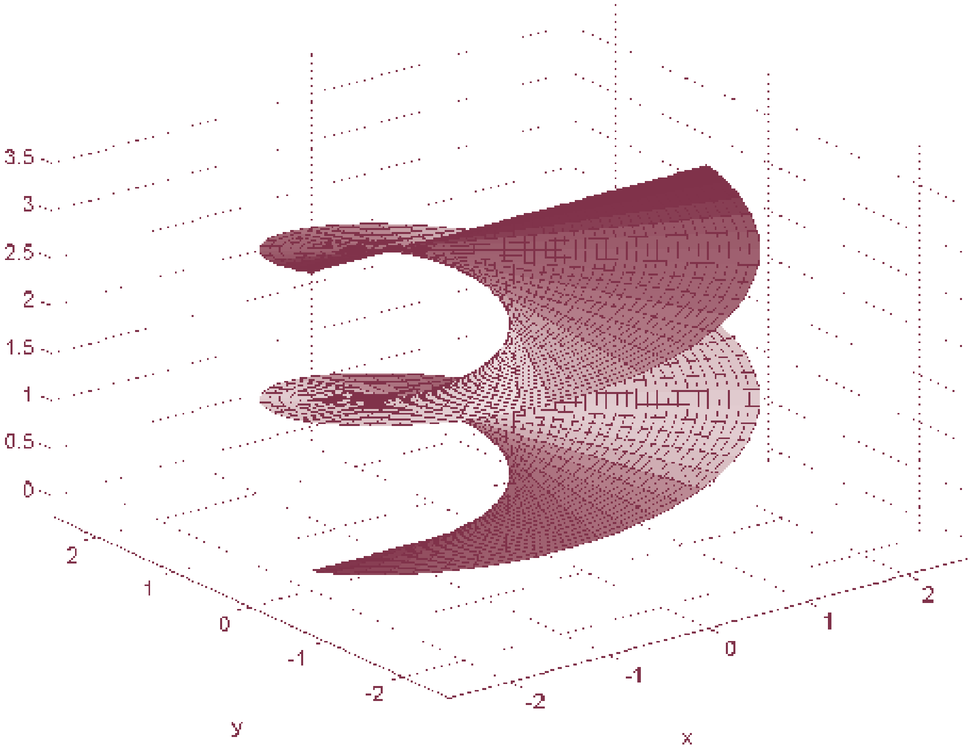

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Some extensions on Fan Ky's inequality

Gao Mingzhe and Mihaly Bencze

Abstract. In this paper we study the inequalities of the determinants of the positive definite matrices and the invertible matrices by applying the integral method and matrix theory such that extensions of Fan Ky's inequality are established. And then an improvement of Fan Ky's inequality is given by using the positive definiteness of Gram matrix.

Mathematics Subject Classification (2010): 15A15, 26D15.

Keywords: Fan Ky's inequality, Gram matrix, positive definite matrix, invertible matrix, characteristic root.

1. Introduction

In view of the importance of the inequality in theory and applications (see [1], [2]), it has been absorbing much interest of mathematicians. The various ways of proving inequalities appear in a great deal of papers. In particular, Kuang enumerated more than 50 methods in the paper [3]. It is obvious that these methods have the characteristic of themselves, technique, theory and applications. The purpose of the present paper is to study the discrete inequalities by applying a thought way on the proof of the inequality of the continuous function, and to try for a new path and to play to throw out a minnow to catch whale role in research and development. Explicitly, the extensions and improvement on the famous Fan Ky inequality are established by applying this method.

For convenience, we introduce some notations and functions.

The determinant of matrix X of order n is denoted by $|X|$ and a unit-matrix of order n is denoted by I. Let $x = (x_1, \ x_1, \ \cdots, x_n)$ be an n-dimension vector, $f(x)$ and $g(x)$ be functions with n variables. Let E be an inner product space, f and g be elements of E. Then the inner product of f and g is defined by the following n-ple integral:

$$(f, \ g) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x)g(x)\mathrm{dx},$$

where $\mathrm{dx} = \mathrm{dx}_1\mathrm{dx}_2 \cdots \mathrm{dx}_n$. And the norm of f is given by $\|f\| = \sqrt{(f, f)}$.

Let $f(x),\ g(x) > 0$ and $r,\ s > 0$. We stipulate that

$$(f^r, g^s) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f^r(x) g^s(x) \mathrm{dx}, \quad \|f\|_r = \left(\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f^r(x) \mathrm{dx} \right)^{\frac{1}{r}}, \quad \|f\|_2 = \|f\|,$$

$S_r(f,h) = \left(f^{\frac{r}{2}}, h\right) \|f\|_r^{-\frac{1}{r}}$.

where h is a variable unit-vector with n variables, i.e. $\|h\| = 1$, and it can be chosen in accordance with our requirements. In particular, $S_r(f,h) = 0$ if h is orthogonal to $f^{\frac{r}{2}}$.

Throughout this paper, we shall frequently use these notations.

2. Statement of main results

Let A, B be two positive definite matrices of order n, $0 \le \lambda \le 1$. Then

$$|A|^{\lambda} |B|^{1-\lambda} \le |\lambda A + (1-\lambda) B|. \tag{2.1}$$

This is the famous Fan Fy's inequality (see [3]). Recently, this inequality has been studied in some papers (such as [4, 5] etc.) Below we will build some extensions and a refinement of (2.1) by using the integral method and matrix theory.

First, we establish some extensions of (2.1).

Theorem 2.1. *Let m be a positive integer greater than 1, $A_i (i = 1, 2, \cdots, m)$ be positive definite matrix of order n, $\sum_{i=1}^{m} \frac{1}{p_i} = 1$ and $p_i > 1$. Then*

$$\prod_{i=1}^{m} |A_i|^{\frac{1}{p_i}} \le \left| \sum_{i=1}^{m} \frac{1}{p_i} A_i \right|. \tag{2.2}$$

In particular, for case $m = 2$, we have

$$|A|^{\frac{1}{p}} |B|^{\frac{1}{q}} \le \left| \frac{1}{p} A + \frac{1}{q} B \right|,$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. Clearly, it is the inequality (2.1)*. It follows that the inequality* (2.2) *is an extension of* (2.1)*.*

Remark 2.2. Inequality (2.1) shows that the function $f : PD \to (0, \infty)$ defined by $f(A) = |A|$, where PD is the set of positive defined matrices of order n is log-concave. So, Theorem 2.1 is Jensen 's inequality for f.

If $p < 1$, applying the reverse Hölder inequality, then the following reverse Fan Ky inequality is obtained:

$$|A|^{\frac{1}{p}} |B|^{\frac{1}{q}} > |\frac{1}{p} A + \frac{1}{q} B|.$$

If A_i $(i = 1, 2, \cdots, m)$ is invertible matrix of order n and A_i' is a transform of A_i, then $A_i A_i'$ is a positive definite matrix of order n and $|A_i A_i'| = |A_i|^2$. Based on Theorem 2.1, the following result is obtained.

Corollary 2.3. *If $A_i (i = 1, 2, \cdots, m)$ is a invertible matrix of order n,then*

$$\prod_{i=1}^{m} |A_i^2|^{\frac{1}{p_i}} \leq \left| \sum_{i=1}^{m} \frac{1}{p_i} A_i A_i' \right|. \tag{2.3}$$

Let $A_i (i = 1, 2, \cdots, m)$ is a symmetrical matrix of order n. Then there exists a sufficiently big k_i such that $k_i I + A_i$ is a positive definite matrix. Let $k = \max\{k_1,\ k_2, \cdots k_m\}$. Then we have the following result.

Corollary 2.4. *With the assumptions as the above-mentioned, then*

$$\prod_{i=1}^{m} |kI + A_i|^{\frac{1}{p_i}} \leq \left| \sum_{i=1}^{m} \frac{1}{p_i} (kI + A_i) \right|. \tag{2.4}$$

Next, we shall establish a refinement of (2.1).

Theorem 2.5. *Let A, B be two positive definite matrices of order n. If $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$, then*

$$|A|^{\frac{1}{p}} |B|^{\frac{1}{q}} \leq \left| \frac{1}{p} A + \frac{1}{q} B \right| (1 - R)^{\frac{2}{r}}, \tag{2.5}$$

where

$$R = (4\pi)^{\frac{n}{2}} \left(\left(\frac{|A|^{\frac{1}{2}}}{|\ A + \pi I|} \right)^{\frac{1}{2}} - \left(\frac{|B|^{\frac{1}{2}}}{|B + \pi I|} \right)^{\frac{1}{2}} \right)^2, \quad r = \max\{p,\ q\}.$$

Remark 2.6. In fact, Theorem 2.5 establishes a refinement of Fan Ky inequality.

If A and B are two invertible matrices of order n, A' and B' are respectively transforms of A and B, then AA' and BB' are positive definite matrices of order n. And notice that $|AA'| = |A|^2$ and $|BB'| = |B|^2$. Based on Theorem 2.5, the following result is obtained.

Corollary 2.7. *With the assumptions as the above-mentioned, then*

$$|A|^{\frac{1}{p}} |B|^{\frac{1}{q}} \leq \left| \frac{1}{p} AA' + \frac{1}{q} BB' \right|^{\frac{1}{2}} \left(1 - \tilde{R}\right)^{\frac{1}{r}}, \tag{2.6}$$

where

$$\tilde{R} = (4\pi)^{\frac{n}{2}} \left(\left(\frac{|A\ |}{|AA' + \pi I|} \right)^{\frac{1}{2}} - \left(\frac{|B\ |}{|BB' + \pi I|} \right)^{\frac{1}{2}} \right)^2, \quad r = \max\{p,\ q\}$$

3. Proofs of main results

In order to apply the integral method and matrix theory to prove our assertions, we need the following lemmas.

Lemma 3.1. *Let D be a positive definite matrix of order n. Then*

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp\left(-xDx'\right)\mathrm{dx}=\left(\frac{\pi^n}{|D|}\right)^{\frac{1}{2}},\tag{3.1}$$

where the vector $x=(x_1,\ x_2,\ \cdots,\ x_n)$,$x'$ is transform of x and $\mathrm{dx}=\mathrm{dx_1dx_2}\cdots\mathrm{dx}_n$. This result is the well known. Its proof is omitted here.

Lemma 3.2. *Let $\frac{1}{p}+\frac{1}{q}=1$ and $p>\ 1$. If $0<\|f\|_p<+\infty$ and $0<\|g\|_q<+\infty$, then*

$$(f,\ g)\le\|f\|_p\|g\|_q(1-R)^{\frac{1}{r}},\tag{3.2}$$

where $R=(S_p(f,h)-S_q(g,h))^2, r=\max\{p,\ q\},\ \|h\|=1$ and $(f^{p/2},h)(g^{q/2},h)\ge 0$.

And the equality in (3.2) holds if and only if $f^{p/2}$ and $g^{q/2}$ are linearly dependent; or h is a linear combination of $f^{p/2}$ and $g^{q/2}$, and $(f^{\frac{p}{2}},h)(g^{\frac{q}{2}},h)=0$ but h is not simultaneously orthogonal to $f^{\frac{p}{2}}$ and $g^{\frac{q}{2}}$.

Proof. First, we consider the case $p=2$. Let f,g and h be three arbitrary functions with n variables. If $\|h\|=1$, then

$$(f,\ g)^2\le\|f\|^2\|g\|^2-(\|f\|u-\|g\|v)^2,\tag{3.3}$$

where $u=(g,h),\quad v=(f,h),\ uv\ge 0$. And the equality in (3.3) holds if and only if f,g and h are linearly dependent; or h is a linear combination of f and g, and $uv=0$ but h is not simultaneously orthogonal to f and g. In fact, consider the Gram determinant constructed by the functions f,g and h:

$$G(f,g,h)=\begin{vmatrix}(f,f)&(f,g)&(f,h)\\(g,f)&(g,g)&(g,h)\\(h,f)&(h,g)&(h,h)\end{vmatrix}.$$

According to the positive definiteness of the Gram matrix, we have $G(f,g,h)\ge 0$, and $G(f,g,h)=0$ if and only if f,g and h are linearly dependent.

Expanding this determinant and using the condition $\|h\|=1$, we obtain

$$\begin{aligned}G(f,g,h)&=\|f\|^2\|g\|^2-(f,g)^2-\{\|f\|^2u^2-2(f,g))uv+\|g\|^2v^2\}\\&\le\|f\|^2\|g\|^2-(f,g)^2-\{\|f\|^2u^2-2(f,g))|uv|+\|g\|^2v^2\}\\&\le\|f\|^2\|g\|^2-(f,g)^2-(\|f\||u|-\|g\||v|)^2\\&\le\|f\|^2\|g\|^2-(f,g)^2-(\|f\|u-\|g\|v)^2\end{aligned}$$

where $u=(g,h)$, $v=(f,h)$ and $uv\ge 0$. And the equality holds if and only if f,g and h are linearly dependent; or h is a linear combination of f and g, and $uv=0$ but h is not simultaneously orthogonal to f and g.
The inequality (3.3) can be written in the following form:

$$(f,g)^2\le\|f\|^2\|g\|^2(1-r_2),\tag{3.4}$$

where $r_2=(S_2(f,h)-S_2(g,h))^2$. Namely, when $p=2$, the inequality (3.2) is valid. It is obvious that the inequality (3.4) is a refinement of the Cauchy inequality and that it is also extensions of the corresponding results of the papers [3, 6, 7].

Next, consider the case $p \neq 2$. Not loss generality, let $p > q > 1$. Since $\frac{1}{p} + \frac{1}{q} = 1$, we have $p > 2$. Let $\alpha = \frac{p}{2}, \beta = \frac{p}{p-2}$. Then $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. By applying Hölder's inequality, we have

$$\begin{aligned}(f,g) &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x)g(x)\mathrm{dx} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left\{f(x)(g(x))^{q/p}\right\}(g(x))^{1-q/p}\mathrm{dx} \\ &\leq \left\{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left(f(x)(g(x))^{q/p}\right)^{\alpha}\mathrm{dx}\right\}^{1/\alpha} \left\{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left\{(g(x))^{1-q/p}\right\}^{\beta}\mathrm{dx}\right\}^{1/\beta} \\ &= \left(f^{p/2}, g^{q/2}\right)^{2/p} \|g\|_q^{q(1-2/p)}. \end{aligned} \tag{3.5}$$

And the equality in (3.5) holds if and only if $f^{p/2}$ and $g^{q/2}$ are linearly dependent. In fact, The equality in (3.5) holds if and only if for any a positive integer k, there exists a positive number c_1, such that $(fg^{q/p})^{\alpha} = c_1(g^{1-q/p})^{\beta}$. After simplifications, we obtain $f^{p/2} = c_1 g^{q/2}$.

If f and g in (3.4) are replaced by $f^{\frac{p}{2}}$ and $g^{\frac{q}{2}}$ respectively, then we have

$$(f^{p/2}, g^{q/2})^2 \leq \|f\|_p^p \|g\|_q^q (1-R), \tag{3.6}$$

where $R = (S_p(f,h) - S_q(g,h))^2$. Substituting (3.6) into(3.5), we obtain after simplifications

$$(f,g) \leq \|f\|_p \|g\|_q (1-R)^{\frac{1}{p}}. \tag{3.7}$$

It is known from (3.4) that the equality in (3.7) holds if and only if $f^{p/2}$ and $g^{q/2}$ are linearly dependent; or h is a linear combination of $f^{p/2}$ and $g^{q/2}$, and $(f^{\frac{p}{2}}, h)(g^{\frac{q}{2}}, h) = 0$ but h is not simultaneously orthogonal to $f^{\frac{p}{2}}$ and $g^{\frac{q}{2}}$. Notice that the symmetry of p and q, it follows that the inequality (3.2) is valid.

It is very easy to prove Theorem 2.1, it is omitted here.

Proof of Theorem 2.5. Let $f(x) = \exp(-\frac{1}{p}(xAx'))$ and $g(x) = \exp(-\frac{1}{q}(xBx')$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. Based on (3.2) and (3.1), we have

$$\begin{aligned}\frac{\pi^{\frac{n}{2}}}{\left|\frac{1}{p}A + \frac{1}{q}B\right|^{\frac{1}{2}}} &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x)g(x)\mathrm{dx} \\ &\leq \left\{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f^p(x)\mathrm{dx}\right\}^{\frac{1}{p}} \left\{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g^q(x)\mathrm{dx}\right\}^{\frac{1}{q}} (1-R)^{\frac{1}{r}} \\ &= \frac{\pi^{\frac{n}{2}}}{\left(|A|^{\frac{1}{p}}|B|^{\frac{1}{q}}\right)^{\frac{1}{2}}}(1-R)^{\frac{1}{r}}. \end{aligned} \tag{3.8}$$

We attain from (3.8) after simplifications

$$|A|^{\frac{1}{p}}|B|^{\frac{1}{q}} \leq |\frac{1}{p}A + \frac{1}{q}B|(1-R)^{\frac{1}{r}} \tag{3.9}$$

where $r = \max\{p,\ q\}$.
We only need to compute R in (3.9). It is known from (3.2) that

$$R = (S_p(f,\ h) - S_q(g,\ h))^2 = \left(\frac{(f^{p/2},\ h)}{\|f\|_p^{p/2}} - \frac{(g^{q/2},\ h\ )}{\|g\|_q^{q/2}} \right)^2,$$

where $h = \exp\left(-\frac{1}{2}xCx'\right)$, Let $C = \pi I$. Then $|C| = \pi^n$, Based on (3.1),we have

$$\|h\| = \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h^2(x)\mathrm{dx} \right\}^{1/2} = 1. \tag{3.10}$$

It is easy to deduce that

$$\left(f^{\frac{p}{2}},\ h\right) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f^{\frac{p}{2}}(x)h\,(x)\,\mathrm{dx}$$

$$= \frac{\pi^{\frac{n}{2}}}{\left|\ \frac{1}{2}\ (A\ +\ \pi I)\right|^{\frac{n}{2}}} = (2\pi)^{\frac{n}{2}} \left(\frac{1}{|A + \pi I|} \right)^{\frac{1}{2}},$$

$$\|f\|_p^{\frac{p}{2}} = \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f^p(x)\mathrm{dx} \right\}^{\frac{1}{2}} = \left\{ \frac{\pi^n}{|A|} \right\}^{\frac{1}{4}},$$

$$S_p\,(f,\ h) = \ \left(f^{\frac{p}{2}},\ \ h\right) \|f\|_p^{-\frac{1}{p}} = (2\pi)^{\frac{n}{2}} \left(\frac{1}{|A + \pi I|} \right)^{\frac{1}{2}} \left\{ \frac{|A|}{\pi^n} \right\}^{\frac{1}{4}}$$

$$= (4\pi)^{\frac{n}{4}} \left(\frac{|A|^{\frac{1}{2}}}{|\ A\ +\ \pi I|} \right)^{\frac{1}{2}}.$$

Similarly, we have $S_q(g,\ h) = (4\pi)^{\frac{n}{4}} \left(\frac{|B|^{\frac{1}{2}}}{|\ B + \pi I|} \right)^{\frac{1}{2}}$.
It follows that

$$R = (S_p(f,h) - S_q(g,h))^2$$

$$= (4\pi)^{\frac{n}{2}} \left(\left(\frac{|A|^{\frac{1}{2}}}{|\ A + \pi I|} \right)^{\frac{1}{2}} - \left(\frac{|B|^{\frac{1}{2}}}{|\ B + \pi I|} \right)^{\frac{1}{2}} \right)^2.$$

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A remark on the proof of Cobzaş-Mustăţa theorem concerning norm preserving extension of convex Lipschitz functions

Iulian Cîmpean

Abstract. In this paper we present an alternative proof of a result concerning norm preserving extension of convex Lipschitz functions due to Ştefan Cobzaş and Costică Mustăţa (see Norm preserving extension of convex Lipschitz functions, Journal of Approximation Theory,24(3)(1987),236-244). Our proof is based on the Choquet Topological lemma, (see J.L.Doob, Classical potential theory and its probabilistic counterpart, Springer Verlag 2001).

Mathematics Subject Classification (2010): 26A16, 26A51.

Keywords: Extension of Lipschitz functions, convex functions, Choquet topological lemma.

1. Introduction

Taking into account a famous result due to Rademacher which states that a Lipschitz function $f : U = \mathring{U} \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is differentiable outside of a Lebesgue null subset of U, one can say that, from the point of view of real analysis the condition of being Lipschitz should be viewed as a weakened version of differentiability. Therefore, the class of Lipschitz functions has been intensively studied. The paper [9] is a very good introduction to the study of Lipschitz topology. One can also consult [16] and [22] for further details about Lipschitz functions.

The problem of the extension of a Lipschitz function is a central one in the theory of Lipschitz functions. Let us mention here just a phrase due to Earl Mickle (see [11]) which sustains our statement: ”In a problem on surface area the writer and Helsel were confronted with the following question: Can a Lipschitz function be extended to a Lipschitz transformation defined in the whole space?” Consequently, there is no surprise that there exist a lot of results in this direction (see for example [1]-[5], [7], [8], [10]-[15], [17]-[21]).

2. Preliminaries

Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is called Lipschitz if there exist a constant number $M \geq 0$ such that

$$|f(x) - f(y)| \leq Md(x, y) \tag{2.1}$$

for all $x, y \in X$.

The smallest constant M verifying (2.1) is called the norm of f and is denoted by $\|f\|_X$.

Denote by Lip$X$ the linear space of all Lipschitz functions on X.

Now let Y be a nonvoid subset of X. A norm preserving extension of a function $f \in$ Lip$Y$ to X is a function $F \in$ Lip$X$ such that

$$F\,|_Y = f$$

and

$$\|f\|_Y = \|F\|_X\,.$$

By a result of McShane [10], every $f \in$ Lip$Y$ has a norm preserving extension $F \in$ Lip$X$. Two of these extensions are given by:

$$F_1(x) = \sup\{f(y) - \|f\|_Y\, d(x, y) \mid y \in Y\} \tag{2.2}$$

$$F_2(x) = \inf\{f(y) + \|f\|_Y\, d(x, y) \mid y \in Y\} \tag{2.3}$$

Every norm preserving extension F of f satisfies:

$$F_1(x) \leq F(x) \leq F_2(x)\,,$$

for all $x \in X$ (see [4]).

It turns out that these results remain true for convex norm preserving extensions.

More precisely, given a normed linear space X and a nonvoid convex subset Y of X, Ş. Cobzaş and C. Mustăţa proved the following two results:

Theorem 2.1. (see [4]) *Every convex function $f \in LipY$ has a convex norm preserving extension F in $LipX$.*

Theorem 2.2. (see [4]) *For every convex function f in $LipY$, there exist two convex functions F_1, F_2, which are norm preserving extensions of f, such that:*

$$F_1(x) \leq F(x) \leq F_2(x)\,,$$

for all $x \in X$ and for every convex norm preserving extension F.

The proof for the last theorem focuses on the existence of F_1, the existence of F_2 following from the fact that the function defined in (3) is also convex.

We will present an alternative proof for the existence of F_1, which is based on the Choquet topological lemma.

3. The result

Lemma 3.1. (Choquet topological lemma) (see [6], Appendix VIII) *Let* $U = \{u_\beta, \beta \in I\}$ *be a family of functions from a second countable Hausdorff space into* $\overline{\mathbb{R}}$*, and if* $J \subseteq I$*, define*

$$u^J = \inf \{u_\beta \mid \beta \in J\}.$$

Then there is a countable subset J *of* I *such that*

$$u^J_+ = u^I_+.$$

In particular, if U *is directed downward, then there is a decreasing sequence* $(u_{\beta_n})_{n\geq 1} \subseteq U$ *with limit* v *such that*

$$v_+ = u^I_+.$$

By f_+, where f is a function from a Hausdorff space into $\overline{\mathbb{R}}$, we denote the lower semicontinuous minorant of f, which majorizes every lower semicontinuous minorant of f. That is

$$f_+(x_0) = f(x_0) \wedge \liminf_{x \to x_0} f(x).$$

Proof. The first assertion of the lemma is proved in [6] (Appendix VIII), so we will prove only the last assertion:

The first conclusion of the lemma assures us of the existence of a countable subset J of I such that

$$u^J_+ = u^I_+, \tag{3.1}$$

which allows us to rewrite the family $\{u_\beta \mid \beta \in J\}$ as a sequence $(u_n)_{n\geq 1}$.

In order to complete the proof, we construct a decreasing sequence $(u_{\alpha_n})_{n\geq 1} \subseteq U$ with limit v such that $v_+ = u^I_+$, as follows:

Let $u_{\alpha_1} = u_1$. For each $n \geq 2$, let u_{α_n} be a function from U such that

$$u_{\alpha_n} \leq \min\left(u_{\alpha_{n-1}}, u_n\right).$$

This construction is possible because U is supposed downward directed. Let v be the limit of this decreasing sequence. Since $u_{\alpha_n} \leq u_n$, we have that $v \leq \inf_{n\geq 1} u_n = u^J$, so that:

$$v_+ \leq u^J_+. \tag{3.2}$$

On the orher hand, $u_{\alpha_n} \geq u^I$, for all $n \geq 1$, so that

$$v_+ \geq u^I_+. \tag{3.3}$$

Now, from (3.1), (3.2) and (3.3) it follows that

$$v_+ = u^J_+ = u^I_+.$$ □

We need another lemma, also used and proved by Ş. Cobzaş and C. Mustăţa:

Lemma 3.2. (see [4]) *The set* $E^c_Y(f)$ *of all convex norm preserving extensions of* f *is downward directed (with respect to the pointwise ordering).*

Now, to prove the existence of F_1, combine the two lemmas as follows:
In Lemma 3.1 take $I = E_Y^c(f)$ and $u_\beta = \beta$, for each $\beta \in I$. Define

$$F_1 = u_+^I.$$

According to the same lemma, there is a decreasing sequence $(u_{\beta_n})_{n\geq 1}$ with limit v, such that $v_+ = u_+^I$. Since $u_{\beta_n} \in E_Y^c(f)$, then v is also in $E_Y^c(f)$, so that

$$v = v_+ = u_+^I = F_1 \in E_Y^c(f).$$

Clearly F_1 minimizes any other $F \in E_Y^c(f)$, which ends the proof.

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Fractional approximation by Cardaliaguet-Euvrard and Squashing neural network operators

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Abstract. This article deals with the determination of the fractional rate of convergence to the unit of some neural network operators, namely, the Cardaliaguet-Euvrard and "squashing" operators. This is given through the moduli of continuity of the involved right and left Caputo fractional derivatives of the approximated function and they appear in the right-hand side of the associated Jackson type inequalities.

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1. Introduction

The Cardaliaguet-Euvrard (3.1) operators were first introduced and studied extensively in [7], where the authors among many other things proved that these operators converge uniformly on compacta, to the unit over continuous and bounded functions. Our "squashing operator" (see [1]) (3.53) was motivated and inspired by the "squashing functions" and related Theorem 6 of [7]. The work in [7] is qualitative where the used bell-shaped function is general. However, our work, though greatly motivated by [7], is quantitative and the used bell-shaped and "squashing" functions are of compact support. We produce a series of Jackson type inequalities giving close upper bounds to the errors in approximating the unit operator by the above neural network induced operators. All involved constants there are well determined. These are pointwise, uniform and L_p, $p \geq 1$, estimates involving the first moduli of continuity of the engaged right and left Caputo fractional derivatives of the function under approximation. We give all necessary background of fractional calculus.

Initial work of the subject was done in [1], where we involved only ordinary derivatives. Article [1] motivated the current work.

2. Background

We need

Definition 2.1. *Let $f \in C(\mathbb{R})$ which is bounded or uniformly continuous, $h > 0$. We define the first modulus of continuity of f at h as follows*

$$\omega_1(f, h) = \sup\{|f(x) - f(y)|; x, y \in \mathbb{R}, \ |x - y| \le h\} \tag{2.1}$$

Notice that $\omega_1(f, h)$ is finite for any $h > 0$, and

$$\lim_{h \to 0} \omega_1(f, h) = 0.$$

We also need

Definition 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$, $\nu \ge 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a, b])$ (space of functions f with $f^{(n-1)} \in AC([a, b])$, absolutely continuous functions), $\forall$ $[a, b] \subset \mathbb{R}$. We call left Caputo fractional derivative (see [8], pp. 49-52) the function*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n - \nu)} \int_a^x (x - t)^{n - \nu - 1} f^{(n)}(t) \, dt, \tag{2.2}$$

$\forall$ $x \ge a$, where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu - 1} dt$, $\nu > 0$. Notice $D_{*a}^{\nu} f \in L_1([a, b])$ and $D_{*a}^{\nu} f$ exists a.e.on $[a, b]$, $\forall$ $b > a$. We set $D_{*a}^{0} f(x) = f(x)$, $\forall$ $x \in [a, \infty)$.

Lemma 2.3. *([5]) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^{n-1}(\mathbb{R})$ and $f^{(n)} \in L_{\infty}(\mathbb{R})$. Then $D_{*a}^{\nu} f(a) = 0$, $\forall$ $a \in \mathbb{R}$.*

Definition 2.4. *(see also [2], [9], [10]). Let $f : \mathbb{R} \to \mathbb{R}$, such that $f \in AC^m([a, b])$, $\forall$ $[a, b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by*

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m - \alpha)} \int_x^b (J - x)^{m - \alpha - 1} f^{(m)}(J) \, dJ, \tag{2.3}$$

$\forall$ $x \le b$. We set $D_{b-}^{0} f(x) = f(x)$, $\forall$ $x \in (-\infty, b]$. Notice that $D_{b-}^{\alpha} f \in L_1([a, b])$ and $D_{b-}^{\alpha} f$ exists a.e.on $[a, b]$, $\forall$ $a < b$.

Lemma 2.5. *([5]) Let $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_{\infty}(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(b) = 0$, $\forall$ $b \in \mathbb{R}$.*

Convention 2.6. *We assume that*

$$D_{*x_0}^{\alpha} f(x) = 0, \ \text{ for } x < x_0, \tag{2.4}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{2.5}$$

for all $x, x_0 \in \mathbb{R}$.

We mention

Proposition 2.7. *(by [3]) Let $f \in C^n(\mathbb{R})$, where $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a, \infty)$.*

Also we have

Proposition 2.8. *(by [3]) Let $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(x)$ is continuous in $x \in (-\infty, b]$.*

We further mention

Proposition 2.9. *(by [3]) Let $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{2.6}$$

for all $x, x_0 \in \mathbb{R} : x \geq x_0$.

Then $D_{*x_0}^{\alpha} f(x)$ is continuous in x_0.

Proposition 2.10. *(by [3]) Let $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x}^{x_0} (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \tag{2.7}$$

for all $x, x_0 \in \mathbb{R} : x_0 \geq x$.

Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0.

Proposition 2.11. *([5]) Let $g \in C_b(\mathbb{R})$ (continuous and bounded), $0 < c < 1$, $x, x_0 \in \mathbb{R}$. Define*

$$L(x, x_0) = \int_{x_0}^{x} (x-t)^{c-1} g(t)\, dt, \text{ for } x \geq x_0, \tag{2.8}$$

and $L(x, x_0) = 0$, for $x < x_0$.

Then L is jointly continuous in $(x, x_0) \in \mathbb{R}^2$.

We mention

Proposition 2.12. *([5]) Let $g \in C_b(\mathbb{R})$, $0 < c < 1$, $x, x_0 \in \mathbb{R}$. Define*

$$K(x, x_0) = \int_{x}^{x_0} (J-x)^{c-1} g(J)\, dJ, \quad \text{for } x \leq x_0, \tag{2.9}$$

and $K(x, x_0) = 0$, for $x > x_0$.

Then $K(x, x_0)$ is jointly continuous in $(x, x_0) \in \mathbb{R}^2$.

Based on Propositions 2.11, 2.12 we derive

Corollary 2.13. *([5]) Let $f \in C^m(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, $x, x_0 \in \mathbb{R}$. Then $D_{*x_0}^{\alpha} f(x)$, $D_{x_0-}^{\alpha} f(x)$ are jointly continuous functions in (x, x_0) from $\mathbb{R}^2$ into $\mathbb{R}$.*

We need

Proposition 2.14. *([5]) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be jointly continuous. Consider*

$$G(x) = \omega_1(f(\cdot, x), \delta)_{[x,+\infty)}, \ \delta > 0, x \in \mathbb{R}. \tag{2.10}$$

(Here ω_1 is defined over $[x, +\infty)$ instead of $\mathbb{R}$.)

Then G is continuous on $\mathbb{R}$.

Proposition 2.15. *([5]) Let $f:\mathbb{R}^2\to\mathbb{R}$ be jointly continuous. Consider*

$$H(x)=\omega_1\left(f(\cdot,x),\delta\right)_{(-\infty,x]},\ \ \delta>0, x\in\mathbb{R}. \tag{2.11}$$

(Here ω_1 is defined over $(-\infty,x]$ instead of $\mathbb{R}$.)

Then H is continuous on $\mathbb{R}$.

By Propositions 2.14, 2.15 and Corollary 2.13 we derive

Proposition 2.16. *([5]) Let $f\in C^m(\mathbb{R})$, $\left\|f^{(m)}\right\|_\infty<\infty$, $m=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $\alpha>0$, $x\in\mathbb{R}$. Then $\omega_1\left(D_{*x}^\alpha f,h\right)_{[x,+\infty)}$, $\omega_1\left(D_{x-}^\alpha f,h\right)_{(-\infty,x]}$ are continuous functions of $x\in\mathbb{R}$, $h>0$ fixed.*

We make

Remark 2.17. Let g be continuous and bounded from $\mathbb{R}$ to $\mathbb{R}$. Then

$$\omega_1(g,t)\leq 2\|g\|_\infty<\infty. \tag{2.12}$$

Assuming that $\left(D_{*x}^\alpha f\right)(t)$, $\left(D_{x-}^\alpha f\right)(t)$, are both continuous and bounded in $(x,t)\in\mathbb{R}^2$, i.e.

$$\left\|D_{*x}^\alpha f\right\|_\infty\leq K_1,\ \forall\ x\in\mathbb{R}; \tag{2.13}$$

$$\left\|D_{x-}^\alpha f\right\|_\infty\leq K_2,\ \forall\ x\in\mathbb{R}, \tag{2.14}$$

where $K_1,K_2>0$, we get

$$\omega_1\left(D_{*x}^\alpha f,\xi\right)_{[x,+\infty)}\leq 2K_1;$$

$$\omega_1\left(D_{x-}^\alpha f,\xi\right)_{(-\infty,x]}\leq 2K_2,\ \forall\ \xi\geq 0, \tag{2.15}$$

for each $x\in\mathbb{R}$.

Therefore, for any $\xi\geq 0$,

$$\sup_{x\in\mathbb{R}}\left[\max\left(\omega_1\left(D_{*x}^\alpha f,\xi\right)_{[x,+\infty)},\omega_1\left(D_{x-}^\alpha f,\xi\right)_{(-\infty,x]}\right)\right]\leq 2\max(K_1,K_2)<\infty. \tag{2.16}$$

So in our setting for $f\in C^m(\mathbb{R})$, $\left\|f^{(m)}\right\|_\infty<\infty$, $m=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $\alpha>0$, by Corollary 2.13 both $\left(D_{*x}^\alpha f\right)(t)$, $\left(D_{x-}^\alpha f\right)(t)$ are jointly continuous in (t,x) on $\mathbb{R}^2$. Assuming further that they are both bounded on $\mathbb{R}^2$ we get (2.16) valid. In particular, each of $\omega_1\left(D_{*x}^\alpha f,\xi\right)_{[x,+\infty)}$, $\omega_1\left(D_{x-}^\alpha f,\xi\right)_{(-\infty,x]}$ is finite for any $\xi\geq 0$.

Clearly here we have that $\sup_{x\in\mathbb{R}}\omega_1\left(D_{*x}^\alpha f,\xi\right)_{[x,+\infty)}\to 0$, as $\xi\to 0+$, and $\sup_{x\in\mathbb{R}}\omega_1\left(D_{x-}^\alpha f,\xi\right)_{(-\infty,x]}\to 0$, as $\xi\to 0+$.

Let us now assume only that $f\in C^{m-1}(\mathbb{R})$, $f^{(m)}\in L_\infty(\mathbb{R})$, $m=\lceil\alpha\rceil$, $\alpha>0$, $\alpha\notin\mathbb{N}$, $x\in\mathbb{R}$. Then, by Proposition 15.114, p. 388 of [4], we find that $D_{*x}^\alpha f\in C([x,+\infty))$, and by [6] we obtain that $D_{x-}^\alpha f\in C((-\infty,x])$.

We make

Remark 2.18. Again let $f\in C^m(\mathbb{R})$, $m=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $\alpha>0$; $f^{(m)}(x)=1$, $\forall\ x\in\mathbb{R}$; $x_0\in\mathbb{R}$. Notice $0<m-\alpha<1$. Then

$$D_{*x_0}^\alpha f(x)=\frac{(x-x_0)^{m-\alpha}}{\Gamma(m-\alpha+1)},\ \forall\ x\geq x_0. \tag{2.17}$$

Let us consider $x, y \geq x_0$, then

$$\left|D^{\alpha}_{*x_0} f(x) - D^{\alpha}_{*x_0} f(y)\right| = \frac{1}{\Gamma(m-\alpha+1)} \left|(x-x_0)^{m-\alpha} - (y-x_0)^{m-\alpha}\right| \leq \frac{|x-y|^{m-\alpha}}{\Gamma(m-\alpha+1)}. \tag{2.18}$$

So it is not strange to assume that

$$\left|D^{\alpha}_{*x_0} f(x_1) - D^{\alpha}_{*x_0} f(x_2)\right| \leq K |x_1 - x_2|^{\beta}, \tag{2.19}$$

$K > 0$, $0 < \beta \leq 1$, $\forall\, x_1, x_2 \in \mathbb{R}$, $x_1, x_2 \geq x_0 \in \mathbb{R}$, where more generally it is $\left\|f^{(m)}\right\|_{\infty} < \infty$. Thus, one may assume

$$\omega_1\left(D^{\alpha}_{x-} f, \xi\right)_{(-\infty,x]} \leq M_1 \xi^{\beta_1}, \text{ and} \tag{2.20}$$

$$\omega_1\left(D^{\alpha}_{*x} f, \xi\right)_{[x,+\infty)} \leq M_2 \xi^{\beta_2},$$

where $0 < \beta_1, \beta_2 \leq 1$, $\forall\, \xi > 0$, $M_1, M_2 > 0$; any $x \in \mathbb{R}$.

Setting $\beta = \min(\beta_1, \beta_2)$ and $M = \max(M_1, M_2)$, in that case we obtain

$$\sup_{x\in\mathbb{R}} \left\{ \max\left(\omega_1\left(D^{\alpha}_{x-} f, \xi\right)_{(-\infty,x]}, \omega_1\left(D^{\alpha}_{*x} f, \xi\right)_{[x,+\infty)} \right) \right\} \leq M \xi^{\beta} \to 0, \text{ as } \xi \to 0+. \tag{2.21}$$

3. Results

3.1. Fractional convergence with rates of the Cardaliaguet-Euvrard neural network operators

We need the following (see [7]).

Definition 3.1. *A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero. The function $b(x)$ may have jump discontinuities. In this work we consider only centered bell-shaped functions of compact support $[-T, T]$, $T > 0$. Call $I := \int_{-T}^{T} b(t)\,dt$. Note that $I > 0$.*

We follow [1], [7].

Example 3.2. (1) $b(x)$ can be the characteristic function over $[-1, 1]$.

(2) $b(x)$ can be the hat function over $[-1, 1]$, i.e.,

$$b(x) = \begin{cases} 1 + x, & -1 \leq x \leq 0, \\ 1 - x, & 0 < x \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

These are centered bell-shaped functions of compact support.

Here we consider functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous.

In this article we study the fractional convergence with rates over the real line, to the unit operator, of the Cardaliaguet-Euvrard neural network operators (see [7]),

$$(F_n(f))(x) := \sum_{k=-n^2}^{n^2} \frac{f\left(\frac{k}{n}\right)}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right), \tag{3.1}$$

where $0 < \alpha < 1$ and $x \in \mathbb{R}$, $n \in \mathbb{N}$. The terms in the sum (3.1) can be nonzero iff

$$\left|n^{1-\alpha}\left(x - \frac{k}{n}\right)\right| \leq T, \text{ i.e. } \left|x - \frac{k}{n}\right| \leq \frac{T}{n^{1-\alpha}}$$

iff

$$nx - Tn^{\alpha} \leq k \leq nx + Tn^{\alpha}. \tag{3.2}$$

In order to have the desired order of numbers

$$-n^2 \leq nx - Tn^{\alpha} \leq nx + Tn^{\alpha} \leq n^2, \tag{3.3}$$

it is sufficient enough to assume that

$$n \geq T + |x|. \tag{3.4}$$

When $x \in [-T, T]$ it is enough to assume $n \geq 2T$ which implies (3.3).

Proposition 3.3. *Let $a \leq b$, $a, b \in \mathbb{R}$. Let $card(k)$ (≥ 0) be the maximum number of integers contained in $[a, b]$. Then*

$$\max(0, (b-a) - 1) \leq card(k) \leq (b-a) + 1. \tag{3.5}$$

Remark 3.4. We would like to establish a lower bound on $card(k)$ over the interval $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$. From Proposition 3.3 we get that

$$card(k) \geq \max(2Tn^{\alpha} - 1, 0).$$

We obtain $card(k) \geq 1$, if

$$2Tn^{\alpha} - 1 \geq 1 \text{ iff } n \geq T^{-\frac{1}{\alpha}}.$$

So to have the desired order (3.3) and $card(k) \geq 1$ over $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$, we need to consider

$$n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right). \tag{3.6}$$

Also notice that $card(k) \to +\infty$, as $n \to +\infty$. We call $b^* := b(0)$ the maximum of $b(x)$.

Denote by $[\cdot]$ the integral part of a number.

Following [1] we have

$$\sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)$$

$$\leq \frac{b^*}{I \cdot n^{\alpha}} \cdot \sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} 1$$

$$\leq \frac{b^*}{I \cdot n^{\alpha}} \cdot (2Tn^{\alpha} + 1) = \frac{b^*}{I} \cdot \left(2T + \frac{1}{n^{\alpha}}\right). \tag{3.7}$$

We will use

Lemma 3.5. *It holds that*

$$S_n(x) := \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I\cdot n^{\alpha}} \cdot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) \to 1, \tag{3.8}$$

pointwise, as $n\to+\infty$, where $x\in\mathbb{R}$.

Remark 3.6. Clearly we have that

$$nx - Tn^{\alpha} \le nx \le nx + Tn^{\alpha}. \tag{3.9}$$

We prove in general that

$$nx - Tn^{\alpha} \le [nx] \le nx \le \lceil nx\rceil \le nx + Tn^{\alpha}. \tag{3.10}$$

Indeed we have that, if $[nx] < nx - Tn^{\alpha}$, then $[nx]+Tn^{\alpha} < nx$, and $[nx]+[Tn^{\alpha}] \le [nx]$, resulting into $[Tn^{\alpha}] = 0$, which for large enough n is not true. Therefore $nx - Tn^{\alpha} \le [nx]$. Similarly, if $\lceil nx\rceil > nx + Tn^{\alpha}$, then $nx+Tn^{\alpha} \ge nx + [Tn^{\alpha}]$, and $\lceil nx\rceil - [Tn^{\alpha}] > nx$, thus $\lceil nx\rceil - [Tn^{\alpha}] \ge \lceil nx\rceil$, resulting into $[Tn^{\alpha}] = 0$, which again for large enough n is not true.

Therefore without loss of generality we may assume that

$$nx - Tn^{\alpha} \le [nx] \le nx \le \lceil nx\rceil \le nx + Tn^{\alpha}. \tag{3.11}$$

Hence $\lceil nx - Tn^{\alpha}\rceil \le [nx]$ and $\lceil nx\rceil \le [nx+Tn^{\alpha}]$. Also if $[nx] \ne \lceil nx\rceil$, then $\lceil nx\rceil = [nx]+1$. If $[nx] = \lceil nx\rceil$, then $nx\in\mathbb{Z}$; and by assuming $n \ge T^{-\frac{1}{\alpha}}$, we get $Tn^{\alpha}\ge 1$ and $nx + Tn^{\alpha} \ge nx+1$, so that $[nx+Tn^{\alpha}] \ge nx+1 = [nx]+1$.

We present our first main result

Theorem 3.7. *We consider $f:\mathbb{R}\to\mathbb{R}$. Let $\beta>0$, $N=\lceil\beta\rceil$, $\beta\notin\mathbb{N}$, $f\in AC^N([a,b])$, $\forall\ [a,b]\subset\mathbb{R}$, with $f^{(N)}\in L_{\infty}(\mathbb{R})$. Let also $x\in\mathbb{R}$, $T>0$, $n\in\mathbb{N}: n \ge \max\left(T+|x|, T^{-\frac{1}{\alpha}}\right)$. We further assume that $D_{*x}^{\beta}f$, $D_{x-}^{\beta}f$ are uniformly continuous functions or continuous and bounded on $[x,+\infty)$, $(-\infty,x]$, respectively.*

Then

1)

$$|F_n(f)(x) - f(x)| \le |f(x)| \cdot \tag{3.12}$$

$$\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1\right| +$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\left(\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right| T^j}{j!n^{(1-\alpha)j}}\right)$$

$$+\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\omega_1\left(D_{*x}^{\beta}f, \frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)} + \omega_1\left(D_{x-}^{\beta}f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

above $\sum_{j=1}^{0} \cdot = 0$,

2)

$$\left| \left(F_n(f)\right)(x) - \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(F_n\left((\cdot - x)^j\right)\right)(x) \right| \leq \tag{3.13}$$

$$\frac{b^*}{I}\left(2T + \frac{1}{n^{\alpha}}\right) \frac{T^{\beta}}{\Gamma(\beta+1)\, n^{(1-\alpha)\beta}} \cdot$$

$$\left\{ \omega_1\left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)} + \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]} \right\} =: \lambda_n(x),$$

3) assume further that $f^{(j)}(x) = 0$, *for* $j = 0, 1, ..., N-1$, *we get*

$$\left|F_n(f)(x)\right| \leq \lambda_n(x), \tag{3.14}$$

4) in case of $N = 1$, *we obtain*

$$\left|F_n(f)(x) - f(x)\right| \leq |f(x)| \cdot \tag{3.15}$$

$$\left| \sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) - 1 \right| +$$

$$\frac{b^*}{I}\left(2T + \frac{1}{n^{\alpha}}\right) \frac{T^{\beta}}{\Gamma(\beta+1)\, n^{(1-\alpha)\beta}} \cdot$$

$$\left\{ \omega_1\left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)} + \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]} \right\}.$$

Here we get fractionally with rates the pointwise convergence of $\left(F_n(f)\right)(x) \to f(x)$, *as* $n \to \infty$, $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$. We have that

$$D_{x-}^{\beta} f(x) = D_{*x}^{\beta} f(x) = 0. \tag{3.16}$$

From [8], p. 54, we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j + \tag{3.17}$$

$$\frac{1}{\Gamma(\beta)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ,$$

for all $x \leq \frac{k}{n} \leq x + Tn^{\alpha-1}$, iff $\lceil nx \rceil \leq k \leq [nx + Tn^{\alpha}]$, where $k \in \mathbb{Z}$.

Also from [2], using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j + \tag{3.18}$$

$$\frac{1}{\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ,$$

for all $x - Tn^{\alpha-1} \leq \frac{k}{n} \leq x$, iff $\lceil nx - Tn^{\alpha} \rceil \leq k \leq [nx]$, where $k \in \mathbb{Z}$.

Notice that $\lceil nx \rceil \leq [nx] + 1$.

Hence we have

$$\frac{f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} + \tag{3.19}$$

$$\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ,$$

and

$$\frac{f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} + \tag{3.20}$$

$$\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ.$$

Therefore we obtain

$$\frac{\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} = \tag{3.21}$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \left(\frac{k}{n}-x\right)^{j} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} \right) +$$

$$\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ,$$

and

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} = \tag{3.22}$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} \left(\frac{k}{n}-x\right)^{j} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} +$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ.$$

We notice here that

$$(F_n(f))(x) := \sum_{k=-n^2}^{n^2} \frac{f\left(\frac{k}{n}\right)}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) = \tag{3.23}$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{f\left(\frac{k}{n}\right)}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right).$$

Adding the two equalities (3.21) and (3.22) we obtain

$$(F_n(f))(x) =$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\frac{k}{n}-x\right)^{j} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} \right) + \theta_n(x), \tag{3.24}$$

where

$$\theta_n(x) := \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \cdot$$

$$\int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ +$$

$$\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ. \tag{3.25}$$

We call

$$\theta_{1n}(x) := \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \cdot$$

$$\int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ, \tag{3.26}$$

and

$$\theta_{2n}(x) := \sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \cdot$$

$$\int_{x}^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ. \tag{3.27}$$

I.e.

$$\theta_n(x) = \theta_{1n}(x) + \theta_{2n}(x). \tag{3.28}$$

We further have

$$(F_n(f))(x) - f(x) = f(x) \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} - 1 \right) + \tag{3.29}$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\frac{k}{n}-x\right)^{j} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} \right) + \theta_n(x),$$

and

$$|(F_n(f))(x) - f(x)| \le |f(x)| \left| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1 \right| +$$

$$\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!} \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left|x-\frac{k}{n}\right|^{j} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} \right) + |\theta_n(x)| \le$$

$$|f(x)| \left| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1 \right| + \tag{3.30}$$

$$\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!} \frac{T^j}{n^{(1-\alpha)j}} \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} \right) + |\theta_n(x)| =: (*).$$

But we have

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) \leq \frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right), \tag{3.31}$$

by (3.7).

Therefore we obtain

$$|(F_n(f))(x) - f(x)| \leq |f(x)| \left| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1 \right| +$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\left(\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right| T^j}{j! n^{(1-\alpha)j}}\right) + |\theta_n(x)|. \tag{3.32}$$

Next we see that

$$\gamma_{1n} := \frac{1}{\Gamma(\beta)} \left| \int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ \right| \leq \tag{3.33}$$

$$\frac{1}{\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left| D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\, dJ \right| \leq$$

$$\frac{1}{\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \omega_1\left(D_{x-}^{\beta} f, |J-x|\right)_{(-\infty,x]} dJ \leq$$

$$\frac{1}{\Gamma(\beta)} \omega_1\left(D_{x-}^{\beta} f, \left|x-\frac{k}{n}\right|\right)_{(-\infty,x]} \int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} dJ \leq$$

$$\frac{1}{\Gamma(\beta)} \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]} \frac{\left(x-\frac{k}{n}\right)^{\beta}}{\beta} \leq$$

$$\frac{1}{\Gamma(\beta+1)} \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]} \frac{T^{\beta}}{n^{(1-\alpha)\beta}}.$$

That is

$$\gamma_{1n} \leq \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}. \tag{3.34}$$

Furthermore

$$|\theta_{1n}(x)| \leq \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} \gamma_{1n} \leq \tag{3.35}$$

$$\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}}\right) \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]} \leq$$

$$\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}}\right)\frac{T^{\beta}}{\Gamma\left(\beta+1\right)n^{(1-\alpha)\beta}}\omega_{1}\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\leq$$

$$\frac{b^{*}}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma\left(\beta+1\right)n^{(1-\alpha)\beta}}\omega_{1}\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}.$$

So that

$$\left|\theta_{1n}\left(x\right)\right|\leq\frac{b^{*}}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma\left(\beta+1\right)n^{(1-\alpha)\beta}}\omega_{1}\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}. \quad (3.36)$$

Similarly we derive

$$\gamma_{2n}:=\frac{1}{\Gamma\left(\beta\right)}\left|\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f\left(J\right)-D_{*x}^{\beta}f\left(x\right)\right)dJ\right|\leq \quad (3.37)$$

$$\frac{1}{\Gamma\left(\beta\right)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left|D_{*x}^{\beta}f\left(J\right)-D_{*x}^{\beta}f\left(x\right)\right|dJ\leq$$

$$\frac{\omega_{1}\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}}{\Gamma\left(\beta+1\right)}\left(\frac{k}{n}-x\right)^{\beta}\leq$$

$$\frac{\omega_{1}\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}}{\Gamma\left(\beta+1\right)}\frac{T^{\beta}}{n^{(1-\alpha)\beta}}.$$

That is

$$\gamma_{2n}\leq\frac{T^{\beta}}{\Gamma\left(\beta+1\right)n^{(1-\alpha)\beta}}\omega_{1}\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}. \quad (3.38)$$

Consequently we find

$$\left|\theta_{2n}\left(x\right)\right|\leq\left(\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}}\right)\cdot$$

$$\frac{T^{\beta}}{\Gamma\left(\beta+1\right)n^{(1-\alpha)\beta}}\omega_{1}\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}\leq \quad (3.39)$$

$$\frac{b^{*}}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma\left(\beta+1\right)n^{(1-\alpha)\beta}}\omega_{1}\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}.$$

So we have proved that

$$\left|\theta_{n}\left(x\right)\right|\leq\frac{b^{*}}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma\left(\beta+1\right)n^{(1-\alpha)\beta}}\cdot \quad (3.40)$$

$$\left\{\omega_{1}\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_{1}\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

Combining (3.32) and (3.40) we have (3.12). □

As an application of Theorem 3.7 we give

Theorem 3.8. *Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in C^N(\mathbb{R})$, with $f^{(N)} \in L_\infty(\mathbb{R})$. Let also $T > 0$, $n \in \mathbb{N} : n \geq \max\left(2T, T^{-\frac{1}{\alpha}}\right)$. We further assume that $D_{*x}^{\beta} f(t)$, $D_{x-}^{\beta} f(t)$ are both bounded in $(x,t) \in \mathbb{R}^2$. Then*

1)

$$\|F_n(f) - f\|_{\infty,[-T,T]} \leq \|f\|_{\infty,[-T,T]} \cdot \tag{3.41}$$

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) - 1 \right\|_{\infty,[-T,T]} +$$

$$\frac{b^*}{I}\left(2T + \frac{1}{n^{\alpha}}\right)\left(\sum_{j=1}^{N-1} \frac{\|f^{(j)}\|_{\infty,[-T,T]} T^j}{j! n^{(1-\alpha)j}}\right) + \frac{b^*}{I}\left(2T + \frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot$$

$$\left\{ \sup_{x\in[-T,T]} \omega_1\left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)} + \sup_{x\in[-T,T]} \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]} \right\},$$

2) in case of $N = 1$, we obtain

$$\|F_n(f) - f\|_{\infty,[-T,T]} \leq \|f\|_{\infty,[-T,T]} \cdot \tag{3.42}$$

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) - 1 \right\|_{\infty,[-T,T]} +$$

$$\frac{b^*}{I}\left(2T + \frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot$$

$$\left\{ \sup_{x\in[-T,T]} \omega_1\left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)} + \sup_{x\in[-T,T]} \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]} \right\}.$$

An interesting case is when $\beta = \frac{1}{2}$.

Assuming further that $\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) - 1 \right\|_{\infty,[-T,T]} \to 0$, as $n \to \infty$, we get fractionally with rates the uniform convergence of $F_n(f) \to f$, as $n \to \infty$.

Proof. From (3.12), (3.15) of Theorem 3.7, and by Remark 2.17.

Also by

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) \leq \frac{b^*}{I}(2T+1), \tag{3.43}$$

we get that

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) - 1 \right\|_{\infty,[-T,T]} \leq \left(\frac{b^*}{I}(2T+1) + 1\right). \tag{3.44}$$

□

One can also apply Remark 2.18 to the last Theorem 3.8, to get interesting and simplified results.

We make

Remark 3.9. Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T,T]$, $T>0$. Let $x\in[-T^*,T^*]$, $T^*>0$, and $n\in\mathbb{N}: n\geq \max\left(T+T^*, T^{-\frac{1}{\alpha}}\right)$, $0<\alpha<1$. Consider $p\geq 1$.

Clearly we get here that

$$\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{In^{\alpha}}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|^{p}\leq\left(\frac{b^*}{I}(2T+1)+1\right)^{p}, \tag{3.45}$$

for all $x\in[-T^*,T^*]$, for any $n\geq\max\left(T+T^*,T^{-\frac{1}{\alpha}}\right)$.

By Lemma 3.5, we obtain that

$$\lim_{n\to\infty}\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{In^{\alpha}}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|^{p}=0, \tag{3.46}$$

all $x\in[-T^*,T^*]$.

Now it is clear, by the bounded convergence theorem, that

$$\lim_{n\to\infty}\left\|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{In^{\alpha}}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{p,[-T^*,T^*]}=0. \tag{3.47}$$

Let $\beta>0$, $N=\lceil\beta\rceil$, $\beta\notin\mathbb{N}$, $f\in C^{N}(\mathbb{R})$, $f^{(N)}\in L_{\infty}(\mathbb{R})$. Here both $D_{*x}^{\alpha}f(t)$, $D_{x-}^{\alpha}f(t)$ are bounded in $(x,t)\in\mathbb{R}^2$.

By Theorem 3.7 we have

$$|F_n(f)(x)-f(x)|\leq\|f\|_{\infty,[-T^*,T^*]}\cdot \tag{3.48}$$

$$\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{In^{\alpha}}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|+$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\left(\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|T^{j}}{j!n^{(1-\alpha)j}}\right)$$

$$+\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\sup_{x\in[-T^*,T^*]}\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\sup_{x\in[-T^*,T^*]}\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

Applying to the last inequality (3.48) the monotonicity and subadditive property of $\|\cdot\|_p$, we derive the following L_p, $p\geq1$, interesting result.

Theorem 3.10. *Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T,T]$, $T>0$. Let $x\in[-T^*,T^*]$, $T^*>0$, and $n\in\mathbb{N}:n\geq\max\left(T+T^*,T^{-\frac{1}{\alpha}}\right)$, $0<\alpha<1$, $p\geq1$. Let $\beta>0$, $N=\lceil\beta\rceil$, $\beta\notin\mathbb{N}$, $f\in C^N(\mathbb{R})$, with $f^{(N)}\in L_\infty(\mathbb{R})$. Here both $D^\beta_{*x}f(t)$, $D^\beta_{x-}f(t)$ are bounded in $(x,t)\in\mathbb{R}^2$. Then*

1)

$$\|F_nf-f\|_{p,[-T^*,T^*]}\leq\|f\|_{\infty,[-T^*,T^*]}\cdot \tag{3.49}$$

$$\left\|\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{1}{In^\alpha}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{p,[-T^*,T^*]}+$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^\alpha}\right)\left(\sum_{j=1}^{N-1}\frac{\|f^{(j)}\|_{p,[-T^*,T^*]}T^j}{j!n^{(1-\alpha)j}}\right)+$$

$$\frac{2^{\frac{1}{p}}T^{*\frac{1}{p}}b^*}{I}\left(2T+\frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\sup_{x\in[-T^*,T^*]}\omega_1\left(D^\beta_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\sup_{x\in[-T^*,T^*]}\omega_1\left(D^\beta_{x-}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

2) When $N=1$, we derive

$$\|F_nf-f\|_{p,[-T^*,T^*]}\leq\|f\|_{\infty,[-T^*,T^*]}\cdot \tag{3.50}$$

$$\left\|\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{1}{In^\alpha}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{p,[-T^*,T^*]}+$$

$$\frac{2^{\frac{1}{p}}T^{*\frac{1}{p}}b^*}{I}\left(2T+\frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\sup_{x\in[-T^*,T^*]}\omega_1\left(D^\beta_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\sup_{x\in[-T^*,T^*]}\omega_1\left(D^\beta_{x-}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

By (3.49), (3.50) we derive the fractional L_p, $p\geq1$, convergence with rates of F_nf to f.

3.2. The "Squashing operators" and their fractional convergence to the unit with rates

We need (see also [1], [7]).

Definition 3.11. *Let the nonnegative function $S:\mathbb{R}\to\mathbb{R}$, S has compact support $[-T,T]$, $T>0$, and is nondecreasing there and it can be continuous only on either $(-\infty,T]$ or $[-T,T]$. S can have jump discontinuities. We call S the "squashing function".*

Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be continuous. Assume that

$$I^{*}:=\int_{-T}^{T}S\left(t\right)dt>0. \tag{3.51}$$

Obviously

$$\max_{x\in[-T,T]}S\left(x\right)=S\left(T\right). \tag{3.52}$$

For $x\in\mathbb{R}$ we define the "squashing operator" ([1])

$$\left(G_{n}\left(f\right)\right)\left(x\right):=\sum_{k=-n^{2}}^{n^{2}}\frac{f\left(\frac{k}{n}\right)}{I^{*}\cdot n^{\alpha}}\cdot S\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right), \tag{3.53}$$

$0<\alpha<1$ and $n\in\mathbb{N}:n\geq\max\left(T+\left|x\right|,T^{-\frac{1}{\alpha}}\right)$. It is clear that

$$\left(G_{n}\left(f\right)\right)\left(x\right)=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{f\left(\frac{k}{n}\right)}{I^{*}\cdot n^{\alpha}}\cdot S\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right). \tag{3.54}$$

Here we study the fractional convergence with rates of $\left(G_{n}f\right)\left(x\right)\rightarrow f\left(x\right)$, as $n\rightarrow+\infty$, $x\in\mathbb{R}$.

Notice that

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}1\leq\left(2Tn^{\alpha}+1\right). \tag{3.55}$$

From [1] we need

Lemma 3.12. *It holds that*

$$D_{n}\left(x\right):=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{I^{*}\cdot n^{\alpha}}\cdot S\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right)\rightarrow1, \tag{3.56}$$

pointwise, as $n\rightarrow+\infty$, where $x\in\mathbb{R}$.

We present our second main result

Theorem 3.13. *We consider $f:\mathbb{R}\rightarrow\mathbb{R}$. Let $\beta>0$, $N=\lceil\beta\rceil$, $\beta\notin\mathbb{N}$, $f\in AC^{N}\left([a,b]\right)$, $\forall\ [a,b]\subset\mathbb{R}$, with $f^{(N)}\in L_{\infty}\left(\mathbb{R}\right)$. Let also $x\in\mathbb{R},T>0,n\in\mathbb{N}:n\geq\max\left(T+\left|x\right|,T^{-\frac{1}{\alpha}}\right)$. We further assume that $D_{*x}^{\beta}f$, $D_{x-}^{\beta}f$ are uniformly continuous functions or continuous and bounded on $[x,+\infty)$, $(-\infty,x]$, respectively.*

Then

1)

$$\left|G_{n}\left(f\right)\left(x\right)-f\left(x\right)\right|\leq\left|f\left(x\right)\right|\cdot \tag{3.57}$$

$$\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{I^{*}n^{\alpha}}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|+$$

$$\frac{S\left(T\right)}{I^{*}}\left(2T+\frac{1}{n^{\alpha}}\right)\left(\sum_{j=1}^{N-1}\frac{\left|f^{(j)}\left(x\right)\right|T^{j}}{j!n^{(1-\alpha)j}}\right)$$

$$+\frac{S(T)}{I^*}\left(2T+\frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\omega_1\left(D_{*x}^\beta f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D_{x-}^\beta f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

above $\sum_{j=1}^{0}\cdot=0,$

2)

$$\left|(G_n(f))(x)-\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(G_n\left((\cdot-x)^j\right)\right)(x)\right|\leq \tag{3.58}$$

$$\frac{S(T)}{I^*}\left(2T+\frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\omega_1\left(D_{*x}^\beta f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D_{x-}^\beta f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}=:\lambda_n^*(x),$$

3) assume further that $f^{(j)}(x)=0$*, for* $j=0,1,...,N-1$*, we get*

$$|G_n(f)(x)|\leq\lambda_n^*(x), \tag{3.59}$$

4) in case of $N=1$*, we obtain*

$$|G_n(f)(x)-f(x)|\leq|f(x)|\cdot \tag{3.60}$$

$$\left|\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{1}{I^*n^\alpha}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|+$$

$$\frac{S(T)}{I^*}\left(2T+\frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\omega_1\left(D_{*x}^\beta f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D_{x-}^\beta f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

Here we get fractionally with rates the pointwise convergence of $(G_n(f))(x)\to f(x)$*, as* $n\to\infty$*,* $x\in\mathbb{R}$*.*

Proof. Let $x\in\mathbb{R}$. We have that

$$D_{x-}^\beta f(x)=D_{*x}^\beta f(x)=0.$$

From [8], p. 54, we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^j+ \tag{3.61}$$

$$\frac{1}{\Gamma(\beta)}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^\beta f(J)-D_{*x}^\beta f(x)\right)dJ,$$

for all $x\leq\frac{k}{n}\leq x+Tn^{\alpha-1}$, iff $\lceil nx\rceil\leq k\leq[nx+Tn^\alpha]$, where $k\in\mathbb{Z}$.

Also from [2], using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}+ \tag{3.62}$$

$$\frac{1}{\Gamma(\beta)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ,$$

for all $x-Tn^{\alpha-1}\leq\frac{k}{n}\leq x$, iff $\lceil nx-Tn^{\alpha}\rceil\leq k\leq[nx]$, where $k\in\mathbb{Z}$.

Hence we have

$$\frac{f\left(\frac{k}{n}\right)S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}+ \tag{3.63}$$

$$\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}\Gamma(\beta)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ,$$

and

$$\frac{f\left(\frac{k}{n}\right)S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}+ \tag{3.64}$$

$$\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}\Gamma(\beta)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ.$$

Therefore we obtain

$$\frac{\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}f\left(\frac{k}{n}\right)S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}= \tag{3.65}$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\left(\frac{k}{n}-x\right)^{j}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}\right)+$$

$$\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}\Gamma(\beta)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ,$$

and

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}f\left(\frac{k}{n}\right)S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}= \tag{3.66}$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}\left(\frac{k}{n}-x\right)^{j}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}+$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}\Gamma(\beta)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ.$$

Adding the two equalities (3.65) and (3.66) we obtain

$$(G_{n}(f))(x)=$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\frac{k}{n}-x\right)^j S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}} \right) + M_n(x), \tag{3.67}$$

where

$$M_n(x) := \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha} \Gamma(\beta)} \cdot$$

$$\int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ +$$

$$\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha} \Gamma(\beta)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ. \tag{3.68}$$

We call

$$M_{1n}(x) := \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} \frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha} \Gamma(\beta)} \cdot$$

$$\int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ, \tag{3.69}$$

and

$$M_{2n}(x) := \sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha} \Gamma(\beta)} \cdot$$

$$\int_{x}^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ. \tag{3.70}$$

I.e.

$$M_n(x) = M_{1n}(x) + M_{2n}(x). \tag{3.71}$$

We further have

$$(G_n(f))(x) - f(x) = f(x) \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}} - 1 \right) + \tag{3.72}$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\frac{k}{n}-x\right)^j S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}} \right) + M_n(x),$$

and

$$|(G_n(f))(x) - f(x)| \le |f(x)| \left| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^* n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1 \right| + \tag{3.73}$$

$$\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!} \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left|x-\frac{k}{n}\right|^j S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}} \right) + |M_n(x)| \le$$

$$|f(x)|\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{I^{*}n^{\alpha}}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|+$$

$$\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|}{j!}\frac{T^{j}}{n^{(1-\alpha)j}}\left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}\right)+|M_{n}(x)|=:(*). \quad (3.74)$$

Therefore we obtain

$$|(G_{n}(f))(x)-f(x)|\leq|f(x)|\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{I^{*}n^{\alpha}}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|+$$

$$\frac{S(T)}{I^{*}}\left(2T+\frac{1}{n^{\alpha}}\right)\left(\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|T^{j}}{j!n^{(1-\alpha)j}}\right)+|M_{n}(x)|. \quad (3.75)$$

We call

$$\gamma_{1n}:=\frac{1}{\Gamma(\beta)}\left|\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ\right|. \quad (3.76)$$

As in the proof of Theorem 3.7 we have

$$\gamma_{1n}\leq\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\omega_{1}\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}. \quad (3.77)$$

Furthermore

$$|M_{1n}(x)|\leq\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}\gamma_{1n}\leq \quad (3.78)$$

$$\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\omega_{1}\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\leq$$

$$\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\omega_{1}\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\leq$$

$$\frac{S(T)}{I^{*}}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\omega_{1}\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}.$$

So that

$$|M_{1n}(x)|\leq\frac{S(T)}{I^{*}}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\omega_{1}\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}. \quad (3.79)$$

We also call

$$\gamma_{2n}:=\frac{1}{\Gamma(\beta)}\left|\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ\right|. \quad (3.80)$$

As in the proof of Theorem 3.7 we get

$$\gamma_{2n} \leq \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)}. \tag{3.81}$$

Consequently we find

$$|M_{2n}(x)| \leq \left(\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}} \right) \cdot$$

$$\frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} \leq \tag{3.82}$$

$$\frac{S(T)}{I^*} \left(2T + \frac{1}{n^{\alpha}} \right) \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)}.$$

So we have proved that

$$|M_n(x)| \leq \frac{S(T)}{I^*} \left(2T + \frac{1}{n^{\alpha}} \right) \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot \tag{3.83}$$

$$\left\{ \omega_1 \left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} + \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\}.$$

Combining (3.75) and (3.83) we have (3.57). □

As an application of Theorem 3.13 we give

Theorem 3.14. *Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in C^N(\mathbb{R})$, with $f^{(N)} \in L_{\infty}(\mathbb{R})$. Let also $T > 0$, $n \in \mathbb{N} : n \geq \max\left(2T, T^{-\frac{1}{\alpha}}\right)$. We further assume that $D_{*x}^{\beta} f(t)$, $D_{x-}^{\beta} f(t)$ are both bounded in $(x,t) \in \mathbb{R}^2$. Then*

1)

$$\|G_n(f) - f\|_{\infty,[-T,T]} \leq \|f\|_{\infty,[-T,T]} \cdot \tag{3.84}$$

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^* n^{\alpha}} S\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right) - 1 \right\|_{\infty,[-T,T]} +$$

$$\frac{S(T)}{I^*} \left(2T + \frac{1}{n^{\alpha}} \right) \left(\sum_{j=1}^{N-1} \frac{\left\| f^{(j)} \right\|_{\infty,[-T,T]} T^j}{j! n^{(1-\alpha)j}} \right) +$$

$$\frac{S(T)}{I^*} \left(2T + \frac{1}{n^{\alpha}} \right) \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot$$

$$\left\{ \sup_{x\in[-T,T]} \omega_1 \left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} + \sup_{x\in[-T,T]} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\},$$

2) in case of $N = 1$, we obtain

$$\|G_n(f) - f\|_{\infty,[-T,T]} \leq \|f\|_{\infty,[-T,T]} \cdot \tag{3.85}$$

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^{*}n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{\infty,[-T,T]} + \frac{S(T)}{I^{*}}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}} \cdot \left\{ \sup_{x\in[-T,T]} \omega_{1}\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)} + \sup_{x\in[-T,T]} \omega_{1}\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]} \right\}.$$

An interesting case is when $\beta=\frac{1}{2}$.

Assuming further that $\left\|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^{*}n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{\infty,[-T,T]} \to 0$, *as* $n\to\infty$, *we get fractionally with rates the uniform convergence of* $G_n(f)\to f$, *as* $n\to\infty$.

Proof. From (3.57), (3.60) of Theorem 3.13, and by Remark 2.17.

Also by

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^{*}n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) \leq \frac{S(T)}{I^{*}}(2T+1), \tag{3.86}$$

we get that

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^{*}n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{\infty,[-T,T]} \leq \left(\frac{S(T)}{I^{*}}(2T+1)+1\right). \tag{3.87}$$

□

One can also apply Remark 2.18 to the last Theorem 3.14, to get interesting and simplified results.

Note 3.15. *The maps* F_n, G_n, $n\in\mathbb{N}$, *are positive linear operators.*

We finish with

Remark 3.16. The condition of Theorem 3.8 that

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{\infty,[-T,T]} \to 0, \tag{3.88}$$

as $n\to\infty$, is not uncommon.

We give an example related to that.

We take as $b(x)$ the characteristic function over $[-1,1]$, that is $\chi_{[-1,1]}(x)$. Here $T=1$ and $I=2$, $n\geq 2$, $x\in[-1,1]$.

We get that

$$\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]} \frac{1}{2n^{\alpha}} \chi_{[-1,1]}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) \overset{(3.2)}{=} \sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]} \frac{1}{2n^{\alpha}} =$$

$$\frac{1}{2n^{\alpha}}\left(\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]}1\right)=\frac{([nx+n^{\alpha}]-\lceil nx-n^{\alpha}\rceil+1)}{2n^{\alpha}}. \tag{3.89}$$

But we have

$$[nx+n^{\alpha}]-\lceil nx-n^{\alpha}\rceil+1\leq 2n^{\alpha}+1,$$

hence

$$\frac{([nx+n^{\alpha}]-\lceil nx-n^{\alpha}\rceil+1)}{2n^{\alpha}}\leq 1+\frac{1}{2n^{\alpha}}. \tag{3.90}$$

Also it holds

$$[nx+n^{\alpha}]-\lceil nx-n^{\alpha}\rceil+1\geq 2n^{\alpha}-2+1=2n^{\alpha}-1,$$

and

$$\frac{([nx+n^{\alpha}]-\lceil nx-n^{\alpha}\rceil+1)}{2n^{\alpha}}\geq 1-\frac{1}{2n^{\alpha}}. \tag{3.91}$$

Consequently we derive that

$$-\frac{1}{2n^{\alpha}}\leq\left(\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]}\frac{1}{2n^{\alpha}}\chi_{[-1,1]}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right)\leq\frac{1}{2n^{\alpha}}, \tag{3.92}$$

for any $x\in[-1,1]$ and for any $n\geq 2$.

Hence we get

$$\left\|\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]}\frac{1}{2n^{\alpha}}\chi_{[-1,1]}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{\infty,[-1,1]}\to 0,\text{ as } n\to\infty. \tag{3.93}$$

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Looking for an exact difference formula for the Dini-Hadamard-like subdifferential

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Abstract. We use in this paper a new concept of a directional subdifferential, namely the *Dini-Hadamard-like ε-subdifferential*, recently introduced in [29], in order to provide a subdifferential formula for the difference of two directionally approximately starshaped functions (a valuable class of nonsmooth functions, see for instance [32]), under weaker conditions than those presented in [7]. As a consequence, we furnish necessary and sufficient optimality conditions for a nonsmooth optimization problem having the difference of two functions as objective.

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1. Introduction

Since the early 1960's there has been a good amount of interest in generalizations of the pointwise derivative for the purposes of optimization. This has lead to many definitions of *generalized gradients*, *subgradients* and other kind of objects under various names. And all this work in order to solve optimization problems where classical differentiability assumptions are no longer appropriate. One of the most widely used *subdifferential* (set of subgradients) is the one who first appeared for convex functions in the context of convex analysis (see for more details [28, 38, 39] and the references therein). It has found many significant theoretical and practical uses in optimization, economics, mechanics and has proven to be a very interesting mathematical construct. But the attempt to extend this success to functions which are no more convex has proven to be more difficult. We mention here two main approaches.

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The first one uses a *generalized directional derivative* f^∂ of $f : X \to \mathbb{R} \cup \{+\infty\}$ of some type and then defines the subdifferential via the formula

$$\partial f(x) := \{x^* \in X^* : x^* \leq f^\partial(x, \cdot)\}, \tag{1.1}$$

where X^* is the topological dual of X. It is worth mentioning here that any subdifferential construction generated by a *polarity* relation like (1.1) is automatically convex regardless of the convexity of the generating directional derivative. As an example, the *Clarke subdifferential*, who in fact uses a positively homogeneous directional derivative, was the first concept of a subdifferential defined for a general nonconvex function and has been introduced in 1973 by Clarke (see for instance [9, 10]) who performed a real pioneering work in the field of *nonsmooth analysis*, spread far beyond the scope of convexity. But unfortunately, as stated in [4], at some *abnormal points* of certain even Lipschitzian nonsmooth functions, the Clarke subdifferential may include some *extraneous subgradients.* And this because, in general, a convex set often provides a subdifferential that is too large for a lot of optimization problems.

The second approach to define general subdifferentials satisfying useful calculus rules is to take limits of some primitive subdifferential constructions which do not possess such calculus. It is important that limiting constructions depend not only on the choice of the primitive objects but also on the character of the limit: *topological* or *sequential*.

The topological way allows one to develop useful subdifferentials in general infinite dimensional settings, but the biggest drawback is the fact that it may lead to broad constructions and in general they have an intrinsically complicated structure, usually following a three-step procedure. Namely, the definition of ∂f for a Lipschitz function which requires considering restrictions to finite-dimensional (or separable) subspaces with intersections over the collection of all such subspaces, then the definition of a normal cone of a set C at a given point x as the cone generated by the subdifferential of the distance function to C and finally the definition of ∂f for an arbitrary lower semicontinuous function by means of the normal cone to the epigraph of f. In this line of development, many infinite dimensional extensions of the nonconvex constructions in [23, 24] were introduced and strongly developed by Ioffe in a series of many publications starting from 1981 (see [17, 18, 19] for the bibliographies and commentaries therein) on the basis of topological limits of Dini-Hadamard ε-subdifferentials. Such constructions, called also *approximate subdifferentials*, are well defined in more general spaces, but all of them (including also their nuclei) may be broader than the Kruger-Mordukhovich extension, even for Lipshictz functions on Banach spaces with Fréchet differentiable renorms.

The sequential way usually leads to more convenient objects, but it requires some special geometric properties of spaces in question (see for instance [5]). Thus, because the convexity is no longer inherent in the procedure, we are able to define smaller subdifferentials and also to exclude some points from the set of stationary points. The sequential nonconvex subdifferential constructions in Banach spaces were first introduced by Kruger and Mordukhovich [20, 21] on the basis of sequential limits of Fréchet ε-normals and subdifferentials. Such limiting normal cone and subdifferential appeared as infinite dimensional extensions of the corresponding finite dimensional constructions in Mordukhovich [23, 24], motivated by applications to optimization and control. Useful properties of those and related constructions were revealed mainly for Banach spaces with Fréchet differentiable renorms. Let us also emphasize that while the subdifferential theory in finite dimensions has been well developed, there still exist many open questions in infinite dimensional spaces.

While the Fréchet epsilon-subdifferential is as a building block for the Mordukhovich subdifferential in Banach spaces, the Dini-Hadamard one lies at the heart of the so called A-subdifferential introduced by Ioffe. Generated with the help of the *lower Dini* (or *Dini-Hadamard*) *directional derivative*, one of the most attractive construction appeared in the 1970's, the Dini-Hadamard subdifferential and its epsilon enlargement are well known in variational analysis and generalized differentiation but they are not widely used due to the lack of calculus. However, as it has been recently observed in [7], an exact difference formula holds for such subdifferentials under natural assumptions (see also [35]). Moreover, necessary and sufficient optimality conditions for cone-constrained optimization problems having a difference of two functions as objective are established, in case the difference function is *calm* and some additional conditions are fulfilled. Our main goal in this paper is to provide the same formula as mentioned above, but without any calmness assumption. To this end we employ the *Dini-Hadamard-like ε-subdifferential* [29], which is defined by the instrumentality of a different kind of *lower limit.* Our analysis relies also on the notion of *spongiously pseuso-dissipativity* of set-valued mappings and involves the notion of a *spongiously local blunt minimizer.*

The reminder of the paper is organized as follows. After introducing in Section 2 some preliminary notions and results especially related to the Dini-Hadamard-like subdifferential, we study in Section 3 some generalized convexity notions in order to provide in the final part of the paper some necessary and sufficient conditions for a point to be a spongiously local blunt minimizer. Finally, we employ the achieved results to the formulation of optimality conditions for a nonsmooth optimization problem having the difference of two functions as objective.

2. Preliminary notions and results

Consider a Banach space X and its topological dual space X^*. We denote the *open ball* with center $\overline{x} \in X$ and radius $\delta > 0$ in X by $B(\overline{x}, \delta)$, while $\overline{B}_X$ and S_X stand for the *closed unit ball* and the *unit sphere* of X, respectively. Having a set $C \subseteq X$, $\delta_C : X \to \mathbb{R} \cup \{+\infty\}$, defined by $\delta_C(x) = 0$ for $x \in C$ and $\delta_C(x) = +\infty$, otherwise, denotes its *indicator function.*

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function. As usual, we denote by $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ the *effective domain* of f and by $\operatorname{epi} f = \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}$ the *epigraph* of f. Dealing with functions that may take infinite values, we adopt the following natural conventions $(+\infty) - (+\infty) = +\infty$ and $0(+\infty) = +\infty$.

For $\varepsilon \geq 0$ the *Fréchet ε-subdifferential* (or the *analytic ε-subdifferential*) of f at $\overline{x} \in \operatorname{dom} f$ is defined by

$$\partial_\varepsilon^F f(\overline{x}) := \left\{ x^* \in X^* : \liminf_{\|h\| \to 0} \frac{f(\overline{x}+h) - f(\overline{x}) - \langle x^*, h\rangle}{\|h\|} \geq -\varepsilon \right\},$$

which means that one has

$$\overline{x}^* \in \partial_\varepsilon^F f(\overline{x}) \Leftrightarrow \quad \begin{array}{l} \text{for all } \alpha > 0 \text{ there exists } \delta > 0 \text{ such that for all } x \in B(\overline{x}, \delta) \\ f(x) - f(\overline{x}) \geq \langle \overline{x}^*, x - \overline{x}\rangle - (\alpha + \varepsilon)\|x - \overline{x}\|. \end{array} \tag{2.1}$$

The following constructions

$$d^- f(\overline{x}; h) := \liminf_{\substack{u \to h \\ t \downarrow 0}} \frac{f(\overline{x}+tu) - f(\overline{x})}{t} = \sup_{\delta > 0} \inf_{\substack{u \in B(h,\delta) \\ t \in (0,\delta)}} \frac{f(\overline{x}+tu) - f(\overline{x})}{t}$$

and (see [17, 18])

$$\partial_\varepsilon^- f(\overline{x}) := \{x^* \in X^* : \langle x^*, h\rangle \leq d^- f(\overline{x}; h) + \varepsilon\|h\| \text{ for all } h \in X\}, \text{ where } \varepsilon \geq 0,$$

are called the *Dini-Hadamard directional derivative* of f at $\overline{x}$ in the direction $h \in X$ and the *Dini-Hadamard ε-subdifferential* of f at $\overline{x}$, respectively.

Similarly, following the two steps procedure of constructing the Dini-Hadamard ε-subdifferential we can define (see [29])

$$D_d^S f(\overline{x}; h) := \sup_{\delta > 0} \inf_{\substack{u \in B(h,\delta) \cap (h + [0,\delta] \cdot B(d,\delta)) \\ t \in (0,\delta)}} \frac{f(\overline{x}+tu) - f(\overline{x})}{t},$$

the *Dini-Hadamard-like directional derivative* of f at $\overline{x}$ in the direction $h \in X$ through $d \in X \setminus \{0\}$ and also, for a given $\varepsilon \geq 0$, the *Dini-Hadamard-like ε-subdifferential* of f at $\overline{x}$

$$\partial_\varepsilon^S f(\overline{x}) := \{x^* \in X^* : \langle x^*, h\rangle \leq D_d^S f(\overline{x}; h) + \varepsilon\|h\| \text{ for all } h \in X \text{ and all } d \in X \setminus \{0\}\}.$$

In case $\varepsilon = 0$, $\partial^- f(\overline{x}) := \partial_0^- f(\overline{x})$ is nothing else than the *Dini-Hadamard subdifferential* of f at $\overline{x}$, while $\partial^S f(\overline{x}) := \partial_0^S f(\overline{x})$ simply denotes the *Dini-Hadamard-like subdifferential* of f at $\overline{x}$. When $\overline{x} \notin \operatorname{dom} f$ we set $\partial_\varepsilon^F f(\overline{x}) = \partial_\varepsilon^- f(\overline{x}) = \partial_\varepsilon^S f(\overline{x}) := \emptyset$ for all $\varepsilon \geq 0$. It is worth emphasizing here that for $\overline{x} \in \operatorname{dom} f$ the following functions $d^- f(\overline{x}; \cdot)$ and $D_d^S f(\overline{x}; \cdot)$ are in general not convex, while $\partial_\varepsilon^- f(\overline{x})$ and $\partial_\varepsilon^S f(\overline{x})$ are always convex sets. Moreover, we notice that $d^- f(\overline{x}; 0)$ is either 0 or $-\infty$ (see [16]).

The function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *calm* at $\overline{x} \in \operatorname{dom} f$ if there exists $c \geq 0$ and $\delta > 0$ such that $f(x) - f(\overline{x}) \geq -c\|x - \overline{x}\|$ for all $x \in B(\overline{x}, \delta)$. As a characterization, for $\overline{x} \in \operatorname{dom} f$ one has (see, for instance, [14]) that f is calm at $\overline{x}$ if and only if $d^- f(\overline{x}; 0) = 0$.

Further, for any $\varepsilon \geq 0$

$$\partial_\varepsilon^F f(\overline{x}) \subseteq \partial_\varepsilon^- f(\overline{x}) \subseteq \partial_\varepsilon^S f(\overline{x}).$$

It is interesting to observe that both inclusions can be even strict (see Example 2.9 belllow and [7] for further remarks and links between the Dini-Hadamard subdifferential and the Fréchet one).

The essential idea behind defining the Dini-Hadamard-like constructions is to employ a directional convergence in place of a usual one. To this aim we say that a sequence (x_n) of X *converges to* $\overline{x}$ *in the direction* $d \in X \setminus \{0\}$ (and we write $(x_n) \underset{d}{\longrightarrow} \overline{x}$) if there exist sequences $(t_n) \to 0$, $t_n \geq 0$ and $(d_n) \to d$ such that $x_n = \overline{x} + t_n d_n$ for each $n \in \mathbb{N}$. Further, a sequence (x_n) is said to *converge directionally* to $\overline{x}$ if there exists $d \in X \setminus \{0\}$ such that $(x_n) \underset{d}{\longrightarrow} \overline{x}$. Our definition, slightly different from the one proposed by Penot in [33], allows us to consider also the constants sequences among the ones which are directionally convergent. Motivated by this observation, we call the *directional lower limit* of f at $\overline{x}$ in the direction $d \in X \setminus \{0\}$ the following limit

$$\liminf_{x \underset{d}{\longrightarrow} \overline{x}} f(x) := \sup_{\delta > 0} \inf_{x \in B(\overline{x},\delta) \cap (\overline{x} + [0,\delta] \cdot B(d,\delta))} f(x).$$

Consequently, since

$$\begin{aligned}\liminf_{\substack{u \underset{d}{\longrightarrow} h \\ t \underset{1}{\longrightarrow} 0}} \frac{f(\overline{x}+tu) - f(\overline{x})}{t} &:= \sup_{\substack{\delta > 0 \\ \delta' > 0}} \inf_{\substack{u \in B(h,\delta) \cap (h + [0,\delta] \cdot B(d,\delta)) \\ t \in (0,\delta') \cap [0,\delta'] \cdot (1-\delta', 1+\delta')}} \frac{f(\overline{x}+tu) - f(\overline{x})}{t} \\ &= \sup_{\delta > 0} \inf_{\substack{u \in B(h,\delta) \cap (h + [0,\delta] \cdot B(d,\delta)) \\ t \in (0,\delta)}} \frac{f(\overline{x}+tu) - f(\overline{x})}{t},\end{aligned}$$

we may (formally) write

$$D_d^S f(\overline{x}; h) = \liminf_{\substack{u \underset{d}{\longrightarrow} h \\ t \underset{1}{\longrightarrow} 0}} \frac{f(\overline{x}+tu) - f(\overline{x})}{t} = \liminf_{\substack{u \underset{d}{\longrightarrow} h \\ t \downarrow 0}} \frac{f(\overline{x}+tu) - f(\overline{x})}{t}.$$

Similarly one can define the *directional upper limit* of f at $\overline{x}$ in the direction $d \in X \setminus \{0\}$, since the lower properties symmetrically induce the corresponding upper ones

$$\limsup_{x \underset{d}{\longrightarrow} \overline{x}} f(x) := -\liminf_{x \underset{d}{\longrightarrow} \overline{x}} (-f)(x) = \inf_{\delta > 0} \sup_{x \in B(\overline{x},\delta) \cap (\overline{x} + [0,\delta] \cdot B(d,\delta))} f(x).$$

Moreover, one can easily observe that

$$\liminf_{x \to \overline{x}} f(x) \leq \liminf_{x \underset{d}{\longrightarrow} \overline{x}} f(x) \leq \limsup_{x \underset{d}{\longrightarrow} \overline{x}} f(x) \leq \limsup_{x \to \overline{x}} f(x) \text{ for all } d \in X \setminus \{0\}. \quad (2.2)$$

The next subdifferential notion we need to recall is the one of *G-subdifferential* and we describe in the following the procedure of constructing it (see [19]). To this aim we consider first the *A-subdifferential* of f at $\overline{x} \in \operatorname{dom} f$, which is defined via topological limits as follows

$$\partial^A f(\overline{x}) := \bigcap_{L \in \mathcal{F}(X)} \overline{\operatorname{Limsup}}_{\substack{x \xrightarrow{f} \overline{x} \\ \varepsilon > 0}} \partial_\varepsilon^- (f + \delta_{x+L})(x),$$

where $\mathcal{F}(X)$ denotes the collection of all finite dimensional subspaces of X and $\overline{\operatorname{Limsup}}$ stands for the *topological counterpart* of the *sequential Painlevé-Kuratowski upper/outer limit* of a set-valued mapping with sequences replaced by nets and where

$x \xrightarrow{f} \overline{x}$ means $x \longrightarrow \overline{x}$ and $f(x) \longrightarrow f(\overline{x})$. More precisely, for a set-valued mapping $F : X \rightrightarrows X^*$, we say that $x^* \in \overline{\mathrm{Limsup}}_{x \to \overline{x}}\, F(x)$ if for each weak*-neighborhood $\mathcal{U}$ of the origin of X^* and for each $\delta > 0$ there exists $x \in B(\overline{x}, \delta)$ such that $(x^* + \mathcal{U}) \cap F(x) \neq \emptyset$.

The *G-normal cone* to a set $C \subseteq X$ at $\overline{x} \in C$ is defined as

$$N^G(C, \overline{x}) := \mathrm{cl}^* \left(\bigcup_{\lambda > 0} \lambda \partial^A d(\overline{x}, C) \right),$$

where $d(\overline{x}, C) := \inf_{c \in C} \|\overline{x} - c\|$ denotes the *distance* from $\overline{x}$ to C and cl^* stands for the *weak***-closure* of a set in X^*, while the *G-subdifferential* of f at $\overline{x} \in \mathrm{dom}\, f$ can be defined now as follows

$$\partial^G f(\overline{x}) := \left\{ x^* \in X^* : \ (x^*, -1) \in N^G(\mathrm{epi}\, f, (\overline{x}, f(\overline{x}))) \right\}.$$

When $\overline{x} \notin \mathrm{dom}\, f$ we set $\partial^A f(\overline{x}) = \partial^G f(\overline{x}) := \emptyset$. Thus, by taking into account [19, Proposition 4.2] one has the inclusion

$$\partial^F f(x) \subseteq \partial^- f(x) \subseteq \partial^G f(x) \ \text{ for all } x \in X. \tag{2.3}$$

One can notice that when f is a convex function it holds $\partial^F f(x) = \partial^- f(x) = \partial^G f(x) = \partial f(x)$ for all $x \in X$, where $\partial f(\overline{x}) := \{x^* \in X^* : f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x}\rangle \ \forall x \in X\}$, for $\overline{x} \in \mathrm{dom}\, f$, and $\partial f(\overline{x}) := \emptyset$, otherwise, denotes the subdifferential of f at $\overline{x}$ in the sense of convex analysis.

It is also worth mentioning that both G- and A-subdifferentials reduce to the basic/limiting/Mordukhovich one whenever X is a finite dimensional space or X is an *Asplund weakly compactly generated* (WCG) space and f is locally Lipschitz at the point in discussion (see [27] and [25, Subsection 3.2.3]).

In what follows, in order to study the behavior of the Dini-Hadamard-like subdifferential we especially need the following result.

Lemma 2.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function and $\overline{x}, h \in X$. Then the following statements are true:*

(i) $D_d^S f(\overline{x}; h) \leq \liminf_{n \to +\infty} \frac{f(\overline{x} + t_n u_n) - f(\overline{x})}{t_n}$, whenever $(u_n) \xrightarrow[d]{} h$ and $(t_n \downarrow 0)$, with $d \in X \setminus \{0\}$.

(ii) If for some $d \in X \setminus \{0\}$, $D_d^S f(\overline{x}; h) = l \in \mathbb{R} \cup \{-\infty\}$, then there exist sequences $(u_n) \xrightarrow[d]{} h$ and $(t_n \downarrow 0)$ such that $\lim_{n \to +\infty} \frac{f(\overline{x} + t_n u_n) - f(\overline{x})}{t_n} = l$.

Proof. To justify (i), since

$$\liminf_{n \to +\infty} \frac{f(\overline{x} + t_n u_n) - f(\overline{x})}{t_n} := \sup_{n \geq 1} \inf_{k \geq n} \frac{f(\overline{x} + t_k u_k) - f(\overline{x})}{t_k},$$

we only have to show that

$$\sup_{\delta > 0} \inf_{\substack{u \in B(h,\delta) \cap (h + [0,\delta] \cdot B(d,\delta)) \\ t \in (0,\delta)}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} \leq \sup_{n \geq 1} \inf_{k \geq n} \frac{f(\overline{x} + t_k u_k) - f(\overline{x})}{t_k}.$$

Let $\delta > 0$ be fixed. Since $(u_n) \underset{d}{\longrightarrow} h$, there exist sequences $(t_n^{'}) \to 0$, $t_n^{'} \geq 0$ and $(d_n) \to d$ such that $u_n = h + t_n^{'} \cdot d_n$ for all $n \in \mathbb{N}$ and thus there exists $k_0 \in \mathbb{N}$ with the property that for each natural number $k \geq k_0$, $u_k \in B(h,\delta) \cap (h + [0,\delta] \cdot B(d,\delta))$ and $t_k \in (0,\delta)$. Hence

$$D_d^S f(\overline{x}; h) \leq \inf_{k \geq k_0} \frac{f(\overline{x} + t_k u_k) - f(\overline{x})}{t_k} \leq \sup_{n \geq 1} \inf_{k \geq n} \frac{f(\overline{x} + t_k u_k) - f(\overline{x})}{t_k}.$$

Taking now the supremum as $\delta > 0$, we obtain the desired conclusion.

(ii) First we study the case $l \in \mathbb{R}$. Using the definition of the directional lower limit, it follows that for any $n \in \mathbb{N}^*$

$$\inf_{\substack{u \in B(h,\delta) \cap (h + [0,\delta] \cdot B(d,\delta)) \\ t \in (0,\delta)}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} \leq l < l + \frac{1}{n}$$

and consequently, there exists $u_n \in B(h, \frac{1}{n}) \cap (h + [0, \frac{1}{n}] \cdot B(d, \frac{1}{n}))$ and $t_n \in (0, \frac{1}{n})$ with $\frac{f(\overline{x}+t_n u_n) - f(\overline{x})}{t_n} < l + \frac{1}{n}$. Further, for each $n \in \mathbb{N}$ we find $t_n^{'} \in [0, \frac{1}{n}]$ and $d_n \in B(d, \frac{1}{n})$ (with $t_0^{'} := 0$ and $d_0 := d$) such that $u_n = h + t_n^{'} \cdot d_n$, which means nothing else that $(u_n) \underset{d}{\longrightarrow} h$. Moreover, since $\lim\limits_{n \to +\infty} \frac{f(\overline{x}+t_n u_n) - f(\overline{x})}{t_n} \leq l$ and due to assertion (i) we get

$$l = D_d^S f(\overline{x}; h) \leq \liminf_{n \to +\infty} \frac{f(\overline{x} + t_n u_n) - f(\overline{x})}{t_n} \leq \lim_{n \to +\infty} \frac{f(\overline{x} + t_n u_n) - f(\overline{x})}{t_n} \leq l.$$

The special case $l = -\infty$ yields for any $n \in \mathbb{N}^*$, $u_n \in B(h, \frac{1}{n}) \cap (h + [0, \frac{1}{n}] \cdot B(d, \frac{1}{n}))$ and $t_n \in (0, \frac{1}{n})$ with the property that $\frac{f(\overline{x}+t_n u_n) - f(\overline{x})}{t_n} < -n$. Thus, we obtain two sequences $(u_n) \underset{d}{\longrightarrow} h$ and $t_n \downarrow 0$ so that $\lim\limits_{n \to +\infty} \frac{f(\overline{x}+t_n u_n) - f(\overline{x})}{t_n} = -\infty$ and finally, the proof of the lemma is complete. □

Remark 2.2. In fact, this result is particularly helpful to conclude that the Dini-Hadamard subdifferential coincide with the Dini-Hadamard-like one in finite dimensions. Indeed, since $d^- f(\overline{x}; h) \leq D_d^S f(\overline{x}; h)$ for all $d \in X \setminus \{0\}$ and consequently $\partial^- f(\overline{x}) \subseteq \partial^S f(\overline{x})$, we only have to prove that the opposite inclusion holds too. To this end, consider $x^* \in \partial^S f(\overline{x})$, $h \in X$ and let us denote for convenience $d^- f(\overline{x}; h) := l$, $l \in \overline{\mathbb{R}}$. Then, in view of Lemma 2.1 above, there exist sequences $(u_n) \to h$ and $(t_n) \downarrow 0$ such that $\lim\limits_{n \to +\infty} \frac{f(\overline{x}+t_n u_n) - f(\overline{x})}{t_n} = l$. Now, due to the finiteness assumption made, we can find $u^{'} \in S_X$ and a subsequence (u_{n_k}) such that

$$(u_{n_k}) \underset{u^{'}}{\longrightarrow} h \quad \text{and} \quad \lim_{k \to +\infty} \frac{f(\overline{x} + t_{n_k} \cdot u_{n_k}) - f(\overline{x})}{t_{n_k}} = l. \tag{2.4}$$

To justify this claim, suppose first that (u_n) has an infinite number of terms not equal to h. Then we can choose a subsequence (u_{n_k}) of (u_n), $u_{n_k} \neq h$ for all $k \in \mathbb{N}$ and we may write $u_{n_k} = h + \|u_{n_k} - h\| \cdot d_{n_k}$ with $d_{n_k} = \frac{u_{n_k} - h}{\|u_{n_k} - h\|}$. Further, since (d_{n_k}) is bounded, there exist $u^{'} \in S_X$ and $(d_{n_{k_l}})$ so that $(d_{n_{k_l}}) \to u^{'}$, and hence $(u_{n_{k_l}}) \underset{u^{'}}{\longrightarrow} h$. If on the contrary u_n has an infinite number of terms equal to h, then we choose a subsequence (u_{n_k}) of (u_n) such that $u_{n_k} = h$ for all $k \in \mathbb{N}$. In this particular case $(u_{n_k}) \underset{u'}{\longrightarrow} h$ for every $u' \in S_X$.

Consequently, relation (2.4) above holds true and hence $l \geq D^S_{u'} f(\overline{x}; h)$, which in turn implies $\langle x^*, h\rangle \leq d^- f(\overline{x}; h)$ and finally $x^* \in \partial^- f(\overline{x})$.

As it was first observed by Penot [33], the concept of a directionally convergent sequence is clearly related to the following notion introduced by Treiman [40].

Definition 2.3. A set $S \subseteq X$ is said to be a *sponge around* $\overline{x} \in X$ if for all $h \in X \setminus \{0\}$ there exist $\lambda > 0$ and $\delta > 0$ such that $\overline{x} + [0, \lambda] \cdot B(h, \delta) \subseteq S$.

Furthermore, the sponges enjoy a nice relationship with the so-called cone-porous sets (see [13, 41] for definition and further remarks). Indeed, accordingly to [11], if S is a sponge around $\overline{x}$ then the complementary set $(X \setminus S) \cup \{\overline{x}\}$ is cone porous in any direction $v \in S_X$. Let us recall also that every neighborhood of a point $\overline{x} \in X$ is also a sponge around $\overline{x}$ and that the converse is not true (see for instance [7, Example 2.2]). However, in case S is a convex set or X is a finite dimensional space (here one can make use of the fact that the unit sphere is compact), then S is also a neighborhood of $\overline{x}$.

Remark 2.4. Trying to answer the question how further can we go with the replacement of a neighborhood by a sponge, it is worth emphasizing that every sponge S around a point $\overline{x} \in X$ has the property (A) and moreover it verifies also (B).

(A) : for all $h \in X \setminus \{0\}$ there exist $\lambda > 0$ and a sponge S' around h such that for all $u \in S'$, $\overline{x} + [0, \lambda] \cdot u \subseteq S$.

(B) : for all $h \in X \setminus \{0\}$ and all $d \in X \setminus \{0\}$ there exists $\delta > 0$ such that for all $u \in B(h, \delta) \cap (h + [0, \delta] \cdot B(d, \delta))$, $\overline{x} + [0, \delta] \cdot u \subseteq S$.

Finally, every set S which satisfies property (B) is a sponge around $\overline{x}$.

Indeed, suppose that S verifies the above property and take an arbitrary $h \in X \setminus \{0\}$. Then there exists $\delta > 0$ such that for all $u \in B(h, \delta) \cap (h + [0, \delta] \cdot B(h, \delta))$, $\overline{x} + [0, \delta] \cdot u \subseteq S$. On the other hand, there exists $\alpha > 0$ ($\alpha < \delta$) so that $h + [0, \alpha] \cdot B(h, \delta) \subseteq B(h, \delta) \cap (h + [0, \delta] \cdot B(h, \delta))$ and therefore $\overline{x} + [0, \delta] \cdot B((\alpha + 1)h, \alpha\delta) \subseteq S$. Consequently, there exist $\alpha' := \delta(\alpha + 1) > 0$ and $\delta' := \frac{\alpha}{\alpha+1} \cdot \delta > 0$ such that $\overline{x} + [0, \alpha'] \cdot B(h, \delta') \subseteq S$ and the conclusion follows easily.

Now we are ready to illustrate the aforementioned relationship between sponges and directionally convergent sequences.

Lemma 2.5. *(*[40, Lemma 2.1]*) A subset S of X is a sponge around $\overline{x}$ if and only if for any sequence (x_n) which converges directionally to $\overline{x}$ there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_0$, $x_n \in S$.*

In what follows we mainly focus on the properties of the Dini-Hadamard-like ε-subdifferential. But first, following the lines of the proof of [7, Lemma 2.1] and taking into account relation (2.2), let us remark that Lemma 2.6 bellow holds true not only for the Dini-Hadamard-like subdifferential but also for the Fréchet subdifferential (which is a building block for the *basic/limiting/Mordukhovich subdifferential* in Banach spaces. We refer the reader to the books [25, 26] for a systematic study) and for the Dini-Hadamard one, as well.

Lemma 2.6. *([29, Lemma 2.3]) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function and $\overline{x} \in \operatorname{dom} f$. Then for all $\varepsilon \geq 0$ it holds*

$$\partial_\varepsilon^S f(\overline{x}) = \partial^S (f + \varepsilon \| \cdot - \overline{x} \|)(\overline{x}). \tag{2.5}$$

Thus, using a classical subdifferential formula provided by the convex analysis, one can easily see that, in case f is convex, $\partial_\varepsilon^S f(\overline{x}) = \partial f(\overline{x}) + \varepsilon \overline{B}_{X^*}$ for all $\varepsilon \geq 0$.

The following notion, introduced by Treiman [40], it turns out to be essential also when characterizing the Dini-Hadamard-like subdifferential.

Definition 2.7. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function, $\overline{x} \in \operatorname{dom} f$ and $\varepsilon \geq 0$. We say that $x^* \in X^*$ is an H_ε-*subgradient *of f at $\overline{x}$ if there exists a sponge S around $\overline{x}$ such that for all $x \in S$*

$$f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle - \varepsilon \|x - \overline{x}\|.$$

Unlike the one obtained in the case of the Dini-Hadamard subdifferential (we refer to [7, Lemma 2.2] for more details and a similar proof), the following lemma does not require any calmness condition (take into account also here the Remark 2.4 above). As a direct consequence, we mention that for any $\gamma \geq \varepsilon \geq 0$ and $x \in X$

$$\partial_\varepsilon^S f(x) \subseteq \partial_\gamma^S f(x). \tag{2.6}$$

Lemma 2.8. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function, $\overline{x} \in \operatorname{dom} f$ and $\varepsilon \geq 0$. The following statements are true:*

(i) If $x^* \in \partial_\varepsilon^S f(\overline{x})$, then x^* is an H_γ-subgradient of f at $\overline{x}$ for all $\gamma > \varepsilon$.

(ii) If x^* is an H_ε-subgradient of f at $\overline{x}$, then $x^* \in \partial_\varepsilon^S f(\overline{x})$.

Moreover, one can even conclude that whenever $\overline{x} \in \operatorname{dom} f$, $\varepsilon \geq 0$ and $\gamma > \varepsilon$ the following set

$$S := \{x \in X : \ f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle - \gamma \|x - \overline{x}\|\} \tag{2.7}$$

is a sponge around $\overline{x}$ not only for those elements $x^* \in \partial_\varepsilon^- f(\overline{x})$ (like in [7, Remark 2.3]) but also for $x^* \in \partial_\varepsilon^S f(\overline{x})$.

Example 2.9. Although in finite dimensions the Dini-Hadamard ε-subdifferential coincide with the corresponding Dini-Hadamard-like one (see for this Remark 2.2, Lemma 2.6 and [7, Lemma 2.1]) this is in general not the case. Indeed, let us consider the function $f : X \to \mathbb{R}$ as being

$$f(x) = \begin{cases} 0, & \text{if } x \in S, \\ a, & \text{otherwise,} \end{cases}$$

where $a < 0$ and S is a sponge around $\overline{x} \in X$ which is not a neighborhood of $\overline{x}$ (for such an example we refer to [7, Example 2.2]). Then taking into account the second assertion of Lemma 2.8, one can easily conclude that for all $\varepsilon \geq 0$, $0 \in \partial_\varepsilon^S f(\overline{x}) \setminus \partial_\varepsilon^- f(\overline{x})$, since 0 is an H_ε-subgradient of f at $\overline{x}$, but f is not calm at $\overline{x}$.

To justify this last assertion, we suppose on the contrary that f is calm at $\overline{x}$. Further, using the aforementioned property of the set S, one can even conclude that for all $n \in \mathbb{N}$ there exists an element $y_n \in B(\overline{x}, \frac{1}{n}) \setminus S$ such that $\|y_n - \overline{x}\| \leq \frac{1}{n}$. But since we may write $y_n = \overline{x} + t_n^{'} \cdot u_n^{'}$ with $t_n^{'} := \sqrt{\|y_n - \overline{x}\|}$ and $u_n^{'} := \frac{y_n - \overline{x}}{\|y_n - \overline{x}\|} \cdot \sqrt{\|y_n - \overline{x}\|}$ and moreover $\lim\limits_{n \to +\infty} \frac{f(y_n) - f(\overline{x})}{t_n^{'}} = -\infty$, we get the following relation $\liminf\limits_{\substack{u \to 0 \\ t \downarrow 0}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} = -\infty$, and consequently $d^- f(\overline{x}, 0) = -\infty$, a contradiction which completes the proof.

The following result provides a variational description for the Dini-Hadamard-like ε-subdifferential, with no additional calmness assumptions. For the reader convenient we list bellow also the proof.

Theorem 2.10. *([29, Theorem 3.1]) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary function and $\overline{x} \in \operatorname{dom} f$. Then for all $\varepsilon \geq 0$ one has*

$$x^* \in \partial_\varepsilon^S f(\overline{x}) \Leftrightarrow \begin{array}{l} \forall \alpha > 0 \text{ there exists } S \text{ a sponge around } \overline{x} \text{ such that} \\ \forall x \in S \ \ f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle - (\alpha + \varepsilon)\|x - \overline{x}\|. \end{array} \tag{2.8}$$

Proof. Consider an $\varepsilon \geq 0$ fixed.

In order to justify the inclusion "$\subseteq$", let $x^* \in \partial_\varepsilon^S f(\overline{x})$ and $\alpha > 0$. Now just observe that using Lemma 2.8 above we can easily obtain the existence of a sponge S around $\overline{x}$ such that for all $x \in S$

$$f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle - (\alpha + \varepsilon)\|x - \overline{x}\|.$$

For the second inclusion "$\supseteq$", let us consider an arbitrary element x^* fulfilling the property in the right-hand side of (2.8). Our goal is to show that

$$D_d^S f(\overline{x}; h) \geq \langle x^*, h \rangle - \varepsilon\|h\| \ \forall h \in X, \forall d \in X \setminus \{0\}. \tag{2.9}$$

Let first $h \in X \setminus \{0\}$ and $d \in X \setminus \{0\}$ be fixed. Then for all $k \in \mathbb{N}$, by taking $\alpha_k := \frac{1}{k}$, there exists S_k a sponge around $\overline{x}$ such that for all $x \in S_k$

$$f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle - \left(\frac{1}{k} + \varepsilon\right)\|x - \overline{x}\|.$$

Thus, for all $k \in \mathbb{N}$ there exists $\delta_k > 0$ such that for all $t \in (0, \delta_k)$ and all $u \in B(h, \delta_k) \cap (h + [0, \delta_k] \cdot B(d, \delta_k))$ one has $\overline{x} + tu \in S_k$ and

$$f(\overline{x} + tu) - f(\overline{x}) \geq \langle x^*, tu \rangle - \left(\frac{1}{k} + \varepsilon\right)\|tu\|,$$

which implies in turn that for all $0 < \delta_k^{'} \leq \delta_k$, all $t \in (0, \delta_k^{'})$ and all $u \in B(h, \delta_k^{'}) \cap (h + [0, \delta_k^{'}] \cdot B(d, \delta_k^{'}))$

$$\frac{f(\overline{x} + tu) - f(\overline{x})}{t} \geq \langle x^*, u \rangle - \left(\frac{1}{k} + \varepsilon\right)\|u\|$$

and consequently, for all $k \in \mathbb{N}$ there exists $\delta_k > 0$ such that for all $0 < \delta_k^{'} \leq \delta_k$

$$\begin{aligned} D_d^S f(\overline{x};h) &\geq \inf_{\substack{u \in B(h,\delta_k^{'}) \cap (h+[0,\delta_k^{'}]\cdot B(d,\delta_k^{'})) \\ t \in (0,\delta_k^{'})}} \frac{f(\overline{x}+tu)-f(\overline{x})}{t} \\ &\geq \inf_{\substack{u \in B(h,\delta_k^{'}) \cap (h+[0,\delta_k^{'}]\cdot B(d,\delta_k^{'})) \\ t \in (0,\delta_k^{'})}} \left[\langle x^*, u\rangle - \left(\frac{1}{k}+\varepsilon\right)\|u\|\right] \\ &\geq \inf_{u \in B(h,\delta_k^{'})} \left[\langle x^*, u\rangle - \left(\frac{1}{k}+\varepsilon\right)\|u\|\right]. \end{aligned}$$

On the other hand, for all $k \in \mathbb{N}$, all $0 < \delta_k^{'} \leq \delta_k$ and all $\delta^{'} \geq \delta_k^{'}$

$$\inf_{u \in B(h,\delta_k^{'})} \left[\langle x^*, u\rangle - \left(\frac{1}{k}+\varepsilon\right)\|u\|\right] \geq \inf_{u \in B(h,\delta^{'})} \left[\langle x^*, u\rangle - \left(\frac{1}{k}+\varepsilon\right)\|u\|\right]$$

and hence, for all $k \in \mathbb{N}$

$$D_d^S f(\overline{x};h) \geq \liminf_{u \to h} \left[\langle x^*, u\rangle - \left(\frac{1}{k}+\varepsilon\right)\|u\|\right] = \langle x^*, h\rangle - \left(\frac{1}{k}+\varepsilon\right)\|h\|.$$

Finally, passing to the limit as $k \to +\infty$, we obtain

$$D_d^S f(\overline{x};h) \geq \langle x^*, h\rangle - \varepsilon\|h\|$$

and thus, the relation (2.9) holds true for all $h \in X \setminus \{0\}$ and all $d \in X \setminus \{0\}$.

For the particular case $h = 0$, let $d \in X\setminus\{0\}$ be an arbitrary element. To complete the proof of the theorem we only have to show that $D_d^S f(\overline{x};0) \geq 0$. Assuming the contrary, accordingly to Lemma 2.1, one gets two sequences $(u_n) \underset{d}{\longrightarrow} 0$ and $(t_n) \downarrow 0$ such that

$$\lim_{n\to+\infty} \frac{f(\overline{x}+t_n u_n)-f(\overline{x})}{t_n} < 0, \tag{2.10}$$

where $(u_n) \underset{d}{\longrightarrow} 0$ means that there exist sequences $(t_n^{'}) \to 0$ ($t_n^{'} \geq 0$ $\forall n \in \mathbb{N}$) and $(d_n) \to d$ such that $u_n = t_n^{'} \cdot d_n$ for all $n \in N$.

On the other hand, there exists a sponge S around $\overline{x}$ such that

$$f(x) - f(\overline{x}) \geq \langle x^*, x-\overline{x}\rangle - \varepsilon\|x-\overline{x}\|$$

for all $x \in S$ and consequently, we can find a natural number n_0 such that for all $n \in \mathbb{N}$, $n \geq n_0$, $\overline{x} + t_n \cdot u_n \in S$ and hence

$$f(\overline{x}+t_n\cdot u_n) - f(\overline{x}) \geq \langle x^*, t_n\cdot u_n\rangle - \varepsilon\|t_n\cdot u_n\|.$$

Therefore, passing to the limit as $k \to +\infty$, we observe that

$$\lim_{n\to+\infty} \frac{f(\overline{x}+t_n u_n)-f(\overline{x})}{t_n} \geq 0,$$

which in fact contradicts the relation (2.10). □

A similar result to Theorem 2.10, by means of the Dini-Hadamard ε-subdifferential, was given in [7], but in a more restrictive framework.

To a more careful look we can see that also in the case of the Dini-Hadamard-like subdifferential it is a sort of calmness condition that is hiding behind. So, we say that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is *weakly calm* at $\overline{x} \in \operatorname{dom} f$ if $D^S_d f(\overline{x}; 0) \geq 0$ for all $d \in X \setminus \{0\}$. Actually, unlike the case of the Dini-Hadmamard subdifferential, this last assumption is automatically fulfilled. It is worth mentioning also here that although the weakly calmness assumption is a more general one, it does coincide with the classical calmness condition in finite dimensions.

Proposition 2.11. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function and $\overline{x} \in \operatorname{dom} f$. If X is finite dimensional then f is calm at $\overline{x}$ if and only if f is weakly calm at $\overline{x}$.*

Proof. Since one can easily check the "if" part of the proposition, it remains us to show just the "only" if one. Suppose on the contrary that f is not calm at $\overline{x}$. Then $d^- f(\overline{x}; 0) = -\infty$ and hence there exist sequences $(u_n) \to 0$ and $(t_n) \downarrow 0$ such that $\lim\limits_{n\to+\infty} \frac{f(\overline{x}+t_n u_n)-f(\overline{x})}{t_n} = -\infty$. Using now the finiteness property of X, the latter clearly yields an element $s \in S_X$ and a subsequence $(u_{n_k}) \underset{s}{\longrightarrow} 0$ with the property that $\lim\limits_{k\to+\infty} \frac{f(\overline{x}+t_{n_k} \cdot u_{n_k})-f(\overline{x})}{t_{n_k}} = -\infty$. Consequently $-\infty \geq D^S_s f(\overline{x}; 0)$, which is a contradiction. □

Finally, to conclude this section, let us present a direct consequence of Theorem 2.10 and [7, Theorem 2.3], interesting in itself.

Corollary 2.12. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function, $\overline{x} \in \operatorname{dom} f$ and $\varepsilon \geq 0$. If $\partial^S_\varepsilon f(\overline{x}) \neq \emptyset$ then f is calm at $\overline{x}$ if and only if $\partial^-_\varepsilon f(\overline{x}) = \partial^S_\varepsilon f(\overline{x})$.*

3. Some generalized convexity notions

Let us mention in the beginning of this section that the Dini-Hadamard-like subdifferential coincides with the Dini-Hadamard one not only in finite dimensional spaces but also in arbitrarily Banach spaces on some particular classes of functions. Furthermore, these classes of functions, which are in fact larger than the one of convex functions, will reveal themselves to be useful in the sequel. We introduce them now.

Definition 3.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function and $\overline{x} \in \operatorname{dom} f$. The function f is said to be*

(i) approximately convex *at $\overline{x}$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in B(\overline{x}, \delta)$ and every $t \in [0, 1]$ one has*

$$f((1-t)y + tx) \leq (1-t)f(y) + tf(x) + \varepsilon t(1-t)\|x - y\|. \tag{3.1}$$

(ii) approximately starshaped *at $\overline{x}$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in B(\overline{x}, \delta)$ and every $t \in [0, 1]$ one has*

$$f((1-t)\overline{x} + tx) \leq (1-t)f(\overline{x}) + tf(x) + \varepsilon t(1-t)\|x - \overline{x}\|. \tag{3.2}$$

(iii) directionally approximately starshaped *at* $\overline{x}$*, if for any* $\varepsilon > 0$ *and any* $u \in S_X$ *there exists* $\delta > 0$ *such that for every* $s \in (0, \delta)$*, every* $v \in B(u, \delta)$ *and every* $t \in [0, 1]$*, when* $x := \overline{x} + sv$*, one has*

$$f((1-t)\overline{x} + tx) \leq (1-t)f(\overline{x}) + tf(x) + \varepsilon t(1-t)\|x - \overline{x}\|. \tag{3.3}$$

While the class of approximately convex functions was initiated and strongly developed by H.V. Ngai, D.T. Luc and M. Théra in [30] (see also [3, 31]), the ones of approximately starshaped and directionally approximately stashaped were introduced and studied in [32]. Actually, they enjoy nice properties and, for instance, the approximate convex functions are stable under finite sums and finite suprema, and moreover the most of the well-known subdifferentials coincide and share several properties of the convex subdifferential (see [30]) on this particular class of functions. Observe also that the class of approximately convex functions is strictly included into the class of approximately starshaped functions, which in turn is contained into the one of directionally approximately starshaped functions (for some examples we refer to [7]). In fact, the last two classes of functions coincide on finite dimensional spaces, as one can easily deduce from the following result.

Proposition 3.2. *(*[7, Proposition 3.1]*) Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a given function and* $\overline{x} \in \operatorname{dom} f$*. Then* f *is directionally approximately starshaped at* $\overline{x}$ *if and only if for any* $\varepsilon > 0$ *there exists a sponge* S *around* $\overline{x}$ *such that for every* $x \in S$ *and* $t \in [0, 1]$ *one has*

$$f((1-t)\overline{x} + tx) \leq (1-t)f(\overline{x}) + tf(x) + \varepsilon t(1-t)\|x - \overline{x}\|. \tag{3.4}$$

It is worth emphasizing here that in view of Remark 2.4, the above characterization via sponges it is also equivalent with the following one.

Proposition 3.3. *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a given function and* $\overline{x} \in \operatorname{dom} f$*. Then* f *is directionally approximately starshaped at* $\overline{x}$ *if and only if for any* $\varepsilon > 0$*,* $h \in X \setminus \{0\}$ *and any* $d \in X \setminus \{0\}$ *there exists* $\delta > 0$ *such that for every* $s \in (0, \delta)$*,* $v \in B(h, \delta) \cap (h + [0, \delta] \cdot B(d, \delta))$ *and every* $t \in [0, 1]$*, with* $x := \overline{x} + sv$*, the relation (*3.4*) above holds true.*

In fact the class of directionally approximately starshaped functions enjoys also the following property, which is more general then the one obtained in [7, Lemma 3.2], or in [1, Lemma 1].

Lemma 3.4. *Let the function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be directionally approximately starshaped at* $\overline{x} \in \operatorname{dom} f$*. Then for every* $\alpha > 0$ *and every* $\varepsilon \geq 0$ *there exists a sponge* S *around* $\overline{x}$ *such that for every* $x \in S$ *one has*

$$f(x) - f(\overline{x}) \geq \langle \overline{x}^*, x - \overline{x} \rangle - (\alpha + \varepsilon)\|x - \overline{x}\| \ \forall \overline{x}^* \in \partial_\varepsilon^S f(\overline{x}), \tag{3.5}$$

$$f(\overline{x}) - f(x) \geq \langle x^*, \overline{x} - x \rangle - (\alpha + \varepsilon)\|x - \overline{x}\| \ \forall x^* \in \partial_\varepsilon^S f(x). \tag{3.6}$$

Proof. Fix $\alpha > 0$, $\varepsilon \geq 0$ and consider the set

$$S := \{x \in X : \ f(x) - f(\overline{x}) \geq \langle \overline{x}^*, x - \overline{x} \rangle - (\alpha + \varepsilon)\|x - \overline{x}\| \ \forall \overline{x}^* \in \partial_\varepsilon^S f(\overline{x})\}.$$

In order to complete the proof of the first inequality, our strategy is to show that S is a sponge around $\overline{x}$.

Indeed, let $h \in X \setminus \{0\}$ and $d \in X \setminus \{0\}$ be arbitrary elements and take $\delta > 0$ so that the relation (3.4) above holds true with $x := \overline{x} + sv$, for any $s \in (0, \delta)$, $v \in B(h, \delta) \cap (h + [0, \delta] \cdot B(d, \delta))$ and any $t \in (0, 1]$. Then,

$$f(\overline{x} + tsv) - f(\overline{x}) \leq t[f(\overline{x} + sv) - f(\overline{x})] + \alpha t(1 - t)\|sv\|$$

and hence, after dividing by t, we take the limit inferior as $t \downarrow 0$ and we obtain

$$\liminf_{t \downarrow 0} \frac{f(\overline{x} + tsv) - f(\overline{x})}{t} \leq f(\overline{x} + sv) - f(\overline{x}) + \alpha\|sv\|.$$

But

$$D_d^S f(\overline{x}; sv) \leq \sup_{\delta > 0} \inf_{\substack{u \in \{sv\} \\ t \in (0, \delta)}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} = \liminf_{t \downarrow 0} \frac{f(\overline{x} + tsv) - f(\overline{x})}{t}$$

and consequently,

$$D_d^S f(\overline{x}; sv) \leq f(\overline{x} + sv) - f(\overline{x}) + \alpha\|sv\|.$$

In other words, for any $h \in X \setminus \{0\}$ and $d \in X \setminus \{0\}$ there exists $\delta > 0$ such that for every $s \in (0, \delta)$ and $v \in B(h, \delta) \cap (h + [0, \delta] \cdot B(d, \delta))$, $\overline{x} + sv \in S$, i.e. S is a sponge around $\overline{x}$, by virtue of Remark 2.4.

Similarly, with $x := \overline{x} + sv$ and $t' := 1 - t$ one has

$$f(x - t'sv) - f(x) \leq t'[f(x - sv) - f(x)] + \alpha t'(1 - t')\|sv\|$$

which implies in turn (following the steps bellow)

$$D_d^S f(x; -sv) \leq f(x - sv) - f(x) + \alpha\|sv\|$$

and finally one obtains that

$$S' := \{x \in X : \ f(\overline{x}) - f(x) \geq \langle x^*, \overline{x} - x\rangle - (\alpha + \varepsilon)\|x - \overline{x}\| \ \forall x^* \in \partial_\varepsilon^S f(x)\}$$

is a sponge around $\overline{x}$, which completes the proof of the second inequality. □

Now we state our main result of this section, thanks to which, the Dini-Hadamard-like subdifferential as well as the Dini-Hadamard one agrees with a great number of well-known subdifferentials such as the Clarke-Rockafellar, the Mordukhovich, the Fréchet and the Ioffe approximate subdifferential on lower semicontinuous and approximately convex functions at a given point of the domain (see for more details [30, Theorem 3.6]).

Proposition 3.5. *Let the function $f : X \to \mathbb{R} \cup \{+\infty\}$ be approximately starshaped at $\overline{x} \in \operatorname{dom} f$. Then for all $\varepsilon \geq 0$ it holds*

$$\partial_\varepsilon^F f(\overline{x}) = \partial_\varepsilon^- f(\overline{x}) = \partial_\varepsilon^S f(\overline{x}).$$

Proof. In view of [32, Lemma 2.6] and [7, Lemma 2.1] the first equality is clearly verified. For the second one, accordingly to Lemma 2.6 above and [7, Lemma 2.1] it is enough to show that it holds true only for $\varepsilon = 0$. To this end we argue why for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in B(\overline{x}, \delta)$ and any $d \in X \setminus \{0\}$

$$D_d^S f(\overline{x}; x - \overline{x}) \leq f(x) - f(\overline{x}) + \varepsilon\|x - \overline{x}\|.$$

This will complete the proof, since given an arbitrary $\overline{x}^* \in \partial^S f(\overline{x})$, the above inequality would provide us the following estimate

$$\langle \overline{x}^*, x - \overline{x} \rangle \le f(x) - f(\overline{x}) + \varepsilon \|x - \overline{x}\|,$$

i.e. the inclusion $\partial^S f(\overline{x}) \subseteq \partial^F f(\overline{x})$ (due to relation (2.1)) and hence the equality.

So, fix an arbitrary $\varepsilon > 0$. Then, since f is approximately starshaped at $\overline{x}$, we choose $\delta > 0$ so that for any $x \in B(\overline{x}, \delta)$ and any $t \in (0, 1]$

$$f(\overline{x} + t(x - \overline{x})) - f(\overline{x}) \le t[f(x) - f(\overline{x})] + \varepsilon t(1 - t)\|x - \overline{x}\|.$$

Then, dividing by t and taking limit inferior as $t \downarrow 0$, one obtains

$$\liminf_{t \downarrow 0} \frac{f(\overline{x} + t(x - \overline{x})) - f(\overline{x})}{t} \le f(x) - f(\overline{x}) + \varepsilon \|x - \overline{x}\|$$

and finally, the desired inequality. □

On the other hand, while [7, Example 3.1] ensures us that the equality $\partial^F f(\overline{x}) = \partial^- f(\overline{x})$ does not hold in case f is only directionally approximately starshaped at $\overline{x} \in \operatorname{dom} f$, Example 2.9 above guarantees the same with the equality $\partial^- f(\overline{x}) = \partial^S f(\overline{x})$, since f is directionally approximately starshaped at $\overline{x}$, but $0 \in \partial^S f(\overline{x}) \setminus \partial^- f(\overline{x})$. Moreover, the function in Example 2.9 shows that in general the class of approximately starshaped functions does not coincide with the one of directionally approximately starshaped functions.

4. Optimality conditions

In what follows we mostly confine ourselves to the study of a subdifferential formula for the difference of two functions. To this end, let us recall first that for two subsets $A, B \subseteq X$ the *star-difference* between them is defined as

$$A \overset{*}{-} B := \{x \in X : \; x + B \subseteq A\} = \bigcap_{b \in B} \{A - b\}.$$

We adopt here the convention $A \overset{*}{-} B := \emptyset$ in case $A = \emptyset, B \neq \emptyset$ and $A \overset{*}{-} B := X$ if $B = \emptyset$. One obviously have $A \overset{*}{-} B + B \subseteq A$ and $A \overset{*}{-} B \subseteq A - B$ if $B \neq \emptyset$. Introduced by Pontrjagin [36] in the context of linear differential games, this notion has been widely used in the field of nonsmooth analysis (see, for instance, [1, 2, 8, 12, 15, 22, 27, 37]).

When dealing with the difference of two functions $g, h : X \to \mathbb{R} \cup \{+\infty\}$ we assume throughout the paper that $\operatorname{dom} g \subseteq \operatorname{dom} h$. This guarantees that the function $f = g - h : X \to \mathbb{R} \cup \{+\infty\}$ is well-defined. Moreover, one can easily observe that $g = f + h$ and $\operatorname{dom} f = \operatorname{dom} g$.

The following simple result yields easily from Theorem 2.10 and due to the fact that the intersection of two sponges around a point is a sponge around that point.

Proposition 4.1. *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be given functions and $f := g - h$. Then for all $\varepsilon, \eta \ge 0$ and all $x \in X$ one has*

$$\partial^S_\varepsilon f(x) \subseteq \partial^S_{\varepsilon+\eta} g(x) \overset{*}{-} \partial^S_\eta h(x). \tag{4.1}$$

In particular, if $\overline{x} \in \operatorname{dom} f$ is a local minimizer of the function $f := g - h$, then

$$0 \in \partial^S g(\overline{x}) \overset{*}{-} \partial^S h(\overline{x})$$

or, equivalently,

$$\partial^S h(\overline{x}) \subseteq \partial^S g(\overline{x}).$$

For similar characterizations to the difference of two functions via the Fréchet subdifferential, by means of the Mordukhovich (basic/limiting) subdifferential and in terms of the Dini-Hadamard one we refer to [27], [25, 26] and [7], respectively.

Although the inclusion (4.1) holds true without no supplementary assumptions on the functions involved, in order to guarantee the reverse one we need to introduce also the following notion.

Definition 4.2. *A set-valued mapping $F : X \rightrightarrows X^*$ is said to be* spongiously pseudo-dissipative *at $\overline{x} \in X$ if for any $\varepsilon > 0$ there exists S a sponge around $\overline{x}$ such that for any $x \in S$ there exist $x^* \in F(x)$ and $\overline{x}^* \in F(\overline{x})$ so that*

$$\langle x^* - \overline{x}^*, x - \overline{x} \rangle \leq \varepsilon \|x - \overline{x}\|$$

or, equivalently, if for any $\varepsilon > 0$ and any $u \in S_X$ there exists $\delta > 0$ such that for any $t \in (0, \delta)$ and $v \in B(u, \delta)$ there exist $x^* \in F(x)$ and $\overline{x}^* \in F(\overline{x})$ so that

$$\langle x^* - \overline{x}^*, v \rangle \leq \varepsilon \|v\|.$$

Actually, there are two ways of extending the *approximately pseudo-dissipativity* introduced by Penot [35]. While the first one was described above by replacing a neighborhood with a sponge, the second one will be presented bellow.

Definition 4.3. *A set-valued mapping $F : X \rightrightarrows X^*$ is said to be* directionally approximately pseudo-dissipative *at $\overline{x} \in X$ if for any $\varepsilon > 0$ and $u \in S_X$ one can find some $\delta > 0$ such that for any $v \in B(u, \delta)$ and any $t \in (0, \delta)$ there exist $x^* \in F(\overline{x} + tv)$ and $\overline{x}^* \in F(\overline{x})$ so that*

$$\langle x^* - \overline{x}^*, x - \overline{x} \rangle \leq \varepsilon.$$

In fact this later conditions are not very restrictive ones, since the following coarse continuity (which has been introduced in [1]) ensures the approximately pseudo-dissipativity and the *spongiously gap-continuity* studied in [7], as well. Let us formulate now this concept.

Definition 4.4. A set-valued mapping $F : X \rightrightarrows Y$ between a topological space X and a metric space Y is said to be *gap-continuous* at $\overline{x} \in X$ if for any $\varepsilon > 0$ one can find some $\delta > 0$ such that for every $x \in B(\overline{x}, \delta)$

$$\operatorname{gap}(F(\overline{x}), F(x)) < \varepsilon,$$

where for two subsets A and B of Y

$$\operatorname{gap}(A, B) := \inf\{d(a, b) : a \in A, b \in B\},$$

with the convention that if one of the sets is empty, then $\operatorname{gap}(A, B) := +\infty$.

When defining a spongiously gap-continuous mapping one only has to replace in the above definition the neighborhood $B(\overline{x}, \delta)$ of $\overline{x}$ with a sponge S around $\overline{x}$. Therefore, every gap-continuous mapping at a point is spongiously gap-continuous and moreover it is also spongiously pseudo-dissipative and directionally approximately pseudo-dissipative at that point, too. Furthermore, every set-valued mapping which is either Hausdorff upper semicontinuous or lower semicontinuous at a given point is gap-continuous at that point (see [34]). Thus, the gap-continuity is a sort of semicontinuity notion which is satisfied in many situations when no other semicontinuity notion holds. Moreover, in case the mapping is single-valued, it coincides with the classical continuity. Clearly, when X is a finite dimensional space then the gap-continuity coincides with the spongiously gap-continuity as well as the approximately pseudo-dissipativity property agrees with the spongiously pseudo-dissipativity and with the directionally approximately pseudo-dissipativity one. It is worth emphasizing also here that the notion of spongiously gap-continuity [7] is equivalent to that of *directionally-gap continuity* introduced later by Penot [35] (we refer the reader to the papers of Penot [35, 34] for more discussions and some criteria ensuring the gap-continuity and also the directionally approximately pseudo-dissipativity). Finally, the following property holds.

Proposition 4.5. *Let $F, G : X \rightrightarrows Y$ be two set-valued mappings. If F is spongiously pseudo-dissipative at $\overline{x} \in X$ and there exists a sponge S around $\overline{x}$ such that $F(x) \subseteq G(x)$ for all $x \in S$, then G is spongiously pseudo-dissipative at $\overline{x}$, too.*

Accordingly to relation (2.6) and the above property, we conclude that for $f : X \to \mathbb{R} \cup \{+\infty\}$ a given function and $\overline{x} \in \operatorname{dom} f$, $\partial^S_\eta f$ is spongiously pseudo-dissipative at $\overline{x}$ for all $\eta > 0$, whenever $\partial^S f$ is spongiously pseudo-dissipative at $\overline{x}$. Hence, following the lines of the proof of [7, Theorem 3.4, Theorem 3.5] we can furnish a formula for the difference of two functions in terms of the Dini-Hadamard-like subdifferential.

Theorem 4.6. *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be two directionally approximately starshaped functions at $\overline{x} \in \operatorname{dom} g$ and $f := g - h$. If for some $\eta \geq 0$ the set-valued mapping $\partial^S_\eta h$ is spongiously pseudo-dissipative at $\overline{x}$, then for all $\varepsilon \geq 0$ it holds*

$$\partial^S_\varepsilon f(\overline{x}) = \partial^S_{\varepsilon+\eta} g(\overline{x}) \overset{*}{-} \partial^S_\eta h(\overline{x}). \tag{4.2}$$

In case the function f is calm at $\overline{x}$ one obtains the result in [7, Theorem 3.5], where the subdifferential in question is the Dini-Hadamard one. For a similar statement in the particular setting $\varepsilon = \eta = 0$, we refer to [35, Theorem 28]. There the function h is assumed to be directionally approximately starshaped, directionally continuous, directionally stable and tangentially convex at $\overline{x}$, a point from $\operatorname{core}(\operatorname{dom} h)$. Similar results expressed by means of the Fréchet subdifferential can be found in [1, Theorem 3] and [35, Theorem 26], where the functions are supposed to be approximately starshaped and a very mild assumption on $\partial^F h$ is required. But taking into account the fact that f may not be calm at $\overline{x}$, or the functions g and h may not be approximately starshaped, or even $\operatorname{core}(\operatorname{dom} h)$ could be empty (for instance,

$\operatorname{core}(\ell^p_+) = \emptyset$ for any $p \in [1, +\infty)$, see [6]), motivates us to formulate results like Theorem 4.6.

Let us mention now some corollaries whose proofs follows the ideas from [7, Corollary 3.7, Corollary 3.8]. Take also into account Proposition 3.5 above.

Corollary 4.7. *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be two directionally approximately starshaped functions at $\overline{x} \in \operatorname{dom} g$ such that $\partial^S h$ is spongiously pseudo-dissipative at $\overline{x}$ and $f := g - h$. Then the following statements are equivalent:*

(i) there exists $\eta \geq 0$ such that $\partial^S_\eta h(\overline{x}) \subseteq \partial^S_\eta g(\overline{x})$;

(ii) $0 \in \partial^S f(\overline{x})$;

(iii) for all $\eta \geq 0$ $\partial^S_\eta h(\overline{x}) \subseteq \partial^S_\eta g(\overline{x})$.

Corollary 4.8. *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be two given functions, $\overline{x} \in \operatorname{dom} g$ and $f := g - h$. Then the following assertions are true:*

(i) If g and h are convex at $\overline{x}$ and ∂h is spongiously pseudo-dissipative at $\overline{x}$, then it holds

$$\partial^S f(\overline{x}) = \partial g(\overline{x}) \overset{*}{-} \partial h(\overline{x}).$$

(ii) If g is convex, h is directionally approximately starshaped at $\overline{x}$ and $\partial^S h$ is spongiously pseudo-dissipative at $\overline{x}$, then for all $\varepsilon \geq 0$ it holds

$$\partial^S_\varepsilon f(\overline{x}) = (\partial g(\overline{x}) + \varepsilon \overline{B}_{X^*}) \overset{*}{-} \partial^S h(\overline{x}).$$

(iii) If g is lower semicontinuous, approximately convex at $\overline{x}$, h is directionally approximately starshaped at $\overline{x}$ and $\partial^S h$ is spongiously pseudo-dissipative at $\overline{x}$, then for all $\varepsilon \geq 0$ it holds

$$\partial^S_\varepsilon f(\overline{x}) = (\partial^S g(\overline{x}) + \varepsilon \overline{B}_{X^*}) \overset{*}{-} \partial^S h(\overline{x}).$$

The following result, which significantly improves the statement in [7, Corollary 3.6], due to Theorem 4.6 and [35, Theorem 26] (see also Proposition 3.5), is meant to reveal that the Dini-Hadamard-like subdifferential coincides with the Dini-Hadamard subdifferential and with the Fréchet one not only on approximately starshaped functions but also on some particular differences of approximately starshaped functions.

Corollary 4.9. *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be two approximately starshaped functions at $\overline{x} \in \operatorname{dom} g$ with the property that there exists $\eta \geq 0$ such that $\partial^S_\eta h$ is approximately pseudo-dissipative at $\overline{x}$ and $f := g - h$. Then for all $\varepsilon \geq 0$ $\partial^F_\varepsilon f(\overline{x}) = \partial^-_\varepsilon f(\overline{x}) = \partial^S_\varepsilon f(\overline{x})$.*

Moreover, in case $\overline{x} \in \operatorname{core}(\operatorname{dom} h)$ and $\partial^- h$ is only directionally approximately pseudo-dissipative at $\overline{x}$, then one can guarantee that for any $\varepsilon \geq 0$, $\partial^-_\varepsilon f(\overline{x}) = \partial^S_\varepsilon f(\overline{x})$ (see for this [35, Lemma 22, Lemma 24, Lemma 27] and Lemma 4.1).

Finally, we characterize the Dini-Hadamard-like subdifferential by means of the so-called *spongiously local ε-blunt minimizers*. Introduced in [7], they came as a generalization to *local ε-blunt minimizers* studied by Amahroq, Penot and Syam in [1].

Definition 4.10. *Let $C \subseteq X$ be a nonempty set, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function, $\overline{x} \in \operatorname{dom} f \cap C$ and $\varepsilon > 0$. We say that $\overline{x}$ is a* spongiously local ε-blunt minimizer of f on the set C *if there exists a sponge S around $\overline{x}$ such that for all $x \in S \cap C$*

$$f(x) \geq f(\overline{x}) - \varepsilon \|x - \overline{x}\|.$$

In case $C = X$, we simply call $\overline{x}$ a spongiously local ε-blunt minimizer of f.

Proposition 4.11. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function and $\overline{x} \in \operatorname{dom} f$. Then:*

$$0 \in \partial^S f(\overline{x}) \Leftrightarrow \quad \overline{x} \text{ is a spongiously local } \varepsilon - \text{blunt minimizer of } f \text{ for all } \varepsilon > 0.$$

In the situation when f is calm at $\overline{x}$ one obtains the result in [7, Proposition 3.9], as a particular case. Similarly, in view of the above discussions and results, we can even furnish optimality conditions for the cone-constrained optimization problem $(\mathcal{P})$ studied in [7], by means of the Dini-Hadamard-like subdifferential and without additional calmness assumptions. For the reader convenient we state this result bellow. To this end, let us consider the following optimization problem

$$(\mathcal{P}) \quad \inf_{x \in \mathcal{A}} f(x).$$
$$\mathcal{A} = \{x \in C : k(x) \ \in -K\},$$

where $C \subseteq X$ is a convex and closed set, K, a subset of a Banach space Z, is a nonempty convex and closed *cone* with $K^* := \{z^* \in Z^* : \langle z^*, z\rangle \geq 0 \text{ for all } z \in K\}$ its *dual cone*, $k : X \to Z$, a given function, is assumed to be K*-convex*, meaning that for all $x, y \in X$ and all $t \in [0, 1]$, $(1-t)k(x) + tk(y) - k((1-t)x + ty) \in K$, and K*-epi closed*, meaning that the K*-epigraph* of k, $\operatorname{epi}_K k := \{(x, z) \in X \times Z : z \in k(x) + K\}$, is a closed set and finally $f := g - h$, where $g, h : X \to \mathbb{R} \cup \{+\infty\}$ are two given functions with $\operatorname{dom} g \subseteq \operatorname{dom} h$. For $z^* \in K^*$, by $(z^*k) : X \to \mathbb{R}$ we denote the function defined by $(z^*k)(x) = \langle z^*, k(x)\rangle$ and we also emphasize that in case $Z = \mathbb{R}$ and $K = \mathbb{R}_+$ the notion of K-epi closedness coincide with the classical lower semicontinuity.

Theorem 4.12. *Let be $\overline{x} \in \operatorname{int}(\operatorname{dom} g) \cap \mathcal{A}$. Suppose that g is lower semicontinuous and approximately convex at $\overline{x}$ and that $\bigcup_{\lambda>0} \lambda(k(C) + K)$ is a closed linear subspace of Z. Then the following assertions are true:*

(a) If $\overline{x}$ is a spongiously local ε-blunt minimizer of f on $\mathcal{A}$ for all $\varepsilon > 0$, then the following relation holds

$$\partial^S h(\overline{x}) \subseteq \partial^S g(\overline{x}) + \bigcup_{\substack{z^* \in K^* \\ (z^*k)(\overline{x}) = 0}} \partial((z^*k) + \delta_C)(\overline{x}). \tag{4.3}$$

(b) Conversely, if h is directionally approximately starshaped at $\overline{x}$, $\partial^S h$ is spongiously pseudo-dissipative at $\overline{x}$ and (4.3) holds, then $\overline{x}$ is a spongiously local ε-blunt minimizer of f on $\mathcal{A}$ for all $\varepsilon > 0$.

It is worth mentioning that accordingly to [35, Lemma 22, Lemma 24 and Lemma 27], our final result remains also true in case $\partial^S h$ is directionally approximately pseudo-dissipative at $\overline{x}$. Moreover, in the particular instance when $K = \{0\}$, $k(x) = 0$ for any $x \in X$, g is lower semicontinuous and approximately convex at $\overline{x} \in \operatorname{int}(\operatorname{dom} g) \cap \mathcal{A}$ and h is convex on C and continuous at $\overline{x}$, and hence directionally approximately pseudo-dissipative at $\overline{x}$ (due to the remarkable dissipativity property of the subdifferential in the sense of convex analysis, see [35, Theorem 6]) then $\overline{x}$ is a spongiously local ε-blunt minimizer of f on $\mathcal{A}$ for all $\varepsilon > 0$ if and only if

$$\partial h(\overline{x}) \subseteq \partial^S g(\overline{x}) + N(\mathcal{A}, \overline{x}).$$

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On new Hermite Hadamard Fejér type integral inequalities

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Abstract. In this paper, we establish several weighted inequalities for some differantiable mappings that are connected with the celebrated Hermite-Hadamard Fejér type integral inequality. The results presented here would provide extensions of those given in earlier works.

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Keywords: Convex function, Hermite-Hadamard inequality, Hermite-Hadamard Fejér inequality.

1. Introduction

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [12]):

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite Hadamard Fejér inequalities. In [4], Fejer gave a weighted generalizatinon of the inequalities (1.1) as the following:

Theorem 1.1. *$f : [a, b] \to \mathbb{R}$, be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leq \frac{1}{b-a}\int_a^b f(x)w(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)dx \tag{1.2}$$

holds, where $w : [a, b] \to \mathbb{R}$ is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$.

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2), (see [1]-[3], [5]-[11], [13], [15] and [16]).

In [2] in order to prove some inequalities related to Hadamard's inequality Dragomir and Agarwal used the following lemma.

Lemma 1.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I$ (I° is the interior of I) with $a < b$. If $f' \in L([a, b])$, then we have*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)\, f'(ta+(1-t)b)dt. \tag{1.3}$$

Theorem 1.3. *([2]) Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, $f' \in L(a, b)$ and $p > 1$. If the mapping $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2(p+1)^{1/p}}\left(\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2}\right)^{(p-1)/p}. \tag{1.4}$$

In [9] some inequalities of Hermite-Hadamard type for differentiable convex mappings were proved using the following lemma.

Lemma 1.4. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I^\circ$ (I° is the interior of I) with $a < b$. If $f' \in L([a, b])$, then we have*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \\ &= (b-a)\left[\int_0^{\frac{1}{2}} t f'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)\, f'(ta+(1-t)b)dt\right]. \end{aligned} \tag{1.5}$$

One more general result related to (1.5) was established in [10]. The main result in [9] is as follows:

Theorem 1.5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I$ with $a < b$. If the mapping $|f'|$ is convex on $[a, b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\left(\frac{|f'(a)| + |f'(b)|}{2}\right). \tag{1.6}$$

In this article, using functions whose derivatives absolute values are convex, we obtained new inequalities of weighted Hermite-Hadamard type. The results presented here would provide extensions of those given in earlier works.

2. Main results

We will establish some new results connected with the left-hand side of (1.2) used the following Lemma. Now, we give the following new Lemma for our results:

Lemma 2.1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $w : [a,b] \to [0,\infty)$ be a differentiable mapping. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx = (b-a)\int_0^1 k(t)f'(ta+(1-t)b)dt \tag{2.1}$$

for each $t \in [0,1]$, where

$$k(t) = \begin{cases} \int_0^t w(as+(1-s)b)ds, & t \in [0,\frac{1}{2}) \\ -\int_t^1 w(as+(1-s)b)ds, & t \in [\frac{1}{2},1]. \end{cases}$$

Proof. It suffices to note that

$$\begin{aligned} I &= \int_0^1 k(t)f'(ta+(1-t)b)dt \\ &= \int_0^{\frac{1}{2}}\left(\int_0^t w(as+(1-s)b)ds\right)f'(ta+(1-t)b)dt \\ &+ \int_{\frac{1}{2}}^1\left(-\int_t^1 w(as+(1-s)b)ds\right)f'(ta+(1-t)b)dt \\ &= I_1 + I_2. \end{aligned}$$

By integration by parts, we get

$$\begin{aligned} I_1 &= \left(\int_0^t w(as+(1-s)b)ds\right)\frac{f(ta+(1-t)b)}{a-b}\Bigg|_0^{\frac{1}{2}} \\ &- \int_0^{\frac{1}{2}} w(ta+(1-t)b)\frac{f(ta+(1-t)b)}{a-b}dt \\ &= \left(\int_0^{\frac{1}{2}} w(as+(1-s)b)ds\right)\frac{f(\frac{a+b}{2})}{a-b} \\ &- \int_0^{\frac{1}{2}} w(ta+(1-t)b)\frac{f(ta+(1-t)b)}{a-b}dt, \end{aligned}$$

and similarly

$$I_2 = \left(\int_{\frac{1}{2}}^1 w(as+(1-s)b)ds\right)\frac{f(\frac{a+b}{2})}{a-b} - \int_{\frac{1}{2}}^1 w(ta+(1-t)b)\frac{f(ta+(1-t)b)}{a-b}dt.$$

Thus, we can write

$$I = I_1 + I_2 = \left(\int_0^1 w(as+(1-s)b)ds\right)\frac{f(\frac{a+b}{2})}{a-b} - \int_0^1 w(ta+(1-t)b)\frac{f(ta+(1-t)b)}{a-b}dt.$$

Using the change of the variable $x = ta + (1-t)b$ for $t \in [0,1]$, and multiplying the both sides by $(b-a)$, we obtain (2.1) which completes the proof. □

Remark 2.2. If we take $w(x) = 1$ in Lemma 2.1, then (2.1) reduces to (1.5).

Now, by using the above lemma, we prove our main theorems:

Theorem 2.3. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $w : [a, b] \to [0, \infty)$ be a differentiable mapping and symmetric to $\frac{a+b}{2}$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right|$$
$$\leq \left(\frac{1}{(b-a)^2}\int_{\frac{a+b}{2}}^b w(x)\left[(x-a)^2-(b-x)^2\right]dx\right)\left(\frac{|f'(a)|+|f'(b)|}{2}\right) \tag{2.2}$$

Proof. From Lemma 2.1 and the convexity of $|f'|$, it follows that

$$\left|\frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right|$$
$$\leq (b-a)\left\{\int_0^{\frac{1}{2}}\left(\int_0^t w(as+(1-s)b)ds\right)\left[t|f'(a)|+(1-t)|f'(b)|\right]dt\right. \tag{2.3}$$
$$\left.+\int_{\frac{1}{2}}^1\left(\int_t^1 w(as+(1-s)b)ds\right)\left[t|f'(a)|+(1-t)|f'(b)|\right]dt\right\}$$
$$= Q_1 + Q_2.$$

By change of the order of integration, we have

$$Q_1 = \int_0^{\frac{1}{2}}\int_0^t w(as+(1-s)b)\left(t|f'(a)|+(1-t)|f'(b)|\right)dsdt$$
$$= \int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(as+(1-s)b)\left(t|f'(a)|+(1-t)|f'(b)|\right)dtds$$
$$= \int_0^{\frac{1}{2}} w(as+(1-s)b)\left[\left(\tfrac{1}{8}-\tfrac{s^2}{2}\right)|f'(a)|+\left(\tfrac{(1-s)^2}{2}-\tfrac{1}{8}\right)|f'(b)|\right]ds.$$

and using the change of the variable $x = as + (1-s)b$ for $s \in [0, 1]$,

$$Q_1 = \frac{1}{8(b-a)^3}\int_{\frac{a+b}{2}}^b w(x)\left[\left((b-a)^2-4(b-x)^2\right)|f'(a)|\right.$$
$$\left.+\left(4(x-a)^2-(b-a)^2\right)|f'(b)|\right]dx \tag{2.4}$$

Similarly, by change of order of the integration, we obtain

$$Q_2 = \int_{\frac{1}{2}}^1 w(as+(1-s)b)\left[\left(\tfrac{s^2}{2}-\tfrac{1}{8}\right)|f'(a)|+\left(\tfrac{1}{8}-\tfrac{(1-s)^2}{2}\right)|f'(b)|\right]ds$$
$$= \frac{1}{8(b-a)^3}\int_a^{\frac{a+b}{2}} w(x)\left[\left(4(b-x)^2-(b-a)^2\right)|f'(a)|\right.$$
$$\left.+\left((b-a)^2-4(x-a)^2\right)|f'(b)|\right]dx.$$

Since $w(x)$ is symmetric to $x = \frac{a+b}{2}$, for $w(x) = w(a+b-x)$, we write

$$Q_2 = Q_1 \tag{2.5}$$

A combination of (2.3), (2.4) and (2.5), we get (2.2). This completes the proof. □

Remark 2.4. If we take $w(x) = 1$ in Theorem 2.3, then (2.2) reduces to (1.6).

Theorem 2.5. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $w : [a,b] \to [0,\infty)$ be a differentiable mapping and symmetric to $\frac{a+b}{2}$. If $|f'|^q$ is convex on $[a,b]$, $q > 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right| \\ &\leq (b-a)\left(\frac{1}{(b-a)^2}\int_{\frac{a+b}{2}}^b \left(x-\frac{a+b}{2}\right)w^p(x)dx\right)^{\frac{1}{p}} \\ &\times\left[\left(\frac{|f'(a)|^q + 2|f'(b)|^q}{24}\right)^{\frac{1}{q}} + \left(\frac{2|f'(a)|^q + |f'(b)|^q}{24}\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{2.6}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.1 and using change of the order of integration, we get

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right| \\ &\leq (b-a)\left\{\left[\int_0^{\frac{1}{2}}\left(\int_0^t w(as+(1-s)b)ds\right)|f'(ta+(1-t)b)|\,dt\right]\right. \\ &\left.+\left[\int_{\frac{1}{2}}^1\left(\int_t^1 w(as+(1-s)b)ds\right)|f'(ta+(1-t)b)|\,dt\right]\right\} \\ &= (b-a)\left\{\left[\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(as+(1-s)b)\,|f'(ta+(1-t)b)|\,dtds\right]\right. \\ &\left.+\left[\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^s w(as+(1-s)b)\,|f'(ta+(1-t)b)|\,dtds\right]\right\}. \end{aligned}$$

By Hölder's inequality, it follows that

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right| \\ &\leq (b-a)\left\{\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w^p(as+(1-s)b)dtds\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} |f'(ta+(1-t)b)|^q\,dtds\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^s w^p(as+(1-s)b)dtds\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^s |f'(ta+(1-t)b)|^q\,dtds\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Since $|f'|^q$ is convex on $[a,b]$, we know that for $t\in[0,1]$

$$|f'(ta+(1-t)b)|^q \le t\,|f'(a)|^q + (1-t)\,|f'(b)|^q,$$

hence

$$\left|\frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right|$$

$$\le (b-a)\left\{\left(\int_0^{\frac12}\int_s^{\frac12} w^p(as+(1-s)b)dtds\right)^{\frac1p}\left(\int_0^{\frac12}\int_s^{\frac12}\left(t\,|f'(a)|^q+(1-t)\,|f'(b)|^q\right)dtds\right)^{\frac1q}\right.$$

$$\left.+\left(\int_{\frac12}^1\int_{\frac12}^s w^p(as+(1-s)b)dtds\right)^{\frac1p}\left(\int_{\frac12}^1\int_{\frac12}^s\left(t\,|f'(a)|^q+(1-t)\,|f'(b)|^q\right)dtds\right)^{\frac1q}\right\}$$

$$= R_1 + R_2. \tag{2.7}$$

where $\frac1p+\frac1q=1$. Now, solving the above integrals with the elementary integrals, respectively, we obtain

$$R_1=\left(\frac{1}{2(b-a)^2}\int_{\frac{a+b}{2}}^b (2x-a-b)\,w^p(x)dx\right)^{\frac1p}\left(\frac{|f'(a)|^q+2\,|f'(b)|^q}{24}\right)^{\frac1q} \tag{2.8}$$

and

$$R_2=\left(\frac{1}{2(b-a)^2}\int_a^{\frac{a+b}{2}} (a+b-2x)\,w^p(x)dx\right)^{\frac1p}\left(\frac{2\,|f'(a)|^q+|f'(b)|^q}{24}\right)^{\frac1q}. \tag{2.9}$$

Since $w(x)$ is symmetric to $x=\frac{a+b}{2}$, we write

$$R_2=\left(\frac{1}{2(b-a)^2}\int_a^{\frac{a+b}{2}} (a+b-2x)\,w^p(a+b-x)dx\right)^{\frac1p}=R_1 \tag{2.10}$$

Using (2.8), (2.9) and (2.10), we get (2.6). Hence, the inequality (2.6) is proved. □

Now, we will give some new results connected with the right-hand side of (1.2) used the following Lemma:

Lemma 2.6. *Let $f: I^\circ\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$, and $w:[a,b]\to[0,\infty)$ be a differentiable mapping. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{1}{b-a}\frac{f(a)+f(b)}{2}\int_a^b w(x)dx-\frac{1}{b-a}\int_a^b f(x)w(x)dx=\frac{(b-a)}{2}\int_0^1 p(t)f'(ta+(1-t)b)dt \tag{2.11}$$

for each $t\in[0,1]$, where

$$p(t)=\int_t^1 w(as+(1-s)b)ds-\int_0^t w(as+(1-s)b)ds.$$

Proof. It suffices to note that

$$J = \int_0^1 p(t) f'(ta + (1-t)b) dt$$

$$= \int_0^1 \left(\int_t^1 w(as + (1-s)b) ds \right) f'(ta + (1-t)b) dt$$

$$+ \int_0^1 \left(- \int_0^t w(as + (1-s)b) ds \right) f'(ta + (1-t)b) dt$$

$$= J_1 + J_2.$$

By integration by parts, we get

$$J_1 = \left(\int_t^1 w(as + (1-s)b) ds \right) \frac{f(ta + (1-t)b)}{a-b} \Big|_0^1$$
$$+ \int_0^1 w(ta + (1-t)b) \frac{f(ta + (1-t)b)}{a-b} dt$$
$$= - \left(\int_0^1 w(as + (1-s)b) ds \right) \frac{f(b)}{a-b}$$
$$+ \int_0^1 w(ta + (1-t)b) \frac{f(ta + (1-t)b)}{a-b} dt,$$

and similarly

$$J_2 = - \left(\int_0^1 w(as + (1-s)b) ds \right) \frac{f(a)}{a-b} + \int_0^1 w(ta + (1-t)b) \frac{f(ta + (1-t)b)}{a-b} dt.$$

Thus, we can write

$$J = J_1 + J_2$$
$$= 2 \int_0^1 w(ta + (1-t)b) \frac{f(ta + (1-t)b)}{a-b} dt - \left(\int_0^1 w(as + (1-s)b) ds \right) \frac{f(a) + f(b)}{a-b}.$$

Using the change of the variable $x = ta + (1-t)b$ for $t \in [0, 1]$, and multiplying the both sides by $\frac{(b-a)}{2}$, we obtain (2.11), which completes the proof. □

Remark 2.7. If we take $w(x) = 1$ in Lemma 2.6, then (2.11) reduces to (1.3).

Theorem 2.8. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $w : [a, b] \to [0, \infty)$ be a differentiable mapping and symmetric to $\frac{a+b}{2}$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{1}{b-a} \frac{f(a) + f(b)}{2} \int_a^b w(x) dx - \frac{1}{b-a} \int_a^b f(x) w(x) dx \right|$$
$$\leq \frac{1}{2} \left[\int_0^1 (g(x))^p dt \right]^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \tag{2.12}$$

where $g(x) = \left| \int_{a+(b-a)t}^{b-(b-a)t} w(x) dx \right|$ *for* $t \in [0, 1]$.

Proof. From Lemma 2.6, we get

$$\left|\frac{1}{b-a}\frac{f(a)+f(b)}{2}\int_a^b w(x)dx-\frac{1}{b-a}\int_a^b f(x)w(x)dx\right|$$
$$\leq\frac{(b-a)}{2}\left[\int_0^1\left|\int_t^1 w(as+(1-s)b)ds-\int_0^t w(as+(1-s)b)ds\right|\left|f'(ta+(1-t)b)\right|dt\right]$$
$$\leq\frac{1}{2}\left[\int_0^1\left|\int_a^{b-(b-a)t} w(x)dx-\int_{b-(b-a)t}^b w(x)dx\right|\left|f'(ta+(1-t)b)\right|dt\right]. \tag{2.13}$$

Since $w(x)$ is symmetric to $x=\frac{a+b}{2}$, we write

$$\int_a^{b-(b-a)t} w(x)dx-\int_{b-(b-a)t}^b w(x)dx=\int_{a+(b-a)t}^{b-(b-a)t} w(x)dx, \tag{2.14}$$

for $t\in[0,\frac{1}{2}]$ and

$$\int_a^{b-(b-a)t} w(x)dx-\int_{b-(b-a)t}^b w(x)dx=-\int_{b-(b-a)t}^{a+(b-a)t} w(x)dx, \tag{2.15}$$

for $t\in[\frac{1}{2},1]$. If we write (2.14) and (2.15) in (2.13), we have

$$\left|\frac{1}{b-a}\frac{f(a)+f(b)}{2}\int_a^b w(x)dx-\frac{1}{b-a}\int_a^b f(x)w(x)dx\right|$$
$$\leq \frac{1}{2}\left[\int_0^1 g(x)\left|f'(ta+(1-t)b)\right|dt\right].$$

where $g(x)=\left|\int_{a+(b-a)t}^{b-(b-a)t} w(x)dx\right|$. By Hölder's inequality, it follows that

$$\left|\frac{1}{b-a}\frac{f(a)+f(b)}{2}\int_a^b w(x)dx-\frac{1}{b-a}\int_a^b f(x)w(x)dx\right|$$
$$\leq \frac{1}{2}\left[\int_0^1 (g(x))^p dt\right]^{\frac{1}{p}}\left[\int_0^1 \left|f'(ta+(1-t)b)\right|^q dt\right]^{\frac{1}{q}}.$$

Since $|f'|^q$ is convex on $[a,b]$, we know that for $t\in[0,1]$

$$\left|f'(ta+(1-t)b)\right|^q\leq t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q,$$

hence

$$\left|\frac{1}{b-a}\frac{f(a)+f(b)}{2}\int_a^b w(x)dx-\frac{1}{b-a}\int_a^b f(x)w(x)dx\right|$$
$$\leq\frac{1}{2}\left[\int_0^1 (g(x))^p dt\right]^{\frac{1}{p}}\left(\int_0^1\left(t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}$$
$$=\frac{1}{2}\left[\int_0^1 (g(x))^p dt\right]^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}$$

which completes the proof. □

Remark 2.9. If we take $w(x) = 1$ in Theorem 2.8, since

$$\int_0^1 \left(\left| \int_{a+(b-a)t}^{b-(b-a)t} dx \right| \right)^p dt = (b-a)^p \int_0^1 |1-2t|^p \, dt = \frac{(b-a)^p}{(p+1)},$$

(2.12) reduces to (1.4).

3. An application

Let d be a division $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of the interval $[a,b]$ and $\xi = (\xi_0, ..., \xi_{n-1})$ a sequence of intermediate points, $\xi_i \in [x_i, x_{i+1}]$, $i = \overline{0, n-1}$. Then the following result holds:

Theorem 3.1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, $f' \in L[a,b]$ and $w : [a,b] \to [0,\infty)$ be a differentiable mapping. If $|f'|$ is convex on $[a,b]$ then we have*

$$\int_a^b f(u)w(u)du = A(f,w,d,\xi) + R(f,w,d,\xi)$$

where

$$A(f,w,d,\xi) := \sum_{i=0}^{n-1} \frac{1}{(x_{i+1}-x_i)} f(\frac{x_i + x_{i+1}}{2}) \left(\int_{x_i}^{x_{i+1}} w(u)du \right).$$

The remainder $R(f,w,d,\xi)$ satisfies the estimation:

$$|R(f,f',d,\xi)|$$

$$\leq \sum_{i=0}^{n-1} \left[\frac{1}{(x_{i+1}-x_i)} \int_{\frac{x_i+x_{i+1}}{2}}^{x_{i+1}} w(u) \left[(u-x_i)^2 - (x_{i+1}-u)^2 \right] du \right] \left(\frac{|f'(x_i)| + |f'(x_{i+1})|}{2} \right) \tag{3.1}$$

for any choice ξ of the intermediate points.

Proof. Apply Theorem 2.3 on the interval $[x_i, x_{i+1}]$, $i = \overline{0, n-1}$ to get

$$\left| \int_{x_i}^{x_{i+1}} f(u)w(u)du - f(\frac{x_i + x_{i+1}}{2}) \left(\int_{x_i}^{x_{i+1}} w(u)du \right) \right|$$

$$\leq \left[\frac{1}{(x_{i+1}-x_i)} \int_{\frac{x_i+x_{i+1}}{2}}^{x_{i+1}} w(u) \left[(u-x_i)^2 - (x_{i+1}-u)^2 \right] du \right] \left(\frac{|f'(x_i)| + |f'(x_{i+1})|}{2} \right).$$

□

Summing the above inequalities over i from 0 to $n-1$ and using the generalized triangle inequality, we get the desired estimation (3.1).

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Approximation in statistical sense by $n-$multiple sequences of fuzzy positive linear operators

Kamil Demirci and Sevda Karakuş

Abstract. Our primary interest in the present paper is to prove a Korovkin-type approximation theorem for $n-$multiple sequences of fuzzy positive linear operators via statistical convergence. Also, we display an example such that our method of convergence is stronger than the usual convergence.

Mathematics Subject Classification (2010): 26E50, 40G15, 41A36.

Keywords: Statistical convergence for $n-$multiple sequences, fuzzy positive linear operators, fuzzy Korovkin theory.

1. Introduction

Anastassiou [3] first introduced the fuzzy analogue of the classical Korovkin theory (see also [1], [2], [5], [12]). Recently, some statistical fuzzy approximation theorems have been obtain by using the concept of statistical convergence (see, [6], [8]). The main motivation of this work is the paper introduced by Duman [9]. In this paper, we prove a Korovkin-type approximation theorem in algebraic and trigonometric case for $n-$multiple sequences of fuzzy positive linear operators defined on the space of all real valued n-variate fuzzy continuous functions on a compact subset of the real n-dimensional space via statistical convergence. Also, we display an example such that our method of convergence is stronger than the usual convergence.

We now recall some basic definitions and notations used in the paper.

A fuzzy number is a function $\mu : \mathbb{R} \to [0,1]$, which is normal, convex, upper semi-continuous and the closure of the set $supp(\mu)$ is compact, where

$$supp(\mu) := \{x \in \mathbb{R} : \ \mu(x) > 0\}.$$

The set of all fuzzy numbers are denoted by $\mathbb{R}_{\mathcal{F}}$. Let

$$[\mu]^0 = \overline{\{x \in \mathbb{R} : \ \mu(x) > 0\}} \text{ and } [\mu]^r = \{x \in \mathbb{R} : \ \mu(x) \geq r\}, \ (0 < r \leq 1).$$

Then, it is well-known [13] that, for each $r \in [0,1]$, the set $[\mu]^r$is a closed and bounded interval of $\mathbb{R}$. For any $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, it is possible to define uniquely the sum

$u \oplus v$ and the product $\lambda \odot u$ as follows:

$$[u \oplus v]^r = [u]^r + [v]^r \text{ and } [\lambda \odot u]^r = \lambda [u]^r, \ (0 \leq r \leq 1).$$

Now denote the interval $[u]^r$ by $\left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$ and $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$ for $r \in [0,1]$. Then, for $u, v \in \mathbb{R}_{\mathcal{F}}$, define

$$u \preceq v \Leftrightarrow u_-^{(r)} \leq v_-^{(r)} \text{ and } u_+^{(r)} \leq v_+^{(r)} \text{ for all } 0 \leq r \leq 1.$$

Define also the following metric $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$ by

$$D(u,v) = \sup_{r \in [0,1]} \ \max \ \left\{ \left| u_-^{(r)} - v_-^{(r)} \right|, \left| u_+^{(r)} - v_+^{(r)} \right| \right\}$$

(see, for details [3]). Hence, $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space [18].

The concept of statistical convergence was introduced by ([10]). A sequence $x = (x_m)$ of real numbers is said to be statistical convergent to some finite number L, if for every $\varepsilon > 0$,

$$\lim_{k \to \infty} \frac{1}{k} \left| \{ m \leq k : \ |x_m - L| \geq \varepsilon \} \right| = 0,$$

where by $m \leq k$ we mean that $m = 1, 2, ..., k$; and by $|B|$ we mean the cardinality of the set $B \subseteq \mathbb{N}$, the set of natural numbers. We recall ([16], p. 290) that "natural (or asymptotic) density" of a set $B \subseteq \mathbb{N}$ is defined by

$$\delta(B) := \lim_{k \to \infty} \frac{1}{k} \left| \{ m \leq k : m \in B \} \right|,$$

provided that the limit on the right-hand side exists. It is clear that a set $B \subseteq \mathbb{N}$ has natural density 0 if and only if complement $B^c := \mathbb{N} \setminus B$ has natural density 1. Some basic properties of statistical convergence may be found in ([7], [11], [17]). These basic properties of statistical convergence were extended to $n-$multiple sequences by ([14], [15]). Let $\mathbb{N}^n$ be the set of $n-$tuples $\mathbf{m} := (m_1, m_2, ..., m_n)$ with non-negative integers for coordinates m_j, where n is a fixed positive integer. Two tuples $\mathbf{m}$ and $\mathbf{k} := (k_1, k_2, ..., k_n)$ are distinct if and only if $m_j \neq k_j$ for at least one j. $\mathbb{N}^n$ is partially ordered by agreeing that $\mathbf{m} \leq \mathbf{k}$ if and only if $m_j \leq k_j$ for each j.

We say that a $n-$multiple sequence $(x_{\mathbf{m}}) = (x_{m_1, m_2, ..., m_n})$ of real numbers is statistically convergent to some number L if for every $\varepsilon > 0$,

$$\lim_{\min \ k_j \to \infty} \frac{1}{|\mathbf{k}|} \left| \{ \mathbf{m} \leq \mathbf{k} : \ |x_{\mathbf{m}} - L| \geq \varepsilon \} \right| = 0,$$

where $|\mathbf{k}| := \prod_{j=1}^{n} (k_j)$. In this case, we write $st - \lim x_{\mathbf{m}} = L$. The "natural (or asymptotic) density" of a set $B \subseteq \mathbb{N}^n$ can be defined as follows:

$$\delta(B) := \lim_{\min \ k_j \to \infty} \frac{1}{|\mathbf{k}|} \left| \{ \mathbf{m} \leq \mathbf{k} : \ \mathbf{m} \in B \} \ \right|,$$

provided that this limit exists ([14]).

2. Statistical fuzzy Korovkin theory

Let the real numbers $a_i; b_i$ so that $a_i < b_i$, for each $i = \overline{1, n}$ and

$$U := [a_1; b_1] \times [a_2; b_2] \times ... \times [a_n; b_n].$$

Let $C(U)$ denote the space of all real valued continuous functions on U endowed with the supremum norm

$$\|f\| = \sup_{\mathbf{x} \in U} |f(\mathbf{x})|, \ (f \in C(U)).$$

Assume that $f : U \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy number valued function. Then f is said to be fuzzy continuous at $\mathbf{x}^0 := (x_1^0, x_2^0, x_3^0, ..., x_n^0) \in U$ whenever $\lim_{\mathbf{m}} x_{\mathbf{m}} = \mathbf{x}^0$, then $\lim_{\mathbf{m}} D(f(x_{\mathbf{m}}), f(\mathbf{x}^0)) = 0$. If it is fuzzy continuous at every point $\mathbf{x} \in U$, we say that f is fuzzy continuous on U. The set of all fuzzy continuous functions on U is denoted by $C_{\mathcal{F}}(U)$. Now let $L : C_{\mathcal{F}}(U) \to C_{\mathcal{F}}(U)$ be an operator. Then L is said to be fuzzy linear if, for every $\lambda_1, \lambda_2 \in \mathbb{R}$ having the same sing and for every $f_1, f_2 \in C_{\mathcal{F}}(U)$, and $\mathbf{x} \in U$,

$$L(\lambda_1 \odot f_1 \oplus \lambda_2 \odot f_2; \mathbf{x}) = \lambda_1 \odot L(f_1; \mathbf{x}) \oplus \lambda_2 \odot L(f_2; \mathbf{x})$$

holds. Also L is called fuzzy positive linear operator if it is fuzzy linear and, the condition $L(f; \mathbf{x}) \preceq L(g; \mathbf{x})$ is satisfied for any $f, g \in C_{\mathcal{F}}(U)$ and all $\mathbf{x} \in U$ with $f(\mathbf{x}) \preceq g(\mathbf{x})$. Also, if $f, g : U \to \mathbb{R}_{\mathcal{F}}$ are fuzzy number valued functions, then the distance between f and g is given by

$$D^*(f, g) = \sup_{\mathbf{x} \in U} \sup_{r \in [0,1]} \max \left\{ \left| f_-^{(r)} - g_-^{(r)} \right|, \left| f_+^{(r)} - g_+^{(r)} \right| \right\}$$

(see for details, [1], [2], [3], [5], [9], [12]). Throughout the paper we use the test functions given by

$$f_0(\mathbf{x}) = 1, \ f_i(\mathbf{x}) = x_i, \ f_{n+i}(\mathbf{x}) = x_i^2, \ i = \overline{1, n}.$$

Theorem 2.1. *Let $\{L_{\mathbf{m}}\}_{\mathbf{m} \in \mathbb{N}^n}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}(U)$ into itself. Assume that there exists a corresponding sequence $\left\{ \widetilde{L}_{\mathbf{m}} \right\}_{\mathbf{m} \in \mathbb{N}^n}$ of positive linear operators from $C(U)$ into itself with the property*

$$\{L_{\mathbf{m}}(f; \mathbf{x})\}_{\pm}^{(r)} = \widetilde{L}_{\mathbf{m}} \left(f_{\pm}^{(r)}; \mathbf{x} \right) \tag{2.1}$$

for all $\mathbf{x} \in U$, $r \in [0, 1]$, $\mathbf{m} \in \mathbb{N}^n$ and $f \in C_{\mathcal{F}}(U)$. Assume further that

$$st - \lim_{\mathbf{m}} \left\| \widetilde{L}_{\mathbf{m}}(f_i) - f_i \right\| = 0 \ \ \textit{for each } i = \overline{0, 2n}. \tag{2.2}$$

Then, for all $f \in C_{\mathcal{F}}(U)$, we have

$$st - \lim_{\mathbf{m}} D^*(L_{\mathbf{m}}(f), f) = 0.$$

Proof. Let $f \in C_{\mathcal{F}}(U)$, $\mathbf{x} = (x_1, ..., x_n) \in U$ and $r \in [0,1]$. By the hypothesis, since $f_{\pm}^{(r)} \in C(U)$, we can write, for every $\varepsilon > 0$, that there exists a number $\delta > 0$ such that $\left| f_{\pm}^{(r)}(\mathbf{u}) - f_{\pm}^{(r)}(\mathbf{x}) \right| < \varepsilon$ holds for every $\mathbf{u} = (u_1 ..., u_n) \in U$ satisfying

$$|\mathbf{u} - \mathbf{x}| := \sqrt{\sum_{i=1}^{n} (u_i - x_i)^2} < \delta.$$

Then we immediately get for all $\mathbf{u} \in U$, that

$$\left| f_{\pm}^{(r)}(\mathbf{u}) - f_{\pm}^{(r)}(\mathbf{x}) \right| \leq \varepsilon + \frac{2M_{\pm}^{(r)}}{\delta^2} \sum_{i=1}^{n} (u_i - x_i)^2,$$

where $M_{\pm}^{(r)} := \left\| f_{\pm}^{(r)} \right\|$. Now, using the linearity and the positivity of the operators $\widetilde{L}_{\mathbf{m}}$, we have, for each $\mathbf{m} \in \mathbb{N}^n$, that

$$\begin{aligned} & \left| \widetilde{L}_{\mathbf{m}}\left(f_{\pm}^{(r)}; \mathbf{x}\right) - f_{\pm}^{(r)}(\mathbf{x}) \right| \\ \leq\ & \varepsilon + \left(\varepsilon + M_{\pm}^{(r)} + \frac{2M_{\pm}^{(r)}}{\delta^2} \sum_{i=1}^{n} x_i^2 \right) \left| \widetilde{L}_{\mathbf{m}}(f_0; \mathbf{x}) - f_0(\mathbf{x}) \right| \\ & + \frac{2M_{\pm}^{(r)}}{\delta^2} \sum_{i=1}^{n} \left\{ \left| \widetilde{L}_{\mathbf{m}}\left(u_i^2; \mathbf{x}\right) - x_i^2 \right| + 2c \left| \widetilde{L}_{\mathbf{m}}(u_i; \mathbf{x}) - x_i \right| \right\} \end{aligned}$$

where $c := \max_{1 \leq i \leq n} \{|a_i|, |b_i|\}$. The last inequality gives that

$$\left| \widetilde{L}_{\mathbf{m}}\left(f_{\pm}^{(r)}; \mathbf{x}\right) - f_{\pm}^{(r)}(\mathbf{x}) \right| \leq \varepsilon + K_{\pm}^{(r)}(\varepsilon) \sum_{i=0}^{2n} \left| \widetilde{L}_{\mathbf{m}}(f_i; \mathbf{x}) - f_i(\mathbf{x}) \right|$$

where $K_{\pm}^{(r)}(\varepsilon) := \max \left\{ \varepsilon + M_{\pm}^{(r)} + \frac{2M_{\pm}^{(r)}}{\delta^2} A, \frac{4M_{\pm}^{(r)}}{\delta^2} c, \frac{2M_{\pm}^{(r)}}{\delta^2} \right\}$ and $A := \sum_{i=1}^{n} x_i^2$ for $x_i \in [a_i, b_i]$, $(i = 1, 2, ...n)$. Also taking supremum over $\mathbf{x} = (x_1 ..., x_n) \in U$, the above inequality implies that

$$\left\| \widetilde{L}_{\mathbf{m}}\left(f_{\pm}^{(r)}\right) - f_{\pm}^{(r)} \right\| \leq \varepsilon + K_{\pm}^{(r)}(\varepsilon) \sum_{i=0}^{2n} \left\| \widetilde{L}_{\mathbf{m}}(f_i) - f_i \right\| \tag{2.3}$$

Now, it follows from (2.1) that

$$\begin{aligned} & D^*(L_{\mathbf{m}}(f), f) \\ =\ & \sup_{\mathbf{x} \in U} \sup_{r \in [0,1]} \max \left\{ \left| \widetilde{L}_{\mathbf{m}}\left(f_{-}^{(r)}; \mathbf{x}\right) - f_{-}^{(r)}(\mathbf{x}) \right|, \left| \widetilde{L}_{\mathbf{m}}\left(f_{+}^{(r)}; \mathbf{x}\right) - f_{+}^{(r)}(\mathbf{x}) \right| \right\} \\ =\ & \sup_{r \in [0,1]} \max \left\{ \left\| \widetilde{L}_{\mathbf{m}}\left(f_{-}^{(r)}\right) - f_{-}^{(r)} \right\|, \left\| \widetilde{L}_{\mathbf{m}}\left(f_{+}^{(r)}\right) - f_{+}^{(r)} \right\| \right\}. \end{aligned}$$

Combining the above equality with (2.3), we have

$$D^*\left(L_{\mathbf{m}}(f), f\right) \leq \varepsilon + K(\varepsilon) \sum_{i=0}^{2n} \left\| \widetilde{L}_{\mathbf{m}}(f_i) - f_i \right\| \tag{2.4}$$

where $K(\varepsilon) := \sup_{r\in[0,1]} \max\left\{K_-^{(r)}(\varepsilon), K_+^{(r)}(\varepsilon)\right\}$.

Now, for a given $r > 0$, choose $\varepsilon > 0$ such that $0 < \varepsilon < r$, and also define the following sets:

$$\begin{aligned} G &:= \left\{\mathbf{m} \in \mathbb{N}^n : D^*\left(L_{\mathbf{m}}(f), f\right) \geq r\right\}, \\ G_i &:= \left\{\mathbf{m} \in \mathbb{N}^n : \left\| \widetilde{L}_{\mathbf{m}}(f_i) - f_i \right\| \geq \frac{r-\varepsilon}{(2n+1)K(\varepsilon)}\right\}, \qquad i = \overline{0, 2n}. \end{aligned}$$

Hence, inequality (2.4) yields that

$$G \subset \bigcup_{i=0}^{2n} G_i$$

which gives,

$$\begin{aligned} &\lim_{\min\ k_j\to\infty} \frac{1}{|\mathbf{k}|} \left|\left\{\mathbf{m} \leq \mathbf{k} : D^*\left(L_{\mathbf{m}}(f), f\right) \geq r\right\}\right| \\ \leq &\lim_{\min\ k_j\to\infty} \frac{1}{|\mathbf{k}|} \left|\left\{\mathbf{m} \leq \mathbf{k} : \left\| \widetilde{L}_{\mathbf{m}}(f_i) - f_i \right\| \geq \frac{r-\varepsilon}{(2n+1)K(\varepsilon)}\right\}\right|, \qquad i = \overline{0, 2n}. \end{aligned}$$

From the hypothesis (2.2), we get

$$\lim_{\min\ k_j\to\infty} \frac{1}{|\mathbf{k}|} \left|\left\{\mathbf{m} \leq \mathbf{k} : D^*\left(L_{\mathbf{m}}(f), f\right) \geq r\right\}\right| = 0.$$

So, the proof is completed. □

If $n = 1$**,** then Theorem 2.1 reduces to result of [6].

Theorem 2.2. *Let* $\{L_m\}_{m\in\mathbb{N}}$ *be a sequence of fuzzy positive linear operators from* $C_{\mathcal{F}}(U)$ *into itself. Assume that there exists a corresponding sequence* $\left\{\widetilde{L}_m\right\}_{m\in\mathbb{N}}$ *of positive linear operators from* $C(U)$ *into itself with the property* (2.1)*. Assume further that*

$$st - \lim_m \left\| \widetilde{L}_m(f_i) - f_i \right\| = 0 \quad \textit{for each } i = 0, 1, 2.$$

Then, for all $f \in C_{\mathcal{F}}(U)$*, we have*

$$st - \lim_m D^*\left(L_m(f),\ f\right) = 0.$$

If $n = 2$**,** then Theorem 2.1 reduces to new result in classical case.

Theorem 2.3. *Let* $\{L_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{N}^2}$ *be a sequence of fuzzy positive linear operators from* $C_{\mathcal{F}}(U)$ *into itself. Assume that there exists a corresponding sequence* $\left\{\widetilde{L}_{\mathbf{m}}\right\}_{\mathbf{m}\in\mathbb{N}^2}$ *of* *positive linear operators from $C(U)$ into itself with the property (2.1). Assume further that*

$$\lim_{\mathbf{m}} \left\| \widetilde{L}_{\mathbf{m}}(f_i) - f_i \right\| = 0 \quad \text{for each } i = 0,1,2,3,4.$$

Then, for all $f \in C_{\mathcal{F}}(U)$, we have

$$\lim_{\mathbf{m}} D^* \left(L_{\mathbf{m}}(f), \ f \right) = 0.$$

We now show that Theorem 2.1 stronger than Theorem 2.3.

Example 2.4. Let $n = 2$, $U := [0,1] \times [0,1]$ *and define the double sequence* $(u_{\mathbf{m}})$ *by*

$$u_{\mathbf{m}} = \begin{cases} \sqrt{m_1 m_2}, & \text{if } m_1 \text{ and } m_2 \text{ are square,} \\ 0, & \textit{otherwise.} \end{cases}$$

We observe that, $st - \lim_{\mathbf{m}} u_{\mathbf{m}} = 0$. But $(u_{\mathbf{m}})$ is neither convergent nor bounded. Then consider the Fuzzy Bernstein-type polynomials as follows:

$$\begin{aligned} B_{\mathbf{m}}^{(\mathcal{F})}(f;\mathbf{x}) &= (1+u_{\mathbf{m}}) \odot \bigoplus_{s=0}^{m_1} \odot \bigoplus_{t=0}^{m_2} \tbinom{m_1}{s}\tbinom{m_2}{t} x_1^s x_2^t (1-x_1)^{m_1-s}(1-x_2)^{m_2-t} \\ &\quad \odot f\left(\tfrac{s}{m_1}, \tfrac{t}{m_2}\right), \end{aligned} \tag{2.5}$$

where $f \in C_{\mathcal{F}}(U)$, $\mathbf{x} = (x_1, x_2) \in U$, $\mathbf{m} \in \mathbb{N}^2$. In this case, we write

$$\begin{aligned} \left\{ B_{\mathbf{m}}^{(\mathcal{F})}(f;\mathbf{x}) \right\}_{\pm}^{(r)} &= \widetilde{B}_{\mathbf{m}}\left(f_{\pm}^{(r)};\mathbf{x}\right) \\ &= (1+u_{\mathbf{m}}) \sum_{s=0}^{m_1}\sum_{t=0}^{m_2} \binom{m_1}{s}\binom{m_2}{t} x_1^s x_2^t (1-x_1)^{m_1-s}(1-x_2)^{m_2-t} \\ &\qquad f_{\pm}^{(r)}\left(\frac{s}{m_1}, \frac{t}{m_2}\right), \end{aligned}$$

where $f_{\pm}^{(r)} \in C(U)$. Then, we get

$$\begin{aligned} \widetilde{B}_{\mathbf{m}}(f_0;\mathbf{x}) &= (1+u_{\mathbf{m}}) f_0(\mathbf{x}), \\ \widetilde{B}_{\mathbf{m}}(f_1;\mathbf{x}) &= (1+u_{\mathbf{m}}) f_1(\mathbf{x}), \\ \widetilde{B}_{\mathbf{m}}(f_2;\mathbf{x}) &= (1+u_{\mathbf{m}}) f_2(\mathbf{x}), \\ \widetilde{B}_{\mathbf{m}}(f_3;\mathbf{x}) &= (1+u_{\mathbf{m}}) \left(f_3(\mathbf{x}) + \frac{x_1 - x_1^2}{m_1} \right) \\ \widetilde{B}_{\mathbf{m}}(f_4;\mathbf{x}) &= (1+u_{\mathbf{m}}) \left(f_4(\mathbf{x}) + \frac{x_2 - x_2^2}{m_2} \right). \end{aligned}$$

So we conclude that

$$st - \lim_{\mathbf{m}} \left\| \widetilde{B}_{\mathbf{m}}(f_i) - f_i \right\| = 0 \quad \text{for each } i = 0,1,2,3,4.$$

By Theorem 2.1, we obtain for all $f \in C_{\mathcal{F}}(U)$, that

$$st - \lim_{\mathbf{m}} D^* \left(B_{\mathbf{m}}^{(\mathcal{F})}(f), \ f \right) = 0.$$

However, since the sequence $(u_{\mathbf{m}})$ *is not convergent, we conclude that Theorem* 2.3 *do not work for the operators* $\left\{B_{\mathbf{m}}^{(\mathcal{F})}(f;\mathbf{x})\right\}$ *in (*2.5*) while our Theorem* 2.1 *still works.*

Remark 2.5. Let $C_{2\pi}(\mathbb{R}^n)$ denote the space of all real valued continuous and 2π-periodic functions on $\mathbb{R}^n$, $(n \in \mathbb{N})$. By $C_{2\pi}^{\mathcal{F}}(\mathbb{R}^n)$ we denote the space of all fuzzy continuous and 2π-periodic functions on $\mathbb{R}^n$. (see for details [4]). If we use the following test functions

$$f_0(\mathbf{x}) = 1, \;\; f_i(\mathbf{x}) = \cos x_i, \; f_{n+i}(\mathbf{x}) = \sin x_i, \;\; i = \overline{1,n},$$

then the proof of Theorem 2.1 can easily be modified to trigonometric case.

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Generalized Salagean-type harmonic univalent functions

Elif Yaşar and Sibel Yalçın

Abstract. The main purpose of this paper is to introduce a generalization of modified Salagean operator for harmonic univalent functions. We define a new subclass of complex-valued harmonic univalent functions by using this operator ,and we investigate necessary and sufficient coefficient conditions, distortion bounds, extreme points and convex combination for the above class of harmonic univalent functions.

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Keywords: Harmonic, univalent, starlike, convex, modified Salagean operator.

1. Introduction

Let H denote the family of continuous complex valued harmonic functions which are harmonic in the open unit disk $U = \{z : |z| < 1\}$ and let A be the subclass of H consisting of functions which are analytic in U. A function harmonic in U may be written as $f = h + \overline{g}$, where h and g are members of A. In this case, f is sense-preserving if $|h'(z)| > |g'(z)|$ in U. See Clunie and Sheil-Small [2]. To this end, without loss of generality, we may write

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k. \tag{1.1}$$

One shows easily that the sense-preserving property implies that $|b_1| < 1$.

Let SH denote the family of functions $f = h + \overline{g}$ which are harmonic, univalent, and sense-preserving in U for which $f(0) = f_z(0) - 1 = 0$.

For the harmonic function $f = h + \overline{g}$, we call h the analytic part and g the co-analytic part of f. Note that SH reduces to the class S of normalized analytic univalent functions in U if the co-analytic part of f is identically zero.

In 1984 Clunie and Sheil-Small [2] investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been

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several related papers on SH and its subclasses such as Avcı and Zlotkiewicz [1], Silverman [7], Silverman and Silvia [8], Jahangiri [3] studied the harmonic univalent functions.

The differential operator D^n $(n \in \mathbb{N}_0)$ was introduced by Salagean [6]. For $f = h + \overline{g}$ given by (1.1), Jahangiri et al. [4] defined the modified Salagean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)},$$

where

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad \text{and} \quad D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k.$$

For $f = h + \overline{g}$ given by (1.1), we define generalization of the modified Salagean operator of f:

$$D^0_\lambda f(z) = D^0 f(z) = h(z) + \overline{g(z)},$$

$$D^1_\lambda f(z) = (1-\lambda) D^0 f(z) + \lambda D^1 f(z), \quad \lambda \geq 0, \tag{1.2}$$

$$D^n_\lambda f(z) = D^1_\lambda \left(D^{n-1}_\lambda f(z) \right). \tag{1.3}$$

If f is given by (1.1) , then from (1.2) and (1.3) we see that

$$D^n_\lambda f(z) = z + \sum_{k=2}^{\infty} \left[\lambda(k-1)+1\right]^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} \left[\lambda(k+1)-1\right]^n \overline{b_k z^k}. \tag{1.4}$$

When $\lambda = 1$, we get modified Salagean differential operator [4]. If we take the co-analytic part of $f = h + \overline{g}$ of the form (1.1) is identically zero, $D^n_\lambda f$ reduces to the Al-Oboudi operator [5].

Denote by $SH(\lambda, n, \alpha)$ the subclass of SH consisting of functions f of the form (1.1) that satisfy the condition

$$\operatorname{Re}\left(\frac{D^{n+1}_\lambda f(z)}{D^n_\lambda f(z)} \right) \geq \alpha, \quad 0 \leq \alpha < 1 \tag{1.5}$$

where $D^n_\lambda f(z)$ is defined by (1.4).

We let the subclass $\overline{SH}(\lambda, n, \alpha)$ consisting of harmonic functions $f_n = h + \overline{g}_n$ in SH so that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \; g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad a_k, \; b_k \geq 0. \tag{1.6}$$

By suitably specializing the parametres, the classes $SH(\lambda, n, \alpha)$ reduces to the various subclasses of harmonic univalent functions. Such as,

(i) $SH(1,0,0) = SH^*(0)$ (Avcı [1], Silverman [7], Silverman and Silvia [8]),

(ii) $SH(1,0,\alpha) = SH^*(\alpha)$ (Jahangiri [3]),

(iii) $SH(1,1,0) = KH(0)$ (Avcı [1], Silverman [7], Silverman and Silvia [8]),

(iv) $SH(1,1,\alpha) = KH(\alpha)$ (Jahangiri [3]),

(v) $SH(1,n,\alpha) = H(n,\alpha)$ (Jahangiri et al. [4]).

The object of the present paper is to investigate the various properties of harmonic univalent functions belonging to the subclass $SH(\lambda, n, \alpha)$. We extend the results of [4], by generalizing the operator.

2. Main results

Theorem 2.1. *Let* $f = h + \overline{g}$ *be so that* h *and* g *are given by* (1.1). *Furthermore, let*

$$\sum_{k=2}^{\infty} [\lambda(k-1)+1]^n [\lambda(k-1)+1-\alpha] |a_k|$$

$$+\sum_{k=1}^{\infty} [\lambda(k+1)-1]^n [\lambda(k+1)-1+\alpha] |b_k| \leq 1-\alpha, \tag{2.1}$$

where $\lambda \geq 1$, $n \in \mathbb{N}_0$, $0 \leq \alpha < 1$. *Then* f *is sense-preserving, harmonic univalent in* U *and* $f \in SH(\lambda, n, \alpha)$.

Proof. If $z_1 \neq z_2$,

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1 - \left|\frac{\sum_{k=1}^{\infty} b_k \left(z_1^k - z_2^k\right)}{(z_1-z_2)+\sum_{k=2}^{\infty} a_k \left(z_1^k - z_2^k\right)}\right|$$

$$> 1 - \frac{\sum_{k=1}^{\infty} k\,|b_k|}{1-\sum_{k=2}^{\infty} k\,|a_k|} \geq 1 - \frac{\sum_{k=1}^{\infty} \frac{[\lambda(k+1)-1]^n [\lambda(k+1)-1+\alpha]}{1-\alpha} |b_k|}{1-\sum_{k=2}^{\infty} \frac{[\lambda(k-1)+1]^n [\lambda(k-1)+1-\alpha]}{1-\alpha} |a_k|} \geq 0,$$

which proves univalence. Note that f is sense preserving in U. This is because

$$\begin{aligned} |h'(z)| &\geq 1 - \sum_{k=2}^{\infty} k\,|a_k|\,|z|^{k-1} > 1 - \sum_{k=2}^{\infty} \frac{[\lambda(k-1)+1]^n [\lambda(k-1)+1-\alpha]}{1-\alpha} |a_k| \\ &\geq \sum_{k=1}^{\infty} \frac{[\lambda(k+1)-1]^n [\lambda(k+1)-1+\alpha]}{1-\alpha} |b_k| > \sum_{k=1}^{\infty} k\,|b_k|\,|z|^{k-1} \geq |g'(z)|\,. \end{aligned}$$

Using the fact that $\mathrm{Re} w \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$$\left|(1-\alpha)D_\lambda^n f(z) + D_\lambda^{n+1} f(z)\right| - \left|(1+\alpha)D_\lambda^n f(z) - D_\lambda^{n+1} f(z)\right| \geq 0. \tag{2.2}$$

Substituting for $D_\lambda^{n+1} f(z)$ and $D_\lambda^n f(z)$ in (2.2), we obtain

$$\left|(1-\alpha)D_\lambda^n f(z) + D_\lambda^{n+1} f(z)\right| - \left|(1+\alpha)D_\lambda^n f(z) - D_\lambda^{n+1} f(z)\right|$$

$$\geq 2(1-\alpha)\,|z| - \sum_{k=2}^{\infty} [\lambda(k-1)+1]^n [\lambda(k-1)+2-\alpha] |a_k|\,|z|^k$$

$$-\sum_{k=1}^{\infty} [\lambda(k+1)-1]^n [\lambda(k+1)-2+\alpha] |b_k|\,|z|^k$$

$$-\sum_{k=2}^{\infty} [\lambda(k-1)+1]^n [\lambda(k-1)-\alpha] |a_k|\,|z|^k$$

$$-\sum_{k=1}^{\infty}\left[\lambda(k+1)-1\right]^{n}\left[\lambda(k+1)+\alpha\right]|b_k|\,|z|^{k}$$

$$\geq 2(1-\alpha)\,|z|\left(1-\sum_{k=2}^{\infty}\frac{\left[\lambda(k-1)+1\right]^{n}\left[\lambda(k-1)+1-\alpha\right]}{1-\alpha}|a_k|\right.$$

$$\left.-\sum_{k=1}^{\infty}\frac{\left[\lambda(k+1)-1\right]^{n}\left[\lambda(k+1)-1+\alpha\right]}{1-\alpha}|b_k|\right).$$

This last expression is non-negative by (2.1), and so the proof is complete. □

Theorem 2.2. *Let* $f_n = h + \overline{g}_n$ *be given by (*1.6*). Then* $f_n \in \overline{SH}(\lambda, n, \alpha)$ *if and only if*

$$\sum_{k=2}^{\infty}\left[\lambda(k-1)+1\right]^{n}\left[\lambda(k-1)+1-\alpha\right]a_k$$

$$+\sum_{k=1}^{\infty}\left[\lambda(k+1)-1\right]^{n}\left[\lambda(k+1)-1+\alpha\right]b_k \leq 1-\alpha, \tag{2.3}$$

where $\lambda \geq 1$, $n \in \mathbb{N}_0$, $0 \leq \alpha < 1$.

Proof. The "if" part follows from Theorem 2.1 upon noting that $\overline{SH}(\lambda, n, \alpha) \subset SH(\lambda, n, \alpha)$. For the "only if" part, we show that $f \notin \overline{SH}(\lambda, n, \alpha)$ if the condition (2.3) does not hold. Note that a necessary and sufficient condition for $f_n = h + \overline{g}_n$ given by (1.6), to be in $\overline{SH}(\lambda, n, \alpha)$ is that the condition (1.5) to be satisfied. This is equivalent to

$$\operatorname{Re}\left\{\frac{(1-\alpha)z-\sum_{k=2}^{\infty}\left[\lambda(k-1)+1\right]^{n}\left[\lambda(k-1)+1-\alpha\right]a_k z^k}{z-\sum_{k=2}^{\infty}\left[\lambda(k-1)+1\right]^{n}a_k z^k+\sum_{k=1}^{\infty}\left[\lambda(k+1)-1\right]^{n}b_k\overline{z}^k}\right.$$

$$\left.\frac{-\sum_{k=1}^{\infty}\left[\lambda(k+1)-1\right]^{n}\left[\lambda(k+1)-1+\alpha\right]b_k\overline{z}^k}{z-\sum_{k=2}^{\infty}\left[\lambda(k-1)+1\right]^{n}a_k z^k+\sum_{k=1}^{\infty}\left[\lambda(k+1)-1\right]^{n}b_k\overline{z}^k}\right\} \geq 0. \tag{2.4}$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$ we must have

$$\frac{(1-\alpha) - \sum_{k=2}^{\infty} [\lambda(k-1)+1]^n [\lambda(k-1)+1-\alpha] a_k r^{k-1}}{1 - \sum_{k=2}^{\infty} [\lambda(k-1)+1]^n a_k r^{k-1} + \sum_{k=1}^{\infty} [\lambda(k+1)-1]^n b_k r^{k-1}}$$

$$\frac{-\sum_{k=1}^{\infty} [\lambda(k+1)-1]^n [\lambda(k+1)-1+\alpha] b_k r^{k-1}}{1 - \sum_{k=2}^{\infty} [\lambda(k-1)+1]^n a_k r^{k-1} + \sum_{k=1}^{\infty} [\lambda(k+1)-1]^n b_k r^{k-1}} \geq 0. \qquad (2.5)$$

If the condition (2.3) does not hold, then the numerator in (2.5) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0,1)$ for which the quotient in (2.5) is negative. This contradicts the required condition for $f_n \in \overline{SH}(\lambda, n, \alpha)$ and so the proof is complete. □

Theorem 2.3. *Let f_n be given by (1.6). Then $f_n \in \overline{SH}(\lambda, n, \alpha)$ if and only if*

$$f_n(z) = \sum_{k=1}^{\infty} \left(X_k h_k(z) + Y_k g_{n_k}(z) \right),$$

where $h_1(z) = z$,

$$h_k(z) = z - \frac{1-\alpha}{[\lambda(k-1)+1]^n [\lambda(k-1)+1-\alpha]} z^k \ (k = 2, 3, ...),$$

$$g_{n_k}(z) = z + (-1)^n \frac{1-\alpha}{[\lambda(k+1)-1]^n [\lambda(k+1)-1+\alpha]} \overline{z}^k \ (k = 1, 2, 3, ...),$$

$$\sum_{k=1}^{\infty} (X_k + Y_k) = 1, \ X_k \geq 0, \ Y_k \geq 0.$$

In particular, the extreme points of $\overline{SH}(\lambda, n, \alpha)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof. For functions f_n of the form (1.6) we have

$$\begin{aligned} f_n(z) &= \sum_{k=1}^{\infty} \left(X_k h_k(z) + Y_k g_{n_k}(z) \right) \\ &= \sum_{k=1}^{\infty} (X_k + Y_k) z - \sum_{k=2}^{\infty} \frac{1-\alpha}{[\lambda(k-1)+1]^n [\lambda(k-1)+1-\alpha]} X_k z^k \\ &\quad + (-1)^n \sum_{k=1}^{\infty} \frac{1-\alpha}{[\lambda(k+1)-1]^n [\lambda(k+1)-1+\alpha]} Y_k \overline{z}^k. \end{aligned}$$

Then

$$\sum_{k=2}^{\infty}\frac{[\lambda(k-1)+1]^n\,[\lambda(k-1)+1-\alpha]}{1-\alpha}\left(\frac{1-\alpha}{[\lambda(k-1)+1]^n\,[\lambda(k-1)+1-\alpha]}X_k\right)$$
$$+\sum_{k=1}^{\infty}\frac{[\lambda(k+1)-1]^n\,[\lambda(k+1)-1+\alpha]}{1-\alpha}\left(\frac{1-\alpha}{[\lambda(k+1)-1]^n\,[\lambda(k+1)-1+\alpha]}Y_k\right)$$
$$=\sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k=1-X_1\le 1,\ \text{and so } f_n\in\overline{SH}(\lambda,n,\alpha).$$

Conversely, if $f_n\in\overline{SH}(\lambda,n,\alpha)$, then

$$a_k\le\frac{1-\alpha}{[\lambda(k-1)+1]^n\,[\lambda(k-1)+1-\alpha]}$$

and

$$b_k\le\frac{1-\alpha}{[\lambda(k+1)-1]^n\,[\lambda(k+1)-1+\alpha]}.$$

Set

$$X_k=\frac{[\lambda(k-1)+1]^n\,[\lambda(k-1)+1-\alpha]}{1-\alpha}a_k,\ (k=2,3,...)$$

$$Y_k=\frac{[\lambda(k+1)-1]^n\,[\lambda(k+1)-1+\alpha]}{1-\alpha}b_k,\ (k=1,2,3,...)$$

and

$$X_1=1-\left(\sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k\right)$$

where $X_1\ge 0$. Then, as required, we obtain

$$f_n(z)=X_1z+\sum_{k=2}^{\infty}X_kh_k(z)+\sum_{k=1}^{\infty}Y_kg_{n_k}(z).$$

□

Theorem 2.4. *Let $f_n\in\overline{SH}(\lambda,n,\alpha)$. Then for $|z|=r<1$ and $\lambda\ge 1$ we have*

$$\begin{aligned}|f_n(z)| &\le (1+b_1)\,r\\ &\quad+\left(\frac{(1-\alpha)}{(\lambda+1)^n\,(\lambda+1-\alpha)}-\frac{(2\lambda-1)^n\,(2\lambda-1+\alpha)}{(\lambda+1)^n\,(\lambda+1-\alpha)}b_1\right)r^2,\end{aligned}$$

and

$$\begin{aligned}|f_n(z)| &\ge (1-b_1)\,r\\ &\quad-\left(\frac{(1-\alpha)}{(\lambda+1)^n\,(\lambda+1-\alpha)}-\frac{(2\lambda-1)^n\,(2\lambda-1+\alpha)}{(\lambda+1)^n\,(\lambda+1-\alpha)}b_1\right)r^2.\end{aligned}$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f_n \in \overline{SH}(\lambda, n, \alpha)$ and $\lambda \geq 1$. Taking the absolute value of f_n we have

$$\begin{aligned}
|f_n(z)| &\leq (1+b_1)\,r + \sum_{k=2}^{\infty}(a_k+b_k)\,r^k \\
&\leq (1+b_1)\,r + \sum_{k=2}^{\infty}(a_k+b_k)\,r^2 \\
&= (1+b_1)\,r + \frac{(1-\alpha)\,r^2}{(\lambda+1)^n(\lambda+1-\alpha)}\sum_{k=2}^{\infty}\frac{(\lambda+1)^n(\lambda+1-\alpha)}{(1-\alpha)}[a_k+b_k] \\
&\leq (1+b_1)\,r + \frac{(1-\alpha)\,r^2}{(\lambda+1)^n(\lambda+1-\alpha)} \\
&\quad \times\sum_{k=2}^{\infty}\left(\frac{[\lambda(k-1)+1]^n[\lambda(k-1)+1-\alpha]}{1-\alpha}a_k\right. \\
&\quad \left.+\frac{[\lambda(k-1)+1]^n[\lambda(k-1)+1-\alpha]}{1-\alpha}b_k\right) \\
&\leq (1+b_1)\,r + \frac{(1-\alpha)\,r^2}{(\lambda+1)^n(\lambda+1-\alpha)} \\
&\quad \times\sum_{k=2}^{\infty}\left(\frac{[\lambda(k-1)+1]^n[\lambda(k-1)+1-\alpha]}{1-\alpha}a_k\right. \\
&\quad \left.+\frac{[\lambda(k+1)-1]^n[\lambda(k+1)-1+\alpha]}{1-\alpha}b_k\right) \\
&\leq (1+b_1)\,r + \frac{(1-\alpha)}{(\lambda+1)^n(\lambda+1-\alpha)}\left(1-\frac{(2\lambda-1)^n(2\lambda-1+\alpha)}{1-\alpha}b_1\right)r^2 \\
&\leq (1+b_1)\,r + \left(\frac{(1-\alpha)}{(\lambda+1)^n(\lambda+1-\alpha)}-\frac{(2\lambda-1)^n(2\lambda-1+\alpha)}{(\lambda+1)^n(\lambda+1-\alpha)}b_1\right)r^2.
\end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 2.4. □

Corollary 2.5. *Let f_n of the form (1.6) be so that $f_n \in \overline{SH}(\lambda, n, \alpha)$. Then*

$$\left\{w : |w| < \frac{(\lambda+1)^n(\lambda+1-\alpha)-1+\alpha}{(\lambda+1)^n(\lambda+1-\alpha)} - \frac{(\lambda+1)^n(\lambda+1-\alpha)-(2\lambda-1)^n(2\lambda-1+\alpha)}{(\lambda+1)^n(\lambda+1-\alpha)}b_1\right\} \subset f_n(U).$$

Theorem 2.6. *The class $\overline{SH}(\lambda, n, \alpha)$ is closed under convex combinations.*

Proof. Let $f_{n_i} \in \overline{SH}(\lambda, n, \alpha)$ for $i = 1, 2, \ldots$, where f_{n_i} is given by

$$f_{n_i}(z) = z - \sum_{k=2}^{\infty} a_{k_i} z^k + (-1)^n \sum_{k=1}^{\infty} b_{k_i} \overline{z}^k.$$ Then by (2.3),

$$\sum_{k=2}^{\infty} \frac{[\lambda(k-1)+1]^n [\lambda(k-1)+1-\alpha]}{1-\alpha} a_{k_i} + \sum_{k=1}^{\infty} \frac{[\lambda(k+1)-1]^n [\lambda(k+1)-1+\alpha]}{1-\alpha} b_{k_i} \leq 1. \tag{2.6}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{k_i} \right) z^k + (-1)^n \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{k_i} \right) \overline{z}^k.$$

Then by (2.6),

$$\begin{aligned}
& \sum_{k=2}^{\infty} \frac{[\lambda(k-1)+1]^n [\lambda(k-1)+1-\alpha]}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i a_{k_i} \right) \\
& + \sum_{k=1}^{\infty} \frac{[\lambda(k+1)-1]^n [\lambda(k+1)-1+\alpha]}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i b_{k_i} \right) \\
= & \sum_{i=1}^{\infty} t_i \left(\sum_{k=2}^{\infty} \frac{[\lambda(k-1)+1]^n [\lambda(k-1)+1-\alpha]}{1-\alpha} a_{k_i} \right. \\
& \left. + \sum_{k=1}^{\infty} \frac{[\lambda(k+1)-1]^n [\lambda(k+1)-1+\alpha]}{1-\alpha} b_{k_i} \right) \leq \sum_{i=1}^{\infty} t_i = 1.
\end{aligned}$$

This is the condition required by (2.3) and so $\sum_{i=1}^{\infty} t_i f_{n_i}(z) \in \overline{SH}(\lambda, n, \alpha)$. □

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On n-weak amenability of a non-unital Banach algebra and its unitization

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Abstract. In [2] the authors asked if a non-unital Banach Algebra $\mathfrak{A}$ is weakly amenable whenever its unitization $\mathfrak{A}^\sharp$ is weakly amenable and whether $\mathfrak{A}^\sharp$ is 2-weakly amenable whenever $\mathfrak{A}$ is 2-weakly amenable. In this paper we give a partial solutions to these questions.

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1. Introduction

The notion of n-weak amenability for a Banach algebra was introduced by Dales, Ghahramani and Gronbæk in [2]. The Banach algebra $\mathfrak{A}$ is called n-weakly amenable if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n)}) = (0)$, where $\mathfrak{A}^{(n)}$ refers to the n-th dual of $\mathfrak{A}$. Also $\mathfrak{A}$ is permanently weakly amenable if $\mathfrak{A}$ is n-weakly amenable for each $n\epsilon\mathbb{N}$. In [2] the authors proved the following(Proposition 1.4):
Let $\mathfrak{A}$ be a non-unital Banach algebra, and $n\epsilon\mathbb{N}$.

(i) Suppose $\mathfrak{A}^\sharp$ is $2n$-weakly amenable. Then $\mathfrak{A}$ is $2n$-weakly amenable.

(ii) Suppose that $\mathfrak{A}$ is $(2n-1)$-weakly amenable. Then $\mathfrak{A}^\sharp$ is $(2n-1)$-weakly amenable.

(iii) Suppose that $\mathfrak{A}$ is commutative. Then $\mathfrak{A}^\sharp$ is n-weakly amenable if and only if $\mathfrak{A}$ is n-weakly amenable.

In this paper we consider the converses to (i) and (ii) and give partial solutions to them. Let us recall some definitions.

Definition 1.1. *([6]) A Banach $\mathfrak{A}$-module $\mathbf{X}$ is called neo-unital if for each $x\epsilon\mathbf{X}$ there are $a, a'\epsilon\mathfrak{A}$ and $y, y'\epsilon\mathbf{X}$ with $x = ay = y'a'$.*

Definition 1.2. *([3]) A Banach algebra $\mathfrak{A}$ is called self-induced if $\mathfrak{A}$ and $\mathfrak{A}\hat{\bigotimes}_{\mathfrak{A}}\mathfrak{A}$ are naturally isomorphic.*

Here $\mathfrak{A}\hat{\bigotimes}_{\mathfrak{A}}\mathfrak{A}=\frac{\mathfrak{A}\hat{\bigotimes}\mathfrak{A}}{\mathbf{K}}$ where $\mathbf{K}$ is the closed linear span of $\{ab\otimes c-a\otimes bc: a,b,c\,\epsilon\mathfrak{A}\}$.
Now we proceed to state and prove our theorem.

Theorem 1.3. *Let $\mathfrak{A}$ be a non-unital Banach algebra and suppose that $\mathfrak{A}$ is self-induced.*
(**i**) *If $\mathfrak{A}^\sharp$ is $(2n-1)$-weakly amenable then $\mathfrak{A}$ is $(2n-1)$-weakly amenable.*
(**ii**) *If $\mathfrak{A}$ is $2n$-weakly amenable then $\mathfrak{A}^\sharp$ is $2n$-weakly amenable.*
(**iii**) $\mathcal{H}^2(\mathfrak{A},\mathfrak{A}^{(2n)})\cong\mathcal{H}^2(\mathfrak{A}^\sharp,{\mathfrak{A}^\sharp}^{(2n)})$

Proof. Clearly $\mathfrak{A}$ is a closed two-sided ideal in $\mathfrak{A}^\sharp$ with codimension one. We consider the corresponding short exact sequence and its iterated duals. That is,

$$0\longrightarrow\mathfrak{A}\xrightarrow{\ i\ }\mathfrak{A}^\sharp\xrightarrow{\ \varphi\ }\mathbb{C}\longrightarrow 0$$

where $i:\mathfrak{A}\longrightarrow\mathfrak{A}^\sharp$ defined by $a\mapsto(a,0)$ and $\varphi:\mathfrak{A}^\sharp\longrightarrow\mathbb{C}$ defined by $(a,\lambda)\mapsto\lambda$.

$$0\longrightarrow\mathbb{C}\longrightarrow{\mathfrak{A}^\sharp}^{(2n-1)}\longrightarrow\mathfrak{A}^{(2n-1)}\longrightarrow 0 \tag{1.1}$$

$$0\longrightarrow\mathfrak{A}^{(2n)}\longrightarrow{\mathfrak{A}^\sharp}^{(2n)}\longrightarrow\mathbb{C}\longrightarrow 0 \tag{1.2}$$

It is easy to see that i is an isometric isomorphism and φ is a character on $\mathfrak{A}^\sharp$ with $\ker\varphi=\mathfrak{A}$. Then we make $\mathbb{C}$ a module over $\mathfrak{A}^\sharp$. Indeed,

$$z\cdot(a,\lambda)=(a,\lambda)\cdot z=\varphi(a,\lambda)z=\lambda z$$

where $(a,\lambda)\epsilon\mathfrak{A}^\sharp$ and $z\epsilon\mathbb{C}$.
Now consider the long exact sequence of cohomology groups concerning to (1.1). That is,

$$\begin{aligned}\ldots\longrightarrow\mathcal{H}^m(\mathfrak{A}^\sharp,\mathbb{C})\longrightarrow\mathcal{H}^m(\mathfrak{A}^\sharp,{\mathfrak{A}^\sharp}^{(2n-1)})\\ \longrightarrow\mathcal{H}^m(\mathfrak{A}^\sharp,\mathfrak{A}^{(2n-1)})\longrightarrow\mathcal{H}^{(m+1)}(\mathfrak{A}^\sharp,\mathbb{C})\longrightarrow\ldots.\end{aligned} \tag{1.3}$$

Obviously $\mathfrak{A}$, $\mathfrak{A}^{(n)}$ and $\mathbb{C}$ are unital Banach $\mathfrak{A}^\sharp$-bimodules. So by [4, Theorem 2.3] we have,

$$\mathcal{H}^m(\mathfrak{A}^\sharp,\mathbb{C})\cong\mathcal{H}^m(\mathfrak{A},\mathbb{C})\qquad and\qquad\mathcal{H}^m(\mathfrak{A}^\sharp,\mathfrak{A}^{(2n-1)})\cong\mathcal{H}^m(\mathfrak{A},\mathfrak{A}^{2n-1}). \tag{1.4}$$

Therefore by substituting (1.4) in (1.3) we get,

$$\begin{aligned}\ldots\longrightarrow\mathcal{H}^m(\mathfrak{A},\mathbb{C})\longrightarrow\mathcal{H}^m(\mathfrak{A}^\sharp,{\mathfrak{A}^\sharp}^{(2n-1)})\\ \longrightarrow\mathcal{H}^m(\mathfrak{A},\mathfrak{A}^{(2n-1)})\longrightarrow\mathcal{H}^{(m+1)}(\mathfrak{A},\mathbb{C})\longrightarrow\ldots.\end{aligned} \tag{1.5}$$

Since $\mathfrak{A}$ is self-induced then $\mathcal{H}^1(\mathfrak{A},\mathbb{C})=\mathcal{H}^2(\mathfrak{A},\mathbb{C})=(0)$ [4,Lemma 2.5](note that $\mathbb{C}$ is an annihilator $\mathfrak{A}$-bimodule). Hence by sequence (1.5) we obtain,

$$\mathcal{H}^1(\mathfrak{A}^\sharp,{\mathfrak{A}^\sharp}^{(2n-1)})\cong\mathcal{H}^1(\mathfrak{A},\mathfrak{A}^{(2n-1)}).$$

Obviously (**i**) holds.
For (**ii**) consider the long exact sequence of cohomology groups corresponding to the short exact sequence (1.2). That is,

$$\ldots \longrightarrow \mathcal{H}^m(\mathfrak{A}^\sharp, \mathfrak{A}^{(2n)}) \longrightarrow \mathcal{H}^m(\mathfrak{A}^\sharp, {\mathfrak{A}^\sharp}^{(2n)})$$

$$\longrightarrow \mathcal{H}^m(\mathfrak{A}^\sharp, \mathbb{C}) \longrightarrow \mathcal{H}^{m+1}(\mathfrak{A}^\sharp, \mathfrak{A}^{(2n)}) \longrightarrow \ldots . \tag{1.6}$$

Like before we have,

$$\mathcal{H}^m(\mathfrak{A}^\sharp, \mathbb{C}) \cong \mathcal{H}^m(\mathfrak{A}, \mathbb{C}) \qquad \textit{and} \qquad \mathcal{H}^m(\mathfrak{A}^\sharp, \mathfrak{A}^{(2n)}) \cong \mathcal{H}^m(\mathfrak{A}, \mathfrak{A}^{(2n)}). \tag{1.7}$$

By substituting (1.7) in (1.6) we get,

$$\ldots \longrightarrow \mathcal{H}^m(\mathfrak{A}, \mathfrak{A}^{(2n)}) \longrightarrow \mathcal{H}^m(\mathfrak{A}^\sharp, \mathfrak{A}^{\sharp(2n)}_\sharp) \longrightarrow \mathcal{H}^m(\mathfrak{A}, \mathbb{C}) \longrightarrow$$

$$\mathcal{H}^{m+1}(\mathfrak{A}, \mathfrak{A}^{(2n)}) \longrightarrow \mathcal{H}^{m+1}(\mathfrak{A}^\sharp, {\mathfrak{A}^\sharp}^{(2n)}) \longrightarrow \mathcal{H}^{m+1}(\mathfrak{A}, \mathbb{C}) \longrightarrow \ldots . \tag{1.8}$$

Now if $\mathfrak{A}$ is $2n$-weakly amenable then self-inducement of $\mathfrak{A}$ and (1.8) imply

$$\mathcal{H}^1(\mathfrak{A}^\sharp, {\mathfrak{A}^\sharp}^{(2n)}) = (0).$$

So (**ii**) holds.
For (**iii**) self-inducement of $\mathfrak{A}$ and (1.8) imply

$$\mathcal{H}^2(\mathfrak{A}, \mathfrak{A}^{(2n)}) \cong \mathcal{H}^2(\mathfrak{A}^\sharp, {\mathfrak{A}^\sharp}^{(2n)})$$

□

A special case occurs when the Banach algebra $\mathfrak{A}$ has a left(right) bounded approximate identity. In this case we have the following result.

Proposition 1.4. *If the Banach algebra $\mathfrak{A}$ has a left(right) bounded approximate identity then the theorem holds.*

Proof. By [5, Proposition II.3.13] $\mathfrak{A} \bigotimes_{\mathfrak{A}} \mathfrak{A} \to \mathfrak{A}^2$ given by $a \otimes b \mapsto ab$ is a topological isomorphism. By [1,§11,corollary 11] $\mathfrak{A}^2 = \mathfrak{A}$. So $\mathfrak{A} \bigotimes_{\mathfrak{A}} \mathfrak{A} \cong \mathfrak{A}$. That is $\mathfrak{A}$ is self-induced. Hence the theorem holds. □

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Approximate character amenability of Banach algebras

Ali Jabbari

Abstract. New notion of character amenability of Banach algebras is introduced. Let $\mathfrak{A}$ be a Banach algebra and $\varphi \in \Delta(\mathfrak{A})$. We say $\mathfrak{A}$ is approximately φ-amenable if there exist a bounded linear functional m on $\mathfrak{A}^*$ and a net (m_α) in $\mathfrak{A}^{**}$, such that $m(\varphi) = 1$ and

$$m(f.a) = \lim_\alpha m_\alpha(f.a) = \lim_\alpha \varphi(a) m_\alpha(f)$$

for all $a \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$. The corresponding class of Banach algebras is larger than that for the classical character amenable (φ-amenable) algebras. General theory is developed for this notion, and we show that this notion is different from that character amenability and φ-amenability.

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1. Introduction

The concept of amenable Banach algebra was introduced by Johnson in 1972 [16], and has proved to be of enormous importance in Banach algebra theory. Johnson showed that locally compact group G is amenable if and only if $L^1(G)$ is amenable as a Banach algebra.

Let $\mathfrak{A}$ be a Banach algebra, and let X be a Banach $\mathfrak{A}$-bimodule. A derivation is a linear map $D : \mathfrak{A} \longrightarrow X$ such that

$$D(ab) = a.D(b) + D(a).b \qquad (a, b \in \mathfrak{A}).$$

Throughout this paper, unless otherwise stated, by a derivation we means that a continuous derivation. For $x \in X$, set $ad_x : a \mapsto a.x - x.a$, $\mathfrak{A} \longrightarrow X$. Then ad_x is the inner derivation induced by x.

Denote the linear space of bounded derivations from $\mathfrak{A}$ into X by $Z^1(\mathfrak{A}, X)$ and the linear subspace of inner derivations by $N^1(\mathfrak{A}, X)$, we consider the quotient space $H^1(\mathfrak{A}, X) = Z^1(\mathfrak{A}, X)/N^1(\mathfrak{A}, X)$, called the first Hochschild cohomology group of $\mathfrak{A}$

with coefficients in X. The Banach algebra $\mathfrak{A}$ is said to be amenable if $H^1(\mathfrak{A}, X^*) = \{0\}$ for all Banach $\mathfrak{A}$-bimodules X.

The concept of approximate amenability of Banach algebras was introduced by F. Ghahramani and R. J. Loy in 2004 [11]. They showed that locally compact group G is amenable if and only if $L^1(G)$ is approximately amenable as a Banach algebra.

The derivation $D : \mathfrak{A} \longrightarrow X$ is approximately inner if there is a net (x_α) in X such that

$$D(a) = \lim_\alpha (a.x_\alpha - x_\alpha.a) \qquad (a \in \mathfrak{A}),$$

so that $D = \lim_\alpha ad_{x_\alpha}$ in the strong-operator topology of $B(\mathfrak{A}, X)$. The Banach algebra $\mathfrak{A}$ is approximately amenable if for any $\mathfrak{A}$-bimodule X, every derivation $D : \mathfrak{A} \longrightarrow X^*$ is approximately inner. There are many alternative formulations of the notion of amenability, of which we note the following, for further details see [6, 5, 9, 12, 13, 14]. Of course, amenable Banach algebra is approximately amenable. Some approximately amenable Banach algebras, which are not amenable constructed in [11]. Further examples have been shown by Ghahramani and Stokke in [13]: the Fourier algebra $A(G)$ is approximately amenable for each amenable, discrete group G, but it is known that these algebras are not always amenable.

The notion of character amenability of Banach algebras was defined by Sangani Monfared in [21]. This notion improved by Kaniuth, Lau, and Pym in [18, 19] and some new results were presented by Azimifard in [2]. Let $\mathfrak{A}$ be an arbitrary Banach algebra and φ be a homomorphism from $\mathfrak{A}$ onto $\mathbb{C}$. We denote that the space of all non-zero multiplicative linear functionals from $\mathfrak{A}$ onto $\mathbb{C}$ by $\Delta(A)$ ($\Delta(A)$ is the maximal ideal space of $\mathfrak{A}$). If $\varphi \in \Delta(\mathfrak{A}) \cup \{0\}$ and X is an arbitrary Banach space, then X can be viewed as Banach left or right $\mathfrak{A}$-module by the following actions

$$a.x = \varphi(a)x \qquad and \qquad x.a = \varphi(a)x \qquad (a \in \mathfrak{A}, x \in X).$$

The Banach algebra $\mathfrak{A}$ is said to be left character amenable (LCA) if for all $\varphi \in \Delta(\mathfrak{A}) \cup \{0\}$ and all Banach $\mathfrak{A}$-bimodules X for which the right module action is given by $a.x = \varphi(a)x$ $(a \in \mathfrak{A}, x \in X)$, every continuous derivation $D : \mathfrak{A} \longrightarrow X^*$ is inner. Right character amenability (RCA) is defined similarly by considering Banach $\mathfrak{A}$-bimodules X for which the left module action is given by $x.a = \varphi(a)x$, and $\mathfrak{A}$ is called character amenable (CA) if it is both left and right character amenable.

Also, according to [18], the Banach algebra $\mathfrak{A}$ is φ-amenable ($\varphi \in \Delta(\mathfrak{A})$) if there exists a bounded linear functional m on $\mathfrak{A}^*$ satisfying $m(\varphi) = 1$ and $m(f.a) = \varphi(a)m(f)$ for all $a \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$. Therefore the Banach algebra $\mathfrak{A}$ is character amenable if and only if $\mathfrak{A}$ is φ-amenable, for every $\varphi \in \Delta(\mathfrak{A}) \cup \{0\}$.

It is clear that if $\mathfrak{A}$ is amenable then $\mathfrak{A}$ is φ-amenable for every $\varphi \in \Delta(\mathfrak{A})$, but converse is not true, because for example let G be a locally compact group and $A(G)$ is a Fourier algebra on G. Fourier algebra $A(G)$ is character amenable (Example 2.6, [18]), but it is not amenable even when G is compact (see [17]).

Recently in [25], authors defined and studied approximate character amenability of Banach algebras. The Banach algebra $\mathfrak{A}$ is said to be approximately left character amenable if for all $\varphi \in \Delta(\mathfrak{A}) \cup \{0\}$ and all Banach $\mathfrak{A}$-bimodules X for which the right module action is given by $a.x = \varphi(a)x$ $(a \in \mathfrak{A}, x \in X)$, every continuous derivation $D : \mathfrak{A} \longrightarrow X^*$ is approximately inner. Approximately right character amenability is defined similarly by considering Banach $\mathfrak{A}$-bimodules X for which the left module action is given by $x.a = \varphi(a)x$, and $\mathfrak{A}$ is called approximately character amenable if it is both approximately left and right character amenable.

In this work, after establishing some background definition (our definition is deferent from [25]) and notation, we discuss some results for general Banach algebras. We generalize the character amenability (φ-amenability) of Banach algebras to approximate character amenability (approximate φ-amenability) of Banach algebras. By approximate character amenability, we provide some results, which they concluded by character amenability. A Banach algebra $\mathfrak{A}$ is φ-amenable, if and only if there exists a bounded net (u_α) in $\mathfrak{A}$ such that $\|a.u_\alpha - \varphi(a).u_\alpha\| \to 0$ for all $a \in \mathfrak{A}$ and $\varphi(u_\alpha) = 1$ for all α (Theorem 1.4 of [18]). By this notion, we construct such net and by this net, we give some results about existing of bounded approximate identity for Banach algebra $\mathfrak{A}$.

Also by giving an example, we show that there exists an approximately character amenable non-character amenable Banach algebra. Finally we consider some hereditary properties of approximate φ-amenability.

2. Main results

Definition 2.1. *Let $\mathfrak{A}$ be a Banach algebra and $\varphi \in \Delta(\mathfrak{A})$. We say $\mathfrak{A}$ is approximately φ-amenable if there exist a bounded linear functional m on $\mathfrak{A}^*$ and a net (m_α) in $\mathfrak{A}^{**}$, such that $m(\varphi) = 1$ and*

$$m(f.a) = \lim_\alpha m_\alpha(f.a) = \lim_\alpha \varphi(a)m_\alpha(f)$$

for all $a \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$.

When $\mathfrak{A}$ is φ-amenable, then it is approximately $\varphi-$amenable. In [18, 19], authors showed that the relation between φ-amenability and amenability of Banach algebras in special case. Banach algebra $\mathfrak{A}$ is φ-amenable if and only if $H^1(\mathfrak{A}, X^*) = \{0\}$ for each Banach $\mathfrak{A}$-bimodule X such that $a.x = \varphi(a)x$, for all $x \in X$ and $a \in \mathfrak{A}$. In the following Theorem, we generalize the Theorem 1.1 of [18] as follows:

Theorem 2.2. *Let $\mathfrak{A}$ be a Banach algebra, and $\varphi \in \Delta(\mathfrak{A})$. Then the following statements are equivalent.*

(i) $\mathfrak{A}$ is approximately φ-amenable.

(ii) If X is a Banach $\mathfrak{A}$-bimodule such that $a.x = \varphi(a).x$ for all $x \in X$ and $a \in \mathfrak{A}$, then every derivation from $\mathfrak{A}$ into X^* is approximately inner.

Proof. $(ii) \to (i)$, let $\varphi \in \Delta(\mathfrak{A})$. It is clear that $\mathfrak{A}^*$ is a Banach $\mathfrak{A}$-bimodule by following action

$$a.f = \varphi(a).f,$$

for all $f \in \mathfrak{A}^*$ and $a \in \mathfrak{A}$. Also since $\varphi \in \mathfrak{A}^*$, so we have

$$a.\varphi = \varphi.a = \varphi(a)\varphi. \tag{2.1}$$

Therefore $\mathbb{C}\varphi$ is a closed $\mathfrak{A}$-submodule of $\mathfrak{A}^*$. Take $X = \mathfrak{A}^* \backslash \mathbb{C}\varphi$, and consider $i : \mathfrak{A}^* \longrightarrow X$ that is a canonical mapping. Let δ be a derivation from $\mathfrak{A}$ into $\mathfrak{A}^{**}$.

Since every derivation from $\mathfrak{A}$ into each Banach $\mathfrak{A}$-bimodule is approximately inner, then there exists a net $(v_\alpha)_\alpha$ in $\mathfrak{A}^{**}$, such that

$$\delta(a) = \lim_\alpha a.v_\alpha - v_\alpha.a.$$

According to (2.1) for given $a \in \mathfrak{A}$, we have

$$\delta(a)(\varphi) \quad = \quad \lim_\alpha (a.v_\alpha)\varphi - (v_\alpha.a)\varphi = \lim_\alpha v_\alpha(\varphi.a) - v_\alpha(\varphi.a) = 0.$$

Therefore $\delta(a) \in i^*X^*$, and since i^* is a monomorphism, thus there exists a unique element $D(a) \in X^*$, such that $i^*(D(a)) = \delta(a)$. This shows that D is a derivation from $\mathfrak{A}$ into X^*. Therefore there exists a net $(\xi_\beta)_\beta$ in X^* such that

$$D(a) = \lim_\beta a.\xi_\beta - \xi_\beta.a,$$

for all $a \in \mathfrak{A}$. Then we have

$$\begin{aligned} \lim_\beta a.(i^*\xi_\beta) - (i^*\xi_\beta).a &= \lim_\beta i^*(a.\xi_\beta - \xi_\beta.a) \\ &= i^*(D(a)) = \delta(a) = \lim_\alpha a.v_\alpha - v_\alpha.a. \end{aligned}$$

Let $m_{\alpha,\beta} = v_\alpha - i^*\xi_\beta$, then $m_{\alpha,\beta} \in \mathfrak{A}^{**}$, $m_{\alpha,\beta}(\varphi) = 1$ and $a.m_{\alpha,\beta} = m_{\alpha,\beta}.a$. Therefore we have

$$m_{\alpha,\beta}(f.a) = m_{\alpha,\beta}(a.f) = \varphi(a)m_{\alpha,\beta}(f) \qquad (a \in \mathfrak{A}, f \in \mathfrak{A}^*).$$

Let I and J be the index sets for nets (v_α) and (ξ_β), respectively. We construct the required net (m_k) using an iterated limit construction (see [20]). Our indexing directed set is defined to be $K = I \times \Pi_{\alpha \in I} J$, equipped with the product ordering, and for each $k = (\alpha, f) \in K$, we define $m_k = m_{\alpha, f(\alpha)}$. Now let $m = \lim_k m_k$ and this complete the proof.

For $(i) \to (ii)$, let (m_α) and m be as in Definition 2.1. Let $D : \mathfrak{A} \longrightarrow X^*$ be a derivation, and let $D' = D^*|_X : X \longrightarrow \mathfrak{A}^*$ and $g_\alpha = (D')^*(m_\alpha) \in X^*$, such that $\lim_\alpha g_\alpha = \lim_\alpha (D')^*(m_\alpha) = (D')^*(m)$. Then, for all $a, b \in \mathfrak{A}$ and $x \in X$,

$$\langle b, D'(a.x)\rangle \quad = \quad \langle a.x, D(b)\rangle = \varphi(a)\langle x, D(b)\rangle = \varphi(a)\langle b, D'(x)\rangle,$$

and, hence $D(a.x) = \varphi(a)D'(x)$. This implies that

$$\begin{aligned} \langle x, g_\alpha.a\rangle &= \langle a.x, g_\alpha\rangle = \langle D'(a.x), m_\alpha\rangle \\ &= \varphi(a)\langle D'(x), m_\alpha\rangle = \varphi(a)\langle x, g_\alpha\rangle. \end{aligned}$$

Since D is a derivation, so we have

$$\begin{aligned} \langle b, D'(x.a)\rangle &= \langle x.a, D(b)\rangle = \langle x, a.D(b)\rangle = \langle ab, D'(x)\rangle - \langle b.x, D(x)\rangle \\ &= \langle b, D'(x).a\rangle - \varphi(b)\langle x, D(a)\rangle \end{aligned}$$

for all $a, b \in \mathfrak{A}$ and $x \in X$. Thus

$$D'(x.a) = D'(x).a - \langle x, D(a)\rangle\varphi,$$

for all $a \in \mathfrak{A}$ and $x \in X$. Consequently

$$\begin{aligned}\langle D'(x.a), m\rangle &= \langle x.a, (D')^*(m)\rangle = \lim_\alpha \langle x, a.g_\alpha\rangle = \lim_\alpha \langle x.a, g_\alpha\rangle = \lim_\alpha \langle D'(x.a), m_\alpha\rangle \\ &= \lim_\alpha \langle D'(x).a, m_\alpha\rangle - \lim_\alpha \langle x, D(a)\rangle\langle \varphi, m_\alpha\rangle \\ &= \lim_\alpha \varphi(a)\langle x, g_\alpha\rangle - \langle x, D(a)\rangle,\end{aligned}$$

and hence $D(a) = \lim_\alpha \varphi(a)g_\alpha - a.g_\alpha$. Therefore we have

$$D(a) = \lim_\alpha a.(-g_\alpha) - (-g_\alpha).a,$$

for all $a \in \mathfrak{A}$. □

According to the Theorem 1.4 of [18], the Banach algebra $\mathfrak{A}$ is φ-amenable if and only if there exists a bounded net (u_α) in $\mathfrak{A}$ such that $\|a.u_\alpha - \varphi(a).u_\alpha\| \to 0$ for all $a \in \mathfrak{A}$ and $\varphi(u_\alpha) = 1$ for all α. Now, by using of technique of Theorem 1.4 of [18], we have the following Theorem.

Theorem 2.3. *Let $\mathfrak{A}$ be a Banach algebra and $\varphi \in \Delta(\mathfrak{A})$. If $\mathfrak{A}$ is approximately φ-amenable then there exists a bounded net (u_α) in $\mathfrak{A}$ such that $\|a.u_\alpha - \varphi(a).u_\alpha\| \to 0$ for all $a \in \mathfrak{A}$.*

Proof. Suppose that $\mathfrak{A}$ is approximately φ-amenable. Let $m \in \mathfrak{A}^{**}$ and net $(m_\alpha) \in \mathfrak{A}^{**}$ be as in Definition 2.1.

Fix α, then by the Goldstaine Theorem (Theorem A.3.29 of [7]) there exists a net $(\nu_{\alpha,\beta}) \subset \mathfrak{A}$, such that $\nu_{\alpha,\beta} \xrightarrow{w^*} m_\alpha$, and $\|\nu_{\alpha,\beta}\| \le \|m_\alpha\| + 1$ for all β. Consider the product space $\mathfrak{A}^{\mathfrak{A}}$ endowed with the product of the norm topologies. Then $\mathfrak{A}^{\mathfrak{A}}$ is a locally convex topological vector space. Define a linear map $T : \mathfrak{A} \longrightarrow \mathfrak{A}^{\mathfrak{A}}$ by

$$T(u) = (au - \varphi(a)u)_{a\in\mathfrak{A}} \qquad (u \in \mathfrak{A}),$$

and a subset C_α of $\mathfrak{A}$ by

$$C_\alpha = \{u \in \mathfrak{A} : \|u\| \le \|m_\alpha\| + 1\}.$$

Then C_α is convex and hence $T(C_\alpha)$ is a convex subset of $\mathfrak{A}^{\mathfrak{A}}$. The fact that $\langle a\nu_{\alpha,\beta} - \varphi(a)\nu_{\alpha,\beta}, f\rangle \longrightarrow 0$, for all $f \in \mathfrak{A}^*$ and $a \in \mathfrak{A}$. This shows that the zero element of $\mathfrak{A}^{\mathfrak{A}}$ is contained in the closure of $T(C_\alpha)$ with respect to the product of the weak topologies. Now this product of the weak topologies coincides with the weak topology on $\mathfrak{A}^{\mathfrak{A}}$ and since $\mathfrak{A}^{\mathfrak{A}}$ is a locally convex space and $T(C_\alpha)$ is convex, the weak closure of $T(C_\alpha)$ equals the closure of $T(C_\alpha)$ in the given topology on $\mathfrak{A}^{\mathfrak{A}}$, that is, the product of the norm topologies (see [24, 23]). It follows that there exists a bounded net (u_γ) in $\mathfrak{A}$ such that $\|a.u_\gamma - \varphi(a).u_\gamma\| \to 0$ for all $a \in \mathfrak{A}$. □

Note that according to the proof of the above Theorem, existence of such net is not unique. for given character amenable Banach algebra $\mathfrak{A}$, set the character amenability constant $C(\mathfrak{A})$ of $\mathfrak{A}$ to be infimum of the norms of nets which obtained by Theorem 1.4 of [18]. We will say $\mathfrak{A}$ is K- character amenable if its character amenability constant is at most K.

We should show that approximate character amenability of Banach algebras is different from character amenability of Banach algebras. To showing this difference, by using of technique of Example 6.1 of [11], we consider the following Example.

Example 2.4. Let $(\mathfrak{A}_n)$ be a sequence of character amenable Banach algebras such that $C(\mathfrak{A}_n) \to \infty$. Then $B = c_0(\mathfrak{A}_n^\sharp)$ is approximately character amenable non-character amenable Banach algebra.

Proof. Let B be character amenable, and let $(U_m)_m$ be the sequence in B which obtained by Theorem 1.4 of [18]. By restricting of $(U_m)_m$ to the n^{th} coordinate of this sequence of bound K yields a sequence $(u_m)_m$ in $\mathfrak{A}_n$ with bound at most K, and since $C(\mathfrak{A}_n) \to \infty$ then B can not be character amenable. Define

$$B_k = \{(x_n) \in c_0(\mathfrak{A}_n^\sharp) : x_n = 0 \textit{ for } n > k\}.$$

For $n \in \mathbb{N}$, let $P_n : c_0(\mathfrak{A}_n^\sharp) \longrightarrow B_n$ be the natural projection of multiplication by $E_n = (e_1, e_2, ..., e_n, 0, 0, ...)$ onto the first n coordinate. Then $P_n(E_n)$ is the identity of B_n, and (E_n) is a central approximate identity for B bounded by 1. Let X be a Banach B-bimodule such that $b.x = \varphi(b)x$, and $x.b = \varphi(b)x$ for all $b \in B$, $x \in X$ $(\varphi \in \Delta(B))$. Now suppose that $D : B \longrightarrow X^*$ is a continuous derivation. By restricting D to some B_n we have a derivation $D_n : B_n \longrightarrow X^*$. Then there exists a bounded sequence (ξ_n) in X^* such that $D_n(b) = b.\xi_n - \xi_n.b$. Then by module actions defined above for $b \in B$ and $x \in X$ we have

$$\langle bE_n.\xi_n, x\rangle = \langle b.\xi_n, x\rangle \qquad \textit{and} \qquad \langle \xi_n.E_n b, x\rangle = \langle \xi_n.b, x\rangle.$$

Since (E_n) is central, so

$$\begin{aligned} D(b) &= D(\lim_n E_n b) = \lim_n D(E_n b) = \lim_n D_n(E_n b) \\ &= \lim_n bE_n.\xi_n - \xi_n.E_n b = \lim_n b.\xi_n - \xi_n.b. \end{aligned}$$

Then D is approximately inner, and hence B is approximately character amenable. □

One of the famous open problems in the amenability and approximate amenability of Banach algebras is: *Does amenability or approximate amenability of Banach algebra such as $\mathfrak{A}$ implies that amenability or approximate amenability of $\mathfrak{A}^{**}$?* This problem in the case of φ-amenability in Proposition 3.4 of [18] is considered and authors proved that the Banach algebra $\mathfrak{A}$ is φ-amenable if and only if $\mathfrak{A}^{**}$ is $\widetilde{\varphi}$-amenable, where $\widetilde{\varphi}$ is the extension of φ to $\mathfrak{A}^{**}$. Now, we generalize this statement in to approximate φ-amenability of Banach algebras.

Theorem 2.5. *Let $\mathfrak{A}$ be a Banach algebra, let $\varphi \in \Delta(\mathfrak{A})$, and let $\widetilde{\varphi}$ denote the extension of φ to $\mathfrak{A}^{**}$. Then $\mathfrak{A}$ is approximately φ-amenable if and only if $\mathfrak{A}^{**}$ is approximately $\widetilde{\varphi}$-amenable.*

Proof. Let $m(f.a) = \lim_\alpha \varphi(a) m_\alpha(f)$, where $m \in \mathfrak{A}^{**}$ and $(m_\alpha)_\alpha \subseteq \mathfrak{A}^{**}$. Suppose that $\hat{m}$ is the Gelfand transform of m and $\hat{m}_\alpha$ is the Gelfand transform of m_α, for all

α. Let $a^{''} \in \mathfrak{A}^{**}$ and $u \in \mathfrak{A}^{***}$, then there exist nets $(a_\gamma) \subseteq \mathfrak{A}$ and $(u_\beta) \subseteq \mathfrak{A}^*$, such that $a_\gamma \xrightarrow{w^*} a^{''}$ and $u_\beta \xrightarrow{w^*} u$. Therefore

$$\begin{aligned}\langle u.a^{''}, \hat{m}\rangle &= \langle m, u.a^{''}\rangle = \lim_\beta \langle m, u_\beta.a^{''}\rangle = \lim_\beta \lim_\gamma \langle m, u_\beta.a_\gamma\rangle \\ &= \lim_\beta \lim_\gamma \lim_\alpha \varphi(a_\gamma)\langle u_\beta, m_\alpha\rangle = \lim_\beta \lim_\alpha \widetilde{\varphi}(a^{''})\langle u_\beta, m_\alpha\rangle \\ &= \lim_\alpha \widetilde{\varphi}(a^{''})\hat{m}_\alpha(u).\end{aligned}$$

Conversely, let $\mathfrak{A}^{**}$ be approximately $\widetilde{\varphi}$-amenable, then there sexist an element M and net $(M_\alpha)_\alpha$ in $\mathfrak{A}^{****}$, such that $M(f.a) = \lim_\alpha \widetilde{\varphi}(a)M_\alpha(f)$. Now with restriction of M and M_α to $\mathfrak{A}^{**}$, we conclude that $\mathfrak{A}$ is approximately φ-amenable. □

In Theorem 2.3 of [21], Monfared showed that character amenability of Banach algebras similarly to amenability of Banach algebras implies that existence of bounded approximate identity for them. By the following Proposition, we show approximate φ-amenability of Banach algebra similar to approximate amenability implies existence of approximate identity for it's.

Proposition 2.6. *Let $\mathfrak{A}$ be a Banach algebra and $\varphi \in \Delta(\mathfrak{A})$. If $\mathfrak{A}$ is approximately φ-amenable, then $\mathfrak{A}$ have right and left approximate identity.*

Proof. In Theorem 2.1 (ii), take $X = \mathfrak{A}^*$, with right action $a.x = \varphi(a).x$ and zero left action. Remain of proof is exactly similar to proof of Lemma 2.2 of [11]. □

Let $\mathfrak{A}$ be an approximately amenable Banach algebra and

$$\sum : 0 \longrightarrow X^* \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be an admissible short exact sequence of left $\mathfrak{A}$-modules. Then by Theorem 2.2 of [11], $\sum$ approximately splits. Proof of the following Theorem is similar to proof of Theorem 2.2 of [11].

Theorem 2.7. *Let $\mathfrak{A}$ be a Banach algebra, and let $\varphi \in \Delta(\mathfrak{A})$. Let*

$$\sum : 0 \longrightarrow X^* \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be an admissible short exact sequence of Banach $\mathfrak{A}$-bimodules such that $a.x = \varphi(a).x$, for all $a \in \mathfrak{A}$ and x is in X, Y and Z. If $\mathfrak{A}$ is approximately φ-amenable, then $\sum$ is approximately splits. That is, there is a net $G_\nu : Z \longrightarrow Y$ of right inverse maps to g such that $\lim_\nu(a.G_\nu - G_\nu.a) = 0$ for $a \in \mathfrak{A}$, and a net $F_\nu : Y \longrightarrow X^*$ of left inverse maps to f such that $\lim_\nu(a.F_\nu - F_\nu.a) = 0$ for $a \in \mathfrak{A}$.

The following Proposition is similar to Proposition 3.3 of [25], but since our proof is different, so we do not remove the proof.

Proposition 2.8. *Let $\mathfrak{A}$ be a Banach algebra, and let J be a closed ideal of $\mathfrak{A}$. Let $\varphi \in \Delta(\mathfrak{A})$ such that $\varphi|_J \neq 0$. If $\mathfrak{A}$ is approximately φ-amenable, then J is approximately $\varphi|_J-$amenable.*

Proof. Let $m \in \mathfrak{A}^{**}$ and net $(m_\alpha)_\alpha$ in $\mathfrak{A}^{**}$ be as in Definition 2.1. Fix α, then there exists a net $(\nu_{\alpha,\beta})$ in $\mathfrak{A}$ such that $\nu_{\alpha,\beta} \longrightarrow m_\alpha$ in w^*-topology. For each $f \in J^\perp$ and any $a \in J$, we have

$$\begin{aligned} m(f.a) &= \lim_\alpha m_\alpha(f.a) = \lim_\beta \lim_\alpha \langle f.a, \nu_{\alpha,\beta}\rangle \\ &= \lim_\beta \lim_\alpha \langle f, a.\nu_{\alpha,\beta}\rangle = 0. \end{aligned}$$

Suppose that $m' \in J^{**}$, such that $m \in \mathfrak{A}^{**}$ is the extending of m'. Without loss of generality, we can suppose that $\varphi(a) = 1$, then

$$m(f.a) = \lim_\alpha m_\alpha(f.a) = \lim_\alpha m_\alpha(f) = 0,$$

for all $f \in J^\perp$. Therefore net $(m_\alpha)_\alpha$ is a net of bounded linear functional on J^*. For each α there exists m'_α in J^{**} such that $m'_\alpha(g) = m_\alpha(f)$, where f is extending of g from J^* into $\mathfrak{A}^*$. By definition of m'_α we have $m'_\alpha(\varphi|_J) = m_\alpha(\varphi) = 1$ and

$$\varphi(a)m'_\alpha(g) = \varphi(a)m_\alpha(f) = m_\alpha(f.a) = m'_\alpha(g.a),$$

for all $g \in J^*$ and $a \in J$. Thus

$$\begin{aligned} m'(g.a) &= m(f.a) = \lim_\alpha m_\alpha(f.a) = \lim_\alpha m'_\alpha(g.a) \\ &= \lim_\alpha \varphi(a)m'_\alpha(g). \end{aligned}$$

□

Proposition 2.9. *Let $\mathfrak{A}$ be a Banach algebra, let J be a closed two-sided ideal in $\mathfrak{A}$, and let $\varphi \in \Delta(\mathfrak{A})$. If $\mathfrak{A}$ is approximately φ-amenable, then $\mathfrak{A}/J$ is approximately $\widetilde{\varphi}$-amenable, where $\widetilde{\varphi}$ is the induction of φ on $\mathfrak{A}/J$.*

Proof. Let $T : \mathfrak{A} \longrightarrow \mathfrak{A}/J$ be a continuous epimorphism. Suppose that X is a Banach $\mathfrak{A}-$bimodule, where $a.x = \varphi(a).x$, for all $a \in \mathfrak{A}, x \in X$. Let $D : \mathfrak{A}/J \longrightarrow X^*$ be a derivation. Then $D \circ T : \mathfrak{A} \longrightarrow X^*$ is a derivation. Define $\widetilde{\varphi} = \varphi \circ T$, thus $\widetilde{\varphi}$ is a homomorphism on $\mathfrak{A}/J$.

Since $\mathfrak{A}$ is approximately φ-amenable, then by Theorem 2.1 (ii), there exists a net $(\xi_\alpha)_\alpha \subset X^*$ such that

$$(D \circ T)(a) = \lim_\alpha T(a).\xi_\alpha - \xi_\alpha.T(a),$$

this shows that D is an approximately inner derivation, by Theorem 2.1, $\mathfrak{A}/J$ is approximately $\widetilde{\varphi}$-amenable. □

Let G be a locally compact group, and let $A(G)$ be the Fourier algebra on G. Ghahramani and Stokke in [13], studied approximate amenability of Fourier algebras on locally compact group G. Character amenability of $A(G)$ is equivalent to amenability of G as a group (Corollary 2.4 of [21]). By Proposition 2.9, we have the following result for $A(G)$.

Corollary 2.10. *Let G be a locally compact group, and H be a closed subgroup of G. If $A(G)$ is approximately character amenable, then $A(H)$ is approximately character amenable.*

Proof. By Lemma 3.8 of [10], $A(H)$ is a quotient of $A(G)$. Then by Proposition 2.9, $A(H)$ is approximately character amenable. □

Proposition 2.11. *Let $\mathfrak{A}$ be a Banach algebra and $\varphi \in \Delta(\mathfrak{A})$. Let J be an ideal in $\mathfrak{A}$ with $J \subseteq \ker\varphi$ and let $\widetilde{\varphi} : \mathfrak{A}/J \longrightarrow \mathbb{C}$ be the homomorphism induced by φ. If J has a right identity and $\mathfrak{A}/J$ is approximately $\widetilde{\varphi}$-amenable then $\mathfrak{A}$ is approximately φ-amenable.*

Proof. Let $T : \mathfrak{A} \longrightarrow \mathfrak{A}/J$ be a epimorphism, such that $\varphi = \widetilde{\varphi} \circ T$. Suppose $m \in \mathfrak{A}^{**}$, such that $T(m)^{**} = n$ and there exists a net $(n_\alpha)_\alpha \subset (J^\perp)^*$, such that $n(a.f) = \lim_\alpha \widetilde{\varphi}(a)n_\alpha(f)$, for all $a \in \mathfrak{A}/J$ and $f \in J^\perp$. For each α there exists $m_\alpha \in \mathfrak{A}^{**}$, such that $T(m_\alpha) = n_\alpha$. For all $x \in \mathfrak{A}$ we have

$$\begin{aligned} T(x)T^{**}(m - me) &= T^{**}(x)T^{**}(m) = T^{**}(x)n = \lim_\alpha \widetilde{\varphi}^{**}(T^{**}(x))n_\alpha \\ &= \lim_\alpha \widetilde{\varphi}(T(x))T^{**}(m_\alpha - m_\alpha e) \\ &= \lim_\alpha \varphi(x)T^{**}(m_\alpha - m_\alpha e), \end{aligned}$$

therefore

$$T^{**}(x(m - me) - \lim_\alpha \varphi(x)(m_\alpha - m_\alpha e)) \longrightarrow 0,$$

Since for each $a \in J$, $(x(m - me) - \lim_\alpha \varphi(x)(m_\alpha - m_\alpha e))a = 0$, thus $x(m - me) - \lim_\alpha \varphi(x)(m_\alpha - m_\alpha e) = 0$. Hence

$$x(m - me) = \lim_\alpha \varphi(x)(m_\alpha - m_\alpha e).$$

□

Let $\mathfrak{A}$ be a non-unital Banach algebra. We denoted the forced unitization of $\mathfrak{A}$ with $\mathfrak{A}^\sharp$ and the adjoined identity element usually be denoted by e unless stated otherwise. Banach algebra $\mathfrak{A}$ is an ideal of $\mathfrak{A}^\sharp$ and $\mathfrak{A}^\sharp = \mathfrak{A} \oplus \mathbb{C}e$. Like as $\mathfrak{A}^\sharp$, for $(\mathfrak{A}^\sharp)^*$ and $(\mathfrak{A}^\sharp)^{**}$, we have $(\mathfrak{A}^\sharp)^* = \mathfrak{A}^* \oplus \mathbb{C}f_0$ and $(\mathfrak{A}^\sharp)^{**} = \mathfrak{A}^{**} \oplus \mathbb{C}m_0$, where $f_0(e) = 1$, $f|_{\mathfrak{A}} = 0$, $m_0(f_0) = 1$ and $m_0|_{\mathfrak{A}^{**}} = 0$.

Proposition 2.12. *Let $\mathfrak{A}$ be a non-unital Banach algebra. Let $\varphi \in \Delta(\mathfrak{A})$ and let φ_e be the unique extension of φ to an element of $\varphi : \mathfrak{A}^\sharp \longrightarrow \mathbb{C}$ be a continuous homomorphism. Then $\mathfrak{A}$ is approximately φ-amenable if and only if $\mathfrak{A}^\sharp$ is approximately φ_e-amenable.*

Proof. See Theorem 3.7 of [25]. □

Proposition 2.13. *Let $\mathfrak{A}$ be a Banach algebra and let $\varphi \in \Delta(\mathfrak{A})$. If $\mathfrak{A}$ is approximately φ-amenable and J is a weakly complemented left ideal of $\mathfrak{A}$. Then J has a right approximate identity and thereupon $\overline{J^2} = J$.*

Proof. Without loss of generality by Proposition 2.12, we can suppose that $\mathfrak{A}$ is unital. Consider the following sequence of left $\mathfrak{A}$-modules

$$\sum : 0 \longrightarrow J \xrightarrow{i} \mathfrak{A} \longrightarrow \mathfrak{A}/J \longrightarrow 0$$

since $(\mathfrak{A}/J)^* \cong J^{\perp}$, then we have

$$\sum\nolimits^{**} : 0 \longrightarrow J^{**} \xrightarrow{i^{**}} \mathfrak{A}^{**} \longrightarrow (J^{\perp})^* \longrightarrow 0.$$

Since J is a weakly complemented left ideal of $\mathfrak{A}$, then $\sum^{**}$ is admissible, and so by Theorem 2.7, there is a net of maps (Q_α) such that $a.Q_\alpha - Q_\alpha.a \longrightarrow 0$, for all $a \in \mathfrak{A}$.

Since $\mathfrak{A}$ is unital, then $\mathfrak{A}^{**}$ has right identity E, then

$$\langle E, Q_\alpha.a\rangle = \langle i^{**}\hat{a}.E, Q_\alpha\rangle = \langle i^{**}\hat{a}, Q_\alpha\rangle = \hat{a}.$$

for all $a \in J$, and thereby we have

$$\langle E, a.Q_\alpha\rangle = \langle a.Q_\alpha - Q_\alpha.a, E\rangle + \hat{a} \longrightarrow \hat{a},$$

and this show that $\langle E, Q_\alpha.a\rangle \longrightarrow \hat{a}$, for all $a \in J$. Therefore by Proposition 2.6, J has a right approximate identity. □

Example 2.14. In Corollary 2.5 of [21], Monfared showed that the measure algebra $M(G)$ is character amenable if and only if G is a discrete amenable group. Now, by Proposition 2.13, $M(G)$ is approximately character amenable if and only if G is discrete and amenable (see [8]).

Let G be a locally compact group. A linear subspace $S^1(G)$ of $L^1(G)$ is said to be a Segal algebra, if it satisfies the following conditions:

(i) $S^1(G)$ is a dense in $L^1(G)$;

(ii) If $f \in S^1(G)$, then $L_x f \in S^1(G)$, i.e. $S^1(G)$ is left translation invariant;

(iii) $S^1(G)$ is a Banach space under some norm $\|.\|_S$ and $\|L_x f\|_s = \|f\|_s$, for all $f \in S^1(G)$ and $x \in G$;

(iv) Map $x \mapsto L_x f$ from G into $S^1(G)$ is continuous.

For more details about Segal algebras see [22]. Character amenability of Segal algebras studied in [1], and many useful results about approximate amenability of such algebras considered in [6]. For amenable locally compact group G, $S^1(G)$ is φ-amenable for all $\varphi \in \Delta(S^1(G))$, and also $S^1(G)$ is character amenable if and only if $S^1(G) = L^1(G)$ (Proposition 3.1 of [1]). Now, we consider the following Corollary for Segal algebras (for prove, we use technique of proof of Theorem 5.5 of [6]).

Corollary 2.15. *Let G be a locally compact group, and let $S^1(G)$ be a Segal algebra on G. If $S^1(G)$ is approximate character amenable then G is an amenable group.*

Proof. Let G be a non-amenable locally compact group. Then by Theorem 5.2 of [26], There is no finite codimensional, closed, left ideal in $L^1(G)$ has right approximate identity. Let $I_0 = \{f \in L^1(G) : \int_G f(x)dx = 0\}$ be the augmentation ideal in $L^1(G)$. Let J be an ideal of $S^1(G)$ such that $J = S^1(G) \cap I_0$. Ideal J is a 1-codimension two-sided closed ideal in $S^1(G)$. If $S^1(G)$ is approximate character amenable, then by Proposition 2.13, J has a right approximate identity. By Proposition 5.4 of [6], I_0 have a right approximate identity, and this is a contradiction. Therefore G is an amenable group. □

By following Proposition, we show that in which condition the approximate character amenability can be transfer.

Proposition 2.16. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras, and let $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a continuous homomorphism with dense range. If $\varphi \in \Delta(\mathfrak{B})$ and $\mathfrak{A}$ is approximately $\varphi \circ h$-amenable, then $\mathfrak{B}$ is approximately φ-amenable.*

Proof. Let $\mathfrak{A}$ be approximately $\varphi \circ h-$amenable, then there exist there exist a bounded linear functional m on $\mathfrak{A}^*$ and a net (m_α) in $\mathfrak{A}^{**}$, such that $m_\alpha(\varphi \circ h) = 1$ and $m(f.a) = \lim_\alpha \varphi(h(a))m_\alpha(f)$ for all $a \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$.

Given $n \in \mathfrak{B}^{**}$ such that $n(g) = m(g.h)$, $g \in \mathfrak{B}^*$. For each $a, a' \in \mathfrak{A}$, we have

$$\begin{aligned}\langle (g.h(a) \circ h, a' \rangle &= \langle g.h(a), h(a') \rangle = \langle g, h(a)h(a') \rangle \\ &= \langle g, h(aa') \rangle = \langle g \circ h, aa' \rangle = \langle (g \circ h).a, a' \rangle.\end{aligned}$$

Therefore $(g.h(a) \circ h = (g \circ h).a$, for all $a \in \mathfrak{A}$. Let $m_\alpha(g \circ h) = n_\alpha(g)$, for $g \in \mathfrak{B}^*$, then

$$\begin{aligned}n(g.h(a)) &= m((g.h(a)) \circ h) = m((g \circ h).a) \\ &= \lim_\alpha \varphi \circ h(a) m_\alpha(g \circ h) = \lim_\alpha \varphi(h(a)) n_\alpha(g),\end{aligned}$$

where $(n_\alpha)_\alpha$ is in $\mathfrak{B}^{**}$. □

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On characterization of dual focal curves of spacelike biharmonic curves with timelike binormal in the dual Lorentzian $\mathbb{D}_1^3$

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Abstract. In this paper, we study dual focal curves of spacelike biharmonic curves with timelike binormal in the dual Lorentzian 3-space $\mathbb{D}_1^3$. We characterize dual focal curves in terms of their dual focal curvatures.

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1. Introduction

The application of dual numbers to the lines of the 3-space is carried out by the principle of transference which has been formulated by Study and Kotelnikov. It allows a complete generalization of the mathematical expression for the spherical point geometry to the spatial line geometry by means of dual-number extension, i.e. replacing all ordinary quantities by the corresponding dual-number quantities.

Harmonic maps $f : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds are the critical points of the energy

$$E(f) = \frac{1}{2}\int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f) = \mathrm{trace}\nabla df. \tag{1.2}$$

Bienergy of a map f by

$$E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [13], showing that the Euler–Lagrange equation associated to E_2 is

$$\begin{aligned}\tau_2(f) &= -\mathcal{J}^f(\tau(f)) = -\Delta\tau(f) - \operatorname{trace}R^N(df,\tau(f))\,df \\ &= 0,\end{aligned} \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of f. The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper, we study dual focal curves in the dual 3-space $\mathbb{D}^3$. We characterize dual focal curves in terms of their focal curvatures.

2. Preliminaries

If a and a^* are real numbers, the combination

$$A = a + \varepsilon a^* \tag{2.1}$$

is called a dual number. Here ε is the dual unit. Dual numbers are considered as polynomials in ε, subject to the rules $\varepsilon \neq 0$, $\varepsilon^2 = 0$, $\varepsilon.1 = 1.\varepsilon = \varepsilon$. W. K. Clifford defined the dual numbers, the set of dual numbers forms a commutative ring having the numbers $\varepsilon a^*(a^*$ real) as divisors of zero, not a field. No number εa^* has an inverse in the algebra. But the other laws of the algebra of dual numbers are the same as the laws of algebra of complex numbers. For example, two dual numbers A and $B = b+\varepsilon b^*$ are added componentwise.

$$A + B = (a + b) + \varepsilon(a^* + b^*), \tag{2.2}$$

and they are multiplied by

$$AB = ab + \varepsilon(a^*b + ab^*). \tag{2.3}$$

For the equality of A and B we have

$$A = B \Leftrightarrow a = b, \quad \text{and} \quad a^* = b^*. \tag{2.4}$$

An oriented line in $\mathbb{E}^3$ may be given by two points x and y on it. If μ is any non-zero constant [2], the parametric equation of the line is:

$$y = x + \mu a, \tag{2.5}$$

a is a unit vector along the line. The moment of a with respect to the origin is

$$a^* = x \times a = y \times a. \tag{2.6}$$

This means that a and a^* are not independent of the choice of the points on the line and these vectors are not independent of one another; satisfy the following equations:

$$< a, a >= 1, \quad < a, a^* >= 0. \tag{2.7}$$

The six components a_i, a_i^* $(i = 1, 2, 3)$ of the vectors a and a^* are known to be Plücker homogeneous line coordinates. These two vectors a and a^* determine an oriented line in $\mathbb{E}^3$. A point z lies on this line if and only if

$$z \times a = a^*. \tag{2.8}$$

The set of oriented lines in $\mathbb{E}^3$ is in one-to-one correspondence with pairs of vectors subject to the conditions (2.7), and so we may expect to represent it as a certain fourdimensional set in $\Re^6$ of sixtuples of real numbers; we may take the space $\mathbb{D}^3$ of triples of dual numbers with coordinates:

$$X_i = x_i + \varepsilon x_i^* \quad (i = 1, 2, 3). \tag{2.9}$$

Each line in $\mathbb{E}^3$ may be represented by a dual unit vector

$$A = a + \varepsilon a^*; \tag{2.10}$$

in $\mathbb{D}^3$. It is clear that this dual unit vector has the property

$$< A, A >=< a, a > +2\varepsilon < a, a^* >= 1. \tag{2.11}$$

The Lorentzian inner product of dual vectors $\hat{\varphi}$ and $\hat{\psi}$ in $\mathbb{D}^3$ is defined by

$$\left\langle \hat{\Omega}, \hat{\psi} \right\rangle = \langle \Omega, \psi \rangle + \varepsilon \left(\langle \Omega, \psi^* \rangle + \langle \Omega^*, \psi \rangle \right), \tag{2.12}$$

with the Lorentzian inner product φ and ψ

$$\langle \Omega, \psi \rangle = -\Omega_1 \psi_1 + \Omega_2 \psi_2 + \Omega_3 \psi_3, \tag{2.13}$$

where $\Omega= (\Omega_1, \Omega_2, \Omega_3)$ and $\psi= (\psi_1, \psi_2, \psi_3)$.

Theorem 2.1. (E. Study) The oriented lines in $\mathbb{E}^3$ are in one-to-one correspondence with points of the dual unit sphere $< X, X >= 1$ in $\mathbb{D}^3$.

3. Dual spacelike biharmonic curves with timelike binormal in the dual Lorentzian space $\mathbb{D}_1^3$

Let $\hat{\gamma} = \gamma + \varepsilon \gamma^* : I \subset R \to \mathbb{D}_1^3$ be a C^4 dual spacelike curve with arc length parameter s. Then the unit tangent vector $\hat{\gamma}' = \hat{\mathbf{T}}$ is defined, and the principal normal is $\hat{\mathbf{N}} = \frac{1}{\hat{\kappa}} \hat{\mathbf{T}}'$, where $\hat{\kappa}$ is never a pure-dual. The function $\hat{\kappa} = \left\| \hat{\mathbf{T}}' \right\| = \kappa + \varepsilon \kappa^*$ is called the dual curvature of the dual curve $\hat{\gamma}$. Then the binormal of $\hat{\gamma}$ is given by the dual vector $\hat{\mathbf{b}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$. Hence, the triple $\left\{ \hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}} \right\}$ is called the Frenet frame fields and the Frenet formulas may be expressed

$$\begin{bmatrix} \hat{\mathbf{T}}' \\ \hat{\mathbf{N}}' \\ \hat{\mathbf{B}}' \end{bmatrix} = \begin{bmatrix} 0 & \hat{\kappa} & 0 \\ -\hat{\kappa} & 0 & \hat{\tau} \\ 0 & \hat{\tau} & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{N}} \\ \hat{\mathbf{B}} \end{bmatrix}, \tag{3.1}$$

where $\hat{\tau} = \tau + \varepsilon \tau^*$ is the dual torsion of the spacelike dual curve $\hat{\gamma}$. Here, we suppose that the dual torsion $\hat{\tau}$ is never pure-dual.

Theorem 3.1. *(see [5]) Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a dual spacelike biharmonic curves with timelike binormal parametrized by arc length. $\hat{\gamma}$ is a dual timelike biharmonic curve if and only if*

$$\begin{array}{c} \kappa = \text{constant and } \kappa^* = \text{constant}, \\ \tau = \text{constant and } \tau^* = \text{constant}, \\ \kappa^2 - \tau^2 + \varepsilon \left(2\kappa\kappa^* - 2\tau\tau^* \right) = 0. \end{array} \tag{3.2}$$

Corollary 3.2. *(see [5]) Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a dual spacelike biharmonic curves with timelike binormal parametrized by arc length. $\hat{\gamma}$ is a dual spacelike elastic biharmonic curves with timelike binormal if and only if*

$$\kappa^2 = \tau^2, \tag{3.3}$$

$$\kappa\kappa^* = \tau\tau^*. \tag{3.4}$$

4. Dual focal curves of dual spacelike biharmonic curves with timelike binormal in the dual Lorentzian space $\mathbb{D}_1^3$

Denoting the dual focal curve by $\hat{\wp}$ we can write

$$\hat{\wp}(s) = (\hat{\gamma} + \hat{\mathfrak{m}}_1\hat{\mathbf{N}} + \hat{\mathfrak{m}}_2\hat{\mathbf{B}})(s), \tag{4.1}$$

where the coefficients $\hat{\mathfrak{m}}_1$, $\hat{\mathfrak{m}}_2$ are smooth functions of the parameter of the curve $\hat{\gamma}$, called the first and second dual focal curvatures of $\hat{\gamma}$, respectively.

The formula (4.1) is separated into the real and dual part, we have

$$\begin{aligned} \wp(s) &= \quad (\gamma + \mathfrak{m}_1\mathbf{N} + \mathfrak{m}_2\mathbf{B})(s), \\ \wp^*(s) &= \quad (\gamma^* + \mathfrak{m}_1\mathbf{N}^* + \mathfrak{m}_1^*\mathbf{N} + \mathfrak{m}_2\mathbf{B}^* + \mathfrak{m}_2^*\mathbf{B})\,(s). \end{aligned} \tag{4.2}$$

Theorem 4.1. *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a unit speed dual spacelike curve and $\hat{\wp}$ its dual focal curve on $\mathbb{D}_1^3$. Then,*

$$\wp = \gamma + \frac{1}{\kappa}\mathbf{N}, \tag{4.3}$$

$$\begin{aligned} \wp^* = \gamma^* + \frac{1}{\kappa}\mathbf{N}^* - \frac{\kappa^*}{\kappa^2}\mathbf{N} + \frac{\kappa'}{\kappa^2\tau}\mathbf{B}^* \\ + \left(\frac{(\kappa^*)'\kappa^2 - 2\kappa\kappa^*\kappa'}{\kappa^4\tau} - \frac{\tau^*\kappa'}{\kappa^2\tau^2}\right)\mathbf{B}. \end{aligned} \tag{4.4}$$

Proof. Assume that $\hat{\gamma}$ is a unit speed dual spacelike curve and $\hat{\wp}$ its dual focal curve on $\mathbb{D}_1^3$.

So, by differentiating of the formula (4.1), we get

$$\hat{\wp}(s)' = (1 - \hat{\kappa}\hat{\mathfrak{m}}_1)\hat{\mathbf{T}} + (\hat{\mathfrak{m}}_1' + \hat{\tau}\hat{\mathfrak{m}}_2)\hat{\mathbf{N}} + (\hat{\tau}\hat{\mathfrak{m}}_1 + \hat{\mathfrak{m}}_2')\hat{\mathbf{B}}. \tag{4.5}$$

Using above equation, the first 2 components vanish, we have using above equation,

$$\begin{aligned} \kappa\mathfrak{m}_1 &= 1, \\ \kappa\mathfrak{m}_1^* + \kappa_1^*\mathfrak{m} &= 0, \\ \mathfrak{m}_1' + \tau\mathfrak{m}_2 &= 0, \\ (\mathfrak{m}_1^*)' + \tau\mathfrak{m}_2^* + \tau^*\mathfrak{m}_2 &= 0. \end{aligned}$$

Considering equations above system, we have

$$\begin{aligned} \mathfrak{m}_1 &= \frac{1}{\kappa}, \\ \mathfrak{m}_1^* &= -\frac{\kappa^*}{\kappa^2}, \\ \mathfrak{m}_2 &= \frac{\kappa'}{\kappa^2\tau}, \\ \mathfrak{m}_2^* &= \frac{(\kappa^*)'\kappa^2 + 2\kappa\kappa^*\kappa'}{\kappa^4\tau} - \frac{\tau^*\kappa'}{\kappa^2\tau^2}. \end{aligned}$$

By means of obtained equations, we express (4.3) and (4.4). This completes the proof.

Corollary 4.2. *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a unit speed dual spacelike curve and $\hat{\wp}$ its dual focal curve on $\mathbb{D}_1^3$. Then, the dual focal curvatures of $\hat{\wp}$ are*

$$\begin{aligned} \mathfrak{m}_1 &= \frac{1}{\kappa}, \\ \mathfrak{m}_1^* &= -\frac{\kappa^*}{\kappa^2}, \\ \mathfrak{m}_2 &= \frac{\kappa'}{\kappa^2\tau}, \\ \mathfrak{m}_2^* &= \frac{(\kappa^*)'\kappa^2 + 2\kappa\kappa^*\kappa'}{\kappa^4\tau} - \frac{\tau^*\kappa'}{\kappa^2\tau^2}. \end{aligned}$$

In the light of Theorem 4.1, we express the following corollary without proof:

Corollary 4.3. *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a unit speed dual spacelike biharmonic curve and $\hat{\wp}$ its dual focal curve on $\mathbb{D}_1^3$. Then,*

$$\begin{aligned} \kappa &= \mp\frac{1}{\mathfrak{m}_1}, \\ \tau &= \mp\frac{1}{\mathfrak{m}_1}, \\ \kappa^* &= \mp\frac{\mathfrak{m}_1^*}{\mathfrak{m}_1^2}, \\ \tau^* &= \mp\frac{\mathfrak{m}_1^*}{\mathfrak{m}_1^2}. \end{aligned}$$

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Variational analysis of a contact problem with friction between two deformable bodies

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Abstract. This paper deals with the study of a nonlinear problem of friction contact between two deformable bodies. The elastic constitutive law is a assumed to be nonlinear and the contact is modeled with Signorini's conditions and version of Coulomb's law of dry friction. We present two variational formulations, noted $\mathbb{P}_1$, $\mathbb{P}_2$, of the considered problem, where $\mathbb{P}_1$ depends on the displacement field and $\mathbb{P}_2$ depends on the stress field. We establish existence and uniqueness results, using arguments of elliptic variational inequalities and a fixed point property and *Lions, Stampachia* theorem.

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1. Introduction

Frictional contact between deformable bodies can be frequently found in industry and everyday life such as train wheels with the rails, a shoe with the floor, tectonic plates, the car's braking system, etc. Considerable progress has been made with the modeling and analysis of static contact problems. The mathematical, mechanical and numerical state of the art can be found in the recent proceedings Raous [21]. Only recently, however, have the quasistatic and dynamic problems been considered. The reason lies in the considerable difficulties that the process of frictional contact presents in the modeling and analysis because of the complicated surface phenomena involved. General models for thermoelastic frictional contact, derived from thermodynamical principles, have been obtained in [25]. Quasistatic contact problems with normal compliance and friction have been considered in [3], where the existence of weak solutions has been proven. The existence of a weak solution to the, technically very complicated, problem with Signorini's contact condition has been established recently in [10]. The quasistatic frictional contact problem for viscoelastic materials can be found in [23] and the one for elastoviscoplastic materials in [22]. Dynamic problems with normal compliance were first considered in [19]. The existence of weak solutions to dynamic thermoelastic contact problems with frictional heat generation have been proven in [1] and when wear is taken into account in [2].

In this work we consider the process of frictional contact which is acted upon by volume forces and surface tractions, between two elastics bodies. The material's constitutive law is assumed to be nonlinear elastic. The contact is modeled with a normal compliance and the friction with the associated Coulomb's law of dry friction. The normal compliance contact condition was proposed in [19] and used in [1] and [15]. This condition allows the interpenetration of the body's surface into the foundation. In [19] normal compliance was justified by considering the interpenetration and deformation of surface asperities. It was assumed to have the form of a power law. We refer to [18] for the existence of static problems with Signorini's and Coulomb's conditions. We use a general expression for the normal compliance, similarly to the one in [2]. In part, the introduction of the normal compliance contact condition, in evolution problems, is motivated by the observation that Signorini's condition, while elegant and easy to explain, leads to discontinuous surface velocities which are associated with infinite tractions on the contact surface. This clearly is physically unrealistic; it leads to severe mathematical and numerical difficulties which do not necessarily represent the physical process. The normal compliance condition predicts large, but finite, contact forces. At any rate, we do not have a completely satisfactory contact condition yet, and maybe it is unrealistic to expect one single condition to model the wide variety of phenomena encountered in frictional contact.

The paper is organized as follows. Section 2 contains the notations and some preliminary material. In Section 3 we describe the model for the process, set it in a variational form, list the assumptions on the problem data and state our main results. In Section 4, basing on the theory of elliptic variational inequalities and application of fixed point theorems, we show the existence and uniqueness of a solution.

2. Notations and preliminaries

In this short section we present the notations and some preliminary material. For further details we refer the reader to [11] or [15]. We denote by $\mathbb{S}_N$ the space of second order symmetric tensors on $\mathbb{R}^N$, or equivalently, the space of the symmetric matrices of order N. The inner products and the corresponding norms on $\mathbb{R}^N$ and $\mathbb{S}_N$ are

$$u^\ell.v^\ell = u_i^\ell.v_i^\ell, \quad \|v^\ell\| = (v^\ell.v^\ell)^{\frac{1}{2}} \quad \forall u^\ell, v^\ell \in \mathbb{R}^N,$$
$$\sigma^\ell.\tau^\ell = \sigma_{ij}^\ell.\tau_{ij}^\ell, \quad \|\tau^\ell\| = (\tau^\ell.\tau^\ell)^{\frac{1}{2}} \quad \forall \sigma^\ell, \tau^\ell \in \mathbb{S}_N.$$

Here and below, $i, j = 1, 2, ..., N$, and the summation convention over repeated indices is adopted. Let two bounded domains Ω^ℓ, $\ell = 1, 2$ of the space $\mathbb{R}^N (N = 2, 3)$ be a bounded domain with a Lipschitz boundary Γ^ℓ and let $\eta^\ell = (\eta_i^\ell)$ denote the normal unit outward vector on Γ^ℓ. We shall use the notations

$$H^\ell = \{u^\ell = (u_i^\ell)/\ u_i^\ell \in \mathbb{L}^2(\Omega^\ell)\}\ ,\quad \mathcal{H}^\ell = \{\sigma^\ell = (\sigma_{ij}^\ell)/\sigma_{ij}^\ell = \sigma_{ji}^\ell \in \mathbb{L}^2(\Omega^\ell)\},$$
$$H_1^\ell = \{u^\ell = (u_i^\ell)/\ u_i^\ell \in H^1(\Omega^\ell)\}\ ,\quad \mathcal{H}_1^\ell = \{\sigma^\ell \in \mathcal{H}^\ell/\sigma_{ij,j}^\ell \in H^\ell\},$$
$$H = H^1 \times H^2 \quad , \quad H_1 = H_1^1 \times H_1^2 \quad , \quad \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2 \quad , \quad \mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2.$$

The spaces $H^\ell, H_1^\ell, \mathcal{H}^\ell$ and $\mathcal{H}_1^\ell$ are real Hilbert spaces endowed with the inner products given by

$$\langle u^\ell, v^\ell\rangle_{H^\ell} = \int_{\Omega^\ell} u_i^\ell v_i^\ell dx\ ,\quad \langle u^\ell, v^\ell\rangle_{H_1^\ell} = \langle u^\ell, v^\ell\rangle_{H^\ell} + \langle \epsilon(u^\ell), \epsilon(v^\ell)\rangle_{\mathcal{H}^\ell},$$
$$\langle \sigma^\ell, \tau^\ell\rangle_{\mathcal{H}^\ell} = \int_{\Omega^\ell} \sigma_{ij}^\ell \tau_{ij}^\ell dx\ ,\quad \langle \sigma^\ell, \tau^\ell\rangle_{\mathcal{H}_1^\ell} = \langle \sigma^\ell, \tau^\ell\rangle_{\mathcal{H}^\ell} + \langle div\sigma^\ell, div\tau^\ell\rangle_{H^\ell},$$

respectively. Here $\epsilon : H_1^\ell \to \mathcal{H}^\ell$ and $div : \mathcal{H}_1^\ell \to H^\ell$ are the *deformation* and *divergence* operators, defined by

$$\epsilon(u^\ell) = \frac{1}{2}\big(\nabla u^\ell + (\nabla u^\ell)^T\big), \qquad div\sigma^\ell = (\sigma_{ij,j}^\ell).$$

The associated norms on the spaces H^ℓ, H_1^ℓ, $\mathcal{H}^\ell$ and $\mathcal{H}_1^\ell$ are denoted by $\|.\|_{H^\ell}, \|.\|_{H_1^\ell}, \|.\|_{\mathcal{H}^\ell}$ and $\|.\|_{\mathcal{H}_1^\ell}$, respectively.

Let $H_{\Gamma^\ell} = H^{\frac{1}{2}}(\Gamma^\ell)^N$ and let $\gamma^\ell : H_1^\ell \to H_{\Gamma^\ell}$ be the trace map. For every element $v^\ell \in H_1^\ell$, we also use the notation v^ℓ for the trace $\gamma^\ell v^\ell$ of v^ℓ on Γ^ℓ and we denote by v_η^ℓ and v_τ^ℓ the *normal* and *tangential* components of v^ℓ on Γ^ℓ given by

$$v_\eta^\ell = v^\ell.\eta^\ell, \quad v_\tau^\ell = v^\ell - v_\eta^\ell \eta^\ell. \tag{2.1}$$

Let H'_{Γ^ℓ} be the dual of H_{Γ^ℓ} and let $\langle .,. \rangle$ denote the duality pairing between H'_{Γ^ℓ} and H_{Γ^ℓ}. For every element $\sigma^\ell \in \mathcal{H}_1^\ell$ let $\sigma^\ell\eta^\ell$ be the element of H'_{Γ^ℓ} given by

$$\langle \sigma^\ell\eta^\ell, \gamma^\ell v^\ell\rangle = \langle \sigma^\ell, \epsilon(v^\ell)\rangle_{\mathcal{H}^\ell} + \langle div\sigma^\ell, v^\ell\rangle_{H^\ell} \quad \forall v^\ell \in H_1^\ell. \tag{2.2}$$

We also denote by σ_η^ℓ and σ_τ^ℓ the *normal* and *tangential* traces of σ^ℓ, respectively. If σ^ℓ is continuously differentiable on $\overline{\Omega}^\ell$, then

$$\sigma_\eta^\ell = (\sigma^\ell\eta^\ell).\eta^\ell, \qquad \sigma_\tau^\ell = \sigma^\ell\eta^\ell - \sigma_\eta^\ell\eta^\ell, \tag{2.3}$$
$$\langle \sigma^\ell\eta^\ell, \gamma^\ell v^\ell\rangle = \int_{\Gamma^\ell} \sigma^\ell\eta^\ell.\gamma^\ell v^\ell da \tag{2.4}$$

for all $v^\ell \in H_1^\ell$, where da is the surface measure element.

3. The model and statement of results

In this section we describe a model for the process, present its variational formulation, list the assumptions on the problem data and state our main results.

Let us consider two elastic bodies, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^N (N = 2, 3)$. The boundary $\Gamma^\ell = \partial\Omega^\ell$ is assumed piecewise continuous, and composed of three complementary parts $\Gamma_1^\ell, \Gamma_2^\ell$ and Γ_3^ℓ. The body $\overline{\Omega}^\ell$ is fixed on the set Γ_1^ℓ of positive measure. The Γ_2^ℓ boundary is submitted to a density of forces noted g^ℓ.

In the initial configuration, both bodies have a common contact portion $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. The Ω^ℓ body is submitted to f^ℓ forces. The normal unit outward vector on Γ^ℓ is denoted $\eta^\ell = (\eta_i^\ell)$. On the contact zone the normal vector $\eta = \eta^1 = -\eta^2$ is assumed to be constant.
We denot by $u^\ell = (u_i^\ell)_{1\leq i\leq N}$ the displacement fields of the body Ω^ℓ, $\sigma^\ell = (\sigma_{ij}^\ell)_{1\leq i,j\leq N}$ the stress field of the body Ω^ℓ and $\epsilon^\ell = \epsilon(u^\ell)$ the linearized strain tensor. The elastic constitutive law of the material is assumed to be

$$\sigma^\ell = F^\ell\left(\epsilon(u^\ell)\right) \quad \text{in} \quad \Omega^\ell \tag{3.1}$$

in which F^ℓ is a given nonlinear function. The elastic equilibrium condition can be written as

$$\begin{cases} div\sigma^\ell + f^\ell = 0 & \text{in} \quad \Omega^\ell, \\ u^\ell = 0 & \text{on} \quad \Gamma_1^\ell, \qquad \ell = 1,2 \\ \sigma^\ell \eta^\ell = g^\ell & \text{on} \quad \Gamma_2^\ell, \end{cases} \tag{3.2}$$

where $u = (u^1, u^2)$. In addition to (3.2) and $\sigma^1\eta^1 = \sigma^2\eta^2$ on Γ_3, we have to satisfy the linearized non-penetration condition. The conditions on the boundary part Γ_3 constrained by *Coulomb* friction unilateral contact conditions incorporate the *Signorini conditions* :

$$[u_\eta] \leq 0, \quad \sigma_\eta \leq 0, \quad \sigma_\eta[u_\eta] = 0, \tag{3.3}$$

$$\begin{cases} |\sigma_\tau| \leq -\mu\sigma_\eta & \text{if} \quad [u_\tau] = 0, \\ \sigma_\tau = \mu\sigma_\eta \frac{[u_\tau]}{|[u_\tau]|} & \text{if} \quad [u_\tau] \neq 0 \end{cases} \tag{3.4}$$

where σ_η and σ_τ is the normal and tangential component, respectively, of the boundary stress, and $[u_\eta] = u_\eta^1 + u_\eta^2$ stands for the jump of the displacements in normal direction: either contact (i.e. $[u_\eta] = 0$)or separation (i.e. $[u_\eta] < 0$)are allowed.in other word ($[u_\eta] \leq 0$) is the nonpenetration condition, $[u_\tau] = u_\tau^1 + u_\tau^2$ stands for the jump of the displacements in tangential direction and $\mu \geq 0$ is the friction coefficient. This is a static version of *Coulomb's* law of dry friction and should be seen either as a mechanical model suitable for the proportional loadings case or as a first approximation of a more realistic model, based on a friction law involving the time derivative of u^1, u^2 (see for instance *Shillor* and *Sofonea*(1997), *Rochdi*(1998)). The friction law (3.4) states that the tangential shear cannot exceed the maximum frictional resistance $-\mu\sigma_\eta$. Then, if the inequality holds, the surfaces adheres and is so-called *stick* state, and the equality holds there is relative sliding, the so-called *slip* state. Therefore, the contact surface Γ_3 is divided into three zones: the stick zone, the slip zone and the zone of separation in which $[u_\eta] < 0$, i.e, there is no contact. The boundaries of these zones are *free boundaries* since they are unknown a priori, and are part of the problem. There is virtually no literature dealing with these free boundaries.

It is possible to express equivalently the contact and friction conditions considering the two following multivalued functions:

$$J_\eta(\xi) = \begin{cases} \{0\} & \text{if } \xi < 0, \\ [0, +\infty[& \text{if } \xi = 0, \\ \emptyset & \text{if } \xi > 0, \end{cases}$$

$$Dir_\tau(v) = \begin{cases} \{\frac{v}{|v|}\} & \text{if } v \in \mathbb{R}^{N-1} \text{ and } v \neq 0, \\ \{\omega \in \mathbb{R}^{N-1} / |\omega| \leq 1 \, ; \, \omega_\eta = 0\} & \text{if } v = 0. \end{cases}$$

J_η and Dir_τ are maximal monotone maps representing sub-gradients of the indicator function of interval $]-\infty, 0]$ and the function $v \mapsto |v_T|$ respectively. With these maps, unilateral contact and *Coulomb* friction conditions can be rewritten as:

$$\begin{cases} -\sigma_\eta \in J_\eta([u_\eta]), \\ -\sigma_\tau \in \mu\sigma_\eta Dir_\tau([u_\tau]). \end{cases}$$

Using (3.1)-(3.4), the mechanical problem non linear of the unilateral contact with *Coulomb* friction between two deformable bodies may be formulated as classically as follows:

Problem$\mathbb{P}$: For $\ell = 1, 2$, find the displacement field $u^\ell : \Omega^\ell \longrightarrow \mathbb{R}^N$ and the stress field $\sigma^\ell : \Omega^\ell \longrightarrow \mathbb{S}_N$ such that

$$\sigma^\ell = F^\ell(\epsilon(u^\ell)) \quad \text{in} \quad \Omega^\ell, \tag{3.5}$$

$$div\sigma^\ell + f^\ell = 0 \quad \text{in} \quad \Omega^\ell, \tag{3.6}$$

$$u^\ell = 0 \quad \text{on} \quad \Gamma_1^\ell, \tag{3.7}$$

$$\sigma^\ell \eta^\ell = g^\ell \quad \text{on} \quad \Gamma_2^\ell, \tag{3.8}$$

$$\begin{cases} (a) & \sigma^1\eta^1 = \sigma^2\eta^2, \\ (b) & [u_\eta] \leq 0, \; \sigma_\eta \leq 0, \; \sigma_\eta[u_\eta] = 0, \\ (c) & |\sigma_\tau| \leq -\mu\sigma_\eta, \\ (d) & |\sigma_\tau| < -\mu\sigma_\eta \Rightarrow [u_\tau] = 0, \\ (e) & |\sigma_\tau| = -\mu\sigma_\eta \Rightarrow \exists \dot{\lambda} \geq 0; \quad \sigma_\tau = -\dot{\lambda}[u_\tau], \end{cases} \quad \text{on} \quad \Gamma_3. \tag{3.9}$$

To obtain a variational formulation for problem (3.5)-(3.9) we need the following additional notations. Let V denote the closed subspace of H_1 given by

$$V = V(\Omega^1) \times V(\Omega^2) \tag{3.10}$$

where

$$V(\Omega^\ell) = \left\{ v^\ell \in H_1^\ell \;\middle|\; v^\ell = 0 \text{ on } \Gamma_1^\ell \right\} \tag{3.11}$$

and let denote the closed subspace of $\mathcal{H}_1$ given by

$$\widehat{\mathcal{H}}_1 = \left\{ \sigma = (\sigma^1, \sigma^2) \in \mathcal{H}_1 \;\middle|\; \sigma^1\eta^1 = \sigma^2\eta^2 \text{ on } \Gamma_3 \right\}. \tag{3.12}$$

Since $meas\Gamma_1^\ell > 0$, the following *Korn's* inequality holds:

$$\|\epsilon(v^\ell)\|_{\mathcal{H}^\ell} \geq c\|v^\ell\|_{H_1^\ell}, \quad \forall v^\ell \in V(\Omega^\ell) \quad \ell = 1.2. \tag{3.13}$$

Here c denotes a positive constant which may depends only on $\Omega^\ell, \Gamma_1^\ell, \ell = 1, 2$. We equip V with the scalar product

$$\langle \nu, \omega \rangle_V = \langle \epsilon(\nu^1), \epsilon(\omega^1) \rangle_{\mathcal{H}^1} + \langle \epsilon(\nu^2), \epsilon(\omega^2) \rangle_{\mathcal{H}^2} \tag{3.14}$$

and $\|.\|_V$ is the associated norm. It follows from *Korn's* inequality (3.13) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent on V. Then $(V, \|.\|_V)$ is a real Hilbert space.

Moreover, by the *Sobolev's* trace theorem and (3.13)we have a positive constant c_0 depending only on the domain $\Omega^\ell, \Gamma_1^\ell, \ell = 1, 2$ and Γ_3 such that

$$\|v^\ell\|_{\mathbb{L}^2(\Gamma_3)^N} \le c_0 \|v^\ell\|_V \qquad \forall v \in V. \tag{3.15}$$

In the study of the mechanical problem (3.5)-(3.9) we assume that operators F^ℓ : $\Omega^\ell \times \mathbb{S}_N \to \mathbb{S}_N$ satisfy

$$\begin{cases} (a) & \text{There exists } m > 0 \text{ such that} \\ & (F^\ell(x,\varepsilon_1) - F^\ell(x,\varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \ge m|\varepsilon_1 - \varepsilon_2|^2 \\ & \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}_N, \text{ a.e. } x \in \Omega^\ell. \\ (b) & \text{There exists } L > 0 \text{ such that} \\ & |F^\ell(x,\varepsilon_1) - F^\ell(x,\varepsilon_2)| \le L|\varepsilon_1 - \varepsilon_2| \\ & \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}_N, \text{ a.e. } x \in \Omega^\ell. \\ (c) & \text{For any } \varepsilon \in \mathbb{S}_N, \;\; x \to F^\ell(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega^\ell. \\ (d) & \text{The mapping } x \mapsto F^\ell(x,0) \in \mathcal{H}^\ell. \end{cases} \tag{3.16}$$

Remark 3.1. Using (3.16) we obtain that for all $\varepsilon^\ell \in \mathcal{H}^\ell$ the function $x \mapsto F^\ell(x, \varepsilon^\ell(x))$ belongs to $\mathcal{H}^\ell$ and hence we may consider F^ℓ as an operator defined on $\mathcal{H}^\ell$ with the range on $\mathcal{H}^\ell$. Moreover, $F^\ell : \mathcal{H}^\ell \to \mathcal{H}^\ell$ is a strongly monotone Lipschitz continuous operator and therefore F^ℓ is invertible and its inverse $(F^\ell)^{-1} : \mathcal{H}^\ell \to \mathcal{H}^\ell$ is also a strongly monotone Lipschitz continuous operator.

We also suppose that the forces and the tractions have the regularity

$$f^\ell \in H^\ell, \qquad g^\ell \in \mathbb{L}^2(\Gamma_2^\ell)^N \tag{3.17}$$

while the coefficient of friction μ is such that

$$\mu \in L^\infty(\Gamma_3), \quad \mu \ge 0 \text{ on } \Gamma_3. \tag{3.18}$$

For $(u, v) \in V$, we define the bilinear form of virtual works produced by the displacement u by

$$a(u,v) = \sum_{\ell=1}^{2} \int_{\Omega^\ell} F^\ell \epsilon(u^\ell).\epsilon(v^\ell) d\Omega^\ell \tag{3.19}$$

and the linear form of virtual works due to volume forces and surface traction by

$$\langle \varphi^\ell, v^\ell \rangle_{V(\overline{\Omega^\ell})} = \int_{\Omega^\ell} f^\ell.v^\ell d\Omega^\ell + \int_{\Gamma_2^\ell} g^\ell.v^\ell \eta^\ell d\Gamma_2^\ell \;,\; \forall v^\ell \in V(\Omega^\ell) \tag{3.20}$$

where $\varphi = (\varphi^1, \varphi^2) \in V$.
and let $j : \mathcal{H}_1^\ell \times V \longrightarrow \mathbb{R}$ be the functional

$$j(\sigma, v) = -\int_{\Gamma_3} \mu \sigma_\eta |[v_\tau]| d\Gamma_3 \tag{3.21}$$

where $|.|$ denotes the Euclidean norm. Let $\sigma \in \mathcal{H}_1^\ell$, the functional $j(\sigma, .)$ is continuous, convex and non-differentiable. Thus, $j(\sigma, .)$ is convex and lower semi-continuous on V. Finally, we denote in the sequel by U_{ad} the set of *geometrically admissible displacement fields* defined by

$$U_{ad} = \left\{ v = (v^1, v^2) \in V \;\mid\; [v_\eta] \le 0 \text{ on } \Gamma_3 \;\right\} \tag{3.22}$$

The set U_{ad} is nonempty ($0 \in U_{ad}$), closed and convex.
For all $g \in \widehat{\mathcal{H}}_1$, let $\Sigma_{ad}(g)$ denote the set of *statically admissible stress fields* given by:

$$\Sigma_{ad}(g) = \Big\{\tau \in \widehat{\mathcal{H}}_1 \mid \sum_{\ell=1}^{2} \langle \tau^\ell, \epsilon(v^\ell)\rangle_{\mathcal{H}^\ell} + j(g,v) \geq \langle g, v\rangle,\ \forall v \in U_{ad}\Big\} \tag{3.23}$$

also, for all $g \in \widehat{\mathcal{H}}_1$ with $g^1_\eta\big|_{\Gamma_3} \leq 0$, the set $\Sigma_{ad}(g)$ is nonempty $\big(g \in \Sigma_{ad}(g)\big)$, closed and convex.
Using (2.1)-(2.4) we have the following result.

Lemma 3.2. *If* (u,σ) *are sufficiently regular functions satisfying* (3.5)*-*(3.9)*, then:*

$$u \in U_{ad}, \qquad \sigma \in \Sigma_{ad}(\sigma), \tag{3.24}$$

$$\sum_{\ell=1}^{2} \langle \sigma^\ell, \epsilon(v^\ell) - \epsilon(u^\ell)\rangle_{\mathcal{H}^\ell} + j(\sigma, v) - j(\sigma,u) \geq \langle \varphi, v-u\rangle_V \quad \forall v \in U_{ad}, \tag{3.25}$$

$$\sum_{\ell=1}^{2} \langle \tau^\ell - \sigma^\ell, \epsilon(u^\ell)\rangle_{\mathcal{H}^\ell} \geq 0 \quad \forall \tau \in \Sigma_{ad}(\sigma). \tag{3.26}$$

Proof. The regularity $u \in U_{ad}$ follows from (3.7) and (3.9). By applying Green formula in (3.6) and from (3.7),(3.8),(3.20), (3.21) we have (3.25). Choosing now $v = 2u \in U_{ad}$ and $v = 0 \in U_{ad}$ in (3.25), we find

$$\sum_{\ell=1}^{2} \langle \sigma^\ell, \epsilon(u^\ell)\rangle_{\mathcal{H}^\ell} + j(\sigma,u) = \langle \varphi, u\rangle_V. \tag{3.27}$$

Using (3.25) we deduce

$$\sum_{\ell=1}^{2} \langle \sigma^\ell, \epsilon(v^\ell)\rangle_{\mathcal{H}^\ell} + j(\sigma,v) \geq \langle \varphi, v\rangle_V \qquad \forall v \in U_{ad}. \tag{3.28}$$

The regularity $\sigma \in \Sigma_{ad}(\sigma)$ is now a consequence of (3.23) and (3.28). Moreover, from (3.23) and (3.27) we obtain

$$\sum_{\ell=1}^{2} \langle \tau^\ell - \sigma^\ell, \epsilon(u^\ell)\rangle_{\mathcal{H}^\ell} \geq \langle \varphi, u\rangle_V - \langle \varphi, u\rangle_V = 0 \qquad \forall \tau \in \Sigma_{ad}.$$

Therefore (3.26). □

Lemma 3.2 and (3.5) lead us to consider the following two variational problems.
Problem $\mathbb{P}_1$: For $\ell = 1,2$, find the displacement fields $u^\ell : \Omega^\ell \longrightarrow \mathbb{R}^N$, such that

$$\begin{cases} u \in U_{ad}, \qquad F^1(\epsilon(u^1)).\eta^1 = F^2(\epsilon(u^2)).\eta^2 \text{ on } \Gamma_3, \\ a(u, v-u) + j(F(\epsilon(u)), v) - j(F(\epsilon(u)), u) \geq \langle \varphi, v-u\rangle_V, \quad \forall v \in U_{ad} \end{cases} \tag{3.29}$$

where

$$F(\epsilon(u)) = F^1(\epsilon(u^1)) \text{ or } F(\epsilon(u)) = F^2(\epsilon(u^2)).$$

Problem $\mathbb{P}_2$: For $\ell = 1, 2$, find the stress fields $\sigma^\ell : \Omega^\ell \longrightarrow \mathbb{S}_N$, such that

$$\sigma \in \Sigma_{ad}(\sigma), \qquad \sum_{\ell=1}^{2} \langle \tau^\ell - \sigma^\ell, \left(F^\ell\right)^{-1}(\sigma^\ell)\rangle_{\mathcal{H}^\ell} \geq 0\,, \quad \forall \tau \in \Sigma_{ad}(\sigma). \tag{3.30}$$

Details of such correspondences can be found in [13]. So, problem (3.29) can be rewritten as the following direct hybrid formulation:

Problem $\overline{\mathbb{P}}_1$: For $\ell = 1, 2$, find the displacement fields $u^\ell : \Omega^\ell \longrightarrow \mathbb{R}^N$, such that

$$u \in U_{ad}, \quad F^1(\epsilon(u^1))\eta^1 = F^2(\epsilon(u^2))\eta^2 \equiv \sigma\eta \quad \text{on } \Gamma_3, \tag{3.31}$$

$$a(u, v) = \langle \varphi, v\rangle + \langle \sigma_\eta, [v_\eta]\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_3} + \langle \sigma_\tau, [v_\tau]\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_3} \quad \forall v \in V, \tag{3.32}$$

$$\langle \sigma_\eta, [v_\eta] - [u_\eta]\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_3} \geq 0 \qquad \forall v \in U_{ad}, \tag{3.33}$$

$$\langle \sigma_\tau, [v_\tau] - [u_\tau]\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_3} - \langle \mu\sigma_\eta, |[v_\tau]| - |[u_\tau]|\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_3} \geq 0 \quad \forall v \in V. \tag{3.34}$$

For each body Ω^ℓ, we define the total potential energy functional J^ℓ by

$$J^\ell(v^\ell) = \frac{1}{2}a(v^\ell, v^\ell) - \langle \varphi^\ell, v^\ell\rangle_{V^\ell}\,, \quad \forall v^\ell \in V^\ell$$

and we set

$$J(v) = J^1(v^1) + J^2(v^2)\,, \quad \forall v \in V \tag{3.35}$$

the total potential energy of the two-body system. With the assumption $mes(\Gamma_1^\ell) > 0$, the functional J is convex, G-differentiable and coercive on V. The following theorem (see e.g. [15], Theorem 3.8) allows us to replace the variational inequality (3.29) by a minimization problem.

Theorem 3.3. *Let $\theta \in \widehat{\mathcal{H}}_1$ and suppose $G : U_{ad} \to \mathbb{R}$ is of the form $G(v) = J(v) + j(\theta, v)$, where $J(.)$ and $j(\theta, .)$ are convex and lower semi-continuous and $J(.)$ is G-differentiable on U_{ad}. Then, if u_θ is a minimizer of G on U_{ad},*

$$\langle DJ(u_\theta), v - u_\theta\rangle + j(\theta, v) - j(\theta, u_\theta) \geq 0, \quad \forall v \in U_{ad}. \tag{3.36}$$

Conversely, if (3.36) holds for $u_\theta \in U_{ad}$, then u_θ is a minimizer of G.

In (3.36), $DJ(u_\theta)$ is the gradient of J. Since J is a quadratic functional, (3.36) is precisely

$$u_\theta \in U_{ad},\ a(u_\theta, v - u_\theta) + j(\theta, v) - j(\theta, u_\theta) \geq \langle \varphi, v - u_\theta\rangle_V\,, \quad \forall v \in U_{ad}. \tag{3.37}$$

With the assumption $mes(\Gamma_1^\ell) > 0$, the functional $J(.) + j(\theta, .)$ is strictly convex and coercive, then there exists a unique solution to (3.36).
With the above preparations, the unilateral contact problem with Coulomb friction can be formulated as the constrained minimization problem.
Problem $\widehat{\mathbb{P}}_1$: For $\ell = 1, 2$, find the displacement fields $u^\ell : \Omega^\ell \longrightarrow \mathbb{R}^N$, such that

$$\begin{cases} u \in U_{ad}, \quad F^1(\epsilon(u^1)).\eta^1 = F^2(\epsilon(u^2)).\eta^2 \text{ on } \Gamma_3, \\ J(u) + j(F(\epsilon(u)), u) \leq J(v) + j(F(\epsilon(u)), v) \qquad \forall v \in U_{ad}. \end{cases} \tag{3.38}$$

Theorem 3.4. *Assume the hypothesis (*3.16*), (*3.17*). Let* $u = (u^1, u^2) \in V$ *be a solution of the variational problem* $\mathbb{P}_1$ *and* $\sigma = (\sigma^1, \sigma^2)$ *is defined by* $\sigma^\ell = F^\ell\left(\epsilon(u^\ell)\right)$, $\ell = 1, 2$, *then* (u, σ) *is a solution of the problem* $\mathbb{P}$.

Proof. For all $\Phi^\ell \in D^\ell \equiv (D(\Omega^\ell))^N$ be arbitrary, $v = u \pm \Phi \in U_{ad}$, where $\Phi = (\Phi^1, \Phi^2)$, and $\Phi^{3-\ell} = 0$, then using (3.29) and $\sigma^\ell = F^\ell\left(\epsilon(u^\ell)\right)$ we have:

$$\begin{aligned} 0 &\le \int_{\Omega^\ell} \sigma^\ell.\epsilon(v^\ell - u^\ell)d\Omega^\ell - \int_{\Omega^\ell} f^\ell.(v^\ell - u^\ell)d\Omega^\ell - \int_{\Gamma_2^\ell} g^\ell(v^\ell - u^\ell)d\Gamma_2^\ell \\ &= \int_{\Gamma^\ell} \sigma^\ell.(v^\ell - u^\ell)d\Gamma^\ell - \int_{\Omega^\ell} (div\sigma^\ell + f^\ell).(v^\ell - u^\ell)d\Omega^\ell \\ &= \pm \int_{\Omega^\ell} (div\sigma^\ell + f^\ell).\Phi^\ell d\Omega^\ell \end{aligned}$$

which implies (3.6).
By applying *Green's* formula and using (3.29) (3.6), we have

$$\sum_{\ell=1}^{2} \langle \sigma^\ell \eta^\ell, (v^\ell - u^\ell)\eta^\ell \rangle_{H'_{\Gamma^\ell} \times H_{\Gamma^\ell}} + j(\sigma, v) - j(\sigma, u) \ge \sum_{\ell=1}^{2} \langle g^\ell, (v^\ell - u^\ell)\eta^\ell \rangle_{H'_{\Gamma_2^\ell} \times H_{\Gamma_2^\ell}}, \quad \forall v \in U_{ad}. \tag{3.39}$$

Taking $v = u \pm (\omega^1, \omega^2) \in U_{ad}$, with $\omega^\ell \in D(\Omega^\ell \cup \Gamma_2^\ell)^N$ and $\omega^{3-\ell} = 0$ in (3.39), it follows that

$$\langle \sigma^\ell \eta^\ell, \omega^\ell \eta^\ell \rangle_{H'_{\Gamma_2^\ell} \times H_{\Gamma_2^\ell}} = \langle g^\ell, \omega^\ell \eta^\ell \rangle_{H'_{\Gamma_2^\ell} \times H_{\Gamma_2^\ell}}$$

which implies (3.8).
Let $(\omega^1, \omega^2) \in H_1$ with $\omega_\eta^\ell = 0$, $\omega^\ell\big|_{\Gamma_1^\ell \cup \Gamma_2^\ell} = 0$ and $\omega_\tau^1\big|_{\Gamma_3} = -\omega_\tau^2\big|_{\Gamma_3}$.
Then $v = u \pm (\omega^1, \omega^2) \in U_{ad}$ and (3.39) gives:

$$\sum_{\ell=1}^{2} \int_{\Gamma_3} \sigma_\tau^\ell.\omega_\tau^\ell d\Gamma_3 = 0.$$

From where, it follows

$$\int_{\Gamma_3} \sigma_\tau^1.\omega_\tau^1 d\Gamma_3 = \int_{\Gamma_3} \sigma_\tau^2.\omega_\tau^1 d\Gamma_3.$$

This implies $\sigma_\tau^1\big|_{\Gamma_3} = \sigma_\tau^2\big|_{\Gamma_3}$ and from (3.29), we have (3.9.a).
Taking $v = u \pm (\omega^1, \omega^2) \in U_{ad}$, with $\omega^\ell \in D(\Omega^\ell \cup \Gamma_3)^N$, $\omega_\tau^\ell = 0$ on Γ_3 and $\omega^{3-\ell} = 0$ in (3.39), it follows that

$$\langle \sigma_\eta^\ell, \omega_\eta^\ell \rangle_{H'_{\Gamma_3} \times H_{\Gamma_3}} \ge 0.$$

Furthermore $\sigma_\eta^\ell \le 0$ on Γ_3.
Now, by $u \in U_{ad}$, we have $[u_\eta] \le 0$ on Γ_3.
Taking now $v \in U_{ad}$ such that $v_\tau = u_\tau$ and $v_\eta = 0$ in (3.39), we obtain:

$$\int_{\Gamma_3} \sigma_\eta [u_\eta] d\Gamma_3 \le 0$$

and from $\sigma_\eta \leq 0$, $[u_\eta] \leq 0$ on Γ_3, we deduce $\sigma_\eta[u_\eta] = 0$ on Γ_3. Therefore, (3.9.b) holds.
Suppose that $v \in U_{ad}$, with $v_\eta = u_\eta$ on Γ_3, and using (3.8),(3.9.a.b) in (3.39), we obtain:

$$\int_{\Gamma_3} (\sigma_\tau[v_\tau] - \mu\sigma_\eta|[v_\tau]|)d\Gamma_3 - \int_{\Gamma_3} (\sigma_\tau[u_\tau] - \mu\sigma_\eta|[u_\tau]|)d\Gamma_3 \geq 0 \tag{3.40}$$

and choosing $v_\tau = 2u_\tau$ $\big($resp. $v_\tau = 0$ $\big)$ in (3.40), we deduce

$$\int_{\Gamma_3} (\sigma_\tau[u_\tau] - \mu\sigma_\eta|[u_\tau]|)d\Gamma_3 = 0. \tag{3.41}$$

Compining (3.40) and (3.41), we have

$$\int_{\Gamma_3} (\sigma_\tau[v_\tau] - \mu\sigma_\eta|[v_\tau]|)d\Gamma_3 \geq 0 \quad \forall v \in U_{ad} \tag{3.42}$$

and let $N = \big\{ x \in \Gamma_3 / |\sigma_\tau| > -\mu\sigma_\eta \big\}$. From $v \in U_{ad}$ with $[v_\tau]|_{\Gamma_3 - N} = 0$ and $[v_\tau]|_N = -\sigma_\tau$ in (3.42), we deduce:

$$\int_N (-|\sigma_\tau|^2 - \mu\sigma_\eta|\sigma_\tau|)d\Gamma_3 \geq 0. \tag{3.43}$$

Since $|\sigma_\tau| > -\mu\sigma_\eta$ and $\sigma_\eta \leq 0$ on N, then $-|\sigma_\tau| - \mu\sigma_\eta < 0$ and $|\sigma_\tau| \neq 0$ on N, which implies

$$-|\sigma_\tau|^2 - \mu\sigma_\eta|\sigma_\tau| > 0 \text{ on } N. \tag{3.44}$$

Using (3.43) and (3.44), we obtain $mes(N) = 0$, we deduce

$$|\sigma_\tau| \leq -\mu\sigma_\eta \quad p.p \text{ on } \Gamma_3$$

and hence (3.9.c) holds.
Using now (3.9.c) and (3.41) we deduce

$$\sigma_\tau[u_\tau] - \mu\sigma_\eta\big|[u_\tau]\big| = 0 \quad p.p \text{ on } \Gamma_3. \tag{3.45}$$

Moreover, from (3.9.c) we obtain

$$0 = \sigma_\tau.[u_\tau] - \mu|[u_\tau]|\sigma_\eta \geq -|\sigma_\tau|.|[u_\tau]| - \mu|[u_\tau]|\sigma_\eta \geq -|[u_\tau]|(|\sigma_\tau| + \mu\sigma_\eta) \geq 0.$$

Therefore,

$$-|[u_\tau]|(|\sigma_\tau| + \mu\sigma_\eta) = 0. \tag{3.46}$$

For $|\sigma_\tau| < -\mu\sigma_\eta$: from (3.46), we deduce $[u_\tau] = 0$, hence (3.9.d) holds.
For $|\sigma_\tau| = -\mu\sigma_\eta$: from (3.45),we deduce

$$\sigma_\tau.[u_\tau] = \mu|[u_\tau]|\sigma_\eta = -|\sigma_\tau|.|[u_\tau]|.$$

So we deduce that there exists a constant $\lambda \geq 0$ such that $[u_\tau] = -\lambda\sigma_\tau$, hence (3.9.e) holds. □

Theorem 3.5. *Assume the hypothesis (3.16),(3.17). Let $\sigma = (\sigma^1, \sigma^2)$ be a solution of the variational problem $\mathbb{P}_2$, and $u = (u^1, u^2) \in V$ is given by $\sigma^\ell = F^\ell\big(\epsilon(u^\ell)\big)$, $\ell = 1, 2$, then u is a solution of the variational problem $\mathbb{P}_1$.*

Proof. Firstly we prove $u \in U_{ad}$. Supposing that $u \notin U_{ad}$, and let u_* the projection of u on U_{ad}, we have the existence of $\alpha \in \mathbb{R}$ such that

$$\langle u_* - u, v \rangle_V > \alpha > \langle u_* - u, u \rangle_V \qquad \forall v \in U_{ad}.$$

We introduce the functional τ_* defined by: $\tau_* = (\epsilon(u_*^1 - u^1), \epsilon(u_*^2 - u^2)) \in \mathcal{H}$, and we use inner products defined by (3.14), we deduce:

$$\sum_{\ell=1}^{2} \langle \tau_*^\ell, \epsilon(v^\ell) \rangle_{\mathcal{H}^\ell} > \alpha > \sum_{\ell=1}^{2} \langle \tau_*^\ell, \epsilon(u^\ell) \rangle_{\mathcal{H}^\ell} \qquad \forall v \in U_{ad}. \tag{3.47}$$

Taking $v = 0 \in U_{ad}$ in (3.47), we obtain $\alpha < 0$, it is easy to verify that

$$\langle \tau_*^1, \epsilon(v^1) \rangle_{\mathcal{H}^1} + \langle \tau_*^2, \epsilon(v^2) \rangle_{\mathcal{H}^2} \geq 0 \qquad \forall v \in U_{ad}. \tag{3.48}$$

Really, we suppose the existence of $v_* = (v_*^1, v_*^2) \in U_{ad}$ where

$$\langle \tau_*^1, \epsilon(v_*^1) \rangle_{\mathcal{H}^1} + \langle \tau_*^2, \epsilon(v_*^2) \rangle_{\mathcal{H}^2} < 0. \tag{3.49}$$

As $\beta v_* \in U_{ad}$, $\forall \beta > 0$, if we replace $v = \beta v_*$ in (3.47) we obtain

$$\beta \left(\langle \tau_*^1, \epsilon(v_*^1) \rangle_{\mathcal{H}^1} + \langle \tau_*^2, \epsilon(v_*^2) \rangle_{\mathcal{H}^2} \right) > \alpha, \quad \forall \beta > 0.$$

And making $\beta \to +\infty$ with (3.49), we have $\alpha \leq -\infty$, this constitutes a contradiction with the fact that α is real. So we deduce (3.48). Now, using (3.30), (3.23) we deduce

$$\sum_{\ell=1}^{2} \langle \sigma^\ell, \epsilon(v^\ell) \rangle_{\mathcal{H}^\ell} + j(\sigma, v) \geq \langle \varphi, v \rangle_V \qquad \forall v \in U_{ad} \tag{3.50}$$

and, using (3.48) we obtain

$$\sum_{\ell=1}^{2} \langle \tau_*^\ell + \sigma^\ell, \epsilon(v^\ell) \rangle_{\mathcal{H}^\ell} + j(\sigma, v) \geq \langle \varphi, v \rangle_V \qquad \forall v \in U_{ad}$$

saying

$$\tau_* + \sigma \in \Sigma_{ad}(\sigma). \tag{3.51}$$

Choosing $\tau = \tau_* + \sigma \in \Sigma_{ad}(\sigma)$ in (3.30), and $\sigma^\ell = F^\ell(\epsilon(u^\ell))$ we obtain

$$\sum_{\ell=1}^{2} \langle \tau_*^\ell, \epsilon(u^\ell) \rangle_{\mathcal{H}^\ell} \geq 0. \tag{3.52}$$

Using now (3.47) and $\alpha < 0$, we find

$$\sum_{\ell=1}^{2} \langle \tau_*^\ell, \epsilon(u^\ell) \rangle_{\mathcal{H}^\ell} < 0. \tag{3.53}$$

The relations (3.52) and (3.53) constitute a contradiction, so we deduce that $u \in U_{ad}$.
It remains to prove the inequality given in (3.29).
Using Riesz's representation theorem we define the nonlinear operator $R : V \to V$ by

$$\langle Rv, w \rangle_V = \sum_{\ell=1}^{2} \langle F^\ell(\epsilon(v^\ell)), \epsilon(w^\ell) \rangle_{\mathcal{H}^\ell}.$$

Then hypotheses (3.16) on F^ℓ imply that R is strictly monotone, coercive and lipschitzian operator, on the other hand the functional $j(\sigma, .)$ is proper, convex and lower continuous on V. Then results from the theory of elliptic variational inequalities [4] of the second kind, we have the existence of $bar\tau = (\bar{\tau}^1, \bar{\tau}^2) \in \mathcal{H}$ such that

$$\sum_{\ell=1}^{2} \langle \bar{\tau}^\ell, \epsilon(v^\ell) - \epsilon(u^\ell)\rangle_{\mathcal{H}^\ell} + j(\sigma, v) - j(\sigma, u) \geq \langle \varphi, v - u\rangle_V, \quad \forall v \in V. \tag{3.54}$$

Taking $v = 2u$ and $v = 0$ in (3.54), then

$$\sum_{\ell=1}^{2} \langle \bar{\tau}^\ell, \epsilon(u^\ell)\rangle_{\mathcal{H}^\ell} + j(\sigma, u) = \langle \varphi, u\rangle_V. \tag{3.55}$$

Subtracting (3.55) from (3.54), this means that $\bar{\tau} \in \Sigma_{ad}(\sigma)$.
Therefore, from (3.30), (3.55) and $\sigma^\ell = F^\ell(\epsilon(u^\ell))$, we derive

$$\langle \varphi, u\rangle_V \geq \sum_{\ell=1}^{2} \langle \sigma^\ell, \epsilon(u^\ell)\rangle_{\mathcal{H}^\ell} + j(\sigma, u).$$

The converse inequality follows from (3.23) since $\sigma \in \Sigma_{ad}(\sigma)$ and $u \in U_{ad}$.
Therefore, we conclude that

$$\langle \varphi, u\rangle_V = \sum_{\ell=1}^{2} \langle \sigma^\ell, \epsilon(u^\ell)\rangle_{\mathcal{H}^\ell} + j(\sigma, u). \tag{3.56}$$

Using again (3.23), we have

$$\sum_{\ell=1}^{2} \langle \sigma^\ell, \epsilon(v^\ell) - \epsilon(u^\ell)\rangle_{\mathcal{H}^\ell} + j(\sigma, v) - j(\sigma, u) \geq \langle \varphi, v - u\rangle_V, \ \forall v \in U_{ad} \tag{3.57}$$

and $\sigma^\ell = F^\ell(\epsilon(u^\ell))$, $\sigma = (\sigma^1, \sigma^2) \in \widehat{\mathcal{H}}_1$ it results that u is a solution of the problem $\mathbb{P}_1$. □

Theorem 3.4 and Theorem 3.5, allow to deduce the following results

Corollary 3.6. *Assume the hypothesis (*3.16*),(*3.17*). Let $\sigma = (\sigma^1, \sigma^2)$ be a solution of the variational problem $\mathbb{P}_2$, and $u = (u^1, u^2) \in V$ is given by $\sigma^\ell = F^\ell\left(\epsilon(u^\ell)\right)$, $\ell = 1, 2$. then (u, σ) is a solution of the problem $\mathbb{P}$.*

Also Theorem 3.4 and Lemma 3.2, allow to deduce the following results

Corollary 3.7. *Assume the hypothesis (*3.16*),(*3.17*). Let $u = (u^1, u^2) \in V$ is a solution of the problem $\mathbb{P}_1$, and setting $\sigma^\ell = F^\ell\left(\epsilon(u^\ell)\right)$, $\ell = 1, 2$ we have $\sigma = (\sigma^1, \sigma^2)$ a solution of the problem $\mathbb{P}_2$.*

Theorem 3.8. *Under the hypotheses (*3.16*)-(*3.17*). Then there exists $C_o > 0$ which depends only on Ω^ℓ, Γ^ℓ and F^ℓ, $\ell = 1, 2$ such that if $\|\mu\|_{L^\infty(\Gamma_3)} \leq C_o$ then there exists a unique solution (u, σ) of problem $\mathbb{P}$. Moreover, the solution satisfies*

$$u \in V, \quad \sigma \in \mathcal{H}_1.$$

Proposition 3.9. *Let $\theta \in \widehat{\mathcal{H}}_1$ and let (u^1_θ, u^2_θ) be the solution of (3.37), then:*

$$\big(F^1(\epsilon(u^1_\theta)), F^2(\epsilon(u^2_\theta))\big) \in \widehat{\mathcal{H}}_1. \tag{3.58}$$

Proof. Let $\omega = (\omega_1, \omega_2)$ where $\omega^\ell \in D(\Omega^\ell \cup \Gamma_3)^N$ and $[\omega\eta] = 0$ on Γ_3. Then $v = u \pm \omega \in U_{ad}$ in (3.37) gives:

$$\sum_{\ell=1}^{2} \langle F^\ell(\epsilon(u^\ell_\theta)), \epsilon(\omega^\ell)\rangle_{\mathcal{H}^\ell} = \langle \varphi, \omega\rangle$$

and using (3.20), with $\omega^\ell \eta^\ell = -\omega^{3-\ell}\eta^{3-\ell}$ on Γ_3, we have

$$\int_{\Gamma_3} \{F^1(\epsilon(u^1_\theta))\eta^1 - F^2(\epsilon(u^2_\theta))\eta^2\}.\omega^1\eta^1 d\Gamma_3 = 0.$$

Therefore, we conclude that $F^1(\epsilon(u^1_\theta))\eta^1 = F^2(\epsilon(u^2_\theta))\eta^2$ on Γ_3
Then (3.58). □

Let us consider now the operator $A : \widehat{\mathcal{H}}_1 \longrightarrow \widehat{\mathcal{H}}_1$ defined by

$$A(\theta) = (F^1(\epsilon(u^1_\theta)), F^2(\epsilon(u^2_\theta)). \tag{3.59}$$

We have the following result.

Proposition 3.10. *There exists $C_0 > 0$, such that, $\|\mu\|_{L^\infty(\Gamma_3)} \leq C_0$, The operator A has a unique fixed point $\theta^* \in \widehat{\mathcal{H}}_1$.*

Proof. Let $\theta_i \in \widehat{\mathcal{H}}_1$, for $i = 1, 2$, and let u_i the solutions of (3.37), we have

$$\begin{cases} a(u_1, u_2 - u_1) + j(\theta_1, u_2) - j(\theta_1, u_1) \geq \langle \varphi, u_2 - u_1\rangle, \\ \\ a(u_2, u_1 - u_2) + j(\theta_2, u_1) - j(\theta_2, u_2) \geq \langle \varphi, u_1 - u_2\rangle. \end{cases}$$

Thus, using (3.19),we deduce that

$$\begin{cases} \sum_{\ell=1}^2 \int_{\Omega^\ell} (F^\ell(\epsilon(u^\ell_1)) - F^\ell(\epsilon(u^\ell_2))(\epsilon(u^\ell_1) - \epsilon(u^\ell_2))d\Omega^\ell \leq \\ \\ \qquad j(\theta_1, u_2) - j(\theta_1, u_1) + j(\theta_2, u_1) - j(\theta_2, u_2). \end{cases} \tag{3.60}$$

From the *Korn's* inequality and (3.16), yields

$$\sum_{\ell=1}^{2} \langle F^\ell(\epsilon(u^\ell_1) - F^\ell(\epsilon(u^\ell_2))\, ,\, \epsilon(u^\ell_1) - \epsilon(u^\ell_2)\rangle \geq C_1 \|u_1 - u_2\|^2_V. \tag{3.61}$$

Using (3.21), we obtain

$$j(\theta_1, u_2) - j(\theta_1, u_1) + j(\theta_2, u_1) - j(\theta_2, u_2) = -\int_{\Gamma_3} \mu(\theta_{1\eta} - \theta_{2\eta})(|[u_{1\tau}]| - |[u_{2\tau}]|)d\Gamma_3.$$

So that

$$j(\theta_1, u_2) - j(\theta_1, u_1) + j(\theta_2, u_1) - j(\theta_2, u_2) \leq C_2 \|\mu\|_{L^\infty(\Gamma_3)} \|\theta_1 - \theta_2\|_{\mathcal{H}_1} . \|u_1 - u_2\|_V$$

and using (3.60),(3.61) and using the trace theorem, we have

$$\|u_1 - u_2\|_V \leq C_3 \|\mu\|_{L^\infty(\Gamma_3)} \|\theta_1 - \theta_2\|_{\mathcal{H}_1}. \tag{3.62}$$

Putting (3.16) and (3.60), it yields:

$$\|A\theta_1 - A\theta_2\|^2_{\mathcal{H}_1} \leq C_4 \sum_{\ell=1}^{2} \|\epsilon(u_1^\ell) - \epsilon(u_2^\ell)\|^2_{\mathcal{H}_1^\ell}. \tag{3.63}$$

Moreover, from (3.62) and (3.63), we obtain:

$$\|A\theta_1 - A\theta_2\|_{\mathcal{H}_1} \leq C_5 \|\mu\|_{L(\Gamma_3)^\infty} \|\theta_1 - \theta_2\|_{\mathcal{H}_1}.$$

We conclude that the operator A is a contradiction if $\|\mu\|_{L(\Gamma_3)^\infty} < \frac{1}{C_5}$. By the Banach fixed point theorem, we obtain that this operator has a unique fixed point $\theta^* \in \widehat{\mathcal{H}}_1$. □

Proposition 3.11. *For each $\theta \in \widehat{\mathcal{H}}_1$, there exists a unique $\sigma_\theta \in \widehat{\mathcal{H}}_1$, such that*

$$\sigma_\theta \in \Sigma_{ad}(\theta), \qquad \sum_{\ell=1}^{2} \langle (F^\ell)^{-1}(\sigma_\theta), \tau^\ell - \sigma_\theta^\ell \rangle_{\mathcal{H}^\ell} \geq 0 \quad \forall \tau \in \Sigma_{ad}(\theta). \tag{3.64}$$

Proof. Let $\sigma \in \widehat{\mathcal{H}}_1$, it is easy to check that the application

$$\tau \longmapsto \sum_{\ell=1}^{2} \langle (F^\ell)^{-1}(\sigma^\ell), \tau^\ell \rangle_{\mathcal{H}^\ell}$$

is a continuous linear form on $\widehat{\mathcal{H}}_1$ (for σ fixe). Moreover, using *Riesz's* representation theorem we may define the operator $E : \widehat{\mathcal{H}}_1 \longrightarrow \widehat{\mathcal{H}}_1$ by the relation

$$\langle E\sigma, \tau \rangle_{\mathcal{H}_1} = \sum_{\ell=1}^{2} \langle (F^\ell)^{-1}(\sigma^\ell), \tau^\ell \rangle_{\mathcal{H}^\ell} \qquad \forall \sigma, \tau \in \widehat{\mathcal{H}}_1. \tag{3.65}$$

Keeping in mind (3.16) and *Korn's* inequality, we deduce that the operator E is strongly monotone and *Lipschitz* contiunous on E. Also , $\Sigma_{ad}(\theta)$ is a closed, convex and nonempty subset of $\widehat{\mathcal{H}}_1$.
According to the *Lions, Stampacchia* theorem, we obtain the existence and uniqueness of the element $\sigma_\theta \in \widehat{\mathcal{H}}_1$ such that

$$\sigma_\theta \in \Sigma_{ad}(\theta), \qquad \langle E\sigma_\theta, \tau - \sigma_\theta \rangle_V \geq 0 \qquad \forall \tau \in \Sigma_{ad}.$$

Then

$$\sigma_\theta \in \Sigma_{ad}(\theta), \qquad \sum_{\ell=1}^{2} \langle (F^\ell)^{-1}(\sigma_\theta), \tau^\ell - \sigma_\theta^\ell \rangle_{\mathcal{H}^\ell} \geq 0 \quad \forall \tau \in \Sigma_{ad}(\theta).$$

□

Let us consider now the operator $B : \widehat{\mathcal{H}}_1 \longrightarrow \widehat{\mathcal{H}}_1$ defined by

$$B\theta = \sigma_\theta : \quad \forall \theta \in \widehat{\mathcal{H}}_1. \tag{3.66}$$

4. Proof of Theorem 3.8

Proof. Existence. Let $u^* = (u^{*1}, u^{*2}) \in V$ the solutions of (3.37) with $\theta = \theta^*$. Taking $v = 0 \in V$ and $v = 2u^* \in V$ in (3.37) we obtain

$$a(u^*, u^*) + j(\theta^*, u^*) = \langle \varphi, u^* \rangle \tag{4.1}$$

and from (3.37), (4.1), we have

$$a(u^*, v^*) + j(\theta^*, v^*) \geq \langle \varphi, v^* \rangle \quad \forall v \in U_{ad}. \tag{4.2}$$

From (3.23), (4.2) and $\theta^* = A(\theta^*)$, it follows that

$$\theta^* \in \Sigma_{ad}(\theta^*). \tag{4.3}$$

Taking now $v = u \pm \phi \in V$ with $\phi = (\phi^1, \phi^2)$ and $\phi^\ell \in (D(\Omega^\ell))^N$, $\phi^{3-\ell} = 0$ in (3.37), it follows that

$$\langle F^\ell(\epsilon(u_\theta^\ell)), \epsilon(\phi^\ell) \rangle_{\mathcal{H}^\ell} = \langle \varphi^\ell, \phi^\ell \rangle. \tag{4.4}$$

Moreover, from (3.20), (4.4) and applying *Green's* formula, we have

$$-div(F^\ell(\epsilon(u_\theta^\ell))) = f^\ell \quad \text{ in } \Omega^\ell. \tag{4.5}$$

Using (4.1) and (3.20), we deduce that

$$\sum_{\ell=1}^{2} \int_{\Omega^\ell} div(F^\ell(\epsilon(u^{*\ell}))).u^{*\ell} d\Omega^\ell + \sum_{\ell=1}^{2} \int_{\Gamma^\ell} F^\ell(\epsilon(u^{*\ell}))\eta^\ell.u^{*\ell}\eta^\ell d\Gamma^\ell + j(\theta^*, u^*)$$

$$= \sum_{\ell=1}^{2} \int_{\Omega^\ell} f^\ell.u^{*\ell} d\Omega^\ell + \sum_{\ell=1}^{2} \int_{\Gamma_2^\ell} g^\ell.u^{*\ell}\eta^\ell d\Gamma_2^\ell.$$

Using now (4.5) and $u^*|_{\Gamma_1^\ell} \equiv 0$, we have

$$\int_{\Gamma_3} \theta^*\eta.[u^*\eta] d\Gamma_3 + j(\theta^*, u^*) = \sum_{\ell=1}^{2} \int_{\Gamma_2^\ell} g^\ell.u^{*\ell}\eta^\ell d\Gamma_2^\ell - \sum_{\ell=1}^{2} \int_{\Gamma_2^\ell} F^\ell(\epsilon(u^{*\ell}))\eta^\ell.u^{*\ell}\eta^\ell d\Gamma^\ell. \tag{4.6}$$

Taking $v = u \pm \phi \in V$ with $\phi = (\phi^1, \phi^2) \in V$, $\phi^{3-\ell} = 0$ and $\phi^\ell\eta^\ell = 0$ on $\Gamma_1^\ell \cup \Gamma_3$ in (3.37), it follows that

$$\int_{\Omega^\ell} F^\ell(\epsilon(u^{*\ell})).\epsilon(\phi^\ell) d\Omega^\ell = \int_{\Omega^\ell} f^\ell.\phi^\ell d\Omega^\ell + \int_{\Gamma_2^\ell} g^\ell.\phi^\ell\eta^\ell d\Gamma_2^\ell. \tag{4.7}$$

By applying *Green's* formula in (4.7) and using (4.5), we obtain

$$F^\ell(\epsilon(u^{*\ell}))\eta^\ell = g^\ell \;\; \texttt{on} \;\; \Gamma_2^\ell. \tag{4.8}$$

Combining (4.6) and (4.8), it follows that

$$\int_{\Gamma_3} \theta^*\eta.[u^*\eta] d\Gamma_3 = -j(\theta^*, u^*) \tag{4.9}$$

and, for any $\tau \in \Sigma_{ad}(\theta^*)$

$$\sum_{\ell=1}^{2} \int_{\Omega^\ell} \tau^\ell.\epsilon(u^{*\ell}) d\Omega^\ell \geq \langle \varphi, u^* \rangle - j(\theta^*, u^*). \tag{4.10}$$

Using (4.5) and (4.8) with $F^1(\epsilon(u^{*1}))\eta^1 = F^2(\epsilon(u^{*2}))\eta^2$ on Γ_3 , we deduce that

$$\sum_{\ell=1}^{2} \int_{\Omega^\ell} F^\ell(\epsilon(u^{*\ell})).\epsilon(u^{*\ell}) d\Omega^\ell = \langle \varphi, u^* \rangle + \int_{\Gamma_3} \theta^* \eta.[u^* \eta] d\Gamma_3. \tag{4.11}$$

Moreover, from (4.10), (4.11) and (4.9), we deduce the inequality in (3.30) witch proves that θ^* is a solution of problem $\mathbb{P}_2$.
It follows from Corollary3.6 that (u^*, θ^*) is a solution to problem$\mathbb{P}$.
Uniqueness. To prove the uniqueness of the solution let (u^*, θ^*) be the solution of problem$\mathbb{P}$ obtained above and let (u, σ) be another solution such that $u \in V$ and $\sigma \in \widehat{\mathcal{H}}_1$.
for all $\theta \in \widehat{\mathcal{H}}_1$. Therefore, choosing $\tilde{\sigma}_\theta = A\theta$ and using (3.37) and (3.59), we get

$$\sum_{\ell=1}^{2} \langle \tilde{\sigma}_\theta^\ell, \epsilon(v^\ell) - \epsilon(u_\theta^\ell) \rangle_{\mathcal{H}^\ell} + j(\theta, v) - j(\theta, u_\theta) \geq \langle \varphi, v - u_\theta \rangle_V, \ \forall v \in V. \tag{4.12}$$

Taking $v = 2u_\theta$ and $v = 0$ in (4.12), we obtain

$$\sum_{\ell=1}^{2} \langle \tilde{\sigma}_\theta^\ell, \epsilon(u_\theta^\ell) \rangle_{\mathcal{H}^\ell} + j(\theta, u_\theta) = \langle \varphi, u_\theta \rangle_V. \tag{4.13}$$

Using now (4.12) and (4.13), we have

$$\tilde{\sigma}_\theta \in \Sigma_{ad}(\theta) \tag{4.14}$$

and from (3.59), (4.13) and (3.23) it follows that

$$\sum_{\ell=1}^{2} \langle (F^\ell)^{-1}(\tilde{\sigma}_\theta^\ell), \tau^\ell - \tilde{\sigma}_\theta^\ell \rangle_{\mathcal{H}^\ell} \geq 0 \qquad \forall \tau \in \Sigma_{ad}(\theta). \tag{4.15}$$

Moreover, from (4.14) and (4.15), it results that $\tilde{\sigma}_\theta$ is a solution of problem (3.64). and by the uniqueness of the solution, we deduce $\tilde{\sigma}_\theta = \sigma_\theta$, then we have

$$A\theta = B\theta : \quad \forall \theta \in \widehat{\mathcal{H}}_1. \tag{4.16}$$

Using now Lemma 3.2, with

$$\theta^* = (F^1(\epsilon(u^{*^1})), F^2(\epsilon(u^{*^2})))$$

and

$$\sigma = (F^1(\epsilon(u^1)), F^2(\epsilon(u^2))),$$

such that

$$\begin{cases} \theta^* \in \Sigma_{ad}(\theta^*), \quad \sum_{\ell=1}^{2} \langle \tau^\ell - \theta^{*^\ell}, (F^\ell)^{-1}(\theta^{*^\ell}) \rangle_{\mathcal{H}^\ell} \geq 0, \ \forall \tau \in \Sigma_{ad}(\theta^*), \\ \sigma \in \Sigma_{ad}(\sigma), \quad \sum_{\ell=1}^{2} \langle \tau^\ell - \sigma^\ell, (F^\ell)^{-1}(\sigma^\ell) \rangle_{\mathcal{H}^\ell} \geq 0, \ \forall \tau \in \Sigma_{ad}(\sigma) \end{cases} \tag{4.17}$$

and from (3.64) and (3.66), we obtain

$$B\theta^* = \theta^*, \quad B\sigma = \sigma. \tag{4.18}$$

Moreover, from (4.18) and (4.16) and proposition3.10, it follows that

$$\theta^* = \sigma. \tag{4.19}$$

Hence

$$F^\ell(\epsilon(u^{*^\ell})) = F^\ell(\epsilon(u^\ell)) \quad \ell = 1, 2. \tag{4.20}$$

Therefore, by (3.16) and (4.20), we have

$$u^* = u.$$

The proof of Theorem 3.8 is complete. □

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More on pairwise extremally disconnected spaces

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Abstract. In [1] the authors, introduced the notion of pairwise extremally disconnected spaces and investigated its fundamental properties. In this paper, we investigate some more properties of pairwise extremally disconnected spaces.

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1. Introduction

The concept of bitopological spaces was first introduced by Kelly [4]. After the introduction of the definition of a bitopological space by Kelly, a large number of topologists have turned their attention to the generalization of different concepts of a single topological space in this space. In [1] the authors, introduced the notion of pairwise extremally disconnected spaces and investigated its fundamental properties. In this paper, we investigate some more properties of pairwise extremally disconnected spaces. Throughout this paper, the triple (X, τ_1, τ_2) where X is a set and τ_1 and τ_2 are topologies on X, will always denote a bitopological space. The τ_i-closure (resp. τ_i-interior) of a subset A of a bitopological space (X, τ_1, τ_2) is denoted by $\tau_i\text{-Cl}(A)$ (resp. $\tau_i\text{-Int}(A)$).

2. Preliminaries

Definition 2.1. *Let A be a subset of a bitopological space (X, τ_1, τ_2). Then A is called*

1. (τ_i, τ_j)*-regular open* [7] *if* $A = \tau_i\text{-Int}(\tau_j\text{-Cl}(A))$,
2. (τ_i, τ_j)*-semiopen* [2] *if* $A \subset \tau_j\text{-Cl}(\tau_i\text{-Int}(A))$,
3. (τ_i, τ_j)*-preopen* [5] *if* $A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}(A))$,
4. (τ_i, τ_j)*-semipreopen* [5] *if* $A \subset \tau_j\text{-Cl}(\tau_i\text{-Int}(\tau_j\text{-Cl}(A)))$,

On each definition above, $i, j = 1, 2$ and $i \neq j$.

The complement of an (i, j)-regular open (resp. (τ_i, τ_j)-semiopen, (τ_i, τ_j)-preopen, (τ_i, τ_j)-semipreopen) set is called an (i, j)-regular closed (resp. (τ_i, τ_j)-semiclosed, (τ_i, τ_j)-preclosed, (τ_i, τ_j)-semipreclosed) set.

Definition 2.2. [2] *Let A be a subset of a bitopological space (X, τ_1, τ_2). Then*

1. *The intersection of all (i, j)-semiclosed sets of X containing A is called the (i, j)-semiclosure of A and is denoted by (i, j)-$s\,\mathrm{Cl}(A)$.*
2. *The union of all (i, j)-semiopen sets of X contained in A is called the (i, j)-semiinterior of A and is denoted by (i, j)-$s\,\mathrm{Int}(A)$.*

Theorem 2.3. *For a subset A of a bitopological space (X, τ_1, τ_2), the following are equivalent:*

1. *A is (τ_i, τ_j)-semiopen,*
2. *$A \subset \tau_j$-$\mathrm{Cl}(\tau_i$-$\mathrm{Int}(A))$,*
3. *τ_j-$\mathrm{Cl}(A) = \tau_j$-$\mathrm{Cl}(\tau_i$-$\mathrm{Int}(A))$.*

Theorem 2.4. [2] *For a set A of a bitopological space (X, τ_1, τ_2), the following are equivalent:*

1. *A is (τ_i, τ_j)-semiclosed,*
2. *τ_j-$\mathrm{Int}(\tau_i$-$\mathrm{Cl}(A)) \subset A$,*
3. *τ_j-$\mathrm{Int}(A) = \tau_j$-$\mathrm{Int}(\tau_i$-$\mathrm{Cl}(A))$.*

Theorem 2.5. [2] *For a subset A of a bitopological space (X, τ_1, τ_2),*

1. *a point $x \in (i, j)$-$s\,\mathrm{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in (i, j)$-$SO(X, x)$.*
2. *(τ_i, τ_j)-$s\,\mathrm{Int}(A) = X \backslash (\tau_i, \tau_j)$-$s\,\mathrm{Cl}(X \backslash A)$,*
3. *(τ_i, τ_j)-$s\,\mathrm{Cl}(A) = X \backslash (\tau_i, \tau_j)$-$s\,\mathrm{Int}(X \backslash A)$.*

Definition 2.6. *A bitopological space (X, τ_1, τ_2) is said to be*

1. *(τ_i, τ_j)-extremally disconnected* [1] *if τ_j-closue of every τ_i-open set is τ_i-open in X,*
2. *pairwise extremally disconnetcted if (X, τ_1, τ_2) is (τ_1, τ_2)-extremally disconnected and (τ_2, τ_1)-extremally deisconnected.*

Theorem 2.7. [1] *A bitopological space (X, τ_1, τ_2) is pairwise extremally disconnected if and only if for each τ_i-open set A and each τ_j-open set B such that $A \cap B = \emptyset$, τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(B) = \emptyset$.*

3. Extremally disconnected bitopological spaces

Theorem 3.1. *The following are equivalent for a bitopological space (X, τ_1, τ_2):*

1. *(X, τ_1, τ_2) is pairwise extremally disconnected.*
2. *For each (τ_j, τ_i)-semiopen set A in X, τ_j-$\mathrm{Cl}(A)$ is τ_i-open set.*
3. *For each (τ_i, τ_j)-semiopen set A in X, (τ_j, τ_i)-$s\,\mathrm{Cl}(A)$ is τ_i-open set.*
4. *For each (τ_i, τ_j)-semiopen set A and each (τ_j, τ_i)-semiopen set B with $A \cap B = \emptyset$, τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(B) = \emptyset$.*
5. *For each (τ_j, τ_i)-semiopen set A in X, τ_j-$\mathrm{Cl}(A) = (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(A)$.*
6. *For each (τ_i, τ_j)-semiopen set A in X, (τ_j, τ_i)-$s\,\mathrm{Cl}(A)$ is τ_j-closed set.*
7. *For each (τ_i, τ_j)-semiclosed set A in X, τ_j-$\mathrm{Int}(A) = (\tau_j, \tau_i)$-$s\,\mathrm{Int}(A)$.*
8. *For each (τ_i, τ_j)-semiclosed set A in X, (τ_j, τ_i)-$s\,\mathrm{Int}(A)$ is τ_j-open set.*

Proof. (1) $\Rightarrow$ (2): Follows from Theorem 2.3. (1) $\Rightarrow$ (5): Since (τ_j,τ_i)-$s\,\mathrm{Cl}(A) \subset \tau_j$-$\mathrm{Cl}(A)$ for any set A of X, it is sufficient to show that (τ_j,τ_i)-$s\,\mathrm{Cl}(A) \supset \tau_j$-$\mathrm{Cl}(A)$ for any (τ_i,τ_j)-semiopen set A of X. Let $x \notin (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A)$. Then there exists a (τ_j,τ_i)-semiopen set W with $x \in W$ such that $W \cap A = \emptyset$. Thus τ_j-$\mathrm{Int}(W)$ and τ_i-$\mathrm{Int}(A)$ are, respectively, τ_j-open and τ_i-open such that τ_j-$\mathrm{Int}(X) \cap \tau_i$-$\mathrm{Int}(A) = \emptyset$. By Theorem 2.7, τ_i-$\mathrm{Cl}(\tau_j$-$\mathrm{Int}(W)) \cap \tau_j$-$\mathrm{Cl}(\tau_i$-$\mathrm{Int}(A)) = \emptyset$ and then by Theorem 2.4, $x \notin \tau_j$-$\mathrm{Cl}(\tau_i$-$\mathrm{Int}(A)) = \tau_j$-$\mathrm{Cl}(A)$. Hence τ_j-$\mathrm{Cl}(A) \subset (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A)$. (5) $\Rightarrow$ (6): Obvious. (6) $\Rightarrow$ (5): For any set A in X, $A \subset (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A) \subset \tau_j$-$\mathrm{Cl}(A)$. Then τ_j-$\mathrm{Cl}(A) = \tau_j$-$\mathrm{Cl}((\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A))$. Since A is (τ_i,τ_j)-semiopen, by (6), (τ_j,τ_i)-$s\,\mathrm{Cl}(A)$ is τ_j-closed. Hence, τ_j-$\mathrm{Cl}(A) = (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A)$. (6) $\Leftrightarrow$ (8): Follows from Theorem 2.5. (7) $\Rightarrow$ (8): Obvious. (8) $\Rightarrow$ (7): For any subset A of X, τ_j-$\mathrm{Int}(A) \subset (\tau_j,\tau_i)$-$s\,\mathrm{Int}(A) \subset A$ and hence τ_j-$\mathrm{Int}(A) = \tau_j$-$\mathrm{Int}((\tau_j,\tau_i)$-$s\,\mathrm{Int}(A))$. Since A is (τ_i,τ_j)-semiclosed, by (8), (τ_j,τ_i)-$s\,\mathrm{Int}(A)$ is τ_j-open. Hence τ_j-$\mathrm{Int}(A) = (\tau_j,\tau_i)$-$s\,\mathrm{Int}(A)$. (1) $\Rightarrow$ (4): Let A be a (τ_i,τ_j)-open set and B a (τ_j,τ_i)-semiopen set such that $A \cap B = \emptyset$. Then τ_i-$\mathrm{Int}(A) \cap \tau_j$-$\mathrm{Int}(B) = \emptyset$ and thus by Theorem 2.7, τ_j-$\mathrm{Cl}(\tau_j$-$\mathrm{Int}(A)) \cap \tau_i$-$\mathrm{Cl}(\tau_j$-$\mathrm{Int}(B)) = \emptyset$. Hence, by Theorem 2.3, τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(B) = \emptyset$. (4) $\Rightarrow$ (2): Let A be a (τ_i,τ_j)-semiopen subset of X. Then $X \backslash \tau_j$-$\mathrm{Cl}(A)$ is (τ_j,τ_i)-semiopen and $A \cap (X \backslash \tau_j$-$\mathrm{Cl}(A))$. Thus, by (4), τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(X \backslash \tau_j$-$\mathrm{Cl}(A)) = \emptyset$ which implies τ_j-$\mathrm{Cl}(A) \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}(A))$. Hence, τ_j-$\mathrm{Cl}(A) = \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}(A))$ and consequently τ_j-$\mathrm{Cl}(A)$ is τ_i-open in X. (5) $\Rightarrow$ (4): Let A be a (τ_i,τ_j)-semiopen set and B be a (τ_j,τ_i)-semiopen set such that $A \cap B = \emptyset$. Then (τ_j,τ_i)-$s\,\mathrm{Cl}(A)$ is (τ_i,τ_j)-semiopen and (τ_i,τ_j)-$s\,\mathrm{Cl}(B)$ is (τ_j,τ_i)-semiopen in X and hence (τ_j,τ_i)-$s\,\mathrm{Cl}(A) \cap (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(B) = \emptyset$. By (5), τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(B) = \emptyset$. (1) $\Rightarrow$ (3): Follows from Theorem 2.3 using the same method as (1) $\Rightarrow$ (5). (3) $\Rightarrow$ (1): Let A be a τ_i-open set in (X,τ_1,τ_2). It is sufficient to prove that τ_j-$\mathrm{Cl}(A) = (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A)$. Obviously, (τ_j,τ_i)-$s\,\mathrm{Cl}(A) \subset \tau_j$-$\mathrm{Cl}(A)$. Let $x \notin (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A)$. Then there exists a (τ_j,τ_i)-semiopen set U with $x \in U$ such that $A \cap U = \emptyset$. Hence (τ_i,τ_j)-$s\,\mathrm{Cl}(U) \subset (\tau_i,\tau_j)$-$s\,\mathrm{Cl}(X \backslash A) = X \backslash A$ and thus (τ_i,τ_j)-$s\,\mathrm{Cl}(U) \cap A = \emptyset$. Since (τ_i,τ_j)-$s\,\mathrm{Cl}(U)$ is a τ_j-open set with $x \in (\tau_i,\tau_j)$-$s\,\mathrm{Cl}(U)$, $x \notin \tau_j$-$\mathrm{Cl}(A)$. Hence τ_j-$\mathrm{Cl}(A) \subset (\tau_j,\tau_i)$-$\mathrm{Cl}(A)$. □

Definition 3.2. [3] *A point x in a bitoplogical space (X,τ_1,τ_2) is said to be (τ_i,τ_j)-θ-cluster point of a set A if for every τ_i-open, say, U containing x, τ_j-$\mathrm{Cl}(U) \cap A \neq \emptyset$. The set of all (τ_i,τ_j)-θ-closure of A and will be denoted by (τ_i,τ_j)-$\mathrm{Cl}_\theta(A)$. A set A is called (τ_i,τ_j)-θ-closed if $A = (\tau_i,\tau_j)$-$\mathrm{Cl}_\theta(A)$.*

Lemma 3.3. *For any (τ_j,τ_i)-preopen set A in a bitopological space (X,τ_1,τ_2), τ_i-$\mathrm{Cl}(A) = (\tau_i,\tau_j)$-$\mathrm{Cl}_\theta(A)$.*

Proof. It is obvious that τ_i-$\mathrm{Cl}(A) \subset (\tau_i,\tau_j)$-$\mathrm{Cl}_\theta(A)$, for any subset A of (X,τ_1,τ_2). Thus, it remains to be shown that (τ_i,τ_j)-$\mathrm{Cl}_\theta(A) \subset \tau_i$-$\mathrm{Cl}(A)$. If $x \notin \tau_i$-$\mathrm{Cl}(A)$, then there exists a τ_i-open set U containing x such that $U \cap A = \emptyset$ and thus $U \cap \tau_i$-$\mathrm{Cl}(A) = \emptyset$. But $U \cap \tau_j$-$\mathrm{Int}(\tau_i$-$\mathrm{Cl}(A)) = \emptyset$ which implies τ_j-$\mathrm{Cl}(U) \cap \tau_j$-$\mathrm{Int}(\tau_i$-$\mathrm{Cl}(A)) = \emptyset$ and so τ_j-$\mathrm{Cl}(U) \cap A = \emptyset$ since A is (τ_j,τ_i)-preopen. Hence $x \notin (\tau_j,\tau_i)$-$\mathrm{Cl}_\theta(A)$ and consequently (τ_j,τ_i)-$\mathrm{Cl}_\theta(A) \subset \tau_i$-$\mathrm{Cl}(A)$. □

Theorem 3.4. *The following are equivalent for a bitopological space (X,τ_1,τ_2):*

1. *(X,τ_1,τ_2) is pairwise extremally disconnected.*

2. *The τ_j-closure of every (τ_i, τ_j)-semipreopen set of X is τ_i-open set.*
3. *The (τ_j, τ_i)-θ-closure of every (τ_i, τ_j)-preopen set of X is τ_i-open set.*
4. *The τ_j-closure of every (τ_i, τ_j)-preopen set of X is τ_i-open set.*

Proof. (1) $\Rightarrow$ (2): Let A be a (τ_i, τ_j)-semipreopen set. Then τ_j-Cl$(A) = \tau_j$-Cl$(\tau_i$-Int$(\tau_j$-Cl$(A)))$. Since (X, τ_1, τ_2) is pairwise extremally disconnected. τ_j-Cl(A) is a τ_i-open set. (2) $\Rightarrow$ (4): Follows from the fact that every (τ_i, τ_j)-preopen set is (τ_i, τ_j)-semipreopen. (4) $\Rightarrow$ (1): Clear. (3) $\Leftrightarrow$ (4): Follows from Lemma 3.3. □

Theorem 3.5. *A bitopological space (X, τ_1, τ_2) is pairwise extremally disconnected if and only if every (τ_i, τ_j)-semiopen set is a (τ_i, τ_j)-preopen set.*

Proof. Let A be a (τ_i, τ_j)-semiopen set. Then $A \subset \tau_j$-Cl$(\tau_i$-Int$(A))$. Since X is pairwise extremally disconnected, τ_j-Cl$(\tau_i$-Int$(A))$ is a τ_i-open set and then $A \subset \tau_j$-Cl$(\tau_i$-Int$(A)) = \tau_i$-Int$(\tau_j$-Cl$(\tau_i$-Int$(A))) \subset \tau_i$-Int$(\tau_j$-Cl$(A))$. Hence A is a (τ_i, τ_j)-preopen set. Conversely, let A be a τ_i-open set. Since τ_j-Cl$(A) = \tau_j$-Cl$(\tau_i$-Int$(A))$, we have τ_j-Cl$(A) = \tau_j$-Cl$(\tau_i$-Int$(\tau_j$-Cl$(A)))$. Then τ_j-Cl(A) is (τ_j, τ_i)-regular closed and hence A is (τ_i, τ_j)-semiopen. By hypothesis, A is (τ_i, τ_j)-propen so that τ_j-Cl$(A) = \tau_i$-Int$(\tau_j$-Cl$(A))$. Then τ_j-Cl(A) is τ_i-open in X and hence X is pairwise extremally disconnected. □

Lemma 3.6. *For a subset A of a bitopological space (X, τ_1, τ_2),*

1. τ_j-Int$(\tau_i$-Cl$(A)) \subset (\tau_i, \tau_j)$-$s\,Cl(A)$,[6]
2. τ_j-Int$((\tau_i, \tau_j)$-$s\,$Cl$(A)) = \tau_j$-Int$(\tau_i$-Cl$(A))$.

Proof. (2) Follows easily from (1). □

Theorem 3.7. *Let A be a subset of a bitopological space (X, τ_1, τ_2). Then A is (τ_i, τ_j)-regular open if and only if A is τ_i-open and τ_j-closed.*

Proof. Let A be a (τ_i, τ_j)-regular open set of a bitoplogical space (X, τ_1, τ_2). Then τ_i-Int$(\tau_j$-Cl$(A)) = A$. Now, $X \backslash \tau_j$-Cl(A) and A are, respectively, τ_j-open and τ_i-open such that $(X \backslash \tau_j$-Cl$(A)) \cap A = \emptyset$. Since (X, τ_1, τ_2) is pairwise extremally disconnected, by Theorem 2.7, τ_i-Cl$(X \backslash \tau_j$-Cl$(A)) \cap \tau_j$-Cl$(A) = \emptyset$. Then τ_i-Cl$(X \backslash \tau_j$-Cl$(A)) = X \backslash \tau_j$-Cl$(A)$ and $X \backslash \tau_j$-Cl(A) is τ_i-closed. Hence, τ_j-Cl(A) is τ_i-open, so that τ_j-Cl$(A) = \tau_i$-Int$(\tau_j$-Cl$(A)) = A$ is τ_i-open and τ_j-closed. The converse is clear. □

Lemma 3.8. *Let A be a subset of a bitopological space (X, τ_1, τ_2). Then we have*

1. *A is (τ_i, τ_j)-preopen if and only if (τ_j, τ_i)-$s\,$Cl$(A) = \tau_i$-Int$(\tau_j$-Cl$(A))$.*
2. *A is (τ_i, τ_j)-preopen if and only if (τ_j, τ_i)-$s\,$Cl(A) is (τ_i, τ_j)-regular open.*
3. *A is (τ_i, τ_j)-regular open if and only if A is (τ_i, τ_j)-preopen and (τ_j, τ_i)-semiclosed.*

Proof. (1) Let A b e a (τ_i, τ_j)-preopen set. Then (τ_j, τ_i)-$s\,$Cl$(A) \subset (\tau_j, \tau_i)$-$s\,Cl(\tau_i$-Int$(\tau_j$-Cl$(A)))$. Since τ_i-Int$(\tau_j$-Cl$(A))$ is (τ_j, τ_i)-semiclosed, (τ_j, τ_i)-$s\,$Cl$(A) \subset \tau_i$-Int$(\tau_j$-Cl$(A))$. Hence, by Lemma 3.6 (1), (τ_j, τ_i)-$s\,$Cl$(A) = \tau_i$-Int$(\tau_j$-Cl$(A))$. The converse is obvious. (2) Let (τ_j, τ_i)-$s\,$Cl(A) be a (τ_i, τ_j)-regular open set. Then we have

(τ_j,τ_i)-$s\,\mathrm{Cl}(A) = \tau_i$-Int($\tau_j$-Cl$(\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A)$) and hence (τ_j,τ_i)-$s\,\mathrm{Cl}(A) \subset \tau_i$-Int($\tau_j$-Cl($\tau_j$-Cl$(A)$))) $= \tau_i$- Int(τ_j-Cl(A)). By Lemma 3.6 (1), we have (τ_j,τ_i)-$s\,\mathrm{Cl}(A) = \tau_i$-Int($\tau_j$-Cl$(A)$). Hence, A is a (τ_i,τ_j)-preopen set from (1). The converse follows from (1). (3) Let A be a (τ_i,τ_j)-preopen and a (τ_j,τ_i)-semiclosed set. Then by (2), A is (τ_i,τ_j)-regular open in X. Conversely, let A be a (τ_i,τ_j)-regular open set. Then $A = \tau_i$-Int(τ_j-Cl(A)) and thus τ_i-Int(τ_j-Cl(A)) $= (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A) = A$. Hence A is (τ_i,τ_j)-preopen and (τ_j,τ_i)-semiclosed. □

Theorem 3.9. *In a bitopological space (X,τ_1,τ_2), the following are equivalent:*

1. *X is pairwise extremally disconnected.*
2. *(τ_j,τ_i)-$s\,\mathrm{Cl}(A) = (\tau_j,\tau_i)$-$\mathrm{Cl}_\theta(A)$ for every (τ_i,τ_j)-preopen (or (τ_i,τ_j)-semiopen) set A in X.*
3. *(τ_j,τ_i)-$s\,\mathrm{Cl}(A) = \tau_j$-Cl$(A)$ for every (τ_i,τ_j)-semipreopen set A in X.*

Proof. (1) ⇒ (2): Since (τ_j,τ_i)-$s\,\mathrm{Cl}(A) \subset (\tau_j,\tau_i)$-$\mathrm{Cl}_\theta(A)$ for any subset A of X, it is sufficient to show that (τ_j,τ_i)-$\mathrm{Cl}_\theta(A) \subset (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A)$ for any (τ_i,τ_j)-preopen or (τ_i,τ_j)-semiopen set A of X. Let $x \notin (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A)$. Then there exists a (τ_j,τ_i)-semiopen set U with $x \in U$ such that $U \cap A = \emptyset$ and thus there exists a τ_j-open set V such that $V \subset U \subset \tau_j$-Cl$(V)$ with $V \cap A = \emptyset$ which implies $V \cap \tau_j$-Cl$(A) = \emptyset$. This means $V \cap \tau_i$-Int(τ_j-Cl(A)) $= \emptyset$ and hence τ_i-Cl$(V) \cap \tau_i$-Int(τ_j-Cl(A)) $= \emptyset$. Now, if A is (τ_i,τ_j)-preopen, then $A \subset \tau_i$-Int(τ_j-Cl(A)) and hence τ_i-Cl$(V) \cap A = \emptyset$. If A is (τ_i,τ_j)-semiopen, since X is pairwise extremally disconnected, τ_i-Cl(V) is τ_j-open and thus τ_i-Cl$(V) \cap \tau_j$-Cl(τ_i-Int(τ_j-Cl(A))) $= \emptyset$ which implies τ_i-Cl$(V) \cap A = \emptyset$. Hence, in any case, $x \notin (\tau_j,\tau_i)$-$\mathrm{Cl}_\theta(A)$. (2) ⇒ (1): First let A be a (τ_i,τ_j)-preopen set in X. By Lemmas 3.8 and 3.3, we have τ_i-Int(τ_j-Cl(A)) $= (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A) = (\tau_j,\tau_i)$-$\mathrm{Cl}_\theta(A) = \tau_j$-Cl$(A)$. Then τ_j-Cl(A) is τ_i-open and hence by Theorem 3.4, X is pairwise extremally disconnected. Next, let A be a (τ_i,τ_j)-semiopen set in X. Then (τ_j,τ_i)-Cl$(A) \subset \tau_j$-Cl$(A) \subset (\tau_j,\tau_i)$-$\mathrm{Cl}_\theta(A) = (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A)$ and thus (τ_j,τ_i)-$s\,\mathrm{Cl}(A) = \tau_j$-Cl$(A)$. Hence, X is pairwise extremally disconnected from Theorem 3.4. (1) ⇒ (3): Let A be a (τ_i,τ_j)-semipreopen set in X. Since X is pairwise extremally disconnected, by Theorem 3.4, τ_j-Cl(A) is τ_i-open in X. Hence, by Lemma 3.8, (τ_j,τ_i)-$s\,\mathrm{Cl}(A) = \tau_j$-Cl$(A)$. (3) ⇒(1): Let U and V, respectively, be τ_i-open and τ_j-open sets such that $U \cap V = \emptyset$. Then $U \subset X\backslash V$ which implies (τ_j,τ_i)-$s\,\mathrm{Cl}(U) \subset (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(X\backslash V) = X\backslash V$ and hence (τ_j,τ_i)-$s\,\mathrm{Cl}(U) \cap V = \emptyset$. Since (τ_j,τ_i)-$s\,\mathrm{Cl}(U)$ is (τ_i,τ_j)-semiopen in X, (τ_j,τ_i)-$s\,\mathrm{Cl}(U) \cap (\tau_i,\tau_j)$-$s\,\mathrm{Cl}(V) = \emptyset$. Then by (3) τ_j-Cl$(U) \cap \tau_i$-Cl$(V) = \emptyset$ and hence by Theorem 2.7, X is pairwise extremally disconnected. □

Theorem 3.10. *In a bitopological space (X,τ_1,τ_2), the following are equivalent:*

1. *X is pairwise extremally disconnected.*
2. *For each (τ_i,τ_j)-semipreopen set A in X and each (τ_j,τ_i)-semiopen set B in X such that $A \cap B = \emptyset$, τ_i-Cl$(A) \cap \tau_j$-Cl$(B) = \emptyset$*
3. *For each (τ_i,τ_j)-preopen set A in X and each (τ_j,τ_i)-semiopen set B in X such that $A \cap B = \emptyset$, τ_i-Cl$(A) \cap \tau - j$-Cl$(B) = \emptyset$.*

Proof. (1)⇒ (2): Let A be a (τ_i,τ_j)-semipreopen set and B a (τ_j,τ_i)-semiopen set such that $A \cap B = \emptyset$. Then $A \cap \tau_j$-Int$(B) = \emptyset$ and hence τ_j-Cl$(A) \cap \tau_j$-Int$(B) = \emptyset$. By Theorem 3.4, τ_j-Cl(A) is a τ_i-open set in X and hence τ_j-Cl$(A) \cap \tau_i$-Cl(τ_j-Int(B)) $= \emptyset$.

Since B is (τ_j, τ_i)-semiopen in X, by Theorem 2.3, τ_i-Cl$(B) = \tau_i$-Cl$(\tau_j$-Int$(B))$. Thus τ_j-Cl$(A) \cap \tau_i$-Cl$(B) = \emptyset$. (2) $\Rightarrow$ (3): Straightforward. (3) $\Rightarrow$ (1): Let A be a τ_i-open set and B a τ_j-open set such that $A \cap B = \emptyset$. Since every τ_i-open set is a (τ_i, τ_j)-semiopen set and every τ_j-open set is a (τ_i, τ_j)-semiopen set and every τ_j-open set is a (τ_j, τ_i)-preopen set, τ_j-Cl$(A) \cap \tau_i$-Cl$(B) = \emptyset$. Hence by Theorem 2.7, X is pairwise extremally disconnected. □

Definition 3.11. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be*

1. *pairwise semicontinuous* [2] *if $f^{-1}(V)$ is a (τ_i, τ_j)-semiopen set in X for each σ_i-open set V in Y.*
2. *pairwise almost open if $f(U)$ is a σ_i-open set in Y for each (τ_i, τ_j)-regular open set U in X.*

Lemma 3.12. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise almost open if and only if for each (τ_j, τ_i)-semiclosed set A in X, $f(\tau_i$-Int$(A)) \subset \sigma_i$-Int$(f(A))$.*

Proof. Let A be a (τ_j, τ_i)-semiclosed set in X. Then τ_i-Int$(\tau_j$-Cl$(A))$ is (τ_i, τ_j)-regular open and hence $f(\tau - i$-Int$(\tau_j$-Cl$(A)))$ is σ_i-open in Y. Now by Theorem 2.4, τ_i-Int$(A) = \tau_i$-Int$(\tau_j$-Cl$(A)) \subset A$ which implies that $f(\tau_i$-Int$(A)) = f(\tau_i$-Int$(\tau_j$-Cl$(A))) = \sigma_i$-Int$(f(\tau_i$-Int$(\tau_j$-Cl$(A)))) \subset \sigma_i$-Int$(f(A))$. Hence $f(\tau_i$-Int$(A)) \subset \sigma_i$-Int$(f(A))$. Conversely, let A be a (τ_i, τ_j)-regular open set in X. Then A is (τ_j, τ_i)-semiclosed and hence $f(\tau_i$-Int$(A)) \subset \sigma_i$-Int$(f(A))$. Now, $A = \tau_i$-Int(A) and thus $f(A) = f(\tau_i$-Int$(A)) \subset \sigma_i$-Int$(f(A))$, so that $f(A)$ is σ_i-open in Y. Hence f is pairwise almost open. □

Lemma 3.13. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise semicontinuous and a pairwise almost open mapping, then $f(A)$ is a (σ_i, σ_j)-preopen set in Y for each (τ_i, τ_j)-preopen set A in X.*

Proof. Let A be a (τ_i, τ_j)-preopen set in X. Since f is pairwise semicontinuous, $f(A) \subset f((\tau_j, \tau_i)$-$s\,Cl(A)) \subset \sigma_j$-Cl$(f(A))$. By Lemma 3.8 (2), (τ_j, τ_i)-$s\,$Cl(A) is (τ_j, τ_i)-regular open set in X and thus $f((\tau_j, \tau_i)$-$s\,$Cl$(A))$ is a (σ_i, σ_j)-preopen set in Y because f is pairwise almost open. By Lemma 3.8 (1), (σ_j, σ_i)-$s\,$Cl$(f((\tau_j, \tau_i)$-$s\,$Cl$(A))) = \sigma_i$-Int$(\sigma_j$-Cl$(f((\tau_j, \tau_i)$-$s\,$Cl$(A))))$. Hence, (σ_j, σ_i)-$s\,$Cl$(f(A)) \subset (\sigma_j, \sigma_i)$-$s\,Cl(f((\tau_j, \tau_i)$-$s\,Cl(A))) = \sigma_i$-Int$(\sigma_j$-Cl$(f((\tau_j, \tau_i)$-$s\,Cl(A)))) \subset \sigma_j$-Cl$(f(A))$. Since σ_i-Int$(\sigma_j$-Cl$(f(A))) = \sigma_i$-Int$(\sigma_j$-Cl$(f((\tau_j, \tau_i)$-$s\,Cl(A))))$, we have $f(A) \subset (\sigma_j, \sigma_i)$-$s\,Cl(f(A)) \subset \sigma_i$-Int$(\sigma_j$-Cl$(f(A)))$. Hence $f(A)$ is (σ_i, σ_j)-preopen in Y. □

Lemma 3.14. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise semicontinuous and a pairwise almost open mapping we have*

1. *$f^{-1}(B)$ is a (τ_i, τ_j)-semiclosed set in X for each (σ_i, σ_j)-semiclosed set in B in Y.*
2. *$f^{-1}(B)$ is a (τ_i, τ_j)-semiopen set in X for each (σ_i, σ_j)-semiopen set in B in Y.*

Proof. (1) Let B be a (σ_i, σ_j)-semiclosed set in Y. Since f is pairwise semicontinuous and σ_i-Cl(B) is a σ_i-closed set, $f^{-1}(\sigma_i$-Cl$(B))$ is (τ_i, τ_j)-semiclosed in X. Hence, τ_i-Int$(\tau_j$-Cl$(f^{-1}(\sigma_i$-Cl$(B)))) \subset \tau_j$- Int$(f^{-1}(\sigma_i$-Cl$(B)))$. Since f is pairwise almost open by Lemma 3.12 $f(\tau_j\text{-Int}(f^{-1}(\sigma_i\text{-Cl}(B)))) \subset \tau_j\text{-Int}(f(f^{-1}(\sigma_i\text{-Cl}(B)))) \subset \sigma_j\text{-Int}(\sigma_i\text{-Cl}(B)) \subset B$. Which implies that $\tau_j\text{-Int}(f^{-1}(\sigma_i\text{-Cl}(B))) \subset f^{-1}(B)$. Now, $\tau_j\text{-Int}(\tau_i\text{-Cl}(f^{-1}(B))) \subset \tau_j\text{-Int}(\tau_i\text{- Cl}(f^{-1}(\sigma_i\text{-Cl}(B)))) \subset \tau_j\text{-Int}(f^{-1}(\sigma_i\text{- Cl}(B))) \subset f^{-1}(B)$. Hence $f^{-1}(B)$ is a (τ_i, τ_j)-semiclosed set in X. (2) Follows easily from (1) by taking the complement. □

Theorem 3.15. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a pairwise semicontinuous and a pairwise almost open surjection. If (X, τ_1, τ_2) is pairwise extremally disconnected, then (Y, σ_1, σ_2) is also pairwise extremally disconnected.*

Proof. Let B be a (σ_i, σ_j)-semiopen set in Y. By Lemma 3.14, $f^{-1}(B)$ is (τ_i, τ_j)-semiopen in X. Since X is pairwise extremally disconnected, by Theorem 3.5, $f^{-1}(B)$ is (τ_i, τ_j)-preopen in X. By Lemma 3.13, B is (σ_i, σ_j)-preopen in Y and hence by Theorem 3.5, Y is pairwise extremally disconnected. □

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Book reviews

Joram Lindenstrauss, David Preiss and Jaroslav Tišer, *Fréchet differentiability of Lipschitz functions and porous sets in Banach spaces,* Annals of Mathematics Studies, vol. 179, Princeton University Press, Princeton, NJ, 2012, ix+425 pp. ISBN: 978-0-691-15355-1 (hardcover) – ISBN: 978-0-691-15356-8 (pbk.).

A classical result proved by Henri Lebesgue in his thesis (around 1900) asserts that every monotone function $f : \mathbb{R} \to \mathbb{R}$ is almost everywhere (a.e.) differentiable. As consequence, the functions with bounded variation and, in particular, the Lipschitz functions have this property. The result was extended in 1919 to Lipschitz functions $f : \mathbb{R}^n \to \mathbb{R}$ by Hans Rademacher. The case of the function $t \mapsto \chi_{[0,t]}$ from the interval $[0, 1]$ to $L^1[0, 1]$, which is nowhere differentiable, shows that some restrictions have to be imposed to the range space Y in order to extend this result to vector functions $f : U \to Y$, $U \subset \mathbb{R}^n$ an open subset and Y a Banach space: if the Banach space Y has the Radon-Nikodým Property (RNP), then the differentiability result is true. In fact, this property is equivalent to the RNP: if every Lipschitz function $f : \mathbb{R} \to Y$ is a.e. differentiable, then the Banach space Y has the RNP. The next step is to extend further the result to Lipschitz functions $f : U \to Y$, where U is an open subset of a Banach space X. It is clear that one needs first an appropriate notion of "almost everywhere" in an infinite dimensional Banach space. Since it is impossible to define a measure μ on an infinite dimensional Banach space X such that the class of sets of μ-measure 0 be a useful class of negligible sets, these must be defined by other means. There are several nonequivalent ways to define negligible sets in infinite dimension, leading to different classes – Haar null sets (J. P. R. Christensen, 1972), cube null sets (P. Mankiewicz, 1973) Gauss null sets (R. R. Phelps, 1978), Aronszajn null sets (N. Aronszajn, 1976), a.o., each of them forming a proper σ-ideal of Borel subsets of the Banach space X. Later, M. Csörnyei (1999) has shown that the classes of Gauss null, cube null and Aronszajn null sets agree and are properly contained in the class of Haar null sets. A good presentation of these result as well as of the a.e. Gâteaux differentiability of Lipschitz functions is given in the sixth chapter of the book by Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Analysis*, Colloq. Publ. vol 48, AMS 2000. A brief presentation of these results is contained also in the second chapter of the present book. Supposing the space X separable and Y an RNP space, then every Lipschitz function $f : U \to Y$, $U \subset X$ open, is Gâteaux differentiable on U, excepting an Aronszajn null set.

The question of a.e. Fréchet differentiability is more delicate and the results are far from being complete. The norm $\|\cdot\|$ of ℓ^1 is nowhere Fréchet differentiable and a similar situation appears when X is separable with nonseparable conjugate, so a natural requirement on the domain space X is to be with separable conjugate (such a space is called an Asplund space). As the authors show in Section 3.6, some results can be extended to the nonseparable case by using separable reduction techniques. The first result, with an involved proof, in this direction was obtained by D. Preiss (1990) – any real-valued Lipschitz function on an Asplund space has a dense set of points of Fréchet differentiability. Another proof, simpler but still involved, was given by Lindenatrauss and Preiss in 2000. The authors consider a general conjecture, namely that for every Asplund space X there exists a nontrivial notion of negligible sets such that every locally Lipschitz function f defined on an open subset G of X and with values in an RNP space Y satisfies the conditions:

(C1) f is a.e. Gâteaux differentiable on G;

(C2) if $S \subset G$ has null complement and f is Gâteaux differentiable on S, then $\mathrm{Lip}(f) = \sup_S \|f'(x)\|$;

(C3) if for some $E \subset G$ the set $\{f'(x) : x \in E\}$ is separable (in $L(X,Y)$), then f is a.e. differentiable on E.

As the authors do mention, a challenging problem in this area is whether a countable family of Lipschitz functions defined on an Asplund space has a common point of Fréchet differentiability. In this book one shows that any Lipschitz mapping from a Hilbert space to $Y = \mathbb{R}^2$ is a.e. differentiable and satisfies a multi-dimensional mean value estimate (which is stronger that (C2)), the first known result of this kind. The result is obtained as a consequence of some more general results, but in the last chapter of the book, Chapter 16, a direct proof, essentially self-contained, based on specific properties of Hilbert spaces, is given. The differentiability result does not hold for $Y = \mathbb{R}^3$, or, more generally, for maps on ℓ^p to $\mathbb{R}^n$ with $n > p$.

Since the distance function $d(\cdot, E)$ to a subset E of a normed space X is nowhere F-differentiable iff the set E is porous, a natural requirement would be the containment of porous sets in these σ-ideals of negligible sets. Lindenstrauss and Preiss (2003) considered Γ-null sets, whose definition involves both Baire category and Lebesgue measure, and proved that each Lipschitz function from a separable Banach space X to an RNP space Y is G-differentiable excepting a Γ-null set, and that such a function is regularly G-differentiable (a notion stronger than Gâteaux differentiability) excepting a σ-porous set. Since a function is a.e. Fréchet differentiable on the set on which it is regularly Gâteaux differentiable, it follows that (C1)–(C3) hold for every Banach space X for which σ-porous sets are Γ-null. Example of such spaces are $C(K)$ with K countable and compact, subspaces of c_0, the Tsirelson space, but not Hilbert spaces.

In the present book, one considers also a more refined concept, namely Γ_n-null sets, which can be viewed as finite dimensional versions of Γ-null sets, one studies the relations between these classes and one presents new classes of Banach spaces for which strong Fréchet differentiability results hold. Of course, some geometric properties of the space X are involved, among which the notions of asymptotic uniform smoothness, asymptotic uniform convexity and their moduli, as well as the modulus of asymptotic smoothness associated to a function. The relevance of these geometric properties for the existence of bump functions with controlled moduli of asymptotic smoothness is discussed in detail in Chapter 8.

Most of the results in this book (Chapters 7–14) are new, leading to a better understanding of this difficult problem - the a.e. Fréchet differentiability of Lipschitz functions. The book is very well organized – each chapter starts with a brief information about its content, followed by an introductory section containing the most important results of the chapter and the proofs of some relevant corollaries. Also the authors explain the intuitive ideas behind the proofs (most of them long and difficult) and to the abstract notions considered in the book.

In conclusion, this is a remarkable book, opening new ways for further investigation of the Fréchet differentiability of Lipschitz functions. It is of great interest for researchers in functional analysis, mainly those interested in the applications of Banach space geometry.

S. Cobzaş

William J. Cook, *In Pursuit of the Traveling Salesman Problem: Mathematics at the Limits of Computation,* Princeton University Press, 2012, xiii + 228 pp., ISBN13: 978-0-691-15270-7

The Traveling Salesman Problem (TSP for short) is simply to formulate but very hard to solve. We are given a collection of cities and the distance to travel between each pair of them and one ask to find the shortest route to visit each city and to return to the starting point. The problem has many practical applications, presented in Chapter 3, *The salesman in action* - first of all routes for traveling salesmen (the GPS system often includes a TSP solver for small instances having a dozen of cities, which usually suffices for daily trips), the routing of buses and vans to pick up or deliver people and packages, to genome study, X-ray crystallography, computer chips, tests for microprocessors, organizing data, and even to music (organizing vast collections of computer-encoded music), speeding up video games, etc.

The difficulty of the problem arises from the big amount of possibilities to be examined in order to find the optimal one. For instance, the 33 cities problem, for which Procter & Gamble offered in 1962 a $10 000 prize for its solution, the number of possibilities are of the order of 10^{35}, for 50 cities the order is 10^{62}. A breakthrough in the field was made in 1954 by an ingenious application by George Dantzig, Ray Fulkerson and Selmer Johnson from RAND Corporation of linear programming to calculate (in a few weeks "by hand") the shortest route for 49 cities. This record lasted until 1975 when Panagiotis Miliotis solved the problem for 80 cities, followed in 1977 by Grötschel with 120 and Padberg and Crowder with 318 cities. In 1987 Grötschel and Padberg raised this number to 2392 cities. The author and Vašek Chvátal, assisted by the computational mathematicians David Applegate and Robert Bixby, started to work on the problem in 1988 and obtained astonishing results, culminating in 2006 with the solution for 85 900 cities, representing locations of the connections that must be cut by a laser to create a customized computer chip. The computer code used to solve the problem, called *Concorde*, is available on the internet. Some of these results are presented in the book by David L. Applegate, Robert E. Bixby, Vasek

Chvátal and William J. Cook, *The Traveling Salesman Problem: A Computational Study,* Princeton University Press, 2007.

It is unknown whether the the complexity of the TSP is polynomial (i.e. if it belongs to the class $\mathcal{P}$), the best estimation being $n^2 \cdot 2^n$. It is known that it belongs to the class $\mathcal{NP}$ (non-deterministic polynomial-time algorithms), and the TSP is $\mathcal{NP}$-complete, meaning that finding a good algorithm for TSP will prove the equality $\mathcal{P} = \mathcal{NP}$, one of the $1 000 000 worth problems from Clay Mathematics Institute. An amusing discussion on the catastrophical consequences of this equality for the mankind, with quotations from the Charles Stross's story "Antibodies" and from Lance Fortnow (Comm. ACM, 2009), can be founded on page 10.

The author presents in a live and entertaining style the historical evolution of the problem and its interaction with other mathematical problems – linear programming (in Ch. 5), the cutting planes method (Ch. 6) and the branch-and-bound method (in Ch. 7) for integer programming – and computational ones – big computing and TPS on large scale (in Ch. 8), and the complexity problem in Chapter 9.

There are a lot of good quotations spread through the book, nice pictures, personal reminiscences and anecdotes. By its non-formal and amazing style, the book addresses a large audience interested to know something about the long and hard chest of generations of mathematicians and computer scientists to solve hard problems and how their solutions will influence our lives.

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