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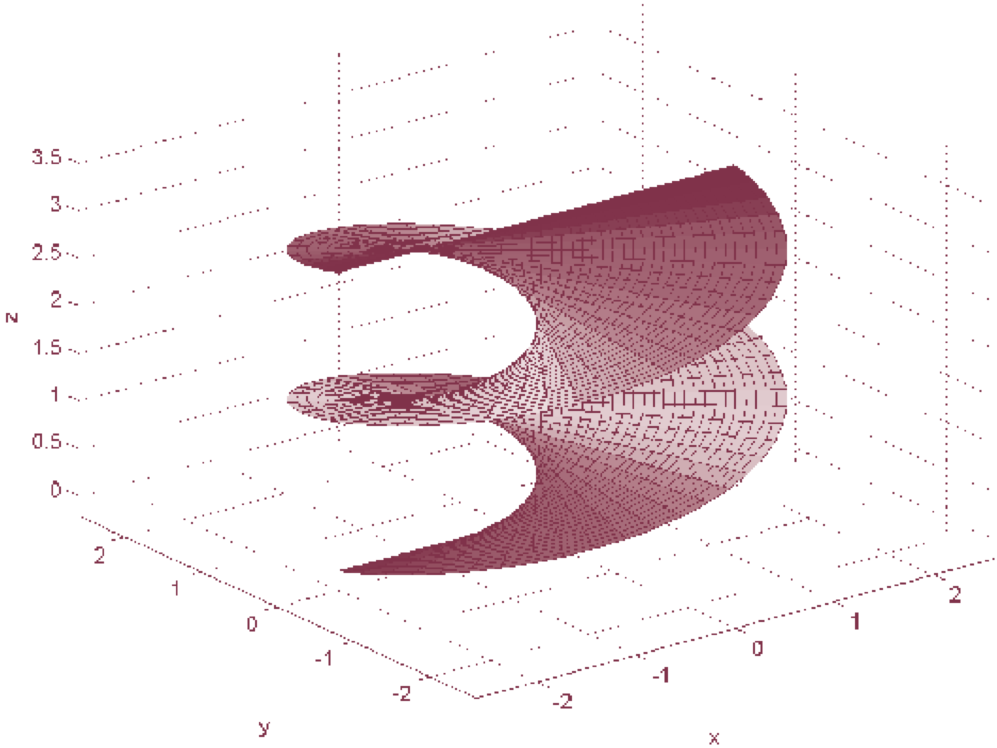

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CONTENTS

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Units in abelian group algebras over indecomposable rings

Peter Danchev

Abstract. Suppose R is a commutative indecomposable unitary ring of prime characteristic p and G is a multiplicative Abelian group such that G_0/G_p is finite. We describe up to isomorphism the unit group $U(RG)$ of the group algebra RG. This extends an earlier result due to Mollov-Nachev (Commun. Algebra, 2006) removing the restriction that G splits.

Mathematics Subject Classification (2010): 16S34, 16U60, 20K21.

Keywords: groups, rings, group rings, indecomposable rings, units, direct decompositions, isomorphisms.

1. Introduction

Throughout the present paper, suppose R is a commutative unitary (i.e., with identity) ring of prime characteristic p and G is an Abelian group written multiplicatively as is customary when exploring group algebras. For such R and G, denote by RG the group algebra of G over R with unit group $U(RG)$ and its normed subgroup $V(RG)$ of units with augmentation 1; note that the decomposition $U(RG) = V(RG) \times U(R)$ holds where $U(R)$ is the unit group of R. As usual, G_0 is the torsion subgroup of G with p-primary component G_p, and $S(RG) = V_p(RG)$ is the p-component of $V(RG)$. Moreover, for any natural number n, ζ_n denotes the primitive nth root of unity and $R[\zeta_n]$ is the free R-module, algebraically generated as a ring by ζ_n, with dimension $[R[\zeta_n] : R]$. As it is well-known, a ring is said to be *indecomposable* if it cannot be decomposed into a direct sum of two or more non-trivial ideals.

The structures of $V(RG)$ and $U(RG)$ have been very intensively studied in the past twenty years (see, e.g., [8]). Some isomorphism description results were obtained in [2] and [11]. The purpose of this work is to improve considerably one of the central results in the latter citation by giving a more direct and conceptual proof (note that some parts of the proof of the corresponding result in [11] are unnecessary complicated). Likewise, our method proposed below gives a new strategy for obtaining other results of this type since it reduces the general case to the p-mixed one.

2. Main results

As noted above, Mollov and Nachev established in ([11], Theorem 5.8) the following assertion.

Theorem (2006). *Let R be a commutative indecomposable ring with identity of prime characteristic p and let G be a splitting Abelian group. Suppose that G_0/G_p is a finite group of exponent n and $n \in U(R)$. Then*

$$U(RG) \cong \coprod_{d/n} \coprod_{\lambda(d)} U(R[\zeta_d]) \times \coprod_{b} G/G_0 \times \coprod_{d/n} \coprod_{\lambda(d)} S(R[\zeta_d](G_p \times G/G_0))$$

where $\lambda(d) = \frac{(G_0/G_p)(d)}{[R[\zeta_d]:R]}$, with $(G_0/G_p)(d)$ the number of elements of G_0/G_p of order d, and $b = \sum_{d/n} \lambda(d)$.

Notice that since $char(R) = p$ is a prime integer, it is self-evident that $exp(G_0/G_p)$ inverts in R, so that the condition $n \in U(R)$ is always fulfilled and hence it is a superfluously stated in the theorem.

The aim that we will pursue is to drop the limitation that G is a splitting group. Specifically, we proceed by proving the following:

Main Theorem. *Suppose R is an indecomposable ring of $char(R) = p$ and G is a group for which G_0/G_p is finite. Then the following isomorphism is true:*

$$U(RG) \cong \coprod_{d/exp(G_0/G_p)} \coprod_{a(d)} [U(R[\zeta_d]) \times [(G/\coprod_{q\neq p} G_q)V_p(R[\zeta_d](G/\coprod_{q\neq p} G_q))]] \qquad (*)$$

where $a(d) = \frac{|\{g\in G_0/G_p : order(g)=d\}|}{[R[\zeta_d]:R]}$.

In particular:

(1) if G is p-splitting, then

$$U(RG) \cong \coprod_{d/exp(G_0/G_p)} \coprod_{a(d)} [U(R[\zeta_d]) \times V_p(R[\zeta_d](G/\coprod_{q\neq p} G_q))] \times$$

$$\times \coprod_{\sum_{d/exp(G_0/G_p)} a(d)} G/G_0.$$

(2) if G_p is a direct sum of cyclic groups, then

$$U(RG) \cong \coprod_{d/exp(G_0/G_p)} \coprod_{a(d)} [U(R[\zeta_d]) \times (V_p(R[\zeta_d](G/\coprod_{q\neq p} G_q))/(G/\coprod_{q\neq p} G_q)_p)] \times$$

$$\times \coprod_{\sum_{d/exp(G_0/G_p)} a(d)} G/\coprod_{q\neq p} G_q.$$

Moreover, the quotient $V_p(R[\zeta_d](G/\coprod_{q\neq p} G_q))/(G/\coprod_{q\neq p} G_q)_p)$ *is a direct sum of cyclic groups and can be characterized via the Ulm-Kaplansky invariants calculated in* [7].

Proof. First observe that $\coprod_{q\neq p} G_q$ is pure in G_0 as its direct factor and hence it is pure in G because G_0 is pure in G. Since $G_0/G_p \cong \coprod_{q\neq p} G_q$ is finite, it is well known that $\coprod_{q\neq p} G_q = F$ is a direct factor even of G, say $G = F \times M$ for some $M \leq G$. It is obvious that $M \cong G/\coprod_{q\neq p} G_q$ is p-mixed with $M_0 = M_p = G_p$.

Next, write $RG = (RM)F$. Since R is indecomposable, it follows from [9] that RM is also indecomposable because there is no prime q which inverts in R such that $M_q \neq 1$. Clearly $exp(F) \in U(R) \subseteq U(RM)$ because $char(R) = char(RM) = p$ and therefore we can apply Theorem 4.4 and Remark 4.5 from [11] to get that $RG \cong \sum_{d/exp(F)} \sum_{a(d)} (RM)[\zeta_d]$, whence $U(RG) \cong \coprod_{d/exp(F)} \coprod_{a(d)} U((RM)[\zeta_d])$. It is straightforward to see that $(RM)[\zeta_d] \cong R[\zeta_d]M$, so that $U((RM)[\zeta_d]) \cong U(R[\zeta_d]M) = V(R[\zeta_d]M) \times U(R[\zeta_d])$. On the other hand, according to [4], [5] or [6], $V(R[\zeta_d]M) = MV_p(R[\zeta_d]M)$ using the fact from [11] that $R[\zeta_d]$ is indecomposable of prime characteristic p as well. This establishes formula $(*)$.

(1) If now G is p-splitting, it is readily seen that it is splitting. Consequently, so is M as its direct factor. Furthermore, it is easily checked that $M \cong G/\coprod_{q\neq p} G_q \cong (G_0 \times G/G_0)/\coprod_{q\neq p} G_q \cong (G_0/\coprod_{q\neq p} G_q) \times (G/G_0) \cong G_p \times (G/G_0)$. Moreover, $M = M_p \times M/M_p \cong G_p \times G/G_0$ because $M/M_p \cong G/\coprod_{q\neq p} G_q/(G/\coprod_{q\neq p} G_q)_0 = G/\coprod_{q\neq p} G_q/G_0/\coprod_{q\neq p} G_q \cong G/G_0$. That is why $MV_p(R[\zeta_d]M) \cong (M/M_p) \times V_p(R[\zeta_d]M) \cong (G/G_0) \times V_p(R[\zeta_d](G/\coprod_{q\neq p} G_q))$ and we are done.

(2) Appealing to [1], $G_p = M_p$ being a direct sum of cyclic groups implies that $V_p(R[\zeta_d]M) = M_p \times T$ for some subgroup T which is a direct sum of cyclic groups. Therefore, $MV_p(R[\zeta_d]M) = M(M_p \times T) = M \times T \cong (G/\coprod_{q\neq p} G_q) \times V_p(R[\zeta_d](G/\coprod_{q\neq p} G_q))/(G/\coprod_{q\neq p} G_q)_p$ since it is easy to see that $M \cap T = M_p \cap T = 1$ because $T = T_p$. This completes the proof. □

Note. In virtue of our lemma from [4], [5] or [6], our result can be generalized to the direct sum (= direct product) of m indecomposable rings, i.e., when the set $id(R)$ of all idempotents of R is finite and contains 2^m elements, that is, $|id(R)| = 2^m$.

That is why, utilizing this approach, the number m in Theorem 5 of [11] can be explicitly computed, namely it is equal exactly to $log_2|id(R)|$.

Remark. The proof of Theorem 2.7 in [10] contains a gap and so it is incomplete. In fact, the authors claimed that they will assume that the splitting group is p-mixed. The reason is that the K-algebras isomorphism $KG \cong KH$ yields that $K(G/\coprod_{q\neq p} G_q) \cong K(H/\coprod_{q\neq p} H_q)$ whenever K is a field of $char(K) = p$. But they need to show that G being splitting ensures that so is $G/\coprod_{q\neq p} G_q$. However, this was already done in [3].

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On the Hausdorff dimension of the graph of a Weierstrass type function

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Abstract. In this note a theorem to compare the box dimension of Weierstrass type functions and their Hausdorff dimension is established. Moreover a method to determine the Hausdorff dimension of the graphs of functions such as the Weierstrass or the Mandelbrot functions is given.

Mathematics Subject Classification (2010): Primary 26A15, Secondary 26A16.

Keywords: Hölder continuous functions, box dimension, Hausdorff dimension, Weierstrass-type functions.

1. Introduction

In this paper the Hausdorff and the box dimension of the graphs of some Weierstrass type functions are compared. Recall that the *Hausdorff dimension* of a set $E \subseteq R^n$ is defined in terms of the *k-dimensional Hausdorff measure* of E, denoted by $H^k(E)$ and given by

$$H^k(E) = \lim_{\delta \to 0} \inf\{\sum_i |E_i|^k, E \subseteq \bigcup E_i, |E_i| < \delta\}, \tag{1.1}$$

where $|E_i|$ denotes the diameter of E_i and the infimum is over all (countable) δ-covers E_i of E (see Falconer [2] and [3]). It is given by:

$$H - \dim E = \inf\{k > 0 : H^k(E) = 0\}. \tag{1.2}$$

There are other classes of covers leading to the Hausdorff dimension; in particular it is possible to consider in definition (1.1) instead of all covers of E, the covers obtained by the family of half-open d-adic cubes in R^n, that is cubes of the form:

$$\{x \in R^n, h_i d^{-m} \leq x_i < (h_i + 1)d^{-m}, for\ \ i = 1, 2, \ldots, n\}$$

where h_i and m are arbitrary integers. Then if the minimum in (1.1) is restricted to the class of these particular covers, one obtains the *net measure* of E, denoted by $N^k(E)$. It is evident that $H^k(E) \leq N^k(E)$, but it is also possible to prove that there exists a constant $A > 0$, only depending on the dimension of the space, n, and on d, such that $N^k(E) \leq AH^k(E)$ for every $E \subseteq R^n$ (see Mattila [4], 5.2 and Falconer [2], 5.1 for binary cubes). One of the most immediate modification of the Hausdorff dimension is given in terms of the upper and lower box dimension of a set, defined in the following way. Let $N_\delta(E)$ be the smallest number of sets of diameter at most δ which cover E. Then the following numbers:

$$\underline{\dim}_B E = \underline{\lim}_{\delta \to 0} \frac{logN_\delta(E)}{-log\delta} \tag{1.3}$$

$$\overline{\dim}_B E = \overline{\lim}_{\delta \to 0} \frac{logN_\delta(E)}{-log\delta} \tag{1.4}$$

are called respectively the *lower and upper box dimensions* or *lower and upper Minkowski dimensions* of E, and, if they agree, their common value is the *box dimension* of E, denoted by $\dim_B E$ or $\Delta(E)$. It is possible to prove that in (1.3) or in (1.4), $N_\delta(E)$ can be substituted by the number of $\delta - mesh$ cubes meeting E, that is the cubes of the form:

$$\{x \in R^n : h_i\delta \leq x_i < h_i + 1)\delta, i = 1, 2, \ldots, n\}$$

where $\delta > 0$ and $h_1, \ldots, h_n$ are integers. In general it is:

$$H - \dim(E) \leq \underline{\dim}_B E \leq \overline{\dim}_B E.$$

If $f : [a, b] \to R$ is a continuous function and if

$$G = \{(a, b) \in R^2 : a \leq x \leq b, y = f(x)\}$$

is its graph, then $H - \dim G \geq 1$ (see Falconer [2] , lemma 1.8); moreover, if f is α-Hölder continuous then: $\overline{\dim}_B G \leq 2 - \alpha$ (see Falconer [2], Theorem 8.1).

Some general discussion about the Hausdorff dimension of the graph of a Hölder continuous function can be found in [5]. In this paper we will consider Weierstrass type functions:

$$f(x) = \sum_{n \in N} \frac{\varphi(b_n x)}{b_n^\delta},$$

where $\varphi : R \to R$ is periodic (with period 1), Lipschitz continuous and b_n is a sequence for which there exists $B > 1$ such that $b_1 \geq B, b_{n+1} \geq Bb_n$ for every $n \in N$. Obviously it is not restrictive to suppose $\varphi(x) \geq 0$ for every $x \in R$, since if this is not the case, it is possible to consider $\psi(x) = \varphi(x) + m$, where $m = min_{x \in [0,1]} \varphi(x)$ and observe that $\psi(x) \geq 0$ for every $x \in R$ and

$$\sum_{n \in N} \frac{\psi(b_n x)}{b_n^\delta} = f(x) + \sum_{n \in N} \frac{m}{b_n^\delta},$$

that is the corresponding function to ψ differs from f by a constant.

In the main theorems of this paper (Theorem 3.2, Remark 3.3, Theorem 3.4) rather general hypotheses for a class of Weierstrass type functions will be established in order the Hausdorff dimension of the graph of f to be equal to the box dimension when this achieves its maximum. To obtain this result some technical ideas (in Lemma 2.2) have been borrowed from an old paper by Besicovitch and Ursell (see [1]). Interesting suggestions have been found also in [6].

2. Three lemmas

In order to establish the main theorem some lemmas are needed:

Lemma 2.1. *Let* $f : [a, b] \to R$ *be an* α*-Hölder continuous function,* ($0 < \alpha \leq 1$)*,* d *a natural number,* $d > 1$*, let* $\{Q_i\}$ *be a finite cover of* G *constituted by meshes, and let* Q *be a* $\frac{1}{d^r}$*-mesh* ($r \in \mathbb{N}$) *from the cover* $\{Q_i\}$*. Let* $\{Q^j\}_{j\in A}$ *be a cover of* $G \bigcap Q$ *constituted by* $\frac{1}{d^k}$*-meshes with fixed* $k > r$*. Then there exists a constant* $C > 0$ *depending only on* f *such that*

$$\sum_{j\in A} |Q^j|^{2-\alpha} \leq Cm(F), \tag{2.1}$$

where F *is the projection of* $G \bigcap Q$ *on the* x*-axis and* m *is the unidimensional Lebesgue measure*

Proof. F is a Lebesgue measurable set, since it is the projection of a Borel set. It is $m(F) = \lim_{s\to\infty} m(A_s)$, where m is the Lebesgue measure and $\{A_s\}_{s\in N}$ is a sequence of open sets, decreasing with respect to the inclusion relation. Let us cover A_s by intervals I_n that are linear $\frac{1}{d^k}$-meshes. The oscillation of f in every one of these intervals is less than or equal to $L(\frac{1}{d^k})^\alpha$, where L is the Hölder coefficient of f. Therefore the part of G whose projection is enclosed in A_s is covered by $\frac{1}{d^k}$-square meshes whose number is at most $\frac{m(A_s)L|I_n|^\alpha}{|I_n|^2}$; let us call these meshes by $Q_{s,n}$; then, if s is fixed, it is:

$$\sum_{n} |Q_{s,n}|^{2-\alpha} \leq m(A_s)L(\sqrt{2})^{2-\alpha}.$$

Since this inequality holds for every $s \in N$, we have, keeping in mind that $\bigcap_{s\in N}\{Q_{s,n}\}$ is a cover $\{Q^j\}_{j\in A}$ of $G \bigcap Q$ constituted by $\frac{1}{d^k}$-meshes:

$$\sum_{j\in A} |Q^j|^{2-\alpha} \leq L(\sqrt{2})^{2-\alpha} \lim_{s\to\infty} m(A_s),$$

whence (2.1) and the lemma is proved. □

In the next lemma, very near to the ideas of a well known paper by Besicovitch and Ursell (see [1], where however a particular case is considered), and in the sequel we will consider a periodic function $\varphi : R \to [0, P]$ with period 1, nonnegative, continuous and piecewise differentiable; assume that $\varphi'_-(x)$ and $\varphi'_+(x)$ are finite and different from 0 for every $x \in R$. Then there exist two constants $c > 0$ and $c_1 > 0$ such that:

$$c \leq |\varphi'(x)| \leq c_1 \tag{2.2}$$

for every $x \in R$ such that φ is differentiable in x.

If φ satisfies all the previous conditions we will refer to it as a *smooth function.*

Lemma 2.2. *Consider, for* $0 < \alpha < 1$, *the following function:*

$$f(x) = \sum_{n \in N} \frac{\varphi(b_n x)}{b_n^\alpha}, \tag{2.3}$$

where φ *is a smooth function and where* $(b_n)_{n \in N}$ *is such that there exists* $B > 1, B \in N$ *for which* $b_{n+1} \geq B b_n$ *for every* $n \in N$ *and*

$$\lim_{n \to \infty} \frac{log b_{n+1}}{log b_n} = 1. \tag{2.4}$$

Let $\{Q_i\}$ *be a cover of* G *constituted by* $\frac{1}{B^u}$*-meshes* ($u \in N$)*, and let* Q *be a* $\frac{1}{B^r}$*-mesh from* $\{Q_i\}$*. Let* $\{Q^j\}_{j \in A}$ *be a cover of* $G \bigcap Q$ *constituted by* $\frac{1}{B^k}$*-meshes with fixed* $k > r$*. Then, if* $B^{1-\alpha} > 1 + \frac{c_1}{c}$*, where* c *and* c_1 *are as in* (2.2)*, there exists a constant* $\lambda > 0$ *such that, for enough large* r*:*

$$\sum_{j \in A} |Q^j|^{2-\alpha} \leq \lambda |Q|^{2-\alpha}. \tag{2.5}$$

Proof. By (2.1), in order to prove (2.5) we have to determine an upper bound for the measure of the set $F = \{x \in R : (x, f(x)) \in Q\}$. To this end observe that, if we consider for every $s \in N$ the function $f_s(x) = \sum_{n \leq s} \frac{\varphi(b_n x)}{b_n^\alpha}$ and if $\varphi(x) \leq 1$ for every $x \in R$ as is not restrictive to suppose, then:

$$|f(x) - f_{k+\nu-1}| \leq \Sigma_{n \geq k+\nu} \frac{\varphi(b_n x)}{b_n^\alpha} \leq \frac{B^\alpha}{b_{k+\nu}^\alpha (B^\alpha - 1)},$$

where, given $r \in N$ and $k > r$, ν has been chosen in such a way that:

$$\frac{1}{b_{k+\nu}^\alpha} \leq \frac{1}{B^r} < \frac{1}{b_{k+\nu-1}^\alpha}.$$

Therefore:

$$|f(x) - f_{k+\nu-1}| \leq \frac{B^\alpha}{B^r (B^\alpha - 1)}. \tag{2.6}$$

Consider the strip S obtained prolonging Q downwards a distance $\frac{B^\alpha}{B^r(B^\alpha-1)}$. By (2.6) if $(x, f(x)) \in Q$ then $(x, f_{k+\nu-1}(x)) \in S$ and, since $f_{k+\nu-1}(x) \leq f_{k+\nu}(x) \leq f(x)$, also $(x, f_{k+\nu}(x)) \in S$. Therefore:

$$F \subseteq F_{k+\nu-1} \bigcap F_{k+\nu},$$

where $F_s = \{x \in R : (x, f_s(x)) \in S\}$ for every $s \in N$. By hypotheses φ is strictly increasing or decreasing in a finite number of intervals of $[0, 1]$: let $M \in N$ be their number.

Now in every interval I in which $f_s'(x)$ is either positive or negative, it is, for every $x \in I$:

$$f_s'(x) = \Sigma_{n \leq s} b_n^{1-\alpha} \varphi'(b_n x) = b_s^{1-\alpha} \Sigma_{n \leq s} (\frac{b_n}{b_s})^{1-\alpha} \varphi'(b_n x),$$

whence, by (2.2), there exists $c_2 = c - \frac{c_1}{B^{1-\alpha}-1} > 0$ by hypothesis, such that:

$$|f_s'(x)| \geq c_2 b_s^{1-\alpha} \tag{2.7}$$

and therefore, for every $s \in N$, the sign of $f_s'(x)$ in I is the same of $\varphi'(b_s x)$. Consequently in every interval of unitary length, the function $f_{k+\nu-1}$ is strictly increasing or decreasing in at most $M([b_{k+\nu-1}]+1)$ intervals, where $[b_{k+\nu-1}]$ denotes the integer part of $b_{k+\nu-1}$; by Lagrange theorem and by (2.7) the length of an interval I where the oscillation of $f_{k+\nu-1}$ is not greater than $\frac{1}{B^r}(1+\frac{B^\alpha}{B^\alpha-1}) = \frac{C_\alpha}{B^r}$ is given by:

$$|I| \leq \frac{C_\alpha}{B^r c_2 b_{k+\nu-1}^{1-\alpha}}.$$

Consider now the intervals J enclosed in the previous ones in which the function $f_{k+\nu}$ has an oscillation less or equal to $\frac{C_\alpha}{B^r}$. As before we have:

$$|J| \leq \frac{C_\alpha}{B^r c_2 b_{k+\nu}^{1-\alpha}}.$$

For every previous interval I there are

$$M([b_{k+\nu}]+1)|I| \leq \frac{MC_\alpha 2b_{k+\nu}}{B^r c_2 b_{k+\nu-1}^{1-\alpha}},$$

such intervals and therefore :

$$m(F) \leq \frac{MC_\alpha 2b_{k+\nu}}{B^r c_2 b_{k+\nu-1}^{1-\alpha}} \frac{C_\alpha}{B^r c_2 b_{k+\nu}^{1-\alpha}} \leq \frac{2MC_\alpha^2 b_{k+\nu}^\alpha}{c_2^2 B^{2r} b_{k+\nu-1}^{1-\alpha}}$$

whence, by the choice of ν :

$$m(F) \leq \frac{2MC_\alpha^2 b_{k+\nu}^\alpha}{c_2^2 B^{r(2-\alpha)} b_{k+\nu-1}^{\alpha^2+1-\alpha}}. \tag{2.8}$$

By (2.4), for every $\varepsilon > 0$ it is possible to determine k_o such that for every $k > k_o$ it is $b_{k+\nu} \leq b_{k+\nu-1}^{1+\varepsilon}$. Now it is possible to determine $\varepsilon > 0$ in such a way that $(1+\varepsilon)\alpha \leq \alpha^2 - \alpha + 1$ and therefore, for enough large k: $b_{k+\nu}^\alpha \leq b_{k+\nu-1}^{\alpha^2-\alpha+1}$. By (2.8) we can conclude that there is a positive constant $\gamma > 0$ such that $m(F) \leq \frac{\gamma}{B^{r(2-\alpha)}}$ for enough large r and, by (2.1), the lemma is proven. □

Remark 2.3. It is worth noticing that it is possible to apply Lemma 2.2 even in situations in which the conditions stated there do not hold, for example if φ is such that it is possible to perform on it a geometrical transformation obtaining a smooth function ψ in such a way that the corresponding functions to φ and ψ have the same geometrical measure properties. For example consider the function

$$\varphi(x) = \frac{1+sin(2\pi x)}{4} \;\; if \;\; -\frac{1}{4} \leq x < \frac{3}{4}, \;\; \varphi(x+1) = \varphi(x) \;\; for \;\; every \;\; x \in R;$$

the hypotheses of Lemma 2.2 are not satisfied, since there exist points x such that $\varphi'(x) = 0$. Consider now the function:

$$\psi(x) = \frac{1}{2} + 2x - \varphi(x) \;\; if \;\; -\frac{1}{4} \leq x < \frac{1}{4}, \;\; \psi(x) = \frac{3}{2} - 2x - \varphi(x) \;\; if \;\; \frac{1}{4} \leq x \leq \frac{3}{4},$$

$$\psi(x+1) = \psi(x) \;\; for \;\; every \;\; x \in R.$$

It is a smooth function since it is nonnegative and continuous in R, piecewise differentiable and $2-\frac{\pi}{2} \leq |\psi'(x)| \leq 2+\frac{\pi}{2}$ for every x where ψ is differentiable.

Consider the function (2.3), where $b_n = d^n$ with a fixed $d \in N$, consider also the function:

$$f_k(x) = \Sigma_{n\leq k}\frac{\varphi(b_n x)}{b_n^\alpha}.$$

Let I be an interval where $\varphi(b_k x)$ is either strictly increasing or strictly decreasing and therefore $\varphi(b_s x)$ for every $s \leq k$ is either strictly increasing or strictly decreasing. But then also $\psi(b_s x)$ for $s \leq k$ is either strictly increasing or decreasing. Therefore it is easy to check that if x' and x" belong to I, then, substituting φ by ψ in f_k, we get:

$$f_k(x') - f_k(x") = (x' - x")\Sigma_{n\leq k}\frac{\pm 2}{d^{n\alpha}} - \frac{\psi(d^k x') - \psi(d^k x")}{d^{k\alpha}}$$

$$-\Sigma_{n<k}\frac{\psi(d^n x') - \psi(d^n x")}{d^{n\alpha}}$$

whence:

$$|f_k(x') - f_k(x")| \geq |x' - x"|\left\{d^{k(1-\alpha)}(2-\frac{\pi}{2}) - \frac{2}{d^\alpha - 1} - (2+\frac{\pi}{2})\frac{d^{k(1-\alpha)}}{d^{1-\alpha}-1}\right\}$$

Let d be large enough that $2\rho = 2 - \frac{\pi}{2} - \frac{2+\frac{\pi}{2}}{d^{1-\alpha}-1} > 0$; then fix k_o in such a way that for every $k > k_o$ it is $d^{k(1-\alpha)}(2 - \frac{\pi}{2} - \frac{2+\frac{\pi}{2}}{d^{1-\alpha}-1} > \frac{4}{d^\alpha-1}$. Then for every $k > k_o$ it is:

$$|f_k(x') - f_k(x")| \geq |x' - x"|\rho d^{k(1-\alpha)}.$$

Then a valuation of the length of an interval I where f_k has an oscillation not greater than $\frac{C_\alpha}{B^r}$ (see the proof of Lemma 2.2, where $B = d \in N$) is given by

$$|I| \leq \frac{C_\alpha}{\rho d^r d^{k(1-\alpha)}}.$$

From this point onwards the proof proceeds as the proof of Lemma 2.2. Another example is given by the function

$$\varphi(x) = \frac{1-cos(2\pi x)}{2} \quad x \in [0,1];\;\; \varphi(x+1) = \varphi(x) \;\; for \;\; every \;\; x \in R;$$

in this case we can consider the related smooth function:

$$\psi(x) = 4x - \varphi(x)\, if \;\; 0 \leq x \leq \frac{1}{2};\;\; \psi(x) = 4(1-x) - \varphi(x) \;\; if \;\; \frac{1}{2} \leq x < 1;$$

$$\psi(x+1) = \psi(x) \;\; for \;\; every \;\; x \in R.$$

Repeating the procedure developed above it is easily seen that Lemma 2.2 holds.

Lemma 2.4. *Let $f : [a, b] \to R$ be an α-Hölder continuous function,* $(0 < \alpha \leq 1)$, *and let $\{Q_i\}$ be a cover of G constituted by $\frac{1}{d^k}$-meshes, with $d \in N$, $d > 1$ and variable $k \in N$. Then there exists a sequence of finite covers $\{Q_i^n\}_{i=1,\dots,k_n}$ of G such that, for every $s \geq 2 - \alpha$ it is:*

$$\sum_{i=1}^{+\infty} |Q_i|^s = \lim_{n\to\infty} \sum_{i=1,\dots,k_n} |Q_i^n|^s.$$

As a consequence, for such values of s it is:

$$N^s(G) = \lim_{\delta\to 0} \inf\{\sum_i |Q_i|^s : G \subseteq \bigcup Q_i, Q_i \ \textit{finite and} \ |Q_i| = \frac{1}{d^k} < \delta.\} \tag{2.9}$$

Proof. If $\{Q_i\}$ is finite then put $\{Q_i^n\} = \{Q_i\}$ for every $n \in N$. Otherwise there exist meshes in $\{Q_i\}$ whose diameter is arbitrarily small and it is possible to execute the following construction.

Let $\frac{1}{d^l}$ be the greatest edge of the meshes appearing in Q_i and let $\{Q_i^1\}_{i=1,\dots,k_1}$ the (finite) cover of G constituted by $\frac{1}{d^l}$-meshes only.

Among all the $\frac{1}{d^l}$-meshes considered above, take only those appearing in $\{Q_i\}$.

Divide the remaining $\frac{1}{d^l}$-meshes in $\frac{1}{d^{l+1}}$-meshes and consider only those having a common point with G. Let $\{Q_i^2\}_{i=1,\dots,k_2}$ be the cover of G constituted by the $\frac{1}{d^l}$-meshes appearing in $\{Q_i\}$ and by the $\frac{1}{d^{l+1}}$-meshes disjoint from the preceding ones and with at least one common point with G.

Iterate the procedure: at step n let $\{Q_i^n\}_{i=1,\dots,k_n}$ be the finite cover of G constituted by all the $\frac{1}{d^l}$-meshes, the $\frac{1}{d^{l+1}}$-meshes, ..., the $\frac{1}{d^{l+n-2}}$-meshes appearing in $\{Q_i\}$ and by the $\frac{1}{d^{l+n-1}}$-meshes disjoint from the preceding ones and having at least one common point with G.

Divide the sum $\sum_{i=1,\dots,k_n} |Q_i^n|^s$ $(s > 0)$ in two parts: in the first one put the contributes of all the elements appearing in the starting cover $\{Q_i\}$; in the second part put the contributes of the remaining elements, let they be, for every $n \in N$, $\{T_i^n\}$ and let h_n be their number. By construction, for every $n \in N$ the diameter $|T_i^n|$ is constant with respect to $i = 1, \dots, h_n$. Moreover:

$$h_n \leq L(\frac{|T_i^n|}{\sqrt{2}})^\alpha \frac{2m(P_n)}{|T_i^n|^2} = L(\sqrt{2})^{2-\alpha}|T_i^n|^{\alpha-2}m(P_n),$$

where $m(P_n)$ is the linear Lebesgue measure of the projection on the x-axis of the set $\bigcup_{i=1,\dots,h_n} T_i^n$. Therefore:

$$\sum_{i=1,\dots,h_n} |T_i^n|^s \leq L(\sqrt{2})^{2-\alpha}|T_i^n|^{s-(2-\alpha)}m(P_n).$$

Since $\{Q_i\}$ is a cover of G, the sequence (P_n) is decreasing with respect to the inclusion relation and $\lim_{n\to\infty} m(P_n) = 0$. It follows that

$\lim_{n\to\infty}\sum_{i=1,\ldots,h_n}|T_i^n|^s = 0$, since $s-(2-\alpha)\geq 0$. On the other hand:

$$\lim_{n\to\infty}\sum_{i=1,\ldots,k_n}|Q_i^n|^s = \sum_{i=1}^{+\infty}|Q_i|^s + \lim_{n\to\infty}\sum_{i=1,\ldots,h_n}|T_i^n|^s$$

and Lemma 2.4 is proven. □

3. The main theorems

By the definition of $H^{2-\alpha}(G)$, if $f:[a,b]\to R$ is α-Hölder continuous, then

$$H^{2-\alpha}(G)\leq \underline{\lim}_{\delta\to 0}N_\delta(G)\delta^{2-\alpha}.$$

Lemmas 2.2 and 2.4 allow us to prove that, under the hypotheses of Lemma 2.2, the last inequality can be inverted. Indeed the following theorem holds:

Theorem 3.1. *Let*

$$f(x)=\sum_{n\in N}\frac{\varphi(b_n x)}{b_n^\alpha},\quad (0<\alpha<1)$$

where φ is a smooth function and where $(b_n)_{n\in N}$ is such that there exists $B>1,\ B\in N$ for which $b_{n+1}\geq Bb_n$ for every $n\in N$ and (2.4) holds. Then, if $B^{1-\alpha}>1+\frac{c_1}{c}$, where c and c_1 are as in (2.2), there exists a constant $\gamma>0$ such that:

$$\underline{\lim}_{\delta\to 0}N_\delta(G)\delta^{2-\alpha}\leq \gamma H^{2-\alpha}(G). \tag{3.1}$$

Proof. Let $\{Q_i\}$ be a finite cover of G constituted by $\frac{1}{d^k}$-meshes, with k variable in N, such that $\frac{1}{d^k}<\delta$. By Lemma 2.2 passing to the g.l.b. we get, for enough small δ:

$$\inf_{\delta_1<\delta}N_{\delta_1}(G)\delta_1^{2-\alpha}\leq\gamma_1\ \ \inf\{\Sigma|Q_i|^{2-\alpha},Q_i\ \ finite, G\subseteq\bigcup Q_i,|Q_i|<\delta\},$$

where δ_1 is the minimum length of the edges of the elements of $\{Q_i\}$ and γ_1 is a suitable constant. Passing to the limit, by Lemma 2.4, one gets:

$$\underline{\lim}_{\delta\to 0}N_\delta(G)\delta^{2-\alpha}\leq\gamma_1 N^{2-\alpha}(G);$$

since there exists a constant $A>0$ such that $N^{2-\alpha}(G)\leq AH^{2-\alpha}(G)$ (see [4], 5.2), the thesis follows. □

Theorem 3.2. *Let*

$$f(x)=\sum_{n\in N}\frac{\varphi(b_n x)}{b_n^\alpha},$$

where $0<\alpha<1$, φ is a smooth function and where $(b_n)_{n\in N}$ is such that there exist two numbers $B>1, B\in N$ and $\mu>0$ for which: $b_{n+1}\geq Bb_n$ for every $n\in N$ and $b_n\geq\mu b_{n+1}$ for every $n\in N$ (whence (7.7)) holds). Then, if B is enough large, the Hausdorff dimension of G is maximum, equal to $2-\alpha$. Moreover there exists a constant $C>0$ such that, for every interval $[a,b]\subseteq R$ it is $H^{2-\alpha}(G)\geq C(b-a)$ if the portion of G whose projection on the x- axis is $[a,b]$ is considered.

Proof. It is easy to see that f is α-Hölder continuous and therefore the Hausdorff dimension of G is less than or equal to $2-\alpha$.

To prove the converse inequality we will use Theorem 3.1 of this Section. Indeed consider, for every $\delta > 0$, the cover of $[0,1]$ constituted by the intervals $[0,\delta[, [\delta, 2\delta[, \ldots, [p\delta, (p+1)\delta[$, with $p = [\frac{1}{\delta}]$. Let $k \in N$ be such that:

$$\frac{2}{b_{k+1}} \leq \delta < \frac{2}{b_k} \tag{3.2}$$

and let $h = \frac{1}{4b_{k+1}}$. Since $\delta \geq \frac{2}{b_{k+1}}$, in every interval of the cover there is an interval whose length is $\frac{1}{b_{k+1}}$, let

$$[\frac{j}{b_{k+1}}, \frac{j+1}{b_{k+1}}[\subseteq [l\delta, (l+1)\delta[$$

for suitable $j \in N$. Therefore, for every $l = 1, 2, \ldots, p$, the oscillation in $[l\delta, (l+1)\delta[$ is not less than

$$|f(\frac{j}{b_{k+1}} + h) - f(\frac{j}{b_{k+1}})| = |\sum_{n=1}^{+\infty} \frac{\varphi[b_n(\frac{j}{b_{k+1}} + h)] - \varphi(b_n \frac{j}{b_{k+1}})}{b_n^\alpha}|.$$

Assume that c is the Lipschitz coefficient of φ and, as is not restrictive, that $\varphi(0) = 0$, φ is positive and increasing in $]0, \frac{1}{4}]$ and therefore $\varphi(\frac{1}{4}) > 0$; we have:

$$|f(\frac{j}{b_{k+1}} + h) - f(\frac{j}{b_{k+1}})| \geq \frac{|\varphi(\frac{1}{4})|}{b_{k+1}^\alpha} - c\sum_{n=1}^{n=k} \frac{\frac{1}{4b_{k+1}}}{b_n^\alpha} - 2\Sigma_{n\geq k+2}\frac{1}{b_n^\alpha}. \tag{3.3}$$

Then it is:

$$|f(\frac{j}{b_{k+1}} + h) - f(\frac{j}{b_{k+1}})| \geq \varphi(\frac{1}{4})4^\alpha h^\alpha - \frac{ch^\alpha}{4^{1-\alpha}}[\frac{1}{b_1^\alpha b_{k+1}^{1-\alpha}} + \ldots + \frac{1}{b_k^\alpha b_{k+1}^{1-\alpha}}]$$

$$-\frac{2}{b_{k+2}^\alpha}[1 + \frac{1}{B^\alpha} + \ldots] \geq 4^\alpha h^\alpha[\varphi(\frac{1}{4}) - \frac{c}{4B^{(k+1)(1-\alpha)}(B^\alpha - 1)} - \frac{2}{B^\alpha - 1}].$$

Therefore, if B is enough large, there exists a constant $C_1 > 0$ such that for every $\delta > 0$ and for every interval $[l\delta, (l+1)\delta[$ with $l = 1, \ldots, p$, we have that the oscillation of f in such an interval is greater than or equal to $C_1 h^\alpha$ (for the method used here to obtain this inequality see the proof of Zhou and He in Lemma 2.5 of [6], where the particular case of $\varphi(x) = sin(x)$ is considered).

Therefore it is $N_\delta(G) \geq \frac{C_1 h^\alpha}{\delta^2}$, whence, by the hypothesis, for every $\delta > 0$:

$$N_\delta(G)\delta^{2-\alpha} \geq \frac{C_1 h^\alpha}{\delta^\alpha} \geq C_2(\frac{b_k}{b_{k+1}})^\alpha \geq C_2\mu^\alpha > 0.$$

It follows that $\underline{\lim}_{\delta\to 0} N_\delta \delta^{2-\alpha} \geq C_2\mu^\alpha > 0$ and the thesis follows, since this inequality implies, by previous Theorem 3.1, that it is also $H^{2-\alpha}(G) > 0$.

Finally if in the preceding proof we consider that part of G whose projection on the x-axis is the interval $[a, b]$ instead of the interval $[0, 1]$, we obtain:

$$H^{2-\alpha}(G) \geq C_2 \mu^\alpha (b - a)$$

whence the thesis. □

As we have seen in Remark 2.3, Lemma 2.2 and therefore also Theorem 3.2, whose proof is essentially based upon Lemma 2.2, can be proved also under other less restrictive hypotheses. For example we claim that:

Theorem 3.3. *If $d \in N$ is enough large, the graph of the Weierstrass function*

$$f(x) = \Sigma_{n \in N} \frac{sin(2\pi d^n x)}{d^{n\alpha}}, \quad (0 < \alpha < 1)$$

has Hausdorff dimension equal to $2 - \alpha$. The same conclusion holds , if d is enough large, for the following function introduced by Mandelbrot (1977):

$$g(x) = \Sigma_{n \in N} \frac{1 - cos(2\pi d^n x)}{d^{n\alpha}}, \quad (0 < \alpha < 1).$$

Proof. Indeed consider the function:

$$\varphi(x) = \frac{1 + sin(2\pi x)}{4}, \quad -\frac{1}{4} \leq x < \frac{3}{4}, \quad \varphi(x + 1) = \varphi(x) \ \ for \ \ every \ \ x \in R;$$

as we have seen in Remark 2.3, we can apply Lemma 2.2 and therefore also Theorem 3.2 to this function, obtaining that the Hausdorff dimension of the graph of the function:

$$\Sigma_{n \in N} \frac{\varphi(d^n x)}{d^{n\alpha}} = \frac{1}{4}(\Sigma_{n \in N} \frac{1}{d^{n\alpha}} + \Sigma_{n \in N} \frac{sin(2\pi d^n x)}{d^{n\alpha}}$$

coincides with $2 - \alpha$ and obviously the same happens for the graph of the function f. With the same procedure we prove the thesis about Mandelbrot function: in this case we condider the function

$$\varphi(x) = \frac{1 - cos(2\pi x)}{2}$$

and the related smooth function ψ given in Remark 2.3. Then Theorem 3.2 is applicable and the thesis is proven. □

Theorem 3.4. *Let*

$$f(x) = \sum_{n \in N} (-1)^n \frac{\varphi(b_n x)}{b_n^\alpha}, \quad (0 < \alpha < 1)$$

where φ and $(b_n)_{n \in N}$ are as in Theorem 3.2. Then, if B is enough large, both the box dimension and the Hausdorff dimension of G are equal to $2 - \alpha$.

Proof. As in the proof of Theorem 3.2 above, it is easy to see that f is α-Hölder continuous and therefore the Hausdorff dimension of G is less than or equal to $2 - \alpha$. To prove the converse inequality let $\delta > 0$ and consider the cover of $[0, 1]$: $[0, \delta[, [\delta, 2\delta[, \ldots, [p\delta, (p + 1)\delta[$, where $p = [\frac{1}{\delta}]$. Let $k \in N$ be such that (3.2) holds and put $h = \frac{1}{4b_{k+1}}$. Then proceed as in the proof of Theorem 3.2, obtaining (3.3). Therefore there exists a constant $C > 0$ such that the oscillation of f in every interval $[l\delta, (l+1)\delta]$, $(0 \leq l \leq p)$, is not less than Ch^α and we can conclude as in the proof of Theorem 3.2. $\square$

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Univariate inequalities based on Sobolev representations

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Abstract. Here we derive very general univariate tight integral inequalities of Chebyshev-Grüss, Ostrowski types, for comparison of integral means and Information theory. These are based on well-known Sobolev integral representations of a function. Our inequalities engage ordinary and weak derivatives of the involved functions. We give also applications. On the way to prove our main results we derive important estimates for the averaged Taylor polynomials and remainders of Sobolev integral representations. Our results expand to all possible directions.

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1. Introduction

This article is greatly motivated by the following theorems:

Theorem 1.1. *(Chebychev, 1882,* [6]*) Let $f, g : [a, b] \to \mathbb{R}$ absolutely continuous functions. If $f', g' \in L_\infty([a, b])$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{(b-a)^2} \left(\int_a^b f(x) dx \right) \left(\int_a^b g(x) dx \right) \right| \tag{1.1}$$
$$\leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty .$$

Theorem 1.2. *(G. Grüss, 1935,* [10]*) Let f, g integrable functions from $[a, b] \to \mathbb{R}$, such that $m \leq f(x) \leq M$, $\rho \leq g(x) \leq \sigma$, for all $x \in [a, b]$, where $m, M, \rho, \sigma \in \mathbb{R}$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{(b-a)^2} \left(\int_a^b f(x) dx \right) \left(\int_a^b g(x) dx \right) \right| \tag{1.2}$$

$$\leq \frac{1}{4}(M-m)(\sigma-\rho).$$

In 1938, A. Ostrowski [13] proved

Theorem 1.3. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)}|f'(t)| < +\infty$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x)\right| \leq \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\cdot(b-a)\|f'\|_\infty, \tag{1.3}$$

for any $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

See also [1], [2], [3] for related works that inspired as well this article.

In this work using the univariate Sobolev type representation formulae, see Theorems 10, 14 and also Corollaries 11, 12, we estimate first their remainders and then the involved averaged Taylor polynomials.

Based on these estimates we derive lots of very tight inequalities on $\mathbb{R}$: of Chebyshev-Grüss type, Ostrowski type, for Comparison of integral means and Csiszar's f-Divergence with applications. Our results involve ordinary and weak derivatives and they go to all possible directions using various norms. All of our tools come from the excellent monograph by V. Burenkov, [5].

2. Basics

Here we follow [5].

For a measurable non empty set $\Omega\subset\mathbb{R}^n$, $n\in\mathbb{N}$ we shall denote by $L_p^{loc}(\Omega)$ $(1\leq p\leq\infty)$ - the set of funtions defined on Ω such that for each compact $K\subset\Omega$ $f\in L_p(K)$.

Definition 2.1. *Let $\Omega\subset\mathbb{R}^n$ be an open set, $\alpha\in\mathbb{Z}_+^n$, $\alpha\neq 0$ and $f,g\in L_1^{loc}(\Omega)$. The function g is a weak derivative of the function f of order α on Ω (briefly $g=D_w^\alpha f$) if $\forall\ \varphi\in C_0^\infty(\Omega)$ (i.e. $\varphi\in C^\infty(\Omega)$ compactly supported in Ω)*

$$\int_\Omega fD^\alpha\varphi dx = (-1)^{|\alpha|}\int_\Omega g\varphi dx. \tag{2.1}$$

Definition 2.2. *$W_p^l(\Omega)$ $(l\in\mathbb{N},\ 1\leq p\leq\infty)$ - Sobolev space, which is the Banach space of functions $f\in L_p(\Omega)$ such that $\forall\ \alpha\in\mathbb{Z}_+^n$ where $|\alpha|\leq l$ the weak derivatives $D_w^\alpha f$ exist on Ω and $D_w^\alpha f\in L_p(\Omega)$, with the norm*

$$\|f\|_{W_p^l(\Omega)} = \sum_{|\alpha|\leq l}\|D_w^\alpha f\|_{L_p(\Omega)}. \tag{2.2}$$

Definition 2.3. *For $l\in\mathbb{N}$, we define the Sobolev type local space*

$$\left(W_1^l\right)^{(loc)}(\Omega) := \{f:\Omega\to\mathbb{R} : f\in L_{loc}^1(\Omega)$$

and all f-distributional partials of orders $\leq l$ belong to

$L_{loc}^1(\Omega)\} = \{f \in L_1^{loc}(\Omega)$: *for each open set* G *compactly embedded into* Ω, $f \in W_1^l(G)\}$.

We use Definitions 2.1, 2.2, 2.3 on $\mathbb{R}$. Next we mention Sobolev's integral representation from [5].

Definition 2.4. *([5], p. 82) Let* $-\infty < a < b < \infty$,

$$\omega \in L_1(a,b), \qquad \int_a^b \omega(x)\,dx = 1. \tag{2.3}$$

Define

$$\Lambda(x,y) := \begin{cases} \int_a^y \omega(u)\,du, & a \le y \le x \le b, \\ -\int_y^b \omega(u)\,du, & a \le x < y \le b. \end{cases} \tag{2.4}$$

Proposition 2.5. *([5], p. 82) Let* f *be absolutely continuous on* $[a,b]$. *Then* $\forall$ $x \in (a,b)$

$$f(x) = \int_a^b f(y)\,\omega(y)\,dy + \int_a^b \Lambda(x,y)\,f'(y)\,dy, \tag{2.5}$$

the simplest case of Sobolev's integral representation.

Remark 2.6. ([5], pp. 82-83) We have that Λ is bounded:

$$\forall\ x,y \in [a,b], \quad |\Lambda(x,y)| \le \|\omega\|_{L_1(a,b)}, \tag{2.6}$$

and if $\omega \ge 0$, then

$$\forall\ x,y \in [a,b], \quad |\Lambda(x,y)| \le \Lambda(b,b) = 1.$$

If ω is symmetric with respect to $\frac{a+b}{2}$, then $\forall\ y \in [a,b]$ we have

$$\left|\Lambda\left(\frac{a+b}{2}, y\right)\right| \le \frac{1}{2}.$$

Examples of ω:

$\omega(x) = \frac{1}{b-a}$, $\forall\ x \in (a,b)$,

also

$\omega(x) = \frac{1}{2m}\left(\chi_{\left(a,a+\frac{1}{m}\right)} + \chi_{\left(b-\frac{1}{m},b\right)}\right)$, where $\chi_{(\alpha,\beta)}$ denotes the characteristic function of (α,β), $m \in \mathbb{N}$ and $m \ge 2(b-a)^{-1}$.

If $f \in \left(W_1^l\right)^{loc}(a,b)$, then f is equivalent to a function, which is locally absolutely continuous on (a,b) (its ordinary derivative, which exists almost everywhere on (a,b), is a weak derivative f_w' of f). Thus (2.5) holds for almost every $x \in (a,b)$ if f' is replaced by f_w'.

In this article sums of the form $\sum_{k=1}^{0} \cdot = 0$.

We mention

Theorem 2.7. *([5], p. 83) Let* $l \in \mathbb{N}$, $-\infty \le a < \alpha < \beta < b \le \infty$ *and*

$$\begin{cases} \omega \in L_1(\mathbb{R}), \quad \textit{(support)}\ \operatorname{supp}\omega \subset [\alpha,\beta], \\ \int_{\mathbb{R}} \omega(x)\,dx = 1 \end{cases} \tag{2.7}$$

Moreover, suppose that the derivative $f^{(l-1)}$ exists and is locally absolutely continuous on (a,b). Then $\forall\ x \in (a,b)$

$$f(x) = \sum_{k=0}^{l-1} \frac{1}{k!} \int_a^b f^{(k)}(y)(x-y)^k \omega(y)\,dy$$

$$+\frac{1}{(l-1)!} \int_a^b (x-y)^{l-1} \Lambda(x,y) f^{(l)}(y)\,dy, \tag{2.8}$$

and

$$f(x) = \sum_{k=0}^{l-1} \frac{1}{k!} \int_\alpha^\beta f^{(k)}(y)(x-y)^k \omega(y)\,dy$$

$$+\frac{1}{(l-1)!} \int_{a_x}^{b_x} (x-y)^{l-1} \Lambda(x,y) f^{(l)}(y)\,dy, \tag{2.9}$$

where $a_x = x$, $b_x = \beta$ for $x \in (a,\alpha]$; $a_x = \alpha$, $b_x = \beta$ for $x \in (\alpha,\beta)$; $a_x = \alpha$, $b_x = x$ for $x \in [\beta, b)$.

If, in particular, $-\infty < a < b < \infty$, $f^{(l-1)}$ exists and is absolutely continuous on $[a,b]$, then (2.8), (2.9) hold $\forall\ x \in [a,b]$ and for any interval $(\alpha,\beta) \subset (a,b)$.

Corollary 2.8. *([5], p. 85) Suppose that $l > 1$, condition (2.7) is replaced by*

$$\begin{cases} \omega \in C^{(l-2)}(\mathbb{R}), & \sup p\omega \subset [\alpha,\beta], \\ \int_{\mathbb{R}} \omega(x)\,dx = 1, \end{cases} \tag{2.10}$$

and the derivative $\omega^{(l-2)}$ is absolutely continuous on $[a,b]$.

Then for the same f as in Theorem 2.7, $\forall\ x \in (a,b)$

$$f(x) = \int_\alpha^\beta \left(\sum_{k=0}^{l-1} \frac{(-1)^k}{k!} \left[(x-y)^k \omega(y) \right]_y^{(k)} \right) f(y)\,dy$$

$$+\frac{1}{(l-1)!} \int_{a_x}^{b_x} (x-y)^{l-1} \Lambda(x,y) f^{(l)}(y)\,dy. \tag{2.11}$$

In particular here

$$\omega(\alpha) = \dots = \omega^{(l-2)}(\alpha) = \omega(\beta) = \dots = \omega^{(l-2)}(\beta) = 0. \tag{2.12}$$

Corollary 2.9. *([5], p. 86) Suppose that $l, m \in \mathbb{N}$, $m < l$. Then for the same f and ω as in Corollary 2.8, $\forall\ x \in (a,b)$*

$$f^{(m)}(x) = \int_\alpha^\beta \left(\sum_{k=0}^{l-m-1} \frac{(-1)^{k+m}}{k!} \left[(x-y)^k \omega(y) \right]_y^{(k+m)} \right) f(y)\,dy \tag{2.13}$$

$$+\frac{1}{(l-m-1)!} \int_{a_x}^{b_x} (x-y)^{l-m-1} \Lambda(x,y) f^{(l)}(y)\,dy.$$

Remark 2.10. ([5], p. 86) The first summand in (2.11) can take the form:

$$\begin{cases} \int_\alpha^\beta \left(\sum_{s=0}^{l-1} \sigma_s (x-y)^s \omega^{(s)}(y) \right) f(y)\, dy, \\ \text{where } \sigma_s := \frac{(-1)^s}{s!} \sum_{k=s}^{l-s-1} \begin{pmatrix} s+k \\ k \end{pmatrix}. \end{cases} \tag{2.14}$$

Similarly we have for the first summand of (2.13) the following form

$$\begin{cases} \int_\alpha^\beta \left(\sum_{s=m}^{l-1} \sigma_{s,m} (x-y)^{s-m} \omega^{(s)}(y) \right) f(y)\, dy, \\ \text{where } \sigma_{s,m} := \frac{(-1)^s}{(s-m)!} \sum_{k=s}^{l-s-1} \begin{pmatrix} s+k \\ k \end{pmatrix}. \end{cases} \tag{2.15}$$

We need

Theorem 2.11. *([5], p. 91) Let $l \in N$, $-\infty \leq a < \alpha < \beta < b \leq \infty$, ω satisfy condition*

$$\omega \in L_1(\mathbb{R}), \sup p\omega \subset [\alpha, \beta], \int_{\mathbb{R}} \omega(x)\, dx = 1, \tag{2.16}$$

and $f \in \left(W_1^l\right)^{loc}(a,b)$. Then for almost every $x \in (a,b)$

$$f(x) = \sum_{k=0}^{l-1} \frac{1}{k!} \int_\alpha^\beta f_w^{(k)}(y) (x-y)^k \omega(y)\, dy$$

$$+ \frac{1}{(l-1)!} \int_{a_x}^{b_x} (x-y)^{l-1} \Lambda(x,y) f_w^{(l)}(y)\, dy, \tag{2.17}$$

where a_x, b_x as in Theorem 2.7.

We denote $f_w^{(0)} := f$.

Remark 2.12. ([5], p. 92) By Theorem 2.11 it follows that if in Corollaries 2.8, 2.9 $f \in \left(W_1^l\right)^{loc}(a,b)$ then equalities (2.11) and (2.13) hold almost everywhere on (a,b), if we replace $f^{(l)}$, $f^{(m)}$ by the weak derivatives $f_w^{(l)}$, $f_w^{(m)}$; respectively.

Next we estimate the remainders of the above mentioned Sobolev representations.

We make

Remark 2.13. Denote by $\overline{f}^{(k)}$ either $f^{(k)}$ or $f_w^{(k)}$, where $k \in \mathbb{N}$. Let $0 \leq m < l$, $m \in \mathbb{Z}_+$. We estimate

$$R_{m,l} f(x) := \frac{1}{(l-m-1)!} \int_\alpha^\beta (x-y)^{l-m-1} \Lambda(x,y) \overline{f}^{(l)}(y)\, dy, \tag{2.18}$$

for $x \in (\alpha, \beta)$, where Λ as in (2.4), see also (2.6).

So we have

$$R_{0,l} f(x) := \frac{1}{(l-1)!} \int_\alpha^\beta (x-y)^{l-1} \Lambda(x,y) \overline{f}^{(l)}(y)\, dy. \tag{2.19}$$

Thus we obtain

$$|R_{m,l}f(x)| \leq \frac{\|\omega\|_{L_1(a,b)} \cdot (\beta-\alpha)^{l-m-1}}{(l-m-1)!} \int_\alpha^\beta \left|\overline{f}^{(l)}(y)\right| dy \tag{2.20}$$

$$= \frac{\|\omega\|_{L_1(a,b)} \cdot \left\|\overline{f}^{(l)}\right\|_{L_1(\alpha,\beta)} \cdot (\beta-\alpha)^{l-m-1}}{(l-m-1)!},$$

$x \in (\alpha, \beta)$.

We also have

$$|R_{m,l}f(x)| \leq \frac{\|\omega\|_{L_1(a,b)}}{(l-m-1)!} \int_\alpha^\beta |x-y|^{l-m-1} \left|\overline{f}^{(l)}(y)\right| dy =: I_1.$$

If $\overline{f}^{(l)} \in L_\infty(\alpha,\beta)$, then

$$I_1 \leq \frac{\|\omega\|_{L_1(a,b)} \left\|\overline{f}^{(l)}\right\|_{L_\infty(\alpha,\beta)}}{(l-m-1)!} \left(\int_\alpha^\beta |x-y|^{l-m-1} dy\right).$$

But

$$\int_\alpha^\beta |x-y|^{l-m-1} dy = \int_\alpha^x (x-y)^{l-m-1} dy + \int_x^\beta (y-x)^{l-m-1} dy$$

$$= \frac{(\beta-x)^{l-m} + (x-\alpha)^{l-m}}{l-m}.$$

Therefore if $\overline{f}^{(l)} \in L_\infty(\alpha,\beta)$, then

$$|R_{m,l}f(x)| \leq \frac{\|\omega\|_{L_1(a,b)} \left\|\overline{f}^{(l)}\right\|_{L_\infty(\alpha,\beta)}}{(l-m)!} \left((\beta-x)^{l-m} + (x-\alpha)^{l-m}\right), \tag{2.21}$$

$x \in (\alpha, \beta)$.

Let now $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. If $\overline{f}^{(l)} \in L_p(\alpha,\beta)$, then

$$I_1 \leq \frac{\|\omega\|_{L_1(a,b)}}{(l-m-1)!} \left(\int_\alpha^\beta |x-y|^{q(l-m-1)} dy\right)^{\frac{1}{q}} \left\|\overline{f}^{(l)}\right\|_{L_p(\alpha,\beta)}.$$

But

$$\int_\alpha^\beta |x-y|^{q(l-m-1)} dy = \int_\alpha^x (x-y)^{q(l-m-1)} dy + \int_x^\beta (y-x)^{q(l-m-1)} dy$$

$$= \frac{(x-\alpha)^{q(l-m-1)+1} + (\beta-x)^{q(l-m-1)+1}}{q(l-m-1)+1}.$$

Hence if $\overline{f}^{(l)} \in L_p(\alpha,\beta)$, then

$$|R_{m,l}f(x)| \leq \frac{\|\omega\|_{L_1(a,b)} \left\|\overline{f}^{(l)}\right\|_{L_p(\alpha,\beta)}}{(l-m-1)!}$$

$$\times\left(\frac{(\beta-x)^{q(l-m-1)+1}+(x-\alpha)^{q(l-m-1)+1}}{q(l-m-1)+1}\right)^{\frac{1}{q}}, \tag{2.22}$$

$x\in(\alpha,\beta)$.

If $\sup p\omega\subset[\alpha,\beta]$, then

$$\|\omega\|_{L_1(a,b)}=\|\omega\|_{L_1(\alpha,\beta)}. \tag{2.23}$$

If $\omega\in C(\mathbb{R})$ and $\sup p\omega\subset[\alpha,\beta]$, then

$$\|\omega\|_{L_1(\alpha,\beta)}\le\|\omega\|_{\infty,[\alpha,\beta]}\cdot(\beta-\alpha). \tag{2.24}$$

We make

Remark 2.14. Here we estimate from the Taylor's averaged polynomial, see (2.9) and (2.17), tha part

$$Q^{l-1}f(x):=\sum_{k=1}^{l-1}\frac{1}{k!}\int_\alpha^\beta\overline{f}^{(k)}(y)(x-y)^k\omega(y)\,dy, \tag{2.25}$$

called also quasi-averaged Taylor polynomial. When $l=1$, then $Q^0f(x)=0$.

We see that

$$\left|Q^{l-1}f(x)\right|\le\sum_{k=1}^{l-1}\frac{1}{k!}\int_\alpha^\beta\left|\overline{f}^{(k)}(y)\right||x-y|^k|\omega(y)|\,dy$$

$$\le\left(\sum_{k=1}^{l-1}\frac{(\beta-\alpha)^k}{k!}\left\|\overline{f}^{(k)}\right\|_{L_1(\alpha,\beta)}\right)\|\omega\|_{L_\infty(\mathbb{R})}, \tag{2.26}$$

given that $\|\omega\|_{L_\infty(\mathbb{R})}<\infty$, $x\in(\alpha,\beta)$.

Similarly, when $\overline{f}^{(k)}\in L_\infty(\alpha,\beta)$, $k=1,...,l-1$, and $\|\omega\|_{L_\infty(\mathbb{R})}<\infty$ we derive

$$\left|Q^{l-1}f(x)\right|\le\sum_{k=1}^{l-1}\frac{\left\|\overline{f}^{(k)}\right\|_{L_\infty(\alpha,\beta)}\|\omega\|_{L_\infty(\mathbb{R})}}{k!}\int_\alpha^\beta|x-y|^k\,dy$$

$$=\left(\sum_{k=1}^{l-1}\left(\frac{(\beta-x)^{k+1}+(x-\alpha)^{k+1}}{(k+1)!}\right)\left\|\overline{f}^{(k)}\right\|_{L_\infty(\alpha,\beta)}\right)\|\omega\|_{L_\infty(\mathbb{R})}, \tag{2.27}$$

$x\in(\alpha,\beta)$.

Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\overline{f}^{(k)}\in L_p(\alpha,\beta)$, $k=1,...,l-1$, and again $\|\omega\|_{L_\infty(\mathbb{R})}<\infty$. Then

$$\left|Q^{l-1}f(x)\right|\le\sum_{k=1}^{l-1}\frac{\left\|\overline{f}^{(k)}\right\|_{L_p(\alpha,\beta)}}{k!}\left(\int_\alpha^\beta|x-y|^{kq}\,dy\right)^{\frac{1}{q}}\|\omega\|_{L_\infty(\mathbb{R})}$$

$$= \left(\sum_{k=1}^{l-1} \left(\frac{(\beta - x)^{(kq+1)} + (x - \alpha)^{(kq+1)}}{kq+1} \right)^{\frac{1}{q}} \frac{\left\| \overline{f}^{(k)} \right\|_{L_p(\alpha,\beta)}}{k!} \right) \|\omega\|_{L_\infty(\mathbb{R})}, \tag{2.28}$$

$x \in (\alpha, \beta)$.

Assume $\omega \in L_1(\mathbb{R})$ and $\overline{f}^{(k)} \in L_\infty(\alpha, \beta)$, $k = 1, ..., l-1$, then

$$\left| Q^{l-1} f(x) \right| \leq \left(\sum_{k=1}^{l-1} \frac{\left\| \overline{f}^{(k)} \right\|_{L_\infty(a,\beta)} (\beta - \alpha)^k}{k!} \right) \|\omega\|_{L_1(\mathbb{R})}, \tag{2.29}$$

$x \in (\alpha, \beta)$.

Assume $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\overline{f}^{(k)} \in L_p(\alpha, \beta)$, $k = 1, ..., l-1$; $\omega \in L_q(\alpha, \beta)$, then

$$\left| Q^{l-1} f(x) \right| \leq \left(\sum_{k=1}^{l-1} \frac{(\beta - \alpha)^k}{k!} \left\| \overline{f}^{(k)} \right\|_{L_p(\alpha,\beta)} \right) \|\omega\|_{L_q(\alpha,\beta)}, \tag{2.30}$$

$x \in (\alpha, \beta)$.

Assume $p, q, r > 1 : \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $\overline{f}^{(k)} \in L_p(\alpha, \beta)$, $k = 1, ..., l-1$; $\omega \in L_q(\alpha, \beta)$, then

$$\left| Q^{l-1} f(x) \right| \leq$$

$$\left(\sum_{k=1}^{l-1} \frac{\left\| \overline{f}^{(k)} \right\|_{L_p(\alpha,\beta)}}{k!} \left(\frac{(\beta - x)^{(kr+1)} + (x - \alpha)^{(kr+1)}}{(kr+1)} \right)^{\frac{1}{r}} \right) \|\omega\|_{L_q(\alpha,\beta)}, \tag{2.31}$$

$x \in (\alpha, \beta)$.

We also make

Remark 2.15. Here $l > 1$, $\omega \in C^{(l-2)}(\mathbb{R})$, $\sup p\omega \subset [\alpha, \beta]$, $\int_{\mathbb{R}} \omega(x)\, dx = 1$, and the derivative $\omega^{(l-2)}$ is absolutely continuous on $[a, b]$. Hence we have that

$$Q^{l-1} f(x) = \sum_{k=1}^{l-1} \frac{(-1)^k}{k!} \int_\alpha^\beta \left(\left[(x - y)^k \omega(y) \right]_y^{(k)} \right) f(y)\, dy, \tag{2.32}$$

$\forall\, x \in (\alpha, \beta)$.

And it holds

$$\left| Q^{l-1} f(x) \right| \leq \sum_{k=1}^{l-1} \frac{1}{k!} \int_\alpha^\beta \left| \left[(x - y)^k \omega(y) \right]_y^{(k)} \right| |f(y)|\, dy, \tag{2.33}$$

$\forall\, x \in (\alpha, \beta)$.

Consequently, $\forall\, x \in (\alpha, \beta)$,

$$\left| Q^{l-1} f(x) \right| \leq$$

$$\begin{cases} \left(\sum_{k=1}^{l-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k)} \right\|_\infty \right) \|f\|_{L_1(\alpha,\beta)}, \text{ if } f \in L_1(\alpha,\beta), \\ \left(\sum_{k=1}^{l-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k)} \right\|_{L_1(\alpha,\beta)} \right) \|f\|_{L_\infty(\alpha,\beta)}, \text{ if } f \in L_\infty(\alpha,\beta), \\ \text{when } p,q>1: \frac{1}{p}+\frac{1}{q}=1, \text{ we have} \\ \left(\sum_{k=1}^{l-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k)} \right\|_{L_q(\alpha,\beta)} \right) \|f\|_{L_p(\alpha,\beta)}, \text{ if } f \in L_p(\alpha,\beta). \end{cases} \tag{2.34}$$

Let $l, m \in \mathbb{N}$, $m < l$, and f, ω as above, $x \in (\alpha, \beta)$.

We consider here

$$Q_m^{l-1} f(x) := \sum_{k=1}^{l-m-1} \frac{(-1)^{k+m}}{k!} \int_\alpha^\beta \left(\left[(x-y)^k \omega(y) \right]_y^{(k+m)} \right) f(y)\, dy. \tag{2.35}$$

When $l = m+1$, then $Q_m^{l-1} f(x) := 0$.

Hence it holds

$$\left| Q_m^{l-1} f(x) \right| \le \sum_{k=1}^{l-m-1} \frac{1}{k!} \int_\alpha^\beta \left| \left(\left[(x-y)^k \omega(y) \right]_y^{(k+m)} \right) \right| |f(y)|\, dy, \tag{2.36}$$

$\forall\, x \in (\alpha, \beta)$.

Consequently, $\forall\, x \in (\alpha, \beta)$,

$$\left| Q_m^{l-1} f(x) \right| \le$$

$$\begin{cases} \left(\sum_{k=1}^{l-m-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k+m)} \right\|_\infty \right) \|f\|_{L_1(\alpha,\beta)}, \text{ if } f \in L_1(\alpha,\beta), \\ \left(\sum_{k=1}^{l-m-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k+m)} \right\|_{L_1(\alpha,\beta)} \right) \|f\|_{L_\infty(\alpha,\beta)}, \text{ if } f \in L_\infty(\alpha,\beta), \\ \text{when } p,q>1: \frac{1}{p}+\frac{1}{q}=1, \text{ we have} \\ \left(\sum_{k=1}^{l-m-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k+m)} \right\|_{L_q(\alpha,\beta)} \right) \|f\|_{L_p(\alpha,\beta)}, \text{ if } f \in L_p(\alpha,\beta). \end{cases} \tag{2.37}$$

We also need

Remark 2.16. Here again $\overline{f}^{(k)}$ means either $f^{(k)}$ or $f_w^{(k)}$, $k \in \mathbb{N}$. We rewrite (2.9), (2.11) and (2.17). For $x \in (\alpha, \beta)$ we get

$$f(x) = \int_\alpha^\beta f(y)\, \omega(y)\, dy + Q^{l-1} f(x) + R_{0,l} f(x). \tag{2.38}$$

Also for $x \in (\alpha, \beta)$ we rewrite (2.13) (see also Remark 2.12) as follows:

$$\overline{f}^{(m)}(x) = (-1)^m \int_\alpha^\beta f(y)\, \omega^{(m)}(y)\, dy + Q_m^{l-1} f(x) + R_{m,l} f(x). \tag{2.39}$$

3. Main results

On our way to prove the general Chebyshev-Grüss type inequalities we establish the general

Theorem 3.1. *For f, g under the assumptions of any of Theorem 2.7, Corollary 2.8 and Theorem 2.11 we obtain that*

$$\Delta(f,g) := \left| \int_\alpha^\beta \omega(x) f(x) g(x)\, dx - \left(\int_\alpha^\beta \omega(x) f(x)\, dx \right) \left(\int_\alpha^\beta \omega(x) g(x)\, dx \right) \right|$$

$$\leq \frac{1}{2} \left[\left(\int_\alpha^\beta |\omega(x)| |g(x)| \left| Q^{l-1} f(x) \right| dx + \int_\alpha^\beta |\omega(x)| |f(x)| \left| Q^{l-1} g(x) \right| dx \right) \right.$$

$$\left. + \left(\int_\alpha^\beta |\omega(x)| |g(x)| |R_{0,l} f(x)|\, dx + \int_\alpha^\beta |\omega(x)| |f(x)| |R_{0,l} g(x)|\, dx \right) \right]. \tag{3.1}$$

Proof. For $x \in (\alpha, \beta)$ we have

$$f(x) = \int_\alpha^\beta f(y) \omega(y)\, dy + Q^{l-1} f(x) + R_{0,l} f(x),$$

and

$$g(x) = \int_\alpha^\beta g(y) \omega(y)\, dy + Q^{l-1} g(x) + R_{0,l} g(x).$$

Hence

$$\omega(x) f(x) g(x)$$

$$= \omega(x) g(x) \int_\alpha^\beta f(y) \omega(y)\, dy + \omega(x) g(x) Q^{l-1} f(x) + \omega(x) g(x) R_{0,l} f(x),$$

and

$$\omega(x) f(x) g(x)$$

$$= \omega(x) f(x) \int_\alpha^\beta g(y) \omega(y)\, dy + \omega(x) f(x) Q^{l-1} g(x) + \omega(x) f(x) R_{0,l} g(x).$$

Therefore

$$\int_\alpha^\beta \omega(x) f(x) g(x)\, dx = \left(\int_\alpha^\beta \omega(x) g(x)\, dx \right) \left(\int_\alpha^\beta f(x) \omega(x)\, dx \right)$$

$$+ \int_\alpha^\beta \omega(x) g(x) Q^{l-1} f(x)\, dx + \int_\alpha^\beta \omega(x) g(x) R_{0,l} f(x)\, dx,$$

and

$$\int_\alpha^\beta \omega(x) f(x) g(x)\, dx = \left(\int_\alpha^\beta \omega(x) f(x)\, dx \right) \left(\int_\alpha^\beta g(x) \omega(x)\, dx \right)$$

$$+ \int_\alpha^\beta \omega(x) f(x) Q^{l-1} g(x)\, dx + \int_\alpha^\beta \omega(x) f(x) R_{0,l} g(x)\, dx.$$

Consequently there hold

$$\int_{\alpha}^{\beta} \omega(x) f(x) g(x) dx - \left(\int_{\alpha}^{\beta} \omega(x) f(x) dx \right) \left(\int_{\alpha}^{\beta} g(x) \omega(x) dx \right)$$

$$= \int_{\alpha}^{\beta} \omega(x) g(x) Q^{l-1} f(x) dx + \int_{a}^{\beta} \omega(x) g(x) R_{0,l} f(x) dx,$$

and

$$\int_{\alpha}^{\beta} \omega(x) f(x) g(x) dx - \left(\int_{\alpha}^{\beta} \omega(x) f(x) dx \right) \left(\int_{\alpha}^{\beta} g(x) \omega(x) dx \right)$$

$$= \int_{\alpha}^{\beta} \omega(x) f(x) Q^{l-1} g(x) dx + \int_{\alpha}^{\beta} \omega(x) f(x) R_{0,l} g(x) dx.$$

Adding the last two equalities and divide by two, we get

$$\int_{\alpha}^{\beta} \omega(x) f(x) g(x) dx - \left(\int_{\alpha}^{\beta} \omega(x) f(x) dx \right) \left(\int_{\alpha}^{\beta} g(x) \omega(x) dx \right)$$

$$= \frac{1}{2} \left[\left(\int_{\alpha}^{\beta} \omega(x) g(x) Q^{l-1} f(x) dx + \int_{\alpha}^{\beta} \omega(x) f(x) Q^{l-1} g(x) dx \right) \right.$$

$$\left. + \left(\int_{\alpha}^{\beta} \omega(x) g(x) R_{0,l} f(x) dx + \int_{\alpha}^{\beta} \omega(x) f(x) R_{0,l} g(x) dx \right) \right],$$

hence proving the claim. □

General Chebyshev-Grüss inequalities follow.

We give

Theorem 3.2. *Let f, g with $f^{(l-1)}$, $g^{(l-1)}$ absolutely continuous on $[a, b] \subset \mathbb{R}$, $l \in \mathbb{N}$; $(\alpha, \beta) \subset (a, b)$. Let also $\omega \in L_1(\mathbb{R})$, $\sup p\omega \subset [\alpha, \beta]$, $\int_{\mathbb{R}} \omega(x) dx = 1$. Then*

$$\left| \int_{\alpha}^{\beta} \omega(x) f(x) g(x) dx - \left(\int_{\alpha}^{\beta} \omega(x) f(x) dx \right) \left(\int_{\alpha}^{\beta} \omega(x) g(x) dx \right) \right| \leq$$

$$\frac{\|\omega\|_{L_1(\mathbb{R})}^2}{2} \left[\left[\|g\|_{\infty,(\alpha,\beta)} \left(\sum_{k=1}^{l-1} \frac{\|f^{(k)}\|_{\infty,(\alpha,\beta)} (\beta - \alpha)^k}{k!!} \right) + \right. \right.$$

$$\left. \|f\|_{\infty,(\alpha,\beta)} \left(\sum_{k=1}^{l-1} \frac{\|g^{(k)}\|_{\infty,(\alpha,\beta)} (\beta - \alpha)^k}{k!} \right) \right] +$$

$$\left. \left[\left(\|g\|_{\infty,(\alpha,\beta)} \left\| f^{(l)} \right\|_{L_1(a,\beta)} + \|f\|_{\infty,(\alpha,\beta)} \left\| g^{(l)} \right\|_{L_1(\alpha,\beta)} \right) \frac{(\beta - \alpha)^{l-1}}{(l-1)!} \right] \right]. \quad (3.2)$$

Proof. By (2.20) and (2.29). □

Theorem 3.3. *Let* $f, g \in C^l([a,b])$, $[a,b] \subset \mathbb{R}$, $l \in \mathbb{N}$, $(\alpha, \beta) \subset (a,b)$. *Let also* $\omega \in L_\infty(\mathbb{R})$, $\operatorname{supp} \omega \subset [\alpha, \beta]$, $\int_{\mathbb{R}} \omega(x)\,dx = 1$. *Then*

$$\left| \int_\alpha^\beta \omega(x) f(x) g(x)\,dx - \left(\int_\alpha^\beta \omega(x) f(x)\,dx \right) \left(\int_\alpha^\beta \omega(x) g(x)\,dx \right) \right| \leq$$

$$\|\omega\|_{L_1(\mathbb{R})} \left[\frac{\|\omega\|_{L_1(\mathbb{R})}}{2} \left\{ \|g\|_{\infty,(\alpha,\beta)} \left(\sum_{k=1}^{l-1} \left(\frac{\|f^{(k)}\|_{\infty,(\alpha,\beta)} (\beta-\alpha)^k}{k!} \right) \right) + \right. \right.$$

$$\left. \|f\|_{\infty,(\alpha,\beta)} \left(\sum_{k=1}^{l-1} \left(\frac{\|g^{(k)}\|_{\infty,(\alpha,\beta)} (\beta-\alpha)^k}{k!} \right) \right) \right\} +$$

$$\left. \left[\|\omega\|_{\infty,(\alpha,\beta)} \frac{(\beta-\alpha)^{l+1}}{(l+1)!} \left(\|g\|_{\infty,(\alpha,\beta)} \left\| f^{(l)} \right\|_{\infty,(\alpha,\beta)} + \|f\|_{\infty,(\alpha,\beta)} \left\| g^{(l)} \right\|_{\infty,(\alpha,\beta)} \right) \right] \right]. \tag{3.3}$$

Proof. By (2.21) and (2.29). □

We further present

Theorem 3.4. *Let* $f, g \in \left(W_1^l\right)^{loc}(a,b)$; $a, b \in \mathbb{R}; (\alpha,\beta) \subset (a,b)$, $l \in \mathbb{N}$; $\omega \in L_\infty(\mathbb{R})$, $\operatorname{supp} \omega \subset [\alpha, \beta]$, $\int_{\mathbb{R}} \omega(x)\,dx = 1$. *Then*

$$\left| \int_\alpha^\beta \omega(x) f(x) g(x)\,dx - \left(\int_\alpha^\beta \omega(x) f(x)\,dx \right) \left(\int_\alpha^\beta \omega(x) g(x)\,dx \right) \right| \leq$$

$$\frac{\|\omega\|_{L_\infty(\mathbb{R})}^2}{2} \left\{ \left[\|g\|_{L_1(\alpha,\beta)} \left(\sum_{k=1}^{l-1} \left(\frac{(\beta-\alpha)^k}{k!} \left\| f_w^{(k)} \right\|_{L_1(\alpha,\beta)} \right) \right) + \right. \right.$$

$$\left. \|f\|_{L_1(\alpha,\beta)} \left(\sum_{k=1}^{l-1} \left(\frac{(\beta-\alpha)^k}{k!} \left\| g_w^{(k)} \right\|_{L_1(\alpha,\beta)} \right) \right) \right] +$$

$$\left. \left[\left(\|g\|_{L_1(\alpha,\beta)} \left\| f_w^{(l)} \right\|_{L_1(\alpha,\beta)} + \|f\|_{L_1(\alpha,\beta)} \left\| g_w^{(l)} \right\|_{L_1(\alpha,\beta)} \right) \frac{(\beta-\alpha)^l}{(l-1)!} \right] \right\}. \tag{3.4}$$

Proof. By (2.20) and (2.26). □

Theorem 3.5. *Let* $f, g \in \left(W_1^l\right)^{loc}(a,b)$; $a, b \in \mathbb{R}; (\alpha,\beta) \subset (a,b)$, $l \in \mathbb{N}$; $\omega \in L_\infty(\mathbb{R})$, $\operatorname{supp} \omega \subset [\alpha, \beta]$, $\int_{\mathbb{R}} \omega(x)\,dx = 1$. *Furthermore assume* $f_w^{(k)}, g_w^{(k)} \in L_\infty(\alpha, \beta)$, $k = 1, ..., l$. *Then*

$$\left| \int_\alpha^\beta \omega(x) f(x) g(x)\,dx - \left(\int_\alpha^\beta \omega(x) f(x)\,dx \right) \left(\int_\alpha^\beta \omega(x) g(x)\,dx \right) \right| \leq$$

$$\frac{\|\omega\|_{L_\infty(\mathbb{R})}^2}{2} \left\{ \left[\|g\|_{L_1(\alpha,\beta)} \left(\sum_{k=1}^{l-1} \frac{\left\| f_w^{(k)} \right\|_{L_\infty(\alpha,\beta)} (\beta-\alpha)^{k+1}}{k!} \right) + \right. \right.$$

$$\|f\|_{L_1(\alpha,\beta)}\left(\sum_{k=1}^{l-1}\frac{\left\|g_w^{(k)}\right\|_{L_\infty(\alpha,\beta)}(\beta-\alpha)^{k+1}}{k!}\right)\Bigg]+$$

$$\left[\left(\|g\|_{L_1(\alpha,\beta)}\left\|f_w^{(l)}\right\|_{L_\infty(\alpha,\beta)}+\|f\|_{L_1(\alpha,\beta)}\left\|g_w^{(l)}\right\|_{L_\infty(\alpha,\beta)}\right)\frac{(\beta-\alpha)^{l+1}}{(l-1)!}\right]\Bigg\}. \tag{3.5}$$

Proof. As in (2.21) and by (2.27). □

Theorem 3.6. *Let* $f,g\in\left(W_1^l\right)^{loc}(a,b)$; $a,b\in\mathbb{R}$; $(\alpha,\beta)\subset(a,b)$, $l\in\mathbb{N}$; $\omega\in L_\infty(\mathbb{R})$, $\operatorname{supp}\omega\subset[\alpha,\beta]$, $\int_{\mathbb{R}}\omega(x)\,dx=1$. *Furthermore assume for* $p>1$ *that* $f_w^{(k)},g_w^{(k)}\in L_p(\alpha,\beta)$, $k=1,...,l$. *Then*

$$\Delta(f,g):=\left|\int_\alpha^\beta\omega(x)f(x)g(x)dx-\left(\int_\alpha^\beta\omega(x)f(x)dx\right)\left(\int_\alpha^\beta\omega(x)g(x)dx\right)\right|$$

$$\leq\frac{\|\omega\|_{L_\infty(\mathbb{R})}^2}{2}\left[\left\{\|g\|_{L_1(\alpha,\beta)}\left(\sum_{k=1}^{l-1}\frac{(\beta-\alpha)^{k+1-\frac{1}{p}}}{k!}\left\|f_w^{(k)}\right\|_{L_p(\alpha,\beta)}\right)+\right.\right.$$

$$\left.\|f\|_{L_1(\alpha,\beta)}\left(\sum_{k=1}^{l-1}\frac{(\beta-\alpha)^{k+1-\frac{1}{p}}}{k!}\left\|g_w^{(k)}\right\|_{L_p(\alpha,\beta)}\right)\right\}+$$

$$\left.\left\{\left(\|g\|_{L_1(\alpha,\beta)}\left\|f_w^{(l)}\right\|_{L_p(\alpha,\beta)}+\|f\|_{L_1(\alpha,\beta)}\left\|g_w^{(l)}\right\|_{L_p(\alpha,\beta)}\right)\frac{(\beta-\alpha)^{l+1-\frac{1}{p}}}{(l-1)!}\right\}\right]. \tag{3.6}$$

Proof. Working as in (2.22) and from (2.30). □

Remark 3.7. When $f,g\in C^l([a,b])$, $l\in\mathbb{N}$, the Theorems 3.4, 3.5, 3.6 are again valid. In this case we replace $f_w^{(k)}$, $g_w^{(k)}$ by $f^{(k)}$, $g^{(k)}$ in all inequalities (3.4), (3.5) and (3.6); $k=1,...,l$.

We continue with

Theorem 3.8. *Let* $l\in\mathbb{N}-\{1\}$, $\omega\in C^{(l-2)}(\mathbb{R})$, $\operatorname{supp}\omega\subset[\alpha,\beta]$, $\int_{\mathbb{R}}\omega(x)\,dx=1$, *and the derivative* $\omega^{(l-2)}$ *is absolutely continuous on* $[a,b]\subset\mathbb{R}$; $(\alpha,\beta)\subset(a,b)$. *Here assume* $f,g\in\left(W_1^l\right)^{loc}(a,b)$, *or* $f,g\in C^l([a,b])$. *Here* $\overline{f}^{(l)}$ *denotes either* $f_w^{(l)}$ *or* $f^{(l)}$, *and* $\Delta(f,g)$ *as in (*3.1*).*

We have the following cases:

i) It holds

$$\Delta(f,g)\leq\frac{\|\omega\|_\infty}{2}\left[2\|g\|_{L_1(\alpha,\beta)}\|f\|_{L_1(\alpha,\beta)}\left(\sum_{k=1}^{l-1}\frac{1}{k!}\sup_{x\in(\alpha,\beta)}\left\|\left[(x-y)^k\omega(y)\right]_y^{(k)}\right\|_\infty\right)\right.$$

$$\left.+\left(\|g\|_{L_1(\alpha,\beta)}\left\|\overline{f}^{(l)}\right\|_{L_1(\alpha,\beta)}+\|f\|_{L_1(\alpha,\beta)}\left\|\overline{g}^{(l)}\right\|_{L_1(\alpha,\beta)}\right)\frac{(\beta-\alpha)^l}{(l-1)!}\|\omega\|_\infty\right]. \tag{3.7}$$

ii) Assume further that $f, g, \overline{f}^{(l)}, \overline{g}^{(l)} \in L_\infty(\alpha, \beta)$. *Then*

$$\Delta(f,g) \le \frac{\|\omega\|_\infty}{2}\Big[\Big\{\Big(\|f\|_{L_\infty(\alpha,\beta)}\|g\|_{L_1(\alpha,\beta)} + \|g\|_{L_\infty(\alpha,\beta)}\|f\|_{L_1(\alpha,\beta)}\Big)\cdot$$

$$\left(\sum_{k=1}^{l-1}\frac{1}{k!}\sup_{x\in(\alpha,\beta)}\left\|\left[(x-y)^k\omega(y)\right]_y^{(k)}\right\|_{L_1(\alpha,\beta)}\right)\Big\}+$$

$$\left\{\left(\|g\|_{L_1(\alpha,\beta)}\left\|\overline{f}^{(l)}\right\|_{L_\infty(\alpha,\beta)} + \|f\|_{L_1(\alpha,\beta)}\left\|\overline{g}^{(l)}\right\|_{L_\infty(\alpha,\beta)}\right)\frac{(\beta-\alpha)^{l+1}}{(l-1)!}\|\omega\|_\infty\right\}\Big]. \tag{3.8}$$

iii) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; *assume further that* $f, g, \overline{f}^{(l)}, \overline{g}^{(l)} \in L_p(\alpha, \beta)$. *Then*

$$\Delta(f,g) \le \frac{\|\omega\|_\infty}{2}\Big[\Big\{\Big(\|g\|_{L_1(\alpha,\beta)}\|f\|_{L_p(\alpha,\beta)} + \|f\|_{L_1(\alpha,\beta)}\|g\|_{L_p(\alpha,\beta)}\Big)\cdot$$

$$\left(\sum_{k=1}^{l-1}\frac{1}{k!}\sup_{x\in(\alpha,\beta)}\left\|\left[(x-y)^k\omega(y)\right]_y^{(k)}\right\|_{L_q(\alpha,\beta)}\right)\Big\}+$$

$$\left\{\left(\|g\|_{L_1(\alpha,\beta)}\left\|\overline{f}^{(l)}\right\|_{L_p(\alpha,\beta)} + \|f\|_{L_1(\alpha,\beta)}\left\|\overline{g}^{(l)}\right\|_{L_p(\alpha,\beta)}\right)\frac{(\beta-\alpha)^{l+1-\frac{1}{p}}}{(l-1)!}\|\omega\|_\infty\right\}\Big]. \tag{3.9}$$

Proof. By (3.1), (2.34) and by Theorems 3.4, 3.5, 3.6. ☐

Next we give a series of Ostrowski type inequalities.

Theorem 3.9. *Let* $l \in \mathbb{N}, [a,b] \subset \mathbb{R}$, $a < \alpha < \beta < b$ *and* $\omega \in L_1(\mathbb{R})$, $\operatorname{supp}\omega \subset [\alpha,\beta]$, $\int_{\mathbb{R}}\omega(x)\,dx = 1$. *Assume* f *on* $[a,b] : f^{(l-1)}$ *exists and is absolutely continuous on* $[a,b]$. *Then for any* $x \in (\alpha,\beta)$ *we get*

$$\left|f(x) - \int_\alpha^\beta f(y)\,\omega(y)\,dy - Q^{l-1}f(x)\right| \le$$

$$\frac{\|\omega\|_{L_1(\mathbb{R})}\left\|f^{(l)}\right\|_{L_1(\alpha,\beta)}(\beta-\alpha)^{l-1}}{(l-1)!} := A_1. \tag{3.10}$$

If additionally we assume $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$, *then* $\forall\, x \in (\alpha,\beta)$, *we get*

$$\left|f(x) - \int_\alpha^\beta f(y)\,\omega(y)\,dy\right| \le$$

$$\left(\sum_{k=1}^{l-1}\left(\frac{(\beta-x)^{k+1}+(x-\alpha)^{k+1}}{(k+1)!}\right)\left\|f^{(k)}\right\|_\infty\right)\|\omega\|_{L_\infty(\mathbb{R})} +$$

$$\frac{\|\omega\|_{L_1(\mathbb{R})}\left\|f^{(l)}\right\|_{L_1(\alpha,\beta)}(\beta-\alpha)^{l-1}}{(l-1)!} := B_1(x). \tag{3.11}$$

Proof. By (2.38), (2.20) and (2.27). ☐

Theorem 3.10. *All as in Theorem* 3.9. *Assume* $f \in C^l([a,b])$. *Then* $\forall\ x \in (\alpha, \beta)$,

$$\left| f(x) - \int_\alpha^\beta f(y)\,\omega(y)\,dy - Q^{l-1} f(x) \right| \leq$$

$$\frac{\|\omega\|_{L_1(\mathbb{R})} \left\| f^{(l)} \right\|_\infty \left((\beta - x)^l + (x - \alpha)^l \right)}{l!} =: A_2(x). \tag{3.12}$$

If additionally we assume $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$, *then* $\forall\ x \in (\alpha, \beta)$,

$$\left| f(x) - \int_\alpha^\beta f(y)\,\omega(y)\,dy \right| \leq$$

$$\left(\sum_{k=1}^{l-1} \left(\frac{(\beta - x)^{k+1} + (x - \alpha)^{k+1}}{(k+1)!} \right) \left\| f^{(k)} \right\|_\infty \right) \|\omega\|_{L_\infty(\mathbb{R})} +$$

$$\frac{\|\omega\|_{L_1(\mathbb{R})} \left\| f^{(l)} \right\|_\infty \left((\beta - x)^l + (x - \alpha)^l \right)}{l!} =: B_2(x). \tag{3.13}$$

Proof. By (2.38), (2.21) and (2.27). □

We continue with

Theorem 3.11. *Let all as in Theorem* 3.9 *or* $f \in \left(W_1^l\right)^{loc}(a,b)$ *and rest as in Theorem* 3.9. *Then* $\forall\ x \in (\alpha, \beta)$ *(or almost every* $x \in (\alpha, \beta)$*, respectively), we get*

$$E(f)(x) := \left| f(x) - \int_\alpha^\beta f(y)\,\omega(y)\,dy - Q^{l-1} f(x) \right| \leq$$

$$\frac{\|\omega\|_{L_1(a,b)} \left\| \overline{f}^{(l)} \right\|_{L_1(\alpha,\beta)} (\beta - \alpha)^{l-1}}{(l-1)!} =: A_3 \tag{3.14}$$

Additionaly, if $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$, $\forall\ x \in (\alpha, \beta)$ *(or almost every* $x \in (\alpha, \beta)$*, respectively), we get*

$$\Delta(f)(x) := \left| f(x) - \int_\alpha^\beta f(y)\,\omega(y)\,dy \right| \leq$$

$$\left(\sum_{k=1}^{l-1} \frac{(\beta - \alpha)^k}{k!} \left\| \overline{f}^{(k)} \right\|_{L_1(\alpha,\beta)} \right) \|\omega\|_{L_\infty(\mathbb{R})}$$

$$+ \frac{\|\omega\|_{L_1(a,b)} \left\| \overline{f}^{(l)} \right\|_{L_1(\alpha,\beta)} (\beta - \alpha)^{l-1}}{(l-1)!} =: B_3. \tag{3.15}$$

Proof. By (2.38), (2.20) and (2.26). □

Theorem 3.12. *Let all as in Theorem* 3.9 *or* $f \in \left(W_1^l\right)^{loc}(a,b)$ *and rest as in Theorem* 3.9*. Assume further* $\overline{f}^{(l)} \in L_\infty(\alpha,\beta)$. *Then* $\forall\ x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get*

$$E(f)(x) \leq \frac{\|\omega\|_{L_1(\mathbb{R})} \left\|\overline{f}^{(l)}\right\|_{L_\infty(\alpha,\beta)} \left((\beta-x)^l + (x-\alpha)^l\right)}{l!} =: A_4(x). \quad (3.16)$$

Additionaly $\overline{f}^{(k)} \in L_\infty(\alpha,\beta)$, $k = 1,...,l-1$ *and if* $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$, *then* $\forall$ $x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get*

$$\Delta(f)(x) \leq \left(\sum_{k=1}^{l-1} \left(\frac{\left((\beta-x)^{k+1} + (x-\alpha)^{k+1}\right)}{(k+1)!}\right) \left\|\overline{f}^{(k)}\right\|_{L_\infty(\alpha,\beta)}\right) \|\omega\|_{L_\infty(\mathbb{R})}$$

$$+\frac{\|\omega\|_{L_1(\mathbb{R})} \left\|\overline{f}^{(l)}\right\|_{L_\infty(\alpha,\beta)} \left((\beta-x)^l + (x-\alpha)^l\right)}{l!} =: B_4(x). \quad (3.17)$$

Proof. By (2.38), (2.21) and (2.27). □

Theorem 3.13. *Let all as in Theorem* 3.9 *or* $f \in \left(W_1^l\right)^{loc}(a,b)$ *and rest as in Theorem* 3.9*. Let* $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*. Assume further* $\overline{f}^{(l)} \in L_p(\alpha,\beta)$. *Then* $\forall\ x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get*

$$E(f)(x) \leq \frac{\|\omega\|_{L_1(\mathbb{R})} \left\|\overline{f}^{(l)}\right\|_{L_p(\alpha,\beta)}}{(l-1)!} \cdot$$

$$\left(\frac{(\beta-x)^{q(l-1)+1} + (x-\alpha)^{q(l-1)+1}}{q(l-1)+1}\right)^{\frac{1}{q}} =: A_5(x). \quad (3.18)$$

Additionally, if $\overline{f}^{(k)} \in L_p(\alpha,\beta)$, $k = 1,...,l-1$ *and* $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$, *then* $\forall\ x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get*

$$\Delta(f)(x) \leq$$

$$\left(\sum_{k=1}^{l-1} \left(\frac{\left((\beta-x)^{(kq+1)} + (x-\alpha)^{(kq+1)}\right)}{kq+1}\right)^{\frac{1}{q}} \frac{\left\|\overline{f}^{(k)}\right\|_{L_p(\alpha,\beta)}}{k!}\right) \|\omega\|_{L_\infty(\mathbb{R})}$$

$$+A_5(x) =: B_5(x). \quad (3.19)$$

Proof. By (2.38), (2.22) and (2.28). □

We further give

Theorem 3.14. *Let all as in Theorem* 3.12*. Here assume* $\omega \in L_1(\mathbb{R})$. *Then* $\forall$ $x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get*

$$\Delta(f)(x) := \left|f(x) - \int_\alpha^\beta f(y)\,\omega(y)\,dy\right| \leq$$

$$\left(\sum_{k=1}^{l-1}\frac{\left\|\overline{f}^{(k)}\right\|_{L_\infty(\alpha,\beta)}(\beta-\alpha)^k}{k!}\right)\|\omega\|_{L_1(\mathbb{R})}+A_4(x)=:B_6(x). \tag{3.20}$$

Proof. By (2.29) and (3.16). □

Theorem 3.15. *Let all be as in Theorem* 3.13. *Here assume* $\omega\in L_q(\alpha,\beta)$, $q>1$. *Then* $\forall\ x\in(\alpha,\beta)$ *(or almost every* $x\in(\alpha,\beta)$*, respectively), we get*

$$\Delta(f)(x)\leq\left(\sum_{k=1}^{l-1}\frac{(\beta-\alpha)^k}{k!}\left\|\overline{f}^{(k)}\right\|_{L_p(\alpha,\beta)}\right)\|\omega\|_{L_q(\alpha,\beta)}$$
$$+A_5(x)=:B_7(x). \tag{3.21}$$

Proof. By (2.30) and (3.18). □

Theorem 3.16. *Let all as in Theorem* 3.9 *or* $f\in\left(W_1^l\right)^{loc}(a,b)$ *and rest as in Theorem* 3.9. *Let* $p,q,r>1:\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$, $\overline{f}^{(k)}\in L_p(\alpha,\beta)$, $k=1,...,l-1$; $\omega\in L_q(\alpha,\beta)$. *Then* $\forall\ x\in(\alpha,\beta)$ *(or almost every* $x\in(\alpha,\beta)$*, respectively), we get*

$$\left|f(x)-\int_\alpha^\beta f(y)\,\omega(y)\,dy-R_{0,l}f(x)\right|\leq$$
$$\left(\sum_{k=1}^{l-1}\frac{\left\|\overline{f}^{(k)}\right\|_{L_p(\alpha,\beta)}}{k!}\left(\frac{(\beta-x)^{(kr+1)}+(x-\alpha)^{(kr+1)}}{(kr+1)}\right)^{\frac{1}{r}}\right)\|\omega\|_{L_q(\alpha,\beta)}=:\Phi(x). \tag{3.22}$$

Proof. By (2.31) and (2.38). □

We also present

Theorem 3.17. *Let* $\mathbb{N}\ni l>1$ *and* $\omega\in C^{(l-2)}(\mathbb{R})$, $\operatorname{supp}\omega\subset[\alpha,\beta]$, $\int_{\mathbb{R}}\omega(x)\,dx=1$, $\omega^{(l-2)}$ *is absolutely continuous on* $[a,b]$, $[\alpha,\beta]\subset(a,b)$; $a,b\in\mathbb{R}$. *Here* $f\in C^l([a,b])$ *or* $f\in\left(W_1^l\right)^{loc}(a,b)$. *For every* $x\in(\alpha,\beta)$ *(or almost every* $x\in(\alpha,\beta)$*, respectively), we get for*

$$\Delta(f)(x):=\left|f(x)-\int_\alpha^\beta f(y)\,\omega(y)\,dy\right|$$

that

i) It holds

$$\Delta(f)(x)\leq\left(\sum_{k=1}^{l-1}\frac{1}{k!}\left\|\left[(x-y)^k\omega(y)\right]_y^{(k)}\right\|_\infty\right)\|f\|_{L_1(\alpha,\beta)}+$$
$$\frac{\|\omega\|_{L_1(\mathbb{R})}\left\|\overline{f}^{(l)}\right\|_{L_1(\alpha,\beta)}(\beta-\alpha)^{l-1}}{(l-1)!}=:C_1(x). \tag{3.23}$$

ii) If $f, \overline{f}^{(l)} \in L_\infty(\alpha,\beta)$, *then*

$$\Delta(f)(x) \le \left(\sum_{k=1}^{l-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k)} \right\|_{L_1(\alpha,\beta)} \right) \|f\|_{L_\infty(\alpha,\beta)} +$$

$$\frac{\|\omega\|_{L_1(\mathbb{R})} \left\| \overline{f}^{(l)} \right\|_{L_\infty(\alpha,\beta)} \left((\beta-x)^l + (x-\alpha)^l \right)}{l!} =: C_2(x). \tag{3.24}$$

iii) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Assume further that* $f, \overline{f}^{(l)} \in L_p(\alpha,\beta)$. *Then*

$$\Delta(f)(x) \le \left(\sum_{k=1}^{l-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k)} \right\|_{L_q(\alpha,\beta)} \right) \|f\|_{L_p(\alpha,\beta)} +$$

$$\frac{\|\omega\|_{L_1(\mathbb{R})} \left\| \overline{f}^{(l)} \right\|_{L_p(\alpha,\beta)}}{(l-1)!} \left(\frac{(\beta-x)^{q(l-1)+1} + (x-\alpha)^{q(l-1)+1}}{q(l-1)+1} \right)^{\frac{1}{q}} =: C_3(x). \tag{3.25}$$

Proof. By (2.34) and Theorems 3.11, 3.12, 3.13. □

We finish Ostrowski type inequalities with

Theorem 3.18. *Let* $l, m \in \mathbb{N}$, $m < l$; $\omega \in C^{(l-2)}(\mathbb{R})$, $\operatorname{supp} \omega \subset [\alpha,\beta]$, $\int_{\mathbb{R}} \omega(x)\,dx = 1$, $\omega^{(l-2)}$ *is absolutely continuous on* $[a,b]$, $[\alpha,\beta] \subset (a,b)$; $a, b \in \mathbb{R}$. *Here* $f \in C^l([a,b])$ *or* $f \in \left(W_1^l\right)^{loc}(a,b)$. *For every* $x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get for*

$$E_\beta(f)(x) := \left| \overline{f}^{(m)}(x) - (-1)^m \int_\alpha^\beta f(y)\,\omega^{(m)}(y)\,dy - Q_m^{l-1} f(x) \right|, \tag{3.26}$$

and

$$\Delta_\beta(f)(x) := \left| \overline{f}^{(m)}(x) - (-1)^m \int_\alpha^\beta f(y)\,\omega^{(m)}(y)\,dy \right| \tag{3.27}$$

that

i) it holds

$$E_\beta(f)(x) \le \frac{\|\omega\|_{L_1(\mathbb{R})} \left\| \overline{f}^{(l)} \right\|_{L_1(\alpha,\beta)} (\beta-\alpha)^{l-m-1}}{(l-m-1)!} =: E_1, \tag{3.28}$$

and

$$\Delta_\beta(f)(x) \le \left(\sum_{k=1}^{l-m-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k+m)} \right\|_\infty \right) \|f\|_{L_1(\alpha,\beta)}$$

$$+E_1 =: G_1(x), \tag{3.29}$$

ii) if $\overline{f}^{(l)} \in L_{\infty}(\alpha,\beta)$, *then*

$$E_{\beta}(f)(x) \leq \frac{\|\omega\|_{L_1(\mathbb{R})} \left\|\overline{f}^{(l)}\right\|_{L_{\infty}(\alpha,\beta)}}{(l-m)!} \left((\beta-x)^{l-m} + (x-\alpha)^{l-m}\right) =: E_2(x), \tag{3.30}$$

if additionally we assume $f \in L_{\infty}(\alpha,\beta)$, *then*

$$\Delta_{\beta}(f)(x) \leq$$

$$\left(\sum_{k=1}^{l-m-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y)\right]_y^{(k)} \right\|_{L_1(\alpha,\beta)}\right) \|f\|_{L_{\infty}(\alpha,\beta)} + E_2(x) =: G_2(x), \tag{3.31}$$

iii) let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *assume further that* $\overline{f}^{(l)} \in L_p(\alpha,\beta)$, *then*

$$E_{\beta}(f)(x) \leq \frac{\|\omega\|_{L_1(\mathbb{R})} \left\|\overline{f}^{(l)}\right\|_{L_p(\alpha,\beta)}}{(l-m-1)!} \cdot$$

$$\left(\frac{(\beta-x)^{q(l-m-1)+1} + (x-\alpha)^{q(l-m-1)+1}}{q(l-m-1)+1}\right)^{\frac{1}{q}} =: E_3(x), \tag{3.32}$$

and if additionally $f \in L_p(\alpha,\beta)$, *then*

$$\Delta_{\beta}(f)(x) \leq \left(\sum_{k=1}^{l-m-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y)\right]_y^{(k+m)} \right\|_{L_q(\alpha,\beta)}\right) \|f\|_{L_p(\alpha,\beta)}$$

$$+E_3(x) =: G_3(x). \tag{3.33}$$

Proof. By (2.20), (2.21), (2.22), (2.37) and (2.39). □

We make

Remark 3.19. In preparation to present comparison of integral means inequalities we consider $(\alpha_1,\beta_1) \subseteq (\alpha,\beta)$. We consider also a weight function $\psi \geq 0$ which is Lebesgue integrable on $\mathbb{R}$ with $\sup p\psi \subset [\alpha_1,\beta_1] \subset [a,b]$, and $\int_{\mathbb{R}} \psi(x)\,dx = 1$. Clearly here $\int_{\alpha_1}^{\beta_1} \psi(x)\,dx = 1$.

E.g. for $x \in (\alpha_1,\beta_1)$, $\psi(x) := \frac{1}{\beta_1-\alpha_1}$, zero elsewhere, etc.

We will apply the following principle: In general a constraint of the form $|F(x) - G| \leq \varepsilon$, where F is a function and G, ε real numbers so that all make sense, implies that

$$\left|\int_{\mathbb{R}} F(x)\,\psi(x)\,dx - G\right| \leq \varepsilon. \tag{3.34}$$

Next we give a series of comparison of integral means inequalities based on Ostrowski type inequalities presented in this article. We use Remark 3.19.

Theorem 3.20. *All as in Theorem* 3.9. *Then*

$$u(f) := \left| \int_{\alpha_1}^{\beta_1} f(x)\psi(x)dx - \int_{\alpha}^{\beta} f(y)\omega(y)dy - \int_{\alpha_1}^{\beta_1} Q^{l-1} f(x)\psi(x)dx \right| \leq A_1, \tag{3.35}$$

and

$$m(f) := \left| \int_{\alpha_1}^{\beta_1} f(x)\psi(x)\,dx - \int_{\alpha}^{\beta} f(y)\,\omega(y)\,dy \right| \leq \left(\sum_{k=1}^{l-1} \frac{(\beta-\alpha)^{k+1}}{(k+1)!} \left\| f^{(k)} \right\|_{\infty} \right) \|\omega\|_{L_\infty(\mathbb{R})} + \frac{\|\omega\|_{L_1(\mathbb{R})} \left\| f^{(l)} \right\|_{L_1(\alpha,\beta)} (\beta-\alpha)^{l-1}}{(l-1)!}. \tag{3.36}$$

Proof. By Remark 3.19, Theorem 3.9, and the fact that the functions $(\beta - x)^{k+1} + (x-\alpha)^{k+1}$, $k = 1, ..., l-1$ are positive and convex with maximum $(\beta-\alpha)^{k+1}$. □

Theorem 3.21. *All as in Theorem* 3.10. *Then*

$$u(f) \leq \frac{\|\omega\|_{L_1(\mathbb{R})} \left\| f^{(l)} \right\|_{\infty} (\beta-\alpha)^{l}}{l!}, \tag{3.37}$$

and

$$m(f) \leq \left(\sum_{k=1}^{l-1} \frac{(\beta-\alpha)^{k+1}}{(k+1)!} \left\| f^{(k)} \right\|_{\infty} \right) \|\omega\|_{L_\infty(\mathbb{R})} + \frac{\|\omega\|_{L_1(\mathbb{R})} \left\| f^{(l)} \right\|_{\infty} (\beta-\alpha)^{l}}{l!}. \tag{3.38}$$

Proof. Just maximize $A_2(x)$ of (3.12) and $B_2(x)$ of (3.13), etc. □

Theorem 3.22. *All as in Theorem* 3.11. *Then*

$$u(f) \leq A_3, \tag{3.39}$$

and

$$m(f) \leq B_3. \tag{3.40}$$

Theorem 3.23. *All as in Theorem* 3.12. *Then*

$$u(f) \leq \frac{\|\omega\|_{L_1(\mathbb{R})} \left\| \overline{f}^{(l)} \right\|_{L_\infty(\alpha,\beta)} (\beta-\alpha)^{l}}{l!}, \tag{3.41}$$

and

$$m(f) \leq \left(\sum_{k=1}^{l-1} \frac{(\beta-\alpha)^{k+1}}{(k+1)!} \left\| \overline{f}^{(k)} \right\|_{L_\infty(\alpha,\beta)} \right) \|\omega\|_{L_\infty(\mathbb{R})} + \frac{\|\omega\|_{L_1(\mathbb{R})} \left\| \overline{f}^{(l)} \right\|_{L_\infty(\alpha,\beta)} (\beta-\alpha)^{l}}{l!}. \tag{3.42}$$

Theorem 3.24. *All as in Theorem* 3.13. *Then*

$$u(f) \leq \int_{\alpha_1}^{\beta_1} A_5(x)\,\psi(x)\,dx, \tag{3.43}$$

and

$$m(f) \leq \int_{\alpha_1}^{\beta_1} B_5(x)\,\psi(x)\,dx. \tag{3.44}$$

Proof. By the principle: if $|F(x) - G| \leq \varepsilon(x)$, then $\left|\int F(x)\,\psi(x)\,dx - G\right| \leq \int \varepsilon(x)\,\psi(x)\,dx$, etc. Here $A_5(x)$ as in (3.18) and $B_5(x)$ as in (3.19). □

Theorem 3.25. *All as in Theorem* 3.14. *Then*

$$m(f) \leq \int_{\alpha_1}^{\beta_1} B_6(x)\,\psi(x)\,dx, \tag{3.45}$$

where $B_6(x)$ *as in (*3.20*).*

Theorem 3.26. *All as in Theorem* 3.15. *Then*

$$m(f) \leq \int_{\alpha_1}^{\beta_1} B_7(x)\,\psi(x)\,dx, \tag{3.46}$$

where $B_7(x)$ *as in (*3.21*).*

Theorem 3.27. *All as in Theorem* 3.16. *Then*

$$\left|\int_{\alpha_1}^{\beta_1} f(x)\,\psi(x)\,dx - \int_{\alpha}^{\beta} f(y)\,\omega(y)\,dy - \int_{\alpha_1}^{\beta_1} (R_{0,l}f(x))\,\psi(x)\,dx\right| \leq$$

$$\int_{\alpha_1}^{\beta_1} \Phi(x)\,\psi(x)\,dx, \tag{3.47}$$

where $\Phi(x)$ *as in (*3.22*).*

We continue with

Theorem 3.28. *All as in Theorem* 3.17. *Then*

i)

$$m(f) \leq \left(\sum_{k=1}^{l-1} \frac{1}{k!} \sup_{x\in[\alpha_1,\beta_1]} \left\| \left[(x-y)^k\,\omega(y)\right]_y^{(k)} \right\|_\infty \right) \|f\|_{L_1(\alpha,\beta)}$$
$$+ \frac{\|\omega\|_{L_1(\mathbb{R})} \left\|\overline{f}^{(l)}\right\|_{L_1(\alpha,\beta)} (\beta-\alpha)^{l-1}}{(l-1)!}, \tag{3.48}$$

ii) if $f, \overline{f}^{(l)} \in L_\infty(\alpha,\beta)$, *then*

$$m(f) \leq \int_{\alpha_1}^{\beta_1} C_2(x)\,\psi(x)\,dx, \tag{3.49}$$

iii) let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; *assume further* $f, \overline{f}^{(l)} \in L_p(\alpha,\beta)$, *then*

$$m(f) \leq \int_{\alpha_1}^{\beta_1} C_3(x)\,\psi(x)\,dx. \tag{3.50}$$

Here $C_2(x)$ *as in (*3.24*) and* $C_3(x)$ *as in (*3.25*).*

We finish the results about comparison of integral means with

Theorem 3.29. *All as in Theorem* 3.18*. Denote by*

$$u_m(f) :=$$
$$\left| \int_{\alpha_1}^{\beta_1} \overline{f}^{(m)}(x)\psi(x)dx - (-1)^m \int_{\alpha}^{\beta} f(y)\omega^{(m)}(y)dy - \int_{\alpha_1}^{\beta_1} \left(Q_m^{l-1} f(x) \right) \psi(x)dx \right|, \tag{3.51}$$

and

$$\rho_m(f) := \left| \int_{\alpha_1}^{\beta_1} \overline{f}^{(m)}(x)\psi(x)dx - (-1)^m \int_{\alpha}^{\beta} f(y)\omega^{(m)}(y)dy \right|. \tag{3.52}$$

i) it holds

$$u_m(f) \leq E_1, \tag{3.53}$$

where E_1 *as in (*3.28*),*

and

$$\rho_m(f) \leq \left(\sum_{k=1}^{l-m-1} \frac{1}{k!} \sup_{x\in[\alpha_1,\beta_1]} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k+m)} \right\|_{\infty} \right) \|f\|_{L_1(\alpha,\beta)} + E_1, \tag{3.54}$$

ii) if $\overline{f}^{(l)} \in L_\infty(\alpha,\beta)$*, then*

$$u_m(f) \leq \frac{\|\omega\|_{L_1(\mathbb{R})} \left\| \overline{f}^{(l)} \right\|_{L_\infty(\alpha,\beta)}}{(l-m)!} (\beta-\alpha)^{l-m}, \tag{3.55}$$

and if additionally assume $f \in L_\infty(\alpha,\beta)$*, then*

$$\rho_m(f) \leq \left(\sum_{k=1}^{l-m-1} \frac{1}{k!} \sup_{x\in[\alpha_1,\beta_1]} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k+m)} \right\|_{L_1(\alpha,\beta)} \right) \|f\|_{L_\infty(\alpha,\beta)}$$
$$+ \frac{\|\omega\|_{L_1(\mathbb{R})} \left\| \overline{f}^{(l)} \right\|_{L_\infty(\alpha,\beta)} (\beta-\alpha)^{l-m}}{(l-m)!}, \tag{3.56}$$

iii) let $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*, assume further* $\overline{f}^{(l)} \in L_p(\alpha,\beta)$*, then*

$$u_m(f) \leq \int_{\alpha_1}^{\beta_1} E_3(x)\psi(x)dx, \tag{3.57}$$

where $E_3(x)$ *as in (*3.32*),*

and if additionally $f \in L_p(\alpha,\beta)$*, then*

$$\rho_m(f) \leq \int_{\alpha_1}^{\beta_1} G_3(x)\psi(x)dx, \tag{3.58}$$

where $G_3(x)$ *as in (*3.33*).*

We need

Background 3.30. Let f be a convex function from $(0, +\infty)$ into $\mathbb{R}$ which is strictly convex at 1 with $f(1) = 0$. Let (X, A, λ) be a measure space, where λ is a finite or a σ-finite measure on (X, A). And let μ_1, μ_2 be two probability measures on (X, A) such that $\mu_1 \ll \lambda$, $\mu_2 \ll \lambda$ (absolutely continuous), e.g. $\lambda = \mu_1 + \mu_2$.

Denote by $p = \frac{d\mu_1}{d\lambda}$, $q = \frac{d\mu_2}{d\lambda}$ the (densities) Radon-Nikodym derivatives of μ_1, μ_2 with respect to λ. Here we suppose that

$$0 < \alpha \leq \frac{p}{q} \leq \beta, \text{ a.e. on } X \text{ and } \alpha \leq 1 \leq \beta.$$

The quantity

$$\Gamma_f(\mu_1, \mu_2) = \int_X q(x) f\left(\frac{p(x)}{q(x)}\right) d\lambda(x), \tag{3.59}$$

was introduced by I. Csiszar in 1967, see [8], and is called f-divergence of the probability measures μ_1 and μ_2. By Lemma 1.1 of [8], the integral (3.59) is well-defined and $\Gamma_f(\mu_1, \mu_2) \geq 0$ with equality only when $\mu_1 = \mu_2$. Furthermore $\Gamma_f(\mu_1, \mu_2)$ does not depend on the choice of λ. The concept of f-divergence was introduced first in [7] as a generalization of Kullback's "information for discrimination" or I-divergence (generalized entropy) [12], [11] and of Rényi's "information gain" (I-divergence of order δ) [14]. In fact the I-divergence of order 1 equals $\Gamma_{u \log_2 u}(\mu_1, \mu_2)$. The choice $f(x) = (u-1)^2$ produces again a known measure of difference of distributions that is called χ^2-divergence. Of course the total variation distance $|\mu_1 - \mu_2| = \int_X |p(x) - q(x)| \, d\lambda(x)$ is equal to $\Gamma_{|u-1|}(\mu_1, \mu_2)$.

Here by supposing $f(1) = 0$ we can consider $\Gamma_f(\mu_1, \mu_2)$, the f-divergence as a measure of the difference between the probability measures μ_1, μ_2. The f-divergence is in general asymmetric in μ_1 and μ_2. But since f is convex and strictly convex at 1 so is

$$f^*(u) = uf\left(\frac{1}{u}\right) \tag{3.60}$$

and as in [8] we obtain

$$\Gamma_f(\mu_2, \mu_1) = \Gamma_{f^*}(\mu_1, \mu_2). \tag{3.61}$$

In Information Theory and Statistics many other divergences are used which are special cases of the above general Csiszar f-divergence, e.g. Hellinger distance D_H, α-distance D_α, Bhattacharyya distance D_B, Harmonic distance D_{Ha}, Jeffrey's distance D_J, triangular discrimination D_Δ, for all these see, e.g. [4], [9]. The problem of finding and estimating the proper distance (or difference or discrimination) of two probability distributions is one of the major ones in Probability Theory.

Here we provide a general probabilistic representation formula for $\Gamma_f(\mu_1, \mu_2)$. Then we present tight estimates for the remainder involving a variety of norms of the engaged functions. Also are implied some direct general approximations for the Csiszar's f-divergence. We give some applications.

We make

Remark 3.31. Here $0 < a < \alpha \leq \frac{p(x)}{q(x)} \leq \beta < b < +\infty$, a e. on X and $\alpha \leq 1 \leq \beta$. Also assume that $f^{(l-1)}$ exists and is absolutely continuous on $[a,b]$, $l \in \mathbb{N}$. Furthermore f is convex from $(0,+\infty)$ into $\mathbb{R}$, strictly convex at 1 with $f(1) = 0$. Let $\omega \in L_1(\mathbb{R})$, $\sup p\omega \subset [\alpha,\beta]$, $\int_{\mathbb{R}} \omega(x)\,dx = 1$.

Then $\forall\ x \in (\alpha,\beta)$ we get by Theorem 2.7, as in (2.38), that

$$f(x) = \int_\alpha^\beta f(y)\,\omega(y)\,dy + Q^{l-1}f(x) + R_{0,l}f(x).$$

Therefore

$$f\left(\frac{p(x)}{q(x)}\right) = \int_\alpha^\beta f(y)\,\omega(y)\,dy + Q^{l-1}f\left(\frac{p(x)}{q(x)}\right) + R_{0,l}f\left(\frac{p(x)}{q(x)}\right),$$

a.e. on X.

Hence

$$q(x)\,f\left(\frac{p(x)}{q(x)}\right) =$$

$$q(x)\int_\alpha^\beta f(y)\,\omega(y)\,dy + q(x)\,Q^{l-1}f\left(\frac{p(x)}{q(x)}\right) + q(x)\,R_{0,l}f\left(\frac{p(x)}{q(x)}\right),$$

a.e. on X.

Therefore we get the representation of f-divergence of μ_1 and μ_2,

$$\Gamma_f(\mu_1,\mu_2) = \int_X q(x)\,f\left(\frac{p(x)}{q(x)}\right)d\lambda(x)$$

$$= \int_\alpha^\beta f(y)\,\omega(y)\,dy + \int_X q(x)\,Q^{l-1}f\left(\frac{p(x)}{q(x)}\right)d\lambda(x)$$

$$+ \int_X q(x)\,R_{0,l}f\left(\frac{p(x)}{q(x)}\right)d\lambda(x). \tag{3.62}$$

Call

$$Q_\Gamma := \int_X q(x)\,Q^{l-1}f\left(\frac{p(x)}{q(x)}\right)d\lambda(x), \tag{3.63}$$

and

$$R_\Gamma := \int_X q(x)\,R_{0,l}f\left(\frac{p(x)}{q(x)}\right)d\lambda(x). \tag{3.64}$$

We estimate Q_Γ and R_Γ.

If $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$, we get by (2.26) that

$$|Q_\Gamma| \leq \left(\sum_{k=1}^{l-1} \frac{(\beta-\alpha)^k}{k!}\left\|f^{(k)}\right\|_{L_1(\alpha,\beta)}\right)\|\omega\|_{L_\infty(\mathbb{R})}. \tag{3.65}$$

Notice if $l = 1$, then always $Q_\Gamma = 0$.

Next if again $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$, then (by (2.27))

$$|Q_\Gamma| \leq \left(\int_X q(x)\left(\sum_{k=1}^{l-1} \frac{\left(\beta - \frac{p(x)}{q(x)}\right)^{k+1} + \left(\frac{p(x)}{q(x)} - \alpha\right)^{k+1}}{(k+1)!}\left\|f^{(k)}\right\|_{L_\infty(\alpha,\beta)}\right)d\lambda(x)\right)$$

$$\cdot \|\omega\|_{L_\infty(\mathbb{R})} \cdot \tag{3.66}$$

Let now $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ and again $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$. Then (by (2.28))

$$|Q_\Gamma| \leq \left(\int_X q(x) \left(\sum_{k=1}^{l-1} \left(\frac{\left(\beta - \frac{p(x)}{q(x)}\right)^{(kq+1)} + \left(\frac{p(x)}{q(x)} - \alpha\right)^{(kq+1)}}{kq+1} \right)^{\frac{1}{q}} \cdot \right.\right.$$

$$\left.\left. \frac{\left\|f^{(k)}\right\|_{L_p(\alpha,\beta)}}{k!} \right) d\lambda(x) \right) \|\omega\|_{L_\infty(\mathbb{R})} \cdot \tag{3.67}$$

Next assume $\omega \in L_1(\mathbb{R})$, then (by (2.29))

$$|Q_\Gamma| \leq \left(\sum_{k=1}^{l-1} \frac{\left\|f^{(k)}\right\|_{L_\infty(\alpha,\beta)} (\beta-\alpha)^k}{k!} \right) \|\omega\|_{L_1(\mathbb{R})} \cdot \tag{3.68}$$

If $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ and $\omega \in L_q(\alpha, \beta)$, then (by (2.30))

$$|Q_\Gamma| \leq \left(\sum_{k=1}^{l-1} \frac{(\beta-\alpha)^k}{k!} \left\|f^{(k)}\right\|_{L_p(\alpha,\beta)} \right) \|\omega\|_{L_q(\alpha,\beta)} \cdot \tag{3.69}$$

Assume $p, q, r > 1 : \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ and $\omega \in L_q(\alpha, \beta)$, then (by (2.31))

$$|Q_\Gamma| \leq \left(\int_X q(x) \left(\sum_{k=1}^{l-1} \frac{\left\|f^{(k)}\right\|_{L_p(\alpha,\beta)}}{k!} \cdot \right.\right.$$

$$\left.\left. \left(\frac{\left(\beta - \frac{p(x)}{q(x)}\right)^{(kr+1)} + \left(\frac{p(x)}{q(x)} - \alpha\right)^{(kr+1)}}{kr+1} \right)^{\frac{1}{r}} \right) d\lambda(x) \right) \|\omega\|_{L_q(\alpha,\beta)} \cdot \tag{3.70}$$

We make

Remark 3.32. (continuation of Remark 3.31) Here $l > 1$, $\omega \in C^{(l-2)}(\mathbb{R})$, $\sup p\omega \subset [\alpha, \beta]$, $\int_{\mathbb{R}} \omega(x)\, dx = 1$, and $\omega^{(l-2)}$ is absolutely continuous on $[a, b]$. Then (by (2.32))

$$Q_\Gamma = \int_X q(x) \left(\sum_{k=1}^{l-1} \frac{(-1)^k}{k!} \int_\alpha^\beta \left(\left[\left(\frac{p(x)}{q(x)} - y \right)^k \omega(y) \right]_y^{(k)} \right) f(y)\, dy \right) d\lambda(x). \tag{3.71}$$

Hence by (2.34) we obtain

$$|Q_\Gamma| \leq \min \text{ of}$$

$$\begin{cases} \left(\int_X q(x)\left(\sum_{k=1}^{l-1}\frac{1}{k!}\left\|\left[\left(\frac{p(x)}{q(x)}-y\right)^k\omega(y)\right]_y^{(k)}\right\|_{\infty}\right)d\lambda(x)\right)\|f\|_{L_1(\alpha,\beta)}, \\ \left(\int_X q(x)\left(\sum_{k=1}^{l-1}\frac{1}{k!}\left\|\left[\left(\frac{p(x)}{q(x)}-y\right)^k\omega(y)\right]_y^{(k)}\right\|_{L_1(\alpha,\beta)}\right)d\lambda(x)\right)\|f\|_{L_\infty(\alpha,\beta)}, \\ \text{when } p,q>1: \frac{1}{p}+\frac{1}{q}=1, \text{ we have} \\ \left(\int_X q(x)\left(\sum_{k=1}^{l-1}\frac{1}{k!}\left\|\left[\left(\frac{p(x)}{q(x)}-y\right)^k\omega(y)\right]_y^{(k)}\right\|_{L_q(\alpha,\beta)}\right)d\lambda(x)\right)\|f\|_{L_p(\alpha,\beta)}. \end{cases} \tag{3.72}$$

We also make

Remark 3.33. (another continuation of Remark 3.31) Here we estimate the remainder R_Γ of (3.62). By (2.20), (3.64) we obtain

$$|R_\Gamma| \leq \frac{\|\omega\|_{L_1(a,b)}\left\|f^{(l)}\right\|_{L_1(\alpha,\beta)}(\beta-\alpha)^{l-1}}{(l-1)!}. \tag{3.73}$$

If $f^{(l)} \in L_\infty(\alpha,\beta)$, then (by (2.21)) we obtain

$$|R_\Gamma| \leq \frac{\|\omega\|_{L_1(a,b)}\left\|f^{(l)}\right\|_{L_\infty(\alpha,\beta)}}{l!}\cdot$$

$$\left(\int_X q(x)\left(\left(\beta-\frac{p(x)}{q(x)}\right)^l+\left(\frac{p(x)}{q(x)}-\alpha\right)^l\right)d\lambda(x)\right). \tag{3.74}$$

Let now $p,q>1: \frac{1}{p}+\frac{1}{q}=1$. Here $f^{(l)} \in L_p(\alpha,\beta)$, then (by (2.22)) we get

$$|R_\Gamma| \leq \frac{\|\omega\|_{L_1(a,b)}\left\|f^{(l)}\right\|_{L_p(\alpha,\beta)}}{(q(l-1)+1)^{\frac{1}{q}}(l-1)!}\cdot$$

$$\left(\int_X q(x)\left(\left(\beta-\frac{p(x)}{q(x)}\right)^{(q(l-1)+1)}+\left(\frac{p(x)}{q(x)}-\alpha\right)^{(q(l-1)+1)}\right)^{\frac{1}{q}}d\lambda(x)\right). \tag{3.75}$$

Finally we see that

$$\Gamma_f(\mu_1,\mu_2)-\int_\alpha^\beta f(y)\,\omega(y)\,dy = Q_\Gamma + R_\Gamma, \tag{3.76}$$

and

$$T := \left|\Gamma_f(\mu_1,\mu_2)-\int_\alpha^\beta f(y)\,\omega(y)\,dy\right| \leq |Q_\Gamma|+|R_\Gamma|. \tag{3.77}$$

Then one by the above estimates of $|Q_\Gamma|$ and $|R_\Gamma|$ can estimate T, in a number of cases.

4. Applications

Example 4.1. Let $V := \{x \in \mathbb{R} : |x - x_0| < \rho\}$, $x_0 \in \mathbb{R}$, and

$$\varphi(x) := \begin{cases} e^{-\left(1-\frac{(x-x_0)^2}{\rho^2}\right)^{-1}}, & \text{if } |x - x_0| < \rho, \\ 0, & \text{if } |x - x_0| \geq \rho. \end{cases} \tag{4.1}$$

Call $c := \int_{\mathbb{R}} \varphi(x)\, dx > 0$, then $\Phi(x) := \frac{1}{c}\varphi(x) \in C_0^\infty(\mathbb{R})$ (space of continuously infinitely many times differentiable functions of compact support) with $\sup p\Phi = \overline{V}$ and $\int_{-\infty}^{\infty} \Phi(x)\, dx = 1$ and $\max|\Phi| \leq cons\tan t \cdot \rho^{-1}$. We call Φ a cut-off function.

One for this article's results by choosing $\omega(x) = \Phi(x)$ or $\omega(x) = \frac{1}{2\rho}$, etc., can give lots of applications. Due to lack of space we avoid it.

Instead, selectively, we give some special cases inequalities. We start with Chebyshev-Grüss type inequalities.

Corollary 4.2. *(to Theorem 3.3) Let $f, g \in C^1([a,b])$, $[a,b] \subset \mathbb{R}$, $(\alpha,\beta) \subset (a,b)$. Let also $\omega \in L_\infty(\mathbb{R})$, $\sup p\omega \subset [\alpha,\beta]$, $\int_{\mathbb{R}} \omega(x)\, dx = 1$. Then*

$$\left| \int_\alpha^\beta \omega(x) f(x) g(x)\, dx - \left(\int_\alpha^\beta \omega(x) f(x)\, dx\right)\left(\int_\alpha^\beta \omega(x) g(x)\, dx\right) \right| \leq$$

$$\|\omega\|_{L_1(\mathbb{R})} \|\omega\|_{\infty,(\alpha,\beta)} \frac{(\beta-\alpha)^2}{2} (\|g\|_{\infty,(\alpha,\beta)} \|f'\|_{\infty,(\alpha,\beta)} + \|f\|_{\infty,(\alpha,\beta} \|g'\|_{\infty,(\alpha,\beta)}). \tag{4.2}$$

If $f = g$, then

$$\left| \int_\alpha^\beta \omega(x) f^2(x)\, dx - \left(\int_\alpha^\beta \omega(x) f(x)\, dx\right)^2 \right| \leq$$

$$\|\omega\|_{L_1(\mathbb{R})} \|\omega\|_{\infty,(\alpha,\beta)} (\beta-\alpha)^2 \|f\|_{\infty,(\alpha,\beta)} \|f'\|_{\infty,(\alpha,\beta)}. \tag{4.3}$$

Corollary 4.3. *(to Theorem 3.4) Let $f \in \left(W_1^1\right)^{loc}(a,b)$; $a, b \in \mathbb{R}$; $(\alpha,\beta) \subset (a,b)$, $\omega(x) := \frac{1}{\beta-\alpha}$ for $x \in [\alpha,\beta]$, and zero elsewhere. Then*

$$\left| \frac{1}{\beta-\alpha} \int_\alpha^\beta f^2(x)\, dx - \frac{1}{(\beta-\alpha)^2} \left(\int_\alpha^\beta f(x)\, dx\right)^2 \right| \leq$$

$$\frac{\|f\|_{L_1(\alpha,\beta)} \left\| f_w^{(1)} \right\|_{L_1(\alpha,\beta)}}{(\beta-\alpha)}. \tag{4.4}$$

We continue with an Ostrowski type inequality.

Corollary 4.4. *(to Theorem 3.11) All as in Theorem 3.11. Case of $l = 1$. Then, for any $x \in (\alpha,\beta)$ (or for almost every $x \in (\alpha,\beta)$, respectively), we get*

$$\left| f(x) - \int_\alpha^\beta f(y)\, \omega(y)\, dy \right| \leq \|\omega\|_{L_1(\mathbb{R})} \left\| \overline{f}' \right\|_{L_1(\alpha,\beta)}. \tag{4.5}$$

Next comes a comparison of means inequality.

Corollary 4.5. *All here as in Corollary* 4.4 *and Remark* 3.19. *Then*

$$\left|\int_{\alpha_1}^{\beta_1} f(x)\,\psi(x)\,dx - \int_{\alpha}^{\beta} f(y)\,\omega(y)\,dy\right| \leq \|\omega\|_{L_1(\mathbb{R})}\left\|\overline{f}'\right\|_{L_1(\alpha,\beta)}. \tag{4.6}$$

Proof. By (4.5). □

We finish with an application of f-divergence.

Remark 4.6. All here as in Background 3.30 and Remark 3.31. Case of $l=1$. By (3.62) we get

$$\Gamma_f(\mu_1,\mu_2) = \int_{\alpha}^{\beta} f(y)\,\omega(y)\,dy + \int_X q(x)\,R_{0,1}f\left(\frac{p(x)}{q(x)}\right)d\lambda(x). \tag{4.7}$$

That is here

$$R_\Gamma = \int_X q(x)\,R_{0,1}f\left(\frac{p(x)}{q(x)}\right)d\lambda(x). \tag{4.8}$$

By (3.73) here we get that

$$|R_\Gamma| \leq \|\omega\|_{L_1(a,b)}\|f'\|_{L_1(\alpha,\beta)}. \tag{4.9}$$

If $f' \in L_\infty(\alpha,\beta)$, then here we get

$$|R_\Gamma| \leq \|\omega\|_{L_1(a,b)}\|f'\|_{L_\infty(\alpha,\beta)}(\beta-\alpha). \tag{4.10}$$

Let now $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$ and assume $f' \in L_p(\alpha,\beta)$, then here we obtain

$$|R_\Gamma| \leq \|\omega\|_{L_1(a,b)}\|f'\|_{L_p(\alpha,\beta)}(\beta-\alpha)^{\frac{1}{q}}. \tag{4.11}$$

Notice also here that

$$K := \Gamma_f(\mu_1,\mu_2) - \int_{\alpha}^{\beta} f(y)\,\omega(y)\,dy = R_\Gamma, \tag{4.12}$$

($l=1$ case).

So the estimates (4.9), (4.10) and (4.11) are also estimates for K.

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Coefficient bounds for certain classes of multivalent functions

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Abstract. In this paper, sharp upper bounds for $\left|a_{p+2} - \eta a_{p+1}^2\right|$ and $|a_{p+3}|$ are derived for a class of Mocanu α-convex p-valent functions defined by an extended linear multiplier differential operator (LMDO) $\mathcal{J}_p^{\delta}(\lambda, \mu, l)$.

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Keywords: Analytic functions, starlike functions, convex functions, p-valent functions, subordination, convolution (or Hadamard product).

1. Introduction

Let $\mathcal{A}_p$ be the class of all functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} := \{z : |z| < 1\}$ and let $\mathcal{A} : = \mathcal{A}_1$. For $f(z)$ given by (1.1) and $g(z)$ given by $g(z) = z^p + \sum\limits_{n=p+1}^{\infty} b_n z^n$, their convolution (or Hadamard product), denoted by $f * g$, is defined by

$$(f * g)(z) : = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

The function $f(z)$ is subordinate to the function $g(z)$, written $f(z) \prec g(z)$, provided there exists analytic function $w(z)$ defined on $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Let φ be an analytic function with positive real part in the unit disk $\mathcal{U}$ with $\varphi(0) = 1$ and $\varphi'(0) > 0$ that maps $\mathcal{U}$ onto a region which is starlike with respect to 1 and symmetric with respect to the real axis. R. M. Ali *et al.* [1] defined and studied the class $\mathcal{S}^*_{b,p}(\varphi)$ consisting of functions in $f \in \mathcal{A}_p$ for which

$$1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right) \prec \varphi(z) \qquad (z \in \mathcal{U}, \quad b \in \mathbb{C}\setminus\{0\}), \tag{1.2}$$

and the class $\mathcal{C}_{b,p}(\varphi)$ of all functions in $f \in \mathcal{A}_p$ for which

$$1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \qquad (z \in \mathcal{U}, \quad b \in \mathbb{C}\setminus\{0\}). \tag{1.3}$$

R. M. Ali *et al.* [1] also defined and studied the class $\mathcal{R}_{b,p}(\varphi)$ to be the class of all functions in $f \in \mathcal{A}_p$ for which

$$1 + \frac{1}{b}\left(\frac{f'(z)}{pz^{p-1}} - 1\right) \prec \varphi(z) \qquad (z \in \mathcal{U}, \quad b \in \mathbb{C}\setminus\{0\}). \tag{1.4}$$

Note that $\mathcal{S}^*_{1,1}(\varphi) = \mathcal{S}^*(\varphi)$ and $\mathcal{C}_{1,1}(\varphi) = \mathcal{C}(\varphi)$, the classes introduced and studied by Ma and Minda [8]. The familiar class $\mathcal{S}^*(\alpha)$ of starlike functions of order α and the class $\mathcal{C}(\alpha)$ of convex functions of order α, $0 \leq \alpha < 1$ are the special case of $\mathcal{S}^*_{1,1}(\varphi)$ and $\mathcal{C}_{1,1}(\varphi)$, respectively, when $\varphi(z) = (1 + (1 - 2\alpha)z)/(1 - z)$.

Owa [9] introduced and studied the class $\mathcal{H}_p(A, B, \alpha, \beta)$ of all functions in $f \in \mathcal{A}_p$ satisfying

$$(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \beta\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha} \prec \frac{1 + Az}{1 + Bz} \tag{1.5}$$

where $z \in \mathcal{U}$, $-1 \leq B < A \leq 1$, $0 \leq \beta \leq 1$, $\alpha \geq 0$.

We note that $\mathcal{H}_1(A, B, \alpha, \beta)$ is a subclass of Bazilevic functions [4].

A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{R}_{(b,p,\alpha,\beta)}(\varphi)$ if

$$1 + \frac{1}{b}\left\{(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \beta\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha} - 1\right\} \prec \varphi(z) \tag{1.6}$$

$(0 \leq \beta \leq 1, \alpha \geq 0)$. The class $\mathcal{R}_{(b,p,\alpha,\beta)}(\varphi)$ was defined and studied by Ramachandran *et al.* [12].

A class of functions which unifies the classes $\mathcal{S}^*_{b,p}(\varphi)$ and $\mathcal{C}_{b,p}(\varphi)$ was introduced by T. N. Shanmugam, S. Owa, C. Ramachandran, S. Sivasubramanian and Y. Nakamura in [14]. They defined this class in the following way.

Let $\varphi(z)$ be a univalent starlike function with respect to 1 which maps the open unit disk $\mathcal{U}$ onto a region in the right half plane and is symmetric with respect to real axis, $\varphi(0) = 1$ and $\varphi'(0) > 0$. A function $f \in \mathcal{A}_p$ is in the class $\mathcal{M}_{(b,p,\alpha,\lambda)}(\varphi)$ if

$$1 + \frac{1}{b}\left[\frac{1}{p}\left((1-\alpha)\frac{zF'(z)}{F(z)} + \alpha\left(1 + \frac{zF''(z)}{F'(z)}\right)\right) - 1\right] \prec \varphi(z) \tag{1.7}$$

$(0 \leq \alpha \leq 1)$, where

$$F(z) := (1-\lambda)f(z) + \lambda zf'(z).$$

T. N. Shangumugam *et al.* [14] obtained certain coefficient inequalities for function $f \in \mathcal{A}_p$ which are in the class $\mathcal{M}_{(b,p,\alpha,\lambda)}(\varphi)$.

For a function f in $\mathcal{A}_p$, the *linear multiplier differential operator (LMDO)* $\mathcal{J}_p^\delta(\lambda,\mu,l)f : \mathcal{A}_p \to \mathcal{A}_p$ was defined by the authors in [5] in the following way.

Definition 1.1. *Let $f \in \mathcal{A}_p$. For the parameters $\delta, \lambda, \mu, l \in \mathbb{R}$; $\lambda \geq \mu \geq 0$ and $\delta, l \geq 0$ the LMDO $\mathcal{J}_p^\delta(\lambda,\mu,l)$ on $\mathcal{A}_p$ is defined by*

$$\mathcal{J}_p^0(\lambda,\mu,l)f(z) = f(z) \tag{1.8}$$

$$\begin{aligned}&(p+l)\mathcal{J}_p^1(\lambda,\mu,l)f(z)\\ &= \lambda\mu z^2 f''(z) + (\lambda-\mu+(1-p)\lambda\mu)\, zf'(z) + (p(1-\lambda+\mu)+l)\, f(z)\end{aligned}$$

$$\begin{aligned}&(p+l)\mathcal{J}_p^2(\lambda,\mu,l)f(z)\\ &= \lambda\mu z^2[\mathcal{J}_p^1(\lambda,\mu,l)f(z)]'' + (\lambda-\mu+(1-p)\lambda\mu)\, z[\mathcal{J}_p^1(\lambda,\mu,l)f(z)]'\\ &\quad + (p(1-\lambda+\mu)+l)\, \mathcal{J}_p^1(\lambda,\mu,l)f(z)\end{aligned}$$

$$\mathcal{J}_p^{\delta_1}(\lambda,\mu,l)(\mathcal{J}_p^{\delta_2}(\lambda,\mu,l)f(z)) = \mathcal{J}_p^{\delta_2}(\lambda,\mu,l)(\mathcal{J}_p^{\delta_1}(\lambda,\mu,l)f(z)), \quad \delta_1,\delta_2 \geq 0$$

for $z \in \mathcal{U}$ and $p \in \mathbb{N} := \{1,2,...\}$.

If f is given by (1.1) then from the definition of the *LMDO* $\mathcal{J}_p^\delta(\lambda,\mu,l)$, we can easily see that

$$\mathcal{J}_p^\delta(\lambda,\mu,l)f(z) = z^p + \sum_{k=p+1}^{\infty} \Phi_p^k(\delta,\lambda,\mu,l)a_k z^k$$

where

$$\Phi_p^k(\delta,\lambda,\mu,l) = \left[\frac{(k-p)(\lambda\mu k+\lambda-\mu)+p+l}{p+l}\right]^\delta.$$

When $p=1$, $l=0$ and $\delta = m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ we get Deniz-Orhan [6] (Also for earlier $0 \leq \mu \leq \lambda \leq 1$ Raducanu-Orhan [11]) differential operator, when $p=1$, $l=0=\mu$ and $\delta = m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ we obtain the differential operator defined by Al-Oboudi [2] and when $p=1$, $l=0=\mu$, $\lambda=1$ and $\delta = m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ we obtain the differential operator defined by Sălăgean [13]. We note that by specializing the parameters δ,λ,μ,l and p, the *LMDO* $\mathcal{J}_p^\delta(\lambda,\mu,l)$ reduces to other several well-known operators of analytic functions. Detailed information can be found in [5].

Now, by making use of the operator $\mathcal{J}_p^\delta(\lambda,\mu,l)$, we define a new subclass of functions belonging to the class $\mathcal{A}_p$.

Definition 1.2. *Let $\varphi(z)$ be a univalent starlike function with respect to 1 which maps the open unit disk $\mathcal{U}$ onto a region in the right half plane and is symmetric with respect to real axis, $\varphi(0)=1$ and $\varphi'(0)>0$. A function $f \in \mathcal{A}_p$ is in the class $\mathcal{M}^\delta_{(b,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$ if*

$$1+\frac{1}{b}\left[\frac{1}{p}\left((1-\alpha)\frac{z(F_{\nu,\delta}(z))'}{F_{\nu,\delta}(z)}+\alpha\left(1+\frac{z(F_{\nu,\delta}(z))''}{(F_{\nu,\delta}(z))'}\right)\right)-1\right] \prec \varphi(z) \tag{1.9}$$

where $0 \leq \alpha \leq 1$; $\delta, \lambda, \mu, l \in \mathbb{R}$; $\delta, l \geq 0$; $0 \leq \mu \leq \lambda$; $p \in \mathbb{N}$, *and*

$$F_{\nu,\delta}(z) = (1-\nu) J_p^{\delta}(\lambda,\mu,l) f(z) + \nu J_p^{\delta+1}(\lambda,\mu,l) f(z) \qquad (0 \leq \nu \leq 1).$$

Note that the class $\mathcal{M}^{\delta}_{(b,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$ reduces to the classes

$$\mathcal{M}^{1}_{(1,1,1,0,0,0,0)}(\varphi) \equiv \mathcal{C}(\varphi),$$

$$\mathcal{M}^{1}_{(1,1,0,0,0,0,0)}(\varphi) \equiv \mathcal{S}^*(\varphi)$$

which were introduced and studied by Ma and Minda [8]. Also,

$$\mathcal{M}^{1}_{(1,p,0,0,0,0,0)}(\varphi) \equiv \mathcal{S}^*_p(\varphi),$$

$$\mathcal{M}^{1}_{(1,p,1,0,0,0,0)}(\varphi) \equiv \mathcal{C}_p(\varphi),$$

$$\mathcal{M}^{1}_{(b,p,0,0,0,0,0)}(\varphi) \equiv \mathcal{S}^*_{b,p}(\varphi)$$

and $\mathcal{M}^{1}_{(b,p,1,0,0,0,0)}(\varphi) \equiv \mathcal{C}_{b,p}(\varphi)$ were introduced and studied by R. M. Ali *et al.* [1]. Also recently for $\delta \in \mathbb{N}_0$ Altuntaş and Kamali [3] were introduced and studied the class $\mathcal{M}^{\delta}_{(b,p,\alpha,1,0,0,\nu)}(\varphi) = \mathcal{M}_{(b,p,\alpha,\nu,\delta)}(\varphi)$

In this paper, we obtain Fekete-Szegö like inequalities and bounds for the coefficient a_{p+3} for the class $\mathcal{M}^{\delta}_{(1,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$. These results can be extended to other classes defined earlier.

Let Ω be the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + ...$$

in the unit disk $\mathcal{U}$ satisfying the condition $|w(z)| < 1$.

We need the following lemmas to prove our main results.

Lemma 1.3. [1] *If* $w \in \Omega$, *then*

$$\left|w_2 - t w_1^2\right| \leq \begin{cases} -t & \text{if} \quad t \leq -1, \\ 1 & \text{if} \quad -1 \leq t \leq 1, \\ t & \text{if} \quad t \geq 1. \end{cases} \tag{1.10}$$

When $t < -1$ *or* $t > 1$, *the equality holds if and only if* $w(z) = z$ *or one of its rotations.*

If $-1 < t < 1$, *then equality holds if and only if* $w(z) = z^2$ *or one of its rotations.*

Equality holds for $t = -1$ *if and only if* $w(z) = \frac{z(z+\lambda)}{1+\lambda z}$ $(0 \leq \lambda \leq 1)$ *or one of its rotations, while for* $t = 1$ *the equality holds if and only if* $w(z) = -\frac{z(z+\lambda)}{1+\lambda z}$ $(0 \leq \lambda \leq 1)$ *or one of its rotations.*

Although the above upper bound is sharp, it can be improved as follows when $-1 < t < 1:$

$$\left|w_2 - t w_1^2\right| + (1+t)\left|w_1\right|^2 \leq 1 \quad (-1 < t \leq 0) \tag{1.11}$$

and

$$\left|w_2 - t w_1^2\right| + (1-t)\left|w_1\right|^2 \leq 1 \quad (0 < t < 1). \tag{1.12}$$

Lemma 1.4. [7] *If $w \in \Omega$, then for any complex number t*

$$\left|w_2 - tw_1^2\right| \le \max\{1; |t|\}. \tag{1.13}$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

Lemma 1.5. [10] *If $w \in \Omega$, then for any real numbers q_1 and q_2 the following sharp estimate holds:*

$$\left|w_3 + q_1 w_1 w_2 + q_2 w_1^3\right| \le H(q_1, q_2) \tag{1.14}$$

where

$$H(q_1, q_2) =$$

$$= \begin{cases} 1 & \text{for} \quad (q_1, q_2) \in D_1 \cup D_2, \\ |q_2| & \text{for} \quad (q_1, q_2) \in \bigcup\limits_{k=3}^{7} D_k, \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \text{for} \quad (q_1, q_2) \in D_8 \cup D_9, \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \text{for} \quad (q_1, q_2) \in D_{10} \cup D_{11} \setminus \{\pm 2, 1\}, \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \text{for} \quad (q_1, q_2) \in D_{12}. \end{cases}$$

The extremal functions, up to rotations, are of the form

$$w(z) = z^3,\ w(z) = z,\ w(z) = w_0(z) = \frac{(z\left[(1-\lambda)\varepsilon_2 + \lambda\varepsilon_1\right] - \varepsilon_1\varepsilon_2 z)}{1 - \left[(1-\lambda)\varepsilon_1 + \lambda\varepsilon_2\right] z},$$

$$w(z) = w_1(z) = \frac{z(t_1 - z)}{1 - t_1 z},\ w(z) = w_2(z) = \frac{z(t_2 + z)}{1 + t_2 z},$$

$$|\varepsilon_1| = |\varepsilon_2| = 1,\ \varepsilon_1 = t_0 - e^{\frac{-i\theta_0}{2}}(a \pm b),\ \varepsilon_2 = -e^{\frac{-i\theta_0}{2}}(ia \pm b),$$

$$a = t_0 \cos\frac{\theta_0}{2},\ b = \sqrt{1 - t_0^2 \sin^2\frac{\theta_0}{2}},\ \lambda = \frac{b \pm a}{2b},$$

$$t_0 = \left[\frac{2q_2(q_1^2+2) - 3q_1^2}{3(q_2-1)(q_1^2+4q_2)}\right],\ t_1 = \left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}},$$

$$t_2 = \left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}},\ \cos\frac{\theta_0}{2} = \frac{q_1}{2}\left[\frac{q_2(q_1^2+8) - 2(q_1^2+2)}{2q_2(q_1^2+2) - 3q_1^2}\right]$$

The sets D_k, $k = 1, 2, ..., 12$, are defined as follows:

$$D_1 = \left\{(q_1, q_2) : |q_1| \le \frac{1}{2}, \quad |q_2| \le 1\right\},$$

$$D_2 = \left\{(q_1, q_2) : \frac{1}{2} \le |q_1| \le 2, \quad \frac{4}{27} \le (|q_1|+1)^3 - (|q_1|+1) \le q_2 \le 1\right\},$$

$$D_3 = \left\{(q_1, q_2) : |q_1| \le \frac{1}{2}, \quad q_2 \le -1\right\},$$

$$D_4 = \left\{(q_1, q_2) : |q_1| \ge \frac{1}{2}, \quad q_2 \le -\frac{2}{3}(|q_1|+1)\right\},$$

$$D_5 = \{(q_1, q_2) : |q_1| \le 2, \quad q_2 \ge 1\},$$

$$D_6=\left\{(q_1,q_2):2\le|q_1|\le 4,\quad q_2\ge\frac{1}{12}(q_1^2+8)\right\},$$

$$D_7=\left\{(q_1,q_2):|q_1|\ge 4,\quad q_2\ge\frac{2}{3}(|q_1|-1)\right\},$$

$$D_8=\left\{(q_1,q_2):\frac{1}{2}\le|q_1|\le 2,-\frac{2}{3}(|q_1|+1)\le q_2\le\frac{4}{27}(|q_1|+1)^3-(|q_1|+1)\right\},$$

$$D_9=\left\{(q_1,q_2):|q_1|\ge 2,\quad -\frac{2}{3}(|q_1|+1)\le q_2\le\frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\right\},$$

$$D_{10}=\left\{(q_1,q_2):2\le|q_1|\le 4,\quad \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\le q_2\le\frac{1}{12}(q_1^2+8)\right\},$$

$$D_{11}=\left\{(q_1,q_2):|q_1|\ge 4,\quad \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\le q_2\le\frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\right\},$$

$$D_{12}=\left\{(q_1,q_2):|q_1|\ge 4,\quad \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\le q_2\le\frac{2}{3}(|q_1|-1)\right\}.$$

2. Coefficient Bounds

By making use of Lemmas 1.3-1.5, we obtain the following results.

Theorem 2.1. *Let $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+...$, where B_n are real with $B_1>0$ and $B_2\ge 0$. Let $0\le\alpha\le 1$; $\delta,\lambda,\mu,l\in\mathbb{R}$; $\delta,l\ge 0$; $0\le\mu\le\lambda$; $p\in\mathbb{N};0\le\nu\le 1$.*

If $f(z)$ given by (1.1) belongs to $\mathcal{M}^{\delta}_{(1,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$, then

$$\left|a_{p+2}-\eta a_{p+1}^2\right|\le$$

$$\begin{cases}\frac{p^2}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}\left\{B_2+pB_1^2\phi(\alpha,p,\mu,\eta,\lambda,\delta,l,\nu)\right\} & \text{if} \quad \eta\le\psi_1,\\ \frac{p^2B_1}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}} & \text{if} \quad \psi_1\le\eta\le\psi_2,\\ -\frac{p^2}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}\left\{B_2+pB_1^2\phi(\alpha,p,\mu,\eta,\lambda,\delta,l,\nu)\right\} & \text{if} \quad \eta\ge\psi_2.\end{cases} \tag{2.1}$$

Further, if $\psi_1\le\eta\le\psi_3$, then

$$\begin{aligned}&\left|a_{p+2}-\eta a_{p+1}^2\right|+\frac{M_1^2N_1^{2\delta}}{M_2N_2^{\delta}}\frac{(p+\alpha)^2}{2p^2B_1^2(p+2\alpha)(p+l)^{\delta+1}}\\&\times\left(B_1-B_2-pB_1^2\phi(\alpha,p,\mu,\eta,\lambda,\delta,l,v)\right)\left|a_{p+1}\right|^2\\ \le\ &\frac{p^2B_1}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}.\end{aligned} \tag{2.2}$$

If $\psi_3 \le \eta \le \psi_2$, then

$$\begin{aligned} & \left|a_{p+2} - \eta a_{p+1}^2\right| + \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^\delta} \frac{(p+\alpha)^2}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}} \\ & \times \left(B_1 + B_2 + pB_1^2 \phi(\alpha, p, \mu, \eta, \lambda, \delta, l, v)\right) \left|a_{p+1}\right|^2 \\ \le \quad & \frac{p^2 B_1}{2(p+2\alpha)} \frac{(p+l)^{\delta+1}}{M_2 N_2^\delta}. \end{aligned} \tag{2.3}$$

For any complex number η,

$$\left|a_{p+2} - \eta a_{p+1}^2\right| \le \frac{p^2 B_1}{2(p+2\alpha)} \frac{(p+l)^{\delta+1}}{M_2 N_2^\delta}$$

$$\times \max\left\{1; \left|\frac{B_2}{B_1} + pB_1 \phi(\alpha, p, \mu, \eta, \lambda, \delta, l, \nu)\right|\right\}$$

where

$$\psi_1 = \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^\delta} \frac{\left\{(B_2 - B_1)(p+\alpha)^2 + pB_1^2(p^2 + 2\alpha p + \alpha)\right\}}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}}$$

$$\psi_2 = \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^\delta} \frac{\left\{(B_2 + B_1)(p+\alpha)^2 + pB_1^2(p^2 + 2\alpha p + \alpha)\right\}}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}}$$

$$\psi_3 = \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^\delta} \frac{\left\{B_2(p+\alpha)^2 + pB_1^2(p^2 + 2\alpha p + \alpha)\right\}}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}}$$

$$\phi(\alpha, p, \mu, \eta, \lambda, \delta, l, \nu) = \frac{p^2 + 2\alpha p + \alpha}{(p+\alpha)^2} - 2\eta p \frac{M_2 N_2^\delta}{M_1^2 N_1^{2\delta}} \frac{(p+2\alpha)(p+l)^{\delta+1}}{(p+\alpha)^2}$$

and $M_c = [p + c\nu(\lambda\mu(p+c) + \lambda - \mu)]$, $N_c = [c(\lambda\mu(p+c) + \lambda - \mu) + p + l]$, $M_c^d = (M_c)^d$, $N_c^d = (N_c)^d$, $c \in \mathbb{N} = \{1, 2, 3, ...\}$.

Further,

$$|a_{p+3}| \le \frac{p^2 B_1}{3(p+3\alpha)} \frac{(p+l)^{\delta+1}}{M_3 N_3^\delta} H(q_1, q_2) \tag{2.4}$$

where $H(q_1, q_2)$ is defined as in Lemma 1.5, with

$$\begin{aligned} q_1 &= \frac{2B_2}{B_1} + \frac{3}{2} \frac{(p^2 + 3\alpha p + 2\alpha)}{(p+\alpha)(p+2\alpha)} pB_1, \\ q_2 &= \frac{B_3}{B_1} + \frac{3}{2} \frac{\left(p^2 + 3\alpha p + 2\alpha\right)}{(p+\alpha)(p+2\alpha)} pB_1 \left[\frac{B_2}{B_1} + pB_1 \frac{\left(p^2 + 2\alpha p + \alpha\right)}{(p+\alpha)^2}\right] \\ & \quad - \frac{\left(p^3 + 3\alpha p^2 + 3\alpha p + \alpha\right)}{(p+\alpha)^3} p^2 B_1^2. \end{aligned}$$

These results are sharp.

Proof. If $f(z) \in \mathcal{M}^\delta_{(1,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$, then there is a Schwarz function

$$w(z) = w_1 z + w_2 z^2 + ... \in \Omega$$

such that

$$\frac{1}{p}\left\{(1-\alpha)\frac{z(F_{\nu,\delta}(z))'}{F_{\nu,\delta}(z)}+\alpha\left(1+\frac{z(F_{\nu,\delta}(z))''}{(F_{\nu,\delta}(z))'}\right)\right\}=\varphi(w(z)) \tag{2.5}$$

where $0 \leq \alpha \leq 1$; $\delta, \lambda, \mu, l \in \mathbb{R}$; $\delta, l \geq 0$; $0 \leq \mu \leq \lambda$; $p \in \mathbb{N}$; $0 \leq \nu \leq 1$; $F_{\nu,\delta}(z) = (1-\nu)J_p^{\delta}(\lambda,\mu,l)f(z) + \nu J_p^{\delta+1}(\lambda,\mu,l)f(z)$ and

$$J_p^{\delta}(\lambda,\mu,l)f(z) = z^p + \sum_{k=p+1}^{\infty}\left[\frac{(k-p)(\lambda\mu k+\lambda-\mu)+p+l}{p+l}\right]^{\delta} a_k z^k.$$

By definition of $J_p^{\delta}(\lambda,\mu,l)f(z)$ and $F_{\nu,\delta}(z)$, we can write

$$\begin{aligned} F_{\nu,\delta}(z) &= z^p + \frac{M_1N_1^{\delta}}{(p+l)^{\delta+1}}a_{p+1}z^{p+1} + \frac{M_2N_2^{\delta}}{(p+l)^{\delta+1}}a_{p+2}z^{p+2} \\ &+ \frac{M_3N_3^{\delta}}{(p+l)^{\delta+1}}a_{p+3}z^{p+3} + \ldots \end{aligned} \tag{2.6}$$

where

$$M_c = [p + c\nu(\lambda\mu(p+c)+\lambda-\mu)],$$
$$N_c = [c(\lambda\mu(p+c)+\lambda-\mu)+p+l],$$

$c \in \mathbb{N} = \{1,2,3,\ldots\}$.

Let

$$T_{p+c} = \frac{M_cN_c^{\delta}}{(p+l)^{\delta+1}}a_{p+c}; \quad c \in \mathbb{N} = \{1,2,3,\ldots\}.$$

Then, we have

$$F_{\nu,\delta}(z) = z^p + T_{p+1}z^{p+1} + T_{p+2}z^{p+2} + T_{p+3}z^{p+3} + \ldots \tag{2.7}$$

and differentiating both sides of the (2.7), we obtain the following equality

$$(F_{\nu,\delta}(z))' = pz^{p-1} + (p+1)T_{p+1}z^p + (p+2)T_{p+2}z^{p+1} + (p+3)T_{p+3}z^{p+2} + \ldots. \tag{2.8}$$

From (2.7) and (2.8), we deduce

$$\frac{z\,(F_{\nu,\delta}(z))'}{F_{\nu,\delta}(z)} = p + T_{p+1}z + (2T_{p+2} - T_{p+1}^2)z^2 + (3T_{p+3} - 3T_{p+2}T_{p+1} + T_{p+1}^3)z^3 + \ldots. \tag{2.9}$$

Similarly, if we take $U_{p+c} = (p+c)T_{p+c}$, we have

$$\begin{aligned} \frac{z\,(F_{\nu,\delta}(z))''}{(F_{\nu,\delta}(z))'} &= p - 1 + \frac{1}{p}U_{p+1}z + \frac{1}{p}(2U_{p+2} - \frac{1}{p}U_{p+1}^2)z^2 \\ &\quad + \frac{1}{p}(3U_{p+3} - \frac{3}{p}U_{p+2}U_{p+1} + \frac{1}{p^2}U_{p+1}^3)z^3 + \ldots. \end{aligned} \tag{2.10}$$

Since

$$\begin{aligned} \frac{1}{p}\left\{(1-\alpha)\frac{z(F_{\nu,\delta}(z))'}{F_{\nu,\delta}(z)}+\alpha\left(1+\frac{z(F_{\nu,\delta}(z))''}{(F_{\nu,\delta}(z))'}\right)\right\} &= \frac{1}{p}\{(1-\alpha)\,[p + T_{p+1}z \\ +(2T_{p+2} - T_{p+1}^2)z^2 + (3T_{p+3} - 3T_{p+2}T_{p+1} + T_{p+1}^3)z^3 + \ldots] \end{aligned} \tag{2.11}$$

$$+\alpha\left[1+p-1+\frac{1}{p}U_{p+1}z+\frac{1}{p}(2U_{p+2}-\frac{1}{p}U_{p+1}^2)z^2\right.$$
$$\left.+\frac{1}{p}(3U_{p+3}-\frac{3}{p}U_{p+2}U_{p+1}+\frac{1}{p^2}U_{p+1}^3)z^3+...\right]\Bigg\}$$
$$=1+\frac{1}{p}(\frac{p+\alpha}{p})T_{p+1}z+\frac{1}{p}\left(\frac{2\,(p+2\alpha)}{p}T_{p+2}-\frac{p^2+2\alpha p+\alpha}{p^2}T_{p+1}^2\right)z^2$$
$$+\frac{1}{p}\left(\frac{3}{p}(p+3\alpha)T_{p+3}-\frac{3}{p^2}(p^2+3\alpha p+2\alpha)T_{p+2}T_{p+1}\right.$$
$$\left.+\frac{1}{p^3}(p^3+3\alpha p^2+3\alpha p+\alpha)T_{p+1}^3\right)z^3+...$$

and

$$\begin{aligned}\varphi(w(z)) &= 1+B_1w_1z+(B_1w_2+B_2w_1^2)z^2\\ &\quad+(B_1w_3+2B_2w_1w_2+B_3w_1^3)z^3+...,\end{aligned} \tag{2.12}$$

by using equality (2.5), we have the equalities that follow.

Firstly, from

$$B_1w_1=\frac{1}{p}(\frac{p+\alpha}{p})\frac{M_1N_1^\delta}{(p+l)^{\delta+1}}a_{p+1}$$

we can write

$$a_{p+1}=\frac{p^2B_1w_1}{(p+\alpha)}\frac{(p+l)^{\delta+1}}{M_1N_1^\delta}. \tag{2.13}$$

Secondly, from

$$B_1w_2+B_2w_1^2=$$
$$=\frac{1}{p}\left(\frac{2(p+2\alpha)}{p}\frac{M_2N_2^\delta}{(p+l)^{\delta+1}}a_{p+2}-\frac{p^2+2\alpha p+\alpha}{p^2}\frac{M_1^2N_1^{2\delta}}{(p+l)^{2(\delta+1)}}a_{p+1}^2\right)$$

we can write

$$a_{p+2}=\frac{p^2B_1}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^\delta}\left\{w_2-w_1^2\left[-\frac{B_2}{B_1}-\frac{pB_1\left(p^2+2\alpha p+\alpha\right)}{(p+\alpha)^2}\right]\right\}. \tag{2.14}$$

Thus, by using (2.13) and (2.14), we can write

$$a_{p+2}-\eta a_{p+1}^2=\frac{p^2B_1}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^\delta}\left\{w_2-w_1^2\left[-\frac{B_2}{B_1}-\frac{pB_1\left(p^2+2\alpha p+\alpha\right)}{(p+\alpha)^2}\right.\right.$$
$$\left.\left.+\ 2\eta p^2\frac{M_2N_2^\delta}{M_1^2N_1^{2\delta}}\frac{B_1(p+2\alpha)\,(p+l)^{\delta+1}}{(p+\alpha)^2}\right]\right\}.$$

Let

$$t=-\frac{B_2}{B_1}-pB_1\frac{\left(p^2+2\alpha p+\alpha\right)}{(p+\alpha)^2}+2\eta p^2\frac{M_2N_2^\delta}{M_1^2N_1^{2\delta}}\frac{B_1(p+2\alpha)\,(p+l)^{\delta+1}}{(p+\alpha)^2}.$$

Therefore, we have

$$a_{p+2}-\eta a_{p+1}^2=\frac{p^2B_1}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}\{w_2-tw_1^2\}. \tag{2.15}$$

By using Lemma 1.3, we can write for $\eta\leq\psi_1$

$$\begin{aligned}\left|a_{p+2}-\eta a_{p+1}^2\right| &\leq \frac{p^2}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}\left\{B_2+pB_1^2\left[\frac{(p^2+2\alpha p+\alpha)}{(p+\alpha)^2}\right.\right.\\ &\qquad \left.\left.-2\eta p\frac{M_2N_2^{\delta}}{M_1^2N_1^{2\delta}}\frac{(p+2\alpha)(p+l)^{\delta+1}}{(p+\alpha)^2}\right]\right\}\\ &= \frac{p^2}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}\left\{B_2+pB_1^2\phi(\alpha,p,\mu,\eta,\lambda,\delta,l,\nu)\right\}\end{aligned}$$

for $\eta\geq\psi_2$

$$\begin{aligned}\left|a_{p+2}-\eta a_{p+1}^2\right| &\leq -\frac{p^2}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}\left\{B_2+pB_1^2\left[\frac{(p^2+2\alpha p+\alpha)}{(p+\alpha)^2}\right.\right.\\ &\qquad \left.\left.-2\eta p\frac{M_2N_2^{\delta}}{M_1^2N_1^{2\delta}}\frac{(p+2\alpha)(p+l)^{\delta+1}}{(p+\alpha)^2}\right]\right\}\\ &= -\frac{p^2}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}\left\{B_2+pB_1^2\phi(\alpha,p,\mu,\eta,\lambda,\delta,l,\nu)\right\}\end{aligned}$$

and for $\psi_1\leq\eta\leq\psi_2$

$$\left|a_{p+2}-\eta a_{p+1}^2\right|\leq\frac{p^2}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}$$

where

$$\begin{aligned}\psi_1 &= \frac{M_1^2N_1^{2\delta}}{M_2N_2^{\delta}}\frac{\left\{(B_2-B_1)(p+\alpha)^2+pB_1^2(p^2+2\alpha p+\alpha)\right\}}{2p^2B_1^2(p+2\alpha)(p+l)^{\delta+1}}\\ \psi_2 &= \frac{M_1^2N_1^{2\delta}}{M_2N_2^{\delta}}\frac{\left\{(B_2+B_1)(p+\alpha)^2+pB_1^2(p^2+2\alpha p+\alpha)\right\}}{2p^2B_1^2(p+2\alpha)(p+l)^{\delta+1}}\\ \psi_3 &= \frac{M_1^2N_1^{2\delta}}{M_2N_2^{\delta}}\frac{\left\{B_2(p+\alpha)^2+pB_1^2(p^2+2\alpha p+\alpha)\right\}}{2p^2B_1^2(p+2\alpha)(p+l)^{\delta+1}}\end{aligned}$$

and

$$\phi(\alpha,p,\mu,\eta,\lambda,\delta,l,\nu)=\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}-2\eta p\frac{M_2N_2^{\delta}}{M_1^2N_1^{2\delta}}\frac{(p+2\alpha)(p+l)^{\delta+1}}{(p+\alpha)^2}.$$

Further, if $\psi_1\leq\eta\leq\psi_3$, then

$$\left|a_{p+2}-\eta a_{p+1}^2\right|+\frac{M_1^2N_1^{2\delta}}{M_2N_2^{\delta}}\frac{(p+\alpha)^2}{2p^2B_1^2(p+2\alpha)(p+l)^{\delta+1}}$$

$$\times\left(B_1-B_2-pB_1^2\phi(\alpha,p,\mu,\eta,\lambda,\delta,l,v)\right)\left|a_{p+1}\right|^2$$

$$\leq \frac{p^2 B_1}{2(p+2\alpha)} \frac{(p+l)^{\delta+1}}{M_2 N_2^\delta}.$$

If $\psi_3 \leq \eta \leq \psi_2$, then

$$\begin{aligned} &\left|a_{p+2} - \eta a_{p+1}^2\right| + \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^\delta} \frac{(p+\alpha)^2}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}} \\ &\times \left(B_1 + B_2 + pB_1^2 \phi(\alpha, p, \mu, \eta, \lambda, \delta, l, v)\right) |a_{p+1}|^2 \\ \leq\ & \frac{p^2 B_1}{2(p+2\alpha)} \frac{(p+l)^{\delta+1}}{M_2 N_2^\delta}. \end{aligned}$$

By using Lemma 1.4, we can write

$$|a_{p+2} - \eta a_{p+1}^2| \leq \frac{p^2 B_1}{2(p+2\alpha)} \frac{(p+l)^{\delta+1}}{M_2 N_2^\delta} \max\left\{1; \left|\frac{B_2}{B_1} + pB_1 \phi(\alpha, p, \mu, \eta, \lambda, \delta, l, \nu)\right|\right\}$$

for any complex number η.

By using equalities (2.11) and (2.12)

$$\begin{aligned} &\frac{1}{p}\left\{\frac{3}{p}(p+3\alpha)\frac{M_3 N_3^\delta}{(p+l)^{\delta+1}} a_{p+3} - \frac{3}{p^2}\left(p^2 + 3\alpha p + 2\alpha\right)\right. \\ &\left.\times \frac{M_2 N_2^\delta}{(p+l)^{\delta+1}} \frac{M_1 N_1^\delta}{(p+l)^{\delta+1}} a_{p+2} a_{p+1} + \frac{1}{p^3}\left(p^3 + 3\alpha p^2 + 3\alpha p + \alpha\right) \frac{M_1^3 N_1^{3\delta}}{(p+l)^{3(\delta+1)}} a_{p+1}^3\right\} \\ &= B_1 w_3 + 2B_2 w_1 w_2 + B_3 w_1^3. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} a_{p+3} = \frac{p^2 B_1}{3(p+3\alpha)} \frac{(p+l)^{\delta+1}}{M_3 N_3^\delta} &\left\{ w_3 + \left(\frac{2B_2}{B_1} + \frac{3}{2} \frac{\left(p^2 + 3\alpha p + 2\alpha\right)}{(p+\alpha)(p+2\alpha)} pB_1\right) w_1 w_2 \right. \\ &+ \left(\frac{B_3}{B_1} + \frac{3}{2} \frac{\left(p^2 + 3\alpha p + 2\alpha\right)}{(p+\alpha)(p+2\alpha)} pB_1 \left[\frac{B_2}{B_1} + pB_1 \frac{\left(p^2 + 2\alpha p + \alpha\right)}{(p+\alpha)^2}\right] \right. \\ &\left.\left. - \frac{\left(p^3 + 3\alpha p^2 + 3\alpha p + \alpha\right)}{(p+\alpha)^3} p^2 B_1^2\right) w_1^3 \right\}. \end{aligned} \tag{2.16}$$

Let

$$\begin{aligned} q_1 &= \frac{2B_2}{B_1} + \frac{3}{2} \frac{\left(p^2 + 3\alpha p + 2\alpha\right)}{(p+\alpha)(p+2\alpha)} pB_1, \\ q_2 &= \frac{B_3}{B_1} + \frac{3}{2} \frac{\left(p^2 + 3\alpha p + 2\alpha\right)}{(p+\alpha)(p+2\alpha)} pB_1 \left[\frac{B_2}{B_1} + pB_1 \frac{\left(p^2 + 2\alpha p + \alpha\right)}{(p+\alpha)^2}\right] \\ &\quad - \frac{\left(p^3 + 3\alpha p^2 + 3\alpha p + \alpha\right)}{(p+\alpha)^3} p^2 B_1^2. \end{aligned}$$

Then, from equality (2.16), we obtain

$$a_{p+3} = \frac{p^2 B_1}{3(p+3\alpha)} \frac{(p+l)^{\delta+1}}{M_3 N_3^\delta} \left\{w_3 + q_1 w_1 w_2 + q_2 w_1^3\right\}.$$

Thus, we can write

$$|a_{p+3}| \leq \frac{p^2 B_1}{3(p+3\alpha)} \frac{(p+l)^{\delta+1}}{M_3 N_3^{\delta}} H(q_1, q_2) \tag{2.17}$$

where $H(q_1, q_2)$ is defined as in Lemma 1.5.

To show that the bounds in (2.1)-(2.3) are sharp, we define the functions $K_{\varphi,n}$ $(n = 2, 3, ...)$ by

$$\frac{1}{p}\left\{(1-\alpha)\frac{z(K_{\varphi,n})'(z)}{(K_{\varphi,n})(z)} + \alpha\left(1 + \frac{z(K_{\varphi,n})''(z)}{(K_{\varphi,n})'}\right)\right\} = \varphi(z^{n-1}) \tag{2.18}$$

$$[K_{\varphi,n}](0) = 0 = [K_{\varphi,n}]'(0) - 1,$$

and the function $F_{\lambda,m}$ and $G_{\lambda,m}$ $(0 \leq \lambda \leq 1,\ m \in \mathbb{N}_0)$ by

$$\frac{1}{p}\left\{(1-\alpha)\frac{z(F_{\lambda,m})'(z)}{(F_{\lambda,m})(z)} + \alpha\left(1 + \frac{z(F_{\lambda,m})''(z)}{(F_{\lambda,m})'}\right)\right\} = \varphi\left(z\frac{z+\lambda}{1+\lambda z}\right) \tag{2.19}$$

$$[F_{\lambda,m}](0) = 0 = [F_{\lambda,m}]'(0) - 1,$$

and

$$\frac{1}{p}\left\{(1-\alpha)\frac{z(G_{\lambda,m})'(z)}{(G_{\lambda,m})(z)} + \alpha\left(1 + \frac{z(G_{\lambda,m})''(z)}{(G_{\lambda,m})'}\right)\right\} = \varphi\left(-z\frac{z+\lambda}{1+\lambda z}\right) \tag{2.20}$$

$$(G_{\lambda,m})(0) = 0 = [G_{\lambda,m}]'(0) - 1.$$

Clearly the functions $K_{\varphi,n}$, $F_{\lambda,m}$, $G_{\lambda,m} \in \mathcal{M}^{\delta}_{(1,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$. Also we write $K_{\varphi} = K_{\varphi,2}$. If $\eta < \psi_1$ or $\eta > \psi_2$, then the equality holds if and only if f is K_{φ} or one of its rotations. When $\psi_1 < \eta < \psi_2$, then the equality holds if and only if f is $K_{\varphi,3}$ or one of its rotations. If $\eta = \psi_1$, then the equality holds if and only if f is $F_{\lambda,m}$ or one of its rotations. If $\eta = \psi_2$, then the equality holds if and only if f is $G_{\lambda,m}$ or one of its rotations. □

Remark 2.2.

1. For $l = \mu = 0$ and $\lambda = 1$ in Theorem 2.1,we get the result obtained by Altuntaş and Kamali [3].
2. For $l = \delta = \mu = \alpha = \nu = 0$ and $\lambda = 1$ in Theorem 2.11, we obtain the result obtained by R. M. Ali et al. [1].
3. For $l = \delta = \mu = \alpha = \nu = 0$ and $p = \lambda = 1$ in Theorem 2.1, we obtain the result obtained by Ma and Minda et al. [8].
4. For $l = \alpha = 0$ and $p = b = 1$ in Theorem 2.1, we obtain the result obtained by Deniz and Orhan et al. [6].

3. Applications to functions defined by convolution

We define $\mathcal{M}^{\delta}_{(b,p,\alpha,\lambda,\mu,l,\nu,g)}(\varphi)$ to be the class of all functions $f \in \mathcal{A}_p$ for which $f * g \in \mathcal{M}^{\delta}_{(b,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$, where g is a fixed function with positive coefficients and the class $\mathcal{M}^{\delta}_{(b,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$ is as in Definition 1.2. In Theorem

2.1 we obtained the coefficient estimate for the class $\mathcal{M}^{\delta}_{(1,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$. Now we obtain the coefficient estimates for the class $\mathcal{M}^{\delta}_{(1,p,\alpha,\lambda,\mu,l,\nu,g)}(\varphi)$.

Theorem 3.1. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ...,$ where B_n are real with $B_1 > 0$ and $B_2 \geq 0$. Let $0 \leq \alpha \leq 1$; $\delta, \lambda, \mu, l \in \mathbb{R}$; $\delta, l \geq 0$; $0 \leq \mu \leq \lambda$; $p \in \mathbb{N}; 0 \leq \nu \leq 1$.*

If $f(z)$ given by (1.1) belongs to $\mathcal{M}^{\delta}_{(1,p,\alpha,\lambda,\mu,l,\nu,g)}(\varphi)$, then

$$\left|a_{p+2} - \eta a_{p+1}^2\right| \leq$$

$$\begin{cases} \frac{p^2}{2(p+2\alpha)g_{p+2}} \frac{(p+l)^{\delta+1}}{M_2 N_2^{\delta}} \left\{B_2 + pB_1^2 \phi(\alpha, p, \mu, \eta, \lambda, \delta, l, \nu)\right\} & \text{if} \quad \eta \leq \psi_1, \\ \frac{p^2 B_1}{2(p+2\alpha)g_{p+2}} \frac{(p+l)^{\delta+1}}{M_2 N_2^{\delta}} & \text{if} \quad \psi_1 \leq \eta \leq \psi_2, \\ -\frac{p^2}{2(p+2\alpha)g_{p+2}} \frac{(p+l)^{\delta+1}}{M_2 N_2^{\delta}} \left\{B_2 + pB_1^2 \phi(\alpha, p, \mu, \eta, \lambda, \delta, l, \nu)\right\} & \text{if} \quad \eta \geq \psi_2. \end{cases}$$

Further, if $\psi_1 \leq \eta \leq \psi_3$, then

$$\begin{aligned} &\left|a_{p+2} - \eta a_{p+1}^2\right| + \frac{g_{p+1}^2}{g_{p+2}} \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^{\delta}} \frac{(p+\alpha)^2}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}} \\ &\times \left(B_1 - B_2 - pB_1^2 \phi(\alpha, p, \mu, \eta, \lambda, \delta, l, v, g)\right) |a_{p+1}|^2 \\ \leq \quad & \frac{p^2 B_1}{2g_{p+2}(p+2\alpha)} \frac{(p+l)^{\delta+1}}{M_2 N_2^{\delta}}. \end{aligned}$$

If $\psi_3 \leq \eta \leq \psi_2$, then

$$\begin{aligned} &\left|a_{p+2} - \eta a_{p+1}^2\right| + \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^{\delta}} \frac{(p+\alpha)^2}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}} \\ &\times \left(B_1 + B_2 + pB_1^2 \phi(\alpha, p, \mu, \eta, \lambda, \delta, l, v, g)\right) |a_{p+1}|^2 \\ \leq \quad & \frac{p^2 B_1}{2g_{p+2}(p+2\alpha)} \frac{(p+l)^{\delta+1}}{M_2 N_2^{\delta}}. \end{aligned}$$

For any complex number η,

$$\begin{aligned} \left|a_{p+2} - \eta a_{p+1}^2\right| &\leq \frac{g_{p+1}^2}{g_{p+2}} \frac{p^2 B_1}{2g_{p+2}(p+2\alpha)} \frac{(p+l)^{\delta+1}}{M_2 N_2^{\delta}} \\ &\times \max\left\{1; \left|\frac{B_2}{B_1} + pB_1 \phi(\alpha, p, \mu, \eta, \lambda, \delta, l, \nu, g)\right|\right\} \end{aligned}$$

where

$$\begin{aligned} \psi_1 &= \frac{g_{p+1}^2}{g_{p+2}} \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^{\delta}} \frac{\left\{(B_2 - B_1)(p+\alpha)^2 + pB_1^2(p^2 + 2\alpha p + \alpha)\right\}}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}} \\ \psi_2 &= \frac{g_{p+1}^2}{g_{p+2}} \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^{\delta}} \frac{\left\{(B_2 + B_1)(p+\alpha)^2 + pB_1^2(p^2 + 2\alpha p + \alpha)\right\}}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}} \\ \psi_3 &= \frac{g_{p+1}^2}{g_{p+2}} \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^{\delta}} \frac{\left\{B_2(p+\alpha)^2 + pB_1^2(p^2 + 2\alpha p + \alpha)\right\}}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}} \end{aligned}$$

$$\phi(\alpha, p, \mu, \eta, \lambda, \delta, l, \nu, g) = \frac{p^2 + 2\alpha p + \alpha}{(p+\alpha)^2} - 2\eta p \frac{g_{p+2}}{g_{p+1}^2} \frac{M_2 N_2^{\delta}}{M_1^2 N_1^{2\delta}} \frac{(p+2\alpha)(p+l)^{\delta+1}}{(p+\alpha)^2}.$$

and $M_c = [p + c\nu(\lambda\mu(p+c) + \lambda - \mu)]$, $N_c = [c(\lambda\mu(p+c) + \lambda - \mu) + p + l]$, $M_c^d = (M_c)^d$, $N_c^d = (N_c)^d$, $c \in \mathbb{N} = \{1, 2, 3, ...\}$.

Further,

$$|a_{p+3}| \le \frac{p^2 B_1}{3g_{p+3}(p+3\alpha)} \frac{(p+l)^{\delta+1}}{M_3 N_3^{\delta}} H(q_1, q_2)$$

where $H(q_1, q_2)$ *is defined as in Lemma* 1.5*,*

$$q_1 = \frac{2B_2}{B_1} + \frac{3}{2}\frac{(p^2+3\alpha p+2\alpha)}{(p+\alpha)(p+2\alpha)} pB_1,$$

$$q_2 = \frac{B_3}{B_1} + \frac{3}{2}\frac{(p^2+3\alpha p+2\alpha)}{(p+\alpha)(p+2\alpha)} pB_2$$
$$+ \left(\frac{3}{2}\frac{(p^2+3\alpha p+2\alpha)(p^2+2\alpha p+\alpha)}{(p+2\alpha)(p+\alpha)^3} - \frac{(p^3+3\alpha p^2+3\alpha p+\alpha)}{(p+\alpha)^3}\right) p^2 B_1^2.$$

These results are sharp.

Proof. The proof is similar to the proof of Theorem 2.1 □

Remark 3.2.

1. For $l = \delta = \mu = \alpha = \nu = 0$ and $\lambda = 1$ in Theorem 3.1, we obtain the result obtained by Ali et al. [1].
2. For $l = \delta = \mu = \alpha = \nu = 0$ and $p = \lambda = 1$ in Theorem 3.1, we obtain the result obtained by Ma and Minda et al. [8].
3. For $l = \alpha = 0$ and $p = b = 1$ in Theorem 2.1, we obtain the result obtained by Deniz and Orhan et al. [6].

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The univalence and the convexity properties for a new integral operator

Virgil Pescar

Abstract. For analytic functions f in the open unit disk $\mathcal{U}$, an integral operator $I_{\alpha,\beta}$ is introduced. The object of the paper is to obtain the conditions of the univalence and the convexity of the integral operator $I_{\alpha,\beta}$.

Mathematics Subject Classification (2010): 30C45.

Keywords: Integral operator, univalence, starlike, convexity.

1. Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of the functions $f \in \mathcal{A}$, which are univalent in $\mathcal{U}$. We denote by $\mathcal{S}^*$ the subclass of $\mathcal{A}$ consisting of all starlike functions in $\mathcal{U}$. Also, we denote by $\mathcal{K}$ the subclass of $\mathcal{A}$ consisting of all convex functions in $\mathcal{U}$.

We consider $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all the convex functions f of the order α, satisfying:

$$Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \quad (z \in \mathcal{U}), \tag{1.1}$$

for some α $(0 \leq \alpha < 1)$. We have $\mathcal{K}(0) = \mathcal{K}$.

Note that $f \in \mathcal{K}$, if and only if $zf' \in \mathcal{S}^*$.

In this work, we introduce a new integral operator, which is defined by

$$I_{\alpha,\beta}(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\alpha} (f'(u))^{\beta}\, du, \tag{1.2}$$

for α, β be complex numbers, $f \in \mathcal{A}$.

For $\beta = 0$, α be a complex number, $f \in \mathcal{A}$, from (1.2) we have the integral operator Kim-Merkes [2],

$$H_\alpha(z) = \int_0^z \left(\frac{f(u)}{u}\right)^\alpha du. \tag{1.3}$$

From (1.2), for $\alpha = 0$, β be a complex number, $f \in \mathcal{A}$, we obtain the integral operator Pfaltzgraff [5],

$$G_\beta(z) = \int_0^z (f'(u))^\beta du. \tag{1.4}$$

2. Preliminary results

We need the following lemmas.

Lemma 2.1. [1]. *If the function f is analytic in $\mathcal{U}$ and*

$$\left(1 - |z|^2\right) \left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{2.1}$$

for all $z \in \mathcal{U}$, then the function f is univalent in $\mathcal{U}$.

Lemma 2.2. *(Schwarz [3]). Let f be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If f has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \ (z \in \mathcal{U}_R), \tag{2.2}$$

the equality (in the inequality (2.2) for $z \neq 0$) can hold if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

3. Main results

Theorem 3.1. *Let α, β be complex numbers, M, L positive real numbers and $f \in \mathcal{A}$, $f(z) = z + a_2 z^2 + \ldots$. If*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq M, \quad (z \in \mathcal{U}), \tag{3.1}$$

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq L, \quad (z \in \mathcal{U}), \tag{3.2}$$

and

$$|\alpha|M + |\beta|L \leq \frac{3\sqrt{3}}{2}, \tag{3.3}$$

then the function

$$I_{\alpha,\beta}(z) = \int_0^z \left(\frac{f(u)}{u}\right)^\alpha (f'(u))^\beta du, \tag{3.4}$$

is in the class $\mathcal{S}$.

Proof. The function $I_{\alpha,\beta}(z)$ is regular in $\mathcal{U}$ and $I_{\alpha,\beta}(0) = I'_{\alpha,\beta}(0) - 1 = 0$. We have:

$$\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)} = \alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + \beta\frac{zf''(z)}{f'(z)}, \tag{3.5}$$

for all $z \in \mathcal{U}$.

From (3.5) we obtain:

$$\left(1-|z|^2\right)\left|\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}\right| \leq \left(1-|z|^2\right)\left[|\alpha|\left|\frac{zf'(z)}{f(z)} - 1\right| + |\beta|\left|\frac{zf''(z)}{f'(z)}\right|\right], \tag{3.6}$$

for all $z \in \mathcal{U}$. By Lemma 2.2, from (3.1) and (3.2) we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq M|z|, \quad (z \in \mathcal{U}), \tag{3.7}$$

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq L|z|, \quad (z \in \mathcal{U}) \tag{3.8}$$

and by (3.6) we obtain

$$\left(1-|z|^2\right)\left|\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}\right| \leq \left(1-|z|^2\right)|z|\left(|\alpha|M + |\beta|L\right), \tag{3.9}$$

for all $z \in \mathcal{U}$. Since

$$\max_{|z|\leq 1}\left[\left(1-|z|^2\right)|z|\right] = \frac{2}{3\sqrt{3}},$$

by (3.3) and (3.9) we have

$$\left(1-|z|^2\right)\left|\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}\right| \leq 1, \quad (z \in \mathcal{U}). \tag{3.10}$$

By Lemma 2.1, we obtain that the integral operator $I_{\alpha,\beta}$ is in the class $\mathcal{S}$. □

Theorem 3.2. *Let α, β be real numbers, with the properties $\alpha \geq 0$, $\beta \geq 0$ and*

$$0 < \alpha + \beta < 1 \tag{3.11}$$

We suppose that the functions $f \in \mathcal{S}^*$ and $g \in \mathcal{S}^*$, where $g(z) = zf'(z)$. Then, the integral operator $I_{\alpha,\beta}$ defined by

$$I_{\alpha,\beta}(z) = \int_0^z \left(\frac{f(u)}{u}\right)^\alpha \left(f'(u)\right)^\beta du, \tag{3.12}$$

is convex by the order $1 - \alpha - \beta$.

Proof. From (3.5) we obtain that:

$$\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}+1=\alpha\frac{zf'(z)}{f(z)}-\alpha+\beta\left(\frac{zf''(z)}{f'(z)}+1\right)-\beta+1 \tag{3.13}$$

and hence, we have

$$Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}+1\right)=\alpha Re\frac{zf'(z)}{f(z)}-\alpha+\beta Re\left(\frac{zf''(z)}{f'(z)}+1\right)-\beta+1, \tag{3.14}$$

for all $z\in\mathcal{U}$.

But $f\in\mathcal{S}^*$ and $g\in\mathcal{S}^*$, where $g(z)=zf'(z)$.

We apply this affirmation in (3.14), we obtain that:

$$Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}+1\right)>1-\alpha-\beta. \tag{3.15}$$

Using the hypothesis $\alpha+\beta<1$, in (3.15), we obtain that $I_{\alpha,\beta}$ is convex function by the order $1-\alpha-\beta$. □

4. Corollaries

Corollary 4.1. *Let α be a complex number, $\alpha\neq 0$ and $f\in\mathcal{A}$,*

$$f(z)=z+a_2z^2+a_3z^3+\dots$$

If

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq\frac{3\sqrt{3}}{2|\alpha|},\quad (z\in\mathcal{U}). \tag{4.1}$$

then the integral operator H_α, defined by (1.3), belongs to the class $\mathcal{S}$.

Proof. For $\beta=0$, from Theorem 3.1 we obtain Corollary 4.1. □

Corollary 4.2. *Let β be a complex number, $\beta\neq 0$ and $f\in\mathcal{A}$,*

$$f(z)=z+a_2z^2+a_3z^3+\dots$$

If

$$\left|\frac{zf''(z)}{f'(z)}\right|\leq\frac{3\sqrt{3}}{2|\beta|},\quad (z\in\mathcal{U}), \tag{4.2}$$

then the integral operator G_β, defined by (1.4), is in the class $\mathcal{S}$.

Proof. We take $\alpha=0$ in Theorem 3.1. □

Corollary 4.3. *If α is a real number, $0<\alpha<1$ and the function $f\in\mathcal{S}^*$, then the integral operator H_α defined in (1.3) is convex by the order $1-\alpha$.*

Proof. For $\beta=0$ in Theorem 3.2, we obtain Corollary 4.3. □

Corollary 4.4. *If β is a real number, $0<\beta<1$ and the function $f\in\mathcal{K}$, then the integral operator G_β, defined by (1.4), is convex by the order $1-\beta$.*

Proof. We take $\alpha=0$ in Theorem 3.2. □

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A class of uniformly convex functions involving a differential operator

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Abstract. The main purpose of this paper is to introduce a new class $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \ \lambda,\ k)$, of functions which are analytic in the open disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. We obtain various results including characterization, coefficients estimates, distortion and covering theorems, radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \ \lambda,\ k)$.

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Keywords: Analytic function, starlike function, convex function, uniformly convex function, convolution product, Cho-Srivastava operator.

1. Introduction and motivations

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

that are analytic in the open unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be a subclass of $\mathcal{A}$ consisting of univalent functions in Δ. By $\mathcal{K}(\beta)$, and $\mathcal{S}^*(\beta)$ respectively, we mean the classes of analytic functions that satisfy the analytic conditions

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta \quad \text{and} \quad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \quad z \in \Delta$$

for $0 \leqq \beta < 1$. In particular, $\mathcal{K} = \mathcal{K}(0)$ and $\mathcal{S}^* = \mathcal{S}^*(0)$ respectively, are the well-known standard class of convex and starlike functions.

The function $f \in \mathcal{A}$ is said to be close-to-convex of order β, $\beta \geqq 0$, with respect to a starlike function g and $\phi \in \mathbb{R}$ if

$$\left|\arg e^{i\phi}\frac{f(z)}{g(z)}\right| \leqq \beta\frac{\pi}{2}, \quad z \in \Delta.$$

Let $\mathcal{CC}(\beta)$ denote the union of all such close-to-convex functions of order β.

Let $\mathcal{T}$ denote the subclass of $\mathcal{S}$ of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geqq 0, \tag{1.1}$$

that are analytic in the open unit disk Δ. This class was introduced and studied in [9]. Analogous to the subclasses $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ of $\mathcal{S}$ respectively, the subclasses of $\mathcal{T}$ denoted by $\mathcal{T}^*(\beta)$ and $\mathcal{C}(\beta)$, $0 \leqq \beta < 1$, were also investigated in [9].

The main class which we investigate in this present paper uses the operator known as the Cho-Srivastava operator. In fact, One important concept that is useful in discussing this operator is the convolution or Hadamard product. Here by convolution we mean the following: For f, g analytic with $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ and $g(z) = b_0 + b_1 z + b_2 z^2 + \cdots$, the (Hadamard) convolution of f and g is defined by $(f * g)(z) = a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \cdots$. It is natural to use the notation $f(z) * g(z)$ for $(f * g)(z)$ and vice versa frequently.

For functions $f \in \mathcal{A}$, we recall the multiplier transformation $I(\lambda, k)$ introduced by Cho and Srivastava [3] defined as

$$I(\lambda, k) f(z) = z + \sum_{n=2}^{\infty} \Psi_n a_n z^n \quad (\lambda \geqq 0;\ k \in \mathbb{Z}) \tag{1.2}$$

where

$$\Psi_n := \left(\frac{n + \lambda}{1 + \lambda} \right)^k \tag{1.3}$$

so that, obviously,

$$I(\lambda, k) \left(I(\lambda, m)\ f(z) \right) = I(\lambda, k + m)\ f(z) \quad (k, m \in \mathbb{Z}). \tag{1.4}$$

For $\lambda = 1$, the operators $I(\lambda, k)$ were studied by Uralegaddi and Somanatha [12]. The operators $I(\lambda, k)$ are closely related to the multiplier transformations studied by Flett [4] and also to the differential and integral operators investigated by Sălăgean [7]. For a detailed analysis of various convolution operators, which are related to the multiplier transformations of Flett [4], refer the work of Li and Srivastava [5] (as well as the references cited by them). Now we define an unified class of analytic function based on this operator.

Definition 1.1. *For* $0 \leqq \gamma \leqq 1$, $0 \leqq \beta < 1$, $\alpha \geqq 0$, *and for all* $z \in \Delta$, *we let the class* $\mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$, *consists of functions* $f \in \mathcal{T}$ *is said to be in the class satisfying the condition*

$$\operatorname{Re} \left\{ \frac{zF'(z)}{F(z)} \right\} > \alpha \left| \frac{zF'(z)}{F(z)} - 1 \right| + \beta, \tag{1.5}$$

with,

$$F(z) := \gamma(1 + \lambda) I(\lambda, k + 1) f(z) + (1 - \gamma(1 + \lambda)) I(\lambda, k) f(z), \tag{1.6}$$

where $I(\lambda, k) f(z)$ *is the Cho-Srivastava operator as defined by* (1.2)

The family $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \ \lambda,\ k)$, unifies various well known classes of analytic univalent functions. We list a few of them. The class $\mathcal{UH}(2,\ 1,\ \lambda,\ \beta,\ 0)$ studied in [1]. Many classes including $\mathcal{UH}(2,\ 1,\ 0,\ \beta,\ 0)$ and $\mathcal{UH}(2,\ 1,\ 1,\ \beta,\ 0)$ given in [11], are particular cases of this class. Further that, the class $\mathcal{UH}(2, 1,\ \lambda,\ 0,\ \beta,\ k)$ is the class of $k-$uniformly convex of order β, was introduced and studied in [10] (also see [2]).

In this present paper, we obtain a characterization, coefficients estimates, distortion theorem and covering theorem, extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \ \lambda,\ k)$,.

2. Characterization and coefficient estimates

Theorem 2.1. *Let* $f \in \mathcal{T}$. *Then* $f \in \mathcal{UH}(\alpha,\ \beta, \gamma,\ \lambda,\ k),\quad 0 \leqq \gamma \leqq 1,\ \ 0 \leqq \beta < 1$ *and* $\alpha \geqq 0$,

$$\sum_{n=2}^{\infty} [n(\alpha+1)-(\alpha+\beta)]\,(\gamma(n-1)+1)\,\Psi_n|a_n| \leqq 1-\beta. \tag{2.1}$$

This result is sharp for the function

$$f(z) = z - \frac{1-\beta}{[n(\alpha+1)-(\alpha+\beta)][\gamma(n-1)+1]\Psi_n}\, z^n \ \ n \geqq 2. \tag{2.2}$$

Proof. We employ the technique adopted by [2]. We have

$$f \in \mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k),$$

if and only if the condition (1.5) is satisfied, which is equivalent to

$$\mathrm{Re}\left\{\frac{zF'(z)(1+ke^{i\theta})-F(z)ke^{i\theta}}{F(z)}\right\} > \beta, \quad -\pi \leqq \theta < \pi. \tag{2.3}$$

Now, letting $G(z) = zF'(z)(1+ke^{i\theta}) - F(z)ke^{i\theta}$, equation (2.3) is equivalent to

$$|G(z)+(1-\beta)F(z)| > |G(z)-(1+\beta)F(z)|,\ 0 \leqq \beta < 1.$$

where $F(z)$ is as defined in (1.6). Now a simple computation gives

$$\begin{aligned}&|G(z)+(1-\beta)F(z)|\\ &\quad\geqq\ (2-\beta)|z| - \sum_{n=2}^{\infty}\Big(n(\alpha+1)-(\alpha+\beta)+1\Big)\Big(\gamma(n-1)+1\Big)\,\Psi_n a_n|z|^n\end{aligned}$$

and similarly,

$$\begin{aligned}&|G(z)-(1+\beta)F(z)|\\ &\quad\leqq \beta|z| + \sum_{n=2}^{\infty}\Big((n(\alpha+1)-(\alpha+\beta)-1)\Big)\Big(\gamma(n-1)+1\Big)\,\Psi_n a_n|z|^n.\end{aligned}$$

Therefore,

$$|G(z)+(1-\beta)F(z)| - |G(z)-(1+\beta)F(z)|$$

$$\geqq 2(1-\beta)|z| - 2\sum_{n=2}^{\infty}\Big((n(\alpha+1)-(\alpha+\beta))\Big)(\gamma(n-1)+1)\,\Psi_n a_n |z|^n \geqq 0,$$

which is equivalent to the result (2.1).

On the other hand, for all $-\pi \leqq \theta < \pi$, we must have

$$\mathrm{Re}\left\{\frac{zF'(z)}{F(z)}(1+ke^{i\theta}) - ke^{i\theta}\right\} > \beta.$$

Now, choosing the values of z on the positive real axis, where $0 \leqq |z| = r < 1$, and using $\mathrm{Re}\{-e^{i\theta}\} \geqq -|e^{i\theta}| = -1$, the above inequality can be written as

$$\mathrm{Re}\left\{\frac{(1-\beta) - \sum_{n=2}^{\infty}\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n a_n r^{n-1}}{1-\sum_{n=2}^{\infty}\Big(\gamma(n-1)+1\Big)\Psi_n a_n r^{n-1}}\right\} \geqq 0.$$

Setting $r \to 1^-$, we get the desired result. □

Many known results can be obtained as particular cases of Theorem 2.1. For details, we refer to [6, 8].

By taking $\alpha = 0, \gamma = 1$, $\lambda = 0$ and $k = 1$ in Theorem 2.1, we get the following interesting result given in [9].

Corollary 2.2. [9] *If $f \in \mathcal{T}$, then $f \in \mathcal{C}(\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\beta)a_n \leqq 1-\beta.$$

Indeed, since $f \in \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$, (2.1), we have

$$\sum_{n=2}^{\infty}\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n a_n \leqq 1-\beta.$$

Hence for all $n \geqq 2$, we have

$$a_n \leqq \frac{1-\beta}{\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n},$$

whenever $0 \leqq \gamma \leqq 1$, $0 \leqq \beta < 1$ and $\alpha \geqq 0$. Hence we state this important observation as a separate theorem.

Theorem 2.3. *If $f \in \mathcal{UH}(q, s, \lambda, \beta, k)$, then*

$$a_n \leqq \frac{1-\beta}{\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n},\quad n \geqq 2, \tag{2.4}$$

where $0 \leqq \gamma \leqq 1$, $0 \leqq \beta < 1$ and $\alpha \geqq 0$. Equality in (2.4) holds for the function

$$f(z) = z - \frac{1-\beta}{\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\Psi_n}. \tag{2.5}$$

This theorem also contains many known results for the special values of the parameters. For example, see [6, 8].

3. Distortion and covering theorems

Theorem 3.1. *If* $f \in \mathcal{UH}(\alpha,\, \beta,\, \gamma,\, \lambda,\, k)$, *then* $f \in \mathcal{T}^*(\delta)$, *where*

$$\delta = 1 - \frac{1-\beta}{\big(\,2(\alpha+1)-(\alpha+\beta)\big)\big(\,\gamma+1\big)\,\Psi_2-(1-\beta)}.$$

This result is sharp with the extremal function being

$$f(z) = z - \frac{1-\beta}{\big(\,2(\alpha+1)-(\alpha+\beta)\big)\big(\,\gamma+1\big)\,\Psi_2} z^2.$$

Proof. It is sufficient to show that (2.1) implies $\sum_{n=2}^{\infty}(n-\delta)a_n \leqq 1-\delta$ [9], that is,

$$\frac{n-\delta}{1-\delta} \leqq \frac{\big(\,n(\alpha+1)-(\alpha+\beta)\big)\big(\,\gamma(n-1)+1\big)\,\Psi_n}{1-\beta},\quad n \geqq 2. \tag{3.1}$$

Since, for $n \geqq 2$, (3.1) is equivalent to

$$\delta \;\leqq\; 1 - \frac{(n-1)(1-\beta)}{\big(\,n(\alpha+1)-(\alpha+\beta)\big)\big(\,\gamma(n-1)+1\big)\,\Psi_n-(1-\beta)} = \Phi(n),$$

and $\Phi(n) \leqq \Phi(2)$, (3.1) holds true for any $0 \leqq \gamma \leqq 1,\;\; 0 \leqq \beta < 1$ and $\alpha \geqq 0$. This completes the proof of the Theorem 3.1. □

As in the previous cases we note this result has many special cases. If we take $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = 1, q = 2, s = 1,\ \lambda = 1$ and $k = 0$ in Theorem 3.1, then we have the following result of [9].

Corollary 3.2. [9] *If* $f \in \mathcal{C}(\beta)$, *then* $f \in \mathcal{T}^*\left(\frac{2}{3-\beta}\right)$. *The result is sharp for the extremal function*

$$f(z) = z - \frac{1-\beta}{2(2-\beta)} z^2.$$

Remark. Since distortion theorem and covering theorem are available for the class $\mathcal{T}^*(\beta)$ [9], we can also obtain the corresponding results for the class $\mathcal{UH}(\alpha,\, \beta,\, \gamma,\, \lambda,\, k)$, from the respective results of $\mathcal{T}^*(\beta)$ by using Theorem 3.1, and we state them without proof.

Theorem 3.3. *Let* Ψ_n *be defined as in* (1.3). *Then, for* $f \in \mathcal{UH}(\alpha,\, \beta,\, \gamma,\, \lambda,\, k)$, *with* $z = re^{i\theta} \in \Delta$, *we have*

$$r - B(\alpha,\, \beta,\, \gamma,\, \lambda)r^2 \leqq |f(z)| \leqq r + B(\alpha,\, \beta,\, \gamma,\, \lambda)r^2, \tag{3.2}$$

where,

$$B(\alpha,\, \beta,\, \gamma,\, \lambda) := \frac{1-\beta}{\Big(\,2(\alpha+1)-(\alpha+\beta)\Big)\Big(\,\gamma+1\Big)\,\Psi_2}.$$

Theorem 3.4. *If* $f \in \mathcal{UH}(\alpha,\, \beta,\, \gamma,\, \lambda,\, k)$, *then for* $|z| = r < 1$

$$1 - B(\alpha,\, \beta,\, \gamma,\, \lambda)r \leqq |f'(z)| \leqq 1 + B(\alpha,\, \beta,\, \gamma,\, \lambda)r\,, \tag{3.3}$$

where $B(\alpha,\, \beta,\, \gamma,\, \lambda)$ *as in Theorem* 3.3.

Note that in Theorem 3.3 and Theorem 3.4 equality holds for the function

$$f(z) = z - \frac{1-\beta}{\Big(2(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma+1\Big)\Psi_2} z^2.$$

4. Extreme points of the class $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \lambda,\ k)$,

Theorem 4.1. *Let $f_1(z) = z$ and*

$$f_n(z) = z - \frac{1-\beta}{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n} z^n, \quad n \geqq 2$$

and Ψ_n be as defined in (1.3). Then $f \in \mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$, if and only if it can be represented in the form

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \quad \mu_n \geqq 0, \quad \sum_{n=1}^{\infty} \mu_n = 1. \tag{4.1}$$

Proof. Suppose $f(z)$ can be written as in (4.1). Then

$$f(z) \quad = \quad z - \sum_{n=2}^{\infty} \mu_n \left\{ \frac{1-\beta}{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n} \right\} z^n.$$

Now,

$$\sum_{n=2}^{\infty} \mu_n \frac{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n(1-\beta)}{(1-\beta)\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n} = \sum_{n=2}^{\infty} \mu_n = 1-\mu_1 \leqq 1.$$

Thus $f \in \mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$. Conversely, let us have $f \in \mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$. Then by using (2.4), we may write

$$\mu_n = \frac{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n}{1-\beta} a_n, \quad n \geqq 2,$$

and $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$. Then $f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)$, with $f_n(z)$ is as in the Theorem. □

Corollary 4.2. *The extreme points of $f \in \mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$, are the functions $f_1(z) = z$ and*

$$f_n(z) = z - \frac{1-\beta}{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n} z^n, \quad n \geqq 2.$$

Remark. As in earlier theorems, we can deduce known results for various other classes and we omit details.

Theorem 4.3. *The class $\mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$ is a convex set.*

Proof. Let the function

$$f_j(z) = \sum_{n=2}^{\infty} a_{n,\,j} z^n, \quad a_{n,\,j} \geqq 0, \quad j = 1, 2\,, \tag{4.2}$$

be the class $\mathcal{UH}(\alpha,\, \beta,\, \gamma,\, \lambda,\, k)$. It sufficient to show that the function $g(z)$ defined by

$$g(z) = \mu f_1(z) + (1-\mu) f_2(z), \quad 0 \leqq \mu \leqq 1,$$

is in the class $\mathcal{UH}(\alpha,\, \beta,\, \gamma,\, \lambda,\, k)$. Since

$$g(z) = z - \sum_{n=2}^{\infty} [\mu a_{n,1} + (1-\mu) a_{n,2}] z^n,$$

an easy computation with the aid of Theorem 2.1 gives,

$$\sum_{n=2}^{\infty} \Big(n(\alpha+1) - (\alpha+\beta)\Big) \Big(\gamma(n-1)+1\Big) \Psi_n [\mu a_{n,1} + (1-\mu) a_{n,2}]$$

$$+(1-\mu) \sum_{n=2}^{\infty} (\, n(\alpha+1) - (\alpha+\beta))(\,\gamma(n-1)+1)\Psi_n$$

$$\leqq \mu(1-\beta) + (1-\mu)(1-\beta) \leqq 1-\beta,$$

which implies that $g \in \mathcal{UH}(\alpha,\, \beta,\, \gamma,\, \lambda,\, k)$. Hence $\mathcal{UH}(\alpha,\, \beta,\, \gamma,\, \lambda,\, k)$ is convex. □

5. Modified Hadamard products

For functions of the form (4.2), we define the modified Hadamard product as

$$(f_1 * f_2)(z) = z - \sum_{n=2}^{\infty} a_{n,\,1}\, a_{n,\,2} z^n. \tag{5.1}$$

Theorem 5.1. *If* $f_j(z) \in \mathcal{UH}(q, s, \lambda, \beta, k)$, $j = 1,\, 2,$ *then*

$$(f_1 * f_2)(z) \in \mathcal{UH}(q, s, \lambda, \beta, k,\, \xi),$$

where

$$\xi = \frac{(2-\beta)\,\big(\,2(\alpha+1) - (\alpha+\beta)\big)\,\big(\,\gamma+1\big)\,\Psi_2 - 2(1-\beta)^2}{(2-\beta)\,\big(\,2(\alpha+1) - (\alpha+\beta)\big)\,\big(\,\gamma+1\big)\,\Psi_2 - (1-\beta)^2},$$

with Ψ_n *be defined as in* (1.3).

Proof. Since $f_j(z) \in \mathcal{UH}(q, s, \lambda, \beta, k)$, $j = 1,\, 2$, we have

$$\sum_{n=2}^{\infty} \Big(n(\alpha+1) - (\alpha+\beta)\Big) \Big(\gamma(n-1)+1\Big) \Psi_n a_{n,\,j} \leqq 1-\beta, \quad j = 1,\, 2.$$

The Cauchy-Schwartz inequality leads to

$$\sum_{n=2}^{\infty} \frac{\big(\, n(\alpha+1) - (\alpha+\beta)\big)\,\big(\,\gamma(n-1)+1\big)\,\Psi_n a_{n,\,j}}{1-\beta} \sqrt{a_{n,\,1}\, a_{n,\,2}} \leqq 1. \tag{5.2}$$

Note that we need to find the largest ξ such that

$$\sum_{n=2}^{\infty} \frac{\big(n(k+1)-(k+\xi)\big)\big(\gamma(n-1)+1\big)\Psi_n a_{n,j}}{1-\xi} a_{n,1}\, a_{n,2} \leqq 1. \tag{5.3}$$

Therefore, in view of (5.2) and (5.3), whenever

$$\frac{n-\xi}{1-\xi}\sqrt{a_{n,1}\, a_{n,2}} \leqq \frac{n-\beta}{1-\beta},\ n \geqq 2$$

holds, then (5.3) is satisfied. We have, from (5.2),

$$\sqrt{a_{n,1}\, a_{n,2}} \leqq \frac{1-\beta}{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n},\ n \geqq 2. \tag{5.4}$$

Thus, if

$$\left(\frac{n-\xi}{1-\xi}\right)\left[\frac{1-\beta}{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n}\right] \leqq \frac{n-\beta}{1-\beta},\ n \geqq 2,$$

or, if

$$\xi \leqq \frac{(n-\beta)\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n - n(1-\beta)^2}{(n-\beta)\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n - (1-\beta)^2},\ n \geqq 2,$$

then (5.2) is satisfied. Note that the right hand side of the above expression is an increasing function on n. Hence, setting $n = 2$ in the above inequality gives the required result. Finally, by taking the function

$$f(z) = z - \frac{1-\beta}{(2-\beta)\big(2(\alpha+1)-(\alpha+\beta)\big)\big(\gamma+1\big)\Psi_2} z^2,$$

we see that the result is sharp. □

6. Radii of close-to-convexity, starlikeness and convexity

Theorem 6.1. *Let the function $f \in \mathcal{T}$ be in the class $\mathcal{UH}(q, s, \lambda, \beta, k)$. Then $f(z)$ is close-to-convex of order ρ, $0 \leqq \rho < 1$ in $|z| < r_1(\alpha, \beta, \gamma, \rho)$, where*

$$r_1(\alpha, \beta, \gamma, \rho) = \inf_n \left[\frac{(1-\rho)\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n}{n(1-\beta)}\right]^{\frac{1}{n-1}},$$

$n \geqq 2$, with Ψ_n be defined as in (1.3)*. This result is sharp for the function $f(z)$ given by* (2.2)*.*

Proof. It is sufficient to show that $|f'(z) - 1| \leqq 1-\rho$, $0 \leqq \rho < 1$, for $|z| < r_1(\alpha, \beta, \gamma, \rho)$, or equivalently

$$\sum_{n=2}^{\infty}\left(\frac{n}{1-\rho}\right) a_n |z|^{n-1} \leqq 1. \tag{6.1}$$

By Theorem 2.1, (6.1) will be true if

$$\left(\frac{n}{1-\rho}\right)|z|^{n-1} \leqq \frac{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n}{1-\beta}$$

or, if

$$|z| \leqq \left[\frac{(1-\rho)\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n}{n(1-\beta)}\right]^{\frac{1}{n-1}}. \tag{6.2}$$

The theorem follows easily from (6.2). □

Theorem 6.2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$. *Then* $f(z)$ *is starlike of order* ρ, $0 \leqq \rho < 1$ *in* $|z| < r_2(\alpha, \beta, \gamma, \rho)$, *where*

$$r_2(\alpha, \beta, \gamma, \rho) = \inf_n \left[\frac{(1-\rho)\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n}{(n-\rho)(1-\beta)}\right]^{\frac{1}{n-1}},$$

$n \geqq 2$, *with* Ψ_n *be defined as in* (1.3). *This result is sharp for the function* $f(z)$ *given by* (2.2).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqq 1-\rho, \text{ or equivalently } \sum_{n=2}^{\infty}\left(\frac{n-\rho}{1-\rho}\right)a_n|z|^{n-1} \leqq 1, \tag{6.3}$$

for $0 \leqq \rho < 1$, and $|z| < r_2(\alpha, \beta, \gamma, \rho)$. Proceeding as in Theorem 6.1, with the use of Theorem 2.1, we get the required result. □

Theorem 6.3. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$. *Then* $f(z)$ *is convex of order* ρ, $0 \leqq \rho < 1$ *in* $|z| < r_3(\alpha, \beta, \gamma, \rho)$, *where*

$$r_3(\alpha, \beta, \gamma, \rho) = \inf_n \left[\frac{(1-\rho)\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n}{n(n-\rho)(1-\beta)}\right]^{\frac{1}{n-1}},$$

$n \geqq 2$, *with* Ψ_n *be defined as in* (1.3). *This result is sharp for the function* $f(z)$ *given by* (2.2).

Proof. It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqq 1-\rho \quad \text{or equivalently} \quad \sum_{n=2}^{\infty}\left(\frac{n(n-\rho)}{1-\rho}\right)a_n|z|^{n-1} \leqq 1, \tag{6.4}$$

for $0 \leqq \rho < 1$ and $|z| < r_3(\alpha, \beta, \gamma, \rho)$. Proceeding as in Theorem 6.1, we get the required result. □

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The double Orlicz sequence spaces $\chi^2_M(p)$ and $\Lambda^2_M(p)$

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Abstract. In this paper, we introduce two general double sequence spaces $\chi^2_M(p)$ and $\Lambda^2_M(p)$ using Orlicz functions. We establish some inclusion relations, topological results and we characterize the duals of these double sequence spaces.

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1. Introduction

Throughout w, χ and Λ denote the classes of gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces are due to Bromwich [4]. Later on, the double sequence spaces were studied by Hardy [8], Moricz [12], Moricz and Rhoades [13], Basarir and Solankan [2], Tripathy [20], Colak and Turkmenoglu [6], Turkmenoglu [22], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : \sup_{m,n \in N} |x_{mn}|^{p_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n \to \infty} |x_{mn} - \ell|^{p_{mn}} = 1 \text{ for some } \ell \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n \to \infty} |x_{mn}|^{p_{mn}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{p_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t),$$

where $p = (p_{mn})$ is the sequence of strictly positive reals p_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $p_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gӧkhan and Colak [27, 28] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zeltser [29] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [30] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Next, Mursaleen [31] and Mursaleen and Edely [32] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [33] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Also Basar and Sever [34] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [35] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p. \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see [1]).

A sequence $x = (x_{mn})$ is said to be double analytic if

$$\sup_{mn} |x_{mn}|^{1/m+n} < \infty.$$

The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. By ϕ, we denote the set of all finite sequences.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only nonzero term is $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metric; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

Orlicz [16] used the idea of Orlicz function to construct the space $\left(L^M\right)$. Lindenstrauss and Tzafriri [10] investigated Orlicz sequence spaces in more details, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p\,(1 \le p < \infty)$. Subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [17], Mursaleen et al. [14], Bektas and Altin [3], Tripathy et al. [21], Rao and Subramanian [18], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [9].

Recalling [16] and [9], an Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex, with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [15] and further discussed by Ruckle [19] and Maddox [11], and many others.

An Orlicz function M is said to satisfy the Δ_2- condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \le KM(u)\,(u \ge 0)$. The Δ_2- condition is equivalent to $M(\ell u) \le K\ell M(u)$, for all values of u and for $\ell > 1$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\tfrac{|x_k|}{\rho}\right) \le 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \le p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

If X is a sequence space, we give the following definitions:

(i) $X' =$ the continuous dual of X;

(ii) $X^\alpha = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X\right\}$;

(iii) $X^\beta = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convegent, for each } x \in X\right\}$;

(iv) $X^\gamma = \left\{a = (a_{mn}) : \sup_{MN} \ge 1 \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, \text{ for each } x \in X\right\}$;

(v) let X be a FK-space $\supset \phi$; then $X^f = \left\{f(\Im_{mn}) : f \in X'\right\}$;

(vi) $X^{\delta} = \left\{ a = (a_{mn}) : \sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X \right\};$
$X^{\alpha}, X^{\beta}, X^{\gamma}$ and X^{δ} are called $\alpha-$ (or Köthe-Toeplitz) dual of X, $\beta-$ (or generalized-Köthe-Toeplitz) dual of X, $\gamma-$ dual of X, $\delta-$ dual of X respectively. X^{α} is defined by Gupta and Kamptan [24]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\alpha} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference spaces of single sequences was introduced by Kizmaz [36] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here w, c, c_0 and ℓ_{∞} denote the classes of all, convergent, null and bounded scalar valued single sequences respectively. The above difference spaces are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k| .$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\},$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

This paper deals with various duals namely α, β, γ, complete paranormed space of $\Lambda^2_M(p)$ and paranormed space of $\chi^2_M(p)$ using Orlicz fuctions.

2. Definitions and preliminaries

Throughout the paper w^2 denotes the spaces of all sequences. $\chi^2_M(p)$ and $\Lambda^2_M(p)$ denote the Pringscheim's sense of double Orlicz space of gai sequences and Pringscheim's sense of double Orlicz space of bounded sequences respectively.

Let w^2 denote the set of all complex double sequences $x = (x_{mn})_{m,n=1}^{\infty}$ and $M : [0, \infty) \to [0, \infty)$ be an Orlicz function, or a modulus function.

Given a double sequence, $x \in w^2$. If $p = (p_{mn})$ is a double sequence of strictly positive real numbers p_{mn} then we write

$$\chi^2_M(p) = \left\{ x \in w^2 : \left(M\left(\frac{((m+n)!\,|x_{mn}|)^{1/m+n}}{\rho} \right)^{p_{mn}} \right) \to 0 \right.$$
$$\left. \text{as } m, n \to \infty \text{ for some } \rho > 0 \right\}$$

and

$$\Lambda^2_M(p) = \left\{ x \in w^2 : \sup_{m,n \geq 1} \left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho} \right)^{p_{mn}} \right) < \infty \text{ for some } \rho > 0 \right\}.$$

The space $\Lambda^2_M(p)$ is a metric space with the metric

$$d(x,y)=\inf\left\{\rho>0:\sup_{m,n\geq 1}\left(M\left(\frac{|x_{mn}-y_{mn}|}{\rho}\right)\right)^{p_{mn}/m+n}\leq 1\right\}.$$

The space $\chi^2_M(p)$ is a metric space with the metric

$$d(x,y)=\inf\left\{\rho>0:\sup_{m,n\geq 1}\left(M\left(\frac{(m+n)!\,|x_{mn}-y_{mn}|}{\rho}\right)\right)^{p_{mn}/m+n}\leq 1\right\}.$$

Throughout the paper we write $\inf p_{mn}$, $\sup p_{mn}$ and $\sum_{mn}$ instead of $\inf_{m,n\geq 1}$, $\sup_{m,n\geq 1}$ and $\sum_{m,n=1}^{\infty}$ respectively.

3. Main results

Theorem 3.1. *For every $p=(p_{mn})$,*

$$\left[\Lambda^2_M(p)\right]^{\beta}=\left[\Lambda^2_M(p)\right]^{\alpha}=\left[\Lambda^2_M(p)\right]^{\gamma}=\eta^2_M(p),$$

where $\eta^2_M(p)=\bigcap_{N\in N-\{1\}}\left\{x=x_{mn}:\sum_{m,n}\left(M\left(\frac{|x_{mn}|N^{m+n/p_{mn}}}{\rho}\right)\right)<\infty\right\}.$

Proof. (1) First we show that $\eta^2_M(p)\subset\left[\Lambda^2_M(p)\right]^{\beta}$.

Let $x\in\eta^2_M(p)$ and $y\in\Lambda^2_M(p)$. Then we can find a positive integer N such that $\left(|y_{mn}|^{1/m+n}\right)^{p_{mn}}<max\left(1,\sup_{m,n\geq 1}\left(|y_{mn}|^{1/m+n}\right)^{p_{mn}}\right)<N$, for all m,n.

Hence we may write

$$\left|\sum_{m,n}x_{mn}y_{mn}\right|\leq\sum_{m,n}|x_{mn}y_{mn}|\leq\sum_{mn}\left(M\left(\frac{|x_{mn}y_{mn}|}{\rho}\right)\right)$$
$$\leq\sum_{m,n}\left(M\left(\frac{|x_{mn}|\,N^{m+n/p_{mn}}}{\rho}\right)\right).$$

Since $x\in\eta^2_M(p)$ the series on the right side of the above inequality is convergent, whence $x\in\left[\Lambda^2_M(p)\right]^{\beta}$. Hence $\eta^2_M(p)\subset\left[\Lambda^2_M(p)\right]^{\beta}$.

Now we show that $\left[\Lambda^2_M(p)\right]^{\beta}\subset\eta^2_M(p)$.

For this, let $x\in\left[\Lambda^2_M(p)\right]^{\beta}$, and suppose that $x\notin\Lambda^2_M(p)$. Then there exists a positive integer $N>1$ such that $\sum_{m,n}\left(M\left(\frac{|x_{mn}|N^{m+n/p_{mn}}}{\rho}\right)\right)=\infty$.

If we define $y_{mn}=N^{m+n/p_{mn}}Sgn x_{mn}\, m,n=1,2,\cdots$, then $y\in\Lambda^2_M(p)$.

But, since

$$\left|\sum_{m,n}x_{mn}y_{mn}\right|=\sum_{mn}\left(M\left(\frac{|x_{mn}y_{mn}|}{\rho}\right)\right)$$
$$=\sum_{m,n}\left(M\left(\frac{|x_{mn}|\,N^{m+n/p_{mn}}}{\rho}\right)\right)=\infty,$$

we get $x \notin \left[\Lambda_M^2(p)\right]^\beta$, which contradicts to the assumption $x \in \left[\Lambda_M^2(p)\right]^\beta$. Therefore $x \in \eta_M^2(p)$. Hence $\left[\Lambda_M^2(p)\right]^\beta = \eta_M^2(p)$.

(ii) and (iii) can be shown in a similar way with (i). □

Theorem 3.2. *Let $p = (p_{mn})$ be an analytic double sequence of strictly positive real numbers p_{mn}. Then*

(i) $\Lambda_M^2(p)$ is a paranormed space with

$$g(x) = \sup_{m,n\geq 1}\left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)$$

if and only if $h = \inf p_{mn} > 0$, where $M = max(1, H)$ and $H = \sup p_{mn}$.

(ii) $\Lambda_M^2(p)$ is a complete paranormed linear metric space if the condition p in (i) is satisfied.

Proof. (i) **Sufficiency.** Let $h > 0$. It is trivial that $g(\theta) = 0$ and $g(-x) = g(x)$. The inequality $g(x+y) \leq g(x) + g(y)$ follows from the inequality (1.1), since $p_{mn}/M \leq 1$ for all positive integers m, n. We also may write $g(\lambda x) \leq max\left(|\lambda|, |\lambda|^{h/M}\right) g(x)$, since $|\lambda|^{p_{mn}} \leq max\left(|\lambda|^h, |\lambda|^M\right)$ for all positive integers m, n and for any $\lambda \in C$, the set of complex numbers. Using this inequality, it can be proved that $\lambda x \to \theta$, when x is fixed and $\lambda \to 0$, or $\lambda \to 0$ and $x \to \theta$, or λ is fixed and $x \to \theta$.

Necessity. Let $\Lambda_M^2(p)$ be a paranormed space with the paranorm

$$g(x) = \sup_{m,n\geq 1}\left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)$$

and suppose that $h = 0$. Since $|\lambda|^{p_{mn}/M} \leq |\lambda|^{h/M} = 1$ for all positive integers m, n and $\lambda \in C$ such that $0 < |\lambda| \leq 1$, we have

$$\sup_{m,n\geq 1}\left(M\left(\frac{|\lambda|^{p_{mn}/M}}{\rho}\right)\right) = 1.$$

Hence it follows that $g(\lambda x) = \sup_{m,n\geq 1}\left(M\left(\frac{|\lambda|^{p_{mn}/M}}{\rho}\right)\right) = 1$ for $x = (\alpha) \in \Lambda_M^2(p)$ as $\lambda \to 0$. But this contradicts the assumption $\Lambda_M^2(p)$ is a paranormed space with $g(x)$.

(ii) The proof is clear. □

Corollary 3.3. *$\Lambda_M^2(p)$ is a complete paranormed space with the natural paranorm if and only if $\Lambda_M^2(p) = \Lambda_M^2$.*

Theorem 3.4. *Let*

$$N_1 = min\left\{n_0 : \sup_{m,n\geq n_0}\left(M\left(\frac{((m+n)!\,|x_{mn}|)^{1/m+n}}{\rho}\right)^{p_{mn}}\right) < \infty\right\},$$

$$N_2 = min\left\{n_0 : \sup_{m,n\geq n_0} p_{mn} < \infty\right\} \text{ and } N = max(N_1, N_2).$$

(i) $\chi_M^2(p)$ is a paranormed space with

$$g(x) = \lim_{N\to\infty} \sup_{m,n\geq N} \left(M\left(\frac{((m+n)!\,|x_{mn}|)^{1/m+n}}{\rho} \right)^{p_{mn}/M} \right) \tag{3.1}$$

if and only if $\mu > 0$, where

$$\mu = \lim_{N\to\infty} \inf_{m,n\geq N} p_{mn} \text{ and } M = max\left(1, \sup_{m,n\geq N} p_{mn}\right).$$

(ii) $\chi_M^2(p)$ is complete with the paranorm (3.1).

Proof. (i) **Necessity.** Let $\chi_M^2(p)$ be a paranormed space with (3.1) and suppose that $\mu = 0$.
Then $\alpha = \inf_{m,n\geq N} p_{mn} = 0$ for all $N \in \mathbb{N}$, and hence we obtain $g(\lambda x) = \lim_{N\to\infty} \sup_{m,n\geq N} |\lambda|^{p_{mn}/M} = 1$ for all $\lambda \in (0,1]$, where $x = (\alpha) \in \chi_M^2(p)$. Whence $\lambda \to 0$ does not imply $\lambda x \to \theta$, when x is fixed. But this contradicts (3.1) to be a paranorm.

Sufficiency. Let $\mu > 0$. It is trivial that $g(\theta) = 0, g(-x) = g(x)$ and $g(x+y) \leq g(x) + g(y)$. Since $\mu > 0$ there exists a positive number β such that $p_{mn} > \beta$ for sufficiently large positive integer m, n. Hence for any $\lambda \in \mathbb{C}$, we may write $|\lambda|^{p_{mn}} \leq max\left(|\lambda|^M, |\lambda|^\beta\right)$ for sufficiently large positive integers $m, n \geq N$. Therefore, we obtain that $g(\lambda x) \leq max\left(|\lambda|, |\lambda|^{\beta/M}\right) g(x)$ using this, one can prove that $\lambda x \to \theta$, whenever x is fixed and $\lambda \to 0$, or $\lambda \to 0$ and $x \to \theta$, or λ is fixed and $x \to \theta$.

(ii) Let $\left(x^{k\ell}\right)$ be a Cauchy sequence in $\chi_M^2(p)$, where

$$x^{k\ell} = \left(x_{mn}^{k\ell}\right)_{mn\in N}.$$

Then for every $\epsilon > 0\,(0 < \epsilon < 1)$ there exists a positive integer s_0 such that

$$g\left(x^{k\ell} - x^{rt}\right) = \lim_{N\to\infty} \sup_{m,n\geq N} \left(M\left(\frac{\left((m+n)!\left|x_{mn}^{k\ell} - x_{mn}^{rt}\right|\right)^{1/m+n}}{\rho} \right)^{p_{mn}/M} \right) < \epsilon/2 \text{ for all } k, \ell, r, t > s_0. \tag{3.2}$$

By (3.2) there exists a positive integer n_0 such that

$$\sup_{m,n\geq N} \left(M\left(\frac{\left((m+n)!\left|x_{mn}^{k\ell} - x_{mn}^{rt}\right|\right)^{1/m+n}}{\rho} \right)^{p_{mn}/M} \right) < \epsilon/2$$

for all $k, \ell, r, t > s_0$ and for $N > n_0$. Hence we obtain

$$\left(M\left(\frac{\left((m+n)!\left|x_{mn}^{k\ell} - x_{mn}^{rt}\right|\right)^{1/m+n}}{\rho} \right)^{p_{mn}/M} \right) < \epsilon/2 < 1 \tag{3.3}$$

so that

$$\left(M\left(\frac{\left((m+n)!\left|x_{mn}^{k\ell}-x_{mn}^{rt}\right|\right)^{1/m+n}}{\rho}\right)\right)$$

$$<\left(M\left(\frac{\left((m+n)!\left|x_{mn}^{k\ell}-x_{mn}^{rt}\right|\right)^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)<\epsilon/2$$

for all $k,\ell,r,t,>s_0$. This implies that $\left(x_{mn}^{k\ell}\right)_{k\ell\in N}$ is a Cauchy sequence in C for each fixed $m,n>n_0$. Hence the sequence $\left(x_{mn}^{k\ell}\right)_{k\ell\in N}$ is convergent to x_{mn} say,

$$\lim_{k,\ell\to\infty} x_{mn}^{k\ell}=x_{mn} \text{ for each fixed } m,n>n_0 \tag{3.4}$$

Getting x_{mn}, we define $x=(x_{mn})$. From (3.3) we obtain

$$g\left(x^{k\ell}-x\right)$$

$$=\lim_{N\to\infty}\sup_{m,n\geq N}\left(M\left(\frac{\left((m+n)!\left|x_{mn}^{k\ell}-x_{mn}\right|\right)^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)<\epsilon/2 \tag{3.5}$$

as $r,t\to\infty$, for $k,\ell>s_0$ by (3.5). This implies that $\lim_{k\ell\to\infty} x^{k\ell}=x$.

Now we show that $x=(x_{mn})\in\chi_M^2(p)$. Since $x^{k\ell}\in\chi_M^2(p)$ for each $(k,1)\in N\times N$ for every $\epsilon>0\,(0<\epsilon<1)$ there exists a positive integer $n_1\in N$ such that

$$\left(M\left(\frac{\left((m+n)!\left|x_{mn}^{k\ell}\right|\right)^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)<\epsilon/2 \text{ for every } m,n>n_1. \tag{3.6}$$

By (3.5) and (3.6) and (3.1) we obtain

$$\left(M\left(\frac{\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)$$

$$\leq\left(M\left(\frac{\left((m+n)!\left|x_{mn}^{k\ell}\right|\right)^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)$$

$$+\left(M\left(\frac{\left((m+n)!\left|x_{mn}^{k\ell}-x_{mn}\right|\right)^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)$$

$$\leq\epsilon/2+\epsilon/2=\epsilon$$

for all $k,\ell>max\,(s_0,s_1)$ and $m,n>max\,(n_0,n_1)$. This implies that $x\in\chi_M^2(p)$. This completes the proof. □

Theorem 3.5. *For every $p=(p_{mn})$, then $\eta_M^2(p)\subset\left[\chi_M^2(p)\right]^\beta \overset{\subset}{\neq} \Lambda^2$.*

Proof. **Case 1.** First we show that $\eta^2_M(p) \subset \left[\chi^2_M(p)\right]^\beta$.
We know that $\chi^2(p) \subset \Lambda^2_M(p)$, $\left[\Lambda^2_M(p)\right]^\beta \subset \left[\chi^2_M(p)\right]^\beta$.
But $\left[\Lambda^2_M(p)\right]^\beta = \eta^2_M(p)$, by Theorem 3.1.
Therefore

$$\eta^2_M(p) \subset [\chi^2_M(p)]^\beta. \tag{3.7}$$

Case 2. Now we show that $\left[\chi^2_M(p)\right]^\beta \underset{\neq}{\overset{\subset}{}} \Lambda^2$.
Let $y = \{y_{mn}\}$ be an arbitrary point in $\left(\chi^2_M(p)\right)^\beta$. If y is not in Λ^2, then for each natural number q, we can find an index $m_q n_q$ such that

$$\left(M\left(\frac{\left((m_q+n_q)!\left|y_{m_q n_q}\right|\right)^{1/m_q+n_q}}{\rho}\right)\right)^{p_{mn}} > q, (1,2,3,\cdots).$$

Define $x = \{x_{mn}\}$ by $\left(M\left(\frac{(m+n)!x_{mn}}{\rho}\right)\right)^{p_{mn}} = \frac{1}{q^{m+n}}$ for $(m,n) = (m_q, n_q)$ for some $q \in \mathbb{N}$; and $\left(M\left(\frac{(m+n)!x_{mn}}{\rho}\right)\right)^{p_{mn}} = 0$ otherwise.
Then x is in $\chi^2_M(p)$, but for infinitely mn,

$$\left(M\left(\frac{(m+n)!\left|y_{mn}x_{mn}\right|}{\rho}\right)\right)^{p_{mn}} > 1. \tag{3.8}$$

Consider the sequence $z = \{z_{mn}\}$, where

$$\left(M\left(\frac{2!z_{11}}{\rho}\right)\right)^{p_{mn}} = \left(M\left(\frac{2!x_{11}}{\rho}\right)\right)^{p_{mn}} - s$$

with

$$s = \sum \left(M\left(\frac{(m+n)!x_{mn}}{\rho}\right)\right)^{p_{mn}};$$

and

$$\left(M\left(\frac{(m+n)!z_{mn}}{\rho}\right)\right)^{p_{mn}} = \left(M\left(\frac{(m+n)!x_{mn}}{\rho}\right)\right)^{p_{mn}} (m,n = 1,2,3,\cdots).$$

Then z is a point of $\chi^2_M(p)$. Also $\sum \left(M\left(\frac{(m+n)!z_{mn}}{\rho}\right)\right)^{p_{mn}} = 0$. Hence z is in $\chi^2_M(p)$.
But, by the equation (3.8), $\sum \left(M\left(\frac{(m+n)!z_{mn}y_{mn}}{\rho}\right)\right)^{p_{mn}}$ does not converge.
$\Rightarrow \sum (m+n)!x_{mn}y_{mn}$ diverges.
Thus the sequence y would not be in $\left(\chi^2_M(p)\right)^\beta$. This contradiction proves that

$$\left(\chi^2_M(p)\right)^\beta \subset \Lambda^2. \tag{3.9}$$

If we now choose $p = (p_{mn})$ constant, $M = id$, where id is the identity and $(1+n)!y_{1n} = (1+n)!x_{1n} = 1$ and $(m+n)!y_{mn} = (m+n)!x_{mn} = 0$ $(m > 1)$ for all n, then obviously $x \in \chi^2_M(p)$ and $y \in \Lambda^2$, but

$$\sum_{m,n=1}^{\infty} (m+n)!x_{mn}y_{mn} = \infty,$$

hence

$$y \notin \left(\chi^2_M(p)\right)^{\beta}. \tag{3.10}$$

From (3.9) and (3.10) we are granted

$$\left(\chi^2_M(p)\right)^{\beta} \overset{\subset}{\neq} \Lambda^2. \tag{3.11}$$

Hence (3.7) and (3.11) we are granted $\eta^2_M(p) \subset \left[\chi^2_M(p)\right]^{\beta} \overset{\subset}{\neq} \Lambda^2$. This completes the proof. □

Theorem 3.6. *Let M be an Orlicz function or modulus function which satisfies the Δ_2-condition. Then $\chi^2(p) \subset \chi^2_M(p)$.*

Proof. Let

$$x \in \chi^2(p). \tag{3.12}$$

Then $\left(((m+n)!\,|x_{mn}|)^{1/m+n}\right)^{p_{mn}} \leq \epsilon$ for sufficiently large m, n and every $\epsilon > 0$.
But then by taking $\rho \geq 1/2$,

$$\left(M\left(\frac{((m+n)!\,|x_{mn}|)^{1/m+n}}{\rho}\right)\right)^{p_{mn}} \leq \left(M\left(\frac{\epsilon}{\rho}\right)\right)$$

(because M is non-decreasing)

$$\leq (M(2\epsilon))$$

$$\Rightarrow \left(M\left(\frac{((m+n)!\,|x_{mn}|)^{1/m+n}}{\rho}\right)\right)^{p_{mn}} \leq KM(\epsilon)$$

(by the Δ_2- condition, for some $k > 0$)

$$\leq \epsilon$$

(by defining $M(\epsilon) < \epsilon/K$)

$$\left(M\left(\frac{((m+n)!\,|x_{mn}|)^{1/m+n}}{\rho}\right)\right)^{p_{mn}} \to 0 \text{ as } m, n \to \infty. \tag{3.13}$$

Hence

$$x \in \chi^2_M(p). \tag{3.14}$$

From (3.12) and (3.14) we get $\chi^2(p) \subset \chi^2_M(p)$. This completes the proof. □

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$\alpha-$tauberian results

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Abstract. In this paper we consider problems that are analoguous to those on summability $(C,1)$ introduced and studied by Hardy. A series $\sum_n x_n$ is said to be summable $(C,1)$ (to sum $S\in\mathbb{C}$) if the sequence $n^{-1}\sum_{k=1}^{n}s_k$ where $s_k=\sum_{i=1}^{k}x_i$ tends to S. Here we extend the Hardy's *tauberian* theorem for *Cesàro means* where it is shown that if the sequence $(x_n)_n$ satisfies $\sup_n\{n\,|x_n-x_{n-1}|\}<\infty$, then $n^{-1}s_n\to\chi$ implies $x_n\to\chi$ for some $\chi\in\mathbb{C}$. In this work, for given sequences λ and μ, we give $\alpha-$tauberian theorems which consists in determining the set of all sequences α such that

$$\frac{1}{\lambda_n}\sum_{k=1}^{n}\mu_k\left(\sum_{i=k}^{\infty}x_i\right)\to l \text{ implies } \frac{x_n}{\alpha_n}\to l' \ \ (n\to\infty)$$

for all $X\in cs$? Then we give simplifications of these theorems in the cases when $\alpha\in\widehat{C_1}$, and $\alpha\in\widehat{\Gamma}$. Finally we deal with the converse of the last condition.

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1. Introduction

In this paper we study problems that are similar to those stated by Hardy [6], Móricz and Rhoades, (cf. [10]), de Malafosse and Rakočević (cf. [5]). In [6] it is said that a series $\sum_{k=1}^{\infty}x_k$ is *summable* $(C,1)$ (to sum $l\in\mathbb{C}$) if

$$\chi_n=\frac{1}{n}\sum_{k=1}^{n}s_k\to l$$

where $s_k=\sum_{i=1}^{k}x_i$. It was shown (cf. [6, p. 132, Theorem 77]) that if a series $\sum_{k=1}^{\infty}x_k$ is *summable* $(C,1)$ to sum S if and only if

$$S=\sum_{k=1}^{\infty}\left(\sum_{i=k}^{\infty}\frac{x_i}{i}\right). \tag{1.1}$$

Móricz and Rhoades gave a generalization of the Hardy theorem using the *weighted mean matrix* $\overline{N}$, (cf. [10, 11]). In de Malafosse and Rakočević (cf. [5]) the series $\sum_{k=1}^{\infty} x_k$ is said to be *summable* (C, λ, μ) (to sum $L \in \mathbb{C}$) for given sequences λ and μ if

$$\chi'_n = \frac{1}{\lambda_n} \sum_{k=1}^{n} \frac{1}{\mu_k} s_k \to L.$$

When $\lambda_n = n$ and $\mu_n = 1$ for all n, summability (C, λ, μ) reduces to summability $(C, 1)$. In the following we extend Hardy's *tauberian* theorem for *Cesàro means* where it is shown that if the sequence $X = (x_n)_n$ satisfies $\sup_n \{n |x_n - x_{n-1}|\} < \infty$, then $n^{-1} s_n \to \chi$ implies $x_n \to \chi$ for some $\chi \in \mathbb{C}$. In this way for given sequences λ and μ we determine the set of all the sequences α such that

$$\frac{1}{\lambda_n} \sum_{k=1}^{n} \mu_k \left(\sum_{i=k}^{\infty} x_i \right) \to l \text{ implies } \frac{x_n}{\alpha_n} \to l' \ \ (n \to \infty) \text{ for all } X \in cs$$

for some l, $l' \in \mathbb{C}$. This statement is called an $\alpha-$tauberian problem. The main result is given by Theorem 4.3.

This paper is organized as follows. In Section 2 we recall some results on the sets of sequences and matrix transformations. In Section 3 we give some properties of the operator Σ^+ defined by $[\Sigma^+ X]_n = \sum_{k=n}^{\infty} x_k$ for all n, on special sets of sequences. In Section 4 we state some α-tauberian theorems in the general case and in the case when $\lambda_n = n$ and $\mu_n = n^{\xi}$ where ξ is a real. Then we give simplifications of α-tauberian theorems when α belongs to special sets of sequences such as $\widehat{C_1}$, or $\widehat{\Gamma}$. Finally we deal with the converse of the previous tauberian results.

2. Preliminary results

In the following we write $A = (a_{nk})_{n,k \geq 1}$ for an infinite matrix of complex numbers. For a given sequence $X = (x_n)_{n \geq 1}$ of complex numbers we define $A_n(X) = \sum_{k=1}^{\infty} a_{nk} x_k$, (provided the series $A_n(X)$ converge) and $AX = (\sum_{k=1}^{\infty} a_{nk} x_k)_{n \geq 1}$. We write s, ℓ_∞, c_0 and c for the sets of all complex, bounded, naught and convergent sequences, respectively, furthermore cs is the set of all convergent series. For E, $F \subset s$, we write (E, F) for the *set of all matrix transformations* that map E to F. For given $\tau \in s$ we define $D_\tau = (\tau_n \delta_{nk})_{n,k \geq 1}$, (where $\delta_{nn} = 1$ for all n and $\delta_{nk} = 0$ otherwise). We define by U^+ the set of all sequences $(u_n)_{n \geq 1} \in s$ with $u_n > 0$ for all n and consider the spaces $s_\alpha = D_\alpha \ell_\infty$, $s_\alpha^0 = D_\alpha c_0$ and $s_\alpha^{(c)} = D_\alpha c$ for $\alpha \in U^+$, see [2, 3]. It can easily be seen that for α, $\beta \in U^+$ and E, $F \subset s$ we have $A \in (D_\alpha E, D_\beta F)$ if and only if $D_{1/\beta} A D_\alpha \in (E, F)$. If $e = (1, 1, ...)$ we put $s_1 = s_e$. Let E and F be any subsets of s. It is well known, see [1] that $(s_1, s_1) = (c_0, s_1) = (c, s_1) = S_1$, where S_1 is the set of all infinite matrices $A = (a_{nk})_{n,k \geq 1}$ such that $\sup_n (\sum_{k=1}^{\infty} |a_{nk}|) < \infty$. For any subset E of s, AE

is the set of all sequences Y such that $Y = AX$ for some $X \in E$. For any subset F of s, the matrix domain $F(A) = F_A$ of A is the set of all sequences X such that $AX \in F$.

In this paper we consider the operators represented by the infinite matrices $C(\lambda)$ and $\Delta(\lambda)$ for $\lambda \in U^+$, see [3]. Recall that $[C(\lambda)]_{n,k} = 1/\lambda_n$ for $k \leq n$ and 0 otherwise. In the following we will use the convention that any term with nonpositive subscript is equal to zero. It can be proved that the matrix $\Delta(\lambda)$ defined by $[\Delta(\lambda)]_{nn} = \lambda_n$, $[\Delta(\lambda)]_{n,n-1} = -\lambda_{n-1}$ and $[\Delta(\lambda)]_{nk} = 0$ for $k \neq n-1$, n, $n \geq 1$, is the inverse of $C(\lambda)$. If $\lambda = e$ we get the well-known operator of the first difference represented by $\Delta(e) = \Delta$ and it is usually written $\Sigma = C(e)$. We have $[\Delta X]_n = x_n - x_{n-1}$ for all $n \geq 1$. Then $\Delta = \Sigma^{-1}$ and Δ, $\Sigma \in S_R = \left(s_{(R^n)_n}, s_{(R^n)_n}\right)$ for $R > 1$. We also use the transpose of $C(\lambda)$ denoted by $C^+(\lambda)$. We easily see that $C^+(\lambda) = \Sigma^+ D_{1/\lambda}$ where Σ^+ is the transpose of Σ.

3. Some properties of the infinite matrix Σ^+ considered as operator in s_α, s_α^0, or $s_\alpha^{(c)}$

In this section we are interested in the study of the set of all sequences X such that

$$\frac{1}{\lambda_n}\sum_{k=1}^{n}\mu_k r_k \to l \text{ for some } l \in \mathbb{C},$$

where $r_k = \sum_{i=k}^{\infty} x_i$.

In the following we will use the characterizations of the sets (E, F), where E, F are either of the sets c or c_0.

We will consider the next conditions

$$A \in S_1, \tag{3.1}$$

$$\lim_{n\to\infty} a_{nk} = l_k \text{ for some } l_k \in \mathbb{C} \text{ and for all } k. \tag{3.2}$$

From [9, Theorem 1.36, p. 160] we immediately deduce the next lemma.

Lemma 3.1. *i) $A \in (c_0, c_0)$ if and only if (3.1) and (3.2) hold with $l_k = 0$;*

ii) $A \in (c, c_0)$ if and only if (3.1), (3.2) hold with $l_k = 0$ and

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk} = 0.$$

iii) $A \in (c_0, c)$ if and only if (3.1) and (3.2) hold;

iv) a) $A \in (c, c)$ if and only if (3.1), (3.2) hold and

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk} = l \textit{ for some } l \in \mathbb{C}. \tag{3.3}$$

b) Let $A \in (c, c)$ and $x \in c$. If (3.3) and (3.2) hold with $l_k = 0$, then

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} x_k = l \lim_{n\to\infty} x_n.$$

Note that the statements given in iv) are direct consequences of Silverman Toeplitz theorem.

We will use the next lemma where $T = (t_{nk})_{n,k\geq 1}$ is called a lower triangular matrix if $t_{nk} = 0$ for $k > n$.

Lemma 3.2. *Let $A = (a_{nk})_{n,k\geq 1}$ be an infinite matrix and T a lower triangular matrix. Then*

$$T(AX) = (TA)X \text{ for all } X \in s(A).$$

Proof. Since $X \in s(A)$ the series $\sum_{k=1}^{\infty} a_{nk} x_k$ is convergent for all n. Then

$$[T(AX)]_n = \sum_{m=1}^{n} t_{nm} \left(\sum_{k=1}^{\infty} a_{mk} x_k \right) = \sum_{k=1}^{\infty} \left(\sum_{m=1}^{n} t_{nm} a_{mk} \right) x_k = [(TA)X]_n$$

for all n and for all $X \in s(A)$. □

In all that follows we use the operator represented by the infinite matrix Σ^+. For the convenience to the reader we note that

$$\Sigma^+ = \begin{pmatrix} 1 & 1 & . & . \\ & 1 & 1 & . \\ \mathbf{0} & & . & . \\ & & & . \end{pmatrix}.$$

We use the following results where Δ^+ is the transpose of Δ.

Lemma 3.3. *i) $\Sigma^+(\Delta^+ X) = X$ for all $X \in c_0$ and $\Delta^+(\Sigma^+ X) = X$ for all $X \in cs$,*

ii) the operator Σ^+ is bijective from cs to c_0 and Δ^+ is bijective from c_0 to cs.

Proof. i) comes from [1, Lemma 3, p. 19]. ii) is a direct consequence of i). □

Lemma 3.2 and Lemma 3.3 lead to define the product $T\Sigma^+$ by $(T\Sigma^+)X = T(\Sigma^+ X)$ for all $X \in cs$ where T is a triangle, that is a lower triangle with $[T]_{nn} \neq 0$ for all n. We note that T is bijective from s to itself and that T^{-1} is again a triangle matrix. In this way we have

Lemma 3.4. *Let T be a triangle, then $T\Sigma^+ \in (cs, Tc_0)$ is bijective and*

$$\left(T\Sigma^+\right)^{-1} = \Delta^+ T^{-1}.$$

Proof. Let $B \in Tc_0$ and consider the equation

$$\left(T\Sigma^+\right) X = B \text{ for } X \in cs. \tag{3.4}$$

Since

$$\left(T\Sigma^+\right) X = T\left(\Sigma^+ X\right) \text{ for all } X \in cs,$$

and $T : c_0 \to Tc_0$ is bijective, equation (3.4) is equivalent to $\Sigma^+ X = T^{-1}B$. Then since $T^{-1}B \in c_0$ and Σ^+ is bijective from cs to c_0, we deduce $T\Sigma^+$ is bijective and (3.4) has a unique solution given by $X = (T\Sigma^+)^{-1} B = \Delta^+ \left(T^{-1}B\right)$. Finally by Lemma 2 it can easily be seen that $X = \Delta^+ \left(T^{-1}B\right) = \left(\Delta^+ T^{-1}\right) B$ for all B and $(T\Sigma^+)^{-1} = \Delta^+ T^{-1}$. □

Let λ, $\mu \in U^+$. In the following we will use the notation $\sigma_n = \sum_{k=1}^n \mu_k$ and define the map

$$\phi_n (X) = \frac{1}{\lambda_n} \left(\sum_{k=1}^n \sigma_k x_k + \sigma_n r_{n+1} \right) \text{ for all } X \in cs \text{ and } n \geq 1.$$

Let us state the next result where $\mathbb{R}^{+*}$ is the set of all reals > 0.

Theorem 3.5. *Let E be a set of sequences.*

i) $c_0 \subset E$ implies $E(\Sigma^+) = cs$;

ii) a) $E(\Sigma^+) \subset \Delta^+ E$;

b) $E \subset c_0$ implies that $E(\Sigma^+) = \Delta^+ E$;

iii) $c_0 (\Sigma^+) \subset s_\alpha^{(c)}$ if and only if $1/\alpha \in \ell_\infty$.

iv) a) Let E be either of the sets s_α, s_α^0, or $s_\alpha^{(c)}$. Then

$$E\left(\Sigma^+\right) = cs \text{ if and only if } 1/\alpha \in \ell_\infty.$$

b) $c_0 (\Sigma^+) = c(\Sigma) = \Delta^+ c_0 = cs$.

v) a) Assume that

$$\sigma/\lambda \in l_\infty \text{ and } \lambda_n \to \infty \ (n \to \infty). \tag{3.5}$$

Then

$$c_0 \left(C(\lambda) D_\mu \Sigma^+\right) = cs,$$

and

$$\frac{1}{\lambda_n} \sum_{k=1}^n \mu_k r_k \to 0 \text{ for all } X \in cs. \tag{3.6}$$

b) The condition $\sup_n (n/\lambda_n) < \infty$ implies $c_0 (C(\lambda) \Sigma^+) = cs$.

vi) Assume that

$$\sigma/\lambda \in \ell_\infty \text{ and } \lambda_n \to l \ (n \to \infty) \text{ for some } l \in \mathbb{R}^{+*} \bigcup \{+\infty\}. \tag{3.7}$$

Then

$$c\left(C(\lambda) D_\mu \Sigma^+\right) = cs,$$

and

$$\frac{1}{\lambda_n} \sum_{k=1}^n \mu_k r_k \to L_X \text{ for some } L_X \in \mathbb{C} \text{ and for all } X \in cs.$$

Proof. i) Necessity. Let $X \in E(\Sigma^+)$. Then $\Sigma^+ X$ exists and $X \in cs$, so $E(\Sigma^+) \subset cs$. Sufficiency. Let $X \in cs$. Then the series $\sum_{k=n}^\infty x_k$ are convergent for all n and $\Sigma^+ X \in c_0$, but the inclusion $c_0 \subset E$ implies $\Sigma^+ X \in E$ and $X \in E(\Sigma^+)$. So we have shown $cs \subset E(\Sigma^+)$. We conclude $E(\Sigma^+) = cs$.

ii) a) If $E(\Sigma^+) = \emptyset$ trivially we have $E(\Sigma^+) \subset \Delta^+ E$. Now assume $E(\Sigma^+) \neq \emptyset$ and let $X \in E(\Sigma^+)$. Then $Y = \Sigma^+ X$ exists, $\Sigma^+ X \in E$ and $X \in cs$. Since $cs \subset c_0$, we have by Lemma 3.3

$$\Delta^+ \left(\Sigma^+ X\right) = \Delta^+ Y = X.$$

We conclude that $X \in \Delta^+ E$ and $E(\Sigma^+) \subset \Delta^+ E$. b) We show that $E \subset c_0$ implies $E(\Sigma^+) \supset \Delta^+ E$. For every $X \in E$ we have $Y = \Delta^+ X \in \Delta^+ E$ and from Lemma 3.3 we have $\Sigma^+ Y = \Sigma^+ (\Delta^+ X) = X$ since $X \in E \subset c_0$. Then $\Sigma^+ Y = X \in E$ and $Y \in E(\Sigma^+)$. So we have shown $\Delta^+ E \subset E(\Sigma^+)$. This result and a) imply b).

iii) Assume $c_0(\Sigma^+) \subset s_\alpha^{(c)}$. Then $I \in \left(c_0(\Sigma^+), s_\alpha^{(c)}\right)$ and since $c_0(\Sigma^+) = c(\Sigma)$ we deduce $\Delta \in \left(c, s_\alpha^{(c)}\right)$, $D_{1/\alpha}\Delta \in (c, c)$ and $1/\alpha \in \ell_\infty$.

iv) a) Using i) we see that it is enough to show that $c_0 \subset E$ if and only if $1/\alpha \in \ell_\infty$ for $E = s_\alpha$, s_α^0, or $s_\alpha^{(c)}$. We have $c_0 \subset s_\alpha$ if and only if $I \in (c_0, s_\alpha)$, that is $D_{1/\alpha} \in (c_0, s_1) = S_1$ and $1/\alpha \in \ell_\infty$. In the same way using the characterizations of (c_0, c_0) and (c_0, c) we deduce $c_0 \subset E$ if and only if $1/\alpha \in \ell_\infty$ for $E = s_\alpha^0$, or $s_\alpha^{(c)}$.

b) Let $X \in c_0(\Sigma^+)$. Then $\Sigma^+ X \in c_0$, $X \in cs$ and so $c_0(\Sigma^+) \subset cs$. Now $X \in cs$ implies $\Sigma^+ X = \left(\sum_{k=n}^\infty x_k\right)_{n\geq 1} \in c_0$ since $\sum_{k=n}^\infty x_k \to 0$ $(n \to \infty)$ and $X \in c_0(\Sigma^+)$. This shows $cs \subset c_0(\Sigma^+)$ and as we have just shown $c_0(\Sigma^+) \subset cs$, so $c_0(\Sigma^+) = cs$. Finally by ii) b) we have $c_0(\Sigma^+) = \Delta^+ c_0$.

v) a) We show $cs \subset c_0(C(\lambda) D_\mu \Sigma^+)$. By Lemma 3.2 we have

$$C(\lambda) D_\mu \left(\Sigma^+ X\right) = \left(C(\lambda) D_\mu \Sigma^+\right) X \quad \text{for all } X \in cs \tag{3.8}$$

since $C(\lambda) D_\mu$ is a triangle and $X \in s(\Sigma^+) = cs$. Now for every $X \in cs$ we have $\Sigma^+ X \in c_0$ and since (3.5) holds we have $C(\lambda) D_\mu \in (c_0, c_0)$ and then $C(\lambda) D_\mu (\Sigma^+ X) \in c_0$ for all $X \in cs$. Finally since (3.8) holds we conclude $(C(\lambda) D_\mu \Sigma^+) X \in c_0$ for all $X \in cs$ and $cs \subset c_0(C(\lambda) D_\mu \Sigma^+)$.

Conversely let $X \in c_0(C(\lambda) D_\mu \Sigma^+)$. By elementary calculations we easily get

$$C(\lambda) D_\mu \Sigma^+ = \begin{pmatrix} \sigma_1/\lambda_1 & . & . & . & . & \sigma_1/\lambda_1 & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \sigma_1/\lambda_n & \sigma_2/\lambda_n & . & \sigma_n/\lambda_n & . & \sigma_n/\lambda_n & . \\ . & . & . & . & . & . & . \end{pmatrix} \tag{3.9}$$

that is

$$\left[C(\lambda) D_\mu \Sigma^+\right]_{nk} = \begin{cases} \sigma_k/\lambda_n \text{ for } k < n, \\ \sigma_n/\lambda_n \text{ for } k \geq n. \end{cases}$$

We deduce

$$\left(C(\lambda) D_\mu \Sigma^+\right) X = (\phi_n(X))_{n\geq 1} \in c_0.$$

Then the series $r_n = \sum_{k=n}^{\infty} x_k$ is convergent for all n and $X \in cs$. This shows $c_0\left(C\left(\lambda\right)D_\mu\Sigma^+\right) \subset cs$. We conclude $c_0\left(C\left(\lambda\right)D_\mu\Sigma^+\right) = cs$. Since

$$\left[C\left(\lambda\right)D_\mu\left(\Sigma^+ X\right)\right]_n = \frac{1}{\lambda_n}\sum_{k=1}^{n}\mu_k r_k \text{ for all } n,$$

statement (3.6) comes from identity (3.8).

v) b) is a direct consequence of v) a) where we put $\mu = e$, furthermore condition $\sup_n\left(n/\lambda_n\right) < \infty$ trivially implies $\lambda_n \to \infty$ $(n \to \infty)$.

vi) can be obtained reasoning as in v) a) by using the characterization of (c_0, c). □

4. $\alpha-$tauberian results

4.1. General case

For given λ, $\mu \in U^+$ the aim of this paper is to determine the set of all sequences $\alpha \in U^+$ such that

$$\frac{1}{\lambda_n}\sum_{k=1}^{n}\mu_k\left(\sum_{j=k}^{\infty}x_j\right) \to l \text{ implies } \frac{x_n}{\alpha_n} \to l' \ \ (n \to \infty) \text{ for all } X \in cs, \qquad (4.1)$$

for some l, $l' \in \mathbb{C}$.

Now state a lemma which is a characterization of condition (4.1).

Lemma 4.1. *For λ, μ, $\alpha \in U^+$ condition (4.1) holds if and only if*

$$\Delta^+ D_{1/\mu}\Delta\left(\lambda\right) \in \left(c\bigcap C\left(\lambda\right)D_\mu c_0, s_\alpha^{(c)}\right). \qquad (4.2)$$

Proof. First condition (4.1) means that

$$C\left(\lambda\right)D_\mu\left(\Sigma^+ X\right) \in c \text{ implies } X \in s_\alpha^{(c)} \text{ for all } X \in cs. \qquad (4.3)$$

Since $\Sigma^+ X \in c_0$ for all $X \in cs$, condition (4.3) is equivalent to the statement

$$Y = C\left(\lambda\right)D_\mu\left(\Sigma^+ X\right) \in c\bigcap C\left(\lambda\right)D_\mu c_0 \text{ implies } X \in s_\alpha^{(c)}. \qquad (4.4)$$

Since $C\left(\lambda\right)D_\mu$ is a triangle and $\Sigma^+ \in (cs, c_0)$ by Lemma 3.2 we have

$$C\left(\lambda\right)D_\mu\left(\Sigma^+ X\right) = \left(C\left(\lambda\right)D_\mu\Sigma^+\right)X \text{ for all } X \in cs.$$

Then by Lemma 3.4 the operator $C\left(\lambda\right)D_\mu\Sigma^+ \in \left(cs, C\left(\lambda\right)s_\mu^0\right)$ is invertible and

$$\left(C\left(\lambda\right)D_\mu\Sigma^+\right)^{-1} = \Delta^+ D_{1/\mu}\Delta\left(\lambda\right),$$

we deduce $Y = C\left(\lambda\right)D_\mu\left(\Sigma^+ X\right)$ if and only if $X = \Delta^+ D_{1/\mu}\Delta\left(\lambda\right)Y$ for all $X \in cs$ and condition (4.4) is equivalent to

$$Y \in c\bigcap C\left(\lambda\right)D_\mu c_0 \text{ implies } X = \Delta^+ D_{1/\mu}\Delta\left(\lambda\right)Y \in s_\alpha^{(c)}$$

and to (4.2). □

To state the next results we need the next lemma.

Lemma 4.2. *Let κ and $\kappa' \in U^{+}$. Then conditions $\kappa + \kappa' \in \ell_\infty$ and $\kappa - \kappa' \in c$ together are equivalent to $\kappa \in \ell_\infty$ and $\kappa - \kappa' \in c$.*

Proof. First we have $\kappa + \kappa' \in \ell_\infty$ if and only if $\kappa, \kappa' \in \ell_\infty$. Then $\kappa - \kappa' \in c$ is equivalent to $\kappa_n = \kappa'_n + L + 0(1)$ $(n \to \infty)$, for some $L \in \mathbb{C}$, which shows that κ is bounded if and only if κ' is bounded. This gives the conclusion. □

In this way it can be easily seen that conditions $\kappa + \kappa' \in \ell_\infty$ and $\kappa - \kappa' \in c$ together are equivalent to $\kappa' \in \ell_\infty$ and $\kappa - \kappa' \in c$.

Now consider the next conditions

$$\frac{1}{\alpha_n}\left(\frac{\lambda_{n-1}}{\mu_n} + \frac{\lambda_{n+1}}{\mu_{n+1}}\right) = O(1) \quad (n \to \infty) \tag{4.5}$$

$$\lim_{n\to\infty}\left\{\frac{1}{\alpha_n}\left[-\frac{\lambda_{n-1}}{\mu_n} + \lambda_n\left(\frac{1}{\mu_n} + \frac{1}{\mu_{n+1}}\right) - \frac{\lambda_{n+1}}{\mu_{n+1}}\right]\right\} = L \text{ for some } L \in \mathbb{C} \tag{4.6}$$

We obtain the following $\alpha-$tauberian theorem.

Theorem 4.3. *Let $\lambda, \mu \in U^{+}$. Then*

i) condition (4.1) holds if α satisfies one of the conditions a) or b), where

a) $1/\alpha \in \ell_\infty$,

b) conditions (4.5) and (4.6) hold.

ii) If there is $L \in \mathbb{R}^{+*} \bigcup \{+\infty\}$ such that

$$\sigma/\lambda \in \ell_\infty \text{ and } \lambda_n \to L \quad (n \to \infty) \tag{4.7}$$

then condition (4.1) holds if and only if $1/\alpha \in \ell_\infty$.

iii) If $(-\lambda_{n-1} + \lambda_n)/\mu_n \to 0$ $(n \to \infty)$ and there is $K' > 0$ such that

$$\frac{\lambda_{n-1} + \lambda_n}{\mu_n} \le K' \text{ for all } n \ge 1 \tag{4.8}$$

then condition (4.1) holds if and only if (4.5) and (4.6) hold.

Proof. i) First we show that a) implies (4.1). Assume $1/\alpha \in \ell_\infty$. Then the condition

$$\frac{1}{\lambda_n}\sum_{k=1}^{n}\mu_k r_k \to l$$

necessary implies $X \in cs$. Then trivially $X \in c_0$ and $(1/\alpha_n)\,x_n \to 0$ $(n \to \infty)$. So we have shown a) implies (4.1).

Next we show that b) implies (4.1). Since trivially $c \bigcap C(\lambda) D_\mu c_0 \subset c$ we have $\left(c, s_\alpha^{(c)}\right) \subset \left(c \bigcap C(\lambda) D_\mu c_0, s_\alpha^{(c)}\right)$. We show that we have

$$\widetilde{\Delta} = \Delta^{+} D_{1/\mu} \Delta(\lambda) \in \left(c, s_\alpha^{(c)}\right) \tag{4.9}$$

which implies (4.2) and (4.1) by Lemma 4.1. Now the calculations of $D_{1/\mu}\Delta(\lambda)$ and $\widetilde{\Delta}$ successively give

$$D_{1/\mu}\Delta(\lambda) = \begin{pmatrix} \frac{\lambda_1}{\mu_1} & & & & \\ -\frac{\lambda_1}{\mu_2} & \frac{\lambda_2}{\mu_2} & & \mathbf{0} & \\ & \cdot & \cdot & & \\ \mathbf{0} & & -\frac{\lambda_{n-1}}{\mu_n} & \frac{\lambda_n}{\mu_n} & \\ & & & \cdot & \cdot \end{pmatrix} \tag{4.10}$$

and

$$\widetilde{\Delta} = \begin{pmatrix} \lambda_1\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right) & -\frac{\lambda_2}{\mu_2} & & & \\ -\frac{\lambda_1}{\mu_2} & \lambda_2\left(\frac{1}{\mu_2}+\frac{1}{\mu_3}\right) & -\frac{\lambda_3}{\mu_3} & \mathbf{0} & \\ & \cdot & \cdot & & \\ \mathbf{0} & & -\frac{\lambda_{n-1}}{\mu_n} & \lambda_n\left(\frac{1}{\mu_n}+\frac{1}{\mu_{n+1}}\right) & -\frac{\lambda_{n+1}}{\mu_{n+1}} \\ & & & \cdot & \cdot \end{pmatrix}. \tag{4.11}$$

Then condition (4.9) means $D_{1/\alpha}\widetilde{\Delta} \in (c,c)$ and from the characterization of (c,c) this condition is equivalent to $\kappa + \kappa' \in \ell_\infty$ and $\kappa - \kappa' \in c$ together where $\kappa = (\kappa_n)_{n\geq 1}$, $\kappa' = (\kappa'_n)_{n\geq 1}$ with

$$\kappa_n = \frac{1}{\alpha_n}\left[\lambda_n\left(\frac{1}{\mu_n}+\frac{1}{\mu_{n+1}}\right)\right] \text{ and } \kappa'_n = \frac{1}{\alpha_n}\left(\frac{\lambda_{n-1}}{\mu_n}+\frac{\lambda_{n+1}}{\mu_{n+1}}\right).$$

Then from Lemma 4.2 condition (4.9) is equivalent to (4.5) and (4.6) and as we have just seen (4.9) implies (4.1). This completes the proof of i).

ii). From Theorem 3.5 vi) we see that (4.1) means that $cs \subset s_\alpha^{(c)}$. Since $cs = c(\Sigma) = \Sigma^{-1}c$ we then have $I \in \left(\Sigma^{-1}c, s_\alpha^{(c)}\right)$ and $D_{1/\alpha}\Sigma^{-1} = D_{1/\alpha}\Delta \in (c,c)$. We have

$$D_{1/\alpha}\Delta = \begin{pmatrix} 1/\alpha_1 & & & \\ \cdot & \cdot & \mathbf{0} & \\ & -1/\alpha_n & 1/\alpha_n & \\ \mathbf{0} & & \cdot & \cdot \end{pmatrix}$$

and from the characterization of (c,c) given in Lemma 3.1 iv) we conclude $D_{1/\alpha}\Delta \in (c,c)$ if and only if $1/\alpha \in \ell_\infty$.

iii) We have $c \subset C(\lambda)D_\mu c_0$. Indeed from the expression of $D_{1/\mu}\Delta(\lambda)$ given by (4.10) it follows that $(C(\lambda)D_\mu)^{-1} = D_{1/\mu}\Delta(\lambda) \in (c,c_0)$ if and only if the hypotheses of iii) hold. Then (4.1) means that $\widetilde{\Delta}Y \in s_\alpha^{(c)}$ for all $Y \in c$ by Lemma 4.1 and $\widetilde{\Delta} \in \left(c, s_\alpha^{(c)}\right)$ that is $D_{1/\alpha}\widetilde{\Delta} \in (c,c)$. Using the characterization of (c,c) given in Lemma 3.1 and Lemma 4.2 we easily conclude that $D_{1/\alpha}\widetilde{\Delta} \in (c,c)$ if and only if (4.5) and (4.6) hold. □

These results lead to the next corollary

Corollary 4.4. *Assume* (4.5) *and* (4.6) *hold. Then condition* (4.1) *holds with* $l' = Ll$.

Proof. This result is a direct consequence of Lemma 3.1 iv) c) and of the proof of i) b) implies (4.1) in Theorem 4.3. □

Example 4.5. If we put $\lambda = e$ in Theorem 4.3 iii), then for given $\mu \in U^+$ with $\sup_n 1/\mu_n < \infty$ we have

$$\sum_{k=1}^{n} \mu_k r_k \to l \text{ implies } \frac{x_n}{\alpha_n} \to l' \ \ (n \to \infty) \ \text{ for all } X \in cs \tag{4.12}$$

if and only if α satisfies

$$\sup_n \left\{ \frac{1}{\alpha_n} \left(\frac{1}{\mu_n} + \frac{1}{\mu_{n+1}} \right) \right\} < \infty.$$

By Corollary 4.4, since $L = 0$ we have $l' = 0$. Particularly if $\mu_n = n$ for all n, (4.12) holds if and only if $1/\alpha_n = O(n)$ $(n \to \infty)$.

In this way we obtain the next result.

Proposition 4.6. *Let* $\lambda \in U^+$ *and assume* $\sup_n (n/\lambda_n) < \infty$. *Then*

i) $c_0 (C(\lambda) \Sigma^+) = cs$.

ii) The condition

$$\frac{1}{\lambda_n} \sum_{k=1}^{n} r_k \to l \textit{ implies } \frac{x_n}{\alpha_n} \to l' \ \ (n \to \infty) \ \textit{ for all } X \in cs \tag{4.13}$$

is equivalent to $1/\alpha \in \ell_\infty$.

Proof. i) is a direct consequence of Theorem 3.5 v) b) since $\sup_n (n/\lambda_n) < \infty$. ii) is a direct consequence of Theorem 4.3 ii). □

4.2. Case when $\lambda_n = n$ and $\mu_n = n^\xi$ where ξ is a real

Now we consider the case when $\lambda_n = n$ and $\mu_n = n^\xi$ with ξ real in condition (4.1), that is

$$\frac{1}{n} \sum_{k=1}^{n} k^\xi r_k \to l \text{ implies } \frac{x_n}{\alpha_n} \to l' \ \ (n \to \infty) \ \text{ for all } X \in cs \tag{4.14}$$

for some l, $l' \in \mathbb{C}$. As another consequence of Theorem 4.3 we obtain the next corollary.

Corollary 4.7. *i) Let* $\xi \geq 1$. *Then condition* (4.14) *holds if and only if*

$$\sup_n \left(\frac{1}{n^{\xi-1} \alpha_n} \right) < \infty. \tag{4.15}$$

ii) If $\xi \leq 0$, *condition* (4.14) *holds if and only if* $1/\alpha \in \ell_\infty$.

Proof. i) is a direct consequence of Theorem 4.3 iii). Indeed for $\lambda_n = n$ and $\mu_n = n^\xi$ we have

$$\frac{\lambda_n + \lambda_{n-1}}{\mu_n} = \frac{2n-1}{n^\xi} = O(1) \quad (n \to \infty).$$

We need to verify (4.5). We have

$$\kappa_n = \frac{n-1}{n^\xi} + \frac{n+1}{(n+1)^\xi} = \frac{1}{n^{\xi-1}} - \frac{1}{n^\xi} + \frac{1}{(n+1)^{\xi-1}} \sim \frac{2}{n^{\xi-1}} \quad (n \to \infty).$$

Then

$$\frac{\kappa_n}{\alpha_n} \sim \frac{2}{n^{\xi-1}\alpha_n} \quad (n \to \infty)$$

and so the condition (4.5) is equivalent to (4.15). To show (4.6), put

$$b_n = -\frac{n-1}{n^\xi} + n\left(\frac{1}{n^\xi} + \frac{1}{(n+1)^\xi}\right) - \frac{1}{(n+1)^{\xi-1}}.$$

We immediately get

$$b_n = \frac{1}{n^\xi}\left[1 - \left(\frac{n}{n+1}\right)^\xi\right] \sim \frac{\xi}{n^{\xi+1}} \quad (n \to \infty)$$

and so there is $C > 0$ such that $b_n/\alpha_n \leq C/n^{\xi+1}\alpha_n$ $(n \to \infty)$. Then by (4.15) we have $1/\alpha_n \leq C' n^{\xi-1}$,

$$\frac{b_n}{\alpha_n} \leq CC' \frac{n^{\xi-1}}{n^{\xi+1}} = O\left(\frac{1}{n^2}\right) \quad (n \to \infty)$$

and $b_n/\alpha_n \to 0$ $(n \to \infty)$. We conclude (4.6) holds and the conditions (4.5) and (4.6) together are equivalent to (4.15).

ii) We only have to apply Theorem 4.3 ii). Indeed for $\xi = -1$ we have

$$\frac{\sigma_n}{n} = \frac{1}{n}\sum_{m=1}^{n}\frac{1}{k} = O(1) \quad (n \to \infty).$$

For $\xi \leq 0$ and $\xi \neq -1$ we have

$$\sum_{k=2}^{n} k^\xi \leq \int_1^n x^\xi dx \leq \frac{n^{\xi+1}}{\xi+1}$$

and we conclude

$$\frac{\sigma_n}{n} = \frac{1}{n}\sum_{k=1}^{n} k^\xi = \frac{1}{n} + \frac{n^\xi}{\xi+1} = O(1) \quad (n \to \infty).$$

□

Remark 4.8. As we have seen in the proof of Theorem 4.3 i) for any real ξ the condition $1/\alpha \in \ell_\infty$ trivially implies condition (4.14).

Example 4.9. Taking $\xi = 1$ in Corollary 4.7 we deduce that for every $X \in cs$ we have

$$\frac{1}{n}\sum_{k=1}^{n} kr_k \to l \text{ implies } \frac{x_n}{\alpha_n} \to l' \ \ (n \to \infty) \tag{4.16}$$

for some l, $l' \in \mathbb{C}$ if and only if $1/\alpha \in \ell_\infty$.

4.3. A simplification of the previous results.

In this subsection we will characterize (4.1) and then rewrite Theorem 4.3 in each of the cases $\mu \in \widehat{C_1}$ and $\lambda \in \widehat{\Gamma}$.

Recall the definitions of the sets $\widehat{C_1}$ and $\widehat{\Gamma}$ defined in [3],

$$\widehat{C_1} = \left\{ X \in U^+ : \quad [C(X)X]_n = \frac{1}{x_n}\left(\sum_{k=1}^{n} x_k\right) = O(1) \ \ (n \to \infty)\right\}$$

and

$$\widehat{\Gamma} = \left\{ X \in U^+ : \quad \lim_{n\to\infty}\left(\frac{x_{n-1}}{x_n}\right) < 1\right\}.$$

It can easily be seen that $\widehat{\Gamma} \subset \widehat{C_1}$ and note that for $a > 1$ we have $(a^n)_{n\geq 1} \in \widehat{\Gamma}$. By [3] if $X \in \widehat{C_1}$ there are $M > 0$ and $\gamma > 1$ such that

$$x_n \geq M\gamma^n \text{ for all } n.$$

From [4, Lemma 11, p. 49] we obtain the next lemma.

Lemma 4.10. *Let $\alpha \in U^+$. Then*

i) $\alpha \in \widehat{C_1}$ if and only if Σ is bijective from s_α^0 to itself,

ii) $\alpha \in \widehat{\Gamma}$ if and only if Σ is bijective from $s_\alpha^{(c)}$ to itself.

Theorem 4.3 can be reduced to the next corollaries.

Corollary 4.11. *Let $\mu \in \widehat{C_1}$.*

i) Let $\lambda \in U^+$ with $\lambda/\mu \in c_0$. Then condition (4.1) holds if and only if (4.5) and (4.6) hold.

ii) Let $\lambda \in U^+$ with $\mu/\lambda \in \ell_\infty$. Then condition (4.1) holds if and only if $1/\alpha \in \ell_\infty$.

Proof. Since $\mu \in \widehat{C_1}$ the operator Σ is bijective from s_μ^0 to itself and

$$C(\lambda)s_\mu^0 = D_{1/\lambda}\Sigma s_\mu^0 = D_{1/\lambda}s_\mu^0 = s_{\mu/\lambda}^0.$$

Now show i). We have $c \subset s_{\mu/\lambda}^0$ since $D_{\lambda/\mu} \in (c, c_0)$ which is equivalent to $\lambda/\mu \in c_0$. By Lemma 4.1 for every Y we have

$$Y \in c\bigcap C(\lambda)s_\mu^0 = c \text{ implies } \Delta^+ D_{1/\mu}\Delta(\lambda)Y \in s_\alpha^{(c)}$$

that is $\Delta^+ D_{1/\mu}\Delta(\lambda) \in \left(c, s_\alpha^{(c)}\right)$. As we have seen in the proof of Theorem 4.3 iii) this means that (4.5) and (4.6) hold.

ii) By Lemma 1 we have $D_{\mu/\lambda} \in (c_0, c)$ if and only if $\mu/\lambda \in \ell_\infty$ and then $s_{\mu/\lambda}^0 \subset c$. Then (4.1) means that $\Delta^+ D_{1/\mu}\Delta(\lambda)Y \in s_\alpha^{(c)}$ for all $Y \in c\bigcap C(\lambda)s_{\mu}^{0} = s_{\mu/\lambda}^{0}$, that is $\Delta^{+}D_{1/\mu}\Delta(\lambda) \in \left(s_{\mu/\lambda}^{0}, s_{\alpha}^{(c)}\right)$ and $D_{1/\alpha}\Delta^{+}D_{1/\mu}\Delta(\lambda)D_{\mu/\lambda} \in (c_0, c)$. Now since

$$\Delta^{+}D_{1/\mu}\Delta(\lambda)D_{\mu/\lambda} = \Delta^{+}D_{1/\mu}\Delta(\mu)$$

we have

$$D_{1/\alpha}\Delta^{+}D_{1/\mu}\Delta(\mu) \in (c_0, c). \tag{4.17}$$

Using the calculation of $\widetilde{\Delta}$ explicited in (4.11) with $\lambda = \mu$ we deduce (3.1) is equivalent to

$$\sup_n \left\{ \frac{1}{\alpha_n} \left[\frac{\mu_{n-1}}{\mu_n} + \left(1 + \frac{\mu_n}{\mu_{n+1}}\right) + 1 \right] \right\} < \infty. \tag{4.18}$$

Now since $\mu \in \widehat{C_1}$ implies there is $M > 1$ such that $\mu_n^{-1}\sum_{k=1}^{n}\mu_k \leq M$ for all $n \geq 1$ and we successively obtain

$$\frac{\mu_{n-1}}{\mu_n} + \left(1 + \frac{\mu_n}{\mu_{n+1}}\right) + 1 \leq \frac{1}{\mu_n}\sum_{k=1}^{n}\mu_k + \frac{1}{\mu_{n+1}}\sum_{k=1}^{n+1}\mu_k + 1 \leq 2M + 1,$$

and

$$\frac{2}{\alpha_n} \leq \frac{1}{\alpha_n}\left[\frac{\mu_{n-1}}{\mu_n} + \left(1 + \frac{\mu_n}{\mu_{n+1}}\right) + 1\right] \leq \frac{1}{\alpha_n}(2M+1) \text{ for all } n \geq 1,$$

thus (4.18) is equivalent to $1/\alpha \in \ell_\infty$. This concludes the proof. □

Now consider the following conditions,

$$\sup_n \left\{ \frac{1}{\alpha_n}\left(\frac{\lambda_n}{\mu_n} + \frac{\lambda_{n+1}}{\mu_{n+1}}\right) \right\} < \infty, \tag{4.19}$$

$$\sup_n \frac{1}{\alpha_n}\frac{\lambda_n}{\mu_n} < \infty, \tag{4.20}$$

$$\lim_{n\to\infty} \frac{1}{\alpha_n}\left(\frac{\lambda_n}{\mu_n} - \frac{\lambda_{n+1}}{\mu_{n+1}}\right) = \chi \text{ for some } \chi \in \mathbb{C}. \tag{4.21}$$

We can state the next corollary.

Corollary 4.12. *Let $\lambda \in \widehat{\Gamma}$, $\mu \in U^{+}$ and assume conditions of Theorem 4.3 iii) hold.*
Then condition (4.1) holds with $l' = l(1-a)\chi$, ($a = \lim_{n\to\infty}\lambda_{n-1}/\lambda_n < 1$) if and only if α satisfies (4.20) and (4.21).

Proof. By conditions of Theorem 4.3 iii) we have $D_{1/\mu}\Delta(\lambda) \in (c, c_0)$ and $\Delta(\lambda)c \subset s_{\mu}^{0}$ and since $C(\lambda) = \Delta(\lambda)^{-1}$ we have $c \subset C(\lambda)s_{\mu}^{0}$. So (4.1) means that

$$X = \Delta^{+}D_{1/\mu}\Delta(\lambda)Y \in s_{\alpha}^{(c)} \text{ for all } Y \in c,$$

that is $\Delta^{+}D_{1/\mu}\Delta(\lambda)c \subset s_{\alpha}^{(c)}$. Now by Lemma 4.10 ii) $\lambda \in \widehat{\Gamma}$ implies $\Delta s_{\lambda}^{(c)} = s_{\lambda}^{(c)}$ and

$$\Delta^{+}D_{1/\mu}\Delta(\lambda)c = \Delta^{+}D_{1/\mu}\Delta s_{\lambda}^{(c)} = \Delta^{+}D_{1/\mu}s_{\lambda}^{(c)} = \Delta^{+}s_{\lambda/\mu}^{(c)}.$$

Then (4.1) is equivalent to $\Delta^+ \in \left(s_{\lambda/\mu}^{(c)}, s_\alpha^{(c)}\right)$ and to (4.19) and (4.21). By Lemma 4.2 where $\kappa_n = \lambda_n/\alpha_n\mu_n$ and $\kappa_n' = \lambda_{n+1}/\alpha_n\mu_{n+1}$ we deduce that $\Delta^+ \in \left(s_{\lambda/\mu}^{(c)}, s_\alpha^{(c)}\right)$ is equivalent to (4.20) and (4.21).

Now show $l' = l\,(1-a)\,\chi$. If $X \in cs$ and

$$\begin{aligned} X &= \Delta^+ D_{1/\mu}\Delta\,(\lambda)\,Y = \Delta^+ D_{1/\mu}\Delta D_\lambda Y \\ &= \Delta^+ D_{1/\mu} D_\lambda \left(D_{1/\lambda}\Delta D_\lambda\right) Y = \Delta^+ D_{\lambda/\mu}\left(D_{1/\lambda}\Delta D_\lambda\right) Y, \end{aligned}$$

and letting $\widehat{Y} = (\widehat{y}_n)_{n\geq 1} = \left(D_{1/\lambda}\Delta D_\lambda\right) Y$, we have

$$\widehat{y}_n = -\frac{\lambda_{n-1}}{\lambda_n} y_{n-1} + y_n.$$

Thus in particular if $Y = e$, then

$$\lim_{n\to\infty} \widehat{y}_n = \lim_{n\to\infty}\left(1 - \frac{\lambda_{n-1}}{\lambda_n}\right) = 1 - a.$$

And if $Y \in c_0$ then, clearly, $\widehat{Y} = \left(D_{1/\lambda}\Delta D_\lambda\right) Y \in c_0$. Consequently, if $Y \in c$ with $l = \lim_{n\to\infty} y_n$, then $\widehat{y}_n - l \to -al + l - l = -al$ $(n\to\infty)$. Then by (4.21), we obtain

$$\begin{aligned} \frac{x_n}{\alpha_n} &= \left(D_{1/\alpha}\Delta^+ D_{\lambda/\mu}\right)_n \left(le + \left(\widehat{Y} - le\right)\right) \\ &= \frac{1}{\alpha_n}\left(\frac{\lambda_n}{\mu_n} - \frac{\lambda_{n+1}}{\mu_{n+1}}\right) l + \frac{1}{\alpha_n}\left(\frac{\lambda_n}{\mu_n} - \frac{\lambda_{n+1}}{\mu_{n+1}}\right)(\widehat{y}_n - l) \to \chi l - \chi a l \ \ (n\to\infty). \end{aligned}$$

This concludes the proof. □

Example 4.13. As a direct application of the preceding we have

$$\frac{1}{(n-1)!}\sum_{k=1}^{n} k! r_k \to 0 \text{ implies } \frac{x_n}{\alpha_n} \to l' \ \ (n\to\infty) \text{ for all } X \in cs \qquad (4.22)$$

if and only if there is $C > 0$ such that $\alpha_n \geq C/n$ for all n. Indeed conditions (4.20) and (4.21) mean that $\sup_n \{1/(n\alpha_n)\} < \infty$ and $\lim_{n\to\infty} 1/\left(n^2\alpha_n\right) = \chi$ for some scalar χ. It can easily be seen that $\sup_n \{1/(n\alpha_n)\} < \infty$ implies $\lim_{n\to\infty} 1/\left(n^2\alpha_n\right) = 0$. Since $\chi = 0$ we have $l' = 0$. This concludes the proof.

Example 4.14. In the same way it can easily be shown that for $1 < \mathbf{a} < \mathbf{b}$ and $\lim_{n\to\infty} \mathbf{a}^n/\mathbf{b}^n\alpha_n = L$, we then have

$$\mathbf{a}^{-n}\sum_{k=1}^{n} \mathbf{b}^k r_k \to l \text{ implies } \frac{x_n}{\alpha_n} \to l\left(1 - \frac{1}{\mathbf{a}}\right)\left(1 - \frac{\mathbf{a}}{\mathbf{b}}\right) L \ \ (n\to\infty)$$

for all $X \in cs$ if and only if $\left(\mathbf{a}^n/\left(\mathbf{b}^n\alpha_n\right)\right)_{n\geq 1} \in c$.

4.4. Study of the converse of tauberian results

For given $\alpha \in U^+$ we will determine the set of all λ, $\mu \in U^+$ such that

$$\frac{x_n}{\alpha_n} \to l \text{ implies } \frac{1}{\lambda_n} \sum_{k=1}^{n} \mu_k r_k \to l' \;\; (n \to \infty) \text{ for all } X \in cs \tag{4.23}$$

and give a characterization of (4.23).

We get the following theorem

Theorem 4.15. *Let λ, μ, $\alpha \in U^+$. Suppose $\alpha \in cs$. Then the sequences λ and μ satisfy condition (4.23) if and only if $1/\lambda \in c$ and*

$$\lim_{n\to\infty} \phi_n(\alpha) = L \text{ for some } L \in \mathbb{C}. \tag{4.24}$$

Proof. First we note that $\alpha \in cs$ if and only if $s_\alpha^{(c)} \subset cs$. Now condition (4.23) means that

$$X \in s_\alpha^{(c)} \bigcap cs = s_\alpha^{(c)} \text{ implies } \left(C(\lambda) D_\mu \Sigma^+\right) X = C(\lambda) D_\mu \left(\Sigma^+ X\right) \in c$$

by Lemma 3.2 which is equivalent to

$$C(\lambda) D_\mu \Sigma^+ D_\alpha \in (c, c). \tag{4.25}$$

We deduce from the proof of Theorem 3.5 (iv) that if we put $C(\lambda) D_\mu \Sigma^+ D_\alpha = (c_{nk})_{n,k\geq 1}$, then

$$c_{nk} = \begin{cases} \dfrac{\sigma_k}{\lambda_n}\alpha_k \text{ for } k < n, \\ \dfrac{\sigma_n}{\lambda_n}\alpha_k \text{ for } k \geq n. \end{cases}$$

So condition (4.25) is equivalent to $1/\lambda \in c$, (4.24) and

$$\sup_n \{\phi_n(\alpha)\} < \infty. \tag{4.26}$$

We conclude the proof since condition (4.24) implies condition (4.26). □

Now to state the next result recall the following result due to Kizmaz.

Lemma 4.16. *([7]) Let $p = (p_n)_{n\geq 1}$ be a strictly increasing sequence. If $pX \in cs$ then $(p_n r_{n+1})_{n\geq 1} \in c_0$.*

Corollary 4.17. *Let $\xi > 0$ be a real, $\alpha \in U^+$ and assume $\left(n^{\xi+1}\alpha_n\right)_{n\geq 1} \in c$ and $\left(n^{\xi}\alpha_n\right)_{n\geq 1} \in cs$. Then*

$$\frac{x_n}{\alpha_n} \to l \text{ implies } \frac{1}{n} \sum_{k=1}^{n} k^\xi r_k \to l' \;\; (n \to \infty).$$

for all $X \in cs$ and for some scalars l, l'.

Proof. We only have to apply Theorem 4.15. For this it suffices to show that

$$\frac{1}{n} \sum_{k=1}^{n} \sigma_k \alpha_k \to l_1 \text{ and } \frac{1}{n}\sigma_n \sum_{k=n+1}^{\infty} \alpha_k \to l_2$$

for some l_1, $l_2 \geq 0$ with $\sigma_n = \sum_{k=1}^{n} k^{\xi}$. First we have

$$\frac{n^{\xi+1}}{\xi+1} \leq \sigma_n \leq \frac{(n+1)^{\xi+1}-1}{\xi+1} \text{ for all } n$$

and then $\sigma_n \sim n^{\xi+1}/(\xi+1)$ $(n \to \infty)$. Since $n^{\xi+1}\alpha_n \to L$ $(n \to \infty)$ we deduce $(\sigma_n \alpha_n)_{n\geq 1} \in c$ and $\left(n^{-1}\sum_{k=1}^{n} \sigma_k \alpha_k\right)_{n\geq 1} \in c$. Then putting $p_n = \sigma_n/n$ we get

$$p_n \sim \frac{1}{n}\frac{n^{\xi+1}}{\xi+1} = \frac{n^{\xi}}{\xi+1} \ (n \to \infty)$$

and by Lemma 4.16 condition $\sum_{n=1}^{\infty} n^{\xi}\alpha_n < \infty$ implies

$$\frac{1}{n}\sigma_n \sum_{k=n+1}^{\infty} \alpha_k \to 0 \ (n \to \infty).$$

This concludes the proof. □

Example 4.18. Let $\gamma > 2$, then $n^{\gamma}x_n \to l$ implies $n^{-1}\sum_{k=1}^{n} kr_k \to l'$ $(n \to \infty)$ for all $X \in cs$.

Indeed it is enough to put $\xi = 1$ and $\alpha_n = n^{-\gamma}$.

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On best simultaneous approximation in operator and function spaces

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Abstract. Let X be a Banach space, $(I, \sum, \mu)$ a finite measure space and $L^1(\mu, X)$ the Banach space of all X-valued μ-integrable functions on the unit interval I equipped with the usual 1-norm. In this paper we prove that for a closed subspace G of X, $L^1(\mu, G)$ is simultaneously Chebyshev in $L^1(\mu, X)$ if and only if G is simultaneously Chebyshev in X. Further results are obtained in the space of bounded linear operators $L(l^1, X)$ and in the space of continuous functions $C^1(I, l^p)$ with respect to the L^1 norm.

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1. Introduction

Let X be a Banach space and $(I, \sum, \mu)$ be a finite measure space. Let us denote by $L^1(\mu, X)$, the Banach space of all X-valued μ-integrable functions on the unit interval I equipped with the usual 1-norm.

For a closed subspace G of X, let us recall that G is simultaneously proximinal in X if for all m-tuples $(x_1, x_2, ..., x_m) \in X^m$, there exists $g \in G$ such that

$$\sum_{i=1}^{m} \|x_i - g\| = \operatorname{dist}(x_1, x_2, ..., x_m, G) = \inf \left\{ \sum_{i=1}^{m} \|x_i - z\| : z \in G \right\}.$$

In this case, g is called a best simultaneous approximation of $(x_1, x_2, ..., x_m)$ in G. If this best approximation is unique for all $(x_1, x_2, ..., x_m) \in X^m$, then G is called simultaneously Chebyshev.

Of course for $m = 1$ the preceding concepts are just best approximation and proximinality.

The problem of best simultaneous approximation can be viewed as a special case of vector valued approximation. Recent results in this area are due to Pinkus [10], where he considered the problem when a finite dimensional subspace is a uniqueness space. Results on best simultaneous approximation in general Banach spaces may be found in [9] and [11]. Related results on $L^p(\mu, X)$, $1 \leq p < \infty$, are given in [12]. In [12], it is shown that if G is a reflexive subspace of a Banach space X, then $L^p(\mu, G)$ is simultaneously proximinal in $L^p(\mu, X)$. If $p = 1$, Abu Sarhan and Khalil [1], proved that if G is a reflexive subspace of the Banach space X or G is a 1-summand subspace of X, then $L^1(\mu, G)$ is simultaneously proximinal in $L^1(\mu, X)$.

It is the aim of this paper to give some sufficient conditions for $L^1(\mu, G)$ to be a Chebyshev subspace of $L^1(\mu, X)$. Further results are obtained in the space of bounded linear operators $L(l^1, X)$ and in the space of continuous functions $C^1(I, l^p)$ with respect to the L^1 norm.

Throughout this paper, X is a Banach space and G is a closed subspace of X.

2. Main results

In [1] it is shown that if $m = 1$ and G is a finite dimensional subspace of a Banach space X, then G is Chebyshev in X if and only if $L^1(\mu, G)$ is Chebyshev in $L^1(\mu, X)$. The main result in this section is: If G is a reflexive subspace of X, then G is simultaneously Chebyshev in X if and only if $L^1(\mu, G)$ is simultaneously Chebyshev in $L^1(\mu, X)$.

Theorem 2.1. *Let G be a reflexive subspace of X. Then G is simultaneously Chebyshev in X if and only if $L^1(\mu, G)$ is simultaneously Chebyshev in $L^1(\mu, X)$.*

Proof. Let $f_1, f_2, ..., f_m \in L^1(\mu, X)$. Since G is reflexive, it follows that [Th.4, 12], there exists $g \in L^1(\mu, G)$ such that

$$\sum_{i=1}^{m} \|f_i - g\|_1 = \operatorname{dist}(f_1, f_2, ..., f_m, L^1(\mu, G)).$$

Thus by [Th.2.2, 2], we have:

$$\sum_{i=1}^{m} \|f_i(t) - g(t)\| = \operatorname{dist}(f_1(t), f_2(t), ..., f_m(t), G),$$

for almost all $t \in I$. But G is simultaneously Chebyshev. So $g(t)$ is unique. Thus g is determined uniquely, and $L^1(\mu, G)$ is simultaneously Chebyshev in $L^1(\mu, X)$.

Conversely. Let $x_1, x_2, ..., x_m \in X$. For $i = 1, 2, ..., m$, consider the functions: $f_i : I \to X$, $f_i(t) = x_i$, for all $t \in I$. Since $L^1(\mu, G)$ is simultaneously Chebyshev in $L^1(\mu, X)$, there exists $g \in L^1(\mu, G)$ such that

$$\operatorname{dist}(f_1, f_2, ..., f_m, L^1(\mu, G)) = \sum_{i=1}^{m} \|f_i - g\|_1 \leq \sum_{i=1}^{m} \|f_i - h\|_1$$

for all $h \in L^1(\mu, G)$. Thus by [Th.2.2, 2] , we have:

$$\sum_{i=1}^{m} \|f_i(t) - g(t)\| \leq \sum_{i=1}^{m} \|f_i(t) - h(t)\| \tag{2.1}$$

for almost all $t \in I$. But since G is reflexive, there exists $w \in G$ such that

$$\sum_{i=1}^{m} \|x_i - w\| \leq \sum_{i=1}^{m} \|x_i - z\|$$

for all $z \in G$, [Lemma 1.12] . Hence the function $b(t) = w$ for all $t \in I$ is a best simultaneous approximation of $f_1, f_2, ..., f_m$ in $L^1(\mu, G)$. Equation (2.1) and since $L^1(\mu, G)$ is simultaneously Chebyshev in $L^1(\mu, X)$ it follows that $g(t) = b(t) = w$ and w is unique. Hence G is simultaneously Chebyshev in X. □

For $0 < p < \infty$, let us denote by $l^p(X)$, the space of all sequences (x_n) in X such that $\sum\limits_{n=1}^{\infty} \|x_n\|^p < \infty$. For $x = (x_n) \in l^p(X)$, let

$$\|x\|_p = \begin{cases} \left(\sum\limits_{k=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \sum\limits_{k=1}^{\infty} \|x_n\|^p & 0 < p < 1 \end{cases}$$

In the space $l^1(X)$, we have the following result:

Theorem 2.2. *G is simultaneously Chebyshev in X if and only if $l^1(G)$ is simultaneously Chebyshev in $l^1(X)$.*

Proof. For $1 \leq i \leq m$, let $x_i = (x_{in}) \in l^1(X)$. If $g_n \in G$ is such that

$$\sum_{i=1}^{m} \|x_{in} - g_n\| \leq \sum_{i=1}^{m} \|x_{in} - z\| \tag{2.2}$$

for all $z \in G$. Using triangle inequality and taking $z = 0$ in (2.2) we get

$$\sum_{i=1}^{m} \|g_n\| - \|x_{in}\| \leq \sum_{i=1}^{m} \|x_{in} - g_n\| \leq \sum_{i=1}^{m} \|x_{in}\|$$

and this implies

$$m \|g_n\| = \sum_{i=1}^{m} \|g_n\| \leq 2 \sum_{i=1}^{m} \|x_{in}\| . \tag{2.3}$$

Thus

$$\sum_{n=1}^{\infty} \|g_n\| \leq \frac{2}{m} \sum_{i=1}^{m} \sum_{n=1}^{\infty} \|x_{in}\| < \infty.$$

Hence the element $g = (g_n) \in l^1(G)$ and g is a best simultaneous approximation of the m-tuple $((x_{in}))_{i=1}^m$ in $l^1(G)$. The uniqueness of g_n implies that $g = (g_n)$ is unique and $l^1(G)$ is simultaneously Chebyshev in $l^1(X)$.

Conversely. Let $x_1, x_2, ..., x_m \in X$. For each $i = 1, 2, ..., m$, consider the sequence $(x_i, 0, ...) \in l^1(X)$. Since $l^1(G)$ is simultaneously Chebyshev in $l^1(X)$, it follows that there exists a sequence of the form $(g, 0, ...)$ in $l^1(G)$ such that

$$\sum_{i=1}^{m} \|(x_i, 0, ...) - (g, 0, ...)\| < \sum_{i=1}^{m} \|(x_i, 0, ...) - (z_1, z_2, ...)\|$$

for all $(z_n) \in l^1(G) \backslash \{(g, 0, ...)\}$. This implies that

$$\sum_{i=1}^{m} \|x_i - g\| < \sum_{i=1}^{m} \|x_i - z\|$$

for all $z \in G \backslash \{g\}$. □

For the space of bounded linear operators, $L(l^1, X)$, from l^1 into X, where l^1 is the space of all summable real sequences it has been proved in [1] that G is proximinal in X if and only if $L(l^1, G)$ is proximinal in $L(l^1, X)$. For the case of simultaneous approximation we have the following result:

Theorem 2.3. *G is simultaneously proximinal in X if and only if $L(l^1, G)$ is simultaneously proximinal in $L(l^1, X)$.*

Proof. Let $T_1, T_2, ..., T_m \in L(l^1, X)$. If (δ_n) is the natural basis of l^1, then $T_i \delta_n \in X$, $i = 1, 2, ..., m$.

Since G is simultaneously proximinal, so for each n there exists $x_n \in G$ such that

$$\sum_{i=1}^{m} \|T_i(\delta_n) - x_n\| = \text{dist } (T_1(\delta_n), T_2(\delta_n), ..., T_m(\delta_n), G).$$

Define $S : l^1 \to G$, $S(\delta_n) = x_n$. Then S is a bounded linear operator from l^1 into G. It is clear that S is linear. To prove that S is bounded, let $y = (\alpha_n) \in l^1$, $\|y\|_1 = \sum_{n=1}^{\infty} |\alpha_n| \le 1$. Then

$$\|S(y)\| = \left\| \sum_{n=1}^{\infty} \alpha_n S(\delta_n) \right\| \le \sum_{n=1}^{\infty} |\alpha_n| \, \|S(\delta_n)\| = \sum_{n=1}^{\infty} |\alpha_n| \, \|x_n\| .$$

Using (2.3) in Theorem 2.2 we get

$$\|S(y)\| \le \sum_{n=1}^{\infty} |\alpha_n| \frac{2}{m} \sum_{i=1}^{m} \|T_i(\delta_n)\| \le \sum_{n=1}^{\infty} |\alpha_n| \frac{2}{m} \sum_{i=1}^{m} \|T_i\| = \frac{2}{m} \sum_{i=1}^{m} \|T_i\| \sum_{n=1}^{\infty} |\alpha_n|$$

Hence S is a bounded linear operator with $\|S\| \leq \frac{2}{m}\sum_{i=1}^{m}\|T_i\|$. Now for any $x = (\beta_n) \in l^1$ we have

$$\begin{aligned}
\sum_{i=1}^{m}\|T_i(x) - S(x)\| &= \sum_{i=1}^{m}\left\|T_i\left(\sum_{n=1}^{\infty}\beta_n\delta_n\right) - S\left(\sum_{n=1}^{\infty}\beta_n\delta_n\right)\right\| \\
&= \sum_{i=1}^{m}\left\|\sum_{n=1}^{\infty}\beta_n T_i(\delta_n) - \sum_{n=1}^{\infty}\beta_n S(\delta_n)\right\| \\
&= \sum_{i=1}^{m}\left\|\sum_{n=1}^{\infty}\beta_n\left(T_i(\delta_n) - S(\delta_n)\right)\right\| \\
&\leq \sum_{i=1}^{m}\sum_{n=1}^{\infty}|\beta_n|\,\|T_i(\delta_n) - S(\delta_n)\| \\
&= \sum_{n=1}^{\infty}|\beta_n|\,\text{dist}\,(T_1(\delta_n), T_2(\delta_n), ..., T_m(\delta_n), G). \\
&\leq \sum_{n=1}^{\infty}|\beta_n|\sum_{i=1}^{m}\|T_i(\delta_n) - g\|
\end{aligned}$$

for every $g \in G$. In particular for every $A \in L(l^1, G)$

$$\begin{aligned}
\sum_{i=1}^{m}\|T_i(x) - S(x)\| &\leq \sum_{n=1}^{\infty}|\beta_n|\sum_{i=1}^{m}\|T_i(\delta_n) - A(\delta_n)\| \\
&\leq \sum_{n=1}^{\infty}|\beta_n|\sum_{i=1}^{m}\|T_i - A\| \\
&= \sum_{i=1}^{m}\|T_i - A\|\sum_{n=1}^{\infty}|\beta_n| = \sum_{i=1}^{m}\|T_i - A\|\,\|x\|.
\end{aligned}$$

Taking supremum over all $x \in l^1, \|x\| = 1$ we get

$$\sum_{i=1}^{m}\|T_i - S\| \leq \sum_{i=1}^{m}\|T_i - A\|.$$

Hence $L(l^1, G)$ is simultaneously proximinal in $L(l^1, X)$.

Conversely. Let $x_1, x_2, ..., x_m \in X$. For each $i = 1, 2, ..., m$, define $T_i : l^1 \to X$,

$$T_i\delta_n = \begin{cases} x_i & n = 1 \\ 0 & n \neq 1 \end{cases}.$$

Then $T_i \in L(l^1, X)$ and $\|T_i\| = \|x_i\|$. By assumption there exists $A \in L(l^1, G)$ such that

$$\sum_{i=1}^{m}\|T_i - A\| \leq \sum_{i=1}^{m}\|T_i - B\|$$

for all $B \in L(l^1, G)$. Hence

$$\sum_{i=1}^{m} \|x_i - A\delta_1\| = \sum_{i=1}^{m} \|(T_i - A)\,\delta_1\| \leq \sum_{i=1}^{m} \|T_i - A\| \leq \sum_{i=1}^{m} \|T_i - B\|\,.$$

If B runs over all functions of the form

$$B\delta_n = \begin{cases} w & n = 1 \\ 0 & n \neq 1 \end{cases}$$

for all $w \in G$, we obtain $\sum_{i=1}^{m} \|x_i - A\delta_1\| \leq \sum_{i=1}^{m} \|x_i - w\|$ for all $w \in G$. Hence G is simultaneously proximinal in X. □

Theorem 2.4. *If $L(l^1, G)$ is simultaneously Chebyshev in $L(l^1, X)$, then G is simultaneously Chebyshev in X.*

Proof. Suppose G is not Chebyshev in X. Then there exist $g_1, g_2 \in G$ and $x_1, x_2, ..., x_m \in X$ such that

$$\sum_{i=1}^{m} \|x_i - g_1\| = \sum_{i=1}^{m} \|x_i - g_2\| = \operatorname{dist}(x_1, x_2, ..., x_m, G).$$

For $i = 1, 2, ..., m$, let

$$T_i\delta_n = \begin{cases} x_i & n = 1 \\ 0 & n \neq 1 \end{cases},$$

and

$$A_1\delta_n = \begin{cases} g_1 & n = 1 \\ 0 & n \neq 1 \end{cases}, \qquad A_2\delta_n = \begin{cases} g_2 & n = 1 \\ 0 & n \neq 1 \end{cases}.$$

Then

$$\sum_{i=1}^{m} \|T_i - A_1\| = \sum_{i=1}^{m} \|T_i - A_2\| = dist(T_1, T_2, ..., T_m, L(l^1, G)).$$

This contradict the fact that $L(l^1, G)$ is simultaneously Chebyshev. □

We remark that the converse of Theorem 2.4 is not true. To see this, let G be a Chebyshev subspace of X and $x_1, x_2, ..., x_m \in X$. For each $i = 1, 2, ..., m$, define $T_i : l^1 \to X$

$$T_i\delta_n = \begin{cases} x_i & n = 1 \\ 0 & n \neq 1 \end{cases},$$

then if $z \in G$ is such that $\sum_{i=1}^{m} \|x_i - z\| =$dist$(x_1, x_2, ..., x_m, G)$, the operator $A : l^1 \to X$,

$$A\delta_n = \begin{cases} z & n = 1 \\ 0 & n \neq 1 \end{cases},$$

is a best simultaneous approximation of $T_1, T_2, ..., T_m$ in $L(l^1, G)$ that is

$$\sum_{i=1}^{m} \|T_i - A\| \leq \sum_{i=1}^{m} \|T_i - B\|$$

for all $B \in L(l^1, G)$. Let $r = \min\limits_{1 \le i \le m} \|x_i - z\|$. Consider the map

$$S : l^1 \to G,\ S\delta_n = \begin{cases} z & n = 1 \\ z_n & n \neq 1 \end{cases}$$

where $|z_n| < r$. Then $\sum_{i=1}^m \|T_i - S\| = \sum_{i=1}^m \|T_i - A\|$. Hence $L(l^1, G)$ is not a Chebyshev subspace of $L(l^1, X)$.

As a corollary from Theorem 2.2 for the Banach space c_0 we have:

Corollary 2.5. *G is simultaneously Chebyshev in X if and only if $L(c_0, G)$ is simultaneously Chebyshev in $L(c_0, X)$.*

Proof. By the result of Grothendieck [6] , page 86, we have $L(c_0, G) = l^1(G)$. the result follows from Theorem 2.2. □

3. Further results

An n-dimensional subspace V_n of $C(I)$, the space of continuous functions on a compact set I, is called a Haar subspace if any $f \in V_n \setminus \{0\}$, f has at most $n - 1$ zero's on I. Haar subspaces on intervals of real numbers are called T-Systems. For each natural number n, let M_n be an n-dimensional Haar subspace. Set

$$U = \left\{ g \in L^1(\mu, l^p) : g = (g_i)\, ,\ g_i \in M_i \right\}.$$

We remark that U is a closed subspace of $L^1(\mu, l^p)$, [1] .

On the space of continuous functions $C^1(I, l^p)$, we have the following result

Theorem 3.1. *For $1 \le p < \infty$, U is proximinal in $C^1(I, l^p)$ with respect to the L^1 norm.*

Proof. Let $p = 1$ and let $S_1, S_2, ...S_m \in C^1(I, l^1)$. Then for each $i = 1, 2, ..., m$, $S_i = (f_{i,k})_{k=1}^\infty$ and $\|S_i\| = \int\limits_I \sum\limits_{k=1}^\infty |f_{i,k}(t)|\, dt$. Hence $\sum_{i=1}^m \|S_i\| = \sum_{i=1}^m \int\limits_I \sum\limits_{k=1}^\infty |f_{i,k}(t)|\, dt$. Using the Monotone Convergence Theorem, we get:

$$\sum_{i=1}^m \|S_i\| = \sum_{k=1}^\infty \sum_{i=1}^m \int_I |f_{i,k}(t)|\, dt = \sum_{k=1}^\infty \sum_{i=1}^m \|f_{i,k}\|_1 .$$

Since for each k, M_k is finite dimensional, there exists $g_k \in M_k$ such that

$$\sum_{i=1}^m \|f_{i,k} - g_k\|_1 \le \sum_{i=1}^m \|f_{i,k} - h_k\|_1$$

for all $h_k \in M_k$. Note that

$$\sum_{i=1}^m \|f_{i,k} - h_k\|_1 \ge \sum_{i=1}^m \|f_{i,k} - g_k\|_1 \ge \sum_{i=1}^m \left| \|f_{i,k}\|_1 - \|g_k\|_1 \right| .$$

for all $h_k \in M_k$. Since $0 \in M_k$, we get:

$$m \|g_k\|_1 \leq 2 \sum_{i=1}^{m} \|f_{i,k}\|_1$$

and so

$$\sum_{k=1}^{\infty} \|g_k\|_1 \leq \frac{2}{m} \sum_{i=1}^{m} \sum_{k=1}^{\infty} \|f_{i,k}\|_1 = \frac{2}{m} \sum_{i=1}^{m} \|f_i\|$$

Hence $g = (g_k) \in U$ and

$$\sum_{i=1}^{m} \|S_i - g\| = \sum_{k=1}^{\infty} \sum_{i=1}^{m} \int_I |f_{i,k}(t) - g_k(t)|\, dt \leq \sum_{k=1}^{\infty} \sum_{i=1}^{m} \int_I |f_{i,k}(t) - h_k(t)|\, dt$$

for all $h_k \in M_k$. In particular we get $\sum_{i=1}^{m} \|S_i - g\| \leq \sum_{i=1}^{m} \|S_i - h\|$ for all $h \in U$. Hence U is proximinal in $C^1(I, l^1)$ with respect to the L^1 norm.

For $1 < p < \infty$, let $S_1, S_2, ...S_m \in C^1(I, l^p)$. Consider the operator

$$\begin{aligned} P_k &: L^1(\mu, l^p) \to L^1(\mu, l_k^p) \\ P_k f &= (f_1, f_2, ..., f_k) \end{aligned}$$

where $f = (f_i)_{i=1}^{\infty}$. Then P_k is continuous. For $1 \leq k < \infty$, set $U_k = \left\{ g = (g_i) \in \prod_{i=1}^{k} M_i \right\}$. Since U_k is finite dimensional, there exists some $\widehat{g} \in U_k$ such that

$$\sum_{i=1}^{m} \|P_k S_i - \widehat{g}\|_1 \leq \sum_{i=1}^{m} \|P_k S_i - h\|_1 \tag{3.1}$$

for all $h \in U_k$,. Let us write g^k for $\widehat{g}$. We shall prove that the sequence$\left(g^k\right)$ must have a subsequence that converges to some $g \in U$.

Since $P_k S_i \to S_i$, then the sets $E_i = \{P_1 S_i,\ P_2 S_i,\ P_3 S_i,\ ...,\ S_i\}$, $i = 1, 2, ..., m$ are weakly compact in $L^1(\mu, l^p)$. Set $\widehat{E} = \left\{g^1,\ g^2,\ g^3,\ ...,\ g^n,\ ...\right\}$. We want to prove that $\widehat{E}$ is weakly relatively compact. Since l^p is reflexive, then by the Dunford Theorem $[4, p.101]$, it is enough to prove that $\widehat{E}$ is bounded and uniformly integrable. Note that

$$\sum_{i=1}^{m} \|P_k S_i - h\|_1 \geq \sum_{i=1}^{m} \left\|P_k S_i - g^k\right\|_1 \geq \sum_{i=1}^{m} \left| \|P_k S_i\|_1 - \left\|g^k\right\|_1 \right|.$$

for all $h \in U_k$. Since $0 \in U_k$, we get

$$m \left\|g^k\right\|_1 \leq 2 \sum_{i=1}^{m} \|P_k S_i\|_1$$

Hence $\widehat{E}$ is bounded.

To see that $\widehat{E}$ is uniformly integrable, first note that for each k

$$\|P_k S_i\|_1 \leq \|S_i\|_1$$

$i = 1, 2, ...m$. Thus $\lim\limits_{\mu(\Omega) \to 0} \int_\Omega \|h(t)\|\, d\mu(t) = 0$ uniformly for h in E_i, $i = 1, 2, ..., m$.

Now let $\epsilon > 0$ be given. By the uniform integrability of E_i there exists $\delta_i > 0$ such that $\int\limits_{\Omega} \|h(t)\|\, d\mu(t) < \frac{\epsilon}{2}$ whenever $\mu(\Omega) < \delta_i$ for all $h \in E_i$. Hence for $\mu(\Omega) < \delta = \min\limits_{1 \le i \le m} (\delta_i)$

$$\int\limits_{\Omega} \left\| g^k(t) \right\| d\mu(t) < \frac{2}{m} \sum_{i=1}^{m} \int\limits_{\Omega} \|P_k S_i\|\, d\mu(t) < \epsilon.$$

Since δ depends only on $E_1, E_2, ..., E_m$ and ϵ it follows that $\widehat{E}$ is uniformly integrable and hence weakly relatively compact. Thus there exists $g \in L^1(\mu, l^p)$ such that $g^k \to g$ weakly.

Since the sequence $\left(g^k\right)$ in U converges weakly to some $g \in L^1(\mu, l^p)$ and U is a closed subspace of $L^1(\mu, l^p)$, hence weakly closed, it follows that $g \in U$.

For $h \in U$, we have $\|P_k h - h\|_1 \to 0$. Hence for each $i = 1, 2, ..., m$, $\|P_k S_i - P_k h\|_1 \underset{k}{\to} \|S_i - h\|_1$. Now let $\varphi \in L^\infty(\mu, l^{p^*}) = \left(L^1(\mu, l^p)\right)^*$, the dual of $L^1(\mu, l^p)$. Then

$$\begin{aligned} \sum_{i=1}^{m} |\langle S_i - g, \varphi \rangle| &= \lim_{k \to \infty} \sum_{i=1}^{m} |\langle P_k S_i - g, \varphi \rangle| \\ &\le \underline{\lim} \sum_{i=1}^{m} \left\| P_k S_i - g^k \right\| \\ &\le \underline{\lim} \sum_{i=1}^{m} \|P_k S_i - P_k h\| \end{aligned}$$

for all $h \in U_k$, since U_k is proximinal. Hence

$$\sum_{i=1}^{m} |\langle S_i - g, \varphi \rangle| \le \sum_{i=1}^{m} \|S_i - h\|.$$

Consequently $\sum_{i=1}^{m} \|S_i - g\| \le \sum_{i=1}^{m} \|S_i - h\|$ for all $h \in U$.

Thus U is proximinal in $C^1(I, l^p)$, with the L^1-norm, $1 < p < \infty$. □

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Fuzzy anti-bounded linear operators

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Abstract. Various types of fuzzy anti-continuity and fuzzy anti-boundedness are defined. Few properties of them are established. The intra and inter relation between various types of fuzzy anti-continuity and fuzzy anti-boundedness are studied.

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1. Introduction

The concept of fuzzy set theory was first introduce by Zadeh[13] in 1965 and thereafter, the concept of fuzzy set theory applied on different branches of pure and applied mathematics in different ways by several authors. The concept of fuzzy norm was introduced by Katsaras [9] in 1984. In 1992, Felbin[7] introduced the idea of fuzzy norm on a linear space. Cheng-Moderson [4] introduced another idea of fuzzy norm on a linear space whose associated metric is same as the associated metric of Kramosil-Michalek [10]. In 2003, Bag and Samanta [1] modified the definition of fuzzy norm of Cheng-Moderson [4] and established the concept of continuity and boundednes of a linear operator with respect to their fuzzy norm in [2].

Later on Jebril and Samanta [8] introduced the concept of fuzzy anti-norm on a linear space depending on the idea of Bag and Samanta [3]. The motivation of introducing fuzzy anti-norm is to study fuzzy set theory with respect to the non-membership function. It is useful in the process of decision making.

In this paper various types of fuzzy anti-continuities and fuzzy anti-boundedness; namely, fuzzy anti-continuity, sequential fuzzy anti-continuity, strong fuzzy anti-continuity, weak fuzzy anti-continuity, strong fuzzy anti-boundedness and weak fuzzy anti-boundedness are defined. The intra-relations among fuzzy anti-continuities and intra-relation among strongly fuzzy anti-bounded and weakly fuzzy anti-bounded are studied. Also, the inter relation between fuzzy anti-continuities and fuzzy anti-boundedness are studied. Also it is established an important property for fuzzy anti-continuity; namely, any linear operator between fuzzy anti-normed linear spaces is strongly and weakly fuzzy anti-continuous if and only if it is strongly and weakly fuzzy anti-bounded respectively.

2. Preliminaries

This section contain some basic definition and preliminary results which will be needed in the sequel.

Definition 2.1. [12] *A binary operation* $\diamond : [0,1] \times [0,1] \to [0,1]$ *is continuous t-conorm if* $\diamond$ *satisfies the following conditions* :
(i) $\diamond$ *is commutative and associative* ,
(ii) $\diamond$ *is continuous* ,
(iii) $a \diamond 0 = a, \ \forall\, a \in [0,1]$,
(iv) $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$, $b \leq d$ *and* $a, b, c, d \in [0,1]$.

A few examples of continuous t-conorm are $a \diamond b = a + b - ab, a \diamond b = \max\{a, b\}, a \diamond b = \min\{a + b, 1\}$.

Definition 2.2. [5] *Let* V *be linear space over the field* $F (= \mathbb{R} or \mathbb{C})$. *A fuzzy subset* ν *of* $V \times \mathbb{R}$ *is called a fuzzy antinorm on* V *with respect to a t-conorm* $\diamond$ *if and only if for all* $x, y \in V$
(i) $\forall\, t \in \mathbb{R}$ *with* $t \leq 0, \nu(x,t) = 1$;
(ii) $\forall\, t \in \mathbb{R}$ *with* $t > 0, \nu(x,t) = 0$ *if and only if* $x = \theta$;
(iii) $\forall\, t \in \mathbb{R}$ *with* $t > 0, \nu(cx,t) = \nu(x, \frac{t}{|c|})$ *if* $c \neq 0, \ c \in F$;
(iv) $\forall\, s, t \in \mathbb{R}$ *with* $\nu(x + y, s + t) \leq \nu(x,s) \diamond \nu(y,t)$;
(v) $\lim\limits_{t \to \infty} \nu(x,t) = 0$.

We further assume that for any fuzzy anti-normed linear space (V, A^*) with respect to a t-conorm $\diamond$,
(vi) $\nu(x,t) < 1, \ \forall\, t > 0 \Rightarrow x = \theta$.
(vii) $\nu(x, \cdot)$ is a continuous function of $\mathbb{R}$ and strictly decreasing on the subset $\{t : 0 < \nu(x,t) < 1\}$ of $\mathbb{R}$.
(viii) $a \diamond a = a, \ \forall\, a \in [0,1]$.

Theorem 2.3. [5] *Let* (V, A^*) *be a fuzzy antinormed linear space satisfying* (vi) *and* (vii) *and* $(viii)$. *Let* $\|x\|_\alpha^* = \wedge\{t : \nu(x,t) \leq 1 - \alpha\}, \alpha \in (0,1)$. *Also, let* $\nu' : V \times \mathbb{R} \to [0,1]$ *be defined by*

$$\nu'(x,t) = \begin{cases} \wedge\{1 - \alpha : \|x\|_\alpha^* \leq t\}, & if \quad (x,t) \neq (\theta, 0) \\ 1, & if \quad (x,t) = (\theta, 0) \end{cases}$$

Then $\nu' = \nu$.

Definition 2.4. [8]. *Let* (V, A^*) *be a fuzzy antinormed linear space. A sequence* $\{x_n\}_n$ *in* V *is said to be convergent to* $x \in V$ *if given* $t > 0, r \in (0,1)$ *there exist an integer* $n_0 \in \mathbb{N}$ *such that*

$$\nu(x_n - x, t) < r \ \forall\, n \geq n_0.$$

Definition 2.5. [8]. *Let (V, A^*) be a fuzzy antinormed linear space. A sequence $\{x_n\}_n$ in V is said to be Cauchy sequence to $x \in V$ if given $t > 0, r \in (0,1)$ there exist an integer $n_0 \in \mathbb{N}$ such that*

$$\nu(x_{n+p} - x_n, t) < r \ \forall\ n \geq n_0, p = 1, 2, 3, \cdots .$$

Definition 2.6. [8]. *A subset A of a fuzzy antinormed linear space (V, A^*) is said to be bounded if and only if there exist $t > 0, r \in (0,1)$ such that*

$$\nu(x,t) < r \ \forall\ x \in A$$

3. Fuzzy anti-continuity

Throughout this section unless otherwise stated (U, A^*) and (V, B^*) are any two fuzzy anti-normed linear spaces over the same field F.

Definition 3.1. *A mapping $T : (U, A^*) \to (V, B^*)$ is said to be* **fuzzy anti-continuous** *at $x_0 \in U$, if for any given $\epsilon > 0$,$\alpha \in (0,1)$ there exist $\delta = \delta(\alpha, \epsilon) > 0$, $\beta = \beta(\alpha, \epsilon) \in (0,1)$ such that for all $x \in U$*

$$\nu_U(x - x_0, \delta) < \beta \Rightarrow \nu_V(T(x) - T(x_0), \epsilon) < \alpha.$$

Definition 3.2. *A mapping $T : (U, A^*) \to (V, B^*)$ is said to be* **sequentially fuzzy anti-continuous** *at $x_0 \in U$, if for any sequence $\{x_n\}_n, x_n \in U$, $\forall\ n$ with $x_n \to x_0$ implies $T(x_n) \to T(x_0)$ in V, that is for all $t > 0$,*

$$\lim_{n\to\infty} \nu_U(x_n - x_0, t) = 0 \Rightarrow \lim_{n\to\infty} \nu_V(T(x_n) - T(x_0), t) = 0.$$

Definition 3.3. *A mapping $T : (U, A^*) \to (V, B^*)$ is said to be* **strongly fuzzy anti-continuous** *at $x_0 \in U$, if for any given $\epsilon > 0$ there exist $\delta = \delta(\alpha, \epsilon) > 0$ such that for all $x \in U$,*

$$\nu_V(T(x) - T(x_0), \epsilon) \leq \nu_U(x - x_0, \delta)$$

Definition 3.4. *A mapping $T : (U, A^*) \to (V, B^*)$ is said to be* **weakly fuzzy anti-continuous** *at $x_0 \in U$, if for any given $\epsilon > 0, \alpha \in (0,1)$ there exist $\delta = \delta(\alpha, \epsilon) > 0$ such that for all $x \in U$,*

$$\nu_U(x - x_0, \delta) \leq 1 - \alpha \Rightarrow \nu_V(T(x) - T(x_0), \epsilon) \leq 1 - \alpha.$$

Theorem 3.5. *If a mapping T from a fuzzy anti-normed linear space (U, A^*) to a fuzzy anti-normed linear space (V, B^*) is strongly fuzzy anti-continuous then it is weakly fuzzy anti-continuous. But not conversely.*

Proof. From the definition it follows obviously. To show the converse result may not be true we consider the following example.

Example 3.6. As in the example of Note 3.3 of [6], we consider the fuzzy anti-normed linear spaces (X, ν_1) and (X, ν_2). Let $f(x) = \frac{x^4}{1+x^4}\ \forall\ x \in \mathbb{R}$. Now from Example 3 of [11] it directly follows that f is not strongly fuzzy anti-continuous. Here we now show that f is weakly fuzzy anti-continuous on X.
Let $x_0 \in X, \epsilon > 0$ and $\delta \in (0,1)$. Now

$\nu_2(f(x)-f(x_0),\epsilon)<1-\alpha$ if $\frac{k|f(x)-f(x_0|}{\epsilon+k|f(x)-f(x_0|}<1-\alpha$
i.e., if

$$\frac{\epsilon}{\epsilon+k\mid\frac{x^4}{1+x^2}-\frac{x_0^4}{1+x_0^2}\mid}\geq\alpha$$

i.e., if

$$\frac{\frac{\epsilon(1+x^2)(1+x_0^2)}{k|x+x_0||x^2x_0^2+x^2+x_0^2|}}{\frac{\epsilon(1+x^2)(1+x_0^2)}{k|x+x_0||x^2x_0^2+x^2+x_0^2|}+\mid x-x_0\mid}\geq\alpha$$

i.e., if

$$\alpha\mid x-x_0\mid\leq(1-\alpha)\frac{\epsilon}{k}\frac{(1+x^2)(1+x_0^2)}{\mid x+x_0\mid\mid x^2x_0^2+x^2+x_0^2\mid}$$
$$\leq(1-\alpha)\frac{\epsilon}{k}$$

So, depending upon $(1-\alpha)\frac{\epsilon}{k}$ we may choose $\delta>0$ such that
$\alpha(\delta+\mid x-x_0\mid)\leq\delta i.e.,\nu_1(x-x_0,\delta)<1-\alpha.$
Thus we see that for every $\epsilon>0,\alpha\in(0,1)\exists\delta>0$ such that

$$\nu_1(x-x_0,\delta)<1-\alpha\Rightarrow\nu_2(f(x)-f(x_0),\epsilon)<1-\alpha.$$

i.e., f is weakly fuzzy anti-continuous at x_0.

Theorem 3.7. *A mapping T from a fuzzy anti-normed linear space (U,A^*) to a fuzzy anti-normed linear space (V,B^*) is fuzzy anti-continuous if and only if it is sequentially fuzzy anti-continuous.*

Proof. The proof of the above theorem is directly follows from Theorem 13 of [11].

Theorem 3.8. *If a mapping T from a fuzzy anti-normed linear space (U,A^*) to a fuzzy anti-normed linear space (V,B^*) is strongly fuzzy anti-continuous then it is sequentially fuzzy anti-continuous.*

Proof. The proof of the above theorem is directly follows from Theorem 12 of [11].

Theorem 3.9. *Let $T:(U,A^*)\to(V,B^*)$ be a linear operator. If T is sequentially fuzzy anti-continuous at a point $x_0\in U$, then it is sequentially fuzzy anti-continuous on U.*

Proof. Let, $x\in U$ be an arbitrary point and let $\{x_n\}_n$ be a sequence in U such that $x_n\to x$. Then $\forall\ t>0$

$$\lim_{n\to\infty}\nu_U(x_n-x,t)=0$$
$$i.e.,\lim_{n\to\infty}\nu_U((x_n-x+x_0)-x_0,t)=0$$

Since T is sequentially fuzzy anti-continuous at x_0 $\forall\ t>0$ we have

$$\lim_{n\to\infty}\nu_V((x_n-x+x_0)-x_0,t)=0$$
$$i.e.,\lim_{n\to\infty}\nu_V(T(x_n)-T(x)+T(x_0)-T(x_0),t)=0$$

$$i.e., \lim_{n\to\infty} \nu_V(T(x_n) - T(x), t) = 0$$

Thus,

$$\lim_{n\to\infty} \nu_U(x_n - x, t) = 0,\ \forall\ t > 0 \Rightarrow \lim_{n\to\infty} \nu_V(T(x_n) - T(x), t) = 0,\ \forall\ t > 0.$$

Hence the proof.

4. Fuzzy anti-boundedness

Definition 4.1. *A mapping $T : (U, A^*) \to (V, B^*)$ is said to be strongly fuzzy anti-bounded on U if and only if there exist a positive real number M such that for all $x \in U$ and for all $t \in \mathbb{R}^+$,*

$$\nu_V(T(x), t) \leq \nu_U(x, \frac{t}{M})$$

Example 4.2. The zero and identity operators are strongly fuzzy anti-bounded.

Example 4.3. It is an example of a strongly fuzzy anti-bounded linear operator other than the zero and the identity operator.

Let $(V, \|.\|)$ be a normed linear space over the field $K(= \mathbb{R} or \mathbb{C})$. Let, $\alpha_1, \alpha_2 \in R$ such that $\alpha_1 > \alpha_2 > 0$. Again, let $\nu_1, \nu_2 : V \times \mathbb{R}^+ \to [0, 1]$ be defined by

$$\nu_1(x, t) = \frac{\alpha_1\|x\|}{t + \alpha_1\|x\|} and \nu_2(x, t) = \frac{\alpha_2\|x\|}{t + \alpha_2\|x\|}$$

Also, define $a \diamond b, = \max\{a, b\}$ for all $a, b \in [0, 1]$.
Now we shall first show that (V, ν_1) and (V, ν_2) are fuzzy anti-normed linear space.
(i) The condition (i) is obvious.
(ii) $\nu_1(x, t) = 0 \Leftrightarrow \frac{\alpha_1\|x\|}{t+\alpha_1\|x\|} = 0 \Leftrightarrow \|x\| = 0 \Leftrightarrow x = \theta.$
(iii) Let $c \in K$ and $c \neq 0$

$$\nu_1(cx, t) = \frac{\alpha_1 \parallel cx \parallel}{t + \alpha_1 \parallel cx \parallel}$$

$$= \frac{\alpha_1 \parallel x \parallel}{\frac{t}{|c|} + \alpha_1 \parallel x \parallel} = \nu_1(x, \frac{t}{\mid c \mid})$$

(iv)

$$\nu_1(x + y, s + t) = \frac{\alpha_1 \parallel x + y \parallel}{s + t + \alpha_1 \parallel x + y \parallel}$$

$$= \frac{1}{\frac{s+t}{\alpha_1\|x+y\|} + 1}$$

$$\leq \frac{1}{\frac{s+t}{\alpha_1\|x\|+\alpha_1\|y\|} + 1}$$

$$= \frac{\alpha_1 \parallel x \parallel + \alpha_1 \parallel y \parallel}{s + t + \alpha_1 \parallel x \parallel + \alpha_1 \parallel y \parallel}$$

Now if

$$\nu_1(x,s) \geq \nu_1(y,t) \Rightarrow \frac{\alpha_1 \parallel x \parallel}{s+\alpha_1 \parallel x \parallel} \geq \frac{\alpha_1 \parallel y \parallel}{t+\alpha_1 \parallel y \parallel};$$

$$\Rightarrow t \parallel x \parallel - s \parallel y \parallel$$

Therefore,

$$\frac{\alpha_1 \parallel x \parallel + \alpha_1 \parallel y \parallel}{s+t+\alpha_1 \parallel x \parallel + \alpha_1 \parallel y \parallel} - \frac{\alpha_1 \parallel x \parallel}{s+\alpha_1 \parallel x \parallel} \leq 0.$$

Thus

$$\nu_1(x+y, s+t) \leq \frac{\alpha_1 \parallel x \parallel + \alpha_1 \parallel y \parallel}{s+t+\alpha_1 \parallel x \parallel + \alpha_1 \parallel y \parallel}$$

$$\leq \frac{\alpha_1 \parallel x \parallel}{s+\alpha_1 \parallel x \parallel} = \nu_1(x,s) \diamond \nu_1(y,t)$$

Again if $\nu_1(y,t) \geq \nu_1(x,s)$ Similarly it can be shown that

$$\nu_1(x+y, s+t) \leq \frac{\alpha_1 \parallel y \parallel}{t+\alpha_1 \parallel y \parallel} = \nu_1(x,s) \diamond \nu_1(y,t)$$

Hence

$$\nu_1(x+y, s+t) \leq \nu_1(x,s) \diamond \nu_1(y,t)$$

(v)

$$\lim_{t\to\infty} \nu_1(x,t) = \lim_{t\to\infty} \frac{\alpha_1 \parallel x \parallel}{t+\alpha_1 \parallel x \parallel} = 0$$

Hence (V, ν_1) is a fuzzy anti-normed linear space. Similarly (V, ν_2) is also fuzzy anti-normed linear space.
We now define a mapping $T : (V, \nu_1) \to (V, \nu_2)$ by $T(x) = rx$ where $r(\neq 0) \in \mathbb{R}$ is fixed. Clearly T is a linear operator.
Let us choose an arbitrary but fixed $M > 0$ such that $M \geq\mid r \mid$ and $x \in V$. Now,

$$M \geq\mid r \mid \Rightarrow \alpha_1 M \parallel x \parallel \geq \alpha_2 \mid r \mid \parallel x \parallel$$

$$\Rightarrow t + \alpha_1 M \parallel x \parallel \geq t + \alpha_2 \mid r \mid \parallel x \parallel \ \ \forall\ t > 0.$$

$$\Rightarrow \frac{t}{t+\alpha_2 \mid r \mid \parallel x \parallel} \geq \frac{t}{t+\alpha_1 M \parallel x \parallel}\ \forall\ t > 0.$$

$$\Rightarrow \frac{t}{t+\alpha_2 \parallel rx \parallel} \geq \frac{\frac{t}{M}}{\frac{t}{M}+\alpha_1 \parallel x \parallel}\ \forall\ t > 0.$$

$$\Rightarrow 1 - \frac{t}{t+\alpha_2 \parallel rx \parallel} \leq 1 - \frac{\frac{t}{M}}{\frac{t}{M}+\alpha_1 \parallel x \parallel}\ \forall\ t > 0.$$

$$\Rightarrow \frac{\alpha_2 \parallel rx \parallel}{t+\alpha_2 \parallel rx \parallel} \leq \frac{\alpha_1 \parallel x \parallel}{\frac{t}{M}+\alpha_1 \parallel x \parallel}\ \forall\ t > 0.$$

i.e.,

$$\nu_2(T(x), t) \leq \nu_1(x, \frac{t}{M})\ \forall\ t > 0 and\ \forall\ x \in V.$$

Hence T is strongly fuzzy anti-bounded on V.

Definition 4.4. *A mapping $T : (U, A^*) \to (V, B^*)$ is said to be weakly fuzzy anti-bounded on U if and only if for any $\alpha \in (0,1)$ there exist $M_\alpha (> 0)$ such that for all $x \in U$ and for all $t \in \mathbb{R}^+$,*

$$\nu_U(x, \frac{t}{M_\alpha}) \leq 1 - \alpha \Rightarrow \nu_V(T(x), t) \leq 1 - \alpha$$

Theorem 4.5. *Let $T : (U, A^*) \to (V, B^*)$ be a linear operator. If T is strongly fuzzy anti-bounded then it is weakly fuzzy anti-bounded. But not conversely.*

Proof. First we suppose that T is strongly fuzzy anti-bounded. Then there exist $M > 0$ such that $\forall\ x \in U$ and $\forall\ t \in \mathbb{R}$,

$$\nu_V(T(x), t) \leq \nu_U(x, \frac{t}{M})$$

Thus for any $\alpha \in (0, 1)$, there exists $M_\alpha (= M)$ such that

$$\nu_U(x, \frac{t}{M_\alpha}) \leq 1 - \alpha \Rightarrow \nu_V(T(x), t) \leq 1 - \alpha$$

Hence T is weakly fuzzy anti-bounded.

The converse of the above theorem is not necessarily true. For example

Example 4.6. Let $(V, \| \cdot \|)$ be a linear space over the field $K (= \mathbb{R} or \mathbb{C})$ and $\nu_1, \nu_2 : V \times \mathbb{R}^+ \to [0, 1]$ be defined by

$$\nu_1(x, t) = \frac{2\|x\|^2}{t^2 + \|x\|^2}, if t > \|x\|$$

$$= 1, \text{if} 0 < t \leq \|x\|$$

$$and \nu_2(x, t) = \frac{\|x\|}{t + \|x\|}$$

Also define $a \diamond b = max\{a, b\}$

Already we have seen that (V, ν_2) is a fuzzy anti-normed linear space. Now we shall prove that (V, ν_1) is a fuzzy anti-normed linear space.

(i) Clearly follows from the definition of ν_1.

(ii) $\nu_1(x, t) = 0 \Leftrightarrow \frac{2\|x\|^2}{t^2 + \|x\|^2} = 0 \Leftrightarrow \|x\| = 0 \Leftrightarrow x = \theta$.

(iii) Let, $c \in K$ and $c \neq 0$. If $t > \|cx\|$,

$$\nu_1(cx, t) = \frac{2\|cx\|^2}{t^2 + \|cx\|^2} = \frac{2|c|^2\|x\|^2}{t^2 + |c|^2\|x\|^2} = \frac{2\|x\|^2}{(\frac{t}{|c|})^2 + \|x\|^2} = \nu_1(x, \frac{t}{|c|})$$

Again if $0 < t \leq \|cx\|$ then $\nu_1(cx, t) = 1$

and $0 < t \leq \|cx\| \Rightarrow 0 < \frac{t}{|c|} \leq \|x\| \Rightarrow \nu_1(x, \frac{t}{|c|}) = 1$

(iv) Let $s, t \in \mathbb{R}^+, x, y \in V$

If $0 < s + t \leq \|x + y\|$, we have the following possibilities

(a) $0 < s \leq \|x\|$ and $0 < t \leq \|y\|$

(b) $0 < s \leq \|x\|$ and $t > \|y\|$

(c) $0 < t \leq \|y\|$ and $s > \|x\|$.

In each case $\nu_1(x + y, s + t) = 1 = \nu_1(x, s) \diamond \nu_1(y, t)$ Again, if $s + t > \|x + y\|$, we have the following four possibilities

(a) $s > \|x\|,\ t \leq \|y\|$

(b) $s \leq \|x\|, t > \|y\|$
(c) $s \leq \|x\|, t \leq \|y\|$
(d) $s > \|x\|, t > \|y\|$
In the cases (a), (b), (c)

$$\nu_1(x+y, s+t) = \frac{2\|x+y\|^2}{(s+t)^2 + \|x+y\|^2}$$
$$< 1 = \nu_1(x,s) \diamond \nu_1(y,t)$$

So, we now suppose that $s > \|x\| and t > \|y\|$ Now, $s+t > \|x\|+\|y\| \geq \|x+y\|$. Therefore,

$$\nu_1(x+y, s+t) = \frac{2\|x+y\|^2}{(s+t)^2 + \|x+y\|^2}$$
$$\leq \frac{2(\|x\|+\|y\|)^2}{(s+t)^2 + (\|x\|+\|y\|)^2}$$

Hence we have

$$\nu_1(x+y, s+t) \leq \frac{2(\|x\|+\|y\|)^2}{(s+t)^2 + (\|x\|+\|y\|)^2}$$
$$\leq \frac{2\|y\|^2}{t^2 + \|y\|^2} = \nu_1(y,t)$$

when $\nu_1(x,s) \leq \nu_1(y,t)$
Similarly,

$$\nu_1(x+y, s+t) \leq \frac{2(\|x\|+\|y\|)^2}{(s+t)^2 + (\|x\|+\|y\|)^2}$$
$$\leq \frac{2\|x\|^2}{t^2 + \|x\|^2} = \nu_1(x,s)$$

when $\nu_1(y,t) \leq \nu_1(x,s)$
Thus

$$\nu_1(x+y, s+t) \leq \nu_1(x,s) \diamond \nu_1(y,t)$$

$$(v) \lim_{t\to\infty} \nu_1(x,t) = \lim_{t\to\infty} \frac{2\|x\|^2}{t^2 + \|x\|^2} = 0$$

Thus we see that (V, ν_1) is a fuzzy anti-normad linear space.
Now we define a linear operator $T : (U, \nu_1) \to (V, \nu_2)$ by $T(x) = x, \; \forall \, x \in V$. Let, $\alpha \in (0,1), x \in V and t \in \mathbb{R}^+$ and choose $M_\alpha = \frac{1}{1-\alpha}$. We now prove that

$$\nu_1(x, \frac{t}{M_\alpha}) \leq 1-\alpha \Rightarrow \nu_2(T(x), t) \leq 1-\alpha.$$

$$\nu_1(x, \frac{t}{M_\alpha}) \leq 1-\alpha$$
$$\Rightarrow \frac{2\| x \|^2}{t^2(1-\alpha)^2 + \| x \|^2} \leq 1-\alpha$$
$$\Rightarrow 1 - \frac{2\| x \|^2}{t^2(1-\alpha)^2 + \| x \|^2} \geq 1-(1-\alpha) = \alpha$$

$$\Rightarrow \frac{t^2(1-\alpha)^2 - \| x \|^2}{t^2(1-\alpha)^2 + \| x \|^2} \geq \alpha$$
$$\Rightarrow t^2(1-\alpha)^3 \geq (1+\alpha)\| x \|^2$$
$$\Rightarrow \| x \| \leq \frac{t(1-\alpha)\sqrt{1-\alpha}}{\sqrt{1+\alpha}}$$
$$\Rightarrow t + \| x \| \leq t\frac{(1-\alpha)\sqrt{1-\alpha} + \sqrt{1+\alpha}}{\sqrt{1+\alpha}}$$
$$\Rightarrow \frac{t}{t+\| x \|} \geq \frac{\sqrt{1+\alpha}}{(1-\alpha)\sqrt{1-\alpha} + \sqrt{1+\alpha}}$$
$$\Rightarrow 1 - \frac{t}{t+\| x \|} \leq 1 - \frac{\sqrt{1+\alpha}}{(1-\alpha)\sqrt{1-\alpha} + \sqrt{1+\alpha}}$$
$$\Rightarrow \frac{\| x \|}{t+\| x \|} \leq \frac{(1-\alpha)\sqrt{1-\alpha}}{(1-\alpha)\sqrt{1-\alpha} + \sqrt{1+\alpha}}$$

Now

$$\frac{(1-\alpha)\sqrt{1-\alpha}}{(1-\alpha)\sqrt{1-\alpha} + \sqrt{1+\alpha}} \leq 1-\alpha \Leftrightarrow \sqrt{1-\alpha} \leq (1-\alpha)\sqrt{1-\alpha} + \sqrt{1+\alpha}$$
$$\Leftrightarrow \alpha\sqrt{1-\alpha} \leq \sqrt{1+\alpha}$$
$$\Leftrightarrow 1+\alpha+\alpha^3 \geq \alpha^2$$

which is true for all $\alpha \in (0,1)$.
Hence

$$\nu_1(x, \frac{t}{M_\alpha}) \leq 1-\alpha \Rightarrow \nu_2(T(x), t) \leq 1-\alpha.$$

Thus T is weakly fuzzy anti-bounded on V.
Now for $t > \|x\|, x \neq \theta$ we have

$$\nu_2(T(x), t) \leq \nu_1(x, \frac{t}{M}) \Leftrightarrow \frac{\|x\|}{t+\|x\|} \leq \frac{2M\|x\|^2}{t^2 + M\|x\|^2}$$
$$\Leftrightarrow t^2\|x\| + M\|x\|^3 \leq 2tM\|x\|^2 + 2M\|x\|^3$$
$$\Leftrightarrow \left(2t\|x\|^2 + \|x\|^3\right) M \geq t^2\|x\|$$
$$M \to \infty \text{ as } t \to \infty$$

Hence T is not strongly fuzzy anti-bounded on V.

Definition 4.7. *A linear operator $T : (U, A^*) \to (V, B^*)$ is said to be uniformly fuzzy anti-bounded if and only if there exist $M > 0$ such that*

$$\|T(x)\|_\alpha^* \geq M \|x\|_\alpha^*, \alpha \in (0,1)$$

where $\{\|\cdot\|_\alpha^* : \alpha \in (0,1)\}$ is ascending family of fuzzy α-norms.

Theorem 4.8. *Let $T : (U, A^*) \to (V, B^*)$ be a linear operator and (U, A^*) and (V, B^*) satisfies (vi), (vii) and (viii). Then T is strongly fuzzy anti-bounded if and only if it is uniformly fuzzy anti-bounded with respect to fuzzy α-norms, $\alpha \in (0,1)$.*

Proof. Let $\{\|\cdot\|_\alpha^* : \alpha \in (0,1)\}$ be ascending family of α-norms. First suppose that T is strongly fuzzy anti-bounded. Then there exist $M > 0$ such that $\forall\ x \in U$ and $\forall\ s \in R$,

$$\nu_V(T(x), t) \leq \nu_U(x, \frac{s}{M}) i.e., \nu_V(T(x), t) \leq \nu_U(Mx, s)$$

$\|Mx\|_\alpha^* > t \Rightarrow \wedge\{s : \nu(Mx, s) \leq 1 - \alpha\} > t.$

$$\Rightarrow \exists s_0 > t \text{ such that } \nu(Mx, s_0) \leq 1 - \alpha$$

$$\Rightarrow \exists s_0 > t \text{ such that } \nu(T(x), s_0) \leq 1 - \alpha$$

$$\Rightarrow \|T(x)\|_\alpha^* \geq s_0 > t$$

Hence $\|T(x)\|_\alpha^* \geq \|Mx\|_\alpha^* = M\,\|x\|_\alpha^*$.
Thus T is uniformly fuzzy anti-bounded.
Conversely, suppose that there exist $M > 0$ such that $\forall\ x \in U$ and $\forall\ \alpha \in (0,1)$

$$\|T(x)\|_\alpha^* \geq M\,\|x\|_\alpha^*$$

Let $p > \nu_U(Mx, s) \Rightarrow p > \wedge\{\alpha \in (0,1) : \|Mx\|_\alpha^* \leq s\}$
$\Rightarrow$ there exist $\alpha_0 \in (0,1)$ such that $p > \alpha_0$ and $\|Mx\|_\alpha^* \leq s$
$\Rightarrow \|T(x)\|_\alpha^* \leq s$
$\Rightarrow \nu_V(T(x), s) \leq 1 - \alpha_0 < p.$

Hence, $\nu_V(T(x), s) \leq \nu_U(Mx, s) = \nu_U(x, \frac{s}{M})$.

Thus T is strongly fuzzy anti-bounded.

Theorem 4.9. *Let $T : (U, A^*) \to (V, B^*)$ be a linear operator. Then,*
(i) *T is strongly fuzzy anti-continuous on U if T is strongly fuzzy anti-continuous at a point $x_0 \in U$.*
(ii) *T is strongly fuzzy anti-continuous if and only if T is strongly fuzzy anti-bounded.*

Proof. **(i)** since, T is strongly fuzzy anti-continuous at $x_0 \in U$, for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\nu_V(T(x) - T(x_0), \epsilon) \leq \nu_U(x - x_0, \delta)$$

Taking $y \in U$ and replacing $x by x + x_0 - y$, we get,
$\nu_V(T(x) - T(x_0), \epsilon) \leq \nu_U(x - x_0, \delta)$
$\Rightarrow \nu_V(T(x + x_0 - y) - T(x_0), \epsilon) \leq \nu_U(x + x_0 - y - x_0, \delta)$
$\Rightarrow \nu_V(T(x) + T(x_0) - T(y) - T(x_0), \epsilon) \leq \nu_U(x - y, \delta)$
$\Rightarrow \nu_V(T(x) - T(y), \epsilon) \leq \nu_U(x - y, \delta)$
Since, $y \in U$ is arbitrary, T is strongly fuzzy anti-continuous on U.
(ii) First we suppose that T is strongly fuzzy anti-bounded. Thus there exist a positive real number M such that for all $x \in U$ and for all $\epsilon \in R^+$,

$$\nu_V(T(x), \epsilon) \leq \nu_U(x, \frac{\epsilon}{M})$$

$$i.e., \nu_V(T(x) - T(\theta), \epsilon) \leq \nu_U(x - \theta, \frac{\epsilon}{M})$$

$$i.e, \nu_V(T(x) - T(\theta), \epsilon) \leq \nu_U(x - \theta, \delta)$$

where $\delta = \frac{\epsilon}{M}$.
Thus T is strongly fuzzy anti-continuous at θ and hence T is strongly fuzzy anti-continuous on U.
Conversely, suppose that T is strongly fuzzy anti-continuous on U. Using fuzzy anti-continuity of T at $x = \theta$ for $\epsilon = 1$ there exist $\delta > 0$ such that for all $x \in U$,

$$\nu_V(T(x) - T(\theta), 1) \leq \nu_U(x - \theta, \delta).$$

If $x \neq \theta$and $t > 0$. Putting $x = ut$
$\nu_V(T(x), t) = \nu_V(uT(u), t) = \nu_V(T(u), 1) \leq \nu_U(u, \delta) = \nu_U(\frac{x}{t}, \delta) = \nu_U(x, \frac{t}{M})$,
where $M = \frac{1}{\delta}$. So, $\nu_V(T(x), t) \leq \nu_U(x, \frac{t}{M})$.
If $x \neq \theta$ and $t \leq 0$ then $\nu_V(T(x), t) = 1 = \nu_U(x, \frac{t}{M})$.
If $x = \theta$ and $t \in R$, then $T(\theta_U) = \theta_V$ and

$$\nu_V(\theta_V, t) = \nu_U(\theta_U, \frac{t}{M}) = 0, \text{ if } t > 0.$$

$$\nu_V(\theta_V, t) = \nu_U(\theta_U, \frac{t}{M}) = 1, \text{ if } t \leq 1.$$

Hence T is strongly fuzzy anti-bounded.

Theorem 4.10. *Let $T : (U, A^*) \to (V, B^*)$ be a linear operator. Then,*
(i) *T is weakly fuzzy anti-continuous on U if T is weakly fuzzy anti-continuous at a point $x_0 \in U$.*
(ii) *T is weakly fuzzy anti-continuous if and only if T is weakly fuzzy anti-bounded.*

Proof. **(i)** Since, T is weakly fuzzy anti-continuous at x_0 in U, for $\epsilon > 0$ and $\alpha \in (0, 1)$ there exist $\delta = \delta(\alpha, \epsilon) > 0$ such that $\forall\ x \in U$

$$\nu_U(x - x_0, \delta) \leq 1 - \alpha \Rightarrow \nu_V(T(x) - T(x_0), \epsilon) \leq 1 - \alpha.$$

Taking $y \in U$ and replacing x by $x + x_0 - y$ we get,

$$\nu_U(x + x_0 - y - x_0, \delta) \leq 1 - \alpha \Rightarrow \nu_V(T(x + x_0 - y) - T(x_0), \epsilon) \leq 1 - \alpha$$

$$i.e., \nu_U(x - y, \delta) \leq 1 - \alpha \Rightarrow \nu_V(T(x) + T(x_0) - T(y) - T(x_0), \epsilon) \leq 1 - \alpha$$

$$i.e., \nu_U(x - y, \delta) \leq 1 - \alpha \Rightarrow \nu_V(T(x) - T(y), \epsilon) \leq 1 - \alpha$$

Since, $y (\in U)$ is arbitrary it follows that T is weakly fuzzy anti-continuous on U.

(ii) First we suppose that T is fuzzy anti-bounded. Thus for any $\alpha \in (0, 1)$ there exist $M_\alpha > 0$ such that $\forall\ t \in R$ and $\forall\ x \in U$ we have

$$\nu_U(x, \frac{t}{M}) \leq 1 - \alpha \Rightarrow \nu_V(T(x), t) \leq 1 - \alpha$$

Therefore,

$$\nu_U(x - \theta, \frac{t}{M}) \leq 1 - \alpha \Rightarrow \nu_V(T(x) - T(\theta), t) \leq 1 - \alpha$$

$$i.e., \nu_U(x - \theta, \frac{\epsilon}{M_\alpha}) \leq 1 - \alpha \Rightarrow \nu_V(T(x) - T(\theta), \epsilon) \leq 1 - \alpha$$

$$i.e., \nu_U(x - \theta, \delta) \leq 1 - \alpha \Rightarrow \nu_V(T(x) - T(\theta), \epsilon) \leq 1 - \alpha$$

where $\delta = \frac{\epsilon}{M_\alpha}$
Thus, T is weakly fuzzy anti-continuous at x_0 and hence weakly fuzzy anti-continuous on U.
Conversely, suppose that T is weakly fuzzy anti-continuous on U. Using continuity of T at θ and taking $\epsilon = 1$ we have for all $\alpha \in (0,1)$ there exists $\delta(\alpha, 1) > 0$ such that for all $x \in U$,

$$\nu_U(x-\theta, \delta) \le 1-\alpha \Rightarrow \nu_V(T(x)-T(\theta), 1) \le 1-\alpha$$

$$i.e., \nu_U(x, \delta) \le 1-\alpha \Rightarrow \nu_V(T(x), 1) \le 1-\alpha.$$

If $x \ne \theta$ and $t > 0$. Putting $x = \frac{u}{t}$ we have,

$$\nu_U(\frac{u}{t}, \delta) \le 1-\alpha \Rightarrow \nu_V(T(\frac{u}{t}), 1) \le 1-\alpha$$

$$i.e., \nu_U(u, t\delta) \le 1-\alpha \Rightarrow \nu_V(T(u), t) \le 1-\alpha$$

$$i.e., \nu_U(u, \frac{t}{M_\alpha}) \le 1-\alpha \Rightarrow \nu_V(T(\frac{u}{t}), 1) \le 1-\alpha$$

where $M_\alpha = \frac{1}{\delta(\alpha,1)}$ If $x \ne \theta$ and $t \le 0$, $\nu_U(x, \frac{t}{M_\alpha}) = \nu_V(T(x), t) = 1$ for any $M_\alpha > 0$.
If $x = \theta$ then for $M_\alpha > 0$,

$$\nu_U(x, \frac{t}{M_\alpha}) = \nu_V(T(x), t) = 0, if t > 0$$

$$\nu_U(x, \frac{t}{M_\alpha}) = \nu_V(T(x), t) = 1, if t \le 0$$

Hence, T is weakly fuzzy anti-bounded.

Theorem 4.11. *Let $T : (U, A^*) \to (V, B^*)$ be a linear operator and (U, A^*) and (V, B^*) satisfies (vi), (vii) and (viii). Then T is weakly fuzzy anti-bounded if and only if T is fuzzy anti-bounded with respect to α-norms.*

Proof. First we suppose that T is weakly fuzzy anti-bounded. Then for all $\alpha \in (0,1)$ there exist $M_\alpha > 0$ such that $\forall\, x \in U, t \in R$ we have

$$\nu_U(x, \frac{t}{M_\alpha}) \le 1-\alpha \Rightarrow \nu_V(T(x), t) \le 1-\alpha$$

Hence we get, $\nu_U(M_\alpha x, t) \le 1-\alpha \Rightarrow \nu_V(T(x), t) \le 1-\alpha$
$i.e., \wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \le t\} \le 1-\alpha \Rightarrow \wedge\{\beta \in (0,1) : \|T(x)\|_\beta^* \le t\} \le 1-\alpha$
Now we show that

$$\wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \le t\} \le 1-\alpha \Leftrightarrow \|M_\alpha x\|_\alpha^* \le t$$

If $x = \theta$ then the relation is obvious.
Suppose $x \ne \theta$.
Now, if

$$\wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \le t\} < 1-\alpha then \|M_\alpha x\|_\alpha^* \le t \tag{4.1}$$

If $\wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \le t\} = 1-\alpha$ then there exists a decreasing sequence $\{\alpha_n\}_n$ in $(0,1)$ such that $\alpha_n \to \alpha$ and $\|M_\alpha x\|_\alpha^* \le t$ Then by Theorem 3.7 we have

$$\|M_\alpha x\|_\alpha^* \le t \tag{4.2}$$

From (4.1) and (4.2) we get

$$\wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \le t\} \le 1-\alpha \Rightarrow \|M_\alpha x\|_\alpha^* \le t \tag{4.3}$$

Next we suppose that $\|M_\alpha x\|_\alpha^* \le t$.
If $\|M_\alpha x\|_\alpha^* < t$ then $\nu_U(M_\alpha x, t) \le 1-\alpha$. i.e.,

$$\wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \le t\} \le 1-\alpha \tag{4.4}$$

If $\|M_\alpha x\|_\alpha^* = t$ i.e.,$\wedge\{s : \nu_U(M_\alpha x, s) \le 1-\alpha\} = t$ then there exist an increasing sequence $\{s_n\}_n$ in $\mathbb{R}^+$ such that $s_n \to t$ and

$$\begin{aligned} \nu_U(M_\alpha x, s_n) \le 1-\alpha &\Rightarrow \lim_{n\to\infty} \nu_U(M_\alpha x, s_n) \le 1-\alpha \\ &\Rightarrow \nu_U(M_\alpha x, \lim_{n\to\infty} s_n) \le 1-\alpha \\ &\Rightarrow \nu_U(M_\alpha x, t) \le 1-\alpha \\ &\Rightarrow \wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \le t\} \le 1-\alpha \end{aligned}$$

Hence from (4.4) it follows that

$$\|M_\alpha x\|_\alpha^* \le t \Rightarrow \wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \le t\} \le 1-\alpha \tag{4.5}$$

From (4.3) and (4.5) we have

$$\wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \le t\} \le 1-\alpha \Leftrightarrow \|M_\alpha x\|_\alpha^* \le t \tag{4.6}$$

In the similar way we can show that

$$\wedge\{\beta \in (0,1) : \|T(x)\|_\beta^* \le t\} \le 1-\alpha \Leftrightarrow \|T(x)\|_\alpha^* \le t \tag{4.7}$$

From (4.6) and (4.7) we have $\nu_U(M_\alpha x, t) \le 1-\alpha \Rightarrow \nu_V(T(x), t) \le 1-\alpha$
Then

$$\|M_\alpha x\|_\alpha^* \le t \Rightarrow \|T(x)\|_\alpha^* \le t$$

This implies that

$$\|T(x)\|_\alpha^* \ge \|M_\alpha x\|_\alpha^*$$

Conversely, suppose that $\forall\ \alpha \in (0,1), \exists M_\alpha > 0$ such that $\forall\ xInU$,

$$\|T(x)\|_\alpha^* \ge \|M_\alpha x\|_\alpha^*$$

Then for $x \ne \theta and \ \forall\ t > 0$,

$$\|M_\alpha x\|_\alpha^* \le t \Rightarrow \|T(x)\|_\alpha^* \le t$$

i.e.,

$$\wedge\{s : \nu_U(M_\alpha x, s) \le 1-\alpha\} \le t \Rightarrow \wedge\{s : \nu_V(T(x), s) \le 1-\alpha\} \le t$$

In the similar way as above we can show that

$$\wedge\{s : \nu_U(M_\alpha x, s) \le 1-\alpha\} \le t \Leftrightarrow \nu_U(M_\alpha x, t) \le 1-\alpha$$

and

$$\wedge\{s : \nu_U(T(x), s) \le 1-\alpha\} \le t \Leftrightarrow \nu_U(T(x)x, t) \le 1-\alpha$$

Thus we have

$$\nu_U(x, \frac{t}{M_\alpha}) \leq 1 - \alpha \Rightarrow \nu_V(T(x), t) \leq 1 - \alpha, \ \forall \ x \in U$$

If $x \neq \theta, t \leq 0$ and if $x = \theta, t > 0$ then the above relation is obvious. Hence the proof.

Theorem 4.12. *Let $T : (U, A^*) \to (V, B^*)$ be a linear operator and (U, A^*) and (V, B^*) satisfies (vi), (vii) and (viii). If U is finite dimensional then T is weakly fuzzy anti-bounded.*

Proof. Since, (U, A^*) and (V, B^*) satisfies (vi) and (viii) we may suppose that $\{\|\cdot\|_\alpha^* : \alpha \in (0,1)\}$ is ascending family of fuzzy α-anti-norms.
Since T is of finite dimension, $T : (U, A^*) \to (V, B^*)$ is bounded linear operator for each $\alpha \in (0,1)$. Thus by Theorem 4.11 it follows that T is weakly fuzzy anti-bounded.

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Transversality and separation of zeroes in second order differential equations

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Abstract. In this paper we consider some second order differential equations in a finite time interval. We give some conditions which ensure that the non-trivial solutions of these differential equations have a finite number of transverse zeroes.

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1. Introduction

The following second order non-autonomous and non-linear differential equation was considered in [1]:

$$(Lu)(t) := -(p(t)u'(t))' + q(t)u(t) = f(t, u(t)), \quad t \in (a, b). \tag{1.1}$$

Here $(a, b) \subseteq \mathbb{R}$, f is a non-linear continuous function, not necessarily Lipschitz continuous function in u, $f(t, 0) \equiv 0$, $p, q \in C^1[a, b]$ and $p(t) > 0$ for all $t \in [a, b]$.

Some sufficient conditions on the non-linearity of f were given which ensure that non-trivial solutions of the second order differential equations of the form (1.1) have a finite number of transverse zeroes ($u(0) = u'(0) = 0$) in a given finite time interval (a, b).

The solution of the equation (1.1) isn't unique when the function f is non-Lipschitz. For example the differential equation

$$-u'' = 24\sqrt{|u|}, \ t \in \mathbb{R}, \tag{1.2}$$

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has at least two solutions, $u_1 \equiv 0$ and u_2 given by

$$u_2(t) = \begin{cases} 0, & t \le 0 \\ -4t^4, & t > 0 \end{cases} . \tag{1.3}$$

Hence there exist non-unique, non-zero solutions possessing a non-transverse zero and, in particular, infinitely many zeroes on any open time interval containing $t = 0$.

In fact, Zeidler in [5] proved that there exist ordinary differential equations which have uncountable many solutions satisfying the conditions of transversality: $u(0) = u'(0) = 0$.

Laister and Beardmore in [1] give only locally conditions on function f, near $u = 0$, and independent of the sign of q which ensure that non-trivial solutions of (1.1) have a finite number of transverse zeroes in a finite time interval ([1], Theorem 2.1).

Let S a finite subset of $[a, b]$. and we denote by $[a, b]_S = [a, b] \setminus S$.

For the case when the equation (1.1) is written in the form

$$(Lu)(t) := -p(t)u''(t) + r(t)u'(t) + q(t)u(t) = f(t, u(t)), \quad t \in (a, b), \tag{1.4}$$

the condition $p \in C^1[a, b]$ can be replaced by $p \in C^1[a, b]_S$, and the situation described above remains true.

For example, with $S = \{0\}$, the differential equation

$$-(sgn\ t + 3)u''(t) = 144\sqrt{|u(t)|},\ t \in \mathbb{R}_S, \tag{1.5}$$

has at least two solutions, $u_1 \equiv 0$ and u_3 given by

$$u_3(t) = \begin{cases} -36(t+2)^4, & t < -2 \\ 0, & -2 \le t < 0 \\ -4t^4, & t > 0 \end{cases} . \tag{1.6}$$

Hence there exist non-unique, non-zero solutions possessing a non-transverse zero and, in particular, infinitely many zeroes on any open interval included in $(-2, 0)$.

2. Main results

We consider a second order differential equation of the form:

$$F(t, u, u', u'') = 0, \quad t \in (a, b) \subseteq \mathbb{R}. \tag{2.1}$$

For the convenience of the reader, following I.A. Rus ([3]), we present the proofs of the next two results:

Theorem 2.1. *We suppose that the following conditions are satisfied:*

1^o *the function F is homogeneous with respect to variables u, u', u'';*

2^o *for all $t_0 \in (a, b)$, $u_0', u_0'' \in \mathbb{R}$ there exists a unique solution of the equation (2.1) such that $u'(t_0) = u_0'$, $u''(t_0) = u_0''$.*

Then, if t_1 and t_2 are two successive zeroes of u_1', where u_1 is a solution of the equation (2.1), every other solution u_2 of the equation (2.1), for which $u_2'(t_1) \neq 0$, $u_2'(t_2) \neq 0$, has in (t_1, t_2) a unique zero.

Proof. We suppose that $u_2'(t) \neq 0$ for all $t \in [t_1, t_2]$. It is not a restriction to assume that

$$u_1'(t) > 0 \quad \text{for} \quad t \in (t_1, t_2) \quad \text{and}$$
$$u_2'(t) > 0 \quad \text{for} \quad t \in [t_1, t_2].$$

Then by Tonelli's Lemma (see [2]) it results that there exist $\lambda > 0$ and $t_0 \in (t_1, t_2)$ such that

$$u_2'(t_0) = \lambda u_1'(t_0) \quad \text{and}$$
$$u_2''(t_0) = \lambda u_1''(t_0).$$

From the conditions $1^o, 2^o$ we get that $u_2(t) \equiv \lambda u_1(t)$, i.e. a contradiction, which proves the theorem. □

Theorem 2.2. *We suppose that:*

1^o *the function F is homogeneous with respect to variables u, u', u'';*

2^o *for all $t_0 \in (a,b)$, $u_0, u_0' \in \mathbb{R}$ there exists a unique solution of the equation (2.1) such that $u(t_0) = u_0$, $u'(t_0) = u_0'$;*

3^o *the equation in t*

$$F(t, \gamma^2, \gamma, 1) = 0$$

hasn't any solution in the interval (a,b), for all $\gamma \in \mathbb{R}^*$.

Then for every solution u of the equation (2.1) the zeroes of u and u' separate each other on the interval $[a,b]$.

Proof. It is sufficient to prove that, if t_1, t_2 are two successive zeroes of u', then u has one zero in the interval (t_1, t_2).

We suppose that $u(t) \neq 0$, for all $t \in [t_1, t_2]$. By Tonelli's Lemma there exist $\lambda \in \mathbb{R}^*$ and $t_0 \in (t_1, t_2)$ such that

$$u(t_0) = \lambda u'(t_0) \quad \text{and} \quad u'(t_0) = \lambda u''(t_0).$$

We obtain that

$$u'(t_0) = \frac{1}{\lambda} u(t_0) \quad \text{and} \quad u''(t_0) = \frac{1}{\lambda^2} u(t_0).$$

Then, from the equation (2.1), we have that

$$F(t_0, u(t_0), u'(t_0), u''(t_0)) = 0$$

or

$$F(t_0, u(t_0), \frac{1}{\lambda} u(t_0), \frac{1}{\lambda^2} u(t_0)) = 0.$$

Because $u(t_0) \neq 0$ and $\lambda \neq 0$, by using the condition 1^o, we obtain that

$$F(t_0, \lambda^2, \lambda, 1) = 0$$

i.e. a contradiction with the condition 3^o, which proves the theorem. □

Corollary 2.3. *We suppose that the conditions of Theorem 2.1. are satisfied. If t_1 and t_2 are two successive transverse zeroes of u_1, where u_1 is a solution of the equation (2.1), then every other solution u_2 of the equation (2.1), for which $u_2'(t_1) \neq 0$, $u_2'(t_2) \neq 0$, has in (t_1, t_2) a unique zero.*

Remark 2.4. In the equation (1.1) we suppose that

1^o the function f is homogeneous in u

2^o for all $t_0 \in (a,b)$, $u_0', u_0'' \in \mathbb{R}$ there exists a unique solution of the equation (1.1) such that $u'(t_0) = u_0'$, $u''(t_0) = u_0''$.

Then, if t_1 and t_2 are two successive zeroes of u_1, where u_1 is a solution of the equation (1.1), every other solution u_2 of the equation (1.1), for which $u_2'(t_1) \neq 0$, $u_2'(t_2) \neq 0$, has in (t_1, t_2) a unique zero.

Remark 2.5. In the equation (1.1) we suppose that

1^o f is homogeneous in u;

2^o for all $t_0 \in (a,b)$, $u_0, u_0' \in \mathbb{R}$ there exists a unique solution of the equation (1.1) such that $u(t_0) = u_0$, $u'(t_0) = u_0'$;

3^o the equation in t

$$p(t) + p'(t)\gamma - q(t)\gamma^2 + f(t, \gamma^2) = 0$$

hasn't any solution in the interval (a,b), for all $\gamma \in \mathbb{R}^*$.

Then for every solution u of the equation (1.1) the zeroes of u and u' separate each other on the interval $[a,b]$.

Theorem 2.6. *We suppose that:*

1^o *the function F is homogeneous with respect to variables u, u', u'';*

2^o *there exists a solution of the equation (2.1) that has a transverse zero in (a,b),*

3^o *the equation in t*

$$F(t, \gamma^2, \gamma, 1) = 0$$

hasn't any solution in the interval (a,b), for all $\gamma \in \mathbb{R}^*$.

Then for every solution u of the equation (2.1) the non-transverse zeroes of u and u' separate each other on the interval $[a,b]$.

Proof. Let u be the solution of the equation (2.1) that has a transverse zero $t_* \in (a,b)$, i.e. $u(t_*) = u'(t_*) = 0$. It is sufficient to prove that if t_1, t_2 are two successive zeroes of u', which aren't transverse zeroes for u, then u has one zero in the interval (t_1, t_2).

We suppose that $u(t) \neq 0$, for all $t \in [t_1, t_2]$. By Tonelli's Lemma there exist $\lambda \in \mathbb{R}^*$ and $t_0 \in (t_1, t_2)$ such that

$$u(t_0) = \lambda u'(t_0) \text{ and } u'(t_0) = \lambda u''(t_0).$$

We obtain that

$$u'(t_0) = \frac{1}{\lambda} u(t_0) \text{ and } u''(t_0) = \frac{1}{\lambda^2} u(t_0).$$

Then, from the equation (2.1), we have that

$$F(t_0, u(t_0), u'(t_0), u''(t_0)) = 0$$

or

$$F(t_0, u(t_0), \frac{1}{\lambda} u(t_0), \frac{1}{\lambda^2} u(t_0)) = 0.$$

Because $u(t_0) \neq 0$ and $\lambda \neq 0$, by using the condition 1^o, we obtain that

$$F(t_0, \lambda^2, \lambda, 1) = 0$$

i.e. a contradiction with the condition 3^o, which proves the theorem. □

Let us consider the following second order non-autonomous differential equation

$$(Lu)(t) := -(p(t)u'(t))' + q(t)u(t) = 0, t \in (a,b), \tag{2.2}$$

where the p and q are such that

$$p, q \in C^1[a,b], p(t) > 0, t \in [a,b]. \tag{2.3}$$

It is well know the following result:

Theorem 2.7. *We suppose that the condition (2.3) holds. If u is any solution of (2.2) satisfying $u(t_0) = u'(t_0) = 0$, for some $t_0 \in [a,b]$, then $u \equiv 0$ on $[a,b]$.*

Corollary 2.8. *Let the hypotheses of Theorem 2.7 hold. If u is any non-trivial solution of (2.2), then u has a finite number of zeroes in $[a,b]$.*

Proof. Suppose that u has an infinite number of zeroes $t_n \in [a,b]$, $n \in \mathbb{N}$. Then by Bolzano-Weierstrass theorem and the continuity of u the exists a subsequence t_{n_j} such that $t_{n_j} \to t_0$ as $j \to \infty$ and $u(t_0) = 0$ for some $t_0 \in [a,b]$. By applying Rolle's theorem to u on $[t_0, t_{n_j}]$ (or $[t_{n_j}, t_0]$) and letting $j \to \infty$ shows that $u'(t_0) = 0$. Hence $u \equiv 0$ on $[a,b]$ by Theorem 2.7, as required. □

Remark 2.9. In the conditions of Theorem 2.7 any non-trivial solution of the equation (2.2) hasn't multiple zeroes.

Theorem 2.10. *Consider the following problem*

$$(Lu)(t) := -(p(t)u'(t))' + q(t)u(t) = f(t, u(t)), \quad t \in (a,b) \tag{2.4}$$

$$u(t_0) = u'(t_0) = 0. \tag{2.5}$$

If there exists $L_f > 0$ such that

$$|f(t,u) - f(t,v)| \le L_f|u - v|, t \in [a,b], \text{ and } u, v \in \mathbb{R}, \tag{2.6}$$

then there exists a unique solution of the problem (2.4)+(2.5).

Proof. The equation (2.4) with the conditions (2.5), $u(t_0) = u'(t_0) = 0$, is equivalent with the following fixed point equation:

$$u = A(u), \tag{2.7}$$

where $u \in C^2[a,b]$ and the operator $A : (C^2[a,b], ||.||_\tau) \to (C^2[a,b], ||.||_\tau)$ is defined by

$$(A(u))(t) = \int_{t_0}^{t} \frac{1}{p(r)} \left(\int_{t_0}^{r} [q(s)u(s) - f(s, u(s))]\, ds \right) dr. \tag{2.8}$$

Here

$$||u||_\tau = \max_{t \in [a,b]} |u(t)| e^{-\tau|t-a|}, \quad \tau > 0.$$

We have

$$|(A(u))(t) - (A(v))(t)| =$$

$$= \left| \int_{t_0}^{t} \frac{1}{p(r)} \left(\int_{t_0}^{r} [q(s)(u(s)-v(s)) - f(s,u(s)) + f(s,v(s))]\, ds \right) dr \right| \leq$$

$$\leq \quad \left| \int_{t_0}^{t} \frac{1}{p(r)} \left| \left(\int_{t_0}^{r} |q(s)|\, |u(s)-v(s)| e^{-\tau|s-t_0|} e^{\tau|s-t_0|} ds \right) \right| dr \right| \leq$$

$$\leq \quad \left| \int_{t_0}^{t} \frac{1}{p(r)} \left| \left(\int_{t_0}^{r} L_f |u(s)-v(s)| e^{-\tau|s-t_0|} e^{\tau|s-t_0|} ds \right) \right| dr \right| \leq$$

$$\leq M_p(M_q + L_f)||u-v||_\tau \left| \int_{t_0}^{t} \left| \int_{t_0}^{r} e^{\tau|s-t_0|} ds \right| dr \right|,$$

where $M_p = \max_{t\in[a,b]} \frac{1}{p(t)}$ and $M_q = \max_{t\in[a,b]} |q(t)|$.

But

$$\left| \int_{t_0}^{r} e^{\tau|s-t_0|} ds \right| \leq \frac{1}{\tau} e^{\tau|r-t_0|},$$

and so,

$$\left| \int_{t_0}^{t} \left| \int_{t_0}^{r} e^{\tau|s-t_0|} ds \right| dr \right| \leq \left| \int_{t_0}^{r} \frac{1}{\tau} e^{\tau|r-t_0|} dr \right| \leq \frac{1}{\tau^2} e^{\tau|t-t_0|}.$$

If follows that

$$|(A(u))(t) - (A(v))(t)| e^{-\tau|t-t_0|} \leq \frac{M_p(M_q+L_f)}{\tau^2} ||u-v||_\tau, \text{ for all } t \in [a,b].$$

Consequently

$$||A(u) - A(v)||_\tau \leq \frac{M_p(M_q+L_f)}{\tau^2} ||u-v||_\tau \text{ for all } u,v \in C^2[a,b].$$

By choosing τ large enough we have that the operator A is a contraction. By using Contraction mapping principle we obtain that the equation (2.4) has, in $C^2[a,b]$, a unique solution satisfying the conditions $u(t_0) = u'(t_0) = 0$. □

Corollary 2.11. *In the conditions of Theorem 2.10, if $f(t,0) = 0$ for all $t \in [a,b]$ then any non-trivial solution $u \in C^2[a,b]$ of the equation (2.4) hasn't transverse zeroes.*

Proof. Suppose that u is a non-trivial solution of the equation (2.4) that have a transverse zero $t_0 \in [a,b]$, i.e. $u(t_0) = u'(t_0) = 0$. From Theorem 2.10 the equation (2.4) with the conditions (2.5) has a unique solution. But, because $f(t,0) = 0$, the function $u(t) = 0$, $t \in [a,b]$, is a solution of the problem (2.4)+(2.5). This is a contradiction with the fact that u is a non-trivial solution of the equation (2.4). □

Remark 2.12. There exist equations of the form (2.4), with $f(t,0) \neq 0$, that have solutions with transverse zeroes and with zeroes with a degree of multiplicity greater than 2. See Example 2.13.

Example 2.13. Let us consider the equation (1.1) where

$$p(t) = t^2 + 1, \quad q(t) = 20, \quad f(t,u) = 11t^2 + \sqrt{|u|}, \quad t \in \mathbb{R}.$$

We have that all the conditions: f is a non-linear continuous function, not necessarily Lipschitz continuous function in u,$p, q \in C^1[a,b]$ and $p(t) > 0$ for all $t \in [a,b]$ are satisfied, except the condition $f(t,0) \equiv 0$. A solution u of this equation given by $u(t) = -t^4$ has a transverse zero $t_0 = 0$, which has degree of multiplicity equal to 4.

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On (h, k)–trichotomy for skew-evolution semiflows in Banach spaces

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Abstract. In this paper we define the notion of (h, k)–trichotomy for skew-evolution semiflows and we emphasize connections between various other concepts of trichotomy on infinite dimensional spaces, as uniform exponential trichotomy, exponential trichotomy and Barreira-Valls exponential trichotomy. The approach is motivated by various examples. Some characterizations for the newly introduced concept are also provided.

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1. Preliminaries

As the dynamical systems that are modelling processes issued from engineering, economics or physics are extremely complex, of great interest is to study the solutions of differential equations by means of associated skew-evolution semiflows, introduced in [10]. They are appropriate to study the asymptotic properties of the solutions for evolution equations of the form

$$\begin{cases} \dot{u}(t) = A(t)u(t), \ t > t_0 \geq 0 \\ u(t_0) = u_0, \end{cases}$$

where $A : \mathbb{R} \to \mathcal{B}(V)$ is an operator, $\mathrm{Dom}A(t) \subset V$, $u_0 \in \mathrm{Dom}A(t_0)$. The case of stability for skew-evolution semiflows is emphasized in [16] and the study of dichotomy for evolution equations is given in [9], where we generalize some concepts given in [1], as well as in [15].

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The exponential dichotomy for evolution equations is one of the domains of the stability theory with an impressive development due to its role in approaching several types of differential equations (see [2], [3], [4], [5], [7] and [8]). Hence, the techniques that describe the stability and instability in Banach spaces have been improved to characterize the dichotomy and its natural generalization, the trichotomy, studied for the case of linear differential equations in the finite dimensional setting in [12]. In fact, the trichotomy supposes the continuous splitting of the state space, at any moment, into three subspaces: the stable one, the instable one and the central manifold. The study of the trichotomy for evolution operators is given in [11]. Some concepts for the stability, instability, dichotomy and trichotomy of skew-evolution semiflows are studied in [14].

In this paper, beside other types of trichotomy, as uniform exponential trichotomy, Barreira-Valls exponential trichotomy, exponential trichotomy, we define, exemplify and characterize the concept of (h,k)–trichotomy for skew-evolution semiflows, as a generalization of the (h,k)–dichotomy given in [6] for evolution operators and in [13] for skew-evolution semiflows. Connections between the trichotomy classes are also emphasized.

2. Notations. Definitions. Examples

Let us denote by X a metric space, by V a Banach space and by $\mathcal{B}(V)$ the space of all bounded linear operators from V into itself. We consider the sets $\Delta = \left\{(t,t_0) \in \mathbb{R}^2_+,\ t \geq t_0\right\}$ and $T = \left\{(t,s,t_0) \in \mathbb{R}^3_+,\ (t,s),(s,t_0) \in \Delta\right\}$. Let I be the identity operator on V. We denote $Y = X \times V$ and $Y_x = \{x\} \times V$, where $x \in X$. Let us define the set $\mathcal{E}$ of all mappings $f : \mathbb{R}_+ \to [1,\infty)$ for which there exists a constant $\alpha \in \mathbb{R}_+$ such that $f(t) = e^{\alpha t}$, $\forall t \geq 0$.

Definition 2.1. A mapping $\varphi : \Delta \times X \to X$ is called *evolution semiflow* on X if following relations hold:

(s_1) $\varphi(t,t,x) = x,\ \forall (t,x) \in \mathbb{R}_+ \times X$;

(s_2) $\varphi(t,s,\varphi(s,t_0,x)) = \varphi(t,t_0,x), \forall (t,s,t_0) \in T, x \in X$.

Definition 2.2. A mapping $\Phi : \Delta \times X \to \mathcal{B}(V)$ is called *evolution cocycle* over an evolution semiflow φ if:

(c_1) $\Phi(t,t,x) = I, \forall (t,x) \in \mathbb{R}_+ \times X$;

(c_2) $\Phi(t,s,\varphi(s,t_0,x))\Phi(s,t_0,x) = \Phi(t,t_0,x), \forall (t,s,t_0) \in T, x \in X$.

Definition 2.3. The mapping $C : \Delta \times Y \to Y$ defined by the relation

$$C(t,s,x,v) = (\varphi(t,s,x), \Phi(t,s,x)v),$$

where Φ is an evolution cocycle over an evolution semiflow φ, is called *skew-evolution semiflow* on Y.

Example 2.4. Let $\mathcal{C} = \mathcal{C}(\mathbb{R},\mathbb{R})$ be the metric space of all continuous functions $x : \mathbb{R} \to \mathbb{R}$, with the topology of uniform convergence on compact subsets of

$\mathbb{R}$. $\mathcal{C}$ is metrizable relative to the metric

$$d(x,y)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{d_n(x,y)}{1+d_n(x,y)},\text{ where } d_n(x,y)=\sup_{t\in[-n,n]}|x(t)-y(t)|.$$

Let $f:\mathbb{R}_+\to(0,\infty)$ be a decreasing function. We denote by X the closure in $\mathcal{C}$ of the set $\{f_t,\ t\in\mathbb{R}_+\}$, where $f_t(\tau)=f(t+\tau)$, $\forall\tau\in\mathbb{R}_+$. We obtain that (X,d) is a metric space and that the mapping

$$\varphi:\Delta\times X\to X,\ \varphi(t,s,x)(\tau)=x_{t-s}(\tau)=x(t-s+\tau)$$

is an evolution semiflow on X. Let $V=\mathbb{R}$. The mapping $\Phi:\Delta\times X\to\mathcal{B}(\mathbb{R})$ given by

$$\Phi(t,s,x)v=e^{\int_s^t x(\tau-s)d\tau}v$$

is an evolution cocycle. Hence, $C=(\varphi,\Phi)$ is a skew-evolution semiflow on Y.

Two classic asymptotic properties for evolution cocycles are given, as in [14], by the next

Definition 2.5. A evolution cocycle Φ is said to have:

(i) *uniform exponential growth* if there exist some constants $M\geq 1$ and $\omega>0$ such that:

$$\|\Phi(t,t_0,x)v\|\leq Me^{\omega(t-s)}\|\Phi(s,t_0,x)v\|, \tag{2.1}$$

for all $(t,s,t_0)\in T$ and all $(x,v)\in Y$.

(ii) *uniform exponential decay* if there exist some constants $M\geq 1$ and $\omega>0$ such that:

$$\|\Phi(s,t_0,x)v\|\leq Me^{\omega(t-s)}\|\Phi(t,t_0,x)v\|, \tag{2.2}$$

for all $(t,s,t_0)\in T$ and all $(x,v)\in Y$.

3. Concepts of trichotomy

Definition 3.1. A continuous mapping $P:Y\to Y$ defined by

$$P(x,v)=(x,P(x)v),\ \forall(x,v)\in Y, \tag{3.1}$$

where $P(x)$ is a linear projection on Y_x, is called *projector* on Y.

Remark 3.2. The mapping $P(x):Y_x\to Y_x$ is linear and bounded and satisfies the relation $P(x)P(x)=P^2(x)=P(x)$ for all $x\in X$.

Definition 3.3. A projector P on Y is called *invariant* relative to a skew-evolution semiflow $C=(\varphi,\Phi)$ if following relation holds:

$$P(\varphi(t,s,x))\Phi(t,s,x)=\Phi(t,s,x)P(x), \tag{3.2}$$

for all $(t,s)\in\Delta$ and all $x\in X$.

Definition 3.4. Three projectors $\{P_k\}_{k\in\{1,2,3\}}$ are said to be *compatible* with a skew-evolution semiflow $C = (\varphi, \Phi)$ if:

(t_1) each of the projectors P_k, $k \in \{1, 2, 3\}$ is invariant on Y;

(t_2) $\forall x \in X$, the projections $P_1(x)$, $P_2(x)$ and $P_3(x)$ verify the relations

$$P_1(x) + P_2(x) + P_3(x) = I \text{ and } P_i(x)P_j(x) = 0,\ \forall i, j \in \{1, 2, 3\},\ i \neq j.$$

In what follows we will denote $C_k(t, s, x, v) = (\varphi(t, s, x), \Phi_k(t, s, x)v)$, $(t, t_0, x, v) \in \Delta \times Y$, $\forall k \in \{1, 2, 3\}$, where $\Phi_k(t, t_0, x) = \Phi(t, t_0, x)P_k(x)$. Let us remind the definitions for various classes of trichotomy, as in [14] and [17].

Definition 3.5. A skew-evolution semiflow $C = (\varphi, \Phi)$ is called *uniformly exponentially trichotomic* if there exist some constants $N \geq 1$, $\nu > 0$ and three projectors $\{P_k\}_{k\in\{1,2,3\}}$ compatible with C such that:

(uet_1)
$$e^{\nu(t-s)} \|\Phi_1(t, t_0, x)v\| \leq N \|\Phi_1(s, t_0, x)v\|; \tag{3.3}$$

(uet_2)
$$e^{\nu(t-s)} \|\Phi_2(s, t_0, x)v\| \leq N \|\Phi_2(t, t_0, x)v\|; \tag{3.4}$$

(uet_3)
$$\begin{aligned}\|\Phi_3(s, t_0, x)v\| &\leq Ne^{\nu(t-s)} \|\Phi_3(t, t_0, x)v\| \leq \\ &\leq N^2e^{2\nu(t-s)} \|\Phi_3(s, t_0, x)v\|,\end{aligned} \tag{3.5}$$

for all $(t, s, t_0) \in T$ and all $(x, v) \in Y$.

Remark 3.6. The constants N and ν are called *trichotomic characteristics* and P_1, P_2, P_3 *associated trichotomic projectors.*

Definition 3.7. A skew-evolution semiflow $C = (\varphi, \Phi)$ is called *exponentially trichotomic* if there exist a mapping $N : \mathbb{R}_+ \to [1, \infty)$, a constant $\nu > 0$ and three projectors $\{P_k\}_{k\in\{1,2,3\}}$ compatible with C such that:

(et_1)
$$e^{\nu(t-s)} \|\Phi_1(t, t_0, x)v\| \leq N(s) \|\Phi_1(s, t_0, x)v\|; \tag{3.6}$$

(et_2)
$$e^{\nu(t-s)} \|\Phi_2(s, t_0, x)v\| \leq N(t) \|\Phi_2(t, t_0, x)v\|; \tag{3.7}$$

(et_3)
$$\begin{aligned}\|\Phi_3(s, t_0, x)v\| &\leq N(t)e^{\nu(t-s)} \|\Phi_3(t, t_0, x)v\| \leq \\ &\leq N(t)N(s)e^{2\nu(t-s)} \|\Phi_3(s, t_0, x)v\|,\end{aligned} \tag{3.8}$$

for all $(t, s, t_0) \in T$ and all $(x, v) \in Y$.

Definition 3.8. A skew-evolution semiflow $C = (\varphi, \Phi)$ is called *Barreira-Valls exponentially trichotomic* if there exist some constants $N \geq 1$, $\alpha, \beta, \mu, \rho > 0$ and three projectors $\{P_k\}_{k\in\{1,2,3\}}$ compatible with C such that:

$(BVet_1)$
$$e^{\alpha(t-s)} \|\Phi_1(t, t_0, x)v\| \leq Ne^{\beta s} \|\Phi_1(s, t_0, x)v\|; \tag{3.9}$$

$(BVet_2)$
$$\|\Phi_2(s, t_0, x)v\| \leq Ne^{-\alpha t}e^{\beta s} \|\Phi_2(t, t_0, x)v\|; \tag{3.10}$$

$(BVet_3)$

$$\|\Phi_3(t,t_0,x)v\| \leq Ne^{\mu t}e^{-\rho s}\|\Phi_3(s,t_0,x)v\| \leq$$
$$\leq N^2 e^{2\mu t}e^{-2\rho s}\|\Phi_3(t,t_0,x)v\|, \tag{3.11}$$

for all $(t,s,t_0) \in T$ and all $(x,v) \in Y$.

Further, let us introduce a more general concept of trichotomy for skew-evolution semiflows, given by the next

Definition 3.9. A skew-evolution semiflow $C = (\varphi, \Phi)$ is (h,k)*–trichotomic* if there exist a constant $N \geq 1$, two continuous mappings $h, k : \mathbb{R}_+ \to \mathbb{R}_+^*$ and three projectors families $\{P_k\}_{k\in\{1,2,3\}}$ compatible with C such that:

(t_1)

$$h(t-s)\|\Phi_1(t,t_0,x)v\| \leq Nk(s)\|\Phi_1(s,t_0,x)v\|; \tag{3.12}$$

(t_2)

$$h(t-s)\|\Phi_2(s,t_0,x)v\| \leq Nk(t)\|\Phi_2(t,t_0,x)v\|; \tag{3.13}$$

(t_3)

$$\|\Phi_3(t,t_0,x)v\| \leq Nk(s)h(t-s)\|\Phi_3(s,t_0,x)v\|; \tag{3.14}$$

$$\|\Phi_3(s,t_0,x)v\| \leq Nk(t)h(t-s)\|\Phi_3(t,t_0,x)v\|; \tag{3.15}$$

for all $(t,s,t_0) \in T$ and all $(x,v) \in Y$.

The concept of (h,k)–trichotomy generalizes the notions of uniform exponential trichotomy, exponential trichotomy and Barreira-Valls exponential trichotomy, as shown in

Remark 3.10. 1) If $h \in \mathcal{E}$ and k is constant in Definition 3.9, then C is uniformly exponentially trichotomic;

2) If $h \in \mathcal{E}$ then C is exponentially trichotomic;

3) If $h, k \in \mathcal{E}$ then C is Barreira-Valls exponentially trichotomic.

In the next particular cases, other (h,k)–asymptotic properties for skew-evolution semiflows are emphasized.

Remark 3.11. (i) For $P_2 = P_3 = 0$ we obtain in Definition 3.9 the property of (h,k)–exponential stability;

(ii) For $P_1 = P_3 = 0$ in Definition 3.9 the property of (h,k)–exponential instability is obtained;

(iii) For $P_3 = 0$ we obtain in Definition 3.9 the property of (h,k)–exponential dichotomy. On the other hand, for $P_3 = 0$, in Definition 3.5, Definition 3.7 and Definition 3.8 the properties of uniform exponential dichotomy, exponential dichotomy, respectively Barreira-Valls exponential dichotomy are obtained (see [13]).

We have following connections between the previously defined classes of trichotomy, given by

Remark 3.12. A uniformly exponentially trichotomic skew-evolution semiflow is Barreira-Valls exponentially trichotomic, which also implies that it is exponentially trichotomic.

The converse statements are not always true, as shown in the next examples.

Example 3.13. Let $f:\mathbb{R}_+\to(0,\infty)$ be a decreasing function with the property that there exists $\lim\limits_{t\to\infty} f(t)=l>0$. We will consider $\lambda>f(0)$. We define the metric space (X,d) and the evolution semiflow as in Example 2.4.

Let us consider $V=\mathbb{R}^3$ with the norm $\|v\|=|v_1|+|v_2|+|v_3|$, where $v=(v_1,v_2,v_3)\in V$. The mapping $\Phi:\Delta\times X\to\mathcal{B}(V)$, defined by

$$\Phi(t,s,x)v=$$
$$=\left(\frac{e^{t\sin t-2t}}{e^{s\sin s-2s}}e^{-\int_s^t x(\tau-s)d\tau}v_1,\frac{e^{3t-2t\cos t}}{e^{3s-2s\cos s}}e^{\int_s^t x(\tau-s)d\tau}v_2,e^{(t-s)x(0)-\int_s^t x(\tau-s)d\tau}v_3\right)$$

is an evolution cocycle over the evolution semiflow φ. We consider the projectors $P_1,\ P_2,\ P_3:Y\to Y$, $P_1(x,v)=(v_1,0,0)$, $P_2(x,v)=(0,v_2,0)$ and $P_3(x,v)=(0,0,v_3)$, where $x\in X$ and $v=(v_1,v_2,v_3)\in V$, compatible with the skew-evolution semiflow $C=(\varphi,\Phi)$.

We obtain

$$|\Phi_1(t,s,x)v|=e^{t\sin t-s\sin s+2s-2t}e^{-\int_s^t x(\tau-s)d\tau}|v_1|\le$$
$$\le e^{-t+3s}e^{-l(t-s)}|v_1|=e^{-(1+l)t}e^{(3+l)s}|v_1|,$$

for all $(t,s,x,v)\in\Delta\times Y$ and

$$|\Phi_2(t,s,x)v|=e^{3t-3s-2t\cos t+2s\cos s+\int_s^t x(\tau-s)d\tau}|v_2|\ge$$
$$\ge e^{t-s}e^{l(t-s)}|v_2|=e^{(1+l)t}e^{-(1+l)s}|v_2|,$$

for all $(t,s,x,v)\in\Delta\times Y$.

We also have, for all $(t,s,x,v)\in\Delta\times Y$,

$$|\Phi_3(t,s,x)v|\le e^{[\lambda-x(0)]t}e^{-[\lambda-x(0)]s}|v_3|$$

and

$$|\Phi_3(t,s,x)v|\ge e^{[l-x(0)]t}e^{-[l-x(0)]s}|v_3|.$$

Hence, the skew-evolution semiflow $C=(\varphi,\Phi)$ is Barreira-Valls exponentially trichotomic with the characteristics

$$N=1,\ \alpha=\beta=3+l,\ \mu=\rho=\min\{\lambda-x(0),x(0)-l\}.$$

Let us suppose now that $C=(\varphi,\Phi)$ is uniformly exponentially trichotomic. According to Definition 3.5, there exist $N\ge 1$ and $\nu>0$ such that

$$e^{t\sin t-s\sin s+2s-2t}e^{-\int_s^t x(\tau-s)d\tau}|v_1|\le Ne^{-\nu(t-s)}|v_1|,\ \forall t\ge s\ge 0$$

and If we consider $t=2n\pi+\frac{\pi}{2}$ and $s=2n\pi$, $n\in\mathbb{N}$, we have

$$e^{2n\pi-\frac{\pi}{2}}\le Ne^{-\nu\frac{\pi}{2}}e^{\int\limits_{2n\pi}^{2n\pi+\frac{\pi}{2}} x(\tau-2n\pi)d\tau}\le Ne^{(-\nu+\lambda)\frac{\pi}{2}},$$

which, for $n\to\infty$, leads to a contradiction.

Hence, we obtain that $C=(\varphi,\Phi)$ is not uniformly exponentially trichotomic.

Example 3.14. We consider the metric space (X,d), the Banach space V, the projectors P_1, P_2, P_3 and the evolution semiflow φ defined as in Example 2.4. Let $g:\mathbb{R}_+\to[1,\infty)$ be a continuous function with

$$g(n)=e^{n\cdot 2^{2n}} \text{ and } g\left(n+\frac{1}{2^{2n}}\right)=e^4, \text{ for all } n\in\mathbb{N}.$$

The mapping $\Phi:\Delta\times X\to\mathcal{B}(V)$, defined by

$$\Phi(t,s,x)v=$$

$$=\left(\frac{g(s)}{g(t)}e^{-(t-s)-\int_s^t x(\tau-s)d\tau}v_1,\frac{g(s)}{g(t)}e^{t-s+\int_s^t x(\tau-s)d\tau}v_2,e^{-(t-s)x(0)+\int_s^t x(\tau)d\tau}v_3\right)$$

is an evolution cocycle over the evolution semiflow φ.

We have that,

$$e^{(1+l)(t-s)}\left\|\Phi_1(t,s,x)v\right\|\leq g(s)\left\|v_1\right\|$$

and

$$e^{(1+l)(t-s)}\left\|v_2\right\|\leq g(s)e^{(1+l)(t-s)}\left\|v_2\right\|\leq g(t)\left\|\Phi_2(t,s,x)v\right\|,$$

for all $(t,s,x,v)\in\Delta\times Y$. We also have

$$|\Phi_3(t,s,x)v|\leq e^{x(0)(t-s)}|v_3|$$

and

$$|\Phi_3(t,s,x)v|\geq e^{-x(0)(t-s)}|v_3|,$$

for all $(t,s,x,v)\in\Delta\times Y$. Thus, $C=(\varphi,\Phi)$ is exponentially trichotomic with

$$\nu=\max\{1+l,\lambda\} \text{ and } N(t)=\sup_{s\in[0,t]}g(s).$$

If we suppose that C is Barreira-Valls exponentially trichotomic, then there exist $N\geq 1$, $\alpha>0$ and $\beta\geq 0$ such that

$$g(s)e^{\alpha t}\leq Ng(t)e^{\beta s+t-s+\int_s^t x(\tau-s)d\tau},$$

for all $(t,s,x)\in\Delta\times X$.

From here, for $t=n+\frac{1}{2^{2n}}$ and $s=n$, it follows that

$$e^{n(2^{2n}+\alpha-\beta)}\leq 81Ne^{\frac{1-\alpha+x(0)}{2^{2n}}},$$

which, for $n\to\infty$, leads to a contradiction.

4. Main results

Let $C:\Delta\times Y\to Y$, $C(t,s,x,v)=(\Phi(t,s,x)v,\varphi(t,s,x))$ be a skew-evolution semiflow on Y. Some characterizations for the concept of (h,k)–trichotomy are obtained. Therefore, let us suppose that $h:\mathbb{R}_+\to\mathbb{R}_+^*$ is a nondecreasing function such that

$$h(u+v)\leq h(u)h(v),\ u,v\in\mathbb{R}_+. \qquad (\chi)$$

Theorem 4.1. *Let $C = (\varphi, \Phi)$ be skew-evolution semiflow such that there exist three projectors $\{P_k\}_{k\in\{1,2,3\}}$ compatible with C such that Φ_1 has uniform exponential growth and Φ_2 has uniform exponential decay. If there exist a constant $K \geq 1$ and two mappings $h, k : \mathbb{R}_+ \to \mathbb{R}_+^*$, where h satisfies condition (χ), such that:*

(i)

$$\int_s^t h(\tau - s) \left\| \Phi_1(\tau, t_0, x)v \right\| d\tau \leq Kk(s) \left\| \Phi_1(s, t_0, x)v \right\|; \tag{4.1}$$

(ii)

$$\int_s^t h(t - \tau) \left\| \Phi_2(\tau, t_0, x)v \right\| d\tau \leq Kk(t) \left\| \Phi_2(t, t_0, x)v \right\|; \tag{4.2}$$

(iii)

$$\int_s^t \frac{1}{h(\tau - s)} \left\| \Phi_3(\tau, t_0, x)v \right\| d\tau \leq Kk(s) \left\| \Phi_3(s, t_0, x)v \right\|; \tag{4.3}$$

$$\int_s^t \frac{1}{h(t - \tau)} \left\| \Phi_3(\tau, t_0, x)v \right\| d\tau \leq Kk(t) \left\| \Phi_3(s, t_0, x)v \right\|, \tag{4.4}$$

for all $(t, s, t_0) \in T$ and all $(x, v) \in Y$, then C is (h, k)–trichotomic.

Proof. Let us suppose that (i) holds. As a first step, we consider $s \in [t-1, t]$. We obtain

$$h(t-s) \left\| \Phi_1(t, t_0, x)v \right\| = \int_{t-1}^t h(t-s) \left\| \Phi_1(t, t_0, x)v \right\| d\tau \leq$$

$$\leq \int_{t-1}^t h(t-\tau)h(\tau - s) \left\| \Phi_1(t, \tau, \varphi(\tau, t_0, x))\Phi_1(s, t_0, x)v \right\| d\tau \leq$$

$$\leq Me^{\omega}h(1) \int_s^t h(\tau - s) \left\| \Phi_1(\tau, t_0, x)v \right\| d\tau \leq KMe^{\omega}h(1)k(s) \left\| \Phi_1(s, t_0, x)v \right\|,$$

for all $(x, v) \in Y$, where M and ω are given by Definition 2.5, as Φ_1 has uniform exponential growth.

As a second step, if $t \in [s, s+1)$, we have

$$h(t-s) \left\| \Phi_1(t, t_0, x)v \right\| \leq Me^{\omega}h(1) \left\| \Phi_1(s, t_0, x)v \right\|,$$

for all $(x, v) \in Y$. Hence, relation (3.12) is obtained, for $N = Me^{\omega}h(1)(K+1)$.

Now, as Φ_2 has uniform exponential decay, an equivalent definition (see [14]) assures the existence of a nondecreasing function $g : [0, \infty) \to [1, \infty)$ with the property $\lim_{t\to\infty} g(t) = \infty$ such that

$$\left\| \Phi(s, t_0, x)v \right\| \leq g(t-s) \left\| \Phi(t, t_0, x)v \right\|,$$

for all $(t, s, t_0) \in T$ and all $(x, v) \in Y$. Let us denote $D = \int_0^1 g(\tau)d\tau$.

If (ii) holds, we obtain

$$Dh(t-s) \left\| \Phi(s, t_0, x)v \right\| = \int_0^1 h(t-s)g(\tau) \left\| \Phi(s, t_0, x)v \right\| d\tau \leq$$

$$\leq \int_0^1 h(t-\tau)h(\tau-s)g(\tau)\left\|\Phi(s,t_0,x)v\right\| d\tau \leq$$
$$\leq h(t)\int_0^1 h(\tau)g(\tau)\left\|\Phi(s,t_0,x)v\right\| d\tau =$$
$$= \int_s^{s+1} h(u-t_0)g(u-s)\left\|\Phi(s,t_0,x)v\right\| du \leq$$
$$\leq \int_0^t h(u-s)\left\|\Phi_2(u,t_0,x)v\right\| du \leq Kk(t)\left\|\Phi_2(t,t_0,x)v\right\|,$$

for all $t \geq s+1 > s \geq 0$ and all $(x,v) \in Y$.

On the other hand, for $t \in [s, s+1)$ we obtain for all $(x,v) \in Y$

$$\left\|\Phi_2(t,t_0,x)v\right\| \geq g(t-s)\left\|\Phi(s,t_0,x)v\right\| \geq g(1)\left\|\Phi(s,t_0,x)(x)v\right\|.$$

We obtain thus relation (3.13).

A similar proof, based on the property (χ) of function h, shows that the inequalities from (iii) imply relations (3.14).

Hence, according to Definition 3.9, C is (h,k)–trichotomic. □

Remark 4.2. Relation (4.1) defines the (h,k)*–integral stability*, while relation (4.1) defines the (h,k)*–integral instability* for skew-evolution semiflow, similar to the notions defined in [17].

In the below mentioned particular cases, we obtain, as in [14], characterizations for other classes of trichotomy.

Corollary 4.3. *In the hypothesis of Theorem* 4.1,

(i) if $h,k \in \mathcal{E}$ and are given by $t \mapsto e^{\alpha t}$ respectively $t \mapsto Me^{\alpha t}$, $M \geq 1$, the skew-evolution semiflow C is uniformly exponentially trichotomic;

(ii) if $h \in \mathcal{E}$, the skew-evolution semiflow C is exponentially trichotomic;

(iii) if $h,k \in \mathcal{E}$ and are given by $t \mapsto e^{\alpha t}$ respectively $t \mapsto Me^{\beta t}$, $M \geq 1$ and $\beta > \alpha$, the skew-evolution semiflow C is Barreira-Valls exponentially trichotomic.

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Some results on the solutions of a functional-integral equation

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Abstract. In this paper we give existence, uniqueness, data dependence and comparison theorems for the solutions of a functional-integral equation of the same type as that considered by L. Olszowy [6]. We apply some results from Picard and weakly Picard operators' theory (see I.A. Rus, [7]).

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1. Introduction

The fixed point theory has a lot of applications in the field of functional-differential equations (see for example [1]-[6], [8]). In the paper [6] has been given theorems on the existence and asymptotic characterization of the solutions of the following problem:

$$y'(t) = f(t, y(H(t)), y'(h(t))), t \in [0, \infty) \tag{1.1}$$

$$y(0) = 0. \tag{1.2}$$

Technique linking measures of noncompactness with the Tichonov' fixed point principle in suitable Fréchet space was used.

As it was shown in [6], the problem (1.1)+(1.2) is equivalent with the following functional- integral equation:

$$x(t) = f(t, \int_0^{H(t)} x(s)ds, x(h(t))), t \in [0, \infty) \tag{1.3}$$

The aim of this paper is to give existence, uniqueness, data dependence and comparison theorems for the solutions of a functional-integral equation

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of the same type as that considered in [6]. We apply some results from Picard and weakly Picard operators' theory (see [7] and [8]).

2. Weakly Picard operators

Here, first we present some notions and results from the weakly Picard operators' theory.

Let (X, d) be a metric space and $A : X \longrightarrow X$ an operator.

We denote by $A^0 := 1_X$, $A^1 := A, ..., A^{n+1} := A \circ A^n$, $n \in \mathbb{N}$, the iterate operators of the operator A. Also:

$$P(X) := \{Y \subset X \ / \ Y \neq \emptyset\},$$

$$I(A) := \{Y \in P(X) \ / \ A(Y) \subset Y\},$$

the family of all nonempty invariant subsets of A,

$$F_A = \{x \in X / A(x) = x\},$$

the fixed point set of the operator A.

Following Rus I.A. [7] and [8], we have:

Definition 2.1. *The operator A is a Picard operator if there exists $x^* \in X$ such that*

1) $F_A = \{x^*\}$*;*

2) *the successive approximation sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in X$.*

Definition 2.2. *A is a weakly Picard operator if the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which generally depends on x_0) is a fixed point of A.*

Definition 2.3. *For an weakly Picard operator $A : X \to X$ we define the operator A^∞ as follows:*

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x), \ \textit{for all } x \in X.$$

Remark 2.4. $A^\infty(X) = F_A$.

We have

Theorem 2.5. (Data dependence theorem) *Let (X, d) be a complete metric space and $A, B : X \longrightarrow X$ two operators. We suppose that:*

(i) A is an α-contraction and let $F_A = \{x_A^*\}$;

(ii) $F_B \neq \emptyset$ and let $x_B^* \in F_B$;

(iii) there exists $\delta > 0$, such that $d(A(x), B(x)) \leq \delta$, for all $x \in X$.

Then

$$d(x_A^*, x_B^*) \leq \frac{\delta}{1-\alpha}.$$

Theorem 2.6. (Characterization theorem) *Let (X,d) be a metric space and $A : X \to X$ an operator. The operator A is a weakly Picard operator if and only if there exists a partition of X, $X = \cup_{\lambda\in\Lambda} X_\lambda$, such that:*

(i) $X_\lambda \in I(A)$;

(ii) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ is a Picard operator, for all $\lambda \in \Lambda$.

Lemma 2.7. *Let $(X,\leq)$ be an ordered metric space and $A : X \to X$ an operator. We suppose that:*

(i) A is a weakly Picard operator;

(ii) A is increasing.

Then the operator A^∞ is increasing.

Lemma 2.8. (Abstract Gronwall lemma) *Let $(X,\leq)$ be an ordered metric space and $A : X \to X$ an operator. We suppose that:*

(i) A is a Picard operator;

(ii) A is increasing.

If we denote by x_A^* the unique fixed point of A, then:

(a) $x \leq A(x)$ implies $x \leq x_A^*$;

(b) $x \geq A(x)$ implies $x \geq x_A^*$.

Lemma 2.9. (Abstract comparison lemma) *Let $(X,\leq)$ be an ordered metric space and the operators $A, B, C : X \to X$ be such that:*

(i) $A \leq B \leq C$;

(ii) the operators A, B, C are weakly Picard operators;

(iii) the operator B is increasing.

Then $x \leq y \leq z$ implies $A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.

3. Existence, uniqueness and data dependence results

Let us consider the following functional-integral equation:

$$x(t) = \alpha + f(t, \int_0^{g(t)} x(s)ds, x(h(t))), t \in [0,T] \tag{3.1}$$

under the following assumptions:

(A_1) $f \in C([0,T] \times \mathbb{R}^2)$;

(A_2) $g, h \in C([0,T],[0,T])$ and $g(t) \leq t$, $h(t) \leq t$, for all $t \in [0,T]$;

(A_3) $\alpha \in \mathbb{R}$ and $f(0,0,\alpha) = 0$;

(A_4) there exists $k_1 > 0$ and $0 < k_2 < 1$, such that

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq k_1 |u_1 - u_2| + k_2 |v_1 - v_2|,$$

$$\text{for all } t \in [0,T] \text{ and all } u_i, v_i \in \mathbb{R},\ i = 1,2.$$

We have

Theorem 3.1. *If all the conditions $(A_1) - (A_4)$ are satisfied, then the equation (3.1) has in $C[0,T]$ a unique solution.*

Proof. On $C[0,T]$, we consider a Bielecki norm $||\cdot||_\tau$, defined by

$$||x||_\tau = \max_{t\in[0,T]} |x(t)|\ e^{-\tau t},$$

where $\tau > 0$, and the operator

$$A : (C[0,T], ||\cdot||_\tau) \to (C[0,T], ||\cdot||_\tau),$$

defined by

$$A(x)(t) := \alpha + f(t, \int_0^{g(t)} x(s)ds, x(h(t))), t \in [0,T].$$

So, we have a fixed point equation:

$$x = A(x).$$

Let $x, z \in C[0,T]$ be. We obtain

$$|A(x)(t) - A(z)(t)| =$$

$$= |f(t, \int_0^{g(t)} x(s)ds, x(h(t))) - f(t, \int_0^{g(t)} z(s)ds, z(h(t)))| \le$$

$$\le k_1 |\int_0^{g(t)} (x(s) - z(s))ds| + k_2|x(h(t)) - z(h(t))| \ \le$$

$$\le k_1 \int_0^{g(t)} |x(s) - z(s)|e^{-\tau s}e^{\tau s}ds + k_2|x(h(t)) - z(h(t))|e^{-\tau h(t)}e^{\tau h(t)} \le$$

$$\le (k_1 \int_0^{g(t)} e^{\tau s}ds + k_2 e^{\tau h(t)})||x - z||_\tau \le$$

$$\le (k_1 \int_0^{t} e^{\tau s}ds + k_2 e^{\tau t})||x - z||_\tau \le$$

$$\le (\frac{k_1}{\tau} + k_2)e^{\tau t}||x - z||_\tau,$$

for all $t \in [0,T]$.

So,

$$|A(x)(t) - A(z)(t)|e^{-\tau t} \le (\frac{k_1}{\tau} + k_2)||x - z||_\tau,$$

for all $t \in [0,T]$.

It follows that

$$||A(x) - A(z)||_\tau \le (\frac{k_1}{\tau} + k_2)||x - z||_\tau,$$

for all $x, z \in C[0,T]$.

We choose τ large enough, such that $\frac{k_1}{\tau} + k_2 < 1$. By applying Contraction mapping principle, we obtain that A is a Picard operator. □

Now, together with (3.1), we consider the following equation:

$$x(t) = \alpha + F(t, \int_0^{g(t)} x(s)ds, x(h(t))), t \in [0, T], \tag{3.2}$$

where $F \in C([0, T] \times \mathbb{R}^2)$ and α, g, h are the same as in (3.1).

We have

Theorem 3.2. *We suppose that:*

(i) the conditions $(A_1)-(A_4)$ *are satisfied and* $x^* \in C[0, T]$ *is the unique solution of the equation (3.1);*

(ii) the equation (3.2) has solutions in $C[0, T]$ *and* $z^* \in C[0, T]$ *is a solution of (3.2);*

(iii) there exists $\eta > 0$ *such that*

$$|f(t, u, v) - F(t, u, v)| \leq \eta, \text{ for all } t \in [0, T] \text{ and all } u, v \in \mathbb{R}.$$

Then

$$||x^* - z^*||_\tau \leq \frac{\eta}{1 - (\frac{k_1}{\tau} + k_2)},$$

where τ *is large enough such that* $\frac{k_1}{\tau} + k_2 < 1$.

Proof. Consider

$$A_F : (C[0, T], || \cdot ||_\tau) \to (C[0, T], || \cdot ||_\tau),$$

$$A_F(x)(t) := \alpha + F(t, \int_0^{g(t)} x(s)ds, x(h(t))), t \in [0, T],$$

the corresponding operator of (3.2).

We have

$$|A(x)(t) - A_F(x)(t)| \leq \eta,$$

for all $t \in [0, T]$, and consequently

$$||A(x) - A_F(x)||_\tau \leq \eta,$$

for all $x \in C[0, T]$. □

Now, we apply Data dependence theorem (Theorem 2.5).

Theorem 3.3. *We suppose that:*

(i) the conditions $(A_1) - (A_4)$ *are satisfied and* $x^* \in C[0, T]$ *is the unique solution of the equation (3.1);*

(ii) $u_i, v_i \in \mathbb{R}$, $i = 1, 2$ *and* $u_1 \leq u_2$, $v_1 \leq v_2$ *implies* $f(t, u_1, v_1) \leq f(t, u_2, v_2)$, *for all* $t \in [0, T]$.

Then

$$x \leq A(x) \text{ implies } x \leq x^*$$

and

$$x \geq A(x) \text{ implies } x \geq x^*.$$

Proof. The operator A is a Picard operator and A is increasing. So, we apply Abstract Gronwall lemma (Lemma 2.8). □

4. Comparison results

Consider the following functional-integral equation:

$$x(t) = x(0) + f(t, \int_0^{g(t)} x(s)ds, x(h(t))), t \in [0, T]. \tag{4.1}$$

The corresponding operator,

$$A_f : (C[0,T], ||\cdot||_\tau) \to (C[0,T], ||\cdot||_\tau),$$

$$A_f \ (x)(t) := x(0) + f(t, \int_0^{g(t)} x(s) \ ds, \ x(h(t))), \quad t \in [0,T],$$

is a continuous operator but it isn't a contraction.

We denote

$$S_f = \{\alpha \in \mathbb{R} / f(0,0,\alpha) = 0\} \ \text{ and } \ X_\alpha := \{x \in C[0,T] / x(0) = \alpha\}.$$

Then

$$\cup_{\alpha \in S_f} X_\alpha \ \text{ is a partition of } C[0,T]$$

and X_α is an invariant subset of A_f if and only if $\alpha \in S_f$.

We have

Theorem 4.1. *We suppose that:*

(i) *the conditions* $(A_1) - (A_4)$ *are satisfied for (4.1);*

(ii) $S_f \neq \emptyset$.

Then

$$A_f|_{\cup_{\alpha \in S_f} X_\alpha} : \cup_{\alpha \in S_f} X_\alpha \to \cup_{\alpha \in S_f} X_\alpha$$

is a weakly Picard operator and $card \ F_{A_f} = card S_f$.

Proof. By using the result of Theorem 3.1, we have that

$$A_f|_{X_\alpha} : X_\alpha \to X_\alpha \ \text{ is a Picard operator, for all } \alpha \in S_f.$$

So, we apply Characterization theorem of the weakly Picard operators (Theorem 2.6). □

Remark 4.2. If the conditions $(A_1) - (A_4)$ are satisfied and $S_f = \{\alpha^*\}$, then the equation (4.1) has in $C[0,T]$ a unique solution.

We have

Theorem 4.3. *We suppose that:*

(i) *all the conditions of Theorem 4.1 are satisfied;*

(ii) $u_i, v_i \in \mathbb{R}$, $i = 1, 2$ *and* $u_1 \leq u_2$, $v_1 \leq v_2$ *implies* $f(t, u_1, v_1) \leq f(t, u_2, v_2)$, *for all* $t \in [0,T]$.

Let x^* *be a solution of the equation (4.1) and* x^{**} *a solution of the following inequality:*

$$x(t) \leq x(0) + f(t, \int_0^{g(t)} x(s)ds, x(h(t))), t \in [0,T].$$

Then

$$x^{**}(0) \leq x^*(0) \ \textit{ implies } x^{**} \leq x^*.$$

Proof. We remark that

$$x^* = A_f(x^*) \quad \text{and} \quad x^{**} \leq A_f(x^{**}).$$

From Lemma 2.7 and the condition (ii) we have that the operator A_f^∞ is increasing. If $\beta \in \mathbb{R}$ then we consider $\widetilde{\beta} \in C[0,T]$ defined by $\widetilde{\beta}(t) = \beta$, for all $t \in [0,T]$. By using the previous considerations and because the operator A_f^∞ is increasing, we obtain:

$$x^{**} \leq A_f^\infty(x^{**}(0)) = A_f^\infty(\widetilde{x^{**}(0)}) \leq A_f^\infty(\widetilde{x^*(0)}) = x^*.$$

□

Now, we consider the following functional-integral equations:

$$x(t) = x(0) + f_i(t, \int_0^{g(t)} x(s)ds,\ x(h(t))), t \in [0,T], \tag{4.2}$$

$i = \overline{1,3}$, where g, h are the same in all three equations.

We have

Theorem 4.4. *We suppose that:*

(i) *the corresponding conditions of Theorem 4.1 are satisfied for all equations (4.2);*

(ii) $f_2(t,\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ *is increasing for all* $t \in [0,T]$;

(iii) $f_1 \leq f_2 \leq f_3$.

Let x_i^* *be a solution of the corresponding equation (4.2),* $i = \overline{1,3}$. *Then*

$$x_1^*(0) \leq x_2^*(0) \leq x_3^*(0) \quad \textit{implies} \quad x_1^* \leq x_2^* \leq x_3^*.$$

Proof. First we remark that the operators A_{f_i}, $i = \overline{1,3}$ are weakly Picard operators (Theorem 4.1). From (ii) we have that the operator A_{f_2} is increasing. From the condition (iii) we have that $A_{f_1} \leq A_{f_2} \leq A_{f_3}$. On the other hand, $x_i^* = A_{f_i}^\infty(\widetilde{x_i^*(0)})$, $i = \overline{1,3}$. Now, the proof follows from Abstract comparison lemma (Lemma 2.9). □

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Generalizations of Krasnoselskii's fixed point theorem in cones

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Abstract. We give some generalizations of Krasnoselskii's fixed point theorem in cones.

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1. Introduction

Firstly we will present the definition of a cone.

Definition 1.1. *Let X be a normed linear space. A nonempty closed, convex set $P \subset X$ is called a cone if it satisfies the following two conditions:*
(i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P, -x \in P$ implies $x = 0$.

After the well known paper of Legget and Williams(see [6]), many authors have given generalizations of the following Krasnoselskii's fixed point theorem:

Theorem 1.2. *(Krasnoselskii) Let $(X, |\ .\ |)$ be a normed linear space, $K \subset X$ a cone and " $\prec$ " the order relation induced by K. Let be $r, R \in R_+$, $0 < r < R$, $K_{r,R} := \{u \in K : r \leq \mid u \mid \leq R\}$ and let $N : K_{r,R} \to K$ be a completely continuous map. Assume that one of the following conditions is satisfied:*
(i) $\mid Nu \mid \geq \mid u \mid$ if $\mid u \mid = r$ and $\mid Nu \mid \leq \mid u \mid$ if $\mid u \mid = R$
(ii) $\mid Nu \mid \leq \mid u \mid$ if $\mid u \mid = r$ and $\mid Nu \mid \geq \mid u \mid$ if $\mid u \mid = R$.
Then N has a fixed point u^* in K with $r \leq \mid u^* \mid \leq R$.

For example, in [8], the author gives the following result. Before to state it, we introduce a few notations. We shall consider two wedges K_1, K_2 of X and the corresponding wedge $K := K_1 \times K_2$ of $X^2 := X \times X$. For

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$r, R \in R_+^2$, $r = (r_1, r_2)$, $R = (R_1, R_2)$, we write $0 < r < R$ if $0 < r_1 < R_1$ and $0 < r_2 < R_2$, and we use the notations:

$$(K_i)_{r_i R_i} := \{u \in K_i : r_i \leq \mid u \mid \leq R_i\} \ (i = 1, 2)$$

$$K_{rR} := \{u \in K : r_i \leq \mid u_i \mid \leq R_i \text{ for } i = 1, 2\}.$$

Clearly, $K_{rR} = (K_1)_{r_1 R_1} \times (K_2)_{r_2 R_2}$.

Theorem 1.3. *([8]) Let* $(X, \mid . \mid)$ *be a normed linear space;* $K_1, K_2 \subset X$ *two wedges;* $K := K_1 \times K_2$; $\alpha_i, \beta_i > 0$ *with* $\alpha_i \neq \beta_i$ *for* $i = 1, 2$, *and let* $r_i = \min\{\alpha_i, \beta_i\}, R_i = \max\{\alpha_i, \beta_i\}$ *for* $i = 1, 2$. *Assume that* $N : K_{rR} \to K$, $N = (N_1, N_2)$, *is a compact map and there exist* $h_i \in K_i \setminus \{0\}$, $i = 1, 2$ *such that for each* $i \in \{1, 2\}$ *the following condition is satisfied in* K_{rR} :

$$N_i u \neq \lambda u_i \ \textit{for} \ \mid u_i \mid = \alpha_i \ \textit{and} \ \lambda > 1;$$

$$N_i u + \mu h_i \neq u_i \ \textit{for} \ \mid u_i \mid = \beta_i \ \textit{and} \ \mu > 0.$$

Then N *has a fixed point* $u = (u_1, u_2)$ *in* K *with* $r_i \leq \mid u_i \mid \leq R_i$ *for* $i = 1, 2$.

Also, in [9], the author gives the following result (Here $(E, \mid . \mid)$ is a normed linear space and $\| . \|$ is another norm on E, $C \subset E$ is a nonempty convex, not necessarily closed set with $0 \notin C$ and $\lambda C \subset C$ for all $\lambda > 0$), assuming that there exist constants $c_1, c_2 > 0$ such that the norms $\mid . \mid$ and $\| . \|$ are topologically equivalent, which is

$$c_1 \mid x \mid \leq \| x \| \leq c_2 \mid x \mid \ \text{ for all } x \in C.$$

Also assume that $\| . \|$ is increasing with respect to C , that is $\| x + y \| > \| x \|$ for all $x, y \in C$.

Theorem 1.4. *([9]) Assume* $0 < c_2 \rho < R$, $\| . \|$ *is increasing with respect to* C, *and the map* $N : D = \{x \in C : \| x \| \leq R\} \to C$ *is compact. In addition assume that the following conditions are satisfied:*

(H1) $\| N(x) \| \geq \| x \|$ *for all* $x \in C$ *with* $\mid x \mid = \rho$,

(H2) $\mid N(x) \mid < \mid x \mid$ *for all* $x \in C$ *with* $\| x \| = R$.

Then N *has at least one fixed point* $x \in C$ *with* $\rho \leq \mid x \mid$ *and* $\| x \| < R$.

For other generalizations and applications of Krasnoselskii's fixed point theorem in cone the reader may see the papers [7] and [1]-[4].

In this paper we are interested to give some new abstract results and we use conditions of type

$$\varphi(u) \geq \varphi(Nu) \text{ if } \mid u \mid = r$$

instead of condition

$$\mid u \mid \geq \mid Nu \mid \ \text{ if } \mid u \mid = r$$

which is assumed in Krasnoselskii's fixed point theorem in cone.

2. The main results

Throughout this paper we consider $(X, | \, . \, |)$ be a normed linear space, $K \subset X$ a positive cone, "$\preceq$" the order relation induced by K and "$\prec$" the strict order relation induced by K.

Theorem 2.1. *Let be $K_{r,R} = \{x \in K : r \leq | \, x \, | \leq R\}$, where $r, R \in R_+$, $r < R$. We assume that $N : K_{r,R} \to K$ is a completely continuous operator and $\varphi : K \to R_+$,$\psi : K \to R$. Also, assume that the following conditions are satisfied:*

$$(i.1) \begin{cases} \varphi(0) = 0 \text{ and there exists } h \in K - \{0\} \text{ such that} \\ \varphi(\lambda h) > 0, \text{ for all } \lambda \in (0, 1], \\ \varphi(x + y) \geq \varphi(x) + \varphi(y) \text{ for all } x, y \in K, \end{cases}$$

$(i.2)$ $\psi(\alpha x) > \psi(x)$ *for all* $\alpha > 1$ *and for all* $x \in K$ *with* $| \, x \, | = R$,

$$(i.3) \begin{cases} \varphi(u) \leq \varphi(Nu) & \text{if} \quad | \, u \, | = r \\ \psi(u) \geq \psi(Nu) & \text{if} \quad | \, u \, | = R. \end{cases}$$

Then N has a fixed point in $K_{r,R}$

Proof. Let $N^* : K \to K$ be given by

$$N^*(u) = \begin{cases} h, & \text{if} \quad u = 0, \\ (1 - \frac{|u|}{r})h + \frac{|u|}{r} N(\frac{r}{|u|}u), & \text{if} \quad 0 < | \, u \, | < r, \\ Nu, & \text{if} \quad r \leq | \, u \, | \leq R, \\ N(\frac{R}{|u|}u), & \text{if} \quad | \, u \, | \geq R. \end{cases}$$

N is completely continuous, so N^* is completely continuous too. From our hypothesis we have that $N^*(K) \subset K$ is a convex and relatively compact set, so from Schauder' s fixed point theorem it follows that there exists $u^* \in K$ with $N^*(u^*) = u^*$. We have to consider three cases.

Case 1. Suppose that $u^* = 0$. We have $0 = N^*(0) = h$, a contradiction with $h \in K \setminus \{0\}$.

Case 2. Suppose that $0 < | \, u^* \, | < r$. We obtain

$$\left(1 - \frac{| \, u^* \, |}{r}\right) h + \frac{| \, u^* \, |}{r} N\left(\frac{r}{| \, u^* \, |} u^*\right) = u^*,$$

$$\left(\frac{r}{| \, u^* \, |} - 1\right) h + N(\frac{r}{| \, u^* \, |} u^*) = \frac{r}{| \, u^* \, |} u^*.$$

Let $\lambda := \frac{r}{|u^*|} - 1$ and $u_0 := \frac{r}{|u^*|} u^*$. Since $| \, u^* \, | < r$ we have that $\frac{r}{|u^*|} > 1$, so $\lambda > 0$. Also, $| \, u_0 \, | = | \, \frac{r}{|u^*|} u^* \, | = \frac{r}{|u^*|} \, | \, u^* \, | = r$, so $| \, u_0 \, | = r$. We obtain

$$\lambda h + N(u_0) = u_0 \tag{2.1}$$

For $\lambda > 0$, from (i1) we obtain that

$$\varphi(N(u_0) + \lambda h) \geq \varphi(N(u_0)) + \varphi(\lambda h) > \varphi(N(u_0)).$$

Then, from (2.1) we obtain $\varphi(u_0) > \varphi(N(u_0))$, a contradiction with (i3).

Case 3. Suppose that $| \, u^* \, | > R$. We have $N(\frac{R}{|u^*|} u^*) = u^*$. Let $u_1 := \frac{R}{|u^*|} u^*$ and $\beta := \frac{|u^*|}{R} > 1$. We have $| \, u_1 \, | = R$ and $N(u_1) = u^* = u_1 \frac{|u^*|}{R}$, so $N(u_1) = \beta u_1$. From (i.2) we obtain $\psi(N(u_1)) = \psi(\beta u_1) > \psi(u_1)$, a contradiction with (i.3). So $r \leq | u^* | \leq R$ and the conclusion follows. $\square$

Remark 2.2. (1) If $X := C[0,1], \eta > 0, I \subset [0,1]$, $I \neq [0,1]$, $\| x \| := \max\limits_{t\in[0,1]} x(t)$ and $K := \{x \in C[0,1] : x \geq 0 \text{ on } [0,1], x(t) \geq \eta \| x \| \text{ for all } t \in I\}$ is a cone, a functional that satisfies (i1) is

$$\varphi(x) := \min_{t\in I} x(t).$$

Indeed, $\varphi(0) = 0$,there exists $h \in K - \{0\}$ such that $\varphi(\lambda h) > 0$, for all $\lambda \in (0,1]$ and

$$\varphi(x+y) = \min_{t\in I}[x(t) + y(t)] \geq \min_{t\in I} x(t) + \min_{t\in I} y(t) = \varphi(x) + \varphi(y).$$

(2) The norm is an example of functional that satisfies (i2).

Theorem 2.3. *Let $K_{r,R} = \{x \in K : r \leq | x | \leq R\}$, where $r, R \in R_+$, $r < R$. We assume that $N : K_{r,R} \to K$ is a completely continuous operator and $\varphi, \psi : K \to R$. Also, we assume that the following conditions are satisfied:*

$(ii.1)$ φ is strictly decreasing,

$(ii.2)$ $\psi(\alpha x) < \psi(x)$ for all $\alpha > 1$ and for all $x \in K$ with $| x | = R$,

$(ii.3)$ $\begin{cases} \varphi(u) \geq \varphi(Nu) & \text{if } | u | = r, \\ \psi(u) \leq \psi(Nu) & \text{if } | u | = R. \end{cases}$

Then N has a fixed point in $K_{r,R}$.

Proof. Let $h \succ 0$ and $N^* : K \to K$,

$$N^*(u) = \begin{cases} h, & \text{if } u = 0 \\ (1 - \frac{|u|}{r})h + \frac{|u|}{r} N(\frac{r}{|u|}u), & \text{if } 0 <| u |< r \\ Nu, & \text{if } r \leq | u | \leq R \\ N(\frac{R}{|u|}u), & \text{if } | u | \geq R. \end{cases}$$

Since N^* is completely continuous, we have, like in Theorem 2.1, that there exists $u^* \in K$ so that $N^*(u^*) = u^*$. We consider three cases.

Case 1. If $u^* = 0$ we obtain $0 = N^*(0) = h$, a contradiction with $h \succ 0$.

Case 2. If $0 <| u^* |< r$. We obtain (2.1) with $\lambda > 0$ and $| u_0 | = r$, like in Theorem 2.1. From $\lambda h \succ 0$, we have that

$$N(u_0) + \lambda h \succ N(u_0),$$

so, from (ii.1), we have that

$$\varphi(N(u_0) + \lambda h) < \varphi(N(u_0))$$

and from (2.1) we obtain

$$\varphi(u_0) < \varphi(N(u_0)) \text{ for } | u_0 | = r,$$

a contradiction with (ii.3).

Case 3. If $| u^* | > R$, we have that

$$N\left(\frac{R}{| u^* |} u^*\right) = u^*,$$

so

$$N\left(\frac{R}{\mid u^* \mid}u^*\right) = \left(\frac{R}{\mid u^* \mid}u^*\right)\frac{\mid u^* \mid}{R}.$$

Let be $u_1 := \frac{R}{|u^*|}u^*$, so $\mid u_1 \mid = R$ and let be $\beta := \frac{|u^*|}{R} > 1$. We obtain $N(u_1) = \beta u_1$, so

$$\psi\left(N(u_1)\right) = \psi\left(\beta u_1\right) \tag{2.2}$$

From (ii.2) we obtain

$$\psi\left(\beta u_1\right) < \psi\left(u_1\right)$$

and from (2.2) we have

$$\psi\left(N(u_1)\right) < \psi\left(u_1\right) \text{ for } \mid u_1 \mid = R,$$

a contradiction with (ii.3). So $r \leq \mid u^* \mid \leq R$ and the conclusion follows. □

Remark 2.4. $\psi(x) := \frac{1}{|x|+1}$ is an example of functional that satisfies (ii.2). Indeed, for $\alpha > 1$ and $\mid x \mid = R$, we have that

$$\psi(\alpha x) = \frac{1}{\alpha \mid x \mid +1} < \frac{1}{\mid x \mid +1} = \psi(x).$$

Also, if $\mid . \mid$ is strictly increasing, i.e., $x < y$ implies $\mid x \mid < \mid y \mid$, then $\varphi(x) := \frac{1}{|x|+1}$ is strictly decreasing, so it satisfies (ii.1).

Theorem 2.5. *Let* $K_{r,R} := \{x \in K : r \leq \mid x \mid \leq R\}$, *where* $r, R \in R_+, r < R$. *We assume that* $N : K_{r,R} \to K$ *is a completely continuous operator and* $\varphi, \psi : K \to R_+$. *Also, we assume that the following conditions are satisfied:*

$$(iii.1)\begin{cases} \varphi(\alpha x) = \alpha\varphi(x), \text{ for all } \alpha > 0 \text{ and for all } x \in K, \\ \varphi(\alpha x) > \varphi(x), \text{ for all } \alpha > 1 \text{ and for all } x \in K \text{ with } \mid x \mid = R, \end{cases}$$

$$(iii.2)\begin{cases} \psi(0) = 0 \text{ and there exists } h \in K \setminus \{0\} \text{ such that} \\ \psi(\lambda h) > 0 \text{ for all } \lambda \in (0, 1], \\ \psi(\alpha x) = \alpha\psi(x) \text{ for all } \alpha > 0 \text{ and for all } x \in K, \\ \psi(x + y) \geq \psi(x) + \psi(y) \text{ for all } x, y \in K, \end{cases}$$

(iii.3)

$$\begin{cases} \varphi(u) \geq \varphi(Nu) & \text{if } \mid u \mid = r, \\ \psi(u) \leq \psi(Nu) & \text{if } \mid u \mid = R. \end{cases}$$

Then N *has a fixed point in* $K_{r,R}$.

Proof. Define $N^* : K_{r,R} \to K$ by

$$N^*(u) := (\frac{R}{\mid u \mid} + \frac{r}{\mid u \mid} - 1)^{-1}N((\frac{R}{\mid u \mid} + \frac{r}{\mid u \mid} - 1)u).$$

Since N is completely continuous, it follows that N^* is completely continuous too. Let

$$\alpha := \frac{R}{\mid u \mid} + \frac{r}{\mid u \mid} - 1$$

and

$$u_0 := \alpha u.$$

We have now,

$$\alpha N^*(u) = N(\alpha u).$$

If $\mid u \mid= r$, then

$$\alpha = \frac{R}{r} \text{ and } \mid u_0 \mid=\mid \alpha u \mid= \frac{R}{r} r = R.$$

So, from (iii.2),

$$\psi(N(u_0)) = \psi(N(\alpha u)) = \psi(\alpha N^*(u)) = \alpha\psi(N^*(u)) \tag{2.3}$$

and from (iii.3),

$$\psi(N(u_0)) \geq \psi(u_0) = \psi(\alpha u) = \alpha\psi(u). \tag{2.4}$$

From (2.3) and (2.4) we obtain that

$$\psi(N^*(u) \geq \psi(u) \text{ if } \mid u \mid= r. \tag{2.5}$$

If $\mid u \mid= R$, then

$$\alpha = \frac{r}{R} \text{ and } \mid u_0 \mid=\mid \alpha u \mid= \frac{r}{R} R = r.$$

Using (iii.3) we obtain that

$$\varphi(\alpha u) = \varphi(u_0) \geq \varphi(N(u_0)) = \varphi(N(\alpha u)) = \varphi(\alpha N^*(u)) \tag{2.6}$$

and from (iii.1),

$$\varphi(\alpha u) = \alpha\varphi(u) \text{ and } \varphi(\alpha N^*(u)) = \alpha\varphi(N^*(u)). \tag{2.7}$$

From (2.6) and (2.7) we deduce that

$$\varphi(u) \geq \varphi(N^*(u)) \text{ if } \mid u \mid= R. \tag{2.8}$$

So, (2.5) and (2.8) imply that φ, ψ and N^* satisfy all the conditions of Theorem 2.1 (with φ and ψ changing their places and N^* instead of N). So N^* has a fixed point u^* in $K_{r,R}$. It follows that

$$N^*(u^*) = u^*, \text{ with } r \leq\mid u^* \mid\leq R,$$

so

$$\frac{1}{\alpha} N(\alpha u^*) = u^*.$$

Making the notation $u_1 := \alpha u^*$, where $\alpha = \frac{R}{|u^*|} + \frac{r}{|u^*|} - 1$, we obtain

$$N(u_1) = u_1 \tag{2.9}$$

and

$$\mid u_1 \mid= \alpha \mid u^* \mid= R + r - \mid u^* \mid .$$

Since

$$\begin{aligned} R + r - &\mid u^* \mid\geq r, \text{ for } r \leq\mid u^* \mid\leq R, \\ R + r - &\mid u^* \mid\leq R, \text{ for } r \leq\mid u^* \mid\leq R, \end{aligned}$$

we have that

$$r \leq\mid u_1 \mid\leq R, \quad \text{that is } u_1 \in K_{r,R}. \tag{2.10}$$

From (2.9) and (2.10) the conclusion follows. □

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Addendum to ”A multiplicity result for nonlocal problems involving nonlinearities with bounded primitive”

Biagio Ricceri

In this addendum, I wish to point out two changes that I expected to do over the galley proofs of [1] which, to the contrary, never reached me.

Namely, in Theorem 1.6, accordingly to Proposition 1.4, the definition of $\hat{\theta}$ has to be changed as follows:

$$\hat{\theta} = \inf_{x \in J^{-1}(]\inf_X J, \sup_X J[\setminus\{0\})} \frac{\psi(x) - \eta(x)}{\varphi(J(x))} \ .$$

When φ can be extended by continuity to $[-\mathrm{osc}_X J, \mathrm{osc}_X J]$, then no change is needed. That is to say, the equality

$$\inf_{x \in J^{-1}(]\inf_X J, \sup_X J[\setminus\{0\})} \frac{\psi(x) - \eta(x)}{\varphi(J(x))} = \inf_{x \in J^{-1}(\mathbf{R}\setminus\{0\})} \frac{\psi(x) - \eta(x)}{\varphi(J(x))}$$

holds.

In this connection, no change is needed in Theorem 1.7 (where $\varphi(t) = e^t - t - 1$), while in the definition of θ^* in Theorem 1.3, the condition $\int_\Omega F(x, u(x))dx \neq 0$ has to be changed in

$$\int_\Omega F(x, u(x))dx \in]\alpha_f, \beta_f[\setminus\{0\} \ .$$

Finally, in (1.6), the inequality

$$\inf_{x \in X}(\psi(x) - \mu(e^{J(x)} - 1)) < 0$$

has to be replaced by

$$\inf_{x \in X\setminus J^{-1}(0)}(\psi(x) - \mu(e^{J(x)} - 1)) < 0$$

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Book reviews

Nonlinear analysis and optimization, Proceedings of the conference held in celebration of Alex Ioffe's 70th and Simeon Reich's 60th birthdays at the Technion, Haifa, June 18–24, 2008. Edited by *Arie Leizarowitz, Boris S. Mordukhovich, Itai Shafrir* and *Alexander J. Zaslavski*, Israel Mathematical Conference Proceedings. American Mathematical Society, Providence, RI; Bar-Ilan University, Ramat Gan, 2010.

I: **Nonlinear Analysis,** Contemporary Mathematics, vol. 513, 263 pp., ISBN-10: 0-8218-4834-8, ISBN-13: 978-0-8218-4834-0.

II: **Optimization**, Contemporary Mathematics, vol. 514, xx+290 pp., ISBN-10: 0-8218-4835-6, ISBN-13: 978-0-8218-4835-7.

A Conference on Nonlinear Analysis and Optimization took place on June 18–24, 2008, at the Technion - Israel Institute of Technology, Haifa, Israel. Continuing the tradition of several previous conferences on related topics, the present one was dedicated to the celebration of Alex Ioffe's 70th and Simeon Reich's 60th birthdays. Alex Ioffe is known for his important contributions to nonlinear analysis, optimization and variational analysis (over 130 publications) as well as for his book (written jointly with V. M. Tikhomirov) "Theory of Extremal Problems" , Moscow 1974 (in Russian), translated in English and German in 1979. Simeon Reich has published more than 300 research papers and two monographs into various areas of mathematics, related to nonlinear analysis and optimization. Taking into account the actuality of the topics as well as the prominence of the two celebrated personalities, the conference attracted a lot of eminent mathematicians from all over the world – more than 70 from 18 countries.

The present volumes collect the written and expanded versions of the contributions of some participants, as well as the contributions of some invited speakers who were unable to attend the conference. The contributions are divided into two volumes - I. *Nonlinear Analysis*, and II. *Optimization*, although it is difficult to trace a clear demarcation line between these two fields, taking into account their strong interdependence, a fact that can be seen by an examination of the titles in the volumes.

The first volume contains 14 contributed papers on various topics from nonlinear analysis such as fixed point theory (T. Domínguez Benavides and S. Phothi, W. Kaczor, T. Kuczumow, and M. Michalska, K. Goebel and B. Sims), nonexpansive mappings, monotone operators and Kirszbraun-Valentine extensions (H. H. Bauschke and X. Wang), nonexpansive operators and convex feasibility problems (A. Cegielski), iterative methods and algorithms for finding fixed points (T. Ibaraki, W. Takahashi, L. Leuştean, G. López, V. Martin-Márquez, and H.-K. Xu), random products of orthogonal projections (R. E. Bruck), Mosco stability for the generalized proximal mapping (D. Butnariu, E. Resmeriţa, and S. Sabach), control for Navier-Stokes equations (V. Barbu), Neumann problem for p-Laplacian (S. Aizicovici, N. S. Papageorgiou and V. Staicu). A paper on biology – an amphibian juvenile-adult model – by A. S. Ackleh, K. Deng and Q. Huang, is also included.

The second volume, on optimization, is concerned with several important topics of great interest in the current research in the area, such as regularity and calmness in nonsmooth analysis (A. Giannessi, A. Moldovan, L. Pellegrini and J.-P. Penot), quadratic optimal control (J. F. Bonnans, N. P. Osmolovskii, and V. Y. Glizer), transportation problems (J.-P. Aubin and S. Martin, G. Buttazzo and G. Carlier), subdifferential calculus (R. Baier and E. Farkhi, J. M. Borwein and S. Sciffer), constrained minimization problems (A. Zaslavski), isoperimetric problems in the calculus of variations (R. A. C. Ferreira and A. C. Torres), Kaldorian business fluctuations (T. Maruyama), time scales (D. Mozyrska and E. Pawluszewicz), Morse indexes for piecewise linear functions (D. Pallaschke and R. Urbanski).

Containing articles on leading themes of current research in nonlinear analysis and optimization, written by prominent researchers in these two areas, these two volumes bring the readers, advanced graduate students and researchers alike, to the frontline of research in important fields of mathematics. Undoubtedly, they must be on the desk of every one working in these areas.

S. Cobzaş

Jiři Matoušek, Thirty-three Miniatures. Mathematical and Algorithmic Applications of Linear Algebra, Student Mathematical Library, Volume 53, American Mathematical Society, Providence, Rhode Island, 2010, x+182 pp; ISBN-13: 978-0-8218-4977-4, ISBN-10: 0-8218-4977-8

This booklet contains a collection of succinct and clever applications of linear algebra to combinatorics, graph theory, geometry and algorithms. In this case, gem or jewel is a good synonym for miniature. Each miniature (chapter, lecture) is dense, concise, carefully polished and written in an attractive style. A motivation of the term is the complete exposition of the result, the length from two to ten pages, and the independence (with few exceptions) from all other chapters. Although Matoušek says in preface that nothing is original, the way of structuring, the selection of topics and the presentation is fully original. Each lecture has a short paragraph ”Sources”, containing annotated bibliographical references.

The text requires only a good undergraduate background in linear algebra and some knowledge in graph theory and geometry.

Among the topics treated, we mention (enumeration reflects the preferences of the reviewer) Fibonacci numbers, error-correcting codes, probabilistic checking of matrix multiplication, tiling a rectangle by squares, counting spanning trees, fast associativity testing, set pairs via exterior products. All topics, both the simple and more advanced, refer to beautiful results and have elegant proofs.

The impact of this book will be two-fold: first, provides the instructors and the students enroled to a linear algebra course with a rich set of examples; second, makes people interested in combinatorics and computer science aware of the existence of linear algebra tools and their strength.

Intended audience: undergraduates, graduate students and research mathematicians interested in combinatorics, graph theory, theoretical computer science and computational geometry, as well as lecturers who want to liven their courses.

Radu Trîmbiţaş

Elias M. Stein and Rami Shakarchi, Functional analysis. Introduction to further topics in analysis. Princeton Lectures in Analysis 4, Princeton University Press, Princeton, NJ, 2011. xviii+423 pp. ISBN: 978-0-691-11387-6.

This is the last of a four volume treatise on analysis published with Princeton University Press between 2003 and 2011 as the series *Princeton Lectures in Analysis.* The previous volumes, I. *Fourier analysis. An introduction* (2003), II. *Complex analysis,* (2003), III. *Real analysis. Measure theory, integration, and Hilbert spaces* (2005), got an enthusiastic reception from the mathematical community.

The aim of the treatise is to present some cornerstone results of analysis, with emphasis on their historical evolution and the interdependence existing between various parts. I think this is best illustrated by the authors in the preface:

> Our goal is to present the various sub-areas of analysis not as separate disciplines, but rather as highly interconnected. It is our view that seeing these relations and their resulting synergies will motivate the reader to attain a better understanding of the subject as a whole. With this outcome in mind, we have concentrated on the main ideas and theorems that have shaped the field (sometimes sacrificing a more systematic approach), and we have been sensitive to the historical order in which the logic of the subject developed.

In the same order of ideas, this last volume, on functional analysis, could be viewed as an illustration of the famous saying of Einar Hille that "a functional analyst is an analyst, first and foremost, and not a degenerate species of a topologist". In the first chapter, Ch. 1. *L^p spaces and Banach spaces*, the basic of Banach spaces (Hahn-Banach theorem, duality, separation by hyperplanes) are exposed in parallel with the corresponding properties of L^p spaces - completeness and dual spaces. In the Appendix the duals of $C(X)$, the space of all continuous functions on a compact metric space X, and of $C_b(X)$, the space of all bounded continuous functions on an arbitrary metric space X, are described.

The study of L^p spaces is continued in the second chapter, 2. *L^p spaces in harmonic analysis*, with Riesz interpolation theorem, the L^p theory of Hilbert transform, the Hardy spaces H^1_r and the space BMO. Chapter 3. *Distributions: Generalized functions*, contains the basic properties of distributions, some important examples, as, for instance, the principal value of $1/x$ and its relation with the Hilbert transform, and applications to fundamental solutions of partial differential equations.

In Chapter 4. *Applications of the Baire category theorem*, one shows that some classes of singular functions as, e.g., continuous nowhere differentiable functions, continuous functions with divergent Fourier, are of second Baire category in the corresponding spaces (i.e., they form large subsets). The open mapping and the closed graph theorems are proved with application to Grothendieck's theorem on closed subspaces of L^p.

The basic of probability theory are presented in Ch. 5. *Rudiments of probability theory*, continued in Ch. 6. *An introduction to Brownian motion*, with stopping time, Markov property and applications to Dirichlet problem. Some basic results on analytic functions of several complex variables are considered in Chapter 7, *A glimpse into several complex variables*, while in the last chapter, 8. *Oscillatory integrals and Fourier analysis,* the presentation of this important area of harmonic analysis is based on the properties of the averaging operators, with applications to dispersion equation, Radon transform and to the counting of lattice points.

Each chapter ends with a consistent set of exercises (some of them with hints) related to and completing the main text. Other ones, more challenging, called Problems, are also included.

By completing this ambitious project, the authors, renowned specialists in harmonic analysis, have done a great service to the mathematical community. Undoubtedly that the volumes will become a standard reference for those interested in analysis understand in a large sense, but also for engineers or physicists needing tools from harmonic analysis in their research.

S. Cobzaş

Henri Anciaux, Minimal submanifolds in pseudo-Riemannian geometry, xv+167 pp., World-Scientific, London-Singapore-Beijing 2011, ISBN: 13 978-981-4291-24-8 and 10 981-4291-24-2.

Since the discovery by Lagrange in 1755 of the differential equation satisfied by a minimal surface in the Euclidean space (the catenoid in $\mathbb{R}^3$), the theory of minimal surfaces attracted the attention of many mathematicians, with successive generalizations, from the Euclidean space to Riemannian manifolds. After the introduction of pseudo-Riemannian manifolds as model of the space-time in the relativity theory, it became clear that this is the most general framework to treat this problem. The present book is the first devoted to a systematic exposition of the theory of minimal submanifolds in pseudo-Riemannian manifolds.

A pseudo-Riemannian manifold is a differentiable manifold $\mathcal{M}$ equipped with a smooth bilinear 2-form g, called the metric, which is non-degenerate in the following sense: $\forall Y \in T_x\mathcal{M},\ g(X,Y) = 0 \Rightarrow X = O$, for every vector X in the tangent space $T_x\mathcal{M}$ and every $x \in \mathcal{M}$. The pseudo-Riemannian space of relativity theory, called the Minkowski space and denoted by $\mathbb{R}^4_1$, is the space $\mathbb{R}^4$ equipped with the metric $\langle\cdot,\cdot\rangle_1 = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$. Taking an orthonormal basis $(e_1,\ldots,e_m)$ in the tangent space $T_x\mathcal{M}$ it turns out that p of these vectors are negative (i.e. $g(e_i,e_i) < 0$) and $m-p$ are positive (i.e. $g(e_i,e_i) > 0$), and the number p does not depend on the point x. The pair $(p, m-p)$ is called the signature of the pseudo-Riemannian space $\mathcal{M}$ (the Minkowski space has signature (1,3)). The basic results on pseudo-Riemannian manifolds, with emphasis on submanifolds and variation formulae for the volume functional, are developed in the first chapter of the book. The second one is devoted to the case of surfaces, meaning two-dimensional submanifolds, in pseudo-Euclidean space, including a variety of examples and the classification of ruled minimal surfaces, the first global result of the book. This discussion is continued in the third chapter, *Equivariant minimal hypersurfaces*, meaning submanifolds of codimension one, with the study of space forms which are the pseudo-Riemannian analogs of the Riemann round sphere.

Chapter 4, *Pseudo-Kähler manifolds*, is devoted to a special class of pseudo-Riemannian manifolds of even dimension, namely the pseudo-Kähler manifolds, which in the positive case yield the Kähler manifolds. The most natural example of a pseudo-Kähler manifold – the complex pseudo-Riemannian space forms – is presented and one proves that the tangent bundle of a pseudo-Kähler manifold admits a pseudo-Kähler structure.

In the fifth chapter, *Complex and Lagrangian submanifolds in pseudo-Kähler manifolds*, one describes several families of minimal submanifolds in pseudo-Kähler manifolds. The main question addressed in the last chapter of the book, 6, *Minimizing properties of minimal submanifolds*, is whether or not a minimal submanifold, which, by definition, is a critical point of the volume, is actually an extremum of the volume functional. After describing several submanifolds satisfying this requirement, one shows that a necessary condition is that the induced metric of both tangent and normal bundle be definite.

Previously pseudo-Riemannian manifolds were treated only in books directed to physical applications, the present one being the first devoted to an exposition of the basic results on pseudo-Riemannian manifolds and of their minimal submanifolds. Exposing in a clear, live and accessible style (the prerequisites are only familiarity with differentiable manifolds) the book will be of great help to young mathematicians and physicists interested in this topic. It can be used also as a base text for advanced graduate courses.

Cornel Pintea