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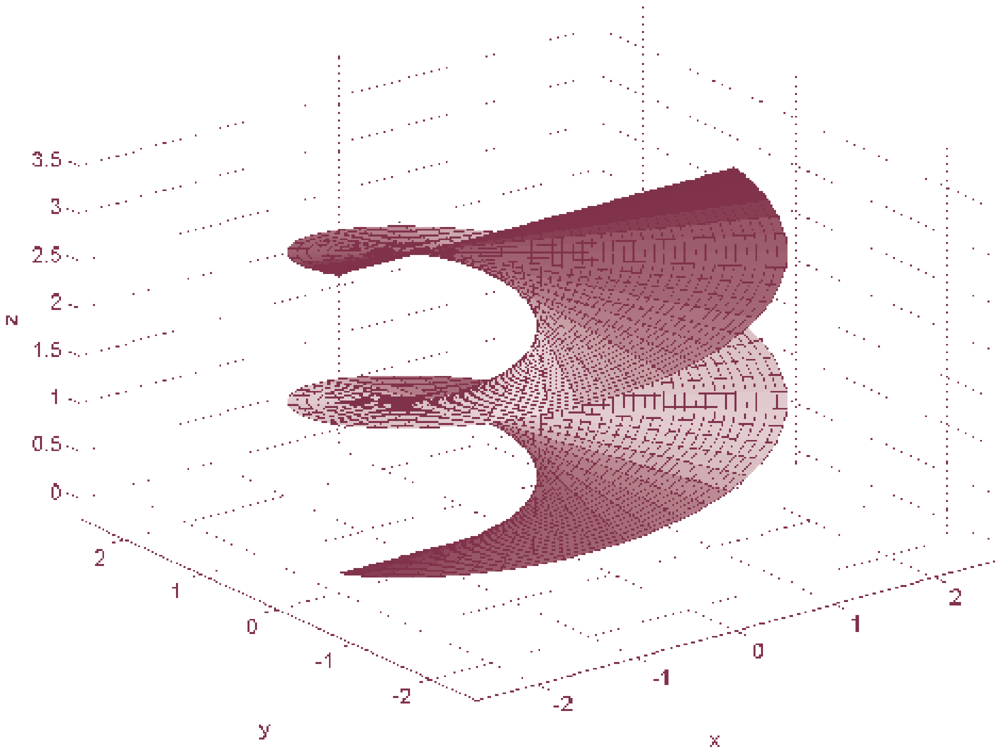

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Generalized projectors and the saturated closure of a π-homomorph of finite π-solvable groups

Rodica Covaci

Abstract. The paper introduces and studies the notion of *generalized projector*, which generalizes the well-known notion of *projector* defined by W. Gaschütz in [8] as a generalization of the *covering subgroups* introduced by the same author in [7]. Let π be an arbitrary set of primes. A new definition for the *saturated closure* of a π-homomorph of finite π-solvable groups, equivalent to that in [3], is given. A property connected with the notion of generalized projector on a class X of finite π-solvable groups, called the *GP-property*, is also introduced. The main results of the paper are the following: 1) a characterization theorem for the saturated closure of the π-homomorphs of finite π-solvable groups with the GP-property by means of the generalized projectors; 2) a theorem showing that if X is a π-homomorph of finite π-solvable groups with the GP-property and $\overline{X}$ is its saturated closure, then X is a Schunck class if and only if $X = \overline{X}$. These results prove that theorems similar to those obtained by J. Weidner in [10] for finite solvable groups can be also obtained in the more general case of finite π-solvable groups.

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1. Preliminaries

In [3], we generalized in the more general case of finite π-solvable groups the results established by J. Weidner in [10] for finite solvable groups, obtaining a characterization of the saturated closure of a homomorph of finite π-solvable groups by means of the semicovering subgroups (introduced by J. Weidner in [10] as a generalization of the covering subgroups defined by W. Gaschütz in

[7]). Following the ideas from [10] and [3], the present paper introduces and studies the notion of *generalized projector*, which generalizes the well-known notion of *projector* defined by W. Gaschütz in [8] as a generalization of the covering subgroups. Using the projectors, a new definition for the *saturated closure* of a π-homomorph of finite π-solvable groups, equivalent to that in [3], is given. We define for a class X of finite π-solvable groups the *GP-property*, which is connected with the generalized projectors. A characterization theorem for the saturated closure of the π-homomorphs of finite π-solvable groups with the GP-property and an important consequence of this characterization are the main results of the paper.

All groups considered in the paper are finite. Denote by π an arbitrary set of primes and by π' the complement to π in the set of all primes.

We remind some definitions and theorems which will be useful for our considerations.

Definition 1.1. *a)* ([9]) *A class X of groups is a* **homomorph** *if X is closed under homomorphisms, i.e. if $G \in X$ and N is a normal subgroup of G, then $G/N \in X$.*

b) A group G is said to be **primitive** *if there exists a stabilizer W of G, i.e. W is a maximal subgroup of G and $core_G W = 1$, where*

$$core_G W = \cap\{W^g \mid g \in G\}.$$

c) ([9]) *A homomorph X is a* **Schunck class** *if X is primitively closed, i.e. if any group G, all of whose primitive factor groups are in X, is itself in X.*

Definition 1.2. *Let X be a class of groups, G a group and H a subgroup of G.*

a) ([8]) *H is an X-***maximal subgroup** *of G if:*

(i) $H \in X$;

(ii) $H \leq H^* \leq G,\ H^* \in X \Rightarrow H = H^*$.

b) ([8]) *H is an X-***projector** *of G if for any normal subgroup N of G, HN/N is X-maximal in G/N.*

c) ([7]) *H is an X-***covering subgroup** *of G if:*

(i) $H \in X$;

(ii) $H \leq K \leq G,\ K_0 \trianglelefteq K,\ K/K_0 \in X \Rightarrow K = HK_0$.

Remark 1.3. *a) Let X be a class of groups and G a group. Then: i) $G \in X$ if and only if G is X-maximal in G ; ii) if G is an X-projector of G, then $G \in X$.*

b) Let X be a homomorph and G a group. Then G is an X-projector of G if and only if $G \in X$.

Theorem 1.4. ([8]) *Let X be a class of groups, G a group and H a subgroup of G.*

a) If H is an X-projector of G and N is a normal subgroup of G, then HN/N is an X-projector of G/N.

b) H is an X-projector of G if and only if:

(i) H is X-maximal in G;

(ii) HM/M is an X-projector of G/M for all minimal normal subgroups M of G.

Theorem 1.5. *Let X be a class of groups, G a group and H a subgroup of G.*

a) If H is an X-covering subgroup or an X-projector of G, then H is X-maximal in G.

b) ([4]) *If X is a homomorph, then H is an X-covering subgroup of G if and only if H is an X-projector in any subgroup K with $H \leq K \leq G$. In particular, any X-covering subgroup of G is an X-projector of G.*

Theorem 1.6. ([1]) *A solvable minimal normal subgroup of a finite group is abelian.*

Introduced by S.A. Čunihin in [6], the π-solvable groups are more general than the solvable groups.

Definition 1.7. *a)* ([6]) *A group G is π-**solvable** if every chief factor M/N of G (i.e. M/N is a minimal normal subgroup of G/N) is either a solvable π-group or a π'-group. In particular, if π is the set of all primes, we obtain the notion of* **solvable group**.

b) ([2]) *A class X of groups is said to be π-**closed** if*

$$G/O_{\pi'}(G) \in X \Rightarrow G \in X,$$

where $O_{\pi'}(G)$ denotes the largest normal π'-subgroup of G.

*c) We say that X is a π-**homomorph** (respectively a π-**Schunck class**) if X is a π-closed homomorph (respectively X is a π-closed Schunck class).*

Theorem 1.8. ([6]) *a) If G is a π-solvable group and N is a normal subgroup of G, then G/N is π-solvable.*

b) If G is a group and N is a normal subgroup of G, such that N and G/N are π-solvable, then G is π-solvable.

Theorem 1.9. ([5]) *Let X be a π-homomorph. The following conditions are equivalent:*

(i) X is a Schunck class;

(ii) if G is a π-solvable group, $G \notin X$ and M is a minimal normal subgroup of G such that $G/M \in X$, then M has a complement in G;

(iii) any π-solvable group G has X-covering subgroups;

(iv) any π-solvable group G has X-projectors.

2. Generalized projectors

In [10], J. Weidner generalizes the notion of covering subgroup given in Definition 1.2.c) by renouncing to the condition (i). In [3], this generalized covering subgroup is called *semicovering subgroup*. Similarly, we will introduce o notion which generalizes the notion of projector.

Definition 2.1. *Let X be a class of groups, G a group and H a subgroup of G. H is called a* **generalized X-projector** *of G if for any normal subgroup N of G, $N \neq 1$, HN/N is X-maximal in G/N.*

It is the aim of this section to prove some properties of the generalized projectors.

Everywhere in this section we denote by X a class of groups, by G an arbitrary finite group and by H a subgroup of G.

Remark 2.2. *If H is an X-projector of G, then H is a generalized X-projector of G.*

Theorem 2.3. *H is an X-projector of G if and only if the following two conditions hold:*

(i) *H is X-maximal in G;*

(ii) *H is a generalized X-projector of G.*

Proof. Let H be an X-projector of G. By Definition 1.2.b), for any normal subgroup N of G we have that HN/N is X-maximal in G/N. In particular, for $N = 1$ we obtain that H is X-maximal in G, and so condition (i) holds. If we take $N \neq 1$ a normal subgroup of G, then HN/N is X-maximal in G/N, and, by Definition 2.1, H is a generalized X-projector of G, which mean that condition (ii) also holds.

Conversely, suppose that conditions (i) and (ii) hold. From (i) follows that for $N = 1$ we have HN/N is X-maximal in G/N. Let now $N \neq 1$ be a normal subgroup of G. By (ii) and Definition 2.1, HN/N is X-maximal in G/N. So HN/N is X-maximal in G/N for any normal subgroup N of G. This means by Definition 1.2.b) that H is an X-projector of G. □

Theorem 2.4. *If H is a generalized X-projector of G and N is a normal subgroup of G, then HN/N is a generalized X-projector of G/N.*

Proof. Let H be a generalized X-projector of G and N a normal subgroup of G. We distinguish two cases:

1° $N = 1$. Since H is a generalized X-projector of G, we have for $N = 1$ that HN/N is a generalized X-projector of G/N.

2° $N \neq 1$. In order to prove that HN/N is a generalized X-projector of G/N, by Definition 2.1 we have to prove that for any normal subgroup L/N of G/N, $L/N \neq 1$, $(HN/N \cdot L/N)/(L/N)$ is X-maximal in $(G/N)/(L/N)$. But

$$(HN/N \cdot L/N)/(L/N) = (HNL/N)/(L/N) = (HL/N)/(L/N) \simeq HL/L$$

and

$$(G/N)/(L/N) \simeq G/L,$$

and so we have to prove that

$$HL/L \text{ is } X\text{-maximal in } G/L.$$

Indeed, from the hypothesis that H is a generalized X-projector of G, by using Definition 2.1 for the normal subgroup L of G, where $L \neq 1$ (since $1 \neq N < L$), we obtain that HL/L is X-maximal in G/L. □

Our last theorem concerning some properties of the generalized projectors is a characterization theorem for the generalized projectors.

Theorem 2.5. *H is a generalized X-projector of G if and only if HM/M is an X-projector of G/M for any minimal normal subgroup M of G.*

Proof. Let H be a generalized X-projector of G and let M be a minimal normal subgroup of G. In order to prove that HM/M is an X-projector of G/M, we use Theorem 2.3 and verify conditions (i) and (ii) from this theorem.

(i) HM/M is X-maximal in G/M. Indeed, H being a generalized X-projector of G and M being normal in G with $M \neq 1$, Definition 2.1 leads to the conclusion that HM/M is X-maximal in G/M.

(ii) HM/M is a generalized X-projector of G/M. Indeed, from the facts that H is a generalized X-projector of G and M is a normal subgroup of G, Theorem 2.4 leads to the conclusion that HM/M is a generalized X-projector of G/M.

Conversely, suppose that HM/M is an X-projector of G/M for any minimal normal subgroup M of G. In order to prove that H is a generalized X-projector of G, we use Definition 2.1. Let N be a normal subgroup of G such that $N \neq 1$. Then there exists a minimal normal subgroup M of G such that $M \subseteq N$. By our hypothesis, HM/M is an X-projector of G/M. From this and from $N/M \trianglelefteq G/M$, we obtain by applying Theorem 1.4.a) that $(HM/M \cdot N/M)/(N/M)$ is an X-projector of $(G/M)/(N/M)$. But

$$(HM/M \cdot N/M)/(N/M) = (HMN/M)/(N/M) = (HN/M)/(N/M) \simeq HN/N$$

and

$$(G/M)/(N/M) \simeq G/N,$$

and so HN/N is an X-projector of G/N, which leads by Theorem 1.5.a) to the conclusion that HN/N is X-maximal in G/N. This means, by Definition 2.1, that H is a generalized X-projector of G. □

Finally in this section, two remarks.

From Theorem 1.5.b) and Remark 2.2, we obtain:

Remark 2.6. *If X is a homomorph, G is a group and H is a subgroup of G, then the following implications hold:*

$$H \text{ is an } X\text{-covering subgroup of } G \Rightarrow H \text{ is an } X\text{-projector of } G \Longrightarrow$$

$$H \text{ is a generalized } X\text{-projector of } G.$$

This shows that if X is a homomorph, then the notion of generalized projector generalizes both the projectors and the covering subgroups.

From the Remarks 1.3.b) and 2.2, follows immediately:

Remark 2.7. *If X is a homomorph and G is a group, then:*

(i) $G \in X \Longleftrightarrow G$ is an X-projector of G;

(ii) $G \in X \Rightarrow G$ is a generalized X-projector of G.

3. The saturated closure of a π-homomorph

Let π be an arbitrary set of primes. From now on, all groups used in our considerations will be finite π-solvable groups.

Definition 3.1. *Let X be a π-homomorph. We call* **the saturated closure** *of X the smallest π-homomorph $\overline{X}$ of finite π-solvable groups such that the following two conditions hold:*

(*i*) $X \subseteq \overline{X}$ *;*

(*ii*) *any finite π-solvable group has $\overline{X}$-projectors.*

Remark 3.2. *a) Theorem 1.9 shows that Definition 3.1 is equivalent with that given in* [3].

b) If X is a π-homomorph and $\overline{X}$ is its saturated closure, then $\overline{X}$ is a π-homomorph and any finite π-solvable group has $\overline{X}$-projectors. It follows by Theorem 1.9 that the saturated closure $\overline{X}$ is a Schunck class. Since $\overline{X}$ is π-closed, we conclude that $\overline{X}$ is a π-Schunck class.

Notation 3.3. *Let X be a class of finite π-solvable groups. We denote by X^* the class of all finite π-solvable groups G such that G is a generalized X-projector of G.*

Let us give some properties of the class X^*, which will be used to prove the main results of the paper. Everywhere X will denote a class of finite π-solvable groups.

Theorem 3.4. *If X is a homomorph, then $X \subseteq X^*$.*

Proof. Let $G \in X$. By Remark 2.7.(*ii*), G is a generalized X-projector of G. It follows that $G \in X^*$. □

Theorem 3.5. *If X is a class of finite π-solvable groups, then X^* is a homomorph.*

Proof. Let $G \in X^*$ and let N be a normal subgroup of G. We show that $G/N \in X^*$. Indeed, from $G \in X^*$ we have that G is a finite π-solvable group and G is a generalized X-projector of G. G being a finite π-solvable group and N being normal in G, it follows by Theorem 1.8.a) that G/N is also a finite π-solvable group. Furthermore, from the facts that G is a generalized X-projector of G and N is a normal subgroup of G, Theorem 2.4 leads to the conclusion that G/N is a generalized X-projector of G/N. It follows that $G/N \in X^*$. □

The property of a class X of finite π-solvable groups we define below is connected with the generalized projectors introduced in Definition 2.1 and will be called therefore the *GP-property*.

Definition 3.6. *A class X of finite π-solvable groups is said to have the* **GP-property** *if X satisfies the following two conditions:*

(*i*) *every finite π-solvable group has generalized X-projectors;*

(ii) if G is a finite π-solvable group, then for any generalized X-projector H of G there exists a minimal normal subgroup M of G such that $M \subseteq H$.

Theorem 3.7. *Let X be a class of finite π-solvable groups with the GP-property and G a finite π-solvable group. The following two conditions are equivalent:*
(i) $G \in X^*$;
(ii) if H is a generalized X-projector of G, then $H = G$.

Proof. Let X be a class with the GP-property and G a finite π-solvable group.

$(i) \Rightarrow (ii)$: Let $G \in X^*$ and H be a generalized X-projector of G. From $G \in X^*$ follows that G is a generalized X-projector of G, which implies by Theorem 2.5 that G/M is an X-projector of G/M for any minimal normal subgroup M of G. By Theorem 1.5.a), we deduce that G/M is X-maximal in G/M, hence $G/M \in X$. On the other side, by applying Theorem 2.5 for the generalized X-projector H of G, we obtain that HM/M is an X-projector of G/M for any minimal normal subgroup M of G, hence HM/M is X-maximal in G/M. From this, since $G/M \in X$, we deduce that $HM/M = G/M$. It follows that $HM = G$ for any minimal normal subgroup M of G. But X is a class with the GP-property and so for the generalized X-projector H of G, there exists a minimal normal subgroup M_0 of G such that $M_0 \subseteq H$. Then $H = HM_0$. But, as we saw above, $HM_0 = G$. It follows that $H = G$.

$(ii) \Rightarrow (i)$: Let H be an arbitrary generalized X-projector of G. Then, by (ii), $H = G$. Hence G is its own generalized X-projector and so $G \in X^*$. □

Theorem 3.8. *If X is a π-homomorph with the GP-property, then X^* is a π-homomorph.*

Proof. Let X be a π-homomorph with the GP-property. By Theorem 3.5, X^* is a homomorph. It remains to prove that X^* is π-closed, i.e. that $G/O_{\pi'}(G) \in X^*$ implies $G \in X^*$. Let $G/O_{\pi'}(G) \in X^*$. We first notice that from $G/O_{\pi'}(G) \in X^*$ follows that $G/O_{\pi'}(G)$ is a finite π-solvable group. Now, $G/O_{\pi'}(G)$ and $O_{\pi'}(G)$ being π-solvable groups, we deduce by Theorem 1.8.b) that G is also a π-solvable group. In order to prove that $G \in X^*$, we use Theorem 3.7. Let H be a generalized X-projector of G. Since $O_{\pi'}(G) \trianglelefteq G$, Theorem 2.4 leads to the conclusion that $HO_{\pi'}(G)/O_{\pi'}(G)$ is a generalized X-projector of $G/O_{\pi'}(G)$. But the class X has the GP-property and $G/O_{\pi'}(G) \in X^*$. By Theorem 3.7, it follows that

$$HO_{\pi'}(G)/O_{\pi'}(G) = G/O_{\pi'}(G).$$

Hence

$$HO_{\pi'}(G) = G. \tag{3.1}$$

We consider two cases:

1° $O_{\pi'}(G) = 1$. In this case, (3.1) gives that $H = G$. But H being a generalized X-projector of G, it follows that G is a generalized X-projector of G. Hence $G \in X^*$.

2° $O_{\pi'}(G) \neq 1$. Then H being a generalized X-projector of G and $O_{\pi'}(G) \trianglelefteq G$, $O_{\pi'}(G) \neq 1$, Definition 2.1 leads to the conclusion that $HO_{\pi'}(G)/O_{\pi'}(G)$ is X-maximal in $G/O_{\pi'}(G)$, which means by applying (3.1) that $G/O_{\pi'}(G)$ is X-maximal in $G/O_{\pi'}(G)$. Hence $G/O_{\pi'}(G) \in X$. But the class X being π-closed, it follows that $G \in X$. By Theorem 3.4, the homomorph X has the property that $X \subseteq X^*$. So $G \in X^*$. □

Theorem 3.9. *If X is a π-homomorph with the GP-property, then any finite π-solvable group has X^*-projectors.*

Proof. Let X be a π-homomorph with the GP-property. Then, by Theorem 3.8, X^* is a π-homomorph. We apply Theorem 1.9 for the π-homomorph X^* and conclude that instead of proving that any finite π-solvable group has X^*-projectors we can prove the equivalent condition (ii) from Theorem 1.9, which becomes in our case: if G is a π-solvable group, $G \notin X^*$ and M is a minimal normal subgroup of G such that $G/M \in X^*$, then M has a complement in G. Let G be a π-solvable group, $G \notin X^*$ and M a minimal normal subgroup of G such that $G/M \in X^*$. We first observe that there exists a subgroup H of G such that H is a generalized X-projector of G and $H \neq G$. Indeed, if we suppose the contrary, then every generalized X-projector H of G is equal to G, which means by Theorem 3.7 that $G \in X^*$, a contradiction with the hypothesis $G \notin X^*$. We complete the proof of the present theorem by showing that H is a complement of M in G, i.e. $HM = G$ and $H \cap M = 1$. Indeed, since H is a generalized X-projector of G and M is normal in G, we conclude by Theorem 2.4 that HM/M is a generalized X-projector of G/M. This and $G/M \in X^*$ imply by Theorem 3.7 that $HM/M = G/M$. Hence $HM = G$. It remains to prove that $H \cap M = 1$. Since M is a minimal normal subgroup of the π-solvable group G, M is either a solvable π-group or a π'-group. Suppose that M is a π'-group. Then $M \leq O_{\pi'}(G)$ and so

$$G/O_{\pi'}(G) \simeq (G/M)/(O_{\pi'}(G)/M). \tag{3.2}$$

Since $G/M \in X^*$ and X^* is a homomorph, (3.2) leads to $G/O_{\pi'}(G) \in X^*$, which implies by the π-closure of X^* that $G \in X^*$, a contradiction with the hypothesis $G \notin X^*$. It follows that M is a solvable π-group. Then, by Theorem 1.6, M is abelian. Let us prove that $H \cap M$ is normal in G. We know that $H \leq G$ and $M \trianglelefteq G$ imply $H \cap M \trianglelefteq H$. Let now $g \in G = HM$ and $x \in H \cap M$. Then $g = hm$, with $h \in H$ and $m \in M$, and we have

$$g^{-1}xg = (hm)^{-1}x\ (hm) = (m^{-1}h^{-1})\ x\ (hm) = m^{-1}(h^{-1}x\ h)\ m. \tag{3.3}$$

From $H \cap M \trianglelefteq H$, we conclude that $h^{-1}x\ h \in H \cap M$. Furthermore, M being abelian, we can commute in (3.3) the elements $h^{-1}x\ h$ and m, both in M, and obtain

$$g^{-1}xg = m^{-1}(h^{-1}x\ h)\ m = m^{-1}m\ (h^{-1}x\ h) = h^{-1}xh \in H \cap M.$$

We proved that $H \cap M$ is normal in G. From this and from $H \cap M \subseteq M$, by using that M is a minimal normal subgroup of G, it follows that $H \cap M = 1$ or $H \cap M = M$. But $H \cap M = M$ leads to $M \subseteq H$, hence $G = HM = H$, a contradiction with $H \neq G$. It follows that $H \cap M = 1$, and the theorem is proved. □

Theorem 3.10. *If X is a π-homomorph with the GP-property, then X^* is a π-Schunck class.*

Proof. Since X is a π-homomorph with the GP-property, Theorem 3.8 shows that X^* is a π-homomorph and Theorem 3.9 shows that any finite π-solvable group has X^*-projectors. By applying Theorem 1.9, we conclude that X^* is a π-Schunck class. □

Theorem 3.11. *Let X be a π-homomorph with the GP-property. If Y is a π-homomorph satisfying the conditions*

(i) $X \subseteq Y$;

(ii) any finite π-solvable group has Y-projectors, then $X^* \subseteq Y$.

Proof. Let $G \in X^*$. Then G is a finite π-solvable group and so, by (ii), there exists an Y-projector H of G. We will prove that H is a generalized X-projector of G. For this, we use Theorem 2.5. and prove that HM/M is an X-projector of G/M for any minimal normal subgroup M of G. Let M be a minimal normal subgroup of G. From $G \in X^*$ follows that G is its own generalized X-projector, and by Theorem 2.5 we have that G/M is an X-projector of G/M, hence by Theorem 1.5.a) G/M is X-maximal in G/M, and so $G/M \in X$. But (i) claims that $X \subseteq Y$. It follows that $G/M \in Y$. Now, H being an Y-projector of G and M being normal in G, Definition 1.2.b) leads to the conclusion that HM/M is Y-maximal in G/M. This and $G/M \in Y$ imply $HM/M = G/M$, hence $HM = G$. But we saw that G/M is an X-projector of G/M, which together with $HM = G$ gives that HM/M is an X-projector of G/M, what we had to prove. It follows that H is a generalized X-projector of G. But $G \in X^*$ and the class X has the GP-property. So we can apply Theorem 3.7 and obtain that $H = G$. From the choice of H as an Y-projector of G, we deduce by Theorem 1.5.a) that H is Y-maximal in G, which implies that $H \in Y$. This and $H = G$ lead to $G \in Y$. The inclusion $X^* \subseteq Y$ is proved. □

Theorem 3.12. *If X is a π-homomorph with the GP-property and $\overline{X}$ is its saturated closure, then*

$$X^* \subseteq \overline{X}\ .$$

Proof. Let X be a π-homomorph with the GP-property and $\overline{X}$ its saturated closure. We can take in Theorem 3.11: $Y = \overline{X}$. Indeed, by Definition 3.1, the saturated closure $\overline{X}$ satisfies conditions (i) and (ii) claimed in Theorem 3.11. By applying Theorem 3.11, we conclude that $X^* \subseteq \overline{X}$. □

From Theorems 3.4 and 3.12 immediately follows:

Corollary 3.13. *If X is a π-homomorph with the GP-property and $\overline{X}$ is its saturated closure, then*

$$X \subseteq X^* \subseteq \overline{X}\ .$$

4. The main results

The main results of this paper, which we prove below, are the following: 1) a characterization theorem for the saturated closure of the π-homomorphs of finite π-solvable groups with the GP-property by means of the generalized projectors; 2) a characterization theorem for Schunck classes of finite π-solvable groups by means of the saturated closure of π-homomorphs of finite π-solvable groups with the GP-property.

Theorem 4.1. *If X is a π-homomorph with the GP-property and $\overline{X}$ is its saturated closure, then*

$$\overline{X} = X^*.$$

Proof. Let X be a π-homomorph with the GP-property and $\overline{X}$ its saturated closure. By applying Theorem 3.12, we obtain that $X^* \subseteq \overline{X}$. In order to prove that $\overline{X} \subseteq X^*$, we use the Definition 3.1 of the saturated closure of X. If we show that X^* verifies conditions (i) and (ii) given in Definition 3.1, then, $\overline{X}$ being the smallest π-homomorph which verifies (i) and (ii), we conclude that $\overline{X} \subseteq X^*$. It is easy to see that X^* verifies condition (i), namely $X \subseteq X^*$, because X is a homomorph and we apply Theorem 3.4. Furthermore, X^* verifies condition (ii), namely any finite π-solvable group has X^*-projectors, as Theorem 3.9 shows. □

Theorem 4.2. *Let X be a π-homomorph with the GP-property and $\overline{X}$ its saturated closure. The following two conditions are equivalent:*

(i) *X is a Schunck class;*

(ii) $X = \overline{X}$.

Proof. Let X be a π-homomorph with the GP-property and $\overline{X}$ its saturated closure.

$(i) \Rightarrow (ii)$: Let X be a Schunck class. We first prove that $X = X^*$. Indeed, X being a homomorph, Theorem 3.4 leads to $X \subseteq X^*$. Furthermore, by applying Theorem 1.9 for the π-homomorph X which is a Schunck class, we conclude that any finite π-solvable group has X-projectors. Let us take in Theorem 3.11 $Y = X$, which is a π-homomorph satisfying the two conditions claimed in this theorem, namely: $X \subseteq X$ and any finite π-solvable group has X-projectors. By applying Theorem 3.11, we obtain that $X^* \subseteq X$. From $X \subseteq X^*$ and $X^* \subseteq X$ follows that

$$X = X^*. \tag{4.1}$$

On the other side, we are in the hypotheses of Theorem 4.1 and so we conclude that

$$\overline{X} = X^*. \tag{4.2}$$

From (4.1) and (4.2) follows that $X = \overline{X}$.

$(ii) \Rightarrow (i)$: Let $X = \overline{X}$. By the Definition 3.1 of the saturated closure $\overline{X}$, any π-solvable group G has $\overline{X}$-projectors. But $X = \overline{X}$. Then any π-solvable group G has X-projectors. We can now apply Theorem 1.9 for the π-homomorph X, and it follows that X is a Schunck class. □

References

[1] Baer, R., *Classes of finite groups and their properties*, Illinois J. Math., **1**(1957), no. 2, 115-187.

[2] Brewster, B., *F-Projectors in finite π-solvable groups*, Arch. Math., **23**(1972), no. 2, 133-138.

[3] Covaci, R., *Saturated closure of homomorphs*, Mathematica, **35**(**58**) (1993), no. 2, 137-139.

[4] Covaci, R., *Projectors and covering subgroups*, Stud. Univ. Babeş-Bolyai Math., **XXVII**(1982), 33-36.

[5] Covaci, R., *A characterization of π-closed Schunck classes*, Stud. Univ. Babeş-Bolyai Math., **XLVIII**(2003), no. 3, 63-69.

[6] Čunihin, S.A., *O teoremah tipa Sylowa*, Dokl. Akad. Nauk SSSR, **66**(1949), no. 2, 165-168.

[7] Gaschütz, W., *Zur Theorie der endlichen auflösbaren Gruppen*, Math. Z., **80**(1963), no. 4, 300-305.

[8] Gaschütz, W., *Selected topics in the theory of soluble groups*, Australian National University, Canberra, January-February 1969.

[9] Schunck, H., *H-Untergruppen in endlichen auflösbaren Gruppen*, Math. Z., **97**(1967), no. 4, 326-330.

[10] Weidner, J., *A new characterization of the saturated closure of a homomorphism closed class of finite solvable groups*, Bull. London Math. Soc., **8**(1976), no. 1, 38-40.

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On the extension and torsion functors of local cohomology of weakly Laskerian and Matlis reflexive modules

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Abstract. Let R be a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of R and M, N two R-modules. The main purpose of this paper is to study the circumstances under which, for fixed integers $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$, the R-modules $\mathrm{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$ and $\mathrm{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$ are weakly Laskerian or Matlis reflexive. In this way, we also get to some results about the associated primes, coassociated primes and Bass numbers of $H^n_{\mathfrak{a}}(M)$.

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1. Introduction

Throughout this paper, we will generally assume that R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ be an ideal of R and M, N be two R-modules. We shall use $V(\mathfrak{a})$ to denote the set of all prime ideals containing $\mathfrak{a}$. Also, we shall use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers.

For a non-negative integer i, the i-th local cohomology module of M with respect to $\mathfrak{a}$ is defined as:

$$H^i_{\mathfrak{a}}(M) = \varinjlim_{n\in\mathbb{N}_0} \mathrm{Ext}^i_R(R/\mathfrak{a}^n, M).$$

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The reader can refer to [6], for the basic properties of local cohomology.

This paper studies the circumstances under which the R-modules $\mathrm{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$ and $\mathrm{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$ are weakly Laskerian or Matlis reflexive, for fixed integers j and n when M, N are certain R-modules. One motivation for our work comes from the concept of cofiniteness for local cohomology modules introduced by Hartshorne in [15]. The local cohomology module $H^n_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite if $\mathrm{Ext}^i_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))$ is finitely generated for all $i \in \mathbb{N}_0$. It is a question of Huneke in [16] that when the local cohomology module $H^n_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite. In this regard, there has been a great deal of work. For instance, we refer the reader to the papers of Huneke and Koh [17], Delfino [8], Delfino and Marley [9], Yoshida [26] and Chiriacescu [7]. A question here arises that for fixed integers j and n, if M, N are certain R-modules, when the R-modules $\mathrm{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$ and $\mathrm{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$ are finitely generated. There has also been a couple of work regarding to this question when M is a finitely generated R-module and $N = R/\mathfrak{b}$ for some ideal $\mathfrak{b}$ of R containing $\mathfrak{a}$ (cf. [11] and [18]). The goal of the present paper is to obtain similar results as above, but for a larger class of modules.

Let E be the minimal injective cogenerator of the category of R-modules and $D(M) = \mathrm{Hom}_R(M, E)$. Recall that, an R-module M is called Matlis reflexive if the canonical map $M \to D(D(M))$ is an isomorphism. Moreover, Divaani-Aazar and Mafi, in [12], introduced and studied another type of modules called weakly Laskerian. A module M is called weakly Laskerian if the set of associated primes of any quotient module of M is finite. Note that, the class of weakly Laskerian modules includes all finitely generated, Artinian, linearly compact and Matlis reflexive modules. Also, the class of Matlis reflexive modules over a complete local ring contains all finitly generated and Artinian modules. Therefore, for fixed integers j and n, it is desirable to ask that when the R-modules $\mathrm{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$ and $\mathrm{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$ are weakly Laskerian or Matlis reflexive which is a generalization of mentioned question "*in some sense*".

In the second section of this paper we list some facts about the weakly Laskerian modules which will be useful in later sections. In the third section, at first, we investigate the above mentioned question for the R-module $\mathrm{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$. In fact, we show that for fixed integers $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$, if N is a finitely generated R-module with $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$ and M is a weakly Laskerian R-module such that

(i) $\mathrm{Ext}^{j+t+1}_R(N, H^{n-t}_{\mathfrak{a}}(M))$ is weakly Laskerian for all $t = 1, \ldots, n$, and
(ii) $\mathrm{Ext}^{j-s-1}_R(N, H^{n+s}_{\mathfrak{a}}(M))$ is weakly Laskerian for all $s = 1, \ldots, \dim M - n$,

then $\mathrm{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$ is also weakly Laskerian. Next, we present some generalizations of [13, Theorem 3.1], [1, Theorem 1.2], [11, Theorem B], [5, Theorem 2.2] and [19, Theorem B(β)], to some extent.

In the forth section, we use an analogue of the above results for $\mathrm{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$. At last, in the final section, when $(R, \mathfrak{m})$ is a complete local ring with respect to $\mathfrak{m}$-adic topology, in a similar way, we study the

Matlis reflexivity of the R-modules $\operatorname{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$ and $\operatorname{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$ and get to some interesting results.

2. Preliminary results

First of all, we recall the definition of weakly Laskerian modules.

Definition 2.1. (i) (*See* [*12, Definition 2.1*].) *An R-module M is called weakly Laskerian if the set of associated primes of any quotient module of M is finite.*

(ii) (*See* [*13, Definition 2.4*].) *An R-module M is called $\mathfrak{a}$-weakly cofinite if* $\operatorname{Supp}_R(M) \subseteq V(\mathfrak{a})$ *and* $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ *is weakly Laskerian for all* $i \in \mathbb{N}_0$.

In the following lemma, we gather together some basic properties of weakly Laskerian modules.

Lemma 2.2. (See [12, Lemma 2.3] and [13, Remark 2.7].)

(i) *Let* $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ *be an exact sequence of R-modules and R-homomorphisms. Then M is weakly Laskerian if and only if L and N are weakly Laskerian. Hence, if* $L \longrightarrow M \longrightarrow N$ *is an exact sequence such that both end terms are weakly Laskerian R-modules, then M is also weakly Laskerian.*

(ii) *Let N be a finitely generated R-module and M be a weakly Laskerian R-module. Then* $\operatorname{Ext}^i_R(N, M)$ *and* $\operatorname{Tor}^R_i(N, M)$ *are weakly Laskerian for all* $i \in \mathbb{N}_0$.

(iii) *Suppose that M is a weakly Laskerian R-module with* $\operatorname{Supp}_R(M) \subseteq V(\mathfrak{a})$. *Then M is $\mathfrak{a}$-weakly cofinite.*

(iv) *If* $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ *is an exact sequence and two of modules in the sequence are $\mathfrak{a}$-weakly cofinite, then so is the third one.*

(v) *The set of associated primes of an $\mathfrak{a}$-weakly cofinite module is finite.*

Remark 2.3. (i) *In the light of* [12, Example 2.2], *the class of weakly Laskerian R-modules includes all finitely generated, Artinian and linearly compact R-modules.*

(ii) *Let E be the minimal injective cogenerator of the category of R-modules. For an R-module M, we let* $D(M) = \operatorname{Hom}_R(M, E)$. *If the canonical map* $M \to D(D(M))$ *is an isomorphism, then M is called Matlis reflexive. Now, by* [3, Theorem 12] *and* (i), *in conjunction with Lemma* 2.2(i), *every Matlis reflexive module is weakly Laskerian.*

Recall that a sequence $x_1, \ldots, x_n$ of elements of R is an $\mathfrak{a}$-filter regular sequence on M if $x_1, \ldots, x_n \in \mathfrak{a}$ and $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_R(M/(x_1, \ldots, x_{i-1})M) \setminus V(\mathfrak{a})$ and for all $i = 1, \ldots, n$. When $i = 1$, this is to be interpreted as

$$x_i \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(M) \setminus V(\mathfrak{a})} \mathfrak{p}.$$

The concept of an $\mathfrak{a}$-filter regular sequence on M is a generalization of the one of a filter regular sequence which has been studied in [21], [23] and has led to some interesting results. Note that both concepts coincide if $\mathfrak{a}$ is the maximal ideal in local ring. Also, note that $x_1, \dots, x_n$ is a weak M-sequence if and only if it is an R-filter regular sequence on M. The following proposition enables one to see quickly that, for a weakly Laskerian R-module M, there exist $\mathfrak{a}$-filter regular sequences on it of any length.

Proposition 2.4. *Let M be a weakly Laskerian R-module and n be a positive integer. Assume that $x_1, \dots, x_n$ is an $\mathfrak{a}$-filter regular sequence on M. Then there exists an element $x_{n+1} \in \mathfrak{a}$ such that $x_1, \dots, x_n, x_{n+1}$ is an $\mathfrak{a}$-filter regular sequence on M.*

Proof. In contrary, suppose that $x_1, \dots, x_n$ is an $\mathfrak{a}$-filter regular sequence on M such that

$$\mathfrak{a} \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M/(x_1,\dots,x_n)M) \setminus V(\mathfrak{a})} \mathfrak{p}.$$

Then, since M is weakly Laskerian R-module, $\mathrm{Ass}_R(M/(x_1, \dots, x_n)M)$ is a finite set. So, Prime Avoidance Theorem provides that $\mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p}$ in the set $\mathrm{Ass}_R(M/(x_1, \dots, x_n)M) \setminus V(\mathfrak{a})$ which is a required contradiction. □

3. Extension functors and local cohomology of weakly Laskerian modules

The first present author, in [18], by using filter regular sequences, established some results about finiteness properties of $\mathrm{Ext}^j_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ and $\mathrm{Tor}^R_j(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ for fixed integers $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$ when $\mathfrak{b}$ is an ideal of R containing $\mathfrak{a}$. Now, in view of Lemma 2.2(i)-(ii), in conjunction with Proposition 2.4, by employing the methods of proofs which are similar to those used in [18], one can establish the following theorem which is generalization of [18, Theorem 3.3], in some sense.

Theorem 3.1. *Fix $j \in \mathbb{N}_0$, $n \in \mathbb{N}$, a finitely generated R-module N with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$ *and a weakly Laskerian R-module M of dimension d. Assume that*

(i) $\mathrm{Ext}^{j+t+1}_R(N, H^{n-t}_{\mathfrak{a}}(M))$ *is weakly Laskerian for all $t = 1, \dots, n$, and*
(ii) $\mathrm{Ext}^{j-s-1}_R(N, H^{n+s}_{\mathfrak{a}}(M))$ *is weakly Laskerian for all $s = 1, \dots, d-n$.*

Then $\mathrm{Ext}^j_R(N, H^n_{\mathfrak{a}}(M))$ *is weakly Laskerian.*

Proof. In view of Proposition 2.4, let $x_1, \dots, x_{n+1}$ be an $\mathfrak{a}$-filter regular sequence on M. By means of [18], for each integer i with $1 \le i \le n$ there exists an exact sequence

$$0 \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{(x_1,\dots,x_i)}(M) \longrightarrow (H^i_{(x_1,\dots,x_i)}(M))_{x_{i+1}} \longrightarrow H^{i+1}_{(x_1,\dots,x_{i+1})}(M) \longrightarrow 0.$$

One can break the above exact sequence into two exact sequences

(1) $\quad 0 \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{(x_1,\dots,x_i)}(M) \longrightarrow L_i \longrightarrow 0 \quad$ and

(2) $\quad 0 \longrightarrow L_i \longrightarrow (H^i_{(x_1,\dots,x_i)}(M))_{x_{i+1}} \longrightarrow H^{i+1}_{(x_1,\dots,x_{i+1})}(M) \longrightarrow 0.$

On the other hand, it is a fact that, for each R-module N and each element x of R, multiplication by x provides an automorphism on N_x. In this regard, since x_i's belong to $\mathfrak{a}$ and N is a finitely generated R-module with $\operatorname{Supp}_R(N) \subseteq V(\mathfrak{a})$, applying the long exact sequences of $\operatorname{Ext}^j_R(N,-)$ on the exact sequence (2) induces the isomorphism

$$\operatorname{Ext}^j_R(N, L_i) \cong \operatorname{Ext}^{j-1}_R(N, H^{i+1}_{(x_1,\dots,x_{i+1})}(M)).$$

Now, several uses of the long exact sequences of $\operatorname{Ext}^j_R(N,-)$ on the exact sequence (1) and our assumptions of the theorem in conjunction with parts (i) and (ii) of Lemma 2.2 imply the result. □

Suppose that M is a finitely generated R-module and n is a positive integer. Marley and Vassilev, in [20, Proposition 2.5], showed that if $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all i with $i \neq n$, then $H^n_{\mathfrak{a}}(M)$ is also $\mathfrak{a}$-cofinite. By using the spectral sequence method, Divaani-Aazar and Mafi established the analogue result for weakly Laskerian modules (see [13, Theorem 3.1]). The following corollary which is a slight generalization of [13, Theorem 3.1], is an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let M be a weakly Laskerian R-module and N be a finitely generated R-module with $\operatorname{Supp}_R(N) \subseteq V(\mathfrak{a})$. Assume that n is a fixed integer such that the R-module $\operatorname{Ext}^s_R(N, H^i_{\mathfrak{a}}(M))$ is weakly Laskerian for all $s \in \mathbb{N}_0$ and all i with $i \neq n$. Then $\operatorname{Ext}^s_R(N, H^n_{\mathfrak{a}}(M))$ is also weakly Laskerian for all $s \in \mathbb{N}_0$.*

The following results are consequences of Theorem 3.1 for special choices of j and n.

Corollary 3.3. *Let M be a weakly Laskerian R-module and N be a finitely generated R-module with $\operatorname{Supp}_R(N) \subseteq V(\mathfrak{a})$. Assume that for a fixed integer n, the R-module $\operatorname{Ext}^s_R(N, H^i_{\mathfrak{a}}(M))$ is weakly Laskerian for all $s \in \mathbb{N}$ and all i with $i < n$. Then*

(i) $\operatorname{Hom}_R(N, H^n_{\mathfrak{a}}(M))$ *is weakly Laskerian and so*

$$\operatorname{Ass}_R(H^n_{\mathfrak{a}}(M)) \cap \operatorname{Supp}_R(N)$$

is finite, and

(ii) $\operatorname{Ext}^1_R(N, H^n_{\mathfrak{a}}(M))$ *is weakly Laskerian.*

Proof. (i) Applying Theorem 3.1 when $j = 0$ ensures that the R-module $\operatorname{Hom}_R(N, H^n_{\mathfrak{a}}(M))$ is weakly Laskerian. The second assertion now follows from the fact that

$$\operatorname{Ass}_R(\operatorname{Hom}_R(N, H^n_{\mathfrak{a}}(M))) = \operatorname{Ass}_R(H^n_{\mathfrak{a}}(M)) \cap \operatorname{Supp}_R(N).$$

(ii) Apply Theorem 3.1 when $j = 1$. □

Note that, by Remarks 2.3(i), the first part of Corollary 3.3 is a generalization of the main results of [5] and [19].

Corollary 3.4. (Compare [1, Theorem 1.2] and [11, Theorem B].) *Let M be a weakly Laskerian R-module and N be a finitely generated R-module with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$. *Let t be a non-negative integer such that*

$$\mathrm{Ext}^s_R(N, H^i_{\mathfrak{a}}(M))$$

is weakly Laskerian for all $s \in \mathbb{N}$ and all i with $i < t$. Then the following statements are equivalent:

(i) $\mathrm{Hom}_R(N, H^{t+1}_{\mathfrak{a}}(M))$ *is weakly Laskerian.*
(ii) $\mathrm{Ext}^2_R(N, H^t_{\mathfrak{a}}(M))$ *is weakly Laskerian.*

Proof. (i)⇒(ii) Apply Theorem 3.1 with $j = 2$ and $n = t$.
(ii)⇒(i) Apply Theorem 3.1 with $j = 0$ and $n = t + 1$. □

By using Theorem 3.1, in conjunction with [6, Corollary 3.3.3], we have the following corollary.

Corollary 3.5. *Let M be a weakly Laskerian R-module, N be a finitely generated R-module and $x, y \in R$ such that $(x, y) \subseteq \sqrt{(0 :_R N)}$. Then, for a fixed integer j, the following statements are equivalent:*

(i) $\mathrm{Ext}^j_R(N, H^2_{(x,y)}(M))$ *is weakly Laskerian.*
(ii) $\mathrm{Ext}^{j+2}_R(N, H^1_{(x,y)}(M))$ *is weakly Laskerian.*

4. Torsion functors and local cohomology of weakly Laskerian modules

In the light of Lemma 2.2(i)-(ii), in conjunction with Proposition 2.4, the methods of proofs used in [18] may be adapted. So, one can establish the following theorem which is a generalization of Theorem 4.1 in [18], in some sense.

Theorem 4.1. *Fix $j \in \mathbb{N}_0$, $n \in \mathbb{N}$, a finitely generated R-module N with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$, *and a weakly Laskerian R-module M of dimension d. Assume that*

(i) $\mathrm{Tor}^R_{j-t-1}(N, H^{n-t}_{\mathfrak{a}}(M))$ *is weakly Laskerian for all $t = 1, \ldots, n$, and*
(ii) $\mathrm{Tor}^R_{j+s+1}(N, H^{n+s}_{\mathfrak{a}}(M))$ *is weakly Laskerian for all $s = 1, \ldots, d - n$.*

Then $\mathrm{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$ *is weakly Laskerian.*

Proof. The proof is similar to that used in the proof of Theorem 3.1 by replacing the functor $\mathrm{Tor}^R_j(N, -)$ in stead of the functor $\mathrm{Ext}^j_R(N, -)$. □

Now, we recall the definition of coassociated prime ideals which is needed in the sequel.

Definition 4.2. (*See* [*25*].) *Let* $(R, \mathfrak{m})$ *be a local ring and* K *be an* R*-module. A prime ideal* $\mathfrak{p}$ *of* R *is said to be a coassociated prime of* K *if* $\mathfrak{p}$ *is an associated prime of* $D(K)$*. We denote the set of coassociated primes of* K *by* $\operatorname{Coass}_R(K)$ *(or simply* $\operatorname{Coass}(K)$*, if there is no ambiguity about the underlying ring).*

Note that $\operatorname{Coass}(K) = \emptyset$ if and only if $K = 0$. Also, for a finitely generated R-module K and arbitrary R-module L, in the light of [24, Theorem 1.22] or [9, Remark p. 50], we have that $\operatorname{Coass}(K \otimes_R L) = \operatorname{Supp}_R(K) \cap \operatorname{Coass}_R(L)$. Now, we present a dual of Corollary 3.3, in some sense.

Corollary 4.3. *Let* n *be a non-negative integer. Let* $(R, \mathfrak{m})$ *be a local ring,* M *a weakly Laskerian* R*-module and* N *be a finitely generated* R*-module with* $\operatorname{Supp}_R(N) \subseteq V(\mathfrak{a})$*. Suppose that* $\operatorname{Tor}_j^R(N, H_{\mathfrak{a}}^i(M))$ *is weakly Laskerian for all* $j \in \mathbb{N}_0$ *and all* i *with* $i > n$*. Then*

(i) $N \otimes_R H_{\mathfrak{a}}^n(M)$ *is weakly Laskerian and so the set*

$$\operatorname{Supp}_R(N) \cap \operatorname{Coass}_R(H_{\mathfrak{a}}^n(M))$$

is finite, and

(ii) $\operatorname{Tor}_1^R(N, H_{\mathfrak{a}}^n(M))$ *is weakly Laskerian.*

Recall that for an R-module K, the cohomological dimension of K with respect to $\mathfrak{a}$ is defined as

$$\operatorname{cd}(\mathfrak{a}, K) = max\{i \in \mathbb{N}_0 \mid H_{\mathfrak{a}}^i(K) \neq 0\}.$$

Now, the following corollary is an immediate consequence of Theorem 4.1.

Corollary 4.4. *Let* M *be a weakly Laskerian* R*-module and* N *be a finitely generated* R*-module with* $\operatorname{Supp}_R(N) \subseteq V(\mathfrak{a})$*. Then for a fixed integer* n*,*

(i) *if* $\operatorname{Tor}_s^R(N, H_{\mathfrak{a}}^i(M))$ *is weakly Laskerian for all* i *with* $i \neq n$*, then* $\operatorname{Tor}_s^R(N, H_{\mathfrak{a}}^i(M))$ *is weakly Laskerian for all integers* i *and* s*.*

(ii) *if* $\operatorname{Tor}_s^R(N, H_{\mathfrak{a}}^i(M))$ *is weakly Laskerian for all* i *with* $i < \operatorname{cd}(\mathfrak{a}, M)$*, then* $\operatorname{Tor}_s^R(N, H_{\mathfrak{a}}^i(M))$ *is weakly Laskerian for all integers* i *and* s*.*

Applying Theorem 4.1 for special integers j and n yields the following corollary.

Corollary 4.5. *Let* n *be a positive integer,* M *a weakly Laskerian* R*-module and* N *a finitely generated* R*-module with* $\operatorname{Supp}_R(N) \subseteq V(\mathfrak{a})$*. Let the* R*-module* $\operatorname{Tor}_s^R(N, H_{\mathfrak{a}}^i(M))$ *be weakly Laskerian for all* i *with* $i > n$ *and all* $s \in \mathbb{N}_0$*. Then the following statements are equivalent:*

(i) $N \otimes_R H_{\mathfrak{a}}^{n-1}(M)$ *is weakly Laskerian.*

(ii) $\operatorname{Tor}_2^R(N, H_{\mathfrak{a}}^n(M))$ *is weakly Laskerian.*

The following corollary is an immediate consequence of Theorem 4.1 which is a dual of Corollary 3.5, in some sense.

Corollary 4.6. *Let* M *be a weakly Laskerian* R*-module,* N *a finitely generated* R*-module and* $x, y \in R$ *such that* $(x, y) \subseteq \sqrt{(0 :_R N)}$*. Then, for a fixed integer* j*, the following statements are equivalent:*

(i) $\mathrm{Tor}_j^R(N, H^2_{(x,y)}(M))$ *is weakly Laskerian.*
(ii) $\mathrm{Tor}_{j-2}^R(N, H^1_{(x,y)}(M))$ *is weakly Laskerian.*

5. Local cohomology of Matlis reflexive modules

Throughout this section, $(R, \mathfrak{m}, k)$ will denote a local complete ring with respect to $\mathfrak{m}$-adic topology. For the reminder of this paper, we focus our attention to Matlis reflexive modules. For basic theory concerning Matlis reflexive modules, the reader is referred to [22, §3.2] and [6, §10].

Remark 5.1. (i) *In view of Matlis duality theorem, the class of Matlis reflexive modules over a complete local ring includes all finitely generated and Artinian modules.*

(ii) *By* [14, Proposition 1.3] *or* [22, Theorem 3.4.13], *M is Matlis reflexive if and only if there is an exact sequence*

$$0 \longrightarrow S \longrightarrow M \longrightarrow A \longrightarrow 0$$

with S finitely generated and A Artinian.

(iii) *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of R-modules and R-homomorphisms. Then B is Matlis reflexive if and only if A and C are Matlis reflexive. This follows by mapping the exact sequence into its double dual and applying the snake lemma.*

In view of Remarks 5.1(iii) and Theorem 3 in [2], one can also gain the following results.

Theorem 5.2. (Compare [18, Theorem 3.3].) *Fix $j \in \mathbb{N}_0$, $n \in \mathbb{N}$, a Matlis reflexive R-module N with $\mathfrak{a} \subseteq \sqrt{(0 :_R N)}$ and a finitely generated R-module M of dimension d. Assume that*

(i) $\mathrm{Ext}_R^{j+t+1}(N, H^{n-t}_{\mathfrak{a}}(M))$ *is Matlis reflexive for all $t = 1, \dots, n$, and*
(ii) $\mathrm{Ext}_R^{j-s-1}(N, H^{n+s}_{\mathfrak{a}}(M))$ *is Matlis reflexive for all $s = 1, \dots, d-n$.*

Then $\mathrm{Ext}_R^j(N, H^n_{\mathfrak{a}}(M))$ is Matlis reflexive.

Theorem 5.3. (Compare [18, Theorem 4.1].) *Fix $j \in \mathbb{N}_0$, $n \in \mathbb{N}$, a Matlis reflexive R-module N with $\mathfrak{a} \subseteq \sqrt{(0 :_R N)}$ and a finitely generated R-module M of dimension d. Assume that*

(i) $\mathrm{Tor}^R_{j-t-1}(N, H^{n-t}_{\mathfrak{a}}(M))$ *is Matlis reflexive for all $t = 1, \dots, n$, and*
(ii) $\mathrm{Tor}^R_{j+s+1}(N, H^{n+s}_{\mathfrak{a}}(M))$ *is Matlis reflexive for all $s = 1, \dots, d-n$.*

Then $\mathrm{Tor}^R_j(N, H^n_{\mathfrak{a}}(M))$ is Matlis reflexive.

Now, we are ready to present the main results of this section.

Theorem 5.4. *Fix $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Let M and N be two Matlis reflexive R-modules with $\mathfrak{a} \subseteq \sqrt{(0 :_R N)}$ such that*

(i) $\mathrm{Ext}_R^{j+t+1}(N, H^{n-t}_{\mathfrak{a}}(M))$ *is Matlis reflexive for all $t = 1, \dots, n$, and*
(ii) $\mathrm{Ext}_R^{j-s-1}(N, H^{n+s}_{\mathfrak{a}}(M))$ *is Matlis reflexive for all $s = 1, \dots, \dim M - n$.*

Then $\mathrm{Ext}_R^j(N, H_{\mathfrak{a}}^n(M))$ *is Matlis reflexive.*

Proof. Since M is Matlis reflexive, in the light of Remarks 5.1(i), there exists an exact sequence

$$0 \longrightarrow S \longrightarrow M \longrightarrow A \longrightarrow 0,$$

with S finitely generated and A Artinian. So, by applying the local cohomology functor $H_{\mathfrak{a}}^0(-)$, one can deduce the exact sequence

$$0 \longrightarrow H_{\mathfrak{a}}^0(S) \longrightarrow H_{\mathfrak{a}}^0(M) \longrightarrow A \xrightarrow{f} H_{\mathfrak{a}}^1(S) \longrightarrow H_{\mathfrak{a}}^1(M) \longrightarrow 0 \tag{1}$$

and the isomorphism

$$H_{\mathfrak{a}}^i(M) \cong H_{\mathfrak{a}}^i(S) \tag{2}$$

for all $i \geqslant 2$. Hence, we have the exact sequence

$$\mathrm{Ext}_R^j(N, H_{\mathfrak{a}}^1(S)) \longrightarrow \mathrm{Ext}_R^j(N, H_{\mathfrak{a}}^1(M)) \longrightarrow \mathrm{Ext}_R^{j+1}(N, \mathrm{Im} f). \tag{3}$$

Since A is Artinian, $\mathrm{Im}f$ is also Artinian. Therefore, by [2, Theorem 3], $\mathrm{Ext}_R^{j+1}(N, \mathrm{Im}f)$ is Matlis reflexive. On the other hand, in view of the isomorphism (2) and [4, Lemma 1], Theorem 5.2 ensures that the R-module $\mathrm{Ext}_R^j(N, H_{\mathfrak{a}}^1(S))$ is Matlis reflexive. So, by Remarks 5.1(iii), the exact sequence (3) proves the theorem when $n = 1$. It remains to prove the claim when $n \geqslant 2$. By means of the isomorphism (2), we need to establish the claim for finitely generated R-module S. Now, in the light of (2), [4, Lemma 1] and Theorem 5.2, we only need to prove that $\mathrm{Ext}_R^{j+n}(N, H_{\mathfrak{a}}^1(S))$ is Matlis reflexive. To do this, we use the exact sequence (1) to get the following exact sequence

$$\mathrm{Ext}_R^{j+n}(N, \mathrm{Im}f) \longrightarrow \mathrm{Ext}_R^{j+n}(N, H_{\mathfrak{a}}^1(S)) \longrightarrow \mathrm{Ext}_R^{j+n}(N, H_{\mathfrak{a}}^1(M)).$$

Note that $\mathrm{Im}f$ is Artinian, so the R-module $\mathrm{Ext}_R^{j+n}(N, \mathrm{Im}f)$ is Matlis reflexive. Now, since both end terms are Matlis reflexive, the R-module $\mathrm{Ext}_R^{j+n}(N, H_{\mathfrak{a}}^1(S))$ is also Matlis reflexive, as desired. So, the proof is complete. □

By using Theorem 5.3 together with straightforward modifications to the arguments in the proof of Theorem 5.4, we can earn the same result for the R-module $\mathrm{Tor}_j^R(N, H_{\mathfrak{a}}^n(M))$ as follows.

Theorem 5.5. *Fix* $j \in \mathbb{N}_0$ *and* $n \in \mathbb{N}$. *Let* M *and* N *be two Matlis reflexive* R*-modules with* $\mathfrak{a} \subseteq \sqrt{(0 :_R N)}$ *such that the following conditions hold:*

(i) $\mathrm{Tor}_{j-t-1}^R(N, H_{\mathfrak{a}}^{n-t}(M))$ *is Matlis reflexive for all* $t = 1, \ldots, n$, *and*
(ii) $\mathrm{Tor}_{j+s+1}^R(N, H_{\mathfrak{a}}^{n+s}(M))$ *is Matlis reflexive for all* $s = 1, \ldots, \dim M - n$.

Then $\mathrm{Tor}_j^R(N, H_{\mathfrak{a}}^n(M))$ *is Matlis reflexive.*

Corollary 5.6. *Fix* $j \in \mathbb{N}_0$ *and* $n \in \mathbb{N}$. *Let* M *be a Matlis reflexive* R*-module of dimension* d *and* N *be a finitely generated* R*-module with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$ *such that*

(i) $\mathrm{Ext}_R^{j+t+1}(N, H_{\mathfrak{a}}^{n-t}(M))$ *is Matlis reflexive for all* $t = 1, \ldots, n$, *and*
(ii) $\mathrm{Ext}_R^{j-s-1}(N, H_{\mathfrak{a}}^{n+s}(M))$ *is Matlis reflexive for all* $s = 1, \ldots, d - n$.

Then $\operatorname{Tor}_j^R(N, D(H_{\mathfrak{a}}^n(M)))$ *is Matlis reflexive.*

Proof. By [6, Theorem 10.2.5], E is Artinian and so is Matlis reflexive. Also, we have the following isomorphism

$$\operatorname{Tor}_j^R(N, D(H_{\mathfrak{a}}^n(M))) \cong \operatorname{Hom}_R(\operatorname{Ext}_R^j(N, H_{\mathfrak{a}}^n(M)), E).$$

This fact together with Theorem 5.4 and [2, Theorem 3] proves the claim. □

We end the paper by the following result about the Bass numbers of local cohomology module $H_{\mathfrak{a}}^n(M)$.

Corollary 5.7. *Let* M *be a Matlis reflexive* R*-module of dimension* d *and* $\mathfrak{p}$ *be a prime ideal of* R*. Assume that*

(i) $\operatorname{Ext}_R^{j+t+1}(R/\mathfrak{p}, H_{\mathfrak{a}}^{n-t}(M))$ *is Matlis reflexive for all* $t = 1, \dots, n$*, and*
(ii) $\operatorname{Ext}_R^{j-s-1}(R/\mathfrak{p}, H_{\mathfrak{a}}^{n+s}(M))$ *is Matlis reflexive for all* $s = 1, \dots, d-n$*.*

Then the j*-th Bass number of* $H_{\mathfrak{a}}^n(M)$ *with respect to* $\mathfrak{p}$ *is finite.*

Proof. If $\mathfrak{p} \not\supseteq \mathfrak{a}$, then $\mathfrak{p} \notin \operatorname{Supp}_R(H_{\mathfrak{a}}^n(M))$. So, there is nothing to prove in this case. In other wise, Theorem 5.4 tells us that $\operatorname{Ext}_R^j(R/\mathfrak{p}, H_{\mathfrak{a}}^n(M))$ is Matlis reflexive. Now, in the case $\mathfrak{p} = \mathfrak{m}$, since $\operatorname{Ext}_R^j(R/\mathfrak{p}, H_{\mathfrak{a}}^n(M))$ is also a k-vector space, it must be finitely generated. Also, If $\mathfrak{p}$ is any non-maximal prime, it follows from Remarks 5.1(ii) that $(\operatorname{Ext}_R^j(R/\mathfrak{p}, H_{\mathfrak{a}}^n(M)))_{\mathfrak{p}}$ is finitely generated over $R_{\mathfrak{p}}$. Thus, in either case, the claim is true. □

References

[1] Asadollahi, J., Schenzel, P., *Some results on associated primes of local cohomology modules*, Japan. J. Math. (N.S.) **29**(2003), no. 2, 285-296.

[2] Belshoff, R., *Some change of ring theorems for Matlis reflexive modules*, Comm. in Algebra, **22**(1994), 3545-3552.

[3] Belshoff, R., Enochs, E., Garcia Rozas, J. R., *Generalized Matlis duality*, Proc. Amer. Math. Soc., **128**(2000), no. 5, 1307-1312.

[4] Belshoff, R., Slattery, S. P., Wickham, C., *The local cohomology modules of Matlis reflexive modules are almost cofinite*, Proc. Amer. Math. Soc., **124**(1996), no. 9, 2649-2654.

[5] Brodmann, M., Lashgari, A., *A finiteness result for associated primes of local cohomology modules*, Proc. Amer. Math. Soc., **128**(2000), 2851-2853.

[6] Brodmann, M., Sharp, R. Y., *Local cohomology - an algebraic introduction with geometric applications*, Cambridge studies in advanced mathematics no. 60, Cambridge University Press, 1998.

[7] Chiriacescu, G., *Cofiniteness of local cohomology modules over regular local rings*, Bull. London Math. Soc., **32**(2000), 1-7.

[8] Delfino, D., *On the cofiniteness of local cohomology modules*, Math. Proc. Camb. Phil. Soc., **115**(1994), 79-84.

[9] Delfino, D., Marley, T., *Cofinite modules and local cohomology*, J. Pure Appl. Algebra, **121**(1997), no. 1, 45-52.

[10] Dibaei, M. T., Yassemi, S., *Associated primes and cofiniteness of local cohomology modules*, Manuscripta Math., **117**(2005), no. 2, 199-205.

[11] Dibaei, M. T., Yassemi, S., *Finiteness of extension functors of local cohomology modules*, Comm. in Algebra, **34**(2006), 3097-3101.

[12] Divaani-Aazar, K.; Mafi, A., *Associated primes of local cohomology modules*, Proc. Amer. Math. Soc., **133**(2005), no. 3, 655-660.

[13] Divaani-Aazar, K., Mafi, A., *Associated primes of local cohomology modules of weakly Laskerian modules*, Comm. in Algebra, **34**(2006), no. 2, 681-690.

[14] Enochs, E., *Flat covers and flat cotorsion modules*, Proc. Amer. Math. Soc., **92**(1984), no. 2, 179-184.

[15] Hartshorne, R., *Affine duality and cofiniteness*, Invent. Math., **9**(1970), 145-164.

[16] Huneke, C., *Problems on local cohomology. Free resolutions in commutative algebra and algebraic geometry*, Res. Notes Math., 2, Jones and Bartlett, Boston, MA, 1992, 93-108.

[17] Huneke, C., Koh, J., *Cofiniteness and vanishing of local cohomology modules*, Math. Proc. Camb. Phil. Soc., **110**(1991), 421-429.

[18] Khashyarmanesh, K., *On the finiteness properties of extension and torsion functors of local cohomology modules*, Proc. Amer. Math. Soc., **135**(2007), no. 5, 1319-1327.

[19] Khashyarmanesh, K., Salarian, Sh., *On the associated primes of local cohomology modules*, Comm. in Algebra, **27(12)**(1999), 6191-6198.

[20] Marley, T., Vassilev, J. C., *Cofiniteness and associated primes of local cohomology modules*, J. Algebra, **256**(2002), no. 1, 180-193.

[21] Schenzel, P., Trung, N. V., Cuong, N. T., *Verallgemeinerte Cohen-Macaulay-Moduln*, Math. Nachr., **85**(1978), 57-73.

[22] Strooker, J. R., *Homological Questions in Local Algebra*, London Mathematical Society Lecture Notes 145, Cambridge University Press, Cambridge, 1990.

[23] Stückrad, J., Vogel, W., *Buchsbaum rings and applications*, VEB Deutscher Verlag der Wissenschaften, Berlin 1986.

[24] Vosconcelos, W. V., *Divisor theory in module categories*, North-Holland Mathematics Studies, No. 14, North-Holland, Amsterdam, 1974.

[25] Yassemi, S., *Coassociated primes*, Comm. in Algebra, **23**(1995), no. 4, 1473-1498.

[26] Yoshida, K. I., *Cofiniteness of local cohomology modules for ideals of dimension one*, Nagoya Math. J., **147**(1997), 179-191.

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Behavior of a rational recursive sequences

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Abstract. We obtain in this paper the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-7}}{\pm 1 \pm x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad n = 0, 1, ...,$$

where the initial conditions are arbitrary nonzero real numbers.

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1. Introduction

In this paper we obtain the solutions of the following recursive sequences

$$x_{n+1} = \frac{x_{n-7}}{\pm 1 \pm x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad n = 0, 1, ..., \tag{1.1}$$

where the initial conditions are arbitrary nonzero real numbers.

Recently there has been a great interest in studying the qualitative properties of rational difference equations. For the systematical studies of rational and nonrational difference equations, one can refer to the papers [1-41] and references therein.

The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations. However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Aloqeili [5] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar [7]-[9] investigated the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + ax_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + ax_n x_{n-1}}, \quad x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}.$$

Elabbasy et al. [11]-[12] investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequences

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}, \quad x_{n+1} = \frac{dx_{n-l}x_{n-k}}{cx_{n-s} - b} + a.$$

Elabbasy et al. [15] gave the solution of the following difference equations

$$x_{n+1} = \frac{x_{n-7}}{\pm 1 \pm x_{n-3}x_{n-7}}.$$

Karatas et al. [26] get the form of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}.$$

Simsek et al. [33] obtained the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

Here, we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [29].

Definition 1.1. *(Equilibrium Point)*

A point $\overline{x} \in I$ is called an equilibrium point of Eq. (1.2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq. (1.2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.2. *(Stability)*

(i) The equilibrium point $\overline{x}$ of Eq. (1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq. (1.2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq. (1.2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ *of Eq. (1.2) is global attractor if for all* $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, *we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ *of Eq. (1.2) is globally asymptotically stable if* $\overline{x}$ *is locally stable, and* $\overline{x}$ *is also a global attractor of Eq. (1.2).*

(v) The equilibrium point $\overline{x}$ *of Eq. (1.2) is unstable if* $\overline{x}$ *is not locally stable.*

The linearized equation of Eq. (1.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}.$$

Theorem 1.3. [28] *Assume that* $p, q \in \mathbb{R}$ *and* $k \in \{0, 1, 2, ...\}$. *Then*

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ...$$

Remark 1.4. *Theorem 1.3 can be easily extended to a general linear equations of the form*

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \;\; n = 0, 1, ..., \tag{1.3}$$

where $p_1, p_2, ..., p_k \in \mathbb{R}$ *and* $k \in \{1, 2, ...\}$. *Then Eq. (1.3) is asymptotically stable provided that*

$$\sum_{i=1}^{k} |p_i| < 1.$$

Definition 1.5. *(Periodicity)*

A sequence $\{x_n\}_{n=-k}^{\infty}$ *is said to be periodic with period* p *if* $x_{n+p} = x_n$ *for all* $n \geq -k$.

2. On the Difference Equation $x_{n+1} = \frac{x_{n-7}}{1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}}$

In this section we give a specific form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-7}}{1 + x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad n = 0, 1, ..., \tag{2.1}$$

where the initial conditions are arbitrary nonzero positive real numbers.

Theorem 2.1. *Let $\{x_n\}_{n=-7}^{\infty}$ be a solution of Eq. (2.1). Then for $n = 0, 1, ...$*

$$x_{8n-7} = \frac{h\prod_{i=0}^{n-1}(1+4ibdfh)}{\prod_{i=0}^{n-1}(1+(4i+1)bdfh)}, \quad x_{8n-3} = \frac{d\prod_{i=0}^{n-1}(1+(4i+2)bdfh)}{\prod_{i=0}^{n-1}(1+(4i+3)bdfh)},$$

$$x_{8n-6} = \frac{g\prod_{i=0}^{n-1}(1+4iaceg)}{\prod_{i=0}^{n-1}(1+(4i+1)aceg)}, \quad x_{8n-2} = \frac{c\prod_{i=0}^{n-1}(1+(4i+2)aceg)}{\prod_{i=0}^{n-1}(1+(4i+3)aceg)},$$

$$x_{8n-5} = \frac{f\prod_{i=0}^{n-1}(1+(4i+1)bdfh)}{\prod_{i=0}^{n-1}(1+(4i+2)bdfh)}, \quad x_{8n-1} = \frac{b\prod_{i=0}^{n-1}(1+(4i+3)bdfh)}{\prod_{i=0}^{n-1}(1+(4i+4)bdfh)},$$

$$x_{8n-4} = \frac{e\prod_{i=0}^{n-1}(1+(4i+1)aceg)}{\prod_{i=0}^{n-1}(1+(4i+2)aceg)}, \quad x_{8n} = \frac{a\prod_{i=0}^{n-1}(1+(4i+3)aceg)}{\prod_{i=0}^{n-1}(1+(4i+4)aceg)},$$

where $x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_{-0} = a,\ \prod_{i=0}^{-1} A_i = 1$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$x_{8n-15} = \frac{h\prod_{i=0}^{n-2}(1+4ibdfh)}{\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}, \quad x_{8n-11} = \frac{d\prod_{i=0}^{n-2}(1+(4i+2)bdfh)}{\prod_{i=0}^{n-2}(1+(4i+3)bdfh)},$$

$$x_{8n-14} = \frac{g\prod_{i=0}^{n-2}(1+4iaceg)}{\prod_{i=0}^{n-2}(1+(4i+1)aceg)}, \quad x_{8n-10} = \frac{c\prod_{i=0}^{n-2}(1+(4i+2)aceg)}{\prod_{i=0}^{n-2}(1+(4i+3)aceg)},$$

$$x_{8n-13} = \frac{f\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}{\prod_{i=0}^{n-2}(1+(4i+2)bdfh)}, \quad x_{8n-9} = \frac{b\prod_{i=0}^{n-2}(1+(4i+3)bdfh)}{\prod_{i=0}^{n-2}(1+(4i+4)bdfh)},$$

$$x_{8n-12} = \frac{e\prod_{i=0}^{n-2}(1+(4i+1)aceg)}{\prod_{i=0}^{n-2}(1+(4i+2)aceg)}, \quad x_{8n-8} = \frac{a\prod_{i=0}^{n-2}(1+(4i+3)aceg)}{\prod_{i=0}^{n-2}(1+(4i+4)aceg)}.$$

Now, it follows from Eq. (2.1) that

$$x_{8n-7} = \frac{x_{8n-15}}{1 + x_{8n-9}x_{8n-11}x_{8n-13}x_{8n-15}}$$

$$= \frac{\frac{h\prod_{i=0}^{n-2}(1+4ibdfh)}{\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}}{1+\frac{b\prod_{i=0}^{n-2}(1+(4i+3)bdfh)}{\prod_{i=0}^{n-2}(1+(4i+4)bdfh)}\frac{d\prod_{i=0}^{n-2}(1+(4i+2)bdfh)}{\prod_{i=0}^{n-2}(1+(4i+3)bdfh)}\frac{f\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}{\prod_{i=0}^{n-2}(1+(4i+2)bdfh)}\frac{h\prod_{i=0}^{n-2}(1+4ibdfh)}{\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}}$$

$$= \frac{h\prod_{i=0}^{n-2}(1+4ibdfh)}{\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}\left(\frac{1}{1+\frac{bdfh}{\prod_{i=0}^{n-2}(1+(4i+4)bdfh)}\prod_{i=0}^{n-2}(1+4ibdfh)}\right)$$

$$= \frac{h\prod_{i=0}^{n-2}(1+4ibdfh)}{\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}\left(\frac{1}{1+\frac{bdfh}{(1+(4n-4)bdfh)}}\left\{\frac{(1+(4n-4)bdfh)}{(1+(4n-4)bdfh)}\right\}\right)$$

$$= \frac{h\prod_{i=0}^{n-2}(1+4ibdfh)}{\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}\left(\frac{1+(4n-4)bdfh}{1+(4n-4)bdfh+bdfh}\right)$$

$$= \frac{h\prod_{i=0}^{n-2}(1+4ibdfh)}{\prod_{i=0}^{n-2}(1+(4i+1)bdfh)}\left(\frac{1+(4n-4)bdfh}{1+(4n-3)bdfh}\right).$$

Hence, we have

$$x_{8n-7} = \frac{h\prod_{i=0}^{n-1}(1+4ibdfh)}{\prod_{i=0}^{n-1}(1+(4i+1)bdfh)}.$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 2.2. *Eq. (2.1) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.*

Proof. For the equilibrium points of Eq. (2.1), we can write

$$\overline{x} = \frac{\overline{x}}{1+\overline{x}^4}.$$

Then

$$\overline{x} + \overline{x}^5 = \overline{x},$$

or

$$\overline{x}^5 = 0.$$

Thus the equilibrium point of Eq. (2.1) is $\overline{x} = 0$. Let $f : (0, \infty)^4 \longrightarrow (0, \infty)$ be a function defined by

$$f(u, v, w, t) = \frac{u}{1 + uvwt}.$$

Therefore it follows that

$$f_u(u, v, w, t) = \frac{1}{(1 + uvwt)^2}, \quad f_v(u, v, w, t) = \frac{-u^2wt}{(1 + uvwt)^2},$$
$$f_w(u, v, w, t) = \frac{-u^2vt}{(1 + uvwt)^2}, \quad f_t(u, v, w, t) = \frac{-u^2vw}{(1 + uvwt)^2},$$

we see that

$$f_u(\overline{x}, \overline{x}, \overline{x}, \overline{x}) = 1, \quad f_v(\overline{x}, \overline{x}, \overline{x}, \overline{x}x) = 0, \quad f_w(\overline{x}, \overline{x}, \overline{x}, \overline{x}) = 0, \quad f_t(\overline{x}, \overline{x}, \overline{x}, \overline{x}) = 0.$$

The proof follows by using Theorem 1.3.

Theorem 2.3. *Every positive solution of Eq. (2.1) is bounded and* $\lim_{n \to \infty} x_n = 0$.

Proof. It follows from Eq. (2.1) that

$$x_{n+1} = \frac{x_{n-7}}{1 + x_{n-1}x_{n-3}x_{n-5}x_{n-7}} \leq x_{n-7}.$$

Then the subsequences $\{x_{8n-7}\}_{n=0}^{\infty}$, $\{x_{8n-6}\}_{n=0}^{\infty}$, $\{x_{8n-5}\}_{n=0}^{\infty}$, $\{x_{8n-4}\}_{n=0}^{\infty}$, $\{x_{8n-3}\}_{n=0}^{\infty}$, $\{x_{8n-2}\}_{n=0}^{\infty}$, $\{x_{8n-1}\}_{n=0}^{\infty}$, $\{x_{8n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0\}$.

Numerical examples

For confirming the results of this section, we consider numerical examples which represent different types of solutions to Eq. (2.1).

Example 2.4. *Consider* $x_{-7} = 2,\ x_{-6} = 7,\ x_{-5} = 3,\ x_{-4} = 2,\ x_{-3} = 6,\ x_{-2} = 9,\ x_{-1} = 5,\ x_0 = 14$. *See Fig. 1.*

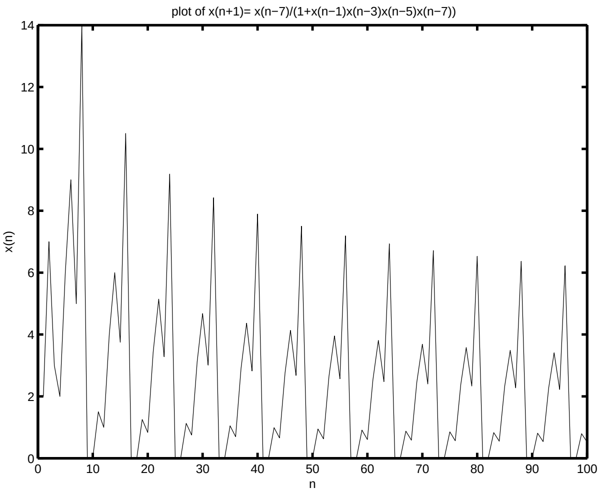

Figure 1.

Example 2.5. *See Fig. 2, since* $x_{-7} = 7,\ x_{-6} = 5,\ x_{-5} = 0.3,\ x_{-4} = 0.2,\ x_{-3} = 4,\ x_{-2} = 1,\ x_{-1} = 1.5,\ x_0 = 2.$

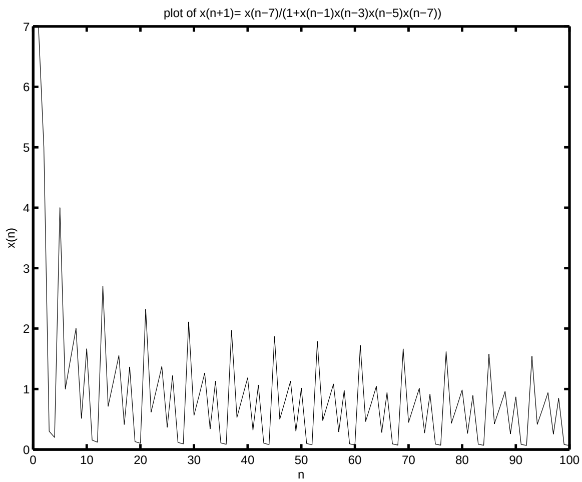

Figure 2.

3. On the Difference Equation $x_{n+1} = \frac{x_{n-7}}{1-x_{n-1}x_{n-3}x_{n-5}x_{n-7}}$

In this section we give a specific form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-7}}{1 - x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad n = 0, 1, ..., \tag{3.1}$$

where the initial conditions are arbitrary nonzero real numbers.

Theorem 3.1. *Let $\{x_n\}_{n=-7}^{\infty}$ be a solution of Eq. (3.1). Then for $n = 0, 1, ...$*

$$x_{8n-7} = \frac{h\prod_{i=0}^{n-1}(1-4ibdfh)}{\prod_{i=0}^{n-1}(1-(4i+1)bdfh)}, \quad x_{8n-3} = \frac{d\prod_{i=0}^{n-1}(1-(4i+2)bdfh)}{\prod_{i=0}^{n-1}(1-(4i+3)bdfh)},$$

$$x_{8n-6} = \frac{g\prod_{i=0}^{n-1}(1-4iaceg)}{\prod_{i=0}^{n-1}(1-(4i+1)aceg)}, \quad x_{8n-2} = \frac{c\prod_{i=0}^{n-1}(1-(4i+2)aceg)}{\prod_{i=0}^{n-1}(1-(4i+3)aceg)},$$

$$x_{8n-5} = \frac{f\prod_{i=0}^{n-1}(1-(4i+1)bdfh)}{\prod_{i=0}^{n-1}(1-(4i+2)bdfh)}, \quad x_{8n-1} = \frac{b\prod_{i=0}^{n-1}(1-(4i+3)bdfh)}{\prod_{i=0}^{n-1}(1-(4i+4)bdfh)},$$

$$x_{8n-4} = \frac{e\prod_{i=0}^{n-1}(1-(4i+1)aceg)}{\prod_{i=0}^{n-1}(1-(4i+2)aceg)}, \quad x_{8n} = \frac{a\prod_{i=0}^{n-1}(1-(4i+3)aceg)}{\prod_{i=0}^{n-1}(1-(4i+4)aceg)},$$

where $jbdfh \neq 1, jaceg \neq 1$ for $j = 1, 2, 3, ...$.

Proof. It is similar to the proof of Theorem 2.1 and will be omitted.

Theorem 3.2. *Eq. (3.1) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.*

Numerical examples

Example 3.3. *Consider $x_{-7} = 7$, $x_{-6} = 5$, $x_{-5} = 3$, $x_{-4} = 2$, $x_{-3} = 4$, $x_{-2} = 1$, $x_{-1} = 11$, $x_0 = 2$. See Fig. 3.*

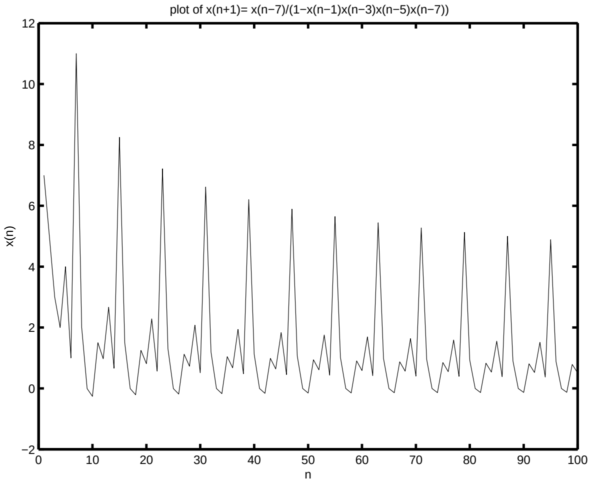

Figure 3.

Example 3.4. *See Fig. 4, since* $x_{-7} = 0.7,\ x_{-6} = 0.5,\ x_{-5} = 0.3,\ x_{-4} = 0.2,\ x_{-3} = 0.4,\ x_{-2} = 0.5,\ x_{-1} = 0.1,\ x_0 = 1.2.$

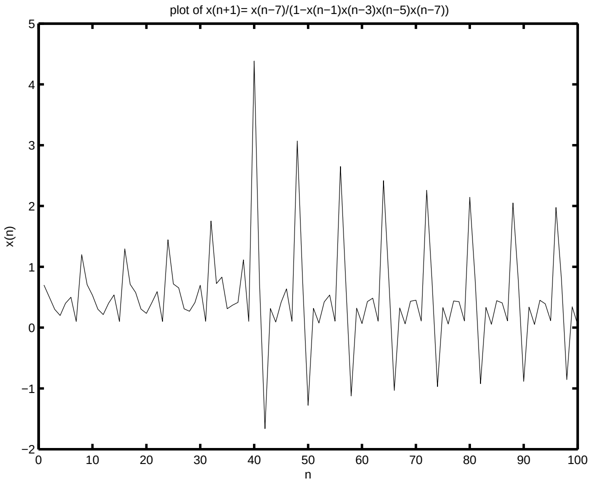

Figure 4.

4. On the Difference Equation $x_{n+1} = \frac{x_{n-7}}{-1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}}$

In this section we investigate the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-7}}{-1 + x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad n = 0, 1, ..., \tag{4.1}$$

where the initial conditions are arbitrary nonzero real numbers with

$$x_{-7}x_{-5}x_{-3}x_{-1} \neq 1, \; x_{-6}x_{-4}x_{-2}x_{0} \neq 1.$$

Theorem 4.1. *Let $\{x_n\}_{n=-7}^{\infty}$ be a solution of Eq. (4.1). Then Eq. (4.1) has unbounded solutions and for $n = 0, 1, ...$*

$$x_{8n-7} = \frac{h}{(-1+bdfh)^n}, \qquad x_{8n-3} = \frac{d}{(-1+bdfh)^n},$$
$$x_{8n-6} = \frac{g}{(-1+aceg)^n}, \qquad x_{8n-2} = \frac{c}{(-1+aceg)^n},$$

$$x_{8n-5} = f\left(-1+bdfh\right)^n, \qquad x_{8n-1} = b\left(-1+bdfh\right)^n,$$
$$x_{8n-4} = e\left(-1+aceg\right)^n, \qquad x_{8n} = a\left(-1+aceg\right)^n.$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$x_{8n-15} = \frac{h}{(-1+bdfh)^{n-1}}, \qquad x_{8n-11} = \frac{d}{(-1+bdfh)^{n-1}},$$
$$x_{8n-14} = \frac{g}{(-1+aceg)^{n-1}}, \qquad x_{8n-10} = \frac{c}{(-1+aceg)^{n-1}},$$

$$x_{8n-13} = f\left(-1+bdfh\right)^{n-1}, \qquad x_{8n-9} = b\left(-1+bdfh\right)^{n-1},$$
$$x_{8n-12} = e\left(-1+aceg\right)^{n-1}, \qquad x_{8n-8} = a\left(-1+aceg\right)^{n-1}.$$

Now, it follows from Eq. (4.1) that

$$x_{8n-7} = \frac{x_{8n-15}}{1+x_{8n-9}x_{8n-11}x_{8n-13}x_{8n-15}}$$

$$= \frac{\frac{h}{(-1+bdfh)^{n-1}}}{-1+b\left(-1+bdfh\right)^{n-1}\frac{d}{(-1+bdfh)^{n-1}}f\left(-1+bdfh\right)^{n-1}\frac{h}{(-1+bdfh)^{n-1}}}$$

$$= \frac{\frac{h}{(-1+bdfh)^{n-1}}}{-1+bdfh}.$$

Hence, we have

$$x_{8n-7} = \frac{h}{(-1+bdfh)^{n-1}}.$$

Similarly

$$x_{8n-4} = \frac{x_{8n-12}}{1+x_{8n-6}x_{8n-8}x_{8n-10}x_{8n-12}}$$

$$= \frac{e\left(-1+aceg\right)^{n-1}}{-1+\frac{g}{(-1+aceg)^n}a\left(-1+aceg\right)^{n-1}\frac{c}{(-1+aceg)^{n-1}}e\left(-1+aceg\right)^{n-1}}$$

$$= \frac{e\left(-1+aceg\right)^{n-1}}{-1+\dfrac{aceg}{(-1+aceg)}}\left(\frac{-1+aceg}{-1+aceg}\right).$$

Hence, we have

$$x_{8n-4} = e\left(-1+aceg\right)^{n}.$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 4.2. *Eq. (4.1) has three equilibrium points which are* $0, \pm\sqrt[4]{2}$ *and these equilibrium points are not locally asymptotically stable.*

Proof. The proof as in Theorem 2.2.

Theorem 4.3. *Eq. (4.1) has a periodic solutions of period eight iff* $aceg = bdfh = 2$ *and will be take the form* $\{h, g, f, e, d, c, b, a, h, g, f, e, d, c, b, a, ...\}$.

Proof. First suppose that there exists a prime period eight solution

$$h, g, f, e, d, c, b, a, h, g, f, e, d, c, b, a, ...,$$

of Eq. (4.1), we see from Eq. (4.1) that

$$h = \frac{h}{(-1+bdfh)^{n}}, \qquad d = \frac{d}{(-1+bdfh)^{n}},$$
$$g = \frac{g}{(-1+aceg)^{n}}, \qquad c = \frac{c}{(-1+aceg)^{n}},$$

$$f = f\left(-1+bdfh\right)^{n}, \qquad b = b\left(-1+bdfh\right)^{n},$$
$$e = e\left(-1+aceg\right)^{n}, \qquad a = a\left(-1+aceg\right)^{n}.$$

or

$$(-1+bdfh)^{n} = 1, \quad (-1+aceg)^{n} = 1.$$

Then

$$bdfh = 2, \quad aceg = 2.$$

Second suppose $aceg = 2$, $bdfh = 2$. Then we see from Eq. (4.1) that

$$x_{8n-7} = h,\ x_{8n-6} = g,\ x_{8n-5} = f,\quad x_{8n-4} = e,\quad x_{8n-3} = d,$$
$$x_{8n-2} = c,\quad x_{8n-1} = b,\quad x_{8n} = a.$$

Thus we have a period eight solution and the proof is complete.

Numerical examples

Example 4.4. *We consider* $x_{-7} = 7$, $x_{-6} = 8$, $x_{-5} = 11$, $x_{-4} = 2$, $x_{-3} = 4$, $x_{-2} = 1$, $x_{-1} = 3$, $x_{0} = 9$. *See Fig. 5.*

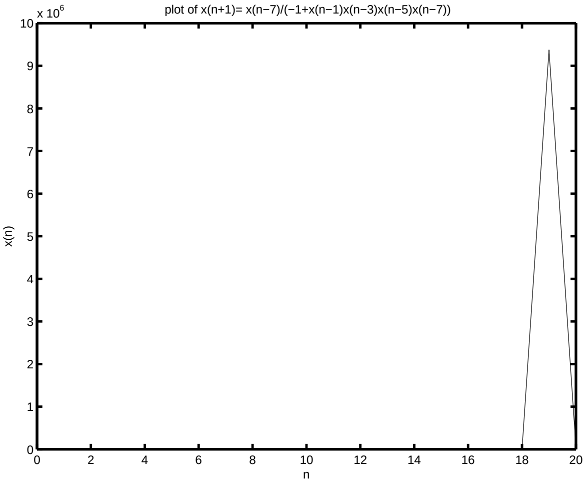

Figure 5.

Example 4.5. *See Fig. 6, since* $x_{-7} = 7,\ x_{-6} = 0.5,\ x_{-5} = 10,\ x_{-4} = 12,\ x_{-3} = 0.4,\ x_{-2} = 1/12,\ x_{-1} = 1/14,\ x_0 = 4.$

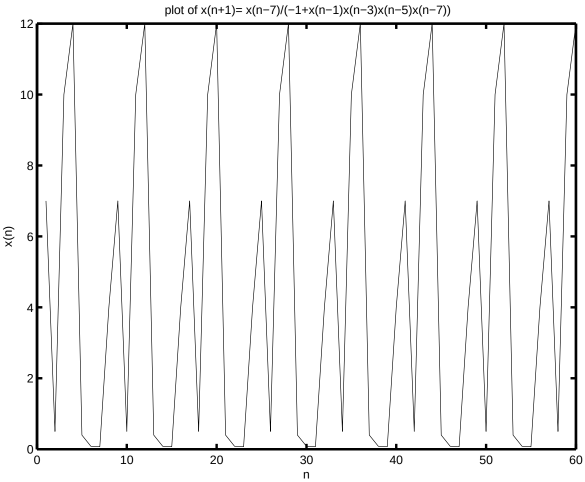

Figure 6.

5. On the Difference Equation $x_{n+1} = \frac{x_{n-7}}{-1-x_{n-1}x_{n-3}x_{n-5}x_{n-7}}$

In this section we investigate the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-7}}{-1 - x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad n = 0, 1, ..., \tag{5.1}$$

where the initial conditions are arbitrary nonzero real numbers with

$$x_{-5}x_{-3}x_{-1} \neq -1,\ x_{-4}x_{-2}x_0 \neq -1.$$

Theorem 5.1. *Let* $\{x_n\}_{n=-7}^{\infty}$ *be a solution of Eq. (5.1). Then Eq. (5.1) has unbounded solutions and for* $n = 0, 1, ...$

$$x_{8n-7} = \frac{(-1)^n h}{(1+bdfh)^n}, \qquad x_{8n-3} = \frac{(-1)^n d}{(1+bdfh)^n},$$
$$x_{8n-6} = \frac{(-1)^n g}{(1+aceg)^n}, \qquad x_{8n-2} = \frac{(-1)^n c}{(1+aceg)^n},$$

$$x_{8n-5} = f(-1)^n (1+bdfh)^n, \qquad x_{8n-1} = b(-1)^n (1+bdfh)^n,$$
$$x_{8n-4} = e(-1)^n (1+aceg)^n, \qquad x_{8n} = a(-1)^n (1+aceg)^n.$$

Theorem 5.2. *Eq. (5.1) has one equilibrium point which is number zero and this equilibrium point is not locally asymptotically stable.*

Theorem 5.3. *Eq. (5.1) has a periodic solutions of period eight iff* $aceg = bdfh = -2$ *and will be take the form* $\{h, g, f, e, d, c, b, a, h, g, f, e, d, c, b, a, ...\}$.

Numerical examples

Example 5.4. *Fig. 7 shows the solution when* $x_{-7} = -7,\ x_{-6} = 8,\ x_{-5} = 11,\ x_{-4} = 2,\ x_{-3} = -4,\ x_{-2} = 1,\ x_{-1} = 3,\ x_0 = -9.$

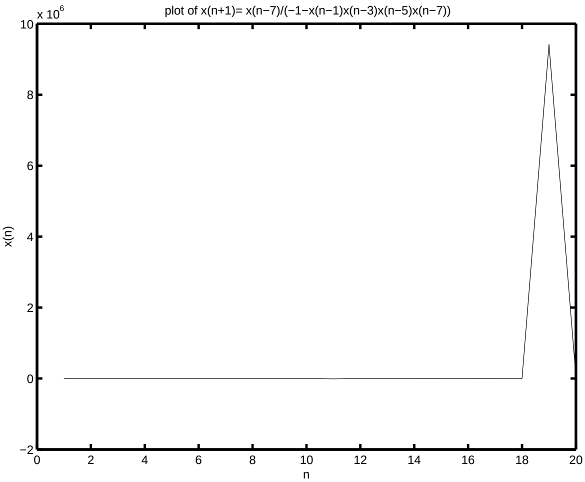

Figure 7.

Example 5.5. *See Fig. 8, since* $x_{-7} = -7,\ x_{-6} = 10,\ x_{-5} = 30,\ x_{-4} = 2,\ x_{-3} = -0.4,\ x_{-2} = 0.6,\ x_{-1} = -1/42,\ x_0 = -1/6$

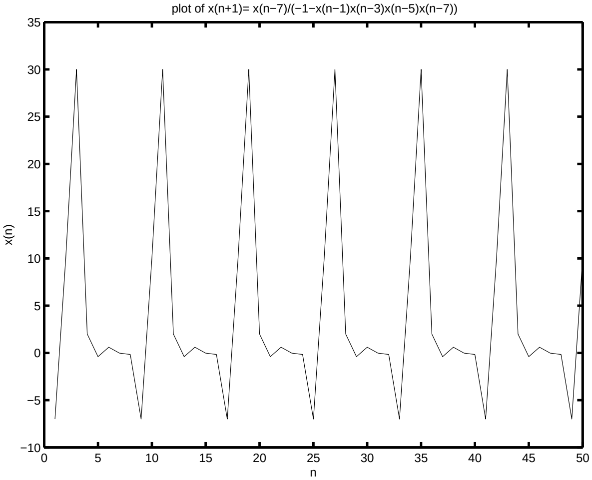

Figure 8.

References

[1] Agarwal, R. P., *Difference Equations and Inequalities. Theory, Methods and Applications*, Marcel Dekker Inc., New York, 1992.

[2] Agarwal, R. P., Elsayed, E. M., *Periodicity and stability of solutions of higher order rational difference equation*, Advanced Studies in Contemporary Mathematics, **17**(2008), no. 2, 181-201.

[3] Agarwal, R. P., Zhang, W., *Periodic solutions of difference equations with general periodicity*, Comput. Math. Appl., **42**(2001), 719-727.

[4] Agarwal, R. P., Popenda, J., *Periodic solutions of first order linear difference equations*, Math. Comput. Modelling, **22**(1995), no. 1, 11-19.

[5] Aloqeili, M., *Dynamics of a rational difference equation*, Appl. Math. Comp., **176**(2006), no. 2, 768-774.

[6] Bozkurt, F., Ozturk, I., Ozen, S., *The global behavior of the difference equation*, Stud. Univ. Babeş-Bolyai Math., **54**(2009), no. 2, 3-12.

[7] Cinar, C., *On the positive solutions of the difference equation* $x_{n+1} = \frac{x_{n-1}}{1+ax_nx_{n-1}}$, Appl. Math. Comp., **158**(2004), no. 3, 809-812.

[8] Cinar, C., *On the solutions of the difference equation* $x_{n+1} = \frac{x_{n-1}}{-1+ax_nx_{n-1}}$, Appl. Math. Comp., **158**(2004), no. 3, 793-797.

[9] Cinar, C., *On the positive solutions of the difference equation* $x_{n+1} = \frac{ax_{n-1}}{1+bx_nx_{n-1}}$, Appl. Math. Comp., **156**(2004), no. 2, 587-590.

[10] Elabbasy, E. M., El-Metwally, H., Elsayed, E. M., *Global attractivity and periodic character of a fractional difference equation of order three*, Yokohama Math. J., **53**(2007), 89-100.

[11] Elabbasy, E. M., El-Metwally, H., Elsayed, E. M., *On the difference equation* $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}$, Adv. Differ. Eq., Volume 2006, Article ID 82579, 1-10.

[12] Elabbasy, E. M., El-Metwally, H., Elsayed, E. M., *Qualitative behavior of higher order difference equation*, Soochow Journal of Mathematics, **33**(2007), no. 4, 861-873.

[13] Elabbasy, E. M., El-Metwally, H., Elsayed, E. M., *On the difference equations* $x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}$, J. Conc. Appl. Math., **5**(2007), no. 2, 101-113.

[14] Elabbasy, E. M., Elsayed, E. M., *On the Global attractivity of difference equation of higher order*, Carpathian Journal of Mathematics, **24**(2008), no. 2, 45-53.

[15] Elabbasy, E. M., Elsayed, E. M., *On the solutions of a class of difference equations of higher order*, International Journal of Mathematics and Statistics, **6**(2010), 57-68.

[16] Elabbasy, E. M., Elsayed, E. M., *Dynamics of a rational difference equation*, Chinese Annals of Mathematics, Series B, **30**(2009), 187-198.

[17] El-Metwally, H., Grove, E. A., Ladas, G., Levins, R., Radin, M., *On the difference equation* $x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}$, Nonlinear Analysis: Theory, Methods & Applications, **47**(2003), no. 7, 4623-4634.

[18] El-Metwally, H., Grove, E. A., Ladas, G., *A global convergence result with applications to periodic solutions*, J. Math. Anal. Appl., **245**(2000), 161-170.

[19] El-Metwally, H., Grove, E. A., Ladas, G., McGrath, *On the difference equation* $y_{n+1} = \frac{y_{n-(2k+1)} + p}{y_{n-(2k+1)} + qy_{n-2l}}$, Proceedings of the 6th ICDE, Taylor and Francis, London, 2004.

[20] Elsayed, E. M., *On the solution of recursive sequence of order two*, Fasciculi Mathematici, **40**(2008), 5-13.

[21] Elsayed, E. M., *Qualitative behavior of difference equation of order three*, Acta Scientiarum Mathematicarum (Szeged), **75**, no. 1-2, 113-129.

[22] Elsayed, E. M., *Qualitative behavior of s rational recursive sequence*, Indagationes Mathematicae, New Series, **19**(2008), no. 2, 189-201.

[23] Elsayed, E. M., *On the Global attractivity and the solution of recursive sequence*, Studia Scientiarum Mathematicarum Hungarica, **47**(2010), no. 3, 401-418.

[24] Elsayed, E. M., *Qualitative properties for a fourth order rational difference equation*, Acta Applicandae Mathematicae, **110**(2010), no. 2, 589-604.

[25] Elsayed, E. M., *Qualitative behavior of difference equation of order two*, Mathematical and Computer Modelling, **50**(2009), 1130-1141.

[26] Karatas, R., Cinar, C., Simsek, D., *On positive solutions of the difference equation* $x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}$, Int. J. Contemp. Math. Sci., **1**(2006), no. 10, 495-500.

[27] Karatas, R., *On solutions of the difference equation* $x_{n+1} = \frac{(-1)^n x_{n-4}}{1 + (-1)^n x_n x_{n-1} x_{n-2} x_{n-3} x_{n-4}}$, Selcuk J. Appl. Math., **8**(2007), no. 1, 51-56.

[28] Kocic, V. L., Ladas, G., *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht, 1993.

[29] Kulenovic, M. R. S., Ladas, G., *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall / CRC Press, 2001.

[30] Saleh, M., Abu-Baha, S., *Dynamics of a higher order rational difference equation*, Appl. Math. Comp., **181**(2006), no. 1, 84-102.

[31] Saleh, M., Aloqeili, M., *On the difference equation* $y_{n+1} = A + \frac{y_n}{y_{n-k}}$ *with* $A < 0$, Appl. Math. Comp., **176**(2006), no. 1, 359-363.

[32] Saleh, M., Aloqeili, M., *On the rational difference equation* $y_{n+1} = A + \frac{y_{n-k}}{y_n}$, Appl. Math. Comp., **171**(2005), no. 2, 862-869.

[33] Simsek, D., Cinar, C., Yalcinkaya, I., *On the recursive sequence* $x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}}$, Int. J. Contemp. Math. Sci., **1**(2006), no. 10, 475-480.

[34] Xianyi, L., Deming, Z., *Global asymptotic stability in a rational equation*, J. Differ. Equations Appl., **9**(2003), no. 9, 833-839.

[35] Yan, X., Li, W., *Global attractivity for a class of nonlinear difference equations*, Soochow J. Math., **29**(2003), no. 3, 327-338.

[36] Yi, T., Zhou, Z., *Periodic solutions of difference equations*, J. Math. Anal. Appl., **286**(2003), 220-229.

[37] Zayed, E. M. E., El-Moneam, M. A., *On the rational recursive sequence* $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}$, Communications on Applied Nonlinear Analysis, **12**(2005), no. 4, 15-28.

[38] Zayed, E. M. E., El-Moneam, M. A., *On the rational recursive sequence* $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-k}}$, Comm. Appl. Nonlinear Analysis, **15**(2008), 47-57.

[39] Zayed, E. M. E., El-Moneam, M. A., *On the rational recursive sequence* $x_{n+1} = \frac{\alpha x_n+\beta x_{n-1}+\gamma x_{n-2}+\delta x_{n-3}}{Ax_n+Bx_{n-1}+Cx_{n-2}+Dx_{n-3}}$, Comm. Appl. Nonlinear Analysis, **12**(2005), 15-28.

[40] Zeng, X. Y., Shi, B., Zhang, D. C., *Stability of solutions for the recursive sequence* $x_{n+1} = (\alpha - \beta x_n)/(\gamma + g(x_{n-k})$, J. Comp. Appl. Math., **176**(2005), 283-291.

[41] Zhang, D. C., Shi, B., Gai, J., *A rational recursive sequence,* Comp. Math. Appl., **41**(2001), 301-306.

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On $A_{p,q}^{lip}(G)$ spaces

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Abstract. In this paper, the space $A_{p,q}^{lip}(G)$ consisting of all complex valued functions $f \in lip(\alpha, 1)$ whose Fourier transform $\widehat{f}$ belongs to $L(p,q)\left(\widehat{G}\right)$ is investigated.

Mathematics Subject Classification (2010): 26A16, 43A25, 46E30.

Keywords: Lipschitz spaces, Fourier transform, Lorentz spaces.

1. Introduction

Let G denote a locally compact Abelian group, with dual group $\widehat{G}$ and Haar measure μ and $\widehat{\mu}$, respectively. The Fourier transform of a function $f \in L^1(G)$ will be denoted by $\widehat{f}$ which is continuous on $\widehat{G}$, vanishes at infinity and satisfies the inequality $\left\|\widehat{f}\right\|_\infty \leq \|f\|_1$. It is known that the space

$$A_p(G) = \left\{ f \in L^1(G) : \widehat{f} \in L^p\left(\widehat{G}\right)\right\}$$

is a Banach algebra for $1 \leq p < \infty$ and for $1 < p < \infty,\ 1 \leq q < \infty$, the space

$$A(p,q)(G) = \left\{ f \in L^1(G) : \widehat{f} \in L(p,q)\left(\widehat{G}\right)\ \right\}$$

is a Segal Algebra with respect to the usual convolution product and the norms defined by $\|f\| = \|f\|_1 + \left\|\widehat{f}\right\|_p$, $\|f\| = \|f\|_1 + \left\|\widehat{f}\right\|_{p,q}$ respectively. These spaces are examined by Larsen-Liu-Wang [15], Lai [11-13], Martin-Yap [16], Yap [23,24] and others.

For the convenience of the reader, we briefly review what we need from the theory of $L(p,q)(G)$ spaces. Let (G, Σ, μ) be a positive measure space and let f be a complex-valued, measurable function on G. For each $y \geq 0$ let

$$\lambda_f(y) = \mu\left\{x \in G :\ |f(x)| > y\right\}.$$

The function λ_f is called the distribution function of f. The rearrangement of f on $(0,\infty)$ is defined by

$$f^*(t) = \inf\{y>0 : \lambda_f(y) \le t\} = \sup\{y>0 : \lambda_f(y) > t\}, \ t>0,$$

where $\inf\phi = \infty$. Also, the average function of f is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds \ , \ t>0.$$

For $p,q \in (0,\infty)$ we define

$$\begin{aligned}\|f\|_{p,q}^* &= \|f\|_{p,q,\mu}^* = \left(\frac{q}{p}\int_0^\infty [f^*(t)]^q\, t^{\frac{q}{p}-1}dt\right)^{\frac{1}{q}} \\ \|f\|_{p,q} &= \|f\|_{p,q,\mu} = \left(\frac{q}{p}\int_0^\infty [f^{**}(t)]^q\, t^{\frac{q}{p}-1}dt\right)^{\frac{1}{q}}.\end{aligned}$$

Also, if $0<p, q=\infty$ we define

$$\|f\|_{p,\infty}^* = \sup_{t>0} t^{\frac{1}{p}} f^*(t) \ \text{ and } \ \|f\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^{**}(t).$$

For $0<p<\infty$ and $0<q\le\infty$, Lorentz spaces are denoted by $L(p,q)(G,\Sigma,\mu)$ (or shortly $L(p,q)(G)$) is defined to be the vector space of all (equivalence classes of) measurable functions f on G such that $\|f\|_{p,q}^* < \infty$. We know that, for $1\le p\le\infty$, $\|f\|_{p,p}^* = \|f\|_p$ and so $L_p(G) = L(p,p)(G)$ where $L_p(G)$ is the usual Lebesgue space. It is also known that if $1<p<\infty$ and $1\le q\le\infty$ then

$$\|f\|_{p,q}^* \le \|f\|_{p,q} \le \frac{p}{p-1}\|f\|_{p,q}^*$$

for each $f\in L(p,q)(G)$ and $\left(L(p,q)(G), \|\cdot\|_{pq}\right)$ is a Banach space [10].

In [4], Chen and Lai showed that there is an approximate identity $\{a_\alpha\}_{\alpha\in I}$ of $L_1(G)$ such that $\|a_\alpha\|_1 = 1$ for each $\alpha\in I$ and $f * a_\alpha \to f$ for every $f\in L(p,q)(G)$, whenever $1<p<\infty$, $1\le q<\infty$. It can be derived from [2],[3] and [20] that $L(p,q)(G)$ is an essential Banach $L_1(G)-$module with the usual convolution and the norm $\|\cdot\|_{p,q}$. Also, in [4], Chen and Lai showed that $(L_1(G), L(p,q)(G))$ is isometrically isomorphic to $L(p,q)(G)$ for $1<p,q<\infty$. One can also review [2-5,10,17,20,22] for more properties of $L(p,q)(G)$ Lorentz spaces.

Throughout the paper G will denote a metrizable locally compact Abelian group with a translation invariant metric d such that for any $y\in G$, $|y| = d(0,y)$ and Haar measure μ.We assume that there is a decreasing countable (open) basis $\{V_n\}_{n\in\mathbb{N}}$ of the identity e of G such that

$$\mu((y+V_n)\triangle V_n)/|y|^\alpha \to 0 \ \text{ as } y\to 0,$$

where $\triangle$ denotes the symmetric difference, $\alpha \in (0,1)$ and $y \in G$. Quek and Yap showed that the above condition is not unduly restrictive. Example of groups that have these properties are $R^k, T^k\,(k \geq 1)$, the 0-dimensional groups, etc.[19]. While χ denotes the characteristic function, it is easy to see that $\{e_n\}_{n\in\mathbb{N}}$ is an approximate identity for $L^1(G)$ which is defined by $e_n = \mu(V_n)^{-1}\chi_{V_n}$. For any $f \in L^1(G)$ and $\delta > 0$, define

$$\omega_1(f;\delta) = \sup\left\{\|\tau_y f - f\|_1 : |y| \leq \delta\right\},$$

where $\tau_y f(x) = f(x-y)$. Following Zygmund [25],Bloom [1] and Quek-Yap [19], we define

$$\begin{aligned} Lip(\alpha,1) &= \left\{f \in L^1(G) : \omega_1(f;\delta) = O(\delta^\alpha)\ \right\} \\ lip(\alpha,1) &= \left\{f \in Lip(\alpha,1) : \omega_1(f;\delta) = o(\delta^\alpha)\ \right\}. \end{aligned}$$

These spaces are called as Lipschitz spaces and the function $\|\cdot\|_{(\alpha,1)}$ defined by

$$\|f\|_{(\alpha,1)} = \|f\|_1 + \sup_{y\neq 0}\frac{\|\tau_y f - f\|_1}{|y|^\alpha}$$

is a norm in both Lipschitz spaces. Quek and Yap in [19],Feichtinger in [6,7] proved a series of results concerning Lipschitz spaces.

2. The space $A^{lip}_{p,q}(G)$

Let G be a metrizable locally compact Abelian group, $\alpha \in (0,1)$ and $1 < p < \infty,\ 1 \leq q < \infty$. We define the vector space $A^{lip}_{p,q}(G)$ by

$$A^{lip}_{p,q}(G) = \left\{f \in lip(\alpha,1)(G) :\ \widehat{f} \in L(p,q)\left(\widehat{G}\right)\right\}.$$

If one endows it with the norm

$$\|f\|^{lip}_{p,q} = \|f\|_{(\alpha,1)} + \left\|\widehat{f}\right\|_{p,q}$$

where $f \in A^{lip}_{p,q}(G)$, then it is easy to see that $A^{lip}_{p,q}(G) = lip(\alpha,1)(G) \cap A(p,q)(G)$ becomes a normed space.

Theorem 2.1. *The space $\left(A^{lip}_{p,q}(G), \|\cdot\|^{lip}_{p,q}\right)$ is a Banach space for $p = q = 1,\ p = q = \infty$ or $1 < p \leq \infty, 1 \leq q \leq \infty$.*

Proof. Assume that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $A^{lip}_{p,q}(G)$. Clearly, $\{f_n\}_{n\in\mathbb{N}}$ and $\left\{\widehat{f_n}\right\}_{n\in\mathbb{N}}$ are also Cauchy sequences in $lip(\alpha,1)(G)$ and $L(p,q)\left(\widehat{G}\right)$, respectively. Since $lip(\alpha,1)(G)$ and $L(p,q)\left(\widehat{G}\right)$ are Banach spaces, there exist $f \in lip(\alpha,1)(G)$ and $g \in L(p,q)\left(\widehat{G}\right)$ such that $\|f_n - f\|_{(\alpha,1)} \to 0$, $\|f_n - f\|_1 \to 0$ and $\left\|\widehat{f_n} - g\right\|_{p,q} \to 0$. Using Lemma 2.2 in

[24], there exists a subsequence $\left\{\widehat{f_{n_k}}\right\}_{n\in\mathbb{N}}$ of $\left\{\widehat{f_n}\right\}_{n\in\mathbb{N}}$ which converges to g almost everywhere. It follows from the inequality

$$\left\|\widehat{f_n}-\widehat{f}\right\|_{\infty} \leq \|f_n - f\|_1 \leq \|f_n - f\|_{(\alpha,1)}$$

that $\left\|\widehat{f_n}-\widehat{f}\right\|_{\infty} \to 0$. Hence it is easly showed that $\left\|\widehat{f_{n_k}}-\widehat{f}\right\|_{\infty} \to 0$. Therefore $\widehat{f} = g$, $\|f_n - f\|_{p,q}^{lip} \to 0$ and $f \in A_{p,q}^{lip}(G)$. Thus $\left(A_{p,q}^{lip}(G), \|\cdot\|_{p,q}^{lip}\right)$ is a Banach space. □

By using the propositions and lemmas proved in [19], one can easly prove the following propositions.

Proposition 2.2. *The space $lip(\alpha,1)(G)$ is a Banach algebra with usual convolution product.*

Proposition 2.3. *The space $lip(\alpha,1)(G)$ is strongly translation and character invariant.*

Proof. It is known that $L^1(G)$ is strongly translation invariant, i.e., $\tau_x f \in L^1(G)$ and $\|\tau_x f\|_1 = \|f\|_1$ for all $x \in G$, $f \in L^1(G)$. Let us take any $f \in lip(\alpha,1)(G)$ and $x \in G$. Then for any $\varepsilon > 0$, there exits a $\delta_\varepsilon > 0$ such that

$$\sup_{|y|\leq\delta} \frac{\|\tau_y f - f\|_1}{\delta^\alpha} < \varepsilon$$

whenever $0 < \delta < \delta_\varepsilon$. For the same $\varepsilon > 0$, we have

$$\sup_{|y|\leq\delta} \frac{\|\tau_y(\tau_x f) - (\tau_x f)\|_1}{\delta^\alpha} = \sup_{|y|\leq\delta} \frac{\|\tau_y f - f\|_1}{\delta^\alpha} < \varepsilon$$

whenever $0 < \delta < \delta_\varepsilon$. Therefore $\omega_1(\tau_x f;\delta) = o(\delta^\alpha)$, $\tau_x f \in lip(\alpha,1)(G)$ and $\|\tau_x f\|_{(\alpha,1)} = \|f\|_{(\alpha,1)}$.

Strongly character invariance of $lip(\alpha,1)(G)$ can be seen in a similar way. □

Proposition 2.4. *The function $x \to \tau_x f$ is continuous from G into $lip(\alpha,1)(G)$ for every $f \in lip(\alpha,1)(G)$.*

Proposition 2.5. *The space $lip(\alpha,1)(G)$ has an approximate identity $\{e_n\}_{n\in\mathbb{N}}$ defined by $e_n = \mu(V_n)^{-1}\chi_{V_n}$.*

Proposition 2.6. *The space $lip(\alpha,1)(G)$ is a homogeneous Banach space.*

Proposition 2.7. *The space $lip(\alpha,1)(G)$ is an essential $L^1(G)-$module.*

Theorem 2.8. *The space $\left(A_{p,q}^{lip}(G), \|\cdot\|_{p,q}^{lip}\right)$ is a Banach module over $L^1(G)$ and $lip(\alpha,1)(G)$. Hence it is a Banach algebra with respect to the usual convolution.*

Proof. Let $f,g \in A^{lip}_{p,q}(G)$ be given. Since the space $lip(\alpha,1)(G)$ is a Banach algebra under convolution, then $f*g \in lip(\alpha,1)(G)$. Since $\widehat{f},\widehat{g} \in L(p,q)\left(\widehat{G}\right)$, we have

$$\begin{aligned}
\lambda_{\widehat{f}\widehat{g}}(y) &= \mu\left\{x \in \widehat{G} : \left|\widehat{f}(x)\,\widehat{g}(x)\right| > y\right\} \\
&\leq \mu\left\{x \in \widehat{G} : \left(\sup \widehat{f}(x)\right)|\widehat{g}(x)| > y\right\} \\
&= \mu\left\{x \in \widehat{G} : K|\widehat{g}(x)| > y\right\} = \lambda_{K\widehat{g}}(y),
\end{aligned}$$

if $\sup\limits_{x\in\widehat{G}} \widehat{f}(x) = K$. Therefore we get

$$\left(\widehat{f}\widehat{g}\right)^{*}(t) = \inf\left\{y > 0 : \lambda_{\widehat{f}\widehat{g}}(y) \leq t\right\} \leq K(\widehat{g})^{*}(t),$$

$$\left(\widehat{f}\widehat{g}\right)^{**}(t) = \frac{1}{t}\int_{0}^{t} \left(\widehat{f}\widehat{g}\right)^{*}(s)\,ds \leq K(\widehat{g})^{**}(t)$$

and so

$$\begin{aligned}
\left\|\widehat{f}\cdot\widehat{g}\right\|_{p,q} &\leq K\|\widehat{g}\|_{p,q} \leq \sup \widehat{f}(x)\,\|\widehat{g}\|_{p,q} \leq \left\|\widehat{f}\right\|_{\infty}\|\widehat{g}\|_{p,q} \\
&\leq \|f\|_{1}\,\|\widehat{g}\|_{p,q}.
\end{aligned}$$

Thus, we obtain $\widehat{f*g} \in L(p,q)\left(\widehat{G}\right)$ and $f*g \in A^{lip}_{p,q}(G)$. Also, we have

$$\begin{aligned}
\|f*g\|^{lip}_{p,q} &= \|f*g\|_{(\alpha,1)} + \left\|\widehat{f*g}\right\|_{p,q} \\
&\leq \|f\|_{(\alpha,1)}\,\|g\|_{(\alpha,1)} + \left\|\widehat{f}\cdot\widehat{g}\right\|_{p,q} \\
&\leq \|f\|_{(\alpha,1)}\,\|g\|_{(\alpha,1)} + \|f\|_{1}\,\|\widehat{g}\|_{p,q} \\
&\leq \|f\|_{(\alpha,1)}\,\|g\|^{lip}_{p,q} \leq \|f\|^{lip}_{p,q}\,\|g\|^{lip}_{p,q},
\end{aligned}$$

for any $f,g \in A^{lip}_{p,q}(G)$. □

By Proposition 2.3 in [19], Proposition 2.2 and Proposition 2.3, the following can be easily proved.

Proposition 2.9. *The space $\left(A^{lip}_{p,q}(G), \|\cdot\|^{lip}_{p,q}\right)$ is strongly translation invariant and the function $x \to \tau_x f$ is continuous from G into $A^{lip}_{p,q}(G)$ for every $f \in A^{lip}_{p,q}(G)$.*

Proposition 2.10. *The space $\left(A^{lip}_{p,q}(G), \|\cdot\|^{lip}_{p,q}\right)$ is a homogeneous Banach space.*

Proposition 2.11. *The space $\left(A^{lip}_{p,q}(G), \|\cdot\|^{lip}_{p,q}\right)$ is strongly character invariant.*

Proposition 2.12. *The space* $\left(A_{p,q}^{lip}(G), \|\cdot\|_{p,q}^{lip}\right)$ *is a semi-simple Banach algebra.*

Proof. Let $f \in A_{p,q}^{lip}(G)$ be given. It will be sufficient to show that $f = 0$ whenever $\left\|\widehat{f}\right\|_{\infty} = 0$. Since $A_{p,q}^{lip}(G)$ is a commutative Banach algebra by Theorem 2.8, it is known that

$$\lim_n \left(\|f^n\|_{p,q}^{lip}\right)^{\frac{1}{n}} = \left\|\widehat{f}\right\|_{\infty}.$$

Moreover, we have

$$\|f^n\|_1^{\frac{1}{n}} \leq \|f^n\|_{(\alpha,1)}^{\frac{1}{n}} \leq \left(\|f^n\|_{p,q}^{lip}\right)^{\frac{1}{n}}$$

and

$$\lim_n \|f^n\|_1^{\frac{1}{n}} \leq \lim_n \left(\|f^n\|_{p,q}^{lip}\right)^{\frac{1}{n}}.$$

If we set

$$\lim_n \|f^n\|_1^{\frac{1}{n}} = \left\|\widehat{f}\right\|_{\infty}^{\prime},$$

then we have the inequality

$$\left\|\widehat{f}\right\|_{\infty}^{\prime} \leq \left\|\widehat{f}\right\|_{\infty}.$$

Since $\left\|\widehat{f}\right\|_{\infty} = 0$, then $\left\|\widehat{f}\right\|_{\infty}^{\prime} = 0$. Also, since $L^1(G)$ is semi-simple [14], then $f = 0$. □

Theorem 2.13. *The space* $A_{p,q}^{lip}(G)$ *is an essential Banach* $L^1(G)-$*module.*

Proof. In view of Lemma 4.1 in [8], it will be sufficient to show that any bounded approximate identity $\{e_\alpha\}_{\alpha\in I}$ of $L^1(G)$ which belongs to

$$\Lambda^K = \left\{f \in L^1(G) : \text{ supp } \widehat{f} \text{ compact}\right\}$$

is also an approximate identity for $A_{p,q}^{lip}(G)$. Let $f \in A_{p,q}^{lip}(G) \subset lip(\alpha, 1) \subset L^1(G)$. By the same Lemma, the bounded approximate identity $\{e_\alpha\}_{\alpha\in I} \subset \Lambda^K$ is also an approximate identity for $L^1(G)$, and so, for any given $\varepsilon > 0$, we have $\|e_\alpha * f - f\|_1 < \varepsilon$ for sufficiently large α. For each $\alpha \in I, \|e_\alpha\|_1 = 1$ implies that $\sup_\alpha \|\widehat{e_\alpha}\|_\infty \leq 1$. Hence, for any $g \in L^1(G)$, the inequality

$$|\widehat{g}|\,|1 - \widehat{e_\alpha}| \leq \|\widehat{g} - \widehat{g}\widehat{e_\alpha}\|_\infty \leq \|g - g * e_\alpha\|_1 \to 0$$

implies uniformly convergence of $\{\widehat{e_\alpha}\}_{\alpha\in I}$ to 1 over compact sets. Since $\widehat{f} \in L(p,q)\left(\widehat{G}\right)$, we can choose a compact set $\widehat{K} \subset \widehat{G}$ such that

$$\left\|\widehat{f} - \widehat{f}\chi_{\widehat{K}}\right\|_{p,q} < \frac{\varepsilon}{8}$$

and the local convergence to 1 implies that one can find an α_0 with

$$\left\| \widehat{e_\alpha} \chi_{\widehat{K}} - \chi_{\widehat{K}} \right\|_\infty < \frac{\varepsilon}{4 \left\| \widehat{f} \right\|_{p,q}} \text{ for all } \alpha > \alpha_0.$$

Altogether,

$$\begin{aligned} \left\| \widehat{f - e_\alpha * f} \right\|_{p,q} &= \left\| \widehat{f} - \widehat{e_\alpha} \cdot \widehat{f} \right\|_{p,q} \qquad (2.1) \\ &\le \left\| \widehat{f} - \widehat{f} \chi_{\widehat{K}} \right\|_{p,q} + \left\| \widehat{f} \chi_{\widehat{K}} - \widehat{f} \chi_{\widehat{K}} \widehat{e_\alpha} \right\|_{p,q} + \left\| \widehat{f} \chi_{\widehat{K}} \widehat{e_\alpha} - \widehat{f} \widehat{e_\alpha} \right\|_{p,q} \\ &\le (1 + \| \widehat{e_\alpha} \|_\infty) \left\| \widehat{f} - \widehat{f} \chi_{\widehat{K}} \right\|_{p,q} + \left\| \widehat{f} \right\|_{p,q} \left\| \widehat{e_\alpha} \chi_{\widehat{K}} - \chi_{\widehat{K}} \right\|_\infty \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} \end{aligned}$$

for all $\alpha > \alpha_0$. Also it is known that

$$\| f - f * e_\alpha \|_{(\alpha,1)} < \frac{\varepsilon}{2} \qquad (2.2)$$

for any $f \in lip\,(\alpha, 1)$ by Proposition 2.4. Finally, by using (2.1) and (2.2), we obtain

$$\begin{aligned} \| f - f * e_\alpha \|_{p,q}^{lip} &= \| f - f * e_\alpha \|_{(\alpha,1)} + \left\| \widehat{f - e_\alpha * f} \right\|_{p,q} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $\alpha > \alpha_0$. Therefore it follows that $\| f - f * e_\alpha \|_{p,q}^{lip} \to 0$ for any $f \in A_{p,q}^{lip}(G)$. Consequently, $A_{p,q}^{lip}(G)$ is an essential Banach module by Module Factorization Theorem in [9]. This means $\left(A_{p,q}^{lip}(G) \right)_e = A_{p,q}^{lip}(G)$. □

References

[1] Bloom, W. R., *A characterization of Lipschitz classes on 0-dimensional groups*, Proc. Amer. Math. Soc., **53**(1953), 149-153.

[2] Blozinski, A. P., *On a convolution theorem for $L(p,q)$ spaces*, Trans. Amer. Math. Soc., **164**(1972), 255-264.

[3] Blozinski, A. P., *Convolution of $L(p,q)$ functions*, Proc. of the Amer. Math. Soc., **32**(1972), no. 1, 237-240.

[4] Chen, Y. K., Lai, H. C., *Multipliers of Lorentz spaces*, Hokkaido Math. J., **4**(1975), 247-260.

[5] Duyar, C., Gürkanlı, A. T., *Multipliers and Relative completion in weighted Lorentz spaces*, Acta Math. Sci., **23**(2003), 467-476.

[6] Feichtinger, H. G., *Multipliers from $L^1(G)$ to a homogeneous Banach space*, J. Math. Anal. Appl., **61**(1977), 341-356.

[7] Feichtinger, H. G., *Konvolutoren von $L^1(G)$ nach Lipschitz-Räumen*, Anz. d.österr. Akad. Wiss., **6**(1979), 148-153.

[8] Feichtinger, H. G., Gürkanlı, A. T., *On a family of weighted convolution algebras*, Internat. J. Math. Sci., **13**(1990), no. 3, 517-526.

[9] Hewitt, E., Ross, K. A., *Abstract Harmonic Analysis,* Vol.1, Springer-Verlag, Berlin, 1963.

[10] Hunt, R. A., *On* $L(p,q)$ *spaces*, L'enseignement Mathematique, **TXII-4**(1966), 249-276.

[11] Lai, H. C., *On some properties of* $A^p(G)-$*algebras,* Proc. Japan Acad., **45**(1969), 572-575.

[12] Lai, H. C., *On the category of* $L^1(G)\cap L^p(G)$ *in* $A^q(G)$, ibid., **45**(1969), 577-581.

[13] Lai, H. C., *Remark on the* $A^p(G)-$*algebras,* ibid., **46**(1970), 58-63.

[14] Larsen, R., *Banach algebras,* Marcel Dekker, Inc., New York, (1973).

[15] Larsen, R., Liu, T. S., Wang, J. K., *On functions with Fourier transforms in* L^p, Michigan Math. Journal, **11**(1964), 369-378.

[16] Martin, J. C., Yap, L. Y. H., *The algebra of functions with Fourier transforms in* L^p, Proc. Amer. Math. Soc., **24**(1970), 217-219.

[17] O'Neil, R., *Convolution operators and* $L(p,q)$ *spaces,* Duke Math. Journal, **30**(1963), 129-142.

[18] Onneweer, C. W., *Generalized Lipschitz spaces and Herz spaces on certain totally disconnected groups*, in Martingale theory in harmonic analysis and Banach spaces, Proc. NSF-CBMS Conf., Cleveland/Ohio 1981, Lect. Notes Math. **939**(1982), 106-121.

[19] Quek, T. S., Yap, L. Y. H., *Multipliers from* $L^1(G)$ *to a Lipschitz space*, J. Math. Anal. Appl., **69**(1979), 531-539.

[20] Saeki, S., Thome, E. L., *Lorentz spaces as* L^1*-modules and multipliers*, Hokkaido Math. J., **23**(1994), 55-92.

[21] Wang, A. C., *Homogeneous Banach Algebras*, Marcel Dekker Inc., New York.

[22] Yap, L. Y. H., *Some remarks on convolution operators and* $L(p,q)$ *spaces,* Duke Math. J., **36**(1969), 647-658.

[23] Yap, L. Y. H., *Ideals in subalgebras of the group algebras,* Studia Math., **35**(1972), 165-175.

[24] Yap, L. Y. H., *On Two classes of subalgebras of* $L^1(G)$, Proc. Japan Acad., **48**(1972), 315-319.

[25] Zygmund, A., *Trigonometric Series,* Vol. I, Cambridge Univ. Press, London/New York, 1968.

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Asymptotic behavior of the solution of nonlinear parametric variational inequalities in notched beams

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Abstract. In this article we study the asymptotic behavior of the solution U_ϵ of a parametric variational inequality governed by a nonlinear differential operator posed in a notched beam (i.e. a thin cylinder with small part of it having a diameter much smaller than the rest) which depends on three positive parameters: ϵ, r_ϵ, and t_ϵ.

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Keywords: homogenization, partial differential equations, notched beam.

1. Introduction

The aim of the paper is to study the asymptotic behavior of the solution of nonlinear variational inequalities in a notched beam (i.e. a thin cylinder with small part of it having a diameter much smaller than the rest). Mathematically, this notched beam is given by

$$\Omega_\epsilon = \{(x_1, x') \in \mathbb{R}^3 : -1 < x_1 < 1, |x'| < \epsilon \text{ if } |x_1| > t_\epsilon, |x'| < \epsilon r_\epsilon \text{ if } |x_1| \leq t_\epsilon\}, \tag{1.1}$$

where ϵ, r_ϵ, and t_ϵ are positive parameters.

Previous work on domains of this type was done by Hale & Vegas [6], Jimbo [7, 8], Cabib, Freddi, Morassi, & Percivale [2], Rubinstein, Schatzman & Sternberg [12], and Casado-Díaz, Luna-Laynez & Murat [3, 4], Kohn & Slastikov [9].

The most recent results are of Casado-Díaz, Luna-Laynez & Murat [4]. They studied the asymptotic behavior of the solution of a diffusion equation in the notched beam Ω_ϵ and obtained at the limit a one-dimensional model.

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In the present article the geometrical setting is the same as in [4], but we consider nonlinear variational inequalities instead of linear variational equalities.

The paper is organized as follows. In Section 2 the geometrical setting is described, the studied problem is given, and the assumptions for our results are formulated. In Section 3 the asymptotic behavior of the solution is studied. The main results are Theorem 3.6 and Theorem 3.7.

2. Setting the problem

Let $\epsilon > 0$ be a parameter, r_ϵ ($r_\epsilon > 0$) and t_ϵ ($t_\epsilon > 0$) be two sequences of real numbers, with

$$r_\epsilon \longrightarrow 0, \qquad t_\epsilon \longrightarrow 0, \quad \text{when } \epsilon \longrightarrow 0.$$

We assume that

$$\frac{t_\epsilon}{r_\epsilon^2} \longrightarrow \mu, \quad \frac{\epsilon}{r_\epsilon} \longrightarrow \nu, \qquad \text{with } 0 \leq \mu \leq +\infty,\ 0 \leq \nu \leq +\infty, \quad \text{when } \epsilon \longrightarrow 0.$$

Let $S \subset \mathbb{R}^2$ be a bounded domain such that $0 \in S$, which is sufficiently smooth to apply the Poincaré-Wirtinger inequality.

Define the following subsets of $\mathbb{R}^3$:

$$\Omega_\epsilon^- = (-1, -t_\epsilon) \times (\epsilon S), \quad \Omega_\epsilon^0 = [-t_\epsilon, t_\epsilon] \times (\epsilon r_\epsilon S), \quad \Omega_\epsilon^+ = (t_\epsilon, 1) \times (\epsilon S),$$

$$\Omega_\epsilon = \Omega_\epsilon^- \cup \Omega_\epsilon^0 \cup \Omega_\epsilon^+, \quad \text{and} \quad \Omega_\epsilon = \Omega_\epsilon^- \cup \Omega_\epsilon^+.$$

Ω_ϵ is a notched beam, the main part of the beam is Ω_ϵ^1 and the notched part Ω_ϵ^0. The plane section of this domain is presented in Figure 1. A point of Ω^ϵ is denoted by $x = (x_1, x') = (x_1, x_2, x_3)$.

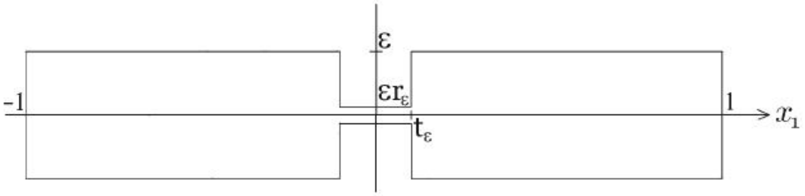

FIGURE 1. The plane section of the notched beam Ω_ϵ

Denote by

$$\Gamma_\epsilon^- = \{-1\} \times (\epsilon S) \quad \text{and} \quad \Gamma_\epsilon^+ = \{1\} \times (\epsilon S)$$

the two bases of the beam, and let

$$\Gamma_\epsilon = \Gamma_\epsilon^- \cup \Gamma_\epsilon^+$$

be the union of the two bases.

Denote

$$\mathcal{V}_\epsilon = \{V \in H^1(\Omega_\epsilon), \quad V = 0 \text{ on } \Gamma_\epsilon\}.$$

We consider the following problem:
find $U_\epsilon \in M_\epsilon$ such that, for all $V_\epsilon \in M_\epsilon$,

$$\int_{\Omega_\epsilon} [A_\epsilon \Phi_\epsilon(x, U_\epsilon, B_\epsilon \nabla U_\epsilon), \nabla(V_\epsilon - U_\epsilon)]\, dx \ + \int_{\Omega_\epsilon} \Psi_\epsilon(x, U_\epsilon, \nabla U_\epsilon)(V_\epsilon - U_\epsilon) dx \tag{2.1}$$

$$+ \int_{\Omega_\epsilon} [G_\epsilon, \nabla(V_\epsilon - U_\epsilon)]\, dx + \int_{\Omega_\epsilon} \Theta_\epsilon(x, U_\epsilon, V_\epsilon - U_\epsilon) \geq 0,$$

with A_ϵ, B_ϵ, Φ_ϵ, Ψ_ϵ, G_ϵ, and Θ_ϵ given functions, M_ϵ a closed, convex, nonempty subset of $\mathcal{V}_\epsilon$.

This problem has applications in Physics. Bruno [1] observed that when a ferromagnet has a thin neck, this will be preferred location for the domain wall. He also notice that if the geometry of the neck varies rapidly enough, it can influence and even dominate the structure of the wall.

We impose the following assumptions:

(B1) The matrix A_ϵ has the following form

$$A_\epsilon(x) = \chi_{\Omega_\epsilon^1}(x) A^1 \left(x_1, \frac{x'}{\epsilon}\right) + \chi_{\Omega_\epsilon^0}(x) A^0 \left(\frac{x_1}{t_\epsilon}, \frac{x'}{\epsilon r_\epsilon}\right),$$

where $A^1, A^0 \in L^\infty((-1,1) \times S)^{3\times 3}$.

(B2) The matrix B_ϵ has the following form

$$B_\epsilon(x) = \chi_{\Omega_\epsilon^1}(x) B^1 \left(x_1, \frac{x'}{\epsilon}\right) + \chi_{\Omega_\epsilon^0}(x) B^0 \left(\frac{x_1}{t_\epsilon}, \frac{x'}{\epsilon r_\epsilon}\right),$$

where $B^1, B^0 \in L^\infty((-1,1) \times S)^{3\times 3}$.

(B3) The functions $\Phi_\epsilon : \Omega_\epsilon \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ and $\Psi_\epsilon : \Omega_\epsilon \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ are Carathédory mappings having the following form

$$\Phi_\epsilon(x, \eta, \xi) = \chi_{\Omega_\epsilon^1}(x) \Phi_\epsilon^1 \left(x_1, \frac{x'}{\epsilon}, \eta, B^1 \left(x_1, \frac{x'}{\epsilon}\right) \xi\right)$$
$$+ \chi_{\Omega_\epsilon^0}(x) \Phi_\epsilon^0 \left(\frac{x_1}{t_\epsilon}, \frac{x'}{\epsilon r_\epsilon}, \eta, B^0 \left(\frac{x_1}{t_\epsilon}, \frac{x'}{\epsilon r_\epsilon} \xi\right) \xi\right);$$

$$\Psi_\epsilon(x, \eta, \xi) = \chi_{\Omega_\epsilon^1}(x) \Psi_\epsilon^1 \left(x_1, \frac{x'}{\epsilon}, \eta, \xi\right) + \chi_{\Omega_\epsilon^0}(x) \Psi_\epsilon^0 \left(\frac{x_1}{t_\epsilon}, \frac{x'}{\epsilon r_\epsilon}, \eta, \xi\right);$$

for a.e. $x \in \Omega_\epsilon$, for all $\eta \in \mathbb{R}$, and $\xi \in \mathbb{R}^3$;
for all $U_\epsilon \in H^1(\Omega_\epsilon)$, $\Phi_\epsilon^1(\cdot, U_\epsilon(\cdot), B_\epsilon^1(\cdot)\nabla U_\epsilon(\cdot)), \Phi_\epsilon^0(\cdot, U_\epsilon(\cdot), B_\epsilon^0(\cdot)\nabla U_\epsilon(\cdot)) \in L^2((-1,1) \times S)^3$; $\Psi_\epsilon^1(\cdot, U_\epsilon(\cdot), \nabla U_\epsilon(\cdot)), \Psi_\epsilon^0(\cdot, U_\epsilon(\cdot), \nabla U_\epsilon(\cdot)) \in L^2((-1,1) \times S)$.

(B4) *Coercivity conditions*
There exist $C_1, C_2 > 0$ and $k_1 \in L^\infty(\Omega_\epsilon)$ such that for all $\xi \in \mathbb{R}^3$, $\eta \in \mathbb{R}$

$$[A_\epsilon(x)\Phi_\epsilon(x, \eta, B_\epsilon(x)\xi), \xi] + \Psi_\epsilon(x, \eta, \xi)\eta \geq C_1 \|\xi\|^2 + C_2 |\eta|^{q_1} - k_1(x) \ \text{ a.e. } x \in \Omega_\epsilon, \tag{2.2}$$

for some $1 < q_1 < 2$, for each $\epsilon > 0$.

(B5) *Growth conditions*
There exist $C > 0$ and $\alpha \in L^\infty(\Omega_\epsilon)$ such that for all $\xi \in \mathbb{R}^3$, $\eta \in \mathbb{R}$

$$\|A_\epsilon(x)\Phi_\epsilon(x, \eta, \xi)\| \leq C\|\xi\| + C|\eta| + \alpha(x) \quad \text{a.e. } x \in \Omega_\epsilon, \tag{2.3}$$

for each $\epsilon > 0$.

There exist $C > 0$ and $\beta \in L^\infty(\Omega_\epsilon)$ such that for all $\xi \in \mathbb{R}^3$, $\eta \in \mathbb{R}$

$$|\Psi_\epsilon(x, \eta, \xi)| \leq C\|\xi\| + C|\eta| + \beta(x) \quad \text{a.e. } x \in \Omega_\epsilon, \tag{2.4}$$

for each $\epsilon > 0$.

(B6) *Monotonicity condition* For all $\xi, \tau \in \mathbb{R}^n$, $\eta \in \mathbb{R}$,

$$[A_\epsilon(x)\phi_\epsilon(x, \eta, B_\epsilon(x)\xi) - A_\epsilon(x)\phi_\epsilon(x, \eta, B_\epsilon(x)\tau), \xi - \tau] \geq 0, \quad \text{a. e. } x \in \Omega_\epsilon,$$

for each $\epsilon > 0$.

(B7) The function $G_\epsilon \in L^2((-1, 1) \times S)^3$ has the following form

$$G_\epsilon(x) = \chi_{\Omega^1_\epsilon}(x) G^1_\epsilon\left(x_1, \frac{x'}{\epsilon}\right) + \chi_{\Omega^0_\epsilon}(x) G^0_\epsilon\left(\frac{x_1}{t_\epsilon}, \frac{x'}{\epsilon r_\epsilon}\right) \quad \text{a.e. } x \in \Omega_\epsilon,$$

where $G^1_\epsilon, G^0_\epsilon \in L^2((-1, 1) \times S)^3$.

(B8) There exists $C > 0$ such that

$$\frac{1}{\epsilon^2} \int_{\Omega_\epsilon} \|G_\epsilon(x)\|^2 \, \mathrm{d}x < C, \tag{2.5}$$

for each $\epsilon > 0$.

(B9) $\Theta_\epsilon : \Omega_\epsilon \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\Theta_\epsilon(x, \cdot, \cdot)$ is upper semi-continuous for almost all $x \in \Omega_\epsilon$; $\Theta_\epsilon(\cdot, y, z)$ is measurable for all $y, z \in \mathbb{R}$; Θ_ϵ is sublinear in its second variable, for each ϵ.

(B10) There exists $g_1, g_2 \in L^\infty(\Omega_\epsilon)$ nonnegative functions such that

$$|\Theta_\epsilon(x, y, z)| \leq g_1(x) + g_2(x)|z| \tag{2.6}$$

for almost all $x \in \Omega_\epsilon$, for all $z \in \mathbb{R}$, for each $\epsilon > 0$.

Remark 2.1. *From Theorem 3.4 in* [10] *it follows that, for all $\epsilon > 0$, the variational inequality (*2.1*) has at least one solution.*

3. Asymptotic behavior of the solution

To study the asymptotic behavior we use the change of variables $y = y_\epsilon(x)$ given by

$$y_1 = x_1 \quad y' = \frac{x'}{\epsilon} \tag{3.1}$$

which transforms the beam (except the notch) in a cylinder of fixed diameter. This change of variable is classical in the study of asymptotic behavior of variational equalities in thin cylinders or beams (see [5], [11], [13]). We denote by Y_ϵ^-, Y_ϵ^0, Y_ϵ^+, Y_ϵ, and Y_ϵ^1 the images of Ω_ϵ^-, Ω_ϵ^0, Ω_ϵ^+, Ω_ϵ, and Ω_ϵ^1 by the change of variables $y = y_\epsilon(x)$, i.e.

$$Y_\epsilon^- = (-1, -t_\epsilon) \times S, \quad Y_\epsilon^0 = [-t_\epsilon, t_\epsilon] \times (r_\epsilon S), \quad Y_\epsilon^+ = (t_\epsilon, 1) \times S,$$

$$Y_\epsilon = Y_\epsilon^- \cup Y_\epsilon^0 \cup Y_\epsilon^+, \quad Y_\epsilon^1 = Y_\epsilon^- \cup Y_\epsilon^+.$$

Denote by Y^-, Y^+, and Y^1 the "limits"of Y_ϵ^-, Y_ϵ^+, and Y_ϵ^1, i.e.

$$Y^- = (-1, 0) \times S, \quad Y^+ = (0, 1) \times S, \quad Y^1 = Y^- \cup Y^+.$$

Note that Y_ϵ^1 is contained in its limit Y^1.

The two bases of the beam Γ_ϵ^- and Γ_ϵ^+ are transformed to Λ^- and Λ^+, respectively, where

$$\Lambda^- = \{-1\} \times S \quad \text{and} \quad \Lambda^+ = \{1\} \times S.$$

Γ_ϵ transforms to $\Lambda = \Lambda^- \cup \Lambda^+$, which doesn't depend on ϵ.

Let $U_\epsilon \in M_\epsilon$ be the solution of the variational inequality (2.1). Define $u_\epsilon \in K_\epsilon$ by

$$u_\epsilon(y) = U_\epsilon(y_\epsilon^{-1}(y)) \quad \text{a.e. } y \in Y_\epsilon, \tag{3.2}$$

K_ϵ being the image of M_ϵ. K_ϵ is a closed, convex, nonempty cone in $\mathcal{D}_\epsilon$, with $\mathcal{D}_\epsilon = \{v \in H^1(Y_\epsilon) \mid v = 0 \text{ on } \Lambda\}$. We need the following two assumptions

(B11) There exists a nonempty, convex cone K in $H^1(Y^1)$ such that
(i) $K \cap H^1((-1,0) \cup (0,1)) \neq \emptyset$;
(ii) $\epsilon_i \to 0$, $u_{\epsilon_i} \in K_{\epsilon_i}$, $u \in H^1((-1,0) \cup (0,1))$, $u_{\epsilon_i} \rightharpoonup u$ (weakly) in $H^1(Y^1)$
imply $u \in K$.

(B12) There exists a nonempty, convex cone L in $L^2((-1,1); H^1(S))$ such that $\epsilon_i \to 0$, $w_{\epsilon_i} \in K_{\epsilon_i}$, $w \in L^2((-1,1); H^1(S))$, $w_{\epsilon_i} \rightharpoonup w$ (weakly) in $L^2((-1,1); H^1(S))$ imply $w \in L$.

By change of variables $y = y_\epsilon(x)$ the operator ∇ transforms to

$$\nabla^\epsilon \cdot = \left(\frac{\partial \cdot}{\partial y_1}, \frac{1}{\epsilon} \frac{\partial \cdot}{\partial y_2}, \frac{1}{\epsilon} \frac{\partial \cdot}{\partial y_3} \right). \tag{3.3}$$

Using the change of variables $y = y_\epsilon(x)$, given by (3.1), the inequality (2.1) transforms to

$$\begin{aligned} &\int_{Y_\epsilon} \left[A_\epsilon(y_\epsilon^{-1}(y)) \Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y)) \nabla^\epsilon u_\epsilon(y)), \nabla^\epsilon(v_\epsilon(y) - u_\epsilon(y)) \right] \, dy \\ &+ \int_{Y_\epsilon} \Psi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), \nabla^\epsilon u_\epsilon(y))(v_\epsilon(y) - u_\epsilon(y)) \, dy \\ &+ \int_{Y_\epsilon} \left[G_\epsilon(y_\epsilon^{-1}(y)), \nabla^\epsilon(v_\epsilon(y) - u_\epsilon(y)) \right] \, dy \\ &+ \int_{Y_\epsilon} \Theta_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), v_\epsilon(y) - u_\epsilon(y)) \, dy \geq 0, \end{aligned} \tag{3.4}$$

for all $v_\epsilon \in K_\epsilon$, where $v_\epsilon(y) = V_\epsilon(y_\epsilon^{-1}(y))$ a. e. $y \in Y_\epsilon$.

Lemma 3.1. *Assume that (B4) holds, $U_\epsilon \in M_\epsilon$, and $u_\epsilon \in K_\epsilon$ is given by (3.2). Then there exist $C_1, C_2 > 0$ and $C_3 \in \mathbb{R}$ such that*

$$\int_{Y_\epsilon} \left[A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y)), \nabla^\epsilon u_\epsilon(y)\right] \mathrm{d}y$$
$$+ \int_{Y_\epsilon} \Psi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), \nabla^\epsilon u_\epsilon(y))u_\epsilon(y) \mathrm{d}y \tag{3.5}$$
$$\geq C_1\|\nabla^\epsilon u_\epsilon\|^2_{L^2(Y_\epsilon)} - C_2\|u_\epsilon\|^{q_1}_{L^2(Y_\epsilon)} - C_3$$

Proof. Putting $\eta = U_\epsilon(x)$ and $\xi = \nabla U_\epsilon(x)$ in coercivity condition (2.2), integrating on Ω_ϵ we get

$$\int_{\Omega_\epsilon} [A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x), B_\epsilon(x)\nabla U_\epsilon(x)), \nabla U_\epsilon(x)] \mathrm{d}x$$
$$+ \int_{\Omega_\epsilon} \Psi_\epsilon(x, U_\epsilon(x), \nabla U_\epsilon(x))U_\epsilon(x) \mathrm{d}x$$
$$\geq C_1 \int_{\Omega_\epsilon} \|\nabla U_\epsilon(x)\|^2 \mathrm{d}x - C_2 \int_{\Omega_\epsilon} |U_\epsilon(x)|^{q_1} \mathrm{d}x - |\Omega_\epsilon|\|k_1\|_\infty.$$

Multiplying by $\frac{1}{\epsilon^2}$ and using the change of variables $y = y_\epsilon(x)$, given by (3.1), we obtain

$$\int_{Y_\epsilon} [A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y)), \nabla^\epsilon u_\epsilon(y)] \mathrm{d}y$$
$$+ \int_{Y_\epsilon} \Psi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), \nabla^\epsilon u_\epsilon(y))u_\epsilon(y) \mathrm{d}y$$
$$\geq C_1 \int_{Y_\epsilon} \|\nabla^\epsilon u_\epsilon(y)\|^2 \mathrm{d}y - C_2 \int_{Y_\epsilon} |u_\epsilon(y)|^{q_1} \mathrm{d}y - \overline{k}_1$$
$$\geq C_1\|\nabla^\epsilon u_\epsilon\|^2_{L^2(Y_\epsilon)} - C_2\|u_\epsilon\|^{q_1}_{L^{q_1}(Y_\epsilon)} - \overline{k}_1,$$

as $q_1 < 2$. □

Lemma 3.2. *Assume that (B5) holds and let $v_\epsilon \in K_\epsilon$, $(v_\epsilon)_\epsilon$ bounded in $H^1(Y_\epsilon)$. Then the following properties hold*

a) There exist k_1, k_2, and k_3 constants such that

$$\int_{Y_\epsilon} \left[A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y)), \nabla^\epsilon v_\epsilon(y)\right] \mathrm{d}y \tag{3.6}$$
$$\leq k_1\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)} + k_2\|u_\epsilon\|_{L^2(Y_\epsilon)} + k_3.$$

b) There exists k_4, k_5, and k_6 such that

$$\int_{Y_\epsilon} \Psi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), \nabla^\epsilon u_\epsilon(y))v_\epsilon(y) \mathrm{d}y \leq k_4\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)} + k_5\|u_\epsilon\|_{L^2(Y_\epsilon)} + k_6. \tag{3.7}$$

Proof. a) Applying the Cauchy-Schwarz inequality and then the growth condition (2.3) for $x = y_\epsilon^{-1}(y)$ we get

$$\int_{Y_\epsilon} \left[A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y)), \nabla^\epsilon v_\epsilon(y)\right] \, \mathrm{d}y$$
$$\leq \int_{Y_\epsilon} \|A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y))\|\|\nabla^\epsilon v_\epsilon(y)\| \, \mathrm{d}y$$
$$\leq \int_{Y_\epsilon} \left(C\|\nabla^\epsilon u_\epsilon(y)\| + C|u_\epsilon(y)| + \overline{\alpha}(y_\epsilon^{-1}(y))\right) \|\nabla^\epsilon v_\epsilon(y)\| \, \mathrm{d}y$$

(by Cauchy-Schwarz inequality)

$$\leq \left(C\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)} + C\|u_\epsilon\|_{L^2(Y_\epsilon)} + \overline{\alpha}\right) \|\nabla^\epsilon v_\epsilon\|_{L^2(Y_\epsilon)},$$

as $(v_\epsilon)_\epsilon$ is bounded.

b) Using the growth condition (2.4) for $x = y_\epsilon^{-1}(y)$ and the Cauchy-Schwarz inequality, we get

$$\int_{Y_\epsilon} \Psi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), \nabla^\epsilon u_\epsilon(y))v_\epsilon(y) \, \mathrm{d}y$$
$$\leq \int_{Y_\epsilon} \left(C\|\nabla^\epsilon u_\epsilon(y)\| + C|u_\epsilon(y)| + \beta(y_\epsilon^{-1}(y))\right) |v_\epsilon(y)| \, \mathrm{d}y$$
$$\leq \left(C\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)} + C\|u_\epsilon\|_{L^2(Y_\epsilon)} + \overline{\beta}\right) \|v_\epsilon\|_{L^2(Y_\epsilon)},$$

as $(v_\epsilon)_\epsilon$ is bounded. □

Lemma 3.3. *If assumption (B10) is satisfied, $U_\epsilon, V_\epsilon \in M_\epsilon$, u_ϵ and v_ϵ are given by (3.2), then there exist $\bar{g}_1, \bar{g}_2 \in \mathbb{R}$ such that*

$$\int_{Y_\epsilon} \Theta_\epsilon(u_\epsilon(y), v_\epsilon(y) - u_\epsilon(y)) \, \mathrm{d}y \leq \bar{g}_1 + \bar{g}_2\|v_\epsilon - u_\epsilon\|_{L^2(Y_\epsilon)}.$$

Proof. Putting $y = U_\epsilon(x)$ and $z = V_\epsilon(x) - U_\epsilon(x)$ in (2.6), multiplying by $\frac{1}{\epsilon^2}$, then integrating over Ω_ϵ, we obtain

$$\frac{1}{\epsilon^2}\int_{\Omega_\epsilon} \Theta_\epsilon(U_\epsilon(x), V_\epsilon(x) - U_\epsilon(x)) \, \mathrm{d}x \leq \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} |\Theta_\epsilon(U_\epsilon(x), V_\epsilon(x) - U_\epsilon(x))| \, \mathrm{d}x$$
$$\leq \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} (g_1(x) + g_2(x)|V_\epsilon(x) - U_\epsilon(x)|) \, \mathrm{d}x$$
$$\leq \bar{g}_1\frac{|\Omega_\epsilon|}{\epsilon^2} + \frac{1}{\epsilon^2}\bar{g}_2\int_{\Omega_\epsilon} |V_\epsilon(x) - U_\epsilon(x)| \, \mathrm{d}x,$$

where $\bar{g}_1 = \|g_1\|_\infty$ and $\bar{g}_2 = \|g_2\|_\infty$. Using the change of variable y_ϵ, the result follows. □

Lemma 3.4. *Let $U_\epsilon \in M_\epsilon$ be the solution of the variational inequality (2.1) and $u_\epsilon \in K_\epsilon$ defined by*

$$u_\epsilon(y) = U_\epsilon(y_\epsilon^{-1}(y)) \quad a.e.\ y \in Y_\epsilon.$$

If assumptions (B1)-(B10) are verified then the following statements hold

2) $(u_\epsilon)_\epsilon$ is bounded in $H^1(Y_\epsilon)$;

1) $\left(\frac{1}{\epsilon}\frac{\partial u_\epsilon}{\partial y_2}\right)_\epsilon$ *and* $\left(\frac{1}{\epsilon}\frac{\partial u_\epsilon}{\partial y_3}\right)_\epsilon$ *are bounded in* $L^2(Y_\epsilon)$*;*

3) $(\sigma_\epsilon)_\epsilon$ *is bounded in* $L^2(Y_\epsilon)^3$*, where*

$$\sigma_\epsilon(y) = A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y)) \quad a.e. \quad y \in Y_\epsilon.$$

Proof. Suppose that $(v_\epsilon)_\epsilon$ is bounded in $H^1(Y_\epsilon)$. From coercivity condition (B4) by Lemma 3.1, then inequality (3.4), we obtain

$$\begin{aligned}
&C_1\|\nabla^\epsilon u_\epsilon\|^2_{L^2(Y_\epsilon)} - C_2\|u_\epsilon\|^{q_1}_{L^2(Y_\epsilon)} - C_3\\
&\leq \int_{Y_\epsilon}\left[A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y)), \nabla^\epsilon u_\epsilon(y)\right]\,\mathrm{d}y\\
&+\int_{Y_\epsilon}\Psi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), \nabla^\epsilon u_\epsilon(y))u_\epsilon(y)\,\mathrm{d}y\\
&\leq \int_{Y_\epsilon}\left[A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y)), \nabla^\epsilon v_\epsilon(y)\right]\,\mathrm{d}y\\
&+\int_{Y_\epsilon}\Psi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), \nabla^\epsilon u_\epsilon(y))v_\epsilon(y)\,\mathrm{d}y\\
&+\int_{Y_\epsilon}\left[G_\epsilon(y_\epsilon^{-1}(y)), \nabla^\epsilon v_\epsilon(y) - \nabla^\epsilon u_\epsilon(y)\right]\,\mathrm{d}y\\
&+\int_{Y_\epsilon}\Theta_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), v_\epsilon(y) - u_\epsilon(y))\,\mathrm{d}y \leq
\end{aligned}$$

(using Lemma 3.2 for the first two terms, the Cauchy-Schwarz inequality and then assumption (2.5) for the third term, assumption (2.6) for the fourth term)

$$\begin{aligned}
&\leq k_1\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)} + k_2\|u_\epsilon\|_{L^2(Y_\epsilon)}\\
&+ C\|\nabla^\epsilon v_\epsilon - \nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)} + c_1'\|v_\epsilon - u_\epsilon\|_{L^2(Y_\epsilon)} + k\\
&\leq c_1\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)} + c_2,
\end{aligned}$$

using the Poincaré inequality , where c_1 and c_2 are constants. On the other hand

$$\begin{aligned}
&C_1\|\nabla^\epsilon u_\epsilon\|^2_{L^2(Y_\epsilon)} - C_2\|u_\epsilon\|^{q_1}_{L^2(Y_\epsilon)} - C_3\\
&\geq c_3\|\nabla^\epsilon u_\epsilon\|^2_{L^2(Y_\epsilon)} - c_4\|\nabla^\epsilon u_\epsilon\|^{q_1}_{L^2(Y_\epsilon)} - c_5,
\end{aligned}$$

by the Poincaré inequality , where c_3, c_4, and c_5 are constants. Thus

$$c_3\|\nabla^\epsilon u_\epsilon\|^2_{L^2(Y_\epsilon)} \leq c_1\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)} + c_4\|\nabla^\epsilon u_\epsilon\|^{q_1}_{L^2(Y_\epsilon)} + c_6,$$

where c_6 is a constant, $q_1 < 2$, and $c_3 > 0$.

It follows that, for $\epsilon \leq 1$, $\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)}$ is bounded.

Then $\left(\frac{1}{\epsilon}\frac{\partial u_\epsilon}{\partial y_2}\right)_\epsilon$ and $\left(\frac{1}{\epsilon}\frac{\partial u_\epsilon}{\partial y_3}\right)_\epsilon$ are bounded in $L^2(Y_\epsilon)$. Using

$$\|\nabla u_\epsilon\|_{L^2(Y_\epsilon)} \leq \|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)},$$

we get that $(u_\epsilon)_\epsilon$ is bounded in $H^1(Y_\epsilon)$, so 2) is true.

To prove 3), we take the square of the first inequality of (B5) and we obtain

$$\|A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x), B_\epsilon(x)\nabla U_\epsilon(x))\|^2 \leq C\|\nabla U_\epsilon(x)\|^2 + C|U_\epsilon(x)|^2 + |\alpha(x)|^2$$

for a.e. $x \in \Omega_\epsilon$.

Multiplying by $\frac{1}{\epsilon^2}$ and integrating on Ω_ϵ we get

$$\begin{aligned}
&\frac{1}{\epsilon^2}\int_{\Omega_\epsilon} \|A_\epsilon(x)\Phi_\epsilon(x, U_\epsilon(x), B_\epsilon(x)\nabla U_\epsilon(x))\|^2 \, \mathrm{d}x \\
&\leq \frac{C}{\epsilon^2}\int_{\Omega_\epsilon} \|\nabla U_\epsilon(x)\|^2 \, \mathrm{d}x + \frac{C}{\epsilon^2}\int_{\Omega_\epsilon} |U_\epsilon(x)|^2 \, \mathrm{d}x + \frac{1}{\epsilon^2}\int_{\Omega_\epsilon} |\alpha|^2 \, \mathrm{d}x \\
&\leq \frac{C}{\epsilon^2}\int_{\Omega_\epsilon} \|\nabla U_\epsilon(x)\|^2 \, \mathrm{d}x + \frac{C}{\epsilon^2}\int_{\Omega_\epsilon} |U_\epsilon(x)|^2 \, \mathrm{d}x + \frac{|\Omega_\epsilon|}{\epsilon^2}\bar{\alpha},
\end{aligned}$$

where $\bar{\alpha}$ is a constant. Using the change of variables $y = y_\epsilon(x)$, we get

$$\begin{aligned}
&\int_{Y_\epsilon} \|A_\epsilon(y_\epsilon^{-1}(y))\Phi_\epsilon(y_\epsilon^{-1}(y), u_\epsilon(y), B_\epsilon(y_\epsilon^{-1}(y))\nabla^\epsilon u_\epsilon(y))\|^2 \, \mathrm{d}y \\
&\leq C\int_{Y_\epsilon} \|\nabla^\epsilon u_\epsilon(y)\|^2 \, \mathrm{d}y + C\int_{Y_\epsilon} |u_\epsilon(y)|^2 \, \mathrm{d}y + \bar{\alpha},
\end{aligned}$$

which can be written as

$$\begin{aligned}
&\|A_\epsilon(y_\epsilon^{-1}(\cdot))\Phi_\epsilon(y_\epsilon^{-1}(\cdot), u_\epsilon, B_\epsilon(y_\epsilon^{-1}(\cdot))\nabla^\epsilon u_\epsilon)\|^2_{L^2(Y_\epsilon)} \\
&\leq C\|\nabla^\epsilon u_\epsilon\|^2_{L^2(Y_\epsilon)} + C\|u_\epsilon(y)\|^2_{L^2(Y_\epsilon)} + \bar{\alpha} \leq \bar{C},
\end{aligned}$$

as $\|\nabla^\epsilon u_\epsilon\|_{L^2(Y_\epsilon)}$ and $\|u_\epsilon\|_{L^2(Y_\epsilon)}$ are bounded. It follows that $(\sigma_\epsilon)_\epsilon$ is bounded in $L^2(Y_\epsilon)$. □

Corollary 3.5. *Let $U_\epsilon \in M_\epsilon$ be the solution of the inequality (2.1) and $u_\epsilon \in K_\epsilon$ given by (3.2). If assumptions (B1) - (B10) are verified then the sequence U_ϵ satisfies*

$$U_\epsilon \in M_\epsilon, \quad \frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon} |\nabla U_\epsilon|^2 dx \leq C. \tag{3.8}$$

Proof. By Lemma 3.4 we get that $(\nabla^\epsilon u_\epsilon)_\epsilon$ is bounded in $L^2(Y_\epsilon)$, i.e. there exists $C > 0$ such that

$$\int_{Y_\epsilon} \|\nabla^\epsilon u_\epsilon(y)\|^2 \, \mathrm{d}y \leq C.$$

Using the change of variables $x = y_\epsilon^{-1}(y)$, we get

$$\frac{1}{\epsilon^2}\int_{\Omega_\epsilon} \|\nabla^\epsilon U_\epsilon(x)\|^2 \, \mathrm{d}x < C,$$

from where the statement of the corollary follows, as

$$|\Omega_\epsilon| = 2\pi|S|^2\epsilon^2(1 - t_\epsilon + t_\epsilon r_\epsilon^2).$$

□

Theorem 3.6. *Let U_ϵ be the solution of the variational inequality (2.1) and $u_\epsilon \in K_\epsilon$ defined by*

$$u_\epsilon(y) = U_\epsilon(y_\epsilon^{-1}(y)) \quad a.e.\ y \in Y_\epsilon.$$

If assumptions (B1)-(B12) are verified, then there exist three functions u, w, and σ^1 with

$$u \in H^1((-1,0)\cup(0,1)) \cap K, \quad u(-1) = u(1) = 0,$$
$$w \in L, \quad \sigma^1 \in L^2(Y^1)^3,$$

such that up to extraction of a subsequence

$$\chi_{Y_\epsilon^1} u_\epsilon \to u \quad in \quad L^2(Y^1); \tag{3.9}$$
$$\chi_{Y_\epsilon^-} \frac{\partial u_\epsilon}{\partial y_1} \rightharpoonup \frac{\partial u}{\partial y_1} \quad in \quad L^2(Y^-);$$
$$\chi_{Y_\epsilon^+} \frac{\partial u_\epsilon}{\partial y_1} \rightharpoonup \frac{\partial u}{\partial y_1} \quad in \quad L^2(Y^+);$$
$$\chi_{Y_\epsilon^1} \frac{1}{\epsilon} \nabla_{y'} u_\epsilon \rightharpoonup \nabla_{y'} w \quad in \quad L^2(Y^1)^2;$$

and

$$\chi_{Y_\epsilon^1} \sigma_\epsilon \rightharpoonup \sigma^1 \quad in \quad L^2(Y^1)^3.$$

Proof. From Lemma 3.4 it follows that there exist three functions $u \in H^1((-1,0)\cup(0,1))$, $w \in L^2((-1,1); H^1(S))$, and $\sigma^1 \in L^2(Y^1)^3$, which satisfy the statement of the lemma. From assumption (B11) we get that $u \in H^1((-1,0)\cup(0,1)) \cap K$, and from (B12) we obtain that $w \in L$. □

Theorem 3.7. *Let U_ϵ be the solution of the variational inequality (2.1) and $u \in H^1((-1,0)\cup(0,1))\cap K$ given in Theorem 3.6. If assumptions (B1)-(B11) are verified, then there exists a subsequence of solutions U_ϵ, also denoted by U_ϵ, such that*

$$\lim_{\epsilon\to 0} \frac{1}{|\Omega_\epsilon|} \int_{\Omega_\epsilon} |U_\epsilon(x) - u(x_1)|^2 \, dx = 0. \tag{3.10}$$

Proof. Let $u_\epsilon \in K_\epsilon$ given by (3.2). From Theorem 3.6 follows that there exists u with

$$u \in H^1((-1,0)\cup(0,1)) \cap K, \quad u(-1) = u(1) = 0,$$

such that up to extraction of a subsequence

$$\chi_{Y_\epsilon^1} u_\epsilon \to u \quad \text{in} \quad L^2(Y^1),$$

which is equivalent with

$$\int_{Y_\epsilon} |u_\epsilon(y) - u(y_1)|^2 \, dy = 0.$$

Using the change of variables $x = y_\epsilon^{-1}(y)$, we get (3.10). □

References

[1] Bruno, P., *Geometrically constrained magnetic wall*, Phys. Rev. Lett., **83**(1999), 2425-2428.

[2] Cabib, E., Freddi, L., Morassi, A., Percivale, D., *Thin notched beams*, J. Elasticity, **64**(2002), 157-178.

[3] Casado-Díaz, J., Luna-Laynez, M., Murat, F., *Asymptotic behavior of diffusion problems in a domain made of two cylinders of different diameters and lengths*, C. R. Acad. Sci. Paris, Sér. I, **338**(2004), 133-138.

[4] Casado-Díaz, J., Luna-Laynez, M., Murat, F., *The diffusion equation in a notched beam*, Calculus of Variations, **31**(2008), 297-323.

[5] Gustafsson, B., Mossino, J., *Non-periodic explicit homogenization and reduction of dimension: the linear case*, IMA Journal of Applied Mathematics, **68**(2003), 269-298.

[6] Hale, J. K., Vegas, J., *A nonlinear parabolic equation with varying domain*, Arch. Rat. Mech. Anal., **86**(1984), 99-123.

[7] Jimbo, S., *Singular perturbation of domains and semilinear elliptic equation*, J. Fac. Sci. Univ. Tokyo, **35**(1988), 27-76.

[8] Jimbo, S., *Singular perturbation of domains and semilinear elliptic equation 2*, J. Diff. Eq., **75**(1988), 264-289.

[9] Kohn, R. V., Slastikov, V. V., *Geometrically Constrained Walls*, Calculus of Variations and Partial Differential Equations, **28**(2007), no. 1, 33-57.

[10] Kolumbán, J., Marchis, I., Szász, T., *Homogenization and reduction of dimension for nonlinear parametric variational inequalities*, Nonlinear Analysis-Theory Methods & Applications, **71**(2009), no. 3-4, 819-828.

[11] Murat, F., Sili, A., *Problèmes monotones dans des cylindres de faible diamètre formés de matériaux hétérogènes*, C. R. Acad. Sci. de Paris, Série 1, **320**(1995), 1199-1204.

[12] Rubinstein, J., Schatzman, M., Sternberg, P., *Ginzburg-Landau model in thin loops with narrow contrictions*, SIAM J. Appl. Math., **64**(2004), 2186-2204.

[13] Sili, A., *Asymptotic behavior of the solutions of monotone problems in flat cylinders*, Asymptotic Analysis, **19**(1999), 19-33.

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The generalized semi-normed difference of double gai sequence spaces defined by a modulus function

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Abstract. In this paper we introduce generalized semi normed difference of double gai sequence spaces defined by a modulus function. We study their different properties and obtain some inclusion relations involving these semi normed difference double gai sequence spaces.

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1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences $(x_{m,n})$, where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces were found in Bromwich [5]. Later on, they were investigated by Hardy [16], Moricz [24], Moricz and Rhoades [25], Basarir and Solankan [3], Tripathy [42], Colak and Turkmenoglu [8], Turkmenoglu [44], and many others.

Let us define the following sets of double sequences

$$\mathcal{M}_u(t) := \left\{ (x_{m,n}) \in w^2 : \sup_{m,n \in N} |x_{m,n}|^{t_{m,n}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{m,n}) \in w^2 : p - \lim_{m,n \to \infty} |x_{m,n} - l|^{t_{m,n}} = 1 \text{ for some } l \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{m,n}) \in w^2 : p - \lim_{m,n \to \infty} |x_{m,n}|^{t_{m,n}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{m,n}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{m,n}|^{t_{m,n}} < \infty \right\},$$
$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t),$$

where $t = (t_{m,n})$ is the sequence of strictly positive reals $t_{m,n}$ for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{m,n} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gŏkhan and Colak [14,15] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [46] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [27] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [28] and Mursaleen and Edely [29] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{j,k})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [2] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also have examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [7] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and have examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [36,40] have studied the space $\chi^2_M(p, q, u)$ and the generalized gai of double sequences and have given some inclusion relations.

We need the following inequality in the sequel of the paper. For $a, b \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p. \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{m,n}$ is called convergent if and only if the double sequence $(s_{m,n})$ is convergent, where $s_{m,n} = \sum_{i,j=1}^{m,n} x_{i,j} (m, n \in \mathbb{N})$ (see[1]).

A sequence $x = (x_{m,n})$ is said to be double analytic if

$$\sup_{m,n} |x_{m,n}|^{1/(m+n)} < \infty.$$

The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{m,n})$ is called double entire sequence if $|x_{m,n}|^{1/(m+n)} \to 0$ as $m, n \to \infty$. The double entire sequences will be denoted by Γ^2. A sequence $x = (x_{m,n})$ is called double gai sequence if $((m + n)! |x_{m,n}|)^{1/(m+n)} \to 0$ as

$m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let ϕ denote the set of all finite sequences.

Consider a double sequence $x = (x_{i,j})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{i,j} \Im_{i,j}$ for all $m, n \in \mathbb{N}$, where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{m,n})$ $(m, n \in \mathbb{N})$ are also continuous.

Orlicz [32] used the idea of Orlicz function to construct the space $\left(L^M\right)$. Lindenstrauss and Tzafriri [21] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p \left(1 \leq p < \infty\right)$. subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [33], Mursaleen et al. [26], Bektas and Altin [4], Tripathy et al. [43], Rao and Subramanian [9], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [17].

Recalling [32] and [17], an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [31] and further discussed by Ruckle [34] and Maddox [23], and many others.

An Orlicz function M is said to satisfy the Δ_2- condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)\,(u \geq 0)$. The Δ_2- condition is equivalent to $M(\ell u) \leq K\ell M(u)$, for all values of u and for $\ell > 1$.

Lindenstrauss and Tzafriri [21] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \, for\, some\, \rho > 0\right\}.$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p \left(1 \leq p < \infty\right)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

If X is a sequence space, we give the following definitions

(i) $X^{'}=$ the continuous dual of X;

(ii) $X^{\alpha} = \left\{a = (a_{m,n}) : \sum_{m,n=1}^{\infty} |a_{m,n}x_{m,n}| < \infty, \text{ for each } x \in X\right\}$;

(iii) $X^{\beta} = \left\{a = (a_{m,n}) : \sum_{m,n=1}^{\infty} a_{m,n}x_{m,n} \text{ is convergent, for each } x \in X\right\}$;

(iv) $X^{\gamma} = \left\{a = (a_{m,n}) : \sup_{m,n} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{m,n}x_{m,n}\right| < \infty, \text{ for each } x \in X\right\}$;

(v) let X be an FK-space $\supset \phi$; then $X^{f} = \left\{ f(\Im_{m,n}) : f \in X^{'} \right\}$;

(vi) $X^{\delta} = \left\{ a = (a_{m,n}) : \sup_{m,n} |a_{m,n}x_{m,n}|^{1/(m+n)} < \infty, \text{ for each } x \in X \right\}$;

X^{α}, X^{β}, X^{γ} are called α- (or Köthe-Toeplitz) dual of X, β- (or generalized-Köthe-Toeplitz) dual of X, γ-dual of X, δ - dual of X respectively. X^{α} is defined by Gupta and Kamptan [18]. It is clear that $x^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\alpha} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [19] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_o$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here w, c, c_o and ℓ_{∞} denote the classes of all, convergent,null and bounded scalar valued single sequences respectively. The above spaces are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k|.$$

The notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{ x = (x_{m,n}) \in w^2 : (\Delta x_{m,n}) \in Z \right\}$$

where $Z = \Lambda^2, \Gamma^2$ and χ^2 respectively.

$$\Delta x_{m,n} = (x_{m,n} - x_{m,n+1}) - (x_{m+1,n} - x_{m+1,n+1})$$

$$= x_{m,n} - x_{m,n+1} - x_{m+1,n} + x_{m+1,n+1}$$

for all $m, n \in \mathbb{N}$.

Let $r \in \mathbb{N}$ be fixed, then

$$Z(\Delta^r) = \{(x_{m,n}) : (\Delta^r x_{m,n}) \in Z\} \; for \, Z = \chi^2, \Gamma^2 \, and \, \Lambda^2$$

where $\Delta^r x_{m,n} = \Delta^{r-1} x_{m,n} - \Delta^{r-1} x_{m,n+1} - \Delta^{r-1} x_{m+1,n} + \Delta^{r-1} x_{m+1,n+1}$.

Now we introduced a generalized difference double operator as follows.

Let $r, \gamma \in \mathbb{N}$ be fixed. Then

$$Z\left(\Delta^r_{\gamma}\right) = \left\{ (x_{m,n}) : \left(\Delta^r_{\gamma} x_{m,n}\right) \in Z \right\} \; for \, Z = \chi^2, \Gamma^2 \, and \, \Lambda^2,$$

where $\Delta^r_{\gamma} x_{m.n} = \Delta^{r-1}_{\gamma} x_{m,n} - \Delta^{r-1}_{\gamma} x_{m,n+1} - \Delta^{r-1}_{\gamma} x_{m+1,n} + \Delta^{r-1}_{\gamma} x_{m+1,n+1}$ and $\Delta^0_{\gamma} x_{m,n} = x_{m,n}$ for all $m, n \in \mathbb{N}$.

The notion of a modulus function was introduced by Nakano [31]. We recall that a modulus f is a function from $[0, \infty) \to [0, \infty)$, such that

(1) $f(x) = 0$ if and only if $x = 0$

(2) $f(x + y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$,

(3) f is increasing,

(4) f is right-continuous at $x = 0$.

Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from condition (4) that f is continuous on $[0, \infty)$.

Also from condition (2), we have $f(nx) \leq nf(x)$ for all $n \in \mathbb{N}$ and $n^{-1} f(x) \leq f\left(xn^{-1}\right)$, for all $n \in \mathbb{N}$.

2. Remark

If f is a modulus function, then the composition $f^s = f \cdot f \cdots f$ ($s\, times$) is also a modulus function, where s is a positive integer.

Let $p = (p_{m,n})$ be a sequence of positive real numbers. We have the following well known inequality, which will be used throughout this paper

$$|a_{m,n} + b_{m,n}|^{p_{m,n}} \leq D \left(|a_{m,n}|^{p_{m,n}} + |b_{m,n}|^{p_{m,n}}\right), \tag{2.1}$$

where $a_{m,n}$ and $b_{m,n}$ are complex numbers, $D = \max\left\{1, 2^{H-1}\right\}$ and $H = \sup_{m,n} p_{m,n} < \infty$.

Spaces of strongly summable sequences were studied at the initial stage by Kuttner [20], Maddox [30] and others. The class of sequences those are strongly Cesàro summable with respect to a modulus was introduced by Maddox [23] as an extension of the definition of strongly Cesàro summable sequences. Cannor [10] further extended this definition to a definition of strongly $A-$ summability with respect to a modulus when A is non-negative regular matrix.

Let $\eta = (\lambda_i)$ be a non-decreasing sequence of positive real numbers tending to infinity and $\lambda_1 = 1$ and $\lambda_{i+1} \leq \lambda_{i+1} + 1$, for all $i \in \mathbb{N}$.

The generalized de la Vallee-Poussin means is defined by $t_i(x) = \lambda_i^{-1} \sum_{k \in I_i} x_k$, where $I_i = [i - \lambda_i + 1, i]$. A sequence $x = (x_k)$ is said to be $(V, \lambda)-$ summable to a number L if $t_i(x) \to L$, as $i \to \infty$ (see [22]).

3. Definitions and preliminaries

Let w^2 denote the set of all complex double sequences. A sequence $x = (x_{m,n})$ is said to be double analytic if $\sup_{m,n} |x_{m,n}|^{1/(m+n)} < \infty$. The vector space of all prime sense double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{m,n})$ is called prime sense double entire sequence if $|x_{m,n}|^{1/(m+n)} \to 0$ as $m, n \to \infty$. The double entire sequences will be denoted by Γ^2. The spaces Λ^2 and Γ^2 are metric spaces with the metric

$$d(x, y) = \sup_{m,n} \left\{|x_{m,n} - y_{m,n}|^{1/(m+n)} : m, n : 1, 2, 3, ...\right\}, \tag{3.1}$$

for all $x = (x_{m,n})$ and $y = (y_{m,n})$ in Γ^2.

A sequence $x = (x_{m,n})$ is called prime sense double gai sequence if $\left((m+n)! \, |x_{m,n}|\right)^{1/(m+n)} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. The space χ^2 is a metric space with the metric

$$\widetilde{d}(x, y) = \sup_{m,n} \left\{\left((m+n)! \, |x_{m,n} - y_{m,n}|\right)^{1/(m+n)} : m, n : 1, 2, 3, ...\right\}, \tag{3.2}$$

for all $x = (x_{m,n})$ and $y = (y_{m,n})$ in χ^2.

Throughout the article E will represent a semi normed space, semi normed by q. We define $w^2(E)$ to be the vector space of all $E-$ valued sequences. Let f be a modulus function $p = (p_{m,n})$ be any sequence of positive real numbers. Let $A = \left(a^{j,k}_{m,n}\right)$ be four dimensional infinite regular matrix of non-negative complex numbers such that $\sup_{j,k} \sum_{m,n=1}^{\infty} a^{j,k}_{m,n} < \infty$.

We define the following sets of sequences

$$\left[V^E_\lambda, A, \Delta^r_\gamma, f, p\right]_{\chi^2}$$

$$=\left\{x \in w^2(E): \lim_{p,q\to\infty} \lambda^{-1}_{pq} \sum_{m,n\in I_{p,q}} a^{j,k}_{m,n}[f(q((m+n)!\Delta^r_\gamma x_{m,n})^{1/(m+n)})]^{p_{m,n}} = 0\right\}$$

uniformly in m, n,

$$\left[V^E_\lambda, A, \Delta^r_\gamma, f, p\right]_{\Gamma^2}$$

$$=\left\{x \in w^2(E) : \lim_{p,q\to\infty} \lambda^{-1}_{pq} \sum_{m,n\in I_{p,q}} a^{j,k}_{m,n}[f(q(\Delta^r_\gamma x_{m,n})^{1/(m+n)})]^{p_{m,n}} = 0\right\}$$

uniformly in m, n,

$$\left[V^E_\lambda, A, \Delta^r_\gamma, f, p\right]_{\Lambda^2}$$

$$=\left\{x \in w^2(E) : \sup_{j,k} \sup_{p,q} \lambda^{-1}_{pq} \sum_{m,n\in I_{p,q}} a^{j,k}_{m,n}[f(q(\Delta^r_\gamma x_{m,n})^{1/(m+n)})]^{p_{m,n}} < \infty\right\}.$$

For $\gamma = 1$, these spaces are denoted by $\left[V^E_\lambda, A, \Delta^r, f, p\right]_Z$, for $Z = \chi^2$, Γ^2 and Λ^2 respectively. We define

$$\left[V^E_\lambda, A, \Delta^r_\gamma, f, p\right]_{\chi^2}$$

$$=\left\{x \in w^2(E) : \lim_{p,q\to\infty} \lambda^{-1}_{pq} \sum_{m,n\in I_{p,q}} [f(q((m+n)!\Delta^r_\gamma x_{mn})^{1/(m+n)})]^{p_{mn}} = 0\right\}.$$

Similarly $\left[V^E_\lambda, \Delta^r_\gamma, f, p\right]_{\Gamma^2}$ and $\left[V^E_\lambda, \Delta^r_\gamma, f, p\right]_{\Lambda^2}$ can be defined.

For $E = \mathbb{C}$, the set of complex numbers, $q(x) = |x|$; $f(x) = x^{1/(m+n)}$; $p_{m,n} = 1$, for all $m, n \in \mathbb{N}$. For $r = 0, \gamma = 0$ the spaces $\left[V^E_\lambda, \Delta^r_\gamma, f, p\right]_Z$, represent the spaces $[V, \lambda]_Z$, for $Z = \chi^2, \Gamma^2$ and Λ^2. These spaces are called as $\lambda-$ strongly gai to zero, $\lambda-$ strongly entire to zero and $\lambda-$ strongly analytic by the de la Vallée-Poussin method. In the special case, where $\lambda_{pq} = pq$, for all $p, q = 1, 2, 3, \cdots$ the sets $[V, \lambda]_{\chi^2}$, $[V, \lambda]_{\Gamma^2}$ and $[V, \lambda]_{\Lambda^2}$ reduce to the sets $w^2_{\chi^2}$, $w^2_{\Gamma^2}$ and $w^2_{\Lambda^2}$.

4. Main results

Theorem 4.1. *Let the sequence $p = (p_{m,n})$ be bounded. Then the set $\left[V^E_\lambda, A, \Delta^r_\gamma, f, p\right]_Z$ is linear space over the complex field $\mathbb{C}$, for $Z = \chi^2$ and Λ^2.*

The proof is easy, consequently we omit it.

Theorem 4.2. *Let f be a modulus function. One has $\left[V^E_\lambda, A, \Delta^r_\gamma, f, p\right]_{\chi^2} \subset \left[V^E_\lambda, A, \Delta^r_\gamma, f, p\right]_{\Lambda^2}$.*

Proof. Let $x = (x_{m,n}) \in \left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\chi^2}$ will represent a semi normed space, semi normed by q. Here there exists a positive integer M_1 such that $q \leq M_1$. Then we have

$$\lambda_{p,q}^{-1} \sum_{m,n \in I_{p,q}} a_{m,n}^{j,k} \left[f\left(q\left(\Delta_\gamma^r x_{m,n}\right)^{1/(m+n)}\right)\right]^{p_{m,n}}$$

$$\leq D\lambda_{pq}^{-1} \sum_{m,n \in I_{pq}} a_{m,n}^{j,k} \left[f\left(q\left((m+n)!\Delta_\gamma^r x_{m,n}\right)^{1/(m+n)}\right)\right]^{p_{m,n}}$$

$$+D\left(M_1, f(1)\right)^H \lambda_{p,q}^{-1} \sum_{m,n \in I_{pq}} a_{m,n}^{j,k}.$$

Thus $x \in \left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\Lambda^2}$. This completes the proof. □

Theorem 4.3. *Let $p = (p_{m,n}) \in \chi^2$, then $\left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\chi^2}$ is a paranormed space with*

$$g(x) =$$
$$\sup_{p,q}\left(\lambda_{p,q}^{-1} \sum_{m,n \in I_{p,q}} a_{m,n}^{j,k} \left[f\left(q\left((m+n)!\Delta_\gamma^r x_{m,n}\right)^{1/(m+n)}\right)\right]^{p_{m,n}}\right)^{1/H},$$

where $H = \max\left(1, \sup_{m,n} p_{m,n}\right)$.

Proof. From Theorem 4.1, for each $x \in \left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\chi^2}$, $g(x)$ exists. Clearly $g(-x) = g(x)$. It is trivial that $\left((m+n)!\Delta_\gamma^r x_{m,n}\right)^{1/(m+n)} = \theta$ for $x = \overline{\theta}$. Hence, we get $g\left(\overline{\theta}\right) = 0$. By Minkowski inequality, we have $g(x+y) \leq g(x) + g(y)$. Now we show that the scalar multiplication is continuous. Let α be any fixed complex number. By definition of f, we deduce that $x \to \theta$ implies $g(\alpha x) \to 0$. Similarly, we have x fixed and $\alpha \to 0$ implies $g(\alpha x) \to 0$. Finally $x \to \theta$ and $\alpha \to 0$ implies $g(\alpha x) \to 0$. This completes the proof. □

Theorem 4.4. *If $r \geq 1$, then the inclusion*

$$\left[V_\lambda^E, A, \Delta_\gamma^{r-1}, f, p\right]_{\chi^2} \subset \left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\chi^2}$$

is strict. In general

$$\left[V_\lambda^E, A, \Delta_\gamma^j, f, p\right]_{\chi^2} \subset \left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\chi^2}$$

for $j = 0, 1, 2, \cdots r-1$ and the inclusions are strict.

Proof. The result follows from the following inequality

$$\lambda_{pq}^{-1} \sum_{m,n \in I_{pq}} a_{m,n}^{j,k} \left[f\left(q\left((m+n)!\Delta_\gamma^r x_{m,n}\right)^{1/(m+n)}\right)\right]^{p_{m,n}}$$

$$\leq D\,\lambda_{pq}^{-1} \sum_{m,n \in I_{pq}} a_{m,n}^{j,k} \left[f\left(q\left((m+n)!x_{m,n}\right)^{1/(m+n)}\right)\right]^{p_{m,n}}$$

$$+D\,\lambda_{pq}^{-1} \sum_{m,n \in I_{p,q}} a_{m,n}^{j,k} \left[f\left(q\left((m+n+1)!x_{m,n+1}\right)^{1/(m+n+1)}\right)\right]^{p_{m,n}}$$

$$+D\,\lambda_{pq}^{-1}\sum_{m,n\in I_{p,q}} a_{m,n}^{j,k}\left[f\left(q\left((m+1+n)!x_{m+1,n}\right)^{1/(m+1+n)}\right)\right]^{p_{m,n}}$$

$$+D\,\lambda_{p,q}^{-1}\sum_{m,n\in I_{p,q}} a_{m,n}^{j,k}\left[f\left(q\left((m+n+2)!x_{m+1,n+1}\right)^{1/(m+n+2)}\right)\right]^{p_{m,n}}.$$

Proceeding inductively, we have

$$\left[V_\lambda^E, A, \Delta_\gamma^j, f, p\right]_{\chi^2} \subset \left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\chi^2} \text{ for } j = 0, 1, 2, \cdots r-1.$$

The inclusion is strict follows from the following example.

Let $E = \mathbb{C}$, $q(x) = |x|$; $\lambda_{pq} = 1$ for all $p, q \in \mathbb{N}$, $p_{m,n} = 2$ for all $m, n \in \mathbb{N}$. Let $f(x) = x$, for all $x \in [0, \infty)$; $a_{m,n}^{j,k} = m^{-2}n^{-2}$ for all $m, n, j, k \in \mathbb{N}$; $\gamma = 1$, $r \geq 1$. Consider the sequence $x = (x_{m,n})$ defined by $x_{m,n} = \frac{1}{(m+n)!}(mn)^{r(m+n)}$ for all $m, n \in \mathbb{N}$. Hence $(x_{m,n}) \in \left[V_\lambda^C, A, \Delta^r, f, p\right]_{\chi^2}$ but $(x_{m,n}) \notin \left[V_\lambda^C, A, \Delta^{r-1}, f, p\right]_{\chi^2}$. □

Theorem 4.5. *Let f be a modulus function and s be a positive integer. Then,*

$$\left[V_\lambda^E, A, \Delta_\gamma^r, f, q\right]_{\Lambda^2} \subset \left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\Lambda^2}.$$

Proof. Let $\epsilon > 0$ be given and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \leq t \leq \delta$. Write $y_{m,n} = f^{s^{-1}}\left(q\left(\Delta_\gamma^r x_{m,n}\right)^{1/(m+n)} - M\right)$ and consider

$$\sum_{m,n\in I_r} a_{m,n}^{j,k}\left[f(y_{m,n})\right]^{p_{m,n}} = \sum_{m,n\in I_r,\, y_{m,n}\leq\delta} a_{m,n}^{j,k}\left[f(y_{m,n})\right]^{p_{m,n}}$$

$$+\sum_{m,n\in I_r} a_{m,n}^{j,k}\left[f(y_{m,n})\right]^{p_{m,n}}.$$

Since f is continuous, we have

$$\sum_{m,n\in I_r,\, y_{m,n}\leq\delta} a_{m,n}^{j,k}\left[f(y_{m,n})\right]^{p_{m,n}} \leq \epsilon^H \sum_{m,n\in I_r,\, y_{m,n}\leq\delta} a_{m,n}^{j,k} \qquad (4.1)$$

and for $y_{m,n} > \delta$, we use the fact that, $y_{m,n} < \frac{y_{m,n}}{\delta} \leq 1 + \frac{y_{m,n}}{\delta}$ and so, by the definition of f, we have for $y_{m,n} > \delta$,

$$f(y_{m,n}) < 2f(1)\frac{y_{m,n}}{\delta}.$$

Hence

$$\frac{1}{\lambda_{pq}}\sum_{m,n\in I_r,\, y_{m,n}\leq\delta} a_{m,n}^{j,k}\left[f(y_{m,n})\right]^{p_{m,n}}$$

$$\leq \max\left(1, \left(2f(1)\delta^{-1}\right)^H\right)\frac{1}{\lambda_{pq}}\sum_{m,n\in I_r,\, y_{m,n}\leq\delta} a_{m,n}^{j,k} y_{m,n}^{p_{m,n}}. \qquad (4.2)$$

From (4.1) and (4.2) we obtain $\left[V_\lambda^E, A, \Delta_\gamma^r, f, q\right]_{\Lambda^2} \subset \left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\Lambda^2}$. This completes the proof. □

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References

[1] Apostol, T., *Mathematical Analysis*, Addison-Wesley , London, 1978.

[2] Altay, B., Basar, F., *Some new spaces of double sequences*, J. Math. Anal. Appl., **309**(2005), no. 1, 70-90.

[3] Basarir, M., Solancan, O., *On some double sequence spaces*, J. Indian Acad. Math., **21**(1999), no. 2, 193-200.

[4] Bektas, C., Altin, Y., *The sequence space $\ell_M(p,q,s)$ on seminormed spaces*, Indian J. Pure Appl. Math., **34**(2003), no. 4, 529-534.

[5] Bromwich, T.J.I'A., *An introduction to the theory of infinite series,* Macmillan and Co. Ltd., New York, 1965.

[6] Burkill, J.C., Burkill, H., *A Second Course in Mathematical Analysis*, Cambridge University Press, Cambridge, New York, 1980.

[7] Basar, F., Sever, Y., *The space $\mathcal{L}_p$ of double sequences*, Math. J. Okayama Univ., **51**(2009), 149-157.

[8] Colak, R., Turkmenoglu, A., *The double sequence spaces $\ell^2_\infty(p), c^2_0(p)$ and $c^2(p)$*, (to appear).

[9] Chandrasekhara Rao, K., Subramanian, N., *The Orlicz space of entire sequences*, Int. J. Math. Math. Sci., **68**(2004), 3755-3764.

[10] Cannor, J. S., *On Stron matrix summability with respect to a modulus and statistical convergence*, Canad Math. Bull., **32**(1989), no. 2, 194-198.

[11] Kreyszig, E., *Introductory Functional Analysis with Applications*, John Wiley and Sons Inc., 1978.

[12] Gupta, M., Kamthan, P. K., *Infinite matrices and tensorial transformations*, Acta Math., **5**(1980), 33-42.

[13] Gupta, M., Kamthan, P. K., *Infinite Matrices and tensorial transformations*, Acta Math. Vietnam, **5**(1980), 33-42.

[14] Gökhan, A., Colak, R., *The double sequence spaces $c^P_2(p)$ and $c^{PB}_2(p)$*, Appl. Math. Comput., **157**(2004), no. 2, 491-501.

[15] Gökhan, A., Colak, R., *Double sequence spaces ℓ^∞_2*, Appl. Math. Comput., **160**(2005), no. 1, 147-153.

[16] Hardy, G. H., *On the convergence of certain multiple series*, Proc. Camb. Phil. Soc., **19**(1917), 86-95.

[17] Krasnoselskii, M. A., Rutickii, Y. B., *Convex functions and Orlicz spaces*, Gorningen, Netherlands, 1961.

[18] Kamthan, P. K., Gupta, M., *Sequence spaces and series*, Lecture notes, Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 1981.

[19] Kizmaz, H., *On certain sequence spaces*, Canad. Math. Bull., **24**(1981), no. 2, 169-176.

[20] Kuttner, B., *Note on strong summability*, J. London Math. Soc., **21**(1946).

[21] Lindenstrauss, J., Tzafriri, L., *On Orlicz sequence spaces*, Israel J. Math., **10**(1971), 379-390.

[22] Leindler, I., *Uber die la Vallee-Pousinche Summierbarkeit Allgemeiner Orthogonalreihen*, Acta Math. Hung., **16**(1965), 375-378.

[23] Maddox, I. J., *Sequence spaces defined by a modulus*, Math. Proc. Cambridge Phil. Soc, **100**(1986), no. 1, 161-166.

[24] Moricz, F., *Extentions of the spaces c and c_0 from single to double sequences*, Acta. Math. Hung., **57**(1991), no. 1-2, 129-136.

[25] Moricz, F., Rhoades, B. E., *Almost convergence of double sequences and strong regularity of summability matrices*, Math. Proc. Camb. Phil. Soc., **104**(1988), 283-294.

[26] Mursaleen, M., Khan, M. A., Qamaruddin, *Difference sequence spaces defined by Orlicz functions*, Demonstratio Math., **XXXII**(1999), 145-150.

[27] Mursaleen, M., Edely, O. H. H., *Statistical convergence of double sequences*, J. Math. Anal. Appl., **288**(2003), no. 1, 223-231.

[28] Mursaleen, M., *Almost strongly regular matrices and a core theorem for double sequences*, J. Math. Anal. Appl., **293**(2004), no. 2, 523-531.

[29] Mursaleen, M., Edely, O. H. H., *Almost convergence and a core theorem for double sequences*, J. Math. Anal. Appl., **293**(2004), no. 2, 532-540.

[30] Maddox, I. J., *Spaces of strongly summable sequences*, Quart. J. Math. Oxford Ser. 2, **18**(1967), 345-355.

[31] Nakano, H., *Concave modulars*, J. Math. Soc. Japan, **5**(1953), 29-49.

[32] Orlicz, W., *Über Raume* $\left(L^M\right)$, Bull. Int. Acad. Polon. Sci. A, 1936, 93-107.

[33] Parashar, S. D., Choudhary, B., *Sequence spaces defined by Orlicz functions*, Indian J. Pure Appl. Math., **25**(1994), no. 4, 419-428.

[34] Ruckle, W. H., *FK spaces in which the sequence of coordinate vectors is bounded*, Canad. J. Math., **25**(1973), 973-978.

[35] Subramanian, N., Nallswamy, R., Saivaraju, N., *Characterization of entire sequences via double Orlicz space*, International Journal of Mathematics and Mathematical Sciences, 2007, Article ID 59681, 10 pages.

[36] Subramanian, N., Misra, U. K., *The semi normed space defined by a double gai sequence of modulus function*, Fasciculi Math., **46**(2010), 111-120.

[37] Subramanian, N., Misra, U. K., *Characterization of gai sequences via double Orlicz space*, Southeast Asian Bulletin of Mathematics, (in press).

[38] Subramanian, N., Tripathy, B. C., Murugesan, C., *The double sequence space of* Γ^2, Fasciculi Math., **40**(2008), 91-103.

[39] Subramanian, N., Tripathy, B. C., Murugesan, C., *The Cesaro of double entire sequences*, International Mathematical Forum, **4**(2009), no. 2, 49-59.

[40] Subramanian, N., Misra, U. K., *The Generalized double of gai sequence spaces*, Fasciculi Math., **43**(2010), 155-164.

[41] Subramanian, N., Misra, U. K., *Tensorial transformations of double gai sequence spaces*, International Journal of Computational and Mathematical Sciences, **3**(2009), 186-188.

[42] Tripathy, B. C., *On statistically convergent double sequences*, Tamkang J. Math., **34**(2003), no. 3, 231-237.

[43] Tripathy, B. C., Et, M., Altin, Y., *Generalized difference sequence spaces defined by Orlicz function in a locally convex space*, J. Analysis and Applications, **1**(2003), no. 3, 175-192.

[44] Turkmenoglu, A., *Matrix transformation between some classes of double sequences*, Jour. Inst. of Math. and Comp. Sci. (Math. Ser.), **12**(1999), no. 1, 23-31.

[45] Wilansky, A., *Summability through Functional Analysis*, North-Holland Mathematics Studies, North-Holland Publishing, Amsterdam, **85**(1984).

[46] Zeltser, M., *Investigation of Double Sequence Spaces by Soft and Hard Analitical Methods*, Dissertationes Mathematicae Universitatis Tartuensis 25, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu, 2001.

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Vanishing viscosity method for quasilinear variational inequalities

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Abstract. In this paper we first define the notion of viscosity solution for the following partial differential quasilinear variational inequalities involving a subdifferential operator:

$$\frac{\partial u(t,x)}{\partial t} + F(t,x,u(t,x)) \cdot Du(t,x) + f(t,x,u(t,x)) \in \partial\varphi(u(t,x)) \text{ in } \mathcal{O}$$

$t \in [0,T]\,, x \in \mathbb{R}^d$, where $\partial\varphi$ is the subdifferential operator of the proper convex lower semicontinuous function $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$. We prove the existence of a viscosity solution $u : \mathcal{O} \to \mathbb{R}^n$, where $\mathcal{O}$ an open set in $[0,T] \times \mathbb{R}^d$.

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1. Introduction

The viscosity solution was first introduced by M.G. Crandall and P.L. Lions [3] in 1983. These generalized solutions need not be differentiable anywhere, as the only regularity required in the definition is continuity (for example see [4]). M.G. Crandall, L.C. Evans, P.L. Lions in [2] give the existence theorem to use the vanishing viscosity method for the nonlinear scalar partial differential equation of the form $F(y, u(y), Du(y)) = 0$ for $y \in \mathcal{O}$, where $\mathcal{O}$ is an open set from $\mathbb{R}^n$, $F : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous. The name viscosity comes from a traditional engineering application where a nonlinear first order PDE is approximated by quasilinear first order equations which are obtained from the initial PDE by adding a regularizing $\epsilon\Delta u_\epsilon$ term, which is called a 'viscosity term', and these approximate equations can be solved by classical or numerical methods and the limit of their solution hopefully solves the initial equation.

L. Maticiuc, E. Pardoux, A. Răşcanu, A. Zălinescu in [6] studied the existence of a viscosity solution of a system of parabolic variational inequalities involving a subdifferential operator. The authors use a stochastic approach in order to prove the existence result (see in [6] pg.6).

The aim of this paper is to give an existence for a viscosity solution $u : \mathcal{O} \to \mathbb{R}^n$, where $\mathcal{O}$ is an open set in $[0, T] \times \mathbb{R}^d$, by the classical vanishing viscosity method for the following partial differential quasilinear variational inequalities involving a subdifferential operator:

$$\frac{\partial u(t,x)}{\partial t} + F(t,x,u(t,x)) \cdot Du(t,x) + f(t,x,u(t,x)) \in \partial\varphi(u(t,x)) \text{ in } \mathcal{O} \tag{1.1}$$

$t \in [0, T], x \in \mathbb{R}^d$, where $\partial\varphi$ is the subdifferential operator of the proper convex lower semicontinuous function $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$. This method can be used just in the quasilinear case .

2. Main results

Throughout this paper $\mathcal{O}$ is an open set in $[0, T] \times \mathbb{R}^d$, where T is a positive number.

We make the following assumptions:

(A.1) the functions

$$F : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^d, \quad f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$$

are continuous.

(A.2) The functions $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ is proper (i.e. $\varphi \neq +\infty$), convex, lower semicontinuous.

We recall that the subdifferential $\partial\varphi$ is defined by

$$\partial\varphi(u) = \{u^* \in R^n : \langle u^*, v - u\rangle \leq \varphi(v) - \varphi(u), \forall v \in R^n\}.$$

It is a common practice to regard sometimes $\partial\varphi$ as a subset of $R^n \times \mathbb{R}^n$ by writing $(u, u^*) \in \partial\varphi(u)$ instead of $u^* \in \partial\varphi(u)$.

We denote by

$$\begin{aligned} \mathrm{Dom}(\varphi) &= \{u \in \mathbb{R}^n : \varphi(u) < +\infty\} \\ \mathrm{Dom}(\partial\varphi) &= \{u \in \mathbb{R}^n : \partial\varphi(u) \neq \emptyset\} \end{aligned}$$

We recall some definitions and results which will be used in the following (see [1] for more details).

Theorem 2.1. *Let $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ be a convex function. Then, for all $u \in Dom(\varphi)$ and $z \in \mathbb{R}^n$, there exist*

$$\begin{aligned} \varphi'_-(u; z) &:= \lim_{t \nearrow 0} \frac{\varphi(u + tz) - \varphi(u)}{t} = \sup_{t<0} \frac{\varphi(u + tz) - \varphi(u)}{t} \\ \varphi'_+(u; z) &:= \lim_{t \searrow 0} \frac{\varphi(u + tz) - \varphi(u)}{t} = \inf_{t>0} \frac{\varphi(u + tz) - \varphi(u)}{t}. \end{aligned} \tag{2.1}$$

Moreover, the following hold:

(a) $\varphi'_-(u;z) \le \varphi'_+(u;z)$, $\forall u \in Dom(\varphi)$ *and* $z \in \mathbb{R}^n$,
(b) $\varphi'_-(u;-z) = -\varphi'_+(u;z)$, $\forall u \in Dom(\varphi)$ *and* $z \in \mathbb{R}^n$,
(c) $\varphi'_-(u,\cdot)$ *is superlinear and* $\varphi'_+(u,z)$ *is sublinear,*
(d) *if* u *and* z *are such that there exists* $\delta > 0$ *such that* $u + tz \in Dom(\varphi)$, $\forall t \in (-\delta, +\delta)$, *then* $\varphi'_-(u,z), \varphi'_+(u,z) \in \mathbb{R}$.

If we take $d = 1$, then we know that, in every point $u \in \mathrm{Dom}(\varphi)$,

$$\partial\varphi(u) \;=\; \mathbb{R} \cap \left[\varphi'_-(u), \varphi'_+(u)\right] \tag{2.2}$$

where $\varphi'_-(u)$ and $\varphi'_+(u)$ are respectively, the left and the right derivative of φ at the point u.

The following proposition generalizes the above characterization to the case of $d \geq 1$:

Proposition 2.2. *Let* $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ *be a proper, convex function and* $u \in Dom(\varphi)$. *The following statements are equivalent:*

(i) $u^\star \in \partial\varphi(u)$;
(ii) $\langle u^*, z\rangle \geq \varphi'_-(u;z), \forall z \in \mathbb{R}^n$;
(iii) $\langle u^*, z\rangle \leq \varphi'_+(u;z), \forall z \in \mathbb{R}^n$.

Let us define, for $u \in \overline{\mathrm{Dom}(\varphi)}$ and $z \in \mathbb{R}^n$,

$$\varphi'_\star(u;z) \;=\; \liminf_{\substack{v\to u\\ v\in \mathrm{Dom}(\partial\varphi)}} \varphi'_-(v;z), \qquad \varphi'^{,\star}(u;z) \;=\; \limsup_{\substack{v\to u\\ v\in \mathrm{Dom}(\partial\varphi)}} \varphi'_+(v;z)$$

For $u \in \mathbb{R}^n$, let (with the usual convention $\inf\emptyset = +\infty$)

$$|\partial\varphi|_0(u) \;=\; \inf|\partial\varphi(u)|.$$

If $u \in \mathrm{Dom}(\partial\varphi)$, then there is a unique $u^\star \in \mathbb{R}^n$, denoted $(\partial\varphi)_0(u)$ such that $|\partial\varphi|_0(u) = |(\partial\varphi)_0(u)|$.

Let $u, v \in \mathbb{R}^d$. The notation $u \cdot v$ denotes the euclidean inner product (also known as the dot product) on $\mathbb{R}^d$. We denote by Du the gradient of u, and Δu the Laplace operator of u:

$$Du(x_1, ..., x_d) \;=\; \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_2} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_1}{\partial x_d} & \cdots & \frac{\partial u_n}{\partial x_d} \end{pmatrix}$$

$$\Delta u(x_1, ..., x_d) \;=\; (\Delta u_1, \Delta u_2, \ldots, \Delta u_n) \;=\; \left(\sum_{i=1}^{d} \frac{\partial^2 u_1}{\partial x_i^2}, \ldots, \sum_{i=1}^{d} \frac{\partial^2 u_n}{\partial x_i^2}\right)$$

We may now define the concept of viscosity solution of (1.1):

Definition 2.3. *Let* $u : \mathcal{O} \to \mathbb{R}^n$ *be a continuous function. We say the function* u *is a viscosity solution of (*1.1*), if:*

$$u(t,x) \in Dom(\partial\varphi), \;\; \forall (t,x) \in \;\mathcal{O}$$

and for all $\Psi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *continuous function, and* $z \in \mathbb{R}^n$*,*

$$\text{if } u \cdot z - \Psi \text{ attains a local maximum at } (t_0, x_0) \in \mathcal{O}, \quad \text{then}$$

we have

$$\frac{\partial \Psi(t_0,x_0)}{\partial t} + F(t_0,x_0,u(t_0,x_0)) \cdot D\Psi(t_0,x_0) + \\ + f(t_0,x_0,u(t_0,x_0)) \cdot z \le \varphi^{`,*}(u(t_0,x_0);z) \tag{2.3}$$

Remark 2.4. *Observe that the Definition* 2.3 *is the particular case of the definition given in (*[6]*) for the quasilinear case.*

The main result is the following:

Theorem 2.5. *Let* $\epsilon > 0$*, and* $F_\epsilon : [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^d$*,* $f_\epsilon : [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ *be a family of continuous functions such that* $F_\epsilon(t,x,p)$*,* $f_\epsilon(t,x,p)$ *converges uniformly on compact subsets of* $\mathcal{O} \times \mathbb{R}^n$ *to some function* $F(t,x,p)$ *and* $f(t,x,p)$*, as* ϵ *tends to* 0*. Finally, suppose that for all* $\epsilon > 0$ $u_\epsilon \in C^2(\mathcal{O})$ *is a solution of*

$$\frac{\partial u_\epsilon(t,x)}{\partial t} - \epsilon \Delta u_\epsilon(t,x) \\ + F_\epsilon(t,x,u_\epsilon(t,x)) \cdot Du_\epsilon(t,x) + f_\epsilon(t,x,u_\epsilon(t,x)) \in \partial\varphi(u_\epsilon(t,x)) \text{ in } \mathcal{O}. \tag{2.4}$$

Then if u_ϵ *converge uniformly on compact subsets of* $\mathcal{O}$ *to some* $u \in C(\mathcal{O})$*, we have*

$$u \text{ is a viscosity solution of } (1.1).$$

Remark 2.6. *By the Proposition* 2.2 *the inequation (*2.4*) can be written in the form*

$$\left(u'_{\epsilon_t} \cdot z\right)(t,x) - \epsilon \Delta u_\epsilon(t,x) \cdot z + F_\epsilon(t,x,u_\epsilon(t,x)) \cdot Du_\epsilon(t,x) \cdot z + \\ + + f_\epsilon(t,x,u_\epsilon(t,x)) \cdot z \le \varphi'_+(u_\epsilon(t,x);z) \text{ in } \mathcal{O}, \quad \text{for all } z \in \mathbb{R}^n \tag{2.5}$$

Proof. Let us check (2.3) first for $\Psi \in C^2(\mathcal{O})$. We assume that $\forall z \in R^n$, $u \cdot z - \Psi$ has a local maximum point at $(t_0, x_0) \in \mathcal{O}$.

Choose $\xi \in C^\infty(\mathcal{O})$, such that

$$0 \le \xi < 1, \text{ if } (t,x) \ne (t_0,x_0), \text{ and } \xi(t_0,x_0) = 1.$$

Obviously, $u \cdot z - (\Psi - \xi)$ has a strict local minimum point at $(t_0, x_0) \in \mathcal{O}$, and thus for ϵ small enough, $u_\epsilon \cdot z - (\Psi - \xi)$ has a local maximum point at some $(t_\epsilon, x_\epsilon) \in \mathcal{O}$, and $(t_\epsilon, x_\epsilon) \to (t_0, x_0)$ as $\epsilon \to 0$.

But at the point $(t_\epsilon, x_\epsilon) = (t_0, x_0)$, we have

$$D(u_\epsilon \cdot z - (\Psi - \xi))(t_\epsilon, x_\epsilon) = 0$$

$$\left(u'_{\epsilon_t} \cdot z\right)(t_\epsilon, x_\epsilon) = \Psi'_t(t_\epsilon, x_\epsilon) - \xi'_t(t_\epsilon, x_\epsilon) \tag{2.6}$$

$$\left(D_x u_\epsilon \cdot z\right)(t_\epsilon, x_\epsilon) = D_x\Psi(t_\epsilon, x_\epsilon) - D_x\xi(t_\epsilon, x_\epsilon) \tag{2.7}$$

By taking (2.6) and (2.7) in (2.4) we have

$$\Psi_t'(t_\epsilon,x_\epsilon) \;-\xi_t'(t_\epsilon,x_\epsilon) \;-\epsilon\Delta u_\epsilon(t_\epsilon,x_\epsilon)\cdot z$$
$$+F_\epsilon(t_\epsilon,x_\epsilon,u_\epsilon(t_\epsilon,x_\epsilon))\cdot(D_x\Psi(t_\epsilon,x_\epsilon)\;-\;D_x\xi(t_\epsilon,x_\epsilon))$$
$$+f_\epsilon(t_\epsilon,x_\epsilon,u_\epsilon(t_\epsilon,x_\epsilon))\cdot z\;\le\;\varphi_+'(u_\epsilon(t_\epsilon,x_\epsilon);z)\text{ in }\mathcal{O},\quad\text{for all } z\in\mathbb{R}^n \tag{2.8}$$

Since, as $\epsilon\to 0$

$$u_\epsilon(t_\epsilon,x_\epsilon)\quad\to\quad u(t_0,x_0),$$
$$D\left(u_\epsilon\cdot z\right)(t_\epsilon,x_\epsilon)=D\left(\Psi\;-\;\xi\right)(t_\epsilon,x_\epsilon)\to D\left(\Psi\;-\;\xi\right)(t_0x_0)=D\Psi(t_0x_0)$$
$$\epsilon\Delta u_\epsilon(t_\epsilon,x_\epsilon)\cdot z\le\epsilon\Delta\left(\Psi\;-\;\xi\right)(t_\epsilon,x_\epsilon)\to 0$$

and F, f are continuous functions, φ is lower semicontinuous, we have

$$\frac{\partial\Psi(t_0,x_0)}{\partial t}\;+\;F(t_0,x_0,u(t_0,x_0))\cdot D\Psi(t_0,x_0)+$$
$$+\,f(t_0,x_0,u(t_0,x_0))\cdot z\le\varphi^{`,*}(u(t_0,x_0);z) \tag{2.9}$$

However, we have to show this for test functions from $C^1(\mathcal{O})$. Let $\Psi\in C^1(\mathcal{O})$, and assume that $\forall z\in R^n$, $u\cdot z-\Psi$ has a local maximum point at $(t_0,x_0)\in\mathcal{O}$.

We have to show that

$$\frac{\partial\Psi(t_0,x_0)}{\partial t}\;+\;F(t_0,x_0,u(t_0,x_0))\cdot D\Psi(t_0,x_0)\;+\;f(t_0,x_0,u(t_0,x_0))\cdot z$$
$$\le\varphi^{`,*}(u(t_0,x_0);z)$$

Let $\Psi_n\in C^1(\mathcal{O})$ such that $\Psi_n\to\Psi$ in $C^1(\mathcal{O})$ and, as before, choose $\xi\in C^\infty(\mathcal{O})$ such that

$$0\le\xi<1,\;\text{ if }\;(t,x)\ne(t_0,x_0),\;\text{ and }\;\xi(t_0,x_0)=1.$$

For n large enough, $u_\epsilon\cdot z-(\Psi_n\;-\;\xi)$ has a local maximum point at some $(t_n,x_n)\in\mathcal{O}$, and $(t_n,x_n)\to(t_0,x_0)$ as $n\to\infty$.

It follows

$$\left(Du\cdot z\right)(t_n,x_n)=D\Psi_n(t_n,x_n)\;-\;D\xi(t_n,x_n) \tag{2.10}$$

Then as shown above, for each n we have

$$\frac{\partial\Psi_n(t_n,x_n)}{\partial t}-\frac{\partial\xi(t_n,x_n)}{\partial t}+F(t_n,x_n,u(t_n,x_n))\cdot(D_x\Psi_n(t_n,x_n)-D_x\xi(t_n,x_n))$$
$$+f(t_n,x_n,u(t_n,x_n))\cdot z\;\le\;\varphi_+'(u(t_n,x_n);z)\text{ in }\mathcal{O},\quad\text{for all } z\in\mathbb{R}^n \tag{2.11}$$

Since, as $n\to\infty$

$$u(t_n,x_n)\quad\to\quad u(t_0,x_0),$$
$$D\left(u\cdot z\right)(t_n,x_n)=D\left(\Psi_n\;-\;\xi\right)(t_n,x_n)\to D\left(\Psi\;-\;\xi\right)(t_0x_0)=D\Psi(t_0x_0)$$

and F,f are continuous functions, φ lower semicontinuous, we have

$$\frac{\partial\Psi(t_0,x_0)}{\partial t}\;+\;F(t_0,x_0,u(t_0,x_0))\cdot D\Psi(t_0,x_0)\;+\;f(t_0,x_0,u(t_0,x_0))\cdot z$$
$$\le\varphi^{`,*}(u(t_0,x_0);z)$$

Therefore u is a viscosity solution of (1.1). □

References

[1] Borwein, J., Lewis, A., *Convex analysis and nonlinear optimization: Theory and Examples*, Springer-Verlag, New-York, 2000.

[2] Crandall, M. G., Evans, L. C., Lions, P. L., *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., **282**(1984), 487-502.

[3] Crandall, M. G., Lions, P. L., *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., **277**(1983), 1-42.

[4] Crandall, M. G., Ishii, H., Lions, P. L., *User's guide to the viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc., **27**(1992), 1-67.

[5] Lions, P. L., *Generalized solutions of Hamilton-Jacobi equations*, Pitman, London, 1982.

[6] Maticiuc, L., Pardoux, E., Rascanu, A., Zalinescu, A., *Viscosity solutions for systems of parabolic variational inequalities*, preprint, http://arxiv.org/abs/0807.4415, 2008.

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Metric relations on mountain slopes

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Abstract. It is well-known that the Ceva and Menelaus theorems are deducible from each other in the Euclidean case. In this paper we show that Ceva's theorem holds whereas Menelaus' theorem fails on Matsumoto's mountain slope geometry.

Mathematics Subject Classification (2010): 58B20, 53A35, 53C22.

Keywords: Non-Euclidean geometry, Finsler surface, geodesics.

1. Introduction

It is well-known that in Euclidean geometry, Ceva's and Menelaus' theorems are dual results, i.e., they are deducible from each other. In the Euclidean context, several extensions of these theorems can be found, see Green [2], Landy [4], Lipman [5], Wernicke [8]. Moreover, Masal'tsev [6] generalized Ceva's theorem to geodesic triangles on Riemannian surfaces of constant curvature (hyperbolic plane, sphere).

A natural question arises in the validity of these two theorems on non-Riemannian surfaces, even with constant curvature. Our aim is to prove that on the *Matsumoto's mountain slope* - which is one of the simplest *non*-Riemannian Finsler surface whose flag curvature is identically 0 - Ceva's theorem holds whereas Menelaus' theorem fails except the case when the slope becomes the horizontal plane.

2. Results

First, we recall the *Matsumoto's mountain slope metric*, see Matsumoto [7] or Kozma-Tamássy [3]. Let us consider an inclined plane (slope) with an angle $\alpha \in [0, \pi/2)$ to the horizontal plane, denoted by (S_α). If a man moves with a constant speed v $[m/s]$ on a horizontal plane, he goes $l_t = vt + \frac{g}{2}t^2 \sin\alpha \cos\theta$

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meters in t seconds on (S_α), where θ is the angle between the straight road and the direct downhill road (θ is measured in clockwise direction). The point here is that the travel speed depends heavily on both the slope of the terrain and the direction of travel, due to the presence of the gravity. The precise law of the above phenomenon - by using the so-called Okubo's technique - can be described relatively to the horizontal plane by the parameterized function

$$F_\alpha(y_1, y_2) = \frac{y_1^2 + y_2^2}{v\sqrt{y_1^2 + y_2^2} + \frac{g}{2} y_1 \sin\alpha}, \qquad (y_1, y_2) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

Here, $g \approx 9.81\ [m/s^2]$ and we assume $g \sin\alpha \leq v$.

For every $\alpha \in (0, \pi/2)$, $(\mathbb{R}^2, F_\alpha)$ is a typical non-Riemannian, Finsler surface. A classification of Finsler manifolds shows that $(\mathbb{R}^2, F_\alpha)$ is a locally Minkowski space with the following additional properties:

(a) its *flag curvature* is identically 0, see Bao-Chern-Shen [1, p. 384];
(b) its geodesics are *straight lines,* see also Bao-Chern-Shen [1, p. 384];
(c) every two points in $(\mathbb{R}^2, F_\alpha)$ determine a unique geodesic which lies them, due to Cartan-Hadamard's and Hopf-Rinow's theorems.

On account of (a)-(c), there is a strong similarity between $(\mathbb{R}^2, F_\alpha)$ and the standard two-dimensional Euclidean space. However, differences appear once we start to measure distances on these spaces. Exploiting the shape of geodesics on $(\mathbb{R}^2, F_\alpha)$, the *distance* (measuring actually the *physical time* to arrive) from $P = (P^1, P^2)$ to $Q = (Q^1, Q^2)$ on $(\mathbb{R}^2, F_\alpha)$ is

$$d_\alpha(P, Q) = F_\alpha(Q^1 - P^1, Q^2 - P^2).$$

Note that usually $d_\alpha(P, Q) \neq d_\alpha(Q, P)$.

Since geodesics are straight lines on $(\mathbb{R}^2, F_\alpha)$, see (b) from above, we may introduce the following two notions:

- $[PQ] = \{t(Q - P) + P : t \in [0, 1]\}$ is the *geodesic segment* lying the points $P, Q \in \mathbb{R}^2$, and
- $[PQ[= \{t(Q - P) + P : t \geq 1\}$ is the *geodesic semi-line* defined by $P, Q \in \mathbb{R}^2$.

Let A, B, C be three arbitrarily fixed points in $(\mathbb{R}^2, F_\alpha)$, and let M, N, P points on the geodesic segments $[BC], [CA], [AB]$, respectively. We consider the following two statements:

(C_1^α): $\frac{d_\alpha(A,P)}{d_\alpha(P,B)} \cdot \frac{d_\alpha(B,M)}{d_\alpha(M,C)} \cdot \frac{d_\alpha(C,N)}{d_\alpha(N,A)} = 1$;
(C_2): The geodesic segments $[AM], [BN], [CP]$ are concurrent.

Theorem 2.1. *For every $\alpha \in [0, \pi/2)$, we have $(C_1^\alpha) \Leftrightarrow (C_2)$.*

Thus, Ceva's theorem holds on the mountain slope $(\mathbb{R}^2, F_\alpha)$ for every $\alpha \in [0, \pi/2)$.

Now, let A, B, C be fixed points in $(\mathbb{R}^2, F_\alpha)$, and fix the points N, P on the geodesic segments $[CA], [AB]$, while M on the geodesic semi-line $[BC[$. We formulate the following two statements (the first being formally the same as (C_1^α)):

(M_1^α): $\frac{d_\alpha(A,P)}{d_\alpha(P,B)} \cdot \frac{d_\alpha(B,M)}{d_\alpha(M,C)} \cdot \frac{d_\alpha(C,N)}{d_\alpha(N,A)} = 1;$
(M_2): The points M, N, P are on the same geodesic (straight line).

Theorem 2.2. *The equivalence $(M_1^\alpha) \Leftrightarrow (M_2)$ holds if and only if $\alpha = 0$.*

Consequently, Menelaus' theorem holds on mountain slopes for every geodesic triangle if and only if the 'slope' becomes the horizontal plane (i.e., $\alpha = 0$), which corresponds exactly to the Euclidean case.

3. Proofs

In the sequel, we denote by d_E the usual two-dimensional Euclidean metric.
Proof of Theorem 2.1. Since $P = \frac{d_E(P,B)}{d_E(A,B)}A + \frac{d_E(A,P)}{d_E(A,B)}B$, we have

$$P - A = \frac{d_E(A,P)}{d_E(A,B)}(B - A).$$

Since F_α is positively homogeneous of degree 1, one has

$$d_\alpha(A,P) = F_\alpha(P^1 - A^1, P^2 - A^2) = \frac{d_E(A,P)}{d_E(A,B)} F_\alpha(B^1 - A^1, B^2 - A^2).$$

A similar calculation for $d_\alpha(P,B)$ implies that

$$\frac{d_\alpha(A,P)}{d_\alpha(P,B)} = \frac{d_E(A,P)}{d_E(P,B)}.$$

Repeating this argument for the other two sides of the triangle, (C_1^α) is equivalent to

$$\frac{d_E(A,P)}{d_E(P,B)} \cdot \frac{d_E(B,M)}{d_E(M,C)} \cdot \frac{d_E(C,N)}{d_E(N,A)} = 1.$$

But, in the Euclidean case, the latter relation is equivalent to the fact that the segments $[AM], [BN], [CP]$ are concurrent, thus the proof is done.
Proof of Theorem 2.2. If $\alpha = 0$, the equivalence $(M_1^\alpha) \Leftrightarrow (M_2)$ is just the well-known Menelaus' theorem in the Euclidean case.

Now, we assume the equivalence $(M_1^\alpha) \Leftrightarrow (M_2)$ holds for every points A, B, C as well as M, N, P in $(\mathbb{R}^2, F_\alpha)$ specified above. We prove that $\alpha = 0$. To see this, we consider the following specific constellation of points: $B = (0,0)$, $C = (1,0)$, $M = (2,0)$, A is arbitrary, while P and N are situated on $[AB]$ and $[AC]$ such that M belongs to the unique geodesic lying them, see (c) from above. Thus, (M_2) holds. Since $(M_2) \Leftrightarrow (M_1^\alpha)$, we have

$$\frac{d_\alpha(A,P)}{d_\alpha(P,B)} \cdot \frac{d_\alpha(B,M)}{d_\alpha(M,C)} \cdot \frac{d_\alpha(C,N)}{d_\alpha(N,A)} = 1. \tag{3.1}$$

On the other hand, (M_2) also implies for the Euclidean metric that

$$\frac{d_E(A,P)}{d_E(P,B)} \cdot \frac{d_E(B,M)}{d_E(M,C)} \cdot \frac{d_E(C,N)}{d_E(N,A)} = 1. \tag{3.2}$$

As in the proof of Theorem 2.1, by using the positive homogeneity of F_α, we deduce

$$\frac{d_\alpha(A,P)}{d_\alpha(P,B)} = \frac{d_E(A,P)}{d_E(P,B)} \quad \text{and} \quad \frac{d_\alpha(C,N)}{d_\alpha(N,A)} = \frac{d_E(C,N)}{d_E(N,A)}.$$

Combining (3.1) and (3.2) with the above relations, we obtain

$$\frac{d_\alpha(B,M)}{d_\alpha(M,C)} = \frac{d_E(B,M)}{d_E(M,C)}.$$

After substitutions, we obtain $\dfrac{F_\alpha(2,0)}{F_\alpha(-1,0)} = 2$. An elementary calculation shows that the latter equation holds only in the case when $g\sin\alpha = 0$, i.e., $\alpha = 0$. This concludes our proof.

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References

[1] Bao, D., Chern, S. S., Shen, Z., *Introduction to Riemann-Finsler Geometry,* Graduate Texts in Mathematics, 200, Springer Verlag, 2000.

[2] Green, H. G., *On the theorems of Ceva and Menelaus,* Amer. Math. Monthly, **64** (1957), 354-357.

[3] Kozma, L., Tamássy, L., *Finsler geometry inspired. Finslerian geometries* (Edmonton, AB, 1998), 9-14, Fund. Theories Phys., 109, Kluwer Acad. Publ., Dordrecht, 2000.

[4] Landy, S., *A generalization of Ceva's theorem to higher dimensions*, Amer. Math. Monthly, **95** (1988), no. 10, 936-939.

[5] Lipman, J., *A generalization of Ceva's theorem,* Amer. Math. Monthly, **67**(1960), 162-163.

[6] Masal'tsev, L. A., *Incidence theorems in spaces of constant curvature,* (Russian) Ukrain. Geom. Sb., **35**(1992), 67-74, 163; translation in J. Math. Sci., **72**(1994), no. 4, 3201-3206.

[7] Matsumoto, M., *A slope of a mountain is a Finsler surface with respect to a time measure,* J. Math. Kyoto Univ., **29**(1989), no. 1, 17-25.

[8] Wernicke, P., *The theorems of Ceva and Menelaus and their extension,* Amer. Math. Monthly, **34**(1927), no. 9, 468-472.

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Jet geometrical objects produced by linear ODEs systems and superior order ODEs

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Abstract. The aim of this paper is to construct a Riemann-Lagrange geometry on 1-jet spaces, in the sense of d-connections, d-torsions, d-curvatures, electromagnetic d-field and geometric electromagnetic Yang-Mills energy, starting from a given linear ODEs system or a given superior order ODE. The case of a non-homogenous linear ODE of superior order is discussed.

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Keywords: 1-jet spaces, jet least squares Lagrangian functions, Riemann-Lagrange geometry, linear ODEs systems, superior order ODEs.

1. Introduction

According to Olver's opinion expressed in [7] and in private discussions, we point out that the 1-jet spaces are main mathematical models necessary for the study of classical or quantum field theories. In such a context, the contravariant differential geometry of the 1-jet spaces was intensively studied by authors like Asanov [1] or Saunders [9].

Situated in the direction initiated by Asanov [1], it has been recently developed the *Riemann-Lagrange geometry of* 1*-jet spaces* [2], [4], which is a geometrical theory on 1-jet spaces analogous with the well known *Lagrange geometry of the tangent bundle* developed by Miron and Anastasiei [3].

It is important to note that the Riemann-Lagrange geometry of the 1-jet spaces allows the regarding of the solutions of a given ODEs (respectively, PDEs) system as *geodesics* [10] (respectively, *generalized harmonic maps* [6] or *potential maps* [11]) in a convenient Riemann-Lagrange geometrical structure on 1-jet spaces. In this way, it was given a final solution for an open problem suggested by Poincaré [8] (*find the geometric structure which transforms the field lines of a given vector field into geodesics*) and generalized by

Udrişte [10] (*find the geometrical structure which converts the solutions of a given first order PDEs system into harmonic maps*).

In this context, using the Riemann-Lagrange geometrical methods, it was constructed an entire contravariant differential geometry on 1-jet spaces, in the sense of d-connections, d-torsions, d-curvatures, electromagnetic d-field and geometric electromagnetic Yang-Mills energy, starting only with a given ODEs [5] (respectively, PDEs [6]) system of order one and a pair of Riemannian metrics.

2. Jet Riemann-Lagrange geometry produced by a non-linear ODEs system of order one and a pair of Riemannian metrics

In this Section we present the main jet Riemann-Lagrange geometrical ideas used for the geometrical study of a given non-linear first order ODEs system. For more details, the reader is invited to consult the works [4], [5] and [11].

Let $T = [a, b] \subset \mathbb{R}$ be a compact interval of the set of real numbers and let us consider the jet fibre bundle of order one

$$J^1(T, \mathbb{R}^n) \to T \times \mathbb{R}^n, \ n \geq 2,$$

whose local coordinates (t, x^i, x_1^i), $i = \overline{1, n}$, transform by the rules

$$\widetilde{t} = \widetilde{t}(t), \ \widetilde{x}^i = \widetilde{x}^i(x^j), \ \widetilde{x}_1^i = \frac{\partial \widetilde{x}^i}{\partial x^j} \frac{dt}{d\widetilde{t}} \cdot x_1^j.$$

Remark 2.1. *From a physical point of view, in the 1-jet space of* **physical events** *the coordinate t has the physical meaning of* **relativistic time***, the coordinates $(x^i)_{i=\overline{1,n}}$ represent* **spatial coordinates** *and the coordinates $(x_1^i)_{i=\overline{1,n}}$ have the physical meaning of* **relativistic velocities***.*

Let $X = \left(X_{(1)}^{(i)}(t, x^k) \right)$ be an arbitrary given d-tensor field on the first order jet space $J^1(T, \mathbb{R}^n)$, which produces the jet non-linear ODEs system of order one (*jet dynamical system*)

$$x_1^i = X_{(1)}^{(i)}(t, x^k(t)), \ \forall \ i = \overline{1, n}, \tag{2.1}$$

where $c(t) = (x^i(t))$ is an unknown curve on $\mathbb{R}^n$ and we use the notations

$$x_1^i \overset{not}{=} \dot{x}^i = \frac{dx^i}{dt}, \ \forall \ i = \overline{1, n}.$$

Suppose now that we fixed *a priori* two Riemannian structures $(T, h_{11}(t))$ and $(\mathbb{R}^n, \varphi_{ij}(x))$, where $x = (x^k)_{k=\overline{1,n}}$, together with their attached Christoffel symbols $H_{11}^1(t)$ and $\gamma_{jk}^i(x)$. Automatically, the jet non-linear ODEs system of order one (2.1), together with the pair of Riemannian metrics

$$\mathcal{P} = (h_{11}(t), \varphi_{ij}(x)),$$

produce the *jet least squares Lagrangian function*

$$JLS_{\mathcal{P}}^{\text{ODEs}} : J^1(T, \mathbb{R}^n) \to \mathbb{R}_+,$$

expressed by

$$JLS_{\mathcal{P}}^{\text{ODEs}}(t,x^k,x_1^k)=h^{11}(t)\varphi_{ij}(x)\left[x_1^i-X_{(1)}^{(i)}(t,x)\right]\left[x_1^j-X_{(1)}^{(j)}(t,x)\right].$$

It is obvious that the *global minimum points* of the *jet least squares energy action*

$$\mathbb{E}_{\mathcal{P}}^{\text{ODEs}}(c(t))=\int_a^b JLS_{\mathcal{P}}^{\text{ODEs}}(t,x^k(t),\dot{x}^k(t))\sqrt{h_{11}(t)}dt$$

are exactly the solutions of class C^2 of the jet non-linear ODEs system of order one (2.1). In other words, we have

Theorem 2.2. *The solutions of class C^2 of the first order ODEs system* (2.1) *verify the second order Euler-Lagrange equations produced by the jet least squares Lagrangian function $JLS_{\mathcal{P}}^{ODEs}$, namely (***jet geometric dinamics***)*

$$\frac{\partial\left[JLS_{\mathcal{P}}^{ODEs}\right]}{\partial x^i}-\frac{d}{dt}\left(\frac{\partial\left[JLS_{\mathcal{P}}^{ODEs}\right]}{\partial \dot{x}^i}\right)=0,\ \forall\ i=\overline{1,n}. \tag{2.2}$$

Remark 2.3. *Conversely, the above statement does not hold good because there exist solutions for the second order Euler-Lagrange ODEs system* (2.2) *which are not global minimum points for the jet least squares energy action $\mathbb{E}_{\mathcal{P}}^{ODEs}$, that is which are not solutions for the jet first order ODEs system* (2.1)*.*

As a conclusion, we believe that we may regard $JLS_{\mathcal{P}}^{\text{ODEs}}$ as a natural geometrical substitute on $J^1(T,\mathbb{R}^n)$ for the jet first order ODEs system (2.1).

But, we point out that a Riemann-Lagrange geometry on $J^1(T,\mathbb{R}^n)$ produced by the jet least squares Lagrangian function $JLS_{\mathcal{P}}^{\text{ODEs}}$, via its second order Euler-Lagrange equations (2.2), geometry in the sense of non-linear connection, generalized Cartan connection, d-torsions and d-curvatures, is now completely done in the papers [4], [5] and [6]. Moreover, a distinguished jet electromagnetic 2-form, characterized by some natural generalized Maxwell equations and a geometric jet Yang-Mills energy [5], is constructed from the jet least squares Lagrangian function $JLS_{\mathcal{P}}^{\text{ODEs}}$.

Definition 2.4. *Any geometrical object on $J^1(T,\mathbb{R}^n)$, which is produced by the jet least squares Lagrangian function $JLS_{\mathcal{P}}^{ODEs}$, via the Euler-Lagrange equations* (2.2)*, is called* **geometrical object produced by the jet first order ODEs system** (2.1) **and the pair of Riemannian metrics** $\mathcal{P}$.

In this context, we give the following jet Riemann-Lagrange geometrical result, which is proved in [5] and, for the multi-time general case, in [6]. For more details, the reader is invited to consult the book [4].

Theorem 2.5. *(i) The* **canonical non-linear connection on** $J^1(T,\mathbb{R}^n)$ **produced by the jet first order ODEs system** (2.1) **and the pair of Riemannian metrics** $\mathcal{P}$ *is*

$$\Gamma_{\mathcal{P}}^{ODEs}=\left(M_{(1)1}^{(i)},N_{(1)j}^{(i)}\right),$$

whose local components are given by

$$M^{(i)}_{(1)1} = -H^1_{11}x^i_1 \text{ and } N^{(i)}_{(1)j} = \gamma^i_{jk}x^k_1 - \frac{1}{2}\left[X^{(i)}_{(1)||j} - \varphi^{ir}X^{(s)}_{(1)||r}\varphi_{sj}\right],$$

where

$$X^{(i)}_{(1)||j} = \frac{\partial X^{(i)}_{(1)}}{\partial x^j} + X^{(m)}_{(1)}\gamma^i_{mj}.$$

(ii) The **canonical generalized Cartan connection** $C\Gamma^{ODEs}_{\mathcal{P}}$ **produced by the jet first order ODEs system** (2.1) **and the pair of Riemannian metrics** $\mathcal{P}$ *has the adapted components*

$$C\Gamma^{ODEs}_{\mathcal{P}} = (H^1_{11}, 0, \gamma^i_{jk}, 0).$$

(iii) The effective adapted components of the **torsion** *d-tensor* $\mathbf{T}^{ODEs}_{\mathcal{P}}$ *of the canonical generalized Cartan connection* $C\Gamma^{ODEs}_{\mathcal{P}}$ **produced by the jet first order ODEs system** (2.1) **and the pair of Riemannian metrics** $\mathcal{P}$ *are*

$$R^{(i)}_{(1)1j} = \frac{1}{2}\left[X^{(i)}_{(1)||j//1} - \varphi^{ir}X^{(s)}_{(1)||r//1}\varphi_{sj}\right]$$

and

$$R^{(i)}_{(1)jk} = r^i_{jkm}x^m_1 - \frac{1}{2}\left[X^{(i)}_{(1)||j||k} - \varphi^{ir}X^{(s)}_{(1)||r||k}\varphi_{sj}\right],$$

where $r^l_{ijk}(x)$ *are the components of the curvature tensor of the Riemannian metric* $\varphi_{ij}(x)$ *and*

$$X^{(i)}_{(1)||j//1} = \frac{\partial X^{(i)}_{(1)||j}}{\partial t} - X^{(i)}_{(1)||j}H^1_{11},$$

$$X^{(i)}_{(1)||j||k} = \frac{\partial X^{(i)}_{(1)||j}}{\partial x^k} + X^{(m)}_{(1)||j}\gamma^i_{mk} - X^{(i)}_{(1)||m}\gamma^m_{jk}.$$

(iv) The effective adapted components of the **curvature** *d-tensor* $\mathbf{R}^{ODEs}_{\mathcal{P}}$ *of the canonical generalized Cartan connection* $C\Gamma^{ODEs}_{\mathcal{P}}$ **produced by the jet first order ODEs system** (2.1) **and the pair of Riemannian metrics** $\mathcal{P}$ *are only* $R^l_{ijk} = r^l_{ijk}$.

(v) The **geometric electromagnetic distinguished 2-form produced by the jet first order ODEs system** (2.1) **and the pair of Riemannian metrics** $\mathcal{P}$ *has the expression*

$$F^{ODEs}_{\mathcal{P}} = F^{(1)}_{(i)j}\delta x^i_1 \wedge dx^j,$$

where

$$\delta x^i_1 = dx^i_1 + M^{(i)}_{(1)1}dt + N^{(i)}_{(1)k}dx^k$$

and, if $h^{11} = 1/h_{11}$, *then*

$$F^{(1)}_{(i)j} = \frac{h^{11}}{2}\left[\varphi_{im}X^{(m)}_{(1)||j} - \varphi_{jm}X^{(m)}_{(1)||i}\right].$$

(vi) The adapted components of the electromagnetic d-form $F_{\mathcal{P}}^{ODEs}$ produced by the jet first order ODEs system (2.1) and the pair of Riemannian metrics $\mathcal{P}$ verify the **generalized Maxwell equations**

$$\begin{cases} F_{(i)j//1}^{(1)} = \dfrac{1}{4}\mathcal{A}_{\{i,j\}}\left\{h^{11}\varphi_{im}\left[X_{(1)||j//1}^{(m)} - \varphi^{mr}X_{(1)||r//1}^{(s)}\varphi_{sj}\right]\right\} \\ \sum_{\{i,j,k\}} F_{(i)j||k}^{(1)} = 0, \end{cases}$$

where $\mathcal{A}_{\{i,j\}}$ represents an alternate sum, $\sum_{\{i,j,k\}}$ means a cyclic sum and

$$F_{(i)j//1}^{(1)} = \frac{\partial F_{(i)j}^{(1)}}{\partial t} + F_{(i)j}^{(1)}H_{11}^{1} \text{ and } F_{(i)j||k}^{(1)} = \frac{\partial F_{(i)j}^{(1)}}{\partial x^k} - F_{(m)j}^{(1)}\gamma_{ik}^{m} - F_{(i)m}^{(1)}\gamma_{jk}^{m}$$

have the geometrical meaning of the horizontal local covariant derivatives "$_{//1}$" and "$_{||k}$" produced by the Berwald linear connection $B\Gamma_0$ on $J^1(T,\mathbb{R}^n)$. For more details, please consult [4].

(vii) The **geometric jet Yang-Mills energy produced by the jet first order ODEs system (2.1) and the pair of Riemannian metrics** *$\mathcal{P}$ is defined by the formula*

$$EYM_{\mathcal{P}}^{ODEs}(t,x) = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[F_{(i)j}^{(1)}\right]^2.$$

Now, let us consider on $T \times \mathbb{R}^n$ the particular pair of Euclidian metrics

$$\Delta = (h_{11}(t) = 1, \varphi_{ij}(x) = \delta_{ij}),$$

where δ_{ij} are the Kronecker symbols. Then we obtain the particular jet least squares Lagrangian function

$$JLS_{\Delta}^{\text{ODEs}} : J^1(T,\mathbb{R}^n) \to \mathbb{R}_+,$$

defined by

$$\begin{aligned} JLS_{\Delta}^{\text{ODEs}}(t,x^k,x_1^k) &= \delta_{ij}\left[x_1^i - X_{(1)}^{(i)}(t,x)\right]\left[x_1^j - X_{(1)}^{(j)}(t,x)\right] = \\ &= \sum_{i=1}^{n}\left[x_1^i - X_{(1)}^{(i)}(t,x)\right]^2. \end{aligned}$$

In this new context, we introduce the following concept:

Definition 2.6. *Any geometrical object on $J^1(T,\mathbb{R}^n)$, which is produced by the jet least squares Lagrangian function JLS_{Δ}^{ODEs}, via its attached second order Euler-Lagrange equations, is called* **geometrical object produced by the jet first order ODEs system (2.1)**.

As a consequence, particularizing the Theorem 2.5 for the pair of Euclidian metrics $\mathcal{P} = \Delta$ and taking into account that we have $H_{11}^1(t) = 0$ and $\gamma_{ij}^k(x) = 0$, we immediately get the following jet geometrical result:

Corollary 2.7. *(i) The* **canonical non-linear connection on** *$J^1(T,\mathbb{R}^n)$* **produced by the jet first order ODEs system (2.1)** *has the local components*

$$\Gamma_{\Delta}^{ODEs} = \left(\bar{M}_{(1)1}^{(i)}, \bar{N}_{(1)j}^{(i)}\right),$$

where

$$\bar{M}_{(1)1}^{(i)} = 0 \ and \ \bar{N}_{(1)j}^{(i)} = -\frac{1}{2}\left[\frac{\partial X_{(1)}^{(i)}}{\partial x^j} - \frac{\partial X_{(1)}^{(j)}}{\partial x^i}\right], \ \forall\ i,j = \overline{1,n}.$$

(ii) All adapted components of the **canonical generalized Cartan connection** $C\Gamma_{\Delta}^{ODEs}$ **produced by the jet first order ODEs system** (2.1) *vanish.*

(iii) The effective adapted components of the **torsion** *d-tensor* $\mathbf{T}_{\Delta}^{ODEs}$ *of the canonical generalized Cartan connection* $C\Gamma_{\Delta}^{ODEs}$ **produced by the jet first order ODEs system** (2.1) *are*

$$\bar{R}_{(1)1j}^{(i)} = \frac{1}{2}\left[\frac{\partial^2 X_{(1)}^{(i)}}{\partial t \partial x^j} - \frac{\partial^2 X_{(1)}^{(j)}}{\partial t \partial x^i}\right], \ \forall\ i,j = \overline{1,n},$$

and

$$\bar{R}_{(1)jk}^{(i)} = -\frac{1}{2}\left[\frac{\partial^2 X_{(1)}^{(i)}}{\partial x^k \partial x^j} - \frac{\partial^2 X_{(1)}^{(j)}}{\partial x^k \partial x^i}\right], \ \forall\ i,j,k = \overline{1,n}.$$

(iv) All adapted components of the **curvature** *d-tensor* $\mathbf{R}_{\Delta}^{ODEs}$ *of the canonical generalized Cartan connection* $C\Gamma_{\Delta}^{ODEs}$ **produced by the jet first order DEs system** (2.1) *vanish.*

(v) The **geometric electromagnetic distinguished 2-form produced by the jet first order ODEs system** (2.1) *has the form*

$$F_{\Delta}^{ODEs} = \bar{F}_{(i)j}^{(1)} \delta x_1^i \wedge dx^j,$$

where

$$\delta x_1^i = dx_1^i + \bar{N}_{(1)k}^{(i)} dx^k, \ \forall\ i = \overline{1,n},$$

and

$$\bar{F}_{(i)j}^{(1)} = \frac{1}{2}\left[\frac{\partial X_{(1)}^{(i)}}{\partial x^j} - \frac{\partial X_{(1)}^{(j)}}{\partial x^i}\right], \ \forall\ i,j = \overline{1,n}.$$

(vi) The adapted components $\bar{F}_{(i)j}^{(1)}$ *of the electromagnetic d-form* F_{Δ}^{ODEs} *produced by the jet first order ODEs system (*2.1*) verify the* **generalized Maxwell equations**

$$\begin{cases} \bar{F}_{(i)j//1}^{(1)} = \frac{1}{4}\mathcal{A}_{\{i,j\}}\left[\frac{\partial^2 X_{(1)}^{(i)}}{\partial t \partial x^j} - \frac{\partial^2 X_{(1)}^{(j)}}{\partial t \partial x^i}\right] = \frac{1}{2}\left[\frac{\partial^2 X_{(1)}^{(i)}}{\partial t \partial x^j} - \frac{\partial^2 X_{(1)}^{(j)}}{\partial t \partial x^i}\right] \\ \sum_{\{i,j,k\}} \bar{F}_{(i)j||k}^{(1)} = 0, \end{cases}$$

where $\mathcal{A}_{\{i,j\}}$ *represents an alternate sum,* $\sum_{\{i,j,k\}}$ *means a cyclic sum and*

$$\bar{F}_{(i)j//1}^{(1)} = \frac{\partial \bar{F}_{(i)j}^{(1)}}{\partial t} \ and \ \bar{F}_{(i)j||k}^{(1)} = \frac{\partial \bar{F}_{(i)j}^{(1)}}{\partial x^k}, \ \forall\ i,j,k = \overline{1,n}.$$

(vii) The **geometric jet Yang-Mills energy produced by the jet first order ODEs system** (2.1) *has the expression*

$$EYM_{\Delta}^{ODEs}(t,x) = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[\bar{F}_{(i)j}^{(1)}\right]^2 = \frac{1}{4}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[\frac{\partial X_{(1)}^{(i)}}{\partial x^j} - \frac{\partial X_{(1)}^{(j)}}{\partial x^i}\right]^2.$$

Remark 2.8. *If we use the matrix notations*

- $J\left(X_{(1)}\right) = \left(\dfrac{\partial X_{(1)}^{(i)}}{\partial x^j}\right)_{i,j=\overline{1,n}}$ *- the* **Jacobian matrix***,*
- $\bar{N}_{(1)} = \left(\bar{N}_{(1)j}^{(i)}\right)_{i,j=\overline{1,n}}$ *- the* **non-linear connection matrix***,*
- $\bar{R}_{(1)1} = \left(\bar{R}_{(1)1j}^{(i)}\right)_{i,j=\overline{1,n}}$, *- the* **temporal torsion matrix***,*
- $\bar{R}_{(1)k} = \left(\bar{R}_{(1)jk}^{(i)}\right)_{i,j=\overline{1,n}}$, $\forall\ k = \overline{1,n}$, *- the* **spatial torsion matrices***,*
- $\bar{F}^{(1)} = \left(\bar{F}_{(i)j}^{(1)}\right)_{i,j=\overline{1,n}}$ *- the* **electromagnetic matrix***,*

*then the following matrix geometrical relations attached to the jet first order ODEs system (*2.1*) hold good:*

1. $\bar{N}_{(1)} = -\dfrac{1}{2}\left[J\left(X_{(1)}\right) - {}^{T}J\left(X_{(1)}\right)\right];$
2. $\bar{R}_{(1)1} = -\dfrac{\partial}{\partial t}\left[\bar{N}_{(1)}\right];$
3. $\bar{R}_{(1)k} = \dfrac{\partial}{\partial x^k}\left[\bar{N}_{(1)}\right],\ \forall\ k = \overline{1,n};$
4. $\bar{F}^{(1)} = -\bar{N}_{(1)};$
5. $EYM_{\Delta}^{ODEs}(t,x) = \dfrac{1}{2}\cdot Trace\left[\bar{F}^{(1)}\cdot\ {}^{T}\bar{F}^{(1)}\right]$, *that is the jet electromagnetic Yang-Mills energy coincides with the square of the norm of the skew-symmetric electromagnetic matrix $\bar{F}^{(1)}$ in the Lie algebra $o(n) = L(O(n))$.*

Remark 2.9. *Note that the spatial torsion matrix $\bar{R}_{(1)k}$ does not coincide for $k = 1$ with the temporal torsion matrix $\bar{R}_{(1)1}$. We have only an overlap of notations.*

3. Jet Riemann-Lagrange geometry produced by a non-homogenous linear ODEs system of order one

In this Section we apply the preceding jet Riemann-Lagrange geometrical results for a non-homogenous linear ODEs system of order one. In this way, let us consider the following non-homogenous linear first order ODEs system locally described, in a convenient chart on $J^1(T,\mathbb{R}^n)$, by the differential equations

$$\frac{dx^i}{dt} = \sum_{k=1}^{n} a^{(i)}_{(1)k}(t)x^k + f^{(i)}_{(1)}(t),\ \forall\, i=\overline{1,n}, \tag{3.1}$$

where the local components $a^{(i)}_{(1)k}$ and $f^{(i)}_{(1)}$ transform after the tensorial rules

$$a^{(i)}_{(1)k} = \frac{\partial x^i}{\partial \widetilde{x}^j}\frac{d\widetilde{t}}{dt}\cdot \widetilde{a}^{(j)}_{(1)k},\ \forall\, k=\overline{1,n},$$

and

$$f^{(i)}_{(1)} = \frac{\partial x^i}{\partial \widetilde{x}^j}\frac{d\widetilde{t}}{dt}\cdot \widetilde{f}^{(j)}_{(1)}.$$

Remark 3.1. *We suppose that the product manifold $T\times\mathbb{R}^n \subset J^1(T,\mathbb{R}^n)$ is endowed* **a priori** *with the pair of Euclidian metrics $\Delta=(1,\delta_{ij})$, with respect to the coordinates (t,x^i).*

It is obvious that the non-homogenous linear ODEs system (3.1) is a particular case of the jet first order non-linear ODEs system (2.1) for

$$X^{(i)}_{(1)}(t,x) = \sum_{k=1}^{n} a^{(i)}_{(1)k}(t)x^k + f^{(i)}_{(1)}(t),\ \forall\, i=\overline{1,n}. \tag{3.2}$$

In order to expose the main jet Riemann-Lagrange geometrical objects that characterize the non-homogenous linear ODEs system (3.1), we use the matrix notation

$$A_{(1)} = \left(a^{(i)}_{(1)j}(t)\right)_{i,j=\overline{1,n}}.$$

In this context, applying our preceding jet geometrical Riemann-Lagrange theory to the non-homogenous linear ODEs system (3.1) and the pair of Euclidian metrics $\Delta=(1,\delta_{ij})$, we get:

Theorem 3.2. *(i) The* **canonical non-linear connection on $J^1(T,\mathbb{R}^n)$ produced by the non-homogenous linear ODEs system** (3.1) *has the local components*

$$\hat{\Gamma} = \left(0, \hat{N}^{(i)}_{(1)j}\right),$$

where $\hat{N}^{(i)}_{(1)j}$ are the entries of the matrix

$$\hat{N}_{(1)} = \left(\hat{N}^{(i)}_{(1)j}\right)_{i,j=\overline{1,n}} = -\frac{1}{2}\left[A_{(1)} - {}^{T}A_{(1)}\right].$$

(ii) All adapted components of the **canonical generalized Cartan connection $C\hat{\Gamma}$ produced by the non-homogenous linear ODEs system** (3.1) *vanish.*

(iii) The effective adapted components $\hat{R}^{(i)}_{(1)1j}$ of the **torsion** *d-tensor* $\hat{\mathbf{T}}$ *of the canonical generalized Cartan connection $C\hat{\Gamma}$* **produced by the non-homogenous linear ODEs system** (3.1) *are the entries of the matrices*

$$\hat{R}_{(1)1} = \left(\hat{R}^{(i)}_{(1)1j}\right)_{i,j=\overline{1,n}} = \frac{1}{2}\left[\dot{A}_{(1)} - {}^{T}\dot{A}_{(1)}\right],$$

where

$$\dot{A}_{(1)} = \frac{d}{dt}\left[A_{(1)}\right].$$

(iv) All adapted components of the **curvature** *d-tensor* $\hat{\mathbf{R}}$ *of the canonical generalized Cartan connection* $C\hat{\Gamma}$ **produced by the non-homogenous linear ODEs system** (3.1) *vanish.*

(v) The **geometric electromagnetic distinguished 2-form produced by the non-homogenous linear ODEs system** (3.1) *is given by*

$$\hat{F} = \hat{F}^{(1)}_{(i)j}\delta x^i_1 \wedge dx^j,$$

where

$$\delta x^i_1 = dx^i_1 - \frac{1}{2}\left[a^{(i)}_{(1)k} - a^{(k)}_{(1)i}\right] dx^k, \ \forall\ i = \overline{1,n},$$

and the adapted components $\hat{F}^{(1)}_{(i)j}$ *are the entries of the matrix*

$$\hat{F}^{(1)} = \left(\hat{F}^{(1)}_{(i)j}\right)_{i,j=\overline{1,n}} = -\hat{N}_{(1)} = \frac{1}{2}\left[A_{(1)} - {}^T A_{(1)}\right],$$

that is

$$\hat{F}^{(1)}_{(i)j} = \frac{1}{2}\left[a^{(i)}_{(1)j} - a^{(j)}_{(1)i}\right].$$

(vi) The **jet Yang-Mills energy produced by the non-homogenous linear ODEs system** (3.1) *is given by the formula*

$$EYM^{NHLODEs}(t) = \frac{1}{4}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[a^{(i)}_{(1)j} - a^{(j)}_{(1)i}\right]^2.$$

Proof. Using the relations (3.2), we easily deduce that we have the Jacobian matrix

$$J\left(X_{(1)}\right) = A_{(1)}.$$

Consequently, applying the Corollary 2.7 to the non-homogenous linear ODEs system (3.1), together with the Remark 2.8, we obtain the required results. □

Remark 3.3. *The entire jet Riemann-Lagrange geometry produced by the non-homogenous linear ODEs system* (3.1) *does not depend on the non-homogeneity terms* $f^{(i)}_{(1)}(t)$*.*

Remark 3.4. *The jet Yang-Mills energy produced by the non-homogenous linear ODEs system* (3.1) *vanishes if and only if the matrix* $A_{(1)}$ *is a symmetric one. In this case, the entire jet Riemann-Lagrange geometry produced by the non-homogenous linear ODEs system* (3.1) *vanish, so it does not offer geometrical informations about the system* (3.1)*. However, it is important to note that in this particular situation we have the symetry of the matrix* $A_{(1)}$*, which implies that the matrix* $A_{(1)}$ *is diagonalizable.*

Remark 3.5. *All torsion adapted components of a non-homogenous linear ODEs system with constant coefficients* $a^{(i)}_{(1)j}$ *are zero.*

4. Jet Riemann-Lagrange geometry produced by a superior order ODE

Let us consider the superior order ODE expressed by

$$y^{(n)}(t) = f(t, y(t), y'(t), ..., y^{(n-1)}(t)),\ n \geq 2, \tag{4.1}$$

where $y(t)$ is an unknown function, $y^{(k)}(t)$ is the derivative of order k of the unknown function $y(t)$ for each $k \in \{0, 1, ..., n\}$ and f is a given differentiable function depending on the distinct variables t, $y(t)$, $y'(t), ...,$ $y^{(n-1)}(t)$.

It is well known the fact that, using the notations

$$x^1 = y,\ x^2 = y',\ ...,\ x^n = y^{(n-1)},$$

the superior order ODE (4.1) is equivalent with the non-linear ODEs system of order one

$$\begin{cases} \dfrac{dx^1}{dt} = x^2 \\ \dfrac{dx^2}{dt} = x^3 \\ . \\ . \\ . \\ \dfrac{dx^{n-1}}{dt} = x^n \\ \dfrac{dx^n}{dt} = f(t, x^1, x^2, ..., x^n). \end{cases} \tag{4.2}$$

But, the first order non-linear ODEs system (4.2) can be regarded, in a convenient local chart, as a particular case of the jet non-linear ODEs system of order one (2.1), taking

$$\begin{array}{llll} X^{(1)}_{(1)}(t, x) = x^2, & X^{(2)}_{(1)}(t, x) = x^3, & \cdots & \\ \cdots & & X^{(n-1)}_{(1)}(t, x) = x^n, & X^{(n)}_{(1)}(t, x) = f(t, x^1, x^2, ..., x^n), \end{array} \tag{4.3}$$

where we suppose that the geometrical object $X = \left(X^{(i)}_{(1)}(t, x)\right)$ behaves like a d-tensor on $J^1(T, \mathbb{R}^n)$.

Remark 4.1. *We assume that the product manifold* $T \times \mathbb{R}^n \subset J^1(T, \mathbb{R}^n)$ *is endowed* **a priori** *with the pair of Euclidian metrics* $\Delta = (1, \delta_{ij})$, *with respect to the coordinates* (t, x^i).

Definition 4.2. *Any geometrical object on* $J^1(T, \mathbb{R}^n)$, *which is produced by the* **first order non-linear ODEs system** (4.2) *is called* **geometrical object produced by the superior order ODE** (4.1).

In this context, the Riemann-Lagrange geometrical behavior on the 1-jet space $J^1(T, \mathbb{R}^n)$ of the superior order ODE (4.1) is described in the following result:

Theorem 4.3. *(i) The* **canonical non-linear connection on** $J^1(T,\mathbb{R}^n)$ **produced by the superior order ODE** (4.1) *has the local components*

$$\check{\Gamma} = \left(0, \check{N}^{(i)}_{(1)j}\right),$$

where $\check{N}^{(i)}_{(1)j}$ *are the entries of the matrix* $\check{N}_{(1)} = \left(\check{N}^{(i)}_{(1)j}\right)_{i,j=\overline{1,n}} =$

$$= -\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 & 0 & -\dfrac{\partial f}{\partial x^1} \\ -1 & 0 & 1 & \cdot & \cdot & 0 & 0 & -\dfrac{\partial f}{\partial x^2} \\ 0 & -1 & 0 & \cdot & \cdot & 0 & 0 & -\dfrac{\partial f}{\partial x^3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 & 1 & -\dfrac{\partial f}{\partial x^{n-2}} \\ 0 & 0 & 0 & \cdot & \cdot & -1 & 0 & 1-\dfrac{\partial f}{\partial x^{n-1}} \\ \dfrac{\partial f}{\partial x^1} & \dfrac{\partial f}{\partial x^2} & \dfrac{\partial f}{\partial x^3} & \cdot & \cdot & \dfrac{\partial f}{\partial x^{n-2}} & -1+\dfrac{\partial f}{\partial x^{n-1}} & 0 \end{pmatrix}.$$

(ii) All adapted components of the **canonical generalized Cartan connection** $C\check{\Gamma}$ **produced by the superior order ODE** (4.1) *vanish.*

(iii) The effective adapted components of the **torsion** *d-tensor* $\check{\mathbf{T}}$ *of the canonical generalized Cartan connection* $C\check{\Gamma}$ **produced by the superior order ODE** (4.1) *are the entries of the matrices*

$$\check{R}_{(1)1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & \cdot & \cdot & 0 & -\dfrac{\partial^2 f}{\partial t\partial x^1} \\ 0 & 0 & \cdot & \cdot & 0 & -\dfrac{\partial^2 f}{\partial t\partial x^2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & -\dfrac{\partial^2 f}{\partial t\partial x^{n-1}} \\ \dfrac{\partial^2 f}{\partial t\partial x^1} & \dfrac{\partial^2 f}{\partial t\partial x^2} & \cdot & \cdot & \dfrac{\partial^2 f}{\partial t\partial x^{n-1}} & 0 \end{pmatrix}$$

and

$$\check{R}_{(1)k} = -\frac{1}{2}\begin{pmatrix} 0 & 0 & \cdot\ \cdot & 0 & -\dfrac{\partial^2 f}{\partial x^k \partial x^1} \\ 0 & 0 & \cdot\ \cdot & 0 & -\dfrac{\partial^2 f}{\partial x^k \partial x^2} \\ \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\ 0 & 0 & \cdot\ \cdot & 0 & -\dfrac{\partial^2 f}{\partial x^k \partial x^{n-1}} \\ \dfrac{\partial^2 f}{\partial x^k \partial x^1} & \dfrac{\partial^2 f}{\partial x^k \partial x^2} & \cdot\ \cdot & \dfrac{\partial^2 f}{\partial x^k \partial x^{n-1}} & 0 \end{pmatrix},$$

where $k \in \{1, 2, ..., n\}$.

(iv) All adapted components of the **curvature** *d-tensor* $\check{\mathbf{R}}$ *of the canonical generalized Cartan connection* $C\check{\Gamma}$ **produced by the superior order ODE** (4.1) *vanish.*

(v) The **geometric electromagnetic distinguished 2-form produced by the superior order ODE** (4.1) *has the form*

$$\check{F} = \check{F}^{(1)}_{(i)j}\delta x^i_1 \wedge dx^j,$$

where

$$\delta x^i_1 = dx^i_1 + \check{N}^{(i)}_{(1)k} dx^k,\ \forall\ i = \overline{1, n},$$

and the adapted components $\check{F}^{(1)}_{(i)j}$ *are the entries of the matrix*

$$\check{F}^{(1)} = \left(\check{F}^{(1)}_{(i)j}\right)_{i,j=\overline{1,n}} = -\check{N}_{(1)}.$$

(vi) The **jet geometric Yang-Mills energy produced by the superior order ODE** (4.1) *is given by the formula*

$$EYM^{SODE}(t, x) = \frac{1}{4}\left[n - 1 - 2\frac{\partial f}{\partial x^{n-1}} + \sum_{j=1}^{n-1}\left(\frac{\partial f}{\partial x^j}\right)^2\right].$$

Proof. By partial derivatives, the relations (4.3) lead to the Jacobian matrix

$$J\left(X_{(1)}\right) = \begin{pmatrix} 0 & 1 & 0 & \cdot\ \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot\ \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot\ \cdot & 0 & 1 \\ \dfrac{\partial f}{\partial x^1} & \dfrac{\partial f}{\partial x^2} & \dfrac{\partial f}{\partial x^3} & \cdot\ \cdot & \dfrac{\partial f}{\partial x^{n-1}} & \dfrac{\partial f}{\partial x^n} \end{pmatrix}.$$

In conclusion, the Corollary 2.7, together with the Remark 2.8, applied to first order non-linear ODEs system (4.2), give what we were looking for. □

5. Riemann-Lagrange geometry produced by a non-homogenous linear ODE of superior order

If we consider the non-homogenous linear ODE of order $n \in \mathbb{N}$, $n \geq 2$, expressed by

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + ... + a_{n-1}(t)y' + a_n(t)y = b(t), \tag{5.1}$$

where $b(t)$ and $a_i(t)$, $\forall\ i = \overline{0,n}$, are given differentiable real functions and $a_0(t) \neq 0$, $\forall\ t \in [a,b]$, then we recover the superior order ODE (4.1) for the particular function

$$f(t,x) = \frac{b(t)}{a_0(t)} - \frac{a_n(t)}{a_0(t)} \cdot x^1 - \frac{a_{n-1}(t)}{a_0(t)} \cdot x^2 - ... - \frac{a_1(t)}{a_0(t)} \cdot x^n, \tag{5.2}$$

where we recall that we have

$$y = x^1, y' = x^2, ..., y^{(n-1)} = x^n.$$

Consequently, we can derive the jet Riemann-Lagrange geometry attached to the non-homogenous linear superior order ODE (5.1).

Corollary 5.1. *(i) The* **canonical non-linear connection on** $J^1(T, \mathbb{R}^n)$ **produced by the non-homogenous linear superior order ODE** (5.1) *has the local components*

$$\tilde{\Gamma} = \left(0, \tilde{N}^{(i)}_{(1)j}\right),$$

where $\tilde{N}^{(i)}_{(1)j}$ *are the entries of the matrix*

$$\tilde{N}_{(1)} = \left(\tilde{N}^{(i)}_{(1)j}\right)_{i,j=\overline{1,n}} =$$

$$= -\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & \cdot\ \cdot & 0 & 0 & \dfrac{a_n}{a_0} \\ -1 & 0 & 1 & \cdot\ \cdot & 0 & 0 & \dfrac{a_{n-1}}{a_0} \\ 0 & -1 & 0 & \cdot\ \cdot & 0 & 0 & \dfrac{a_{n-2}}{a_0} \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot\ \cdot & 0 & 1 & \dfrac{a_3}{a_0} \\ 0 & 0 & 0 & \cdot\ \cdot & -1 & 0 & 1+\dfrac{a_2}{a_0} \\ -\dfrac{a_n}{a_0} & -\dfrac{a_{n-1}}{a_0} & -\dfrac{a_{n-2}}{a_0} & \cdot\ \cdot & -\dfrac{a_3}{a_0} & -1-\dfrac{a_2}{a_0} & 0 \end{pmatrix}.$$

(ii) All adapted components of the **canonical generalized Cartan connection** $C\tilde{\Gamma}$ **produced by the non-homogenous linear superior order ODE** (5.1) *vanish.*

(iii) All adapted components of the **torsion** *d-tensor* $\tilde{\mathbf{T}}$ *of the canonical generalized Cartan connection* $C\tilde{\Gamma}$ **produced by the non-homogenous linear superior order ODE** (5.1) *are zero, except the temporal components*

$$\tilde{R}^{(i)}_{(1)1n} = -\tilde{R}^{(n)}_{(1)1i} = \frac{a'_{n-i+1}a_0 - a_{n-i+1}a'_0}{2a_0^2}, \ \forall\ i = \overline{1, n-1},$$

where we denoted by " $'$ " the derivatives of the functions $a_k(t)$.

(iv) All adapted components of the **curvature** *d-tensor* $\tilde{\mathbf{R}}$ *of the canonical generalized Cartan connection* $C\tilde{\Gamma}$ **produced by the non-homogenous linear superior order ODE** (5.1) *vanish.*

(v) The **geometric electromagnetic distinguished 2-form produced by the non-homogenous linear superior order ODE** (5.1) *has the expression*

$$\tilde{F} = \tilde{F}^{(1)}_{(i)j}\delta x^i_1 \wedge dx^j,$$

where

$$\delta x^i_1 = dx^i_1 + \tilde{N}^{(i)}_{(1)k}dx^k, \ \forall\ i = \overline{1, n},$$

and the adapted components $\tilde{F}^{(1)}_{(i)j}$ *are the entries of the matrix*

$$\tilde{F}^{(1)} = \left(\tilde{F}^{(1)}_{(i)j}\right)_{i,j=\overline{1,n}} = -\tilde{N}_{(1)}.$$

(vi) The **jet geometric Yang-Mills electromagnetic energy produced by the non-homogenous linear superior order ODE** (5.1) *has the form*

$$EYM^{NHLSODE}(t) = \frac{1}{4}\left[n - 1 + 2\frac{a_2}{a_0} + \sum_{j=2}^{n}\frac{a_j^2}{a_0^2}\right].$$

Proof. We apply the Theorem 4.3 for the particular function (5.2) and we use the relations

$$\frac{\partial f}{\partial x^j} = -\frac{a_{n-j+1}}{a_0}, \ \forall\ j = \overline{1, n}.$$

□

Remark 5.2. *The entire jet Riemann-Lagrange geometry produced by the non-homogenous linear superior order ODE (*5.1*) is independent by the term of non-homogeneity* $b(t)$*. In author's opinion, this fact emphasizes that the most important role in the study of the ODE (*5.1*) is played by its attached homogenous linear superior order ODE.*

Example 5.3. *The law of motion without friction (***harmonic oscillator***) of a material point of mass* $m > 0$*, which is placed on a spring having the constant of elasticity* $k > 0$*, is given by the homogenous linear ODE of order two*

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \tag{5.3}$$

where the coordinate y *measures the distance from the mass's equlibrium point and* $\omega^2 = k/m$*. It follows that we have*

$$n = 2,\ a_0(t) = 1,\ a_1(t) = 0 \text{ and } a_2(t) = \omega^2,$$

that is the **harmonic oscillator** *second order ODE (*5.3*) provides the* **jet geometric Yang-Mills electromagnetic energy**

$$EYM^{Harmonic\ Oscillator} = \frac{1}{4}\left(1+\omega^2\right)^2.$$

Open problem. There exists a real physical interpretation for the previous jet geometric Yang-Mills electromagnetic energy attached to the harmonic oscillator?

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References

[1] Asanov, G. S., *Jet extension of Finslerian gauge approach*, Fortschritte der Physik, **38**(1990), no. 8, 571-610.

[2] Balan, V., *Generalized Maxwell and Lorentz equations on first order geometrized jet spaces*, Proceeding of the International Conference in Geometry and Topology, Cluj-Napoca, Romania, 1-5 October 2002; University of Cluj-Napoca Editors, 2004, 11-24.

[3] Miron, R., Anastasiei, M., *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, 1994.

[4] Neagu, M., *Riemann-Lagrange Geometry on* 1*-Jet Spaces*, Matrix Rom, Bucharest, 2005.

[5] Neagu, M., Nicola, I. R., *Geometric dynamics of calcium oscillations ODEs systems*, Balkan Journal of Geometry and Its Applications, **9**(2004), no. 2, 36-67.

[6] Neagu, M., Udrişte, C., *From PDEs systems and metrics to multi-time field theories and geometric dynamics*, Seminarul de Mecanică, Sisteme Dinamice Diferenţiale, **79**(2001), Timişoara, Romania.

[7] Olver, P. J., *Applications of Lie Groups to Differential Equations*, Springer-Verlag, 1986.

[8] Poincaré, H., *Sur les courbes definies par les equations différentielle*, C. R. Acad. Sci., Paris, **90**(1880), 673-675.

[9] Saunders, D. J., *The Geometry of Jet Bundles*, Cambridge University Press, New York, London, 1989.

[10] Udrişte, C., *Geometric Dynamics*, Kluwer Academic Publishers, 2000.

[11] Udrişte, C., Postolache, M., *Atlas of Magnetic Geometric Dynamics*, Geometry Balkan Press, Bucharest, 2001.

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Pappus's harmonic theorem in the Einstein relativistic velocity model of hyperbolic geometry

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Abstract. In this note, we present a proof of Pappus's harmonic theorem in the Einstein relativistic velocity model of hyperbolic geometry.

Mathematics Subject Classification (2010): 51K05, 51M10, 30F45, 20N99, 51B10.

Keywords: Hyperbolic geometry, hyperbolic triangle, Pappus's harmonic theorem, gyrovector, Einstein relativistic velocity model.

1. Introduction

Hyperbolic geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis for geometry. It is also known as a type of non-Euclidean geometry, being in many respects similar to Euclidean geometry. Hyperbolic geometry includes such concepts as: distance, angle and both of them have many theorems in common. There are known many main models for hyperbolic geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. The hyperbolic geometry is a non-Euclidian geometry. Here, in this study, we present a proof of Pappus's harmonic theorem in the Einstein relativistic velocity model of hyperbolic geometry. Pappus's harmonic theorem states that if $A'B'C'$ is the cevian triangle of point M with respect to the triangle ABC such that the lines $B'C'$ and BC meet at A'', then $\frac{A''B}{A''C} = \frac{A'B}{A'C}$ [4].

Let D denote the complex unit disc in complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of D is

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyrotranslation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then is true gyrocommutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyrovector space $(G, \oplus, \otimes)$ is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$

(G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$

(G4) $\frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$

(G5) $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$

(G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of onedimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(G7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$

(G8) $\|\mathbf{a} \oplus \mathbf{b}\| \le \|\mathbf{a}\| \oplus \|\mathbf{b}\|$

Theorem 1.1. (The Hyperbolic Theorem of Ceva in Einstein Gyrovector Space). *Let $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ be three non-gyrocollinear points in an Einstein gyrovector space $(V_s, \oplus, \otimes)$. Furthermore, let $\mathbf{a}_{123}$ be a point in their gyroplane, which is off the gyrolines $\mathbf{a}_1\mathbf{a}_2, \mathbf{a}_2\mathbf{a}_3$, and $\mathbf{a}_3\mathbf{a}_1$. If $\mathbf{a}_1\mathbf{a}_{123}$ meets $\mathbf{a}_2\mathbf{a}_3$ at $\mathbf{a}_{23}$, etc., then*

$$\frac{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}} \|\ominus\mathbf{a}_1 \oplus \mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}} \|\ominus\mathbf{a}_2 \oplus \mathbf{a}_{12}\|} \frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}} \|\ominus\mathbf{a}_2 \oplus \mathbf{a}_{23}\|}{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}} \|\ominus\mathbf{a}_3 \oplus \mathbf{a}_{23}\|} \frac{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}} \|\ominus\mathbf{a}_3 \oplus \mathbf{a}_{13}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}} \|\ominus\mathbf{a}_1 \oplus \mathbf{a}_{13}\|} = 1,$$

(here $\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{s^2}}}$ is the gamma factor).

(see [6, p. 461])

Theorem 1.2. (The Hyperbolic Theorem of Menelaus in Einstein Gyrovector Space). *Let* $\mathbf{a}_1, \mathbf{a}_2,$ *and* $\mathbf{a}_3$ *be three non-gyrocollinear points in an Einstein gyrovector space* $(V_s, \oplus, \otimes)$. *If a gyroline meets the sides of gyrotriangle* $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ *at points* $\mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{23},$ *then*

$$\frac{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}}\,\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}}\,\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}\|}\,\frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}}\,\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}\|}{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}}\,\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}\|}\,\frac{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}}\,\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}}\,\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}\|} = 1.$$

(see [6, p. 463])

Theorem 1.3. (The Gyrotriangle Bisector Theorem). *Let* ABC *be a gyrotriangle in an Einstein gyrovector space* $(V_s, \oplus, \otimes)$, *and let* P *be a point lying on side* BC *of the gyrotriangle such that* AP *is a bisector of gyroangle* $\measuredangle BAC$. *Then,*

$$\frac{\gamma_{|BP|}\,|BP|}{\gamma_{|PC|}\,|PC|} = \frac{\gamma_{|AB|}\,|AB|}{\gamma_{|AC|}\,|AC|}.$$

(see [7, p. 150])

For further details we refer to the recent book of A.Ungar [6].

Definition 1.4. *The symmetric of the median with respect to the internal bisector issued from the same vertex is called symmedian.*

Theorem 1.5. *If the gyroline* AP *is a symmedian of a gyrotriangle* ABC, *and the point* P *is on the gyroside* BC, *then*

$$\frac{\gamma_{|CP|}\,|CP|}{\gamma_{|BP|}\,|BP|} = \left(\frac{\gamma_{|CA|}\,|CA|}{\gamma_{|BA|}\,|BA|}\right)^2.$$

(See [3])

Definition 1.6. *We call antibisector of a triangle, the izotomic of a internal bisector of a triangle interior angle.*

2. Main results

In this section, we present a proof of Pappus's harmonic theorem in the Einstein relativistic velocity model of hyperbolic geometry.

Theorem 2.1. (Pappus's harmonic theorem for hyperbolic gyrotriangle). *If* $A'B'C'$ *is the cevian gyrotriangle of gyropoint* M *with respect to the gyrotriangle* ABC *such that the gyrolines* $B'C'$ *and* BC *meet at* A'', *then*

$$\frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|} = \frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|}.$$

Proof. If we use Theorem 1.1 in the gyrotriangle ABC (see Figure 1), we have

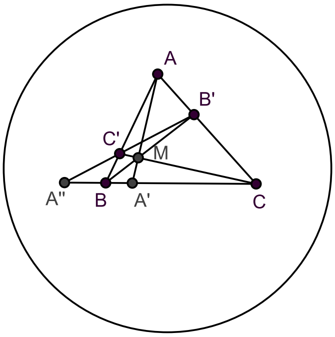

Figure 1

$$\frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|} \cdot \frac{\gamma_{|B'C|}|B'C|}{\gamma_{|B'A|}|B'A|} \cdot \frac{\gamma_{|C'A|}|C'A|}{\gamma_{|C'B|}|C'B|} = 1. \tag{2.1}$$

If we use Theorem 1.2 in the gyrotriangle ABC, cut by the gyroline $A'A''$, we get

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} \cdot \frac{\gamma_{|B'C|}|B'C|}{\gamma_{|B'A|}|B'A|} \cdot \frac{\gamma_{|C'A|}|C'A|}{\gamma_{|C'B|}|C'B|} = 1. \tag{2.2}$$

From the relations (2.1) and (2.2) we have $\frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|} = \frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|}$. □

Corollary 2.2. *If $A'B'C'$ is the cevian gyrotriangle of gyropoint M with respect to the gyrotriangle ABC such that the gyrolines $B'C'$ and BC meet at A'', and AA' is a bisector of gyroangle $\measuredangle BAC$, then*

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|}.$$

Proof. If we use Theorem 1.3 in the triangle ABC, we get

$$\frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|} = \frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|}. \tag{2.3}$$

If we use Theorem 2.1 in the triangle ABC, we get

$$\frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|} = \frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|}. \tag{2.4}$$

From the relations (2.3) and (2.4) we have $\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|}$. □

Corollary 2.3. *If $A'B'C'$ is the cevian gyrotriangle of gyropoint M with respect to the gyrotriangle ABC such that the gyrolines $B'C'$ and BC meet at A'', and*

AA' is a bisector of gyroangle $\measuredangle BAC$, and AA_1 is a antibisector of gyroangle $\measuredangle BAC$, then

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \left(\frac{\gamma_{|A_1B|}|A_1B|}{\gamma_{|A_1C|}|A_1C|}\right)^{-1}.$$

Proof. Because the gyroline AA_1 is a isotomic line of the bisector AA', then

$$\frac{\gamma_{|A_1B|}|A_1B|}{\gamma_{|A_1C|}|A_1C|} = \frac{\gamma_{|A'C|}|A'C|}{\gamma_{|A'B|}|A'B|} = \frac{\gamma_{|AC|}|AC|}{\gamma_{|AB|}|AB|}. \tag{2.5}$$

If we use Corollary 2.2, we have

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|}. \tag{2.6}$$

From the relations (2.5) and (2.6), we have

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \left(\frac{\gamma_{|A_1B|}|A_1B|}{\gamma_{|A_1C|}|A_1C|}\right)^{-1}. \tag{2.7}$$

□

Corollary 2.4. *If $A'B'C'$ is the cevian gyrotriangle of gyropoint M with respect to the gyrotriangle ABC such that the gyrolines $B'C'$ and BC meet at A'', and AA' is a symmedian of gyroangle $\measuredangle BAC$, and the point A' is on the gyroside BC, then*

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \left(\frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|}\right)^{2}.$$

Proof. If we use Theorem 1.5, we have

$$\frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|} = \left(\frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|}\right)^{2}. \tag{2.8}$$

If we use Theorem 2.1, we have

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|}. \tag{2.9}$$

From the relations (2.8) and (2.9), we get $\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \left(\frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|}\right)^{2}$. □

Theorem 2.5. *If $A'B'C'$ is the cevian gyrotriangle of gyropoint M with respect to the gyrotriangle ABC such that the gyrolines $B'C'$ and BC meet at A'', and AA' is a bisector of gyroangle $\measuredangle BAC$, the gyrolines $A'C'$ and BB' meet at D, $A'B'$ and CC' meet at E, AD and BC meet at D', and AE and BC meet in E', then*

$$\frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|} = \frac{\gamma_{|D'B|}|D'B|}{\gamma_{|D'A'|}|D'A'|} \cdot \frac{\gamma_{|E'A'|}|E'A'|}{\gamma_{|E'C|}|E'C|}.$$

Proof. If we use Theorem 1.1 in the gyrotriangle ABA' (see Figure 2),

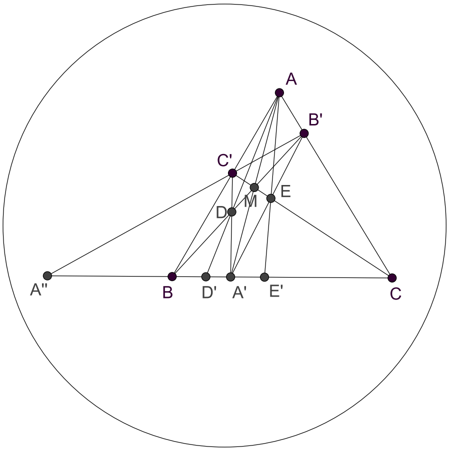

Figure 2

we have

$$\frac{\gamma_{|D'B|}|D'B|}{\gamma_{|D'A'|}|D'A'|} \cdot \frac{\gamma_{|C'A|}|C'A|}{\gamma_{|C'B|}|C'B|} \cdot \frac{\gamma_{|MA'|}|MA'|}{\gamma_{|MA|}|MA|} = 1. \tag{2.10}$$

If we use Theorem 1.2 in the gyrotriangle ABA', cut by the gyroline CC', we get

$$\frac{\gamma_{|CB|}|CB|}{\gamma_{|CA'|}|CA'|} \cdot \frac{\gamma_{|C'A|}|C'A|}{\gamma_{|C'B|}|C'B|} \cdot \frac{\gamma_{|MA'|}|MA'|}{\gamma_{|MA|}|MA|} = 1. \tag{2.11}$$

From the relations (2.10) and (2.11), we have

$$\frac{\gamma_{|D'B|}|D'B|}{\gamma_{|D'A'|}|D'A'|} = \frac{\gamma_{|CB|}|CB|}{\gamma_{|CA'|}|CA'|}. \tag{2.12}$$

Similarly, we obtain that

$$\frac{\gamma_{|E'C|}|E'C|}{\gamma_{|E'A'|}|E'A'|} = \frac{\gamma_{|BC|}|BC|}{\gamma_{|BA'|}|BA'|}. \tag{2.13}$$

If ratios the equations (2.12) and (2.13) among themselves, respectively, then

$$\frac{\gamma_{|D'B|}|D'B|}{\gamma_{|D'A'|}|D'A'|} \cdot \frac{\gamma_{|E'A'|}|E'A'|}{\gamma_{|E'C|}|E'C|} = \frac{\gamma_{|BA'|}|BA'|}{\gamma_{|CA'|}|CA'|}. \tag{2.14}$$

If we use Theorem 1.3 and the Corollary 2.2 in the triangle ABC, we get

$$\frac{\gamma_{|A'B|}|A'B|}{\gamma_{|A'C|}|A'C|} = \frac{\gamma_{|AB|}|AB|}{\gamma_{|AC|}|AC|} = \frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|}. \tag{2.15}$$

From the relations (2.14) and (2.15), we get

$$\frac{\gamma_{|D'B|}|D'B|}{\gamma_{|D'A'|}|D'A'|} \cdot \frac{\gamma_{|E'A'|}|E'A'|}{\gamma_{|E'C|}|E'C|} = \frac{\gamma_{|A''B|}|A''B|}{\gamma_{|A''C|}|A''C|}.$$

□

References

[1] Barbu, C., *Teoreme fundamentale din geometria triunghiului*, Ed. Unique, Bacău, 2008.

[2] Barbu, C., *Smarandache's cevian triangle theorem in the Einstein relativistic velocity model of hyperbolic geometry*, Progress in Physics, 2010.

[3] Barbu, C., *Steiner's Isogonals Theorem in the Einstein Relativistic Model of Hyperbolic Geometry*, Annales Mathematicae et Informaticae (to appear).

[4] Coxeter, H., Greitzer, S., *Geometry revisited*, Washington, DC, Math. Assoc. Amer., 1967, 67-68.

[5] Smarandache, F., *Eight Solved and Eight Open Problems in Elementary Geometry*, arXiv: 1003.2153.

[6] Ungar, A. A., *Analytic hyperbolic geometry and Albert Einstein's Special Theory of Relativity*, World Scientific Publishing Co., Hackensack (NJ), 2008.

[7] Ungar, A. A., *A Gyrovector Space Approach to Hyperbolic Geometry*, Morgan & Claypool Publishers, 2009.

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The Sălăgean integral operator and strongly starlike functions

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Abstract. Let A denote the class of analytic functions $f(z)$ defined in the unit disc $U = \{z : |z| < 1\}$ and satisfying the conditions $f(0) = f'(0) - 1 = 0$. We introduce some new subclasses of strongly starlike functions defined by the Sălăgean integral operator and study their properties.

Mathematics Subject Classification (2010): 30C45.

Keywords: Strongly starlike function, strongly convex function, Sălăgean integral operator.

1. Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in A$ is said to be starlike of order γ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \;\; (z \in U) \tag{1.2}$$

for some $\gamma\,(0 \leq \gamma < 1)$. We denote by $S^*(\gamma)$ the subclass of A consisting of functions which are starlike of order γ in U. Also, a function $f(z) \in A$ is said to be convex of order γ if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma \;\; (z \in U) \tag{1.3}$$

for some γ $(0 \leq \gamma < 1)$. We denote by $C(\gamma)$ the subclass of A consisting of all functions which are convex of order γ in U.

It follows from (1.2) and (1.3) that

$$f(z) \in C(\gamma) \iff zf'(z) \in S^*(\gamma), \tag{1.4}$$

the classes $S^*(\gamma)$ and $C(\gamma)$ were introduced by Robertcen [8].
If $f(z) \in A$ satisfies

$$\left|\arg(\frac{zf'(z)}{f(z)} - \gamma)\right| < \frac{\pi}{2}\beta \quad (z \in U) \tag{1.5}$$

for some $\gamma\,(0 \leq \gamma < 1)$ and $\beta(0 < \beta \leq 1)$, then $f(z)$ is said to be strongly starlike of order β and type γ in U. We denote this by $f(z) \in S^*(\beta, \gamma)$.

If $f(z) \in A$ satisfies

$$\left|\arg(1 + \frac{zf''(z)}{f'(z)} - \gamma)\right| < \frac{\pi}{2}\beta \quad (z \in U) \tag{1.6}$$

for some $\gamma\,(0 \leq \gamma < 1)$ and $\beta(0 < \beta \leq 1)$, then we say that $f(z)$ is strongly convex of order β and type γ in U. We denote by $C(\beta, \gamma)$ the class of all such functions (see also Liu [3] and Nurokawa et al. [7]). In particular, the classes $S^*(\beta, 0)$ and $C(\beta, 0)$ have been extensively studied by Mocanu [5] and Nunokawa [6].

It follows from (1.5) and (1.6) that

$$f(z) \in C(\beta, \gamma) \iff zf'(z) \in S^*(\beta, \gamma). \tag{1.7}$$

Also, we note that $S^*(1, \gamma) = S^*(\gamma)$ and $C(1, \gamma) = C(\gamma)$.

For a function $f(z) \in A$, we define the integral operator $I^n f(z)$, $n \in N_0 = N \cup \{0\}$, where $N = \{1, 2, ...\}$, by

$$I^0 f(z) = f(z), \tag{1.8}$$

$$I^1 f(z) = I\,f(z) = \int_0^z f(t)\,t^{-1} dt, \tag{1.9}$$

and (in general)

$$I^n f(z) = I(I^{n-1} f(z)). \tag{1.10}$$

It is easy to see that:

$$(i) \qquad I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k \quad (n \in N_0), \tag{1.11}$$

and

$$(ii) \qquad z(I^n f(z))' = I^{n-1} f(z). \tag{1.12}$$

The integral operator $I^n f(z)\,(f \in A)$ was introduced by Sălăgean [9] and studied by Aouf et al. [1]. We call the operator I^n by Sălăgean integral operator. The relation (1.12) plays an important and significant role in obtaining our results.

Using the Sălăgean integral operator, we introduce and study the properties of some new classes of analytic functions, defined as follows:

$$S_n^*(\beta,\gamma) = \{ f(z) \in A : I^n f(z) \in S^*(\beta,\gamma), \frac{z(I^n f(z))'}{I^n f(z)} \neq \gamma \text{ for all } z \in U\}$$

and

$$C_n(\beta,\gamma) = \{f(z) \in A : I^n f(z) \in C(\beta,\gamma), 1 + \frac{z(I^n f(z))''}{(I^n f(z))'} \neq \gamma \text{ for all } z \in U\}.$$

Clearly,

$$f(z) \in C_n(\beta,\gamma) \Longleftrightarrow zf'(z) \in S_n^*(\beta,\gamma). \tag{1.13}$$

We note that:

(*i*) $S_n^*(\beta,\gamma) = S^*(\beta,\gamma)$ and $C_0^*(\beta,\gamma) = C(\beta,\gamma)$;

and

(*ii*) $S_0^*(1,\gamma) = S^*(\gamma)$ and $C_0^*(1,\gamma) = C(\gamma)$.

2. Main Results

In order to give our results, we need the following lemma, which is due to Nunokawa [6].

Lemma 2.1. *Let a function* $p(z) = 1 + c_1 z + c_2 z^2 + \ldots.$ *be analytic in* U *and* $p(z) \neq 0$ $(z \in U)$. *If there exists a point* $z_0 \in U$ *such that*

$$|\arg f(z)| < \frac{\pi}{2}\beta, \quad (|z| < |z_0|) \text{ and } |\arg p(z_0)| = \frac{\pi}{2}\beta \ (0 < \beta \leq 1),$$

then we have $\dfrac{z p_0'(z)}{p(z_0)} = ik\beta$, *where*

$$k \geq \frac{1}{2}(a + \frac{1}{a}) \quad (\text{when} \quad \arg p(z_0) = \frac{\pi}{2}\beta),$$

$$k \leq \frac{-1}{2}(a + \frac{1}{a}) \quad (\text{when} \quad \arg p(z_0) = \frac{-\pi}{2}\beta),$$

and $(p(z_0))^{\frac{1}{\beta}} = \pm ia \ (a > 0)$.

Theorem 2.2. $S_n^*(\beta,\gamma) \subset S_{n+1}^*(\beta,\gamma)$ *for each* $n \in N_0$.

Proof. Let $f(z) \in S_n^*(\beta,\gamma)$. Then we put

$$\frac{z(I^{n+1} f(z))'}{I^{n+1} f(z)} = \gamma + (1-\gamma)\, p(z), \tag{2.1}$$

where $p(z) = 1 + c_1 z + c_2 z^2 + \ldots..$ is analytic in U and $p(z) \neq 0$ for all $z \in U$. Using (1.12) and (2.1), we have

$$\frac{I^n f(z)}{I^{n+1} f(z)} = \gamma + (1-\gamma)\, p(z). \tag{2.2}$$

Differentiating (2.2) with respect to z logarithmically, we obtain

$$\begin{aligned}\frac{z(I^n f(z))'}{I^n f(z)} &= \frac{z(I^{n+1} f(z))'}{I^{n+1} f(z)} + \frac{(1-\gamma)zp'(z)}{\gamma+(1-\gamma)p(z)}\\ &= \gamma+(1-\gamma)p(z)+\frac{(1-\gamma)zp'(z)}{\gamma+(1-\gamma)p(z)},\end{aligned}$$

or

$$\frac{z(I^n f(z))'}{I^n f(z)} - \gamma = (1-\gamma)\,p(z)+\frac{(1-\gamma)zp'(z)}{\gamma+(1-\gamma)p(z)}. \tag{2.3}$$

Suppose that there exists a point $z_0 \in U$ such that

$$|\arg\ f(z)| < \frac{\pi}{2}\beta\ (|z|<|z_0|)\ \text{ and }\ |\arg\, p(z_0)| = \frac{\pi}{2}\beta.$$

Then, applying Lemma 2.1, we can write that

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta\ \text{ and }\ (p(z_0))^{\frac{1}{\beta}} = \pm\, ia\ (a>0).$$

Therefore, if $\arg p(z_0) = -\frac{\pi}{2}\beta$, then

$$\begin{aligned}\frac{z_0(I^n f(z_0))'}{I^n f(z_0)} - \gamma &= (1-\gamma)\,p(z_0)\left[1+\frac{\frac{zp'(z_0)}{p(z_0)}}{\gamma+(1-\gamma)p(z_0)}\right]\\ &= (1-\gamma)a^\beta e^{-\frac{i\Pi\beta}{2}}\left[1+\frac{ik\beta}{\gamma+(1-\gamma)a^\beta e^{-\frac{i\Pi\beta}{2}}}\right].\end{aligned}$$

This implies that

$$\arg\left\{\frac{z_0(I^n f(z_0))'}{I^n f(z_0)}-\gamma\right\} = -\frac{\pi}{2}\beta+\arg\left\{1+\frac{ik\beta}{\gamma+(1-\gamma)a^\beta e^{-\frac{i\Pi\beta}{2}}}\right\}$$

$$= \frac{-\pi}{2}\beta +$$

$$\tan^{-1}\left\{\frac{k\beta[\gamma+(1-\gamma)a^\beta\cos\left(\frac{\Pi}{2}\beta\right)]}{\gamma^2+2\gamma(1-\gamma)a^\beta\cos\left(\frac{\Pi}{2}\beta\right)+(1-\gamma)^2a^{2\beta}-k\beta(1-\gamma)a^\beta\sin\left(\frac{\Pi}{2}\beta\right)}\right\}$$

$$\le \frac{-\pi}{2}\beta\ \ (\text{where } k \le \frac{-1}{2}(a+\frac{1}{a}) \le -1),$$

which contradicts the condition $f(z) \in S^*_n(\beta,\gamma)$.

Similarly, if $\arg\ p(z_0) = \frac{\Pi}{2}\beta$, then we obtain that

$$\left|\arg\left\{\frac{z_0(I^n f(z_0))'}{I^n f(z_0)}-\gamma\right\}\right| \ge \frac{\pi}{2}\beta,$$

which also contradicts the hypothesis that $f(z) \in S^*_n(\beta,\gamma)$.

Thus the function $p(z)$ has to satisfy $|\arg p(z)| < \frac{\Pi}{2}\beta\ (z \in U)$. This shows that

$$\left|\arg\left\{\frac{z\,(I^{n+1} f(z))'}{I^{n+1}\, f(z)}-\gamma\right\}\right| < \frac{\pi}{2}\beta \quad (z\in U),$$

or $f(z) \in S^*_{n+1}(\beta,\gamma)$. This completes the proof of Theorem 2.2.

Theorem 2.3. $C_n(\beta,\gamma) \subset C_{n+1}(\beta,\gamma)$ *for each* $n \in N_0$.

Proof. $f(z) \in C_n(\beta,\gamma) \Longleftrightarrow I^n f(z) \in C(\beta,\gamma) \Longleftrightarrow z(I^n f(z))' \in S^*(\beta,\gamma)$
$\Longleftrightarrow I^n(zf^{'}(z)) \in S^*(p,\gamma) \Longleftrightarrow zf^{'}(z) \in S_n^*(\beta,\gamma)$
$\Longrightarrow zf^{'}(z) \in S_{n+1}^*(\beta,\gamma) \Longleftrightarrow I^{n+1}(zf^{'}(z)) \in S^*(\beta,\gamma)$
$\Longleftrightarrow z(I^{n+1}f(z))' \in S^*(\beta,\gamma) \Longleftrightarrow I^{n+1}f(z) \in C(\beta,\gamma)$
$\Longleftrightarrow f(z) \in C_{n+1}(\beta,\gamma)$.
This completes the proof of Theorem 2.3.

For $c > -1$ and $f(z) \in A$, we define the integral operator $L_c(f)$ as

$$L_c f(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt. \tag{2.4}$$

The operator $L_c(f)$ when $c \in N$ was studied by Bernardi [2]. For $c = 1$, $L_1(f)$ was introduced by Libera [4].

Theorem 2.4. *Let* $c > -\gamma$ *and* $0 \le \gamma < 1$. *If* $f(z) \in S_n^*(\beta,\gamma)$ *with* $\dfrac{z(I^n L_c f(z))^{'}}{I^n L_c f(z)} \neq \gamma$ *for all* $z \in U$, *then we have* $L_c(f) \in S_n^*(\beta,\gamma)$.

Proof. Set

$$\frac{z(I^n L_c\ f(z))^{'}}{I^n L_c f(z)} = \gamma + (1-\gamma)\,p(z), \tag{2.5}$$

where $p(z)$ is analytic in U, $p(0) = 1$, and $p(z) \neq 0\,(z \in U)$. From (2.4), we have

$$z(I^n L_c f(z))^{'} = (c+1)\,I^n f(z) - cI^n L_c f(z). \tag{2.6}$$

Using (2.5) and (2.6), we have

$$(c+1)\frac{I^n f(z)}{I^n L_c f(z)} = c + \gamma + (1-\gamma)p(z). \tag{2.7}$$

Differentiating both sides of (2.7) with respect to z logarithmically, we obtain

$$\frac{z(I^n f(z))'}{I^n f(z)} - \gamma = (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{c+\gamma+(1-\gamma)p(z)}.$$

Suppose that there exists a point $z_0 \in U$ such that

$$|\arg p(z)| < \frac{\pi}{2}\beta\ \ (|z| < |z_0|) \text{ and } |\arg\ p(z_0)| = \frac{\pi}{2}\beta.$$

Then, applying Lemma 2.1, we can write that

$$\frac{z_0 p^{'}(z_0)}{p(z_0)} = ik\beta \text{ and } (p(z_0))^{\frac{1}{\beta}} = \pm ia\ (a > 0).$$

If $\arg p(z_0) = \frac{\Pi}{2}\beta$, then

$$\frac{z_0(I^n f(z_0))^{'}}{I^n f(z_0)} - \gamma = (1-\gamma)p(z_0)\left[1 + \frac{\dfrac{z_0 p^{'}(z_0)}{p(z_0)}}{c+\gamma+(1-\gamma)p(z_0)}\right]$$

$$= (1-\gamma)a^{\beta}e^{i\frac{\Pi\beta}{2}}\left[1+\frac{ik\beta}{c+\gamma+(1-\gamma)a^{\beta}e^{i\frac{\Pi\beta}{2}}}\right].$$

This shows that

$$\arg\left\{\frac{z_0(I^n f(z_0))'}{I^n f(z_0)}-\gamma\right\} = \frac{\pi}{2}\beta + \arg\left[1+\frac{ik\beta}{c+\gamma+(1-\gamma)a^{\beta}e^{i\frac{\Pi\beta}{2}}}\right] = \frac{\Pi}{2}\beta$$

$$+\tan^{-1}\left\{\frac{k\beta[c+\gamma+(1-\gamma)a^{\beta}\cos(\frac{\Pi\beta}{2})]}{(c+\gamma)^2+2(c+\gamma)(1-\gamma)a^{\beta}\cos(\frac{\Pi\beta}{2})+(1-\gamma)^2a^{2\beta}+k\beta(1-\gamma)a^{\beta}\sin(\frac{\Pi\beta}{2})}\right\}$$

$$\geq \frac{\pi}{2}\beta \quad (\text{where } k \geq \frac{1}{2}(a+\frac{1}{a}) \geq 1),$$

which contradicts the condition $f(z) \in S_n^*(\beta,\gamma)$.

Similarly, we can prove the case $\arg\ p(z_0) = -\frac{\Pi}{2}\beta$. Thus we conclude that the function $p(z)$ has to satisfy $|\arg\ p(z)| < \frac{\Pi}{2}\beta$ for all $z \in U$. This gives that

$$\left|\arg\left\{\frac{z(I^n L_c f(z))'}{I^n L_c f(z)}-\gamma\right\}\right| < \frac{\pi}{2}\beta \ (z \in U),$$

or $L_c f(z) \in S_n^*(\beta,\gamma)$. This completes the proof of Theorem 2.4.

Theorem 2.5. *Let* $c > -\gamma$ *and* $0 \leq \gamma < 1$. *If* $f(z) \in C_n(\beta,\gamma)$ *and*

$$1+\frac{z(I^n L_c f(z))''}{(I^n L_c f(z))'} \neq \gamma$$

for all $z \in U$, *then we have* $L_c f(z) \in C_n(\beta,\gamma)$.

Proof. $f(z) \in C_n(\beta,\gamma) \iff zf'(z) \in S_n^*(\beta,\gamma) \implies L_c(zf'(z)) \in S_n^*(\beta,\gamma) \iff z(L_c f(z))' \in S_n^*(\beta,\gamma) \iff L_c f(z) \in C_n(\beta,\gamma)$.
This completes the proof of Theorem 2.5.

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References

[1] Aouf, M. K., Al-Oboudi, F. M., Haidan, M. M., *An application of certain integral operator*, Math. (Cluj), **47(70)**(2005), no. 2, 121-124.

[2] Bernardi, S. D., *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., **135**(1969), 429-446.

[3] Liu, J. L., *The Noor integral and strongly starlike functions*, J. Math. Anal. Appl., **261**(2001), 441-447.

[4] Libera, R. J., *Some classes of regular univalent functions*, Proc. Amer. Math Soc., **16**(1965), 755-758.

[5] Mocanu, P. T., *Alpha-convex integral operators and strongly starlike functions*, Stud. Univ. Babes-Bolyai Math., **34**(1989), 18-24.

[6] Nunokawa, M., *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan Acad. Ser. A Math. Sci., **69**(1993), 234-237.

[7] Nunokawa, M., Owa, S., Saitoh, H., Ikeda, A., Koike, N., *Some results for strongly starlike functions*, J. Math. Anal. Appl., **212**(1997), 98-106.

[8] Robertson, M. S., *On the theory of univalent functions*, Ann. Math., **37**(1936), 374-408.

[9] Sălăgean, G. S., *Subclasses of univalent functions*, Lecture Notes in Math. 1013, Springer-Verlag, Berlin - Heidelberg - New York, 1983, 362-372.

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The extensions for the univalence conditions of certain general integral operators

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Abstract. In this paper, we generalize certain integral operators given by Pescar [8] and determine conditions for univalence of these general integral operators.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$$

and satisfy the following usual normalization condition

$$f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$.

In [6] and [7], Pescar gave the following univalence conditions for the functions $f \in \mathcal{A}$.

Theorem 1.1. [6] *Let α be a complex number, $\Re(\alpha) > 0$, and c be a complex number, $|c| \leq 1$, $c \neq -1$ and $f(z) = z + \cdots$ a regular function in $\mathbb{U}$. If*

$$\left| c\,|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$, *then the function*

$$F_{\alpha}(z) = \left(\alpha \int_0^z t^{\alpha-1} f'(t)dt\right)^{\frac{1}{\alpha}} = z + \cdots$$

is regular and univalent in $\mathbb{U}$.

Theorem 1.2. [7] *Let* α *be a complex number,* $\Re(\alpha) > 0$, *and* c *be a complex number,* $|c| \leq 1$, $c \neq -1$ *and* $f \in \mathcal{A}$. *If*

$$\frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1 - |c|,$$

for all $z \in \mathbb{U}$, *then for any complex number* β, $\Re(\beta) \geq \Re(\alpha)$, *the function*

$$F_{\beta}(z) = \left(\beta \int_0^z t^{\beta-1} f'(t)dt\right)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

On the other hand, for the functions $f \in \mathcal{A}$, Ozaki and Nunokawa [5] proved another univalence condition asserted by Theorem 1.3.

Theorem 1.3. [5] *Let* $f \in \mathcal{A}$ *satisfy the condition*

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| \leq 1 \quad (z \in \mathbb{U}). \tag{1.1}$$

Then f *is univalent in* $\mathbb{U}$.

Furthermore in [8], Pescar determined necessary conditions for univalence of some integral operators.

Theorem 1.4. [8] *Let the function* $g \in \mathcal{A}$ *satisfy (*1.1*),* M *be a positive real number fixed and* c *be a complex number. If*

$$\alpha \in \left[\frac{2M+1}{2M+2}, \frac{2M+1}{2M}\right],$$

$$|c| \leq 1 - \left|\frac{\alpha-1}{\alpha}\right|(2M+1), \; c \neq -1$$

and

$$|g(z)| \leq M$$

for all $z \in \mathbb{U}$, *then the function*

$$G_{\alpha}(z) = \left(\alpha \int_0^z (g(t))^{\alpha-1} dt\right)^{\frac{1}{\alpha}} \tag{1.2}$$

is in the class $\mathcal{S}$.

Theorem 1.5. [8] *Let $g \in \mathcal{A}$, α be a real number, $\alpha \geq 1$, and c be a complex number, $|c| \leq \frac{1}{\alpha}$, $c \neq -1$. If*

$$\left|\frac{g''(z)}{g'(z)}\right| \leq 1 \quad (z \in \mathbb{U}),$$

then the function

$$H_\alpha(z) = \left(\alpha \int_0^z (tg'(t))^{\alpha-1}\, dt\right)^{\frac{1}{\alpha}} \tag{1.3}$$

is in the class $\mathcal{S}$.

Theorem 1.6. [8] *Let $g \in \mathcal{A}$ satisfies (1.1), α be a complex number, $M > 1$ fixed, $\Re(\alpha) > 0$ and c be a complex number, $|c| < 1$. If*

$$|g(z)| \leq M$$

for all $z \in \mathbb{U}$, then for any complex number β

$$\Re(\beta) \geq \Re(\alpha) \geq \frac{2M+1}{|\alpha|(1-|c|)},$$

the function

$$H_\beta(z) = \left(\beta \int_0^z t^{\beta-1} \left(\frac{g(t)}{t}\right)^{\frac{1}{\alpha}} dt\right)^{\frac{1}{\beta}} \tag{1.4}$$

is in the class $\mathcal{S}$.

Finally, Breaz and Breaz [1] considered the following family of integral operators and proved that the function $G_{n,\alpha}$ defined by

$$G_{n,\alpha}(z) = \left([n(\alpha-1)+1] \int_0^z \prod_{j=1}^n (g_j(t))^{\alpha-1}\, dt\right)^{\frac{1}{n(\alpha-1)+1}} \quad (g_1, \ldots, g_n \in \mathcal{A}) \tag{1.5}$$

is univalent in $\mathbb{U}$. For some recent investigations of the integral operator $G_{n,\alpha}$, see the works by Breaz et al. [2] and [3].

Now we introduce two new general integral operators as follows:

$$H_{n,\alpha}(z) := \left([n(\alpha-1)+1] \int_0^z \prod_{j=1}^n \left(tg_j'(t)\right)^{\alpha-1} dt\right)^{\frac{1}{n(\alpha-1)+1}} \quad (g_1, \ldots, g_n \in \mathcal{A}), \tag{1.6}$$

$$H_{n,\beta}(z) := \left([n(\beta-1)+1] \int_0^z t^{n(\beta-1)} \prod_{j=1}^n \left(\frac{g_j(t)}{t}\right)^{\frac{1}{\alpha}} dt\right)^{\frac{1}{n(\beta-1)+1}} \quad (g_1, \ldots, g_n \in \mathcal{A}). \tag{1.7}$$

Remark 1.7. *For $n = 1$, the integral operators in (1.5), (1.6) and (1.7) would reduce to the integral operators in (1.2), (1.3) and (1.4), respectively.*

In this paper, we investigate univalence conditions involving the general family of integral operators defined by (1.5), (1.6) and (1.7). For this purpose, we need the following result.

General Schwarz Lemma. [4] *Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m \quad (z \in \mathbb{U}_R).$$

The equality can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

2. Main Results

Theorem 2.1. *Let $M > 0$ and the functions $g_j \in \mathcal{A}$ $(j \in \{1, \ldots, n\})$ satisfies the inequality* (1.1)*. Also let*

$$\alpha \in \mathbb{R} \quad \left(\alpha \in \left[\frac{(2M+1)\,n}{(2M+1)\,n+1}, \frac{(2M+1)\,n}{(2M+1)\,n-1}\right]\right) \quad \text{and} \quad c \in \mathbb{C}.$$

If

$$|c| \leq 1 - \left|\frac{\alpha - 1}{n(\alpha - 1) + 1}\right| (2M+1)\,n, \quad c \neq -1 \tag{2.1}$$

and

$$|g_j(z)| \leq M \quad (z \in \mathbb{U};\ j \in \{1, \ldots, n\}),$$

then the function $G_{n,\alpha}$ defined by (1.5) *is in the class $\mathcal{S}$.*

Proof. Define a function

$$h(z) = \int_0^z \prod_{j=1}^n \left(\frac{g_j(t)}{t}\right)^{\alpha - 1} dt.$$

Then we obtain

$$h'(z) = \prod_{j=1}^n \left(\frac{g_j(z)}{z}\right)^{\alpha - 1}.$$

Also, a simple computation yields

$$\frac{zh''(z)}{h'(z)} = (\alpha - 1) \sum_{j=1}^n \left(\frac{zg_j'(z)}{g_j(z)} - 1\right),$$

which readily shows that

$$\begin{aligned}
&\left| c\,|z|^{2[n(\alpha-1)+1]} + \left(1-|z|^{2[n(\alpha-1)+1]}\right) \frac{zh''(z)}{[n(\alpha-1)+1]\,h'(z)} \right| \\
\leq\ & |c| + \frac{1}{|n(\alpha-1)+1|}\left|\frac{zh''(z)}{h'(z)}\right| \\
\leq\ & |c| + \left|\frac{\alpha-1}{n(\alpha-1)+1}\right| \sum_{j=1}^{n}\left(\left|\frac{z^2 g_j'(z)}{(g_j(z))^2}\right|\left|\frac{g_j(z)}{z}\right| + 1\right).
\end{aligned}$$

Since

$$|g_j(z)| \leq M \ \ (z \in \mathbb{U};\ j \in \{1,\ldots,n\}),$$

by using the inequality (1.1) and the general Schwarz lemma, we obtain

$$\begin{aligned}
&\left| c\,|z|^{2[n(\alpha-1)+1]} + \left(1-|z|^{2[n(\alpha-1)+1]}\right) \frac{zh''(z)}{[n(\alpha-1)+1]\,h'(z)} \right| \\
\leq\ & |c| + \left|\frac{\alpha-1}{n(\alpha-1)+1}\right| (2M+1)\,n,
\end{aligned}$$

which, by (2.1), yields

$$\left| c\,|z|^{2[n(\alpha-1)+1]} + \left(1-|z|^{2[n(\alpha-1)+1]}\right) \frac{zh''(z)}{[n(\alpha-1)+1]\,h'(z)} \right| \leq 1 \quad (z \in \mathbb{U}).$$

Applying Theorem 1.1, we conclude that the function $G_{n,\alpha}$ defined by (1.5) is in the class $\mathcal{S}$. □

Remark 2.2. *Setting $n = 1$ in Theorem 2.1, we have Theorem 1.4.*

Theorem 2.3. *Let $g_j \in \mathcal{A}$ $(j \in \{1,\ldots,n\})$, α be a real number, $\alpha \geq 1$, and c be a complex number with*

$$|c| \leq \frac{1}{n(\alpha-1)+1}, \quad c \neq -1. \tag{2.2}$$

If

$$\left|\frac{g_j''(z)}{g_j'(z)}\right| \leq 1 \quad (z \in \mathbb{U};\ j \in \{1,\ldots,n\}), \tag{2.3}$$

then the function $H_{n,\alpha}$ defined by (1.6) is in the class $\mathcal{S}$.

Proof. Define a function

$$h(z) = \int_0^z \prod_{j=1}^{n} \left(g_j'(t)\right)^{\alpha-1} dt.$$

Then we obtain

$$h'(z) = \prod_{j=1}^{n} \left(g_j'(z)\right)^{\alpha-1}.$$

Also, a simple computation yields

$$\frac{zh''(z)}{h'(z)} = (\alpha-1)\sum_{j=1}^{n}\frac{zg_j''(z)}{g_j'(z)},$$

which readily shows that

$$\begin{aligned} & \left| c\left|z\right|^{2[n(\alpha-1)+1]} + \left(1-\left|z\right|^{2[n(\alpha-1)+1]}\right) \frac{zh''(z)}{[n(\alpha-1)+1]\, h'(z)} \right| \\ \leq \; & |c| + \frac{1}{n(\alpha-1)+1} \left| \frac{zh''(z)}{h'(z)} \right| \\ \leq \; & |c| + \left(\frac{\alpha-1}{n(\alpha-1)+1} \right) \sum_{j=1}^{n} \left| \frac{zg_j''(z)}{g_j'(z)} \right| . \end{aligned}$$

By (2.2) and (2.3), we obtain

$$\left| c\left|z\right|^{2[n(\alpha-1)+1]} + \left(1-\left|z\right|^{2[n(\alpha-1)+1]}\right) \frac{zh''(z)}{[n(\alpha-1)+1]\, h'(z)} \right| \leq 1 \quad (z \in \mathbb{U}).$$

Applying Theorem 1.1, we conclude that the function $H_{n,\alpha}$ defined by (1.6) is in the class $\mathcal{S}$. □

Remark 2.4. *Setting $n = 1$ in Theorem 2.3, we have Theorem 1.5.*

Theorem 2.5. *Let $M > 0$ and the functions $g_j \in \mathcal{A}$ $(j \in \{1, \ldots, n\})$ satisfies the inequality (1.1). Also let α be a complex number, $\Re(\alpha) > 0$, and c be a complex number, $|c| < 1$. If*

$$|g_j(z)| \leq M \, (z \in \mathbb{U};\ j \in \{1, \ldots, n\}),$$

then for any complex number β with

$$\Re\left(n(\beta-1)+1\right) \geq \Re(\alpha) \geq \frac{(2M+1)\, n}{|\alpha|\,(1-|c|)}, \tag{2.4}$$

the function $H_{n,\beta}$ defined by (1.7) is in the class $\mathcal{S}$.

Proof. Define a function

$$h(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{g_j(t)}{t} \right)^{\frac{1}{\alpha}} dt.$$

Then we obtain

$$h'(z) = \prod_{j=1}^{n} \left(\frac{g_j(z)}{z} \right)^{\frac{1}{\alpha}}.$$

Also, a simple computation yields

$$\frac{zh''(z)}{h'(z)} = \frac{1}{\alpha} \sum_{j=1}^{n} \left(\frac{zg_j'(z)}{g_j(z)} - 1 \right),$$

which readily shows that

$$\frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)} \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{1}{|\alpha|\, \Re(\alpha)} \sum_{j=1}^{n} \left(\left| \frac{z^2 g_j'(z)}{(g_j(z))^2} \right| \left| \frac{g_j(z)}{z} \right| + 1 \right).$$

Since

$$|g_j(z)| \leq M \ \ (z \in \mathbb{U};\ j \in \{1, \ldots, n\}),$$

by using the inequality (1.1) and the general Schwarz lemma, we obtain

$$\frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{1}{|\alpha|\,\Re(\alpha)}\,(2M+1)\,n,$$

which, by (2.4), yields

$$\frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1-|c| \quad (z \in \mathbb{U}).$$

Applying Theorem 1.2, we conclude that the function $H_{n,\beta}$ defined by (1.7) is in the class $\mathcal{S}$. □

Remark 2.6. *Setting* $n = 1$ *in Theorem* 2.5*, we have Theorem 1.6.*

References

[1] Breaz, D., Breaz, N., *Univalence of an integral operator*, Mathematica (Cluj), **47(70)**(2005), no. 1, 35-38.

[2] Breaz, D., Breaz, N., *An integral univalent operator II*, Proceedings of the International Conference on Theory and Applications of Mathematics and Informatics. Part A. Acta Univ. Apulensis Math. Inform. **10**(2005), 117-122.

[3] Breaz, D., Breaz, N., Srivastava, H. M., *An extension of the univalent condition for a family of integral operators*, Appl. Math. Lett., **22**(2009), no. 1, 41-44.

[4] Nehari, Z., *Conformal Mapping*, Dover, New York, NY, USA, 1975.

[5] Ozaki, S., Nunokawa, M., *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc., **33**(1972), 392-394.

[6] Pescar, V., *A new generalization of Ahlfors's and Becker's criterion of univalence*, Bull. Malaysian Math. Soc., **19**(1996), no. 2, 53-54.

[7] Pescar, V., *Univalence criteria of certain integral operators*, Acta Cienc. Indica Math., **29**(2003), no.1, 135-138.

[8] Pescar, V., *On the univalence of some integral operators*, Gen. Math., **14**(2006), no. 2, 77-84.

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On a differential operator for multivalent functions

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Abstract. In this article we define a differential operator for multivalent functions in the unit disk. Further, we introduce some classes of functions defined by this operator. Partial sums are also considered.

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1. Introduction

Let $T(p)$ denote the class of functions f of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p}, \quad (p \in \mathbb{N}, \ \ z \in U). \tag{1.1}$$

which are analytic and p-valent (multivalent) in the open unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Let be given two functions $f, g \in T(p)$,

$$f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p}$$

and

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n.$$

Then their *convolution* or *Hadamard product* $f(z) * g(z)$ is defined by

$$f(z) * g(z) = z^p + \sum_{n=1}^{\infty} a_n b_n z^{n+p}, \ \ (z \in U).$$

Define a function $\varphi_p(a,c;z)$ as follows

$$\varphi_p(a,c;z) := z^p + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p}, \ \ c \neq 0, -1, -2, ...$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n = 0 \\ a(a+1)...(a+n-1), & n = \{1, 2, ...\}. \end{cases}$$

Assume that $a = k + p > 0$ and $c = 1$ where $k = 0, 1, 2, ...$ in $\varphi_p(a,c;z)$ so we obtain the function

$$\varphi_p(k+p, 1; z) = z^p + \sum_{n=1}^{\infty} \frac{(k+p)_n}{(1)_n} z^{n+p}. \tag{1.2}$$

Next we define the following differential operator $\mathcal{D}^k_{\lambda,p} : T(p) \to T(p)$ by

$$\begin{aligned} D^0 f(z) &= f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p} \\ D^1_{\lambda,p} f(z) &= (1+\lambda p) f(z) - \lambda z f'(z) = z^p + \sum_{n=1}^{\infty} (1-\lambda n) a_n z^{n+p} \\ &\vdots \\ D^k_{\lambda,p} f(z) &= z^p + \sum_{n=1}^{\infty} (1-\lambda n)^k a_n z^{n+p}, \ \ (z \in U), \end{aligned} \tag{1.3}$$

where

$$\Big(p \in \mathbb{N}, \ \ k \in \mathbb{N}_0, \ 0 \le \lambda < \frac{1}{n}, \quad n \in \mathbb{N}\Big).$$

Again by applying convolution product on (1.2) and (1.3) we have the following operator

$$\begin{aligned} \mathcal{D}^k_{\lambda,p} f(z) &= \frac{z^p}{(1-z)^{k+p}} * D^k_{\lambda,p} f(z) \\ &= z^p + \sum_{n=1}^{\infty} \frac{(k+p)_n}{(1)_n} (1-\lambda n)^k a_n z^{n+p} \\ &= z^p + \sum_{n=1}^{\infty} C(n,k) (1-\lambda n)^k a_n z^{n+p}, \ \ (z \in U), \end{aligned} \tag{1.4}$$

where $C(n,k) := \frac{(k+p)_n}{(1)_n}$.

Remark 1.1. The symbol $\mathcal{D}^k_{\lambda,p} f(z)$, when $\lambda = 0$, $p = 1$, was introduced by Ruscheweyh [1] and when $\lambda = 0$ by Goel and Sohi [2].

A function $f \in T(p)$ is said to be *p-valent starlike of order* $\mu, 0 \le \mu < p$ if

$$\Re\Big\{\frac{zf'(z)}{f(z)}\Big\} > \mu, \ \ (z \in U).$$

The class of p-valent starlike functions of order μ is denoted by $S_p^*(\mu)$. A function $f \in T(p)$ is said to be *p-valent convex of order* $\mu, 0 \le \mu < p$ if

$$\Re\Big\{1 + \frac{zf''(z)}{f'(z)}\Big\} > \mu, \ \ (z \in U).$$

The class of p-valent convex functions of order μ is denoted by $C_p(\mu)$.
A function $f \in T(p)$ is said *to be in the class $S_p^*(\mu, \lambda)$ of order* μ, where $0 \le \mu < p$ if

$$\Re\Big\{\frac{z[\mathcal{D}^k_{\lambda,p} f(z)]'}{\mathcal{D}^k_{\lambda,p} f(z)}\Big\} > \mu, \ \ (z \in U).$$

A function $f \in T(p)$ is said *to be in the class $C_p(\mu, \lambda)$ of order* μ, where $0 \le \mu < p$ if

$$\Re\Big\{1 + \frac{z[\mathcal{D}^k_{\lambda,p} f(z)]''}{[\mathcal{D}^k_{\lambda,p} f(z)]'}\Big\} > \mu, \ \ (z \in U).$$

For $0 \le \alpha < p$ and $\beta \ge 0$, let $S_p^*(\alpha, \beta, \lambda)$ be the subclass of $T(p)$ consisting of functions of the form (1.1) satisfying the analytic criterion

$$\Re\Big\{\frac{z[\mathcal{D}^k_{\lambda,p} f(z)]'}{\mathcal{D}^k_{\lambda,p} f(z)} - \alpha\Big\} > \beta\Big|\frac{z[\mathcal{D}^k_{\lambda,p} f(z)]'}{\mathcal{D}^k_{\lambda,p} f(z)} - p\Big|, \ \ (z \in U). \tag{1.5}$$

Also, for $0 \le \alpha < p$ and $\beta \ge 0$, let $C_p(\alpha, \beta, \lambda)$ be the subclass of $T(p)$ satisfying the analytic criterion

$$\Re\Big\{1 + \frac{z[\mathcal{D}^k_{\lambda,p} f(z)]''}{[\mathcal{D}^k_{\lambda,p} f(z)]'} - \alpha\Big\} > \beta\Big|\frac{z[\mathcal{D}^k_{\lambda,p} f(z)]''}{[\mathcal{D}^k_{\lambda,p} f(z)]'} - (p-1)\Big|, \ \ (z \in U). \tag{1.6}$$

The main goal of this work is to determine sufficient conditions for the analytic functions to belong to these general classes. Sharp results involving partial sums $f_{m+p}(z)$ of functions $f(z)$ in the classes $S_p^*(\alpha, \beta, \lambda)$ and $C_p(\alpha, \beta, \lambda)$ are obtained.

2. The classes $S_p^*(\alpha, \beta, \lambda)$ and $C_p(\alpha, \beta, \lambda)$

In this section we obtain sufficient conditions for functions $f(z)$ to be in the classes $S_p^*(\alpha, \beta, \lambda)$ and $C_p(\alpha, \beta, \lambda)$.

Theorem 2.1. A sufficient condition for a function $f(z)$ of the form (1.1) to be in $S_p^*(\alpha, \beta, \lambda)$ is

$$\sum_{n=1}^{\infty}[(1+\beta)n + (p-\alpha)]C(n,k)(1-\lambda n)^k|a_n| < p - \alpha, \ \ (z \in U), \tag{2.1}$$

for $0 \le \alpha < p$, $\beta \ge 0$ and $0 \le \lambda < \frac{1}{n}, \quad n \in \mathbb{N}$.

Proof. It suffices to show that

$$\beta\Big|\frac{z[\mathcal{D}^k_{\lambda,p} f(z)]'}{\mathcal{D}^k_{\lambda,p} f(z)} - p\Big| - \Re\Big\{\frac{z[\mathcal{D}^k_{\lambda,p} f(z)]'}{\mathcal{D}^k_{\lambda,p} f(z)} - p\Big\} \le p - \alpha, \ \ (z \in U).$$

We have

$$\beta\Big|\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]'}{\mathcal{D}^k_{\lambda,p}f(z)}-p\Big|-\Re\Big\{\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]'}{\mathcal{D}^k_{\lambda,p}f(z)}-p\Big\}\le(1+\beta)\Big|\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]'}{\mathcal{D}^k_{\lambda,p}f(z)}-p\Big|$$

$$\le\frac{(1+\beta)\sum_{n=1}^{\infty}nC(n,k)(1-\lambda n)^k|a_n||z|^{n+p}}{1-\sum_{n=1}^{\infty}C(n,k)(1-\lambda n)^k|a_n||z|^{n+p}}$$

$$\le\frac{(1+\beta)\sum_{n=1}^{\infty}nC(n,k)(1-\lambda n)^k|a_n|}{1-\sum_{n=p+1}^{\infty}C(n,k)(1-\lambda n)^k|a_n|}.$$

This last expression is bounded above by $(p-\alpha)$ if

$$\sum_{n=1}^{\infty}[(1+\beta)n+(p-\alpha)]C(n,k)(1-\lambda n)^k|a_n|<p-\alpha,$$

and the proof is complete.

By setting $\beta=\lambda=0$ (Goel-Sohi operator [2]) in Theorem 2.1, we obtain the following result:

Corollary 2.2. Let f be given by (1.1) and satisfying

$$\sum_{n=2}^{\infty}(n+p-\alpha)C(n,k)|a_n|\le p-\alpha,\quad(0\le\alpha<p, z\in U)$$

then $f\in S_p^*(\alpha)$ (p-valent starlike).

By letting $\beta=\lambda=0$ and $p=1$ (Ruscheweyh operator [1]) in Theorem 2.1, we obtain the following result:

Corollary 2.3. Let f be given by (1.1) and satisfying

$$\sum_{n=2}^{\infty}(n+1-\alpha)C(n,k)|a_n|\le 1-\alpha,\quad(0\le\alpha<1, z\in U)$$

then $f\in S^*(\alpha)$ (starlike).

In the same manner we can obtain the next result.

Theorem 2.4. A sufficient condition for a function f of the form (1.1) to be in $C_p(\alpha,\beta,\lambda)$ is

$$\sum_{n=1}^{\infty}(n+p)[n(1+\beta)+(p-\alpha)]C(n,k)(1-\lambda n)^k|a_n|<p(p-\alpha),\ \ (p\in\mathbb{N},\ z\in U),\tag{2.2}$$

for $0\le\alpha<p$ and $\beta\ge0$.

Proof. It suffices to show that

$$\beta\Big|\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]''}{[\mathcal{D}^k_{\lambda,p}f(z)]'}-(p-1)\Big|-\Re\Big\{\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]''}{[\mathcal{D}^k_{\lambda,p}f(z)]'}-(p-1)\Big\}\le p-\alpha,$$

$$(p\in\mathbb{N},\ 0\le\alpha<p,\ \beta\ge0,\ z\in U).$$

Then we have

$$\beta\Big|\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]''}{[\mathcal{D}^k_{\lambda,p}f(z)]'}-(p-1)\Big|-\Re\Big\{\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]''}{[\mathcal{D}^k_{\lambda,p}f(z)]'}-(p-1)\Big\}$$

$$\leq (1+\beta)\Big|\frac{z[\mathcal{D}^k_{\lambda,p}f(z)]''}{[\mathcal{D}^k_{\lambda,p}f(z)]'}-(p-1)\Big|$$

$$\leq \frac{(1+\beta)\sum_{n=1}^{\infty}n(n+p)C(n,k)(1-\lambda n)^k|a_n||z|^{n+p-1}}{p|z|^{p-1}-\sum_{n=1}^{\infty}(n+p)C(n,k)(1+\lambda n)^k|a_n||z|^{n+p-1}}$$

$$\leq \frac{(1+\beta)\sum_{n=1}^{\infty}n(n+p)C(n,k)(1+\lambda n)^k|a_n|}{p-\sum_{n=1}^{\infty}(n+p)C(n,k)(1-\lambda n)^k|a_n|}.$$

This last expression is bounded above by $(p-\alpha)$ if

$$\sum_{n=1}^{\infty}(n+p)[n(1+\beta)+(p-\alpha)]C(n,k)(1-\lambda n)^k|a_n|<p(p-\alpha),\ \ (p\in\mathbb{N}).$$

This completes the proof.

3. Partial sums

In this section, applying methods used by Silverman [3] and Silvia [4], we will investigate the ratio of a function $f(z)$ of the form (1.1) to its sequence of partial sums

$$f_{m+p}(z)=z^p+\sum_{n=1}^{m}a_nz^{n+p},\quad (z\in U) \tag{3.1}$$

when the coefficients are small enough in order to satisfy either condition (2.1) or (2.2). More precisely, we will determine sharp lower bounds for

$$\Re\Big\{\frac{f(z)}{f_{m+p}(z)}\Big\},\Re\Big\{\frac{f_{m+p}(z)}{f(z)}\Big\},\Re\Big\{\frac{f'(z)}{f'_{m+p}(z)}\Big\}\text{ and }\Re\Big\{\frac{f'_{m+p}(z)}{f'(z)}\Big\}.$$

In the sequel, we will make use of the fact that

$$\Re\Big\{\frac{(1+w(z))}{(1-w(z))}\Big\}>0,\ \ (z\in U)$$

if and only if $w(z)=\sum_{n=1}^{\infty}c_nz^n$ satisfies the inequality $|w(z)|<|z|$.

Theorem 3.1. Let f given by (1.1) and satisfies (2.1). Then

$$\Re\Big\{\frac{f(z)}{f_{m+p}(z)}\Big\}>1-\frac{p-\alpha}{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}, \tag{3.2}$$

$$\Big(z\in U,\ p>\alpha,m=0,1,2,...,\quad 0\leq\lambda<\frac{1}{m+p+1}\Big).$$

The result is sharp for every m with the extremal function

$$f(z)=z^p+\frac{p-\alpha}{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}z^{m+p+1}, \tag{3.3}$$

$$(z\in U,\ \ m\geq 0,\ \ p>\alpha).$$

Proof. Assume that $f \in T(p)$ satisfies (2.1). By setting

$$w(z) = \tfrac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{p-\alpha}\Big\{\tfrac{f(z)}{f_{m+p}(z)}$$

$$-\Big(1-\tfrac{p-\alpha}{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}\Big)\Big\}$$

$$:= 1+\frac{H_{m+p+1}\sum_{n=m+1}^{\infty}a_nz^n}{1+\sum_{n=1}^{m}a_nz^n},$$

where

$$H_{m+p+1} := \tfrac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{p-\alpha}.$$

Thus we find that

$$\Big|\frac{w(z)-1}{w(z)+1}\Big| \le \frac{H_{m+p+1}\sum_{n=m+1}^{\infty}|a_n|}{2-2\sum_{n=1}^{m}|a_n|-H_{m+p+1}\sum_{n=m+1}^{\infty}|a_n|}$$

$$\le 1, \quad (z \in U)$$

if and only if

$$2H_{m+p+1}\sum_{n=m+1}^{\infty}|a_n| \le 2-2\sum_{n=1}^{m}|a_n|$$

which is equivalent to

$$\sum_{n=1}^{m}|a_n| + H_{m+p+1}\sum_{n=m+1}^{\infty}|a_n| \le 1. \tag{3.4}$$

In order to see that

$$f(z) = z^p + \frac{z^{m+p+1}}{H_{m+p+1}}, \quad (z \in U)$$

gives a sharp result, we observe that for

$$z = re^{\frac{\pi i}{m+p}}, \quad (z \in U)$$

that

$$\frac{f(z)}{f_{m+p}(z)} = 1+\frac{z^{m+p}}{H_{m+p+1}} \to 1-\frac{1}{H_{m+p+1}} \quad as \quad z \to 1^-.$$

This completes the proof.

Theorem 3.2. Let f given by (1.1) satisfying (2.1). Then

$$\Re\Big\{\tfrac{f_{m+p}(z)}{f(z)}\Big\} > \tfrac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{(p-\alpha)+[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}, \tag{3.5}$$

$$\Big(z \in U,\ p>\alpha,\ \ m=0,1,2,...,\quad 0\le\lambda<\frac{1}{m+p+1}\Big).$$

The result is sharp for every m with an extremal function given by (3.3).

Proof. Assume that $f \in T(p)$ and satisfies (2.1). Write

$$w(z) = \left(1 + \tfrac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{p-\alpha}\right)\left\{ \tfrac{f_{m+p}(z)}{f(z)} \right.$$

$$\left. - \tfrac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{(p-\alpha)+[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k} \right\}$$

$$= 1 - \frac{\left(1 + H_{m+p+1}\right) \sum_{n=m+1}^{\infty} a_n z^n}{1 + \sum_{n=1}^{m} a_n z^n}$$

where H_{m+p+1} is defined in Theorem 3.1. This yields that

$$\left|\frac{w(z)-1}{w(z)+1}\right| \le \frac{(1 + H_{m+p+1}) \sum_{n=m+1}^{\infty} |a_n|}{2 - 2\sum_{n=p+1}^{m} |a_n| - (1 + H_{m+p+1}) \sum_{n=m+1}^{\infty} |a_n|}$$

$$\le 1, \quad (z \in U)$$

if and only if

$$2[(1 + H_{m+p+1}) \sum_{n=m+1}^{\infty} |a_n|] \le 2 - 2\sum_{n=2}^{m} |a_n|$$

or

$$\sum_{n=p+1}^{m} |a_n| + (1 + H_{m+p+1}) \sum_{n=m+1}^{\infty} |a_n| \le 1, \tag{3.6}$$

which gives (3.5). The bound in (3.5) is sharp for all $m \in \mathbb{N}$ with the extremal function given by (3.3). This completes the proof.

Theorem 3.3. Let f given by (1.1) satisfies (2.1). Then

$$\Re\left\{ \tfrac{f'(z)}{f'_{m+p}(z)} \right\} \ge 1 - \tfrac{(m+p+1)(p-\alpha)}{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}, \tag{3.7}$$

$$\left(z \in U, \ p > \alpha, \ m = 0, 1, 2, ..., 0 \le \lambda < \frac{1}{m+p+1}\right).$$

Proof. Assume that $f \in T(p)$ satisfies (2.1). Write

$$w(z) = \tfrac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{p-\alpha} \left\{ \tfrac{f'(z)}{f'_{m+p}(z)} \right.$$

$$\left. -\left(1 - \tfrac{(m+p+1)(p-\alpha)}{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}\right) \right\}$$

$$= \frac{1 + \frac{H_{m+p+1}}{(m+p+1)} \sum_{n=m+1}^{\infty} \frac{n+p}{p} a_n z^n + \sum_{n=1}^{\infty} \frac{n+p}{p} a_n z^n}{1 + \sum_{n=1}^{m} \frac{n+p}{p} a_n z^n}$$

$$= 1 + \frac{\frac{H_{m+p+1}}{(m+p+1)} \sum_{n=m+1}^{\infty} \frac{n+p}{p} a_n z^n}{1 + \sum_{n=1}^{m} \frac{n+p}{p} a_n z^n},$$

where H_{m+p+1} is defined in Theorem 3.1. This implies

$$\Big|\frac{w(z)-1}{w(z)+1}\Big| \le \frac{\frac{H_{m+p+1}}{m+p+1}\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n|}{2-2\sum_{n=1}^{m}\frac{n+p}{p}|a_n| - \frac{H_{m+p+1}}{m+p+1}\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n|}$$

$$\le 1, \quad (z\in U)$$

if and only if

$$2[\frac{H_{m+p+1}}{m+p+1}\sum_{n=m+1}^{\infty}\frac{n}{p}|a_n|] \le 2-2\sum_{n=p+1}^{m}\frac{n}{p}|a_n|,$$

i.e.

$$\sum_{n=1}^{m+p}\frac{n}{p}|a_n| + \frac{H_{m+p+1}}{m+p+1}\sum_{n=m+1}^{\infty}\frac{n}{p}|a_n| \le 1.$$

We therefore obtain (3.7). The result is sharp with functions given by (3.3). The proof of the Theorem 3.3 is completed.

Theorem 3.4. Let f given by (1.1) satisfying (2.1). Then

$$\Re\Big\{\frac{f'_m(z)}{f'(z)}\Big\} \ge \frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{(m+p+1)(p-\alpha)+[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}, \tag{3.8}$$

$$\Big(z\in U,\ \ p>\alpha,\ \ m=0,1,2,...,\quad 0\le\lambda<\frac{1}{m+p+1}\Big).$$

Proof. Assume that $f\in T(p)$ satisfies (2.1). Consider

$$w(z)=\Big((m+p+1)+\frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{p-\alpha}\Big)\Big\{\frac{f'_m(z)}{f'(z)}$$

$$-\frac{[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{(m+p+1)(p-\alpha)+[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}\Big\}$$

$$=1-\frac{(1+\frac{H_{m+p+1}}{m+p+1})\sum_{n=m+1}^{\infty}\frac{n+p}{p}a_nz^n}{1+\sum_{n=2}^{m}\frac{n+p}{p}a_nz^n}.$$

This implies that

$$\Big|\frac{w(z)-1}{w(z)+1}\Big| \le \frac{(1+\frac{H_{m+p+1}}{m+p+1})\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n|}{2-2\sum_{n=1}^{m}\frac{n+p}{p}|a_n|-(1+\frac{H_{m+p+1}}{m+p+1})\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n|}$$

$$\le 1, \quad (z\in U)$$

if and only if

$$2[(1+\frac{H_{m+p+1}}{m+p+1})\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n|] \le 2-2\sum_{n=1}^{m}\frac{n+p}{p}|a_n|,$$

i.e.

$$\sum_{n=1}^{m}\frac{n+p}{p}|a_n|+(1+\frac{H_{m+p+1}}{m+p+1})\sum_{n=m+1}^{\infty}\frac{n+p}{p}|a_n|\leq 1.$$

We therefore obtain (3.8). The result is sharp with functions given by (3.3). The proof of Theorem 3.4 is complete.

In the same manner as the proof of Theorems 3.1-3.4, we can show the following results:

Theorem 3.5. Let f given by (1.1) satisfying (2.2). Then

$$\Re\left\{\frac{f(z)}{f_{m+p}(z)}\right\}>1-\frac{p(p-\alpha)}{(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}. \tag{3.9}$$

The result is sharp for every m with the extremal function

$$f(z)=z^p+\frac{p(p-\alpha)}{(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}z^{m+p+1}, \tag{3.10}$$

$$\left(z\in U,\ \ p>\alpha,\ \ m=0,1,2,...,\quad 0\leq\lambda<\frac{1}{m+p+1}\right).$$

Theorem 3.6. Let f given by (1.1) satisfies (2.2). Then

$$\Re\left\{\frac{f_{m+p}(z)}{f(z)}\right\}>\frac{(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{p(p-\alpha)+(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}, \tag{3.11}$$

$$\left(z\in U,\ \ p>\alpha,\ \ m=0,1,2,...,\quad 0\leq\lambda<\frac{1}{m+p+1}\right).$$

The result is sharp for every m with the extremal function given by (3.10).

Theorem 3.7. Let f given by (1.1) satisfies (2.2). Then

$$\Re\left\{\frac{f'(z)}{f'_{m+p}(z)}\right\}\geq 1-\frac{p(m+p+1)(p-\alpha)}{(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}, \tag{3.12}$$

$$\left(z\in U,\ \ p>\alpha,\ \ m=0,1,2,...,\quad 0\leq\lambda<\frac{1}{m+p+1}\right).$$

Theorem 3.8. Let f given by (1.1) satisfies (2.2). Then

$$\Re\left\{\frac{f'_m(z)}{f'(z)}\right\}$$

$$\geq\frac{(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k}{p(m+p+1)(p-\alpha)+(m+p+1)[(1+\beta)(m+p+1)+(p-\alpha)]C(m+p+1,k)(1-\lambda(m+p+1))^k} \tag{3.13}$$

where

$$\left(z\in U,\ \ p>\alpha,\ \ m=0,1,2,...,\quad 0\leq\lambda<\frac{1}{m+p+1}\right).$$

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References

[1] Ruscheweyh, St., *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49**(1975), 109-115.

[2] Goel, R. M., Sohi, N.S., *A new criterion for p-valent functions*, Proc. Amer. Math. Soc., **78**(1980), 353-357.

[3] Silverman, H., *Partial sums of starlike and convex functions*, J. Math. Anal. Appl., **209**(1997), 221-227.

[4] Silvia, E. M., *Partial sums of convex functions of order α*, Houston J. Math., **11**(1985), no. 3, 397-404.

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On the stability of the bivariate geometric composed distribution's characterization

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Abstract. Let $(X_j, Y_j), j = 1, 2...$ be nonnegative i.i.d random vectors and (N_1, N_2) be independent of (X_j, Y_j), $j = 1, 2, ...$ with Bivariate Geometric Distribution. The vector $(Z_1 = \sum_{j=1}^{N_1} X_j; Z_2 = \sum_{j=1}^{N_2} Y_j)$ is called the Bivariate Geometric Composed vector. In [3], a characterization for distribution function of this vector was showed and in this paper we shall consider the stability of this characterization.

Mathematics Subject Classification (2010): 60E10, 62E10.

Keywords: Characterization, stability of characterization, composed random variables, geometric summation.

1. Introduction

At first, we recall a well-known characterization of the univariate geometric composed distribution. Let $X_1, X_2, ...$ be nonegative i.i.d random variables (r.v's) $P(X_j > x) = \overline{F}(x), EX_j = 1 (j = 1, 2, ..)$ and let N be independent of $X_j, (j = 1, 2, ...)$ with the Geometric distribution, i.e.

$$P(N = k) = p(1 - p)^{k-1} \quad (k = 1, 2, ...)$$

The random variable $Z = \sum_{j=1}^{N} X_j$ is called the Geometric Composed random variable. We denote $\overline{G}_p(x) = P\{pZ > x\}$. In [1], Renyi has given characteristics of this Geometric Composed Distribution. In [2], some stabilities of this Renyi's characteristic theorem was considered by two Vietnamese authors. In [3] (1985), A. Kovat (Hungarian) expanded this Renyi's characteristic theorem for the case of two dimensions.

We consider the Bivariate Geometric Composed distribution as the following definition (See [3]).

Let A_1, A_2 be arbitrary events and $p = (p_1, p_2, p_{12})$, means the probabilities

$$P(A_1\overline{A_2}) = p_1; P(\overline{A_1}A_2) = p_2; P(A_1A_2) = p_{12} \tag{1.1}$$

and $q = 1 - p_1 - p_2 - p_{12} = 1 - P(\overline{A_1 \cup A_2})$.

Let N_1, N_2 be the serial numbers of necessary trials for occuring at first of the event A_1, A_2 resp. occur at first. Then we will say that the random vector (N_1, N_2) has bivariate geometric distribution and we can obtain the following distribution of (N_1, N_2):

$$P\{N_1 = k_1; N_2 = k_2\} = \begin{cases} q^{k_2-1}p_2(1-p_1-p_{12})^{k_1-k_2-1}(p_1+p_{12} & if \;\; k_1 > k_2 \\ q^{k_1-1}p_{12} & if \;\; k_1 = k_2 \\ q^{k_1-1}p_1(1-p_2-p_{12})^{k_2-k_1-1}(p_1+p_{12} & if \;\; k_1 < k_2 \end{cases} \tag{1.2}$$

Let $(X_j, Y_j), j = 1, 2, ...$ be nonegative i.i.d. random vectors, $P\{X_j > x; Y_j > y\} = \overline{F}(x, y)$, $\varphi(t_1, t_2) = E\{e^{it_1X_j + it_2Y_j}\}; EX_j = 1; EY_j = 1 (j = 1, 2, ..)$

Let (N_1, N_2) be independent of (X_j, Y_j) (j=1,2,...) and (N_1, N_2) has Bivariate geometric distribution. The random vector $(Z_1 = \sum_{j=1}^{N_1} X_j; Z_2 = \sum_{j=1}^{N_2} Y_j)$ is called the Bivariate Geometric Composed random vector.

Put

$$\overline{G}_p(x, y) = P\{(p_1 + p_{12})Z_1 > x; (p_2 + p_{12})Z_2 > y\}. \tag{1.3}$$

The following characteristic theorem was showed in [3].

Theorem 1.1 *$\overline{G}_p(x, y) = \overline{F}(x, y)$ if and only if*

$$\varphi(t_1, t_2) = [1 - it_1 - it_2 + \sum_{r=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k}a_{n,k}t_1^n t_2^k]^{-1}, \tag{1.4}$$

where

$$a_{1,1} = \frac{p_1 + p_2}{p_1 + p_2 + p_{12} - (p_1 + p_{12})(p_2 + p_{12})},$$

$$a_{1,2} = \frac{p_2 - a_{1,1}(p_2 + p_{12})(1 - p_1 - p_{12})}{p_1 + p_2 + p_{12} - (p_1 + p_{12})(p_2 + p_{12})^2},$$

$$a_{2,1} = \frac{p_1 - a_{1,1}(p_1 + p_{12})(1 - p_2 - p_{12})}{p_1 = p_2 + p_{12} - (p_1 + p_{12})^2(p_2 + p_{12})}, \tag{1.5}$$

$$\begin{aligned} a_{n,k} = {} & [p_1 + p_2 + p_{12} - (p_1 + p_{12})^n(p_2 + p_{12})^k]^{-1} \\ & \cdot \{a_{n-1,k-1}[(p_1 + p_{12})^{n-1}(p_2 + p_{12})^{k-1} - p_{12}] \\ & + a_{n,k-1}[(p_1 + p_{12})^n(p_2 + p_{12})^{k-1} - p_2 - p_{12}] \\ & + a_{n-1,k}[(p_1 + p_{12})^{n-1}(p_2 + p_{12})^k - p_1 - p_{12}]\} \end{aligned}$$

Now, we shall consider the stability of this characteristic theorem.

2. Stability theorems

Suppose that X and Y are two n-dimensional random vectors with the characteristic functions $\varphi_X(t)$ and $\varphi_Y(t)$ respectively. In [4], the metric $\lambda(X;Y)$ was defined as follows

$$\lambda(X;Y) = \lambda(\varphi_X;\varphi_Y) = \sup_{T>0}\{max\{v(X,Y;T); \frac{1}{T}\}\} \tag{2.1}$$

where

$$v(X,Y;T) = \frac{1}{2}max\{|\varphi_X(t) - \varphi_Y(t)|; ||t|| < T\} \tag{2.2}$$

and $\varphi_X(t) = Ee^{i(t,X)}$, where $(.,.)$ denotes the scalar product in the space $\mathbb{R}^n$ and $||t|| = \sqrt{(t,t)}$ with $t \in \mathbb{R}^n$.

Theorem 2.1. *Let us consider the 2-dimensional characteristic function*

$$\varphi_0(t_1,t_2) = [1 - it_1 - it_2 + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k}a_{n,k}t_1^n t_2^k]^{-1}, \tag{2.3}$$

where $a_{n,k}$ was given in (1.5).

If X_j and Y_j (with $j = 1,..,n$) has the same ϵ-exponential distribution, i.e. $\exists\ T_1(\epsilon) > 0, T_2(\epsilon) > 0$ (such that $T_1(\epsilon) \to \infty$ and $T_2(\epsilon) \to \infty$ when $\epsilon \to 0$) and such that

$$|\varphi_{X_j}(t_1) - \frac{1}{1-it_1}| \le \epsilon \quad \forall t_1, \quad |t_1| \le T_1(\epsilon), \quad \forall j, \tag{2.4}$$

$$|\varphi_{Y_j}(t_2) - \frac{1}{1-it_2}| \le \epsilon \quad \forall t_2, \quad |t_2| \le T_2(\epsilon), \quad \forall j, \tag{2.5}$$

then, for every characteristic function $\varphi(t_1,t_2)$ of the random vector (X_j,Y_j), we always have the estimation

$$\lambda(\varphi;\varphi_0) = \lambda[\varphi(t_1,t_2);\varphi_0(t_1,t_2)] \le max(C_1\epsilon; \frac{1}{T^*(\epsilon)}), \tag{2.6}$$

where $T^*(\epsilon) = min[T_1(\epsilon);T_2(\epsilon)]$ and C is a constant independent of ϵ.

Proof of the Theorem 2.1. From the proof of Theorem 2 in [3] or see [5], we have

$$\varphi(t_1,t_2) = \varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2][p_{12}+p_1\varphi(0,t_2)+p_2\varphi(t_1,0)+q\varphi(t_1,t_2)]$$

and

$$\varphi(t_1,t_2) = \frac{\varphi[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2][p_{12}+p_1\varphi(0,t_2)+p_2\varphi(t_1,0)]}{1-q\varphi[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2]}. \tag{2.7}$$

Thus, we shall have the estimation

$$|\varphi(t_1,t_2) - \varphi_0(t_1,t_2)|$$
$$= |\frac{\varphi[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2][p_{12}+p_1\varphi(0,t_2)+p_2\varphi(t_1,0)]}{1-q\varphi[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2]} - \varphi_0(t_1,t_2)|. \tag{2.8}$$

But from (2.4) and (2.5), $\exists\ T^*(\epsilon) = min\{T_1(\epsilon);T_2(\epsilon)]$ such that

$$\varphi(0,t_2) = \frac{1}{1-it_2} + r_2(t_2) \ \ where\ \ |r_2(t_2)| \le \epsilon, \ \ \forall t_2, \ \ |t_2| \le T^*(\epsilon) \tag{2.9}$$

$$\varphi(t_1,0)=\frac{1}{1-it_1}+r_1(t_1) \;\; where \;\; |r_1(t_1)|\le\epsilon, \;\; \forall t_1, \;\; |t_1|\le T^*(\epsilon). \quad (2.10)$$

On the other hand, from formula (2.8) of the proof of the Theorem 2 in [3], we obtain also the following equality

$$\varphi_0(t_1,t_2)=\frac{\varphi_0[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2][p_{12}+\frac{p_1}{1-it_1}+\frac{p_2}{1-it_2}]}{1-q\varphi_0[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2]}. \quad (2.11)$$

Taking into account (2.8), (2.9), (2.10) and (2.11) we get

$$|\varphi(t_1,t_2)-\varphi_0(t_1,t_2)|=|\frac{\varphi_0[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2]}{1-q\varphi_0[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2]}||r^*(t_1,t_2)|, \quad (2.12)$$

where $r^*(t_1,t_2)=p_1r_1(t_1)+p_2r_2(t_2)$ and from (2.9) and (2.10) we notice that

$$|r^*(t_1,t_2)|=|p_1r_1(t_1)+p_2r_2(t_2)|\le C\epsilon,$$

for all $|t_1|\le T_1(\epsilon), |t_2|\le T_2(\epsilon)$.

On the other hand, we always have the inequalities:

$$|1-qz|\ge|1-q|z||\ge 1-q \quad (2.13)$$

for all complex number $z, |z|\le 1$.
So, we have

$$|\varphi(t_1,t_2)-\varphi_0(t_1,t_2)|\le\frac{r^*(t_1,t_2)}{1-q}\le\frac{C\epsilon}{1-q}=C_1\epsilon, \quad (2.14)$$

where C_1 is a constant of ϵ. The proof Theorem 2.1 is completed.

Let us denote the characteristic function corresponding to $\overline{G_p}(x,y)$ by $\psi_p(t_1,t_2)$. Now, we consider the second stability theorem.

Theorem 2.2. *If both X_j and Y_j have ϵ-exponential distribution $(j=1,2,...,n)$ as described in Theorem 2.1, then we have the inequality*

$$\lambda(\psi_p,\varphi_0)=\lambda[\psi_p(t_1,t_2);\varphi_0(t_1,t_2)]\le max\{C_2\epsilon;\frac{1}{T^*(\epsilon)}\} \quad (2.15)$$

Proof of Theorem 2.2. At first, denoting by $\psi(t_1,t_2)$ the characteristic function of (Z_1,Z_2), then

$$\psi_p(t_1,t_2)=\psi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2].$$

But, in the proof of Theorem 1 in [3], we have

$$\psi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2]$$

$$=\frac{\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2][p_{12}+p_1\psi(0,t_2)+p_2\psi(t_1,0)]}{1-\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2]}; \quad (2.16)$$

in [2], we have already proved that if X_j is ϵ-exponentially distributed then

$$|\psi(t_1,0)-\frac{1}{1-it_1}|=|r_1(t_1)|\le max_{|t_1|\le T_1(\epsilon)}\{\frac{\epsilon}{2};\frac{1}{T_1(\epsilon)}\} \quad \forall t_1, \quad |t_1|\le T_1(\epsilon) \quad (2.17)$$

and, more, if Y_j is ϵ-exponentially distributed then

$$|\psi(0,t_2) - \frac{1}{1-it_2}| = |r_2(t_2)| \leq max_{|t_2|\leq T_1(\epsilon)}\{\frac{\epsilon}{2}; \frac{1}{T_2(\epsilon)}\} \quad \forall t_2, \quad |t_2| \leq T_2(\epsilon), \tag{2.18}$$

and from (2.16), (2.17) and (2.18) it follows that

$$\psi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2]$$
$$= \frac{\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2][p_{12}+\frac{p_1}{1-it_1}+\frac{p_2}{1-it_2}]}{1-q\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2]}$$
$$+\frac{\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2][p_1r_1(t_1)+p_2r_2(t_2)]}{1-q\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2]} \tag{2.19}$$

Therefore

$$|\psi_p(t_1,t_2) - \varphi_0(t_1,t_2)|$$
$$\leq ||\frac{\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2][p_{12}+\frac{p_1}{1-it_1}+\frac{p_2}{1-it_2}]}{1-q\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2]} - \varphi_0(t_1,t_2)|$$
$$+|\frac{\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2]}{1-q\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2]}||p_1r_1(t_1)+p_2r_2(t_2)| = J_1 + J_2. \tag{2.20}$$

Taking into account (2.9), (2.10) and (2.13), we get

$$J_2 \leq max\{C_2\epsilon; \frac{1}{T^*(\epsilon)}\} \tag{2.21}$$

where $T^*(\epsilon) = min\{T_1(\epsilon); T_2(\epsilon)\}$ and C_2 is a constant of ϵ.

According to the proof of Theorem 2 in [3], we have

$$\varphi_0(t_1,t_2) = \frac{\varphi[(p_{12}+p_1)t_1;(p_{12}+p_2)t_2][p_{12}+\frac{p_1}{1-it_1}+\frac{p_2}{1-it_2}]}{1-q\varphi[(p_{12}+p_1)t_1,(p_{12}+p_2)t_2]}. \tag{2.22}$$

Thus, $J_1 = 0$ and we have:

$$J_1 + J_2 \leq max\{C_2\epsilon; \frac{1}{T^*(\epsilon)}\}. \tag{2.23}$$

where C_2 is a constant independent of ϵ. Therefore it follows that

$$\lambda(\psi_P;\varphi_0) \leq max\{C_2\epsilon; \frac{1}{T*(\epsilon)}\} \tag{2.24}$$

References

[1] Renyi, A., *A characterization of the Poisson Process* (Hungarian), Magyar Tud. Akad. Math. Kut. Int. Ky, **1**(1976), 519-527.

[2] Tran Kim Thanh, Nguyen Huu Bao, *On the geometric composed variable and the estimate of the stable degree of the Renny's characteristis theorem*, ACTA Matematica Vietnamica, **21**(1996), 1996, no. 2, 269-277.

[3] Kovat, A., *On Bivariate Geometric Compouding*, Prov. of the 5^{th} Pannonian Sym. on Math, Stat., Visegrad, Hungary 1985.

[4] Klebanov, L., Yanushkerichius, R., *On the ϵ-independence of statisties $X_1 + X_2$ and $X_1 - X_2$*, (Russian), Lecturos Matematikor Rikinys, **3**(1985), no. 3, 1985

[5] Galambos, Kotz, J., *Characterizations of Probability Distribution*, Lecture Notes in Mathematics, 675, Springer Verlag.

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Fractional stochastic differential equations: A semimartingale approach

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Abstract. The aim of this paper is to study some class of fractional stochastic equations from the approach given in [2]. The existence and uniqueness for equations with deterministic volatility are proved. The explicit solutions of some important equations are found and the ruin probability in the asset liability management (ALM) model is investigated as well.

Mathematics Subject Classification (2010): 65C30, 26A33.

Keywords: Fractional Brownian motion, stochastic differential equation, semimartingale approach, ruin probability.

1. Introduction

The first problem in the study of fractional stochastic equations is how to define in some sense the fractional stochastic integration. For this, many attempts have been made by various authors. And there are definitions obtained from some kinds of approximation approach as those of D. Nualart and al.[1], Tran Hung Thao and Christine Thomas-Agnan [12, 11] and P. Carmona, L. Coutin and G. Montseny [2, 4]. This paper is based on the results given by the last mentioned authors.

By definition, a fractional Brownian motion (fBm) W^H is a centered Gaussian process with the covariance function given by

$$R_H(t,s) := E[W_t^H W_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) .$$

In [2], the authors proved that W_t^H can be approximated by semimartingales $W_t^{H,\varepsilon}$

$$W_t^H = \int_0^t K(t,s) dB_s \ , \ \ t \geq 0$$

$$W_t^{H,\varepsilon} = \int_0^t K(t+\varepsilon, s) dB_s \,,$$

where B is a standard Brownian montion and the kernel $K(t,s)$ is given by

$$K(t,s) = C_H \left[\frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}} (t-s)^{H-\frac{1}{2}} - (H - \frac{1}{2}) \int_s^t \frac{u^{H-\frac{3}{2}}}{s^{H-\frac{1}{2}}} (u-s)^{H-\frac{1}{2}} du \right].$$

Under suitable conditions on the function f, they proved that the integral $\int_0^t f_s dW_s^{H,\varepsilon}$ converges in $L^2(\Omega)$ as $\varepsilon \to 0$, and then the fractional stochastic integral $\int_0^t f_s dW_s^H$ is defined as a limit of $\int_0^t f_s dW_s^{H,\varepsilon}$.

In this paper we are interested in a class of fractional stochastic differential equations with deterministic volatility of the following form

$$\begin{cases} dX_t = a(t, X_t)\, dt + \sigma(t)\, dW_t^H \\ X_t|_{t=0} = X_0 \;,\;\; t \in [0,T]\,. \end{cases} \tag{1.1}$$

The existence and uniqueness of the solution of (1.1) are established via a study of its corresponding approximation equation.

The organization of the paper is as follows: Section 2 contains some basic results on the semimartingale approach given in [2, 4]. In Section 3, we prove the existence, uniqueness and Lipschitzian continuity of the solution of the approximation equations, one of main results of this paper is formulated in Theorem 3.5. In Section 4, the explicit solutions for the equation of Ornstein-Uhlenbeck type and for the fractional stochastic differential equation with polynomial drift are found. Finally, in Section 5 we study the ruin probability in the ALM model.

2. Preliminaries

For the sake of convenience, we recall some important results from [2, 4] which will be the basis of this paper.

Theorem 2.1. *For every $\varepsilon > 0$, $W_t^{H,\varepsilon}$ is a $\mathcal{F}_t$-semimartingale with the following decomposition*

$$W_t^{H,\varepsilon} = \int_0^t K(s+\varepsilon, s) dB_s + \int_0^t \varphi_s^\varepsilon ds, \tag{2.1}$$

where $(\mathcal{F}_t, 0 \le t \le T)$ is the natural filtration associated to B or W^H and

$$\varphi_s^\varepsilon = \int_0^s \partial_1 K(s+\varepsilon, u) dB_u \,,$$

$$\partial_1 K(t,s) = \frac{\partial K(t,s)}{\partial t} = C_H \frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}}(t-s)^{H-\frac{3}{2}}\,.$$

Hypothesis (H): *Assume that f is an adapted process belonging to the space $\mathbf{D}_B^{1,2}\big(L^2([0,T],\mathbb{R},du)\big)$ and that there exists β fulfilling $\beta + H > 1/2$ and $p > 1/H$ such that*

(i) $\sup\limits_{0<s<u<T} \frac{E\left[(f_u-f_s)^2+\int_0^T (D_r^B f_u - D_r^B f_s)^2 dr\right]}{|u-s|^{2\beta}}$ *is finite,*

(ii) $\sup\limits_{0<s<T} f_s$ *belongs to* $L^p(\Omega)$.

Remark 2.2. *The space $\mathbf{D}_B^{1,2}\big(L^2([0,T],\mathbb{R},du)\big)$ is defined as follows:*

For $h \in L^2([0,T],\mathbb{R})$, we denote by $B(h)$ the Wiener integral

$$B(h) = \int_0^T h(t)dB_t.$$

Let $\mathcal{S}$ denote the dense subset of $L^2(\Omega,\mathcal{F},P)$ consisting of those classes of random variables of the form

$$F = f(B(h_1),...,B(h_n)), \tag{2.2}$$

where $n \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^n, L^2([0,T],\mathbb{R})), h_1,...,h_n \in L^2([0,T],\mathbb{R})$. If F has the form (2.2), we define its derivative as the process $D^B F := \{D_t^B F, t \in [0,T]\}$ given by

$$D_t^B F = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(B(h_1),...,B(h_n))h_k(t).$$

We shall denote by $\mathbf{D}_B^{1,2}\big(L^2([0,T],\mathbb{R},du)\big)$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2} := \big[E|F|^2\big]^{\frac{1}{2}} + E\left[\int_0^T |D_u^B F|^2 du\right]^{\frac{1}{2}}.$$

Definition 2.3. *For a process f fulfilling* **Hypothesis (H)**. *The fractional stochastic integral of f with respect to W^H is defined by*

$$\int_0^t f_s\, dW_s^H = \int_0^t f_s K(t,s)\, dB_s + \int_0^t \int_s^t (f_u - f_s)\, \partial_1 K(u,s) du \delta B_s + \int_0^t du \int_0^u D_s^B f_u\, \partial_1 K(u,s) ds\,, \tag{2.3}$$

where the second integral in the right-hand side is a Skorohod integral (we refer to [10] *for more details about the Skorohod integral).*

Remark 2.4. *Suppose that f be an adapted process belonging to the space $\mathbf{D}_B^{1,2}\big(L^2([0,T],\mathbb{R},du)\big)$, then*

$$\int_0^t f_s\,dW_s^{H,\varepsilon} = \int_0^t f_s K(t+\varepsilon,s)\,dB_s + \int_0^t\int_s^t (f_u - f_s)\,\partial_1 K(u+\varepsilon,s)du\delta B_s + \int_0^t du\int_0^u D_s^B f_u\,\partial_1 K(u+\varepsilon,s)ds$$

and under the **Hypothesis (H)**, $\int_0^t f_s\,dW_s^{H,\varepsilon} \to \int_0^t f_s\,dW_s^H$ *in* $L^2(\Omega)$ *as* $\varepsilon \to 0$.

Remark 2.5. *If f is a deterministic function such that*

$$\int_0^t f_s^2 K^2(t,s)\,ds < \infty\,,$$

then

$$\int_0^t f_s\,dW_s^H = \int_0^t f_s K(t,s)\,dB_s + \int_0^t\int_s^t (f_u - f_s)\,\partial_1 K(u,s)dudB_s.$$

3. The main result

In this section we study the existence and uniqueness of the solution of (1.1) by considering its corresponding approximation equation which is defined immediately below.

Definition 3.1. *The stochastic differential equation*

$$\begin{cases} dX_t^\varepsilon = a(t,X_t^\varepsilon)\,dt + \sigma(t)\,dW_t^{H,\varepsilon} \\ X_t^\varepsilon|_{t=0} = X_0\ ,\ \ t\in[0,T] \end{cases} \tag{3.1}$$

*is called the approximation equation corresponding to the fractional stochastic differential equation (*1.1*).*

Noting that (3.1) is a stochastic differential equation driven by a semimartingale, the conditions for uniqueness and existence of the solution of it is well known. For more details, from (2.1) we can rewrite the equation (3.1) as follows

$$dX_t^\varepsilon = \Big(a(t,X_t^\varepsilon) + \sigma(t)\varphi_t^\varepsilon\Big)\,dt + K(t+\varepsilon,t)\sigma(t)\,dB_t\,. \tag{3.2}$$

The stochastic process $\sigma(t)\varphi_t^\varepsilon$ is not bounded. However, we can establish the existence and uniqueness of the solution of equation (3.2) by considering the sequence of stopped times

$$\tau_M = \inf\{t \in [0,T] : \int_0^t (\varphi_s^\varepsilon)^2 ds > M\} \wedge T\,, \tag{3.3}$$

and consider the sequence of corresponding stopped equations. The existence and uniqueness of the solution of the stopped equations is well known (see, for instance, [7, 8]). Then by taking limit when $M \to \infty$, we have the following theorem

Theorem 3.2. *Assume that the functions* $a : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}\,, \sigma : [0,T] \longrightarrow \mathbb{R}$ *are measurable with respect to all their arguments and the following conditions hold:*
(A1) There exists a constant $K > 0$ *such that for* $x, y \in \mathbb{R}$ *and* $t \in [0,T]$

$$|a(t,x) - a(t,y)| \leq K|x-y| \ \ , \ \ |a(t,x)| \leq K(1+|x|)\,. \tag{3.4}$$

(A2) For all $t \in [0,T]$

$$\int_0^t \sigma_s^2 K^2(t,s)\, ds < \infty\,, \tag{3.5}$$

(A3) The initial value X_0 *is square-integrable random variable and it is independent of W.*

*Then equation (*3.1*) has unique solution* $\sigma(W_s, 0 \leq s \leq t)$*-adapted* X_t^ε *on* $[0,T]$*. Moreover, in the case* $H > 1/2$

$$\sup_{0\leq t\leq T} E|X_t^\varepsilon|^2 \leq C\,, \tag{3.6}$$

where C *is some positive constant not depending on* ε*.*

Proposition 3.3. *Assume that conditions for the existence and uniqueness of the solutions of both fractional stochastic differential equation (*1.1*) and approximation equation (*3.1*) hold. Then the sequence of solutions of the approximation equation (*3.1*) converges in* $L^2(\Omega)$ *to the solution of (*1.1*) as* $\varepsilon \to 0$.

Proof. We have

$$E|X_t^\varepsilon - X_t|^2 \leq 2E|\int_0^t a(s, X_s^\varepsilon)\, ds - \int_0^t a(s, X_s)\, ds|^2 + 2E|\int_0^t \sigma(s)\, dW_s^{H,\varepsilon} - \int_0^t \sigma(s)\, dW_s^H|^2 \tag{3.7}$$

According to Remark 2.4 we can see that

$$E|\int_0^t \sigma(s)\, dW_s^{H,\varepsilon} - \int_0^t \sigma(s)\, dW_s^H|^2 := C(t,\varepsilon) \to 0 \ \text{ as } \ \varepsilon \to 0\,.$$

Now, using the Lipschitz continuity assumption (3.4) we get

$$\int_0^t E|a(s,X_s^\varepsilon)-a(s,X_s)|^2 ds \le K^2\int_0^t E|X_t^\varepsilon - X_t|^2 ds.$$

Thus, the conclusion of this proposition is easily achieved by applying Gronwall's lemma. □

Due to the above Proposition, the solution of equation (1.1) can be considered as the limit in $L^2(\Omega)$ of the solutions of the equations (3.1), and so, if this limit exists then the equation (1.1) has an unique solution.

Let $\varepsilon = \frac{1}{n}$, $n \ge 1$, we recall from Remark 2.4 and Remark 2.5 that

$$\int_0^t \sigma_s\, dW_s^{H,\frac{1}{n}} = \int_0^t \sigma_s K(t+\frac{1}{n},s)\, dB_s + \int_0^t\int_s^t (\sigma_u-\sigma_s)\,\partial_1 K(u+\frac{1}{n},s) du dB_s$$

Let us now consider a sequence of approximation equations

$$\begin{cases} dX_t^n = a(t,X_t^n)\, dt + \sigma(t)\, dW_t^{H,\frac{1}{n}} \\ X_t^n|_{t=0} = X_0\ ,\ \ t\in[0,T] \end{cases} \tag{3.8}$$

or

$$X_t^n = X_0 + \int_0^t a(s,X_s^n)\, ds + \int_0^t \sigma_s K(t+\frac{1}{n},s)\, dB_s + \int_0^t\int_s^t (\sigma_u-\sigma_s)\,\partial_1 K(u+\frac{1}{n},s) du dB_s\,. \tag{3.9}$$

Theorem 3.4. *Let $H\in(\frac{1}{2},1)$ and the coefficients of equation (3.8) satisfy the assumptions (A1), (A2), (A3) from Theorem 3.1. Then*
I. The solution of (3.8) is Lipschitz continuous in $L^2(\Omega)$, i.e

$$E|X_t^n - X_s^n|^2 \le C|t-s|\,. \tag{3.10}$$

II. For every $t\in[0,T]$, the sequence $\{X_t^n\}_{n\ge1}$ of the solutions of the equations (3.8) is a fundamental sequence in $L^2(\Omega)$.

Proof. I. We consider $E|X_{t+\tau}^n - X_t^n|^2$ for $0\le t\le t+\tau\le T$:

$$E|X_{t+\tau}^n - X_t^n|^2 \le 3E\Big(\int_t^{t+\tau} a(s,X_s^n)\, ds\Big)^2$$

$$+3E\Big(\int_0^{t+\tau}\sigma(s)K(t+\tau+\frac{1}{n},s)\, dB_s - \int_0^t \sigma(s)K(t+\frac{1}{n},s)\, dB_s\Big)^2$$

$$+3E\Big(\alpha \int_0^{t+\tau}\int_s^{t+\tau} [\sigma(u)-\sigma(s)]\partial_1 K(u+\frac{1}{n},s)du dB_s$$

$$-\alpha \int_0^{t}\int_s^{t} [\sigma(u)-\sigma(s)]\partial_1 K(u+\frac{1}{n},s)du dB_s\Big)^2 := 3I_1+3I_2+3I_3.$$

First, it follows from (3.4), (3.6) that

$$I_1 \le K^2 \int_t^{t+\tau} E(1+X_s^n)^2\,ds \le 2K^2(1+C)\tau\,. \tag{3.11}$$

Next, we can estimate I_2 as

$$I_2 \le 2E\Big(\int_0^t \sigma(s)[K(t+\tau+\frac{1}{n},s)-K(t+\frac{1}{n},s)]\,dB_s\Big)^2$$

$$+\,2E\Big(\int_t^{t+\tau} \sigma(s)K(t+\tau+\frac{1}{n},s)\,dB_s\Big)^2 \tag{3.12}$$

$$= 2\int_0^t \sigma^2(s)[K(t+\tau+\frac{1}{n},s)-K(t+\frac{1}{n},s)]^2\,ds$$

$$+\,2\int_t^{t+\tau} \sigma^2(s)K^2(t+\tau+\frac{1}{n},s)\,ds$$

$$\le 2\|\sigma\|_\infty^2 E|W^H_{t+\tau+\frac{1}{n}}-W^H_{t+\frac{1}{n}}|^2 + 2\int_t^{t+\tau} \sigma^2(s)K^2(t+\tau+\frac{1}{n},s)\,ds$$

$$= 2\|\sigma\|_\infty^2\tau^{2H} + 2\int_t^{t+\tau} \sigma^2(s)K^2(t+\tau+\frac{1}{n},s)\,ds \le C_1\tau$$

where $\|\sigma\|_\infty = \sup\limits_{0\le s\le T}|\sigma(s)|$, C_1 is a positive finite constant depending on σ.

Similarly, for I_3 we have

$$I_3 \le 2E\Big(\int_0^t\int_t^{t+\tau} [\sigma(u)-\sigma(s)]\partial_1 K(u+\frac{1}{n},s)du dB_s\Big)^2+$$

$$+2E\Big(\int_t^{t+\tau}\int_s^{t+\tau} [\sigma(u)-\sigma(s)]\partial_1 K(u+\frac{1}{n},s)du dB_s\Big)^2$$

$$= 2\int_0^t \Big(\int_t^{t+\tau} [\sigma(u) - \sigma(s)]\partial_1 K(u + \frac{1}{n}, s)du\Big)^2 ds$$

$$+2\int_t^{t+\tau} \Big(\int_s^{t+\tau} [\sigma(u) - \sigma(s)]\partial_1 K(u + \frac{1}{n}, s)du\Big)^2 ds$$

$$\leq 8\|\sigma\|_\infty^2 \int_0^t [K(t+\tau+\frac{1}{n}, s) - K(t+\frac{1}{n}, s)]^2 ds$$

$$+8\|\sigma\|_\infty^2 \int_t^{t+\tau} [K(t+\tau+\frac{1}{n}, s) - K(s+\frac{1}{n}, s)]^2 ds$$

$$= 8\|\sigma\|_\infty^2 \tau^{2H} + 8\|\sigma\|_\infty^2 \int_t^{t+\tau} [K(t+\tau+\frac{1}{n}, s) - K(s+\frac{1}{n}, s)]^2 ds\,.$$

Hence,

$$I_3 \leq C_2 \tau\,. \tag{3.13}$$

Finally, (3.10) follows from the inequalities (3.11)-(3.13).

II. We now are ready to prove the rest of the theorem. Consider $E|X_t^n - X_t^m|^2$:

$$\begin{aligned} E|X_t^n - X_t^m|^2 &\leq 3\int_0^t E[a(s, X_s^n) - a(s, X_s^m)]^2\, ds \\ &+ 3E\Big(\int_0^t [\sigma(s)K(t+\frac{1}{n}, s) - \sigma(s)K(t+\frac{1}{m}, s)]\, dB_s\Big)^2 \\ &+ 3E\Big(\int_0^t \int_s^t \big\{[\sigma(u) - \sigma(s)]\partial_1 K(u+\frac{1}{n}, s) \\ &- [\sigma(u) - \sigma(s)]\partial_1 K(u+\frac{1}{m}, s)\big\} du dB_s\Big)^2 \\ &:= 3(J_1 + J_2 + J_3)\,. \end{aligned} \tag{3.14}$$

$$J_1 \leq K^2 \int_0^t E|X_s^n - X_s^m|^2 ds\,. \tag{3.15}$$

$$J_2 = \int_0^t [\sigma(s)K(t+\frac{1}{n},s) - \sigma(s)K(t+\frac{1}{m},s)]^2\,ds$$

$$\leq \|\sigma\|_\infty^2 \int_0^T [K(t+\frac{1}{n},s) - K(t+\frac{1}{m},s)]^2\,ds$$

$$= \|\sigma\|_\infty^2 \{R(t+\frac{1}{n},t+\frac{1}{n}) + R(t+\frac{1}{m},t+\frac{1}{m}) - 2R(t+\frac{1}{n},t+\frac{1}{m})\}$$

$$\leq C_3|\frac{1}{n}-\frac{1}{m}|^{2H-1} := c_1(m,n), \quad (3.16)$$

where C_3 is a finite positive constant depending on σ, and $R(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$ is the covariance function of the fBm W^H. We have

$$J_3 = \int_0^t \left(\int_s^t [\sigma(u)-\sigma(s)][\partial_1 K(u+\frac{1}{n},s) - \partial_1 K(u+\frac{1}{m},s)]du \right)^2 ds \quad (3.17)$$

$$\leq 8\|\sigma\|_\infty^2 \int_0^t \left(K(t+\tfrac{1}{n},s) - K(t+\tfrac{1}{m},s) \right)^2 ds$$

$$+ 8\|\sigma\|_\infty^2 \int_0^t \left(K(s+\frac{1}{n},s) - K(s+\frac{1}{m},s) \right)^2 ds \leq 16c_1(m,n).$$

Put $g(t) = E|X_t^n - X_t^m|^2$, then combining (3.14)-(3.17) yields

$$g(t) \leq 3K^2 \int_0^t g(s)ds + c(m,n)\,, \quad (3.18)$$

where $c(m,n) = 3(c_1(m,n) + 16c_1(m,n)) \to 0$ as $m \to \infty, n \to \infty$.

From (3.18) and by applying Gronwall's lemma we get

$$g(t) \leq c(m,n)\,e^{3K^2 t}\,,$$

or

$$E|X_t^n - X_t^m|^2 \leq c(m,n)\,e^{3K^2 t}\,.$$

And, as a consequence, the solutions $\{X_t^n, 0 \leq t \leq T\}_{n\geq 1}$ of equations (3.8) form a fundamental sequence in $L^2(\Omega)$. □

Now we can state the following theorem

Theorem 3.5. *Suppose that $H \in (\frac{1}{2}, 1)$. Consider the fractional stochastic differential equation*

$$\begin{cases} dX_t = a(t, X_t)\, dt + \sigma(t)\, dB_t \\ X_t|_{t=0} = X_0 \ , \ \ t \in [0, T], \end{cases} \tag{3.19}$$

where $\sigma(t)$ is a deterministic function. If the coefficients $a(t,x)\,, \sigma(t)$ satisfy the assumptions (A1), (A2) from Theorem 3.1, then (3.19) has a unique solution. Moreover, this solution is Lipschitz continuous in $L^2(\Omega)$, i.e

$$E|X_t - X_s|^2 \leq C|t-s|\,.$$

Remark 3.6. *If $a(t,x)$ is Lipschitzian with respect to x and under assumption (A2) then the existence of the solution (3.19) can be proved by applying the fixed point theorem in some Banach space after constructing an appropriate contraction operator in this space. For the uniqueness, it suffices to use Gronwall's lemma.*

4. Explicit solution for some important classes of stochastic differential equations

From practical point of view, it is important to find the explicit expression for the solution of each specific model. In the rest of this paper, we will see that the semimatingale approach has more advantages for this.

4.1. The Ornstein-Uhlenbeck type equations

The fractional Ornstein-Uhlenbeck processes are studied in [3]. Let us use semimartingale approach to find the solution for a class of Ornstein-Uhlenbeck type equations of following form:

$$\begin{cases} dX_t = (\alpha(t)X_t + \beta(t))\, dt + \sigma(t)\, dW_t^H \\ X_t|_{t=0} = X_0 \ , \ \ t \in [0, T], \end{cases} \tag{4.1}$$

where $\alpha(t), \beta(t), \sigma(t)$ are deterministic functions.

The approximation equation corresponding to (4.1) is

$$\begin{cases} dX_t^\varepsilon = (\alpha(t)X_t^\varepsilon + \beta(t))\, dt + \sigma(t)\, dW_t^{H,\varepsilon} \ , \ \varepsilon > 0 \\ X_t^\varepsilon|_{t=0} = X_0 \ , \ \ t \in [0, T] \end{cases}$$

or equivalently,

$$\begin{cases} dX_t^\varepsilon = (\alpha(t)X_t^\varepsilon + \beta(t) + \sigma(t)\varphi_t^\varepsilon)\, dt + K(t+\varepsilon, t)\sigma(t)\, dB_t \\ X_t^\varepsilon|_{t=0} = X_0 \ , \ \ t \in [0, T]\,. \end{cases} \tag{4.2}$$

This is a semilinear stochastic differential equation.Therefore, its solution is given by

$$X^{\varepsilon}(t) = e^{\int_0^t \alpha(u)du}\Big(X_0 + \int_0^t \beta(s)\, e^{-\int_0^s \alpha(u)du} ds + \int_0^t \sigma(s)\varphi_s^{\varepsilon}\, e^{-\int_0^s \alpha(u)du} ds$$
$$+ \int_0^t K(s+\varepsilon, s)\sigma(s)\, e^{-\int_0^s \alpha(u)du} dB_s\Big).$$

Using (2.1) we can rewrite the solution $X^{\varepsilon}(t)$ into the following form

$$X^{\varepsilon}(t) = e^{\int_0^t \alpha(u)du}\Big(X_0 + \int_0^t \beta(s)\, e^{-\int_0^s \alpha(u)du} ds$$
$$+ \int_0^t \sigma(s)\, e^{-\int_0^s \alpha(u)du} dW_s^{H,\varepsilon}\Big). \quad (4.3)$$

By taking limit when $\varepsilon \to 0$ we get the following theorem.

Theorem 4.1. *Suppose that X_0 is a square-integrable random variable independent of W^H. Then the solution of* (4.1) *is unique and given by*

$$X_t = e^{\int_0^t \alpha(u)du}\big(X_0 + \int_0^t \beta(s) e^{-\int_0^s \alpha(u)du} ds + \sigma \int_0^t e^{-\int_0^s \alpha(u)du} dW_t^H\big).$$

4.2. Fractional stochastic differential equations with polynomial drift

Let us consider the fractional stochastic differential equation in a complete probability space $(\Omega, \mathcal{F}, P)$

$$\begin{cases} dX_t = \big(a\,X_t^n + b\,X_t\big)\,dt + c\,X_t\,dW_t^H \\ X_t|_{t=0} = X_0. \end{cases} \quad (4.4)$$

The initial value X_0 is a measurable random variable independent of $\{B_t : 0 \le t \le T\}$.

This equation is a generalization of many important equations such as the Black-Sholes model in mathematical finance ($a = 0$), the Ginzburg-Landau equation in the theoretical physics ($n = 3$), the Verlhust equation in population study ($n = 2$).

We consider now a corresponding approximation equation with the same initial condition $X_t^{\varepsilon}|_{t=0} = X_0$

$$dX_t^{\varepsilon} = \big(a\,(X_t^{\varepsilon})^n + b\,X_t^{\varepsilon}\big)\,dt + c\,X_t^{\varepsilon}\,dW_t^{H,\varepsilon}\ ,\ \varepsilon > 0. \quad (4.5)$$

Using (2.1) again we get

$$dX_t^{\varepsilon} = \big(a\,(X_t^{\varepsilon})^n + b\,X_t^{\varepsilon} + c\varphi_t^{\varepsilon}\,X_t^{\varepsilon}\big)\,dt + c\,X_t^{\varepsilon}\,dB_t\,. \quad (4.6)$$

In order to find the explicit expression for the solution of the equation (4.5) we will carry out several steps.

Step 1. Put $Y_t^\varepsilon = e^{-U_t}, U_t = \int_0^t cK(s+\varepsilon, s)ds$. According to the Itô formula we have:

$$dY_t^\varepsilon = \frac{1}{2} Y_t^\varepsilon c^2 K^2(t+\varepsilon, t)dt - Y_t^\varepsilon cK(t+\varepsilon,t)dB_t. \tag{4.7}$$

Step 2. We consider $Z_t^\varepsilon = X_t^\varepsilon Y_t^\varepsilon$ and then the integration-by-part formula gives us

$$dZ_t^\varepsilon = X_t^\varepsilon dY_t^\varepsilon + Y_t^\varepsilon dX_t^\varepsilon - d[X^\varepsilon, Y^\varepsilon]_t$$

or

$$dZ_t^\varepsilon = \left\{ [-\frac{1}{2}c^2K^2(t+\varepsilon,t)+b+c\varphi_t^\varepsilon] Z_t^\varepsilon + a\, e^{(n-1)\int_0^t cK(s+\varepsilon,s)ds} (Z_t^\varepsilon)^n \right\} dt. \tag{4.8}$$

For every fixed $\omega \in \Omega$, the equation (4.8) is an ordinary Bernoulli equation of the form:

$$(Z_t^\varepsilon)' = P(t)(Z_t^\varepsilon)^n + Q(t) Z_t^\varepsilon$$

and the solution Z_t^ε is given by

$$Z_t^\varepsilon = e^{\int_0^t Q(u)du} \left(Z_0^{1-n} + \int_0^t (1-n)P(s) e^{(n-1)\int_0^s Q(u)du} ds \right)^{\frac{1}{1-n}}$$

where $P(t) = a\, e^{(n-1)\int_0^t cK(s+\varepsilon,s)ds}, Q(t) = -\frac{1}{2}c^2K^2(t+\varepsilon,t) + b + c\varphi_t^\varepsilon$, the initial condition $Z_0^\varepsilon = X_0^\varepsilon Y_0^\varepsilon = X_0$.

Finally, the solution $X_t^\varepsilon = \frac{Z_t^\varepsilon}{Y_t^\varepsilon}$ of the equation (4.5) is given by

$$X_t^\varepsilon = e^{bt - \frac{1}{2}\int_0^t c^2K^2(s+\varepsilon,s)ds + c\, W_t^{H,\varepsilon}}$$
$$\times \left(X_0^{1-n} + (1-n)a \int_0^t e^{(n-1)\left(bs - \frac{1}{2}\int_0^s c^2K^2(u+\varepsilon,u)du + c\, W_s^{H,\varepsilon}\right)} ds \right)^{\frac{1}{1-n}}. \tag{4.9}$$

Noting that the solution of (4.4) is a limit in $L^2(\Omega)$ of the solution of (4.5). Hence, by taking limit when $\varepsilon \to 0$ we get the following theorem.

Theorem 4.2. *Suppose that X_0 is a random variable independent of W^H such that $E[X_0^2] < \infty$. Then the solution of (4.4) exists and is unique and given by*

$$X_t = e^{bt + c\, W_t^H} \left(X_0^{1-n} + (1-n)a \int_0^t e^{(n-1)\,(bs + c\, W_s^H)} ds \right)^{\frac{1}{1-n}}.$$

5. The ruin probability in the Asset Liability Management model

In this section, we consider the asset X_t and the liability Y_t satisfing the following stochastic differential equations

$$\begin{cases} dX_t = \mu_1 X_t dt + \sigma_1 X_t dW_t^{(1)}, \\ dY_t = \mu_2 Y_t dt + \sigma_2 Y_t dW_t^{(2)}, \\ X|_{t=0} = X_0 \ , \ Y|_{t=0} = Y_0 < X_0, \end{cases} \tag{5.1}$$

where $\mu_1, \mu_2, \sigma_1, \sigma_2$ are non-negative parameters, $W_t^{(1)} = \int_0^t K(t,s)dB_t^{(1)}, W_t^{(2)} = \int_0^t K(t,s)dB_t^{(2)}$ are two fractional Brownian motions with correlation coefficient $|\rho| \leq 1$.

It follows from Theorem 4.2 that

$$X_t = X_0 e^{\mu_1 t + \sigma_1 W_t^{(1)}} \ , \ Y_t = Y_0 e^{\mu_2 t + \sigma_2 W_t^{(2)}}$$

and

$$\frac{X_t}{Y_t} = \frac{X_0}{Y_0} \exp\left((\mu_1 - \mu_2)t + \sigma_1 W_t^{(1)} - \sigma_2 W_t^{(2)}\right).$$

Noting that $B^{(1)}, B^{(2)}$ have correlation coefficient ρ, because $W^{(1)}, W^{(2)}$ have correlation coefficient ρ. Hence

$$\sigma_2 B_t^{(2)} - \sigma_1 B_t^{(1)}$$

is equivalent in distribution to the process σB_t, where B_t is a standard Brownian motion and

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \tag{5.2}$$

We obtain

$$\sigma_1 W_t^{(1)} - \sigma_2 W_t^{(2)} = \int_0^t K(t,s)d(\sigma_1 B_s^{(1)} - \sigma_2 B_s^{(2)})$$

$$= -\sigma \int_0^t K(t,s)dB_s =: -\sigma W_t^H$$

and

$$\frac{X_t}{Y_t} = \frac{X_0}{Y_0} \exp(\mu t - \sigma W_t^H) \ , \tag{5.3}$$

where $\mu = \mu_1 - \mu_2$, W_t^H is a fractional Brownian motion with index H.

We now can study the lifetime τ of a bank or of an insurance company that is naturally defined as the first value of t such that $X_t < Y_t$:

$$\tau = \inf\{t : \ln \frac{X_t}{Y_t} < 0\}.$$

and the ruin probability on a finite time horizon $[0, t]$ is defined as

$$\varphi(X_0, Y_0, t) := P(\tau < t) = P(\ln \frac{X_s}{Y_s} < 0 \text{ for some } s < t),$$

and on an infinite time horizon,

$$\varphi(X_0, Y_0) := \lim_{t\to\infty} \varphi(X_0, Y_0, t).$$

By the relation (5.3) we obtain

$$\begin{aligned}\varphi(X_0, Y_0) &= P(\ln \frac{X_t}{Y_t} < 0 \text{ for some } t \geq 0)\\ &= P(-\mu t + \sigma W_t^H > u \text{ for some } t \geq 0)\\ &= P\big(\sup_{t\geq 0}(-\mu t + \sigma W_t^H) > u\big),\end{aligned}$$

where $u = \ln \frac{X_0}{Y_0}$. In order to estimate $\varphi(X_0, Y_0)$ we use the following result of Dębicki [5, Corollary 4.1]:

Proposition 5.1. *For* $\frac{1}{2} \leq H \leq 1$

$$\lim_{u\to\infty} \frac{1}{u^{2-2H}} \ln P\big(A(W^H, c) > u\big) = -h \qquad (5.4)$$

where $A(W^H, c) = \sup\{W_t^H - ct : t \geq 0\}$ *and*

$$h = \frac{1}{2}(\frac{c}{H})^{2H}(\frac{1}{1-H})^{2-2H}.$$

Now we can state the following theorem

Theorem 5.2. *If* $\mu_1 \geq \mu_2$*, then the ruin probability for the ALM model (*5.1*) satisfies the following relation:*

$$\lim_{u\to\infty} \frac{\ln \varphi(X_0, Y_0)}{u^{2-2H}} = -\frac{\mu^{2H}}{2H^2\sigma^2}(\frac{H}{1-H})^{2-2H}, \qquad (5.5)$$

where $\mu = \mu_1 - \mu_2$ *,* $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$ *and* $u = \ln \frac{X_0}{Y_0}$.

Proof. We have

$$\varphi(X_0, Y_0) = P\big(\sup_{t\geq 0}(W_t^H - \frac{\mu}{\sigma}t) > \frac{u}{\sigma}\big)$$

from Proposition 5.1. The theorem is completed. □

References

[1] Alòs, E., Mazet, O., Nualart, D., *Stochastic calculus with respect to fractional Brownian motion with Hurst parameter less than* $\frac{1}{2}$, Stochastic Processes and their Applications, **86**(2000), no. 1, 121-139.

[2] Carmona, P., Coutin, L., Montseny, G., *Stochastic integration with respect to fractional Brownian motion*, Ann. Inst. H. Poincaré Probab. Statist., **39(1)**(2003), 27-68.

[3] Cheridito, P., Kawaguchi, H., Maejima, M., *Fractional Ornstein-Uhlenbeck processes*, Electronic Journal of Probability, **8**(2003), no. 3, 1-14.

[4] Coutin, L., *An Introduction to Stochastic Calculus with Respect to Fractional Brownian motion*, In Séminaire de Probabilités XL, Springer-Verlag Berlin Heidelberg, 2007, 3-65.

[5] Dębicki, K., Michna, Z., Rolski, T., *On the supremum from Gaussian processes over infinite horizon*, Probability and Math. Stat., **18**(1998), 83-100.

[6] Dung, N. T., *A class of fractional stochastic differential equations*, Vietnam Journal of Mathematics, **36**(2008), no. 3, 271-279.

[7] Gihman, I. I., Skorohod, A. V., *Stochastic Differential Equations*, Springer, 1972.

[8] Kloeden, P. E., Platen, E., *Numerical Solution of Stochastic Differential Equations*, Springer, 1995.

[9] Léon, *Fubini theorem for anticipating stochastic integrals in Hilbert space*, Appl. Math. Optim., **27(3)**(1993), 313-327.

[10] Nualart, D., *The Malliavin Calculus and Related Topics*, 2nd edition, Springer, 2006.

[11] Thao, T. H., Thomas-Agnan, C., *Evolution des cours gouvernée par un processus de type ARIMA fractionnaire,* Stud. Univ. Babeş-Bolyai Math., **XVIII**(2003), no. 2.

[12] Thao, T. H., *An approximate approach to fractional analysis for finance,* Nonlinear Analysis, **7**(2006), 124-132.

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On uniform exponential stability of backwards evolutionary processes

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Abstract. The exponential stability of a special class of evolution families is analyzed. Extensions of the well-known theorems due to Datko and Barbashin are obtained, in both continuous and discrete-time.

Mathematics Subject Classification (2010): 34D20.

Keywords: Backwards evolutionary processes, uniform exponential stability, Barbashin's theorem.

1. Introduction

In 1967 E. A. Barbashin [1, Th. 5.1] obtained a stability result for exponentially bounded evolution families generated by differential systems in Banach spaces, a result that remains true in the case of evolution families with exponential growth. In 1970 R. Datko [4] proved that the C_0-semigroup $\{T_t\}_{t\geq 0}$ is exponentially stable if and only if its trajectories $(T(\cdot)x)$ are in L^2 for all x in X. This result was generalized by A. Pazy [12], who proved that the exponential stability property is equivalent with $T(\cdot)x \in L^p$, for $1 \leq p < \infty$ and for all x in X, where X is a Banach space.

Later, a well-known Datko result from 1972 [5] states that an exponentially bounded, strongly continuous evolution family $\mathbf{U} = \{U(t,t_0)\}_{t\geq t_0\geq 0}$ with exponential growth is exponentially stable if and only if there exist $k, p > 0$ such that

$$\left(\int_t^\infty ||U(\tau,t)x||^p d\tau\right)^{\frac{1}{p}} \leq k||x||, \text{ for all } t \geq 0, \text{ and } x \in X.$$

This result was extended by J.L. Daleckij and M.G. Krein [3] for evolutionary processes generated by differential systems in Banach spaces and instead of R. Datko's method, it has been used a characterization theorem for the exponential stability of differential systems [3, Th. 6.1, pg 132]. S. Rolewicz [13] noticed that the theorem used by J.L. Daleckij and M.G. Krein in [3] remains true in the case of evolutionary processes with exponential growth (without stating the proof, though). This theorem, along with the Baire Cathegory Principle allowed S. Rolewicz [13] to extend Datko's result from 1972 to the fact that $\{U(t,t_0)\}_{t\geq t_0\geq 0}$ is exponentially stable if and only if there exists $N:(0,\infty)\times(0,\infty)\to\mathbb{R}_+$ with the property that $N(\alpha,u)$ is continuous and increasing for all α, and $N(\alpha,u)$ is increasing for all u, $N(\alpha,0)=0$ for all $\alpha>0$, $N(\alpha,u)>0$ for all $u>0$ and for all $x\in X$ there exists $\alpha(x)>0$ such that

$$\sup_{t\geq 0}\int_t^\infty N(\alpha(x),||U(\tau,t)||)d\tau<\infty.$$

Another extension of the result due to Datko [4] and Pazy [12] was obtained by W. Littman [7] in 1989. V. Pata [11] came with a new proof and a generalization of the result due to Datko [5] for the case of strongly continuous semigroups of bounded linear operators.

The classical ideas of J. L. Massera and J. J. Schäffer ([8],[9]) on exponential stability and other asymptotic properties of the solutions of differential equations have also been developed in the last years. Other results for the stability of nonlinear evolution families were obtained by A. Ichikawa [6] and in 2007, a strong variant of a result due to E. A. Barbashin [1] was obtained by C. Buşe, M. Megan, M. S. Prajea and P. Preda [2] on the dual space of the Banach space X. Some Datko [5] type results for the asymptotic behavior of skew-evolution semiflows in Banach spaces were given by M. Megan and C. Stoica [10] in 2008.

The purpose of the present paper is to give a characterization for the exponential stability of a special class of evolution families, called the backwards evolution families, and thus to reformulate the result due to E. A. Barbashin [1].

2. Preliminaries

Let us consider X a Banach space, $\mathcal{B}(X)$ the Banach algebra of all linear and bounded operators acting on X and $\Delta=\{(t,t_0)\in\mathbb{R}^2 : t\geq t_0\geq 0\}$. We denote the norm of vectors on X and operators on $\mathcal{B}(X)$ by $||\cdot||$.

Definition 2.1. *A family of linear and bounded operators*

$$\mathbf{\Phi}=\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}:\Delta\to\mathcal{B}(X)$$

is called a backwards evolutionary process if the following properties hold:

i) $\Phi(t,t)=I$, *for all* $t\geq 0$*;*
ii) $\Phi(\tau,t_0)\ \Phi(t,\tau)=\Phi(t,t_0)$, *for all* $t\geq\tau\geq t_0\geq 0$*;*
iii) $\Phi(\cdot,t_0)x:[t_0,\infty)\to X$ *is continuous for all* $t_0\geq 0$ *and* $x\in X$
$\Phi(t,\cdot)x:[0,t]\to X$ *is continuous for all* $t\geq 0$ *and* $x\in X$*;*

iv) there exist $M \geq 1$ *and* $\omega \in \mathbb{R}$ *such that:*

$$||\Phi(t,t_0)|| \leq Me^{\omega(t-t_0)},$$

for all $t \geq t_0 \geq 0$.

Example 2.2. *Take* $X = \mathbb{R}$ *and the equation:*

$$(A) \qquad \dot{x}(t) = A(t)x(t), \ t \geq 0.$$

We consider the Cauchy problem associated:

$$(B) \qquad \begin{cases} \dot{U}(t) = A(t)U(t) \\ U(0) = I. \end{cases}$$

where $A \in \mathcal{M}(2,\mathbb{R})$ *and* $\mathcal{M}(2,\mathbb{R})$ *denotes the set of all* 2*-by-*2 *real matrices and* $t \geq 0$.

The unique solution of the Cauchy problem (B) *will be denoted by* $U(t)$ *and* $\Phi(t,t_0) = U^{*-1}(t_0)U^*(t)$ *represents the backwards evolutionary process generated by the equation* (A).

Example 2.3. *Let* $X = \mathbb{R}$. *Then*

$$\Phi(t,t_0) = \frac{\sin t + 1}{\sin t_0 + 1}$$

defines a backwards evolutionary process.

Example 2.4. *Let* $X = \mathbb{R}$. *Then*

$$\Phi(t,t_0) = \frac{t^2+1}{t_0^2+1}$$

defines a backwards evolutionary process.

Definition 2.5. *Let* $\Phi = \{\Phi(t,t_0)\}_{t \geq t_0 \geq 0}$ *be a backwards evolutionary process.* Φ *is called uniformly exponentially stable if there exist* $N, \nu > 0$ *such that:*

$$||\Phi(t,t_0)|| \leq Ne^{-\nu(t-t_0)}, \ \textit{for all } t \geq t_0 \geq 0.$$

3. The main result

In order to establish sufficient conditions for the uniform exponential stability of the backwards evolutionary process, we will use a result due to J. L. Massera and J. J. Schäffer [8]:

Lemma 3.1. *Take* $f, g : \mathbb{R}_+ \to \mathbb{R}_+$, *g continuous, such that*

i) $f(t) \leq g(t-t_0)f(t_0)$, *for all* $t \geq t_0 \geq 0$;

ii) $\inf_{t\geq 0} g(t) < 1$.

Then there exist $N, \nu > 0$ *such that*

$$f(t) \leq Ne^{-\nu(t-t_0)}f(t_0), \ \textit{for all } t \geq t_0 \geq 0.$$

The following theorem is a strong variant of a result due to E. A. Barbashin [1], for the case of backwards evolutionary processes:

Theorem 3.2. *Let Φ be a backwards evolutionary process. Φ is uniformly exponentially stable if and only if there exist $p, k > 0$ such that:*

$$\Big(\int_{t_0}^{t} ||\Phi(t,\tau)x||^p d\tau\Big)^{\frac{1}{p}} \le k||x||,$$

for all $x \in X$ and $t \ge t_0$.

Proof. The *necessity* is immediate, and for the *sufficiency* let $t \ge t_0 + 1$ and $r(t) = Me^{\omega t}$, where

$$||\Phi(t,t_0)x|| \le Me^{\omega(t-t_0)}||x||, \text{ for all } t \ge t_0.$$

Then

$$||\Phi(t,t_0)x||^p \int_{t_0}^{t} r^{-p}(\tau - t_0)d\tau \le \int_{t_0}^{t} ||\Phi(\tau,t_0)||^p \; ||\Phi(t,\tau)x||^p \; r^{-p}(\tau - t_0)d\tau$$

$$\le \int_{t_0}^{t} ||\Phi(t,\tau)x||^p d\tau \le k^p||x||^p.$$

But

$$\int_{t_0}^{t} r^{-p}(\tau - t_0)d\tau = \int_{0}^{t-t_0} r^{-p}(s)ds \ge \int_{0}^{1} r^{-p}(s)ds.$$

We denote by

$$\int_{0}^{1} r^{-p}(s)ds = \alpha > 0.$$

For $\sup_{||x||=1}$ it implies that

$$||\Phi(t,t_0)|| \le \frac{k}{\alpha^{\frac{1}{p}}}, \text{ for all } t \ge t_0 + 1.$$

If $t \in [t_0, t_0 + 1]$ then

$$||\Phi(t,t_0)|| \le Me^{\omega}$$

and therefore

$$||\Phi(t,t_0)|| \le \max\{Me^{\omega}, \frac{k}{\alpha^{\frac{1}{p}}}\} = L, \text{ for all } t \ge t_0 \tag{3.1}$$

Take now $t \ge t_0 \ge 0$ and $\tau \in [t_0, t]$. It follows that

$$||\Phi(t,t_0)x|| = ||\Phi(\tau,t_0)\;\Phi(t,\tau)x|| \le L||\Phi(t,\tau)x||.$$

Thus,

$$(t - t_0)||\Phi(t,t_0)x||^p \le L^p \int_{t_0}^{t} ||\Phi(t,\tau)x||^p d\tau \le L^p k^p ||x||^p.$$

For $\sup_{||x||=1}$ in the above inequality we obtain that

$$(t - t_0)^{\frac{1}{p}}||\Phi(t,t_0)|| \le Lk. \tag{3.2}$$

Adding the inequalities (3.1) and (3.2) it results that

$$||\Phi(t,t_0)|| \le \frac{(1+k)L}{1 + (t - t_0)^{\frac{1}{p}}}, \text{ for all } t \ge t_0.$$

Therefore, we have obtained that

$$||\Phi(t,t_0)|| \leq ||\Phi(\tau,t_0)||\ ||\Phi(t,\tau)|| \leq \frac{(1+k)L}{1+(t-\tau)^{\frac{1}{p}}}\ ||\Phi(\tau,t_0)||.$$

By denoting

$$f(t) = ||\Phi(t,t_0)|| \text{ and } g(t-\tau) = \frac{(1+k)L}{1+(t-\tau)^{\frac{1}{p}}},$$

from Lemma 3.1 it follows that there exist $N,\nu > 0$ such that

$$||\Phi(t,t_0)|| \leq Ne^{-\nu(t-\tau)}||\Phi(\tau,t_0)||.$$

Taking $\tau = t_0$ we obtain that

$$||\Phi(t,t_0)|| \leq Ne^{-\nu(t-t_0)} \text{ for all } t \geq t_0.$$ □

Remark 3.3. *We give now another proof for the sufficiency of Theorem* 3.2*, with a direct method:*

Let $t \geq t_0 + 1$ *and* $\tau \in [t_0, t_0+1]$*. Then*

$$||\Phi(t,t_0)x|| \leq ||\Phi(\tau,t_0)||\ ||\Phi(t,\tau)x|| \leq Me^{\omega}||\Phi(t,\tau)x||.$$

For $\sup_{||x||=1}$ *we obtain that*

$$||\Phi(t,t_0)|| \leq Me^{\omega}||\Phi(t,\tau)||.$$

Thus,

$$\begin{aligned} ||\Phi(t,t_0)|| &\leq Me^{\omega}\left(\int_{t_0}^{t_0+1} ||\Phi(t,\tau)||^p d\tau\right)^{\frac{1}{p}} \\ &\leq Me^{\omega}\left(\int_{t_0}^{t} ||\Phi(t,\tau)||^p d\tau\right)^{\frac{1}{p}} \\ &\leq Me^{\omega}k. \end{aligned}$$

It follows that

$$||\Phi(t,t_0)|| \leq Me^{\omega}\max\{1,k\}, \text{ for all } t \geq t_0 \geq 0.$$

Denoting $L' = Me^{\omega}\max\{1,k\}$ *we obtain the condition (3.1) from the Theorem* 3.2*.*

The next steps in the proof of the sufficiency are as in Theorem 3.2*.*

The discrete correspondent of Theorem 3.2 is given:

Theorem 3.4. *Let* Φ *be a backwards evolutionary process.* Φ *is uniformly exponentially stable if and only if there exist* $p, l > 0$ *such that:*

$$\left(\sum_{k=n_0}^{n} ||\Phi(n,k)x||^p\right)^{\frac{1}{p}} \leq l||x||, \text{ for all } n \geq n_0, \text{ and } x \in X.$$

Proof. The *necessity* is immediate.
Sufficiency. From the hypothesis we have that

$$||\Phi(n,n_0)x|| \leq l||x||, \text{ for all } n \geq n_0, \text{ and } x \in X.$$

For $k \in \{n_0, n_0+1, \cdots, n\}$ it follows that

$$\begin{array}{rcl} \sum_{k=n_0}^{n} ||\Phi(n,n_0)x||^p & \leq & \sum_{k=n_0}^{n} ||\Phi(k,n_0)||^p \; ||\Phi(n,k)x||^p \\ & \leq & l^p \sum_{k=n_0}^{n} ||\Phi(n,k)x||^p \\ & \leq & l^{2p}||x||, \end{array}$$

for all $n \geq n_0$ and $x \in X$. Thus,

$$(n-n_0+1)||\Phi(n,n_0)x||^p \leq l^{2p}||x||.$$

For $\sup_{||x||=1}$ we have that

$$(n-n_0+1)||\Phi(n,n_0)||^p \leq l^{2p},$$

which implies that

$$||\Phi(n,n_0)|| \leq \frac{l^2}{(n-n_0+1)^{\frac{1}{p}}}.$$

Therefore, it follows that there exists $m_0 \in \mathbb{N}^*$ such that

$$||\Phi(n_0+m_0,n_0)|| \leq \frac{1}{2}, \text{ for all } n_0 \in \mathbb{N}.$$

For $n \geq n_0$ it results that there exist $q \in \mathbb{N}$ and $r \in \{0, 1, \cdots, m_0-1\}$ such that:

$$\begin{array}{rcl} ||\Phi(n,n_0)|| & = & ||\Phi(n_0+qm_0+r,n_0)|| \\ & \leq & ||\Phi(n_0+qm_0,n_0)|| \; ||\Phi(n_0+qm_0+r,n_0+qm_0)|| \\ & \leq & L\left(\frac{1}{2}\right)^q \\ & = & L(e^{-\nu m_0})^q \\ & = & Le^{-\nu(m_0q+r)}e^{\nu r} \\ & = & Le^{\nu r}e^{-\nu(n-n_0)} \\ & \leq & Le^{\nu m_0}e^{-\nu(n-n_0)} \\ & = & 2Le^{-\nu(n-n_0)} \end{array}$$

Denoting $\nu = \frac{1}{m_0}\ln 2$ and $N = 2L$, it follows that:

$$||\Phi(n,n_0)|| \leq Ne^{-\nu(n-n_0)}, \text{ for all } n \geq n_0.$$

Let now $t \geq t_0+1, n = [t], n_0 = [t_0]$. Thus $n \geq n_0+1$ and we obtain that:

$$\begin{array}{rcl} ||\Phi(t,t_0)|| & = & ||\Phi(n_0+1,t_0) \; \Phi(n,n_0+1) \; \Phi(t,n)|| \\ & \leq & M^2e^{2\omega}||\Phi(n,n_0+1)|| \\ & \leq & M^2e^{2\omega}Ne^{-\nu(n-n_0-1)} \\ & = & M^2e^{2\omega}Ne^{-\nu(t-t_0)}e^{\nu(t-t_0-n+n_0+1)} \\ & \leq & M^2e^{2\omega}Ne^{2\nu}e^{-\nu(t-t_0)} \\ & = & M^2Ne^{2\omega+2\nu}e^{-\nu(t-t_0)}, \end{array}$$

for all $t \geq t_0+1$.

For $t_0 \leq t < t_0+1$ it results that

$$||\Phi(t,t_0)|| \leq Me^{\omega}e^{\nu}e^{-\nu(t-t_0)}.$$

Denoting $N = \max\{Me^{\omega+\nu}N, 1\}$ we obtain that:

$$||\Phi(t,t_0)|| \leq Ne^{-\nu(t-t_0)}, \text{ for all } t \geq t_0 \geq 0. \qquad \square$$

References

[1] Barbashin, E. A., *Introduction in the theory of stability,* (Russian), Izd. Nauka, Moskow, 1967.

[2] Buşe, C., Megan, M., Prajea, M. S., Preda, P., *The strong variant of a Barbashin theorem on stability of solutions for non-autonomous differential equations in Banach spaces*, Integr. Eq. and Operator Theory, **59**(2007), 491-500.

[3] Daleckij, J. L., Krein, M. G., *Stability of Differential Equations in Banach Space*, Amer. Math. Soc., Providence, R.I., 1974.

[4] Datko, R., *Extending a Theorem of Lyapunov to Hilbert Spaces*, J. Math. Anal. Appl., **32**(1970), 610-616.

[5] Datko, R., *Uniform Asymptotic Stability of Evolutionary Processes in Banach Space*, SIAM J. Math. Anal., **3**(1972), 428-445.

[6] Ichikawa, A., *Equivalence of L_p stability for a class of nonlinear semigroups*, Nonlinear Analysis, **8**(1984), no. 7, 805-815.

[7] Littman, W., *A generalization of the theorem Datko-Pazy*, Lecture Notes in Control and Inform. Sci., **130**(1989), 318-323.

[8] Massera, J. L., Schäffer, J. J., *Linear Differential Equations and Functional Analysis*, J. Annals of Mathematics, **67**(1958), no. 3, 517-573.

[9] Massera, J. L., Schäffer, J. J., *Linear Differential Equations and Function Spaces*, Academic Press, New York, 1966.

[10] Megan, M., Stoica, C., *Exponential instability of skew-evolution semiflows in Banach spaces*, Stud. Univ. Babes-Bolyai Math., **LIII**(2008), no. 1, 17-24.

[11] Pata, V., *A Remark on the Decay of Strongly Continuous Semigroups of Bounded Linear Operators*, Analisi funsionale, Istituto Lombardo (Rend. Sc.) A, **131**(1997), 143-149.

[12] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[13] Rolewicz, S., *On uniform N-equistability*, Journal of Math. Anal. Appl., **115**(1986), 434-441.

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Simulated results for deterministic model of HIV dynamics

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Abstract. In this paper, an algorithm based on He's variational iteration method (shortly, VIM) is developed to approximate the solution of a non-linear mathematical model of HIV dynamics. Using a system of ordinary differential equations, the model describes the viral dynamics of HIV-1. Some plots of the solution are depicted and used to investigate the influence of certain key parameters on the spread of the disease. The results shows that the VIM has the advantages of being more concise for numerical purposes. Furthermore, this work opens a new direction of research whereby He's VIM applications might offer more insight into the modeling of dynamical systems in life sciences.

Mathematics Subject Classification (2010): 35R99, 49M27.

Keywords: Iteration method, HIV-1 dynamics, mathematical epidemiology, ODE models.

1. Introduction

Mathematical modeling of many biological or physical systems leads to nonlinear ordinary differential equations. An effective method is required to analyze the mathematical model which provides solutions conforming to physical reality. Therefore, we must be able to solve nonlinear ordinary differential equations. Common analytic procedures linearize the system or assume that nonlinearities are relatively insignificant. Such procedures change the actual problem to make it tractable by the conventional methods. In short, the physical problem is transformed to a purely mathematical one, for which the solution is readily available. This changes, sometimes seriously, the solution, which means that the problem being solved is no longer a proper representation of the physical problem whose solution is desired. However, in spite of the extensive development in the mathematical and statistical techniques applied to modeling infectious diseases, little has been done to apply approximate methods to solve epidemic models. We try to obtain some analytical results to the deterministic model posed in this paper. In particular, we discuss mathematical and statistical ideas representing HIV internal virus dynamics. Simulation results from initial attempts in the areas of applied mathematics and statistics will be presented.

The human immuno-deficiency virus (HIV) infection which can lead to acquired immuno-deficiency syndrome (AIDS), has become an important infectious disease in both developed and developing nations. Mathematical models have been used extensively in research into the epidemiology of HIV/AIDS, to help improve our understanding of the major contributing factors in a given epidemic.

The key markers of the disease progression due to HIV and ADIS are the CD4+ $T-$cell and viral levels in the plasma. Modeling the interaction between HIV-1 virus and CD4 cells has been a major area of research for many years [17, 3, 18]. In recent years, a few studies of HIV dynamics have been conducted to describe the effects of various epidemiological factors [1, 5, 15, 2, 19, 16]. In particular, in [1], the authors present an overview of some concepts and methodologies that are useful on modeling HIV pathogenesis. A dynamical system modeling the HIV infection was used in [5] to show the impact of the viral diversity on the immune response and disease dynamics. In [15], the authors considered a non-linear mathematical model for HIV epidemic that spreads in a variable size population through both horizontal and vertical transmission. Using stability theory and computer simulation, they showed that by controlling the rate of vertical transmission, the spread of the disease can be reduced significantly. In [4], the author introduce a novel class of HIV models that incorporates mutation, the mutation is modeled by an integral operator whose kernel describes the transition probability between different strains. Numerical aspects of computer simulations are discussed.

Instead of finding a small parameter for solving nonlinear problems through perturbation method, a new analytical method called He's variational iteration method will be used in this paper to solve the epidemic model problem. The VIM is useful to obtain exact and approximate solutions for linear and nonlinear differential equations. It has been used to solve effectively, easily and accurately a large class of nonlinear problems with approximations.

The organization of the paper is as follows: In section 2, we describe a $3-$dimensional model for internal HIV dynamics. In section 3, we review the procedure of VIM. To show the efficiency of the method, in section 4, we apply the method on the model system appeared in section 2. Simulation results are presented in section 5.

2. HIV Model System

Mathematical models have come to play an important part in biological systems. Mathematics makes it possible to make predictions about the behavior

Symbol	Description
$x(t)$	concentration of uninfected cells.
$y(t)$	concentration of infected cells.
$z(t)$	concentration of virus particles.
$(1-\gamma)$	reverse transcriptase inhibitor drug effects.
$(1-\eta)$	protease inhibitor drug effect.
λ	total rate of production of healthy cells per unit time.
κ	per capita death rate of healthy cells.
β	transmission coefficient between uninfected cells and the infective virus particles.
N	average number of infective virus particles produced by an infected cell in the absence of HAART during its entire infectious lifetime.
u	per capita death rate of infective virus particles.
a	death rate of infected cells.

TABLE 1. Variables and parameters in system (2.1)

of the system. Following [14], we introduce a 3−dimensional model to describe the viral dynamics in the presence of HIV-1 infection and Highly Active Antiretrovital Treatment (HAART). The equations of the model represents the variation rate of uninfected cells, infected cells, and virus particles. The model is thus described by the following

$$\begin{aligned} \tfrac{dx(t)}{dt} &= \lambda - \kappa x(t) - (1-\gamma)\beta x(t)z(t) \\ \tfrac{dy(t)}{dt} &= (1-\gamma)\beta x(t)z(t) - ay(t) \\ \tfrac{dz(t)}{dt} &= (1-\eta)Nay(t) - uz(t) - (1-\gamma)\beta x(t)z(t) \end{aligned} \tag{2.1}$$

with suitable initial conditions. The variables $x(t), y(t)$ and $z(t)$ are functions of time $t \in [0, \infty)$. We summarize in Table 1 the biological meaning of the variables and parameters occurring in this model. This model captures mathematically the viral dynamics of HIV-1 virus interacting with CD4 cells. It can be seen that a model of such a simple nature is able to adequately reflect the disease progression from the initial infection to an asymptomatic stage where the set-point is reached.

We assume that the cells and the virus are uniformly distributed on the organism. Note that when a single infective virus particle infects a single uninfected cell the virus particle is absorbed into the infected cell and effectively dies. Hence, the term $(1-\gamma)\beta x(t)z(t)$ appears in all the three equations. In system (2.1), the first equation represents the dynamics of the concentration of healthy cells $x(t)$; λ represents the rate (assumed to be constant) at which new $x(t)$ cells are generated. In the case of active HIV infection, the concentration of healthy cells decreases proportionally to the product $(1-\gamma)\beta x(t)z(t)$, where β represents a coefficient that depends on various factors, including the velocity of penetration of virus into cells, and the frequency of encounters between uninfected cells and free virus. The second equation in system (2.1) describes the dynamics of the concentration of infected cells $y(t)$; $(1-\gamma)\beta$ is the rate of infections; a is the death rate of infected cells. Therefore, the average lifetime of an infected cell is $1/a$. The third equation describes the concentration of free virion $z(t)$, which are produced by the infected cells at a rate $(1-\eta)Na$, and u is the death rate of the virion. The parameters of the model and their values are defined in Tables 1 and 2. Regarding equilibrium points and stability for system (2.1), a qualitative investigation [14] of the system described by equations (2.1) reveals that the model system has a unique disease-free equilibrium given by $(\lambda/\kappa, 0, 0)$.

A value for R_0, the basic reproduction number, is also useful to study further behavior of the system. This number tells us how many secondary infective virus particle will result from the introduction of one infected cell which was infected by the original infective virus particle. Hence

$$R_0 = \frac{(1-\gamma)\beta\lambda N(1-\eta)}{\kappa u + \beta\lambda(1-\gamma)}.$$

R_0 can also be interpreted as the expected number of secondary infected particles caused by a single infected virus particle entering the disease-free population at equilibrium $(\lambda/\kappa, 0, 0)$. $R_0 = 1$, means that each infected cell will infect one uninfected cell. Usually, $R_0 < 1$ implies that an epidemic will not result from the introduction of one infected cell, whereas $R_0 > 1$ implies that an epidemic will occur, and $R_0 = 1$ requires further investigation. However, as will be seen, the model (2.1) may imply something further, namely that the threshold value of R_0 must be brought far below one in order to avoid an epidemic, and if this does not happen, an endemic equilibrium may be established. R_0 is also useful for establishing the existence of equilibrium points, and in performing stability analysis for the system. To discuss the local behavior of the system around the equilibrium point, we introduce the following theorem

Theorem 2.1. *The solution of the model system (*2.1*) is asymptotically stable at the equilibrium point* $(\lambda/\kappa, 0, 0)$ *provided that* $R_0 < 1$.

Proof. The Jacobian of the system (2.1) is

$$J(x,y,z) = \begin{bmatrix} -\kappa-(1-\gamma)\beta z(t) & 0 & -(1-\gamma)\beta x(t) \\ (1-\gamma)\beta z(t) & -a & (1-\gamma)\beta x(t) \\ -(1-\gamma)\beta z(t) & (1-\eta)Na & -u-(1-\gamma)\beta x(t) \end{bmatrix}$$

Substituting the equilibrium point $(\lambda/\kappa, 0, 0)$, the Jacobian matrix becomes

$$J(\lambda/\kappa, 0, 0) = \begin{bmatrix} -\kappa & 0 & -(1-\gamma)\beta\lambda/\kappa \\ 0 & -a & (1-\gamma)\beta\lambda/\kappa \\ 0 & (1-\eta)Na & -u-(1-\gamma)\beta\lambda/\kappa \end{bmatrix}$$

The eigenvalues of this matrix are $\lambda_1 = -\kappa$,

$$\lambda_2 = \frac{-a\kappa - \beta\lambda + \beta\gamma\lambda - \kappa u + \sqrt{M - 4a\kappa(-\beta(-1+\gamma)\lambda(1+(-1+\eta)N) + \kappa u)}}{2\kappa}$$

and,

$$\lambda_3 = -\frac{a\kappa + \beta\lambda - \beta\gamma\lambda + \kappa u + \sqrt{M - 4a\kappa(-\beta(-1+\gamma)\lambda(1+(-1+\eta)N) + \kappa u)}}{2\kappa}$$

where $M = (a\kappa + \beta(\lambda - \gamma\lambda) + \kappa u)^2$. λ_1 is clearly real and negative. Also, as

$$R_0 = \frac{(1-\gamma)\beta\lambda N(1-\eta)}{\kappa u + \beta\lambda(1-\gamma)} < 1,$$

then $(1-\gamma)\beta\lambda N(1-\eta)$ is less than $\kappa u + \lambda(1-\gamma)\beta\lambda$, and so λ_2, λ_3 meets the necessary criteria. The system (2.1) shows local asymptotic stability at the equilibrium point $(\lambda/\kappa, 0, 0)$.

To examine the sensitivity of R_0 to the parameters, say N and u, the normalized forward sensitivity index [6] with respect to the parameters N, u are calculated as

$$\mu_N = \frac{\frac{\partial R_0}{R_0}}{\frac{\partial N}{N}} = \frac{N}{R_0}\frac{\partial R_0}{\partial N} = \frac{N}{R_0}\frac{(1-\gamma)\beta\lambda(1-\eta)}{\kappa u} = 1.$$

Thus, R_0 and N are directly proportional. Also,

$$|\mu_u| = |\frac{\frac{\partial R_0}{R_0}}{\frac{\partial u}{u}}| = |\frac{u}{R_0}\frac{\partial R_0}{\partial u}| = |\frac{-\kappa u}{\kappa u + \beta\lambda(1-\gamma)}| < 1.$$

Therefore, R_0 is most sensitive to changes in N. So, in section 5, we choose to focus on changing the parameters N and u.

3. Basic Idea of VIM

In 1978, Inokuti et al [8] proposed a general Lagrange multiplier method to solve nonlinear problems. Ji-Huan He has modified the method of Inokuti, and propose the variational iteration method (VIM) [9, 12]. This method has been employed to solve a large variety of linear and nonlinear problems with approximations converging rapidly to accurate solutions. Some advantages of this technique are

1. The initial condition can be chosen freely with some unknown parameters.
2. The unknown parameters in the initial condition can be easily identified.
3. The calculation is simple and straightforward.

This approach is successfully and effectively applied to various equations, see for example [9, 12, 13], and the reference therein.

The idea of this method is constructing a correction functional by a general Lagrange multiplier. The multiplier in the functional should be chosen such that its correction solution is superior to its initial approximation, called trial function, and is the best within the flexibility of trial function, accordingly we can identify the multiplier by the variational theory [9, 12]. A complete review of the VIM is available in [10].

The initial approximation can be freely chosen with possible unknowns, which can be determined by imposing the boundary/initial conditions. To illustrate the procedure of this approach, we consider the following general differential equation

$$\mathbf{L}u(t) + \mathbf{N}u(t) = f(t). \tag{3.1}$$

where $\mathbf{L}$ is a linear operator, $\mathbf{N}$ is a nonlinear operator, and $f(t)$ is an inhomogeneous term. According to the variational iteration method [9, 12], the terms of a sequence $\{u_n\}$ are constructed such that this sequence converges to the exact solution, u_n's are calculated by a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)\left\{\mathbf{L}u_n(\tau) + \mathbf{N}(\tilde{u})(\tau) - f(\tau)\right\} d\tau \tag{3.2}$$

where λ is general Lagrangian multipliers, which can be identified optimally via the variational theory [9], the subscript n denotes the nth order approximation. The second term, involving the integral, on the right-hand side of equation (3.2) is called the correction. Under suitable restricted variational assumption (i.e.,$\tilde{u}_n$ is considered as a restricted variation), we can assume that the above correctional functional are stationary (i.e., $\delta\tilde{u}_n = 0$). The successive approximations $u_{n+1}(t), n \geq 0$ of the solution $u(t)$ will be readily obtained upon using Lagrange multipliers, and by using the selective function u_0. The initial condition $u(0)$ is usually used for selecting the zeroth approximation u_0. With λ determined, then several approximations $u_n(t), n \geq 0$, follow immediately, the exact solution may be obtained by using

$$u(t) = \lim_{n\to\infty} u_n(t).$$

For linear problems, its exact solution can be obtained by only one iteration step, this is due to the fact that the Lagrange multipliers can be exactly identified, see [9]. He's technique provides a sequence of functions which converges to the exact solution of the problem [12].

In fact, the solution of the differential equation (3.1) is considered as the fixed point of the functional (3.2) under suitable choice of the initial approximation. For the convergence proof of (3.2), we state the following known result that is useful to support the convergence of our iteration.

Theorem 3.1. [7] *For a Banach space X, suppose the nonlinear mapping $A : X \to X$ satisfy*

$$\| A[u] - A[\bar{u}] \| \leq \gamma \| u - \bar{u} \|, \quad u, \bar{u} \in X$$

for some constant $\gamma < 1$. Then A has a unique fixed point. Furthermore, the sequence $u_{n+1} = A[u_n]$ with arbitrary choice of $u_0 \in X$, converges to the fixed point of A, and

$$\| u_k - u_j \| \leq \| u_1 - u_0 \| \sum_{\ell=j-1}^{k-2} \gamma^{\ell}.$$

According to this Theorem, for the nonlinear mapping

$$A[u] = u(t) + \int_0^t \Big[\mathbf{L}u(\tau) + \mathbf{N}(u(\tau)) - f(\tau)\Big] d\tau.$$

A sufficient condition for the convergence of the VIM is strictly contraction of A. Furthermore, the sequence (3.2) converges to the fixed point of A, which is also the solution of the differential equation in Equation (3.1). In what follows, we will apply the VIM to solve the epidemic model (2.1), to illustrate the strength of the method and to establish approximations of high accuracy for these models.

4. Applications

To show the efficiency of the method described in the previous section, in this section, we apply the VIM to solve the system of nonlinear ordinary differential equations (2.1). According to the VIM, we can construct the correction functionals as follows:

$$x_{n+1}(t) = x_n(t) + \int_0^t \lambda_1(\tau)\Big\{x'_n(\tau) - \lambda + \kappa x_n(\tau) + (1-\gamma)\beta x_n(\tau)\tilde{z}_n(\tau)\Big\} d\tau$$

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda_2(\tau)\Big\{y'_n(\tau) - (1-\gamma)\beta\tilde{x}_n(\tau)\tilde{z}_n(\tau) + ay(\tau)\Big\} d\tau$$

$$z_{n+1}(t) = z_n(t)$$

$$+ \int_0^t \lambda_3(\tau)\Big\{z'_n(\tau) - (1-\eta)Na\tilde{y}_n(\tau) + uz_n(\tau) + (1-\gamma)\beta\tilde{x}_n(\tau)\tilde{z}_n(\tau)\Big\} d\tau \quad (4.1)$$

where λ_1, λ_2 and λ_3 are the general Lagrange multipliers, and $\tilde{x}_n, \tilde{y}_n$ and $\tilde{z}_n$ denote restricted variations, i.e., $\delta\tilde{x}_n = \delta\tilde{y}_n = \delta\tilde{z}_n = 0$. Making the above correction functional stationary

$$\delta x_{n+1}(t) = \delta x_n(t) + \delta\int_0^t \lambda_1(\tau)\Big\{x'_n(\tau) - \lambda + \kappa x_n(\tau) + (1-\gamma)\beta x_n(\tau)\tilde{z}_n(\tau)\Big\} d\tau$$

$$= \delta x_n(t) + \delta\int_0^t \lambda_1(\tau)\Big\{x'_n(\tau) + \kappa x_n(\tau)\Big\} d\tau$$

$$= \delta x_n(t) + \lambda_1(\tau)\delta x_n(\tau)\Big|_{\tau=t} + \int_0^t (\kappa\lambda_1 - \lambda'_1)(\tau)\delta x_n(\tau) d\tau = 0,$$

also,

$$\delta y_{n+1}(t) = \delta y_n(t) + \delta \int_0^t \lambda_2(\tau)\Big\{y_n'(\tau) - (1-\gamma)\beta\tilde{x}_n(\tau)\tilde{z}_n(\tau) + ay(\tau)\Big\}d\tau$$

$$= \delta y_n(t) + \delta \int_0^t \lambda_2(\tau)\Big\{y_n'(\tau) + ay_n(\tau)\Big\}d\tau$$

$$= \delta y_n(t) + \lambda_2(\tau)\delta y_n(\tau)\Big|_{\tau=t} + \int_0^t (a\lambda_2 - \lambda_2')(\tau)\delta y_n(\tau)d\tau = 0,$$

and,

$$\delta z_{n+1}(t) = \delta z_n(t)$$

$$+\delta \int_0^t \lambda_3(\tau)\Big\{z_n'(\tau) - (1-\eta)Na\tilde{y}_n(\tau) + uz_n(\tau) + (1-\gamma)\beta\tilde{x}_n(\tau)\tilde{z}_n(\tau)\Big\}d\tau$$

$$= \delta z_n(t) + \delta \int_0^t \lambda_3(\tau)\Big\{z_n'(\tau) + uz_n(\tau)\Big\}d\tau$$

$$= \delta z_n(t) + \lambda_3(\tau)\delta z_n(\tau)\Big|_{\tau=t} + \int_0^t (u\lambda_3 - \lambda_3')(\tau)\delta z_n(\tau)d\tau = 0,$$

yield the following stationary conditions

$$\begin{aligned} &\lambda_1'(\tau) - \kappa\lambda_1(\tau) = 0, \;\; 1 + \lambda_1(\tau)\Big|_{\tau=t} = 0 \\ &\lambda_2'(\tau) - a\lambda_2(\tau) = 0, \;\; 1 + \lambda_2(\tau)\Big|_{\tau=t} = 0 \\ &\lambda_3'(\tau) - u\lambda_3(\tau) = 0, \;\; 1 + \lambda_3(\tau)\Big|_{\tau=t} = 0 \end{aligned} \tag{4.2}$$

The general Lagrange multipliers can be identified by solving the system of equations in (4.2), to obtain $\lambda_1(\tau) = -e^{\kappa(\tau-t)}$, $\lambda_2(\tau) = -e^{a(\tau-t)}$, $\lambda_3(\tau) = -e^{u(\tau-t)}$. Substituting these values back into the correction functional Equation (4.1) results into the following iteration formula:

$$x_{n+1}(t) = x_n(t) - \int_0^t e^{\kappa(\tau-t)}\Big\{x_n'(\tau) - \lambda + \kappa x_n(\tau) + (1-\gamma)\beta x_n(\tau)z_n(\tau)\Big\}d\tau$$

$$y_{n+1}(t) = y_n(t) - \int_0^t e^{a(\tau-t)}\Big\{y_n'(\tau) - (1-\gamma)\beta x_n(\tau)z_n(\tau) + ay(\tau)\Big\}d\tau$$

$$z_{n+1}(t) = z_n(t)$$

$$-\int_0^t e^{u(\tau-t)}\Big\{z_n'(\tau) - (1-\eta)Nay_n(\tau) + uz_n(\tau) + (1-\gamma)\beta x_n(\tau)z_n(\tau)\Big\}d\tau. \tag{4.3}$$

We start with initial approximations $x_0(t) = N_1, y_0(t) = N_2, z_0(t) = N_3$. We can use $x_{n+1}(t)$ obtained in the first equation of (4.3) into the second equation of (4.3), and so on for other variables, this increases the convergence rate. By the above iteration formula (4.3), we can obtain a few first terms being calculated.

$$\begin{aligned} x_1(t) &= 9.999995 \times 10^7 - 9.989995 \times 10^6 \; e^{-0.1t} \\ y_1(t) &= 1. + 9999e^{-0.5t} \\ z_1(t) &= 49999.9 - 39999.9e^{-5t} \end{aligned} \tag{4.4}$$

While,

$$\begin{aligned} x_2(t) &= 1 \times 10^7 + 399.6e^{-5.1t} - 408.162e^{-5t} - 9.99 \times 10^6 e^{-0.1t} \\ &\quad - 24994.9e^{-3.60822\times10^{-16}t} - e^{-0.1t}(-25003.5 - 2497.5t) \\ y_2(t) &= 4999.99 - 434.347e^{-5.1t} + 444.443e^{-5t} + 11233.7e^{-0.5t} - 6243.73e^{-0.1t} \\ z_2(t) &= -494.9 + 19979.9e^{-5.1t} - 39999.9e^{-5t} + 55550e^{-0.5t} + 509.693e^{-0.1t} \\ &\quad - e^{-5t}(25544.7 - 1999.9t) \end{aligned}$$

Continuing in this manner, the rest of components of the iteration formulas can be obtained using symbolic packages such as *Mathematica*. In our case, only three terms from the iteration formula are used to obtain the approximation for our solutions.

5. Simulation Results and Discussion

To illustrate the use of the VIM, we describe some numerical experiments made to get a better understanding of the solutions behavior for the model system (2.1). The parameter values used here have all been taken from a published paper [14] and the reference therein, which are quoted here as in Table 2. The computer simulations were performed using the first three iterations $(x_3(t), y_3(t), z_3(t))$ for each variable, with the parameters values appeared in Table 2. Simulation results for the model, are displayed in Figures $1-6$. As can be clearly seen, Figure 1 shows the uninfected cells, it is found that uninfected cells first increases with time, and then after almost 40 days reaches it equilibrium position, which is $\lambda/\kappa = 1 \times 10^7$. As seen from Figure 2 that infected cells decreases exponentially as all infectives will develop AIDS and will die out. Figure 3, show the virus particles, we observe that immediately after infection, the amount of virus particles rises dramatically. After a few days (usually six to eight days), the virus concentration falls to the virus particles. Our further graphs $4-6$ dealing mainly with the existence of steady state for some values of $R_0 < 1$.

It should be pointed out that the parameters in the model are independent of each other, since each of them plays an independent role. These parameters have definite meaning, so the results of simulation can hardly coincide with the actual situation of the epidemic if the parameters cannot be adjusted to proper values.

Parameter	Values in Simulation
λ	10^6 day^{-1} dm^3
κ	0.1 day^{-1}
u	5 day^{-1}
a	0.5 day^{-1}
η	0.5
β	1×10^{-8} day^{-1} dm^3
N	100 per cell
γ	0.5
$N_1 = N_2 = N_3$	10000

TABLE 2. Parameters in system (2.1) with their values

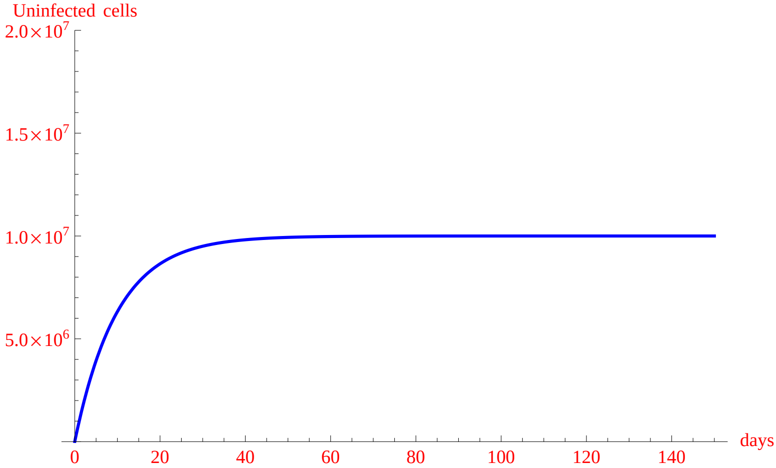

FIGURE 1. Simulated behavior of uninfected cells with parameter values given in Table 2, $R_0 = 0.49$, the steady state $(\lambda/\kappa, 0, 0)$ is asymptotically stable.

References

[1] Adams, B. M., Banks, H. T., Davidian, M., Hee-Dae Kwon, Tran, H. T., Wynne, S. N., Rosenberg, E. S., *HIV dynamics, Modeling, data analysis, and optimal treatment protocols*, Comput. Math. Appl., **184**(2005), no. 1, 10-49.

[2] Tripathi, A., *Ram Naresh and Dileep Sharma: Modeling the effect of screening of unaware infectives on the spread of HIV infection*, Appl. Math. Comput., **184**(2007), 1053-1068.

[3] Bhatia, N. P., Szego, G. P., *Stability Theory of Dynamical Systems*, Springer, New York, 1970.

[4] Pastore, D. H., *On the numerical simulations of a class of HIV models*, Math. Comp. Mode., in press, 2008.

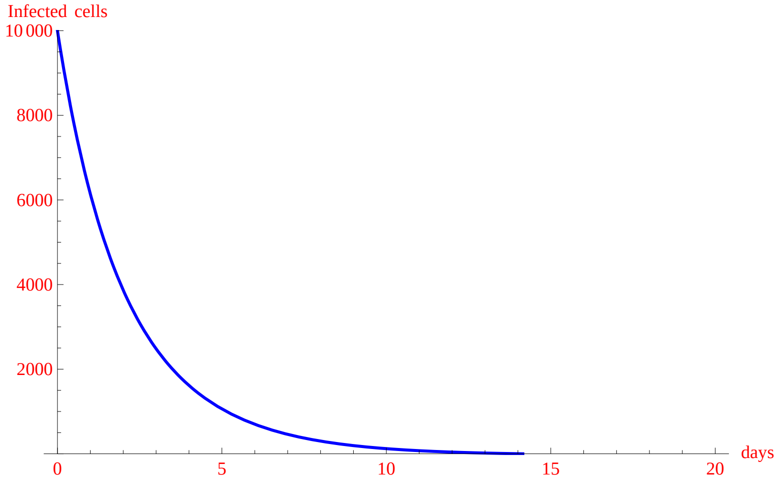

FIGURE 2. Simulated behavior of infected cells with parameter values as in Table 2, $R_0 = 0.49$

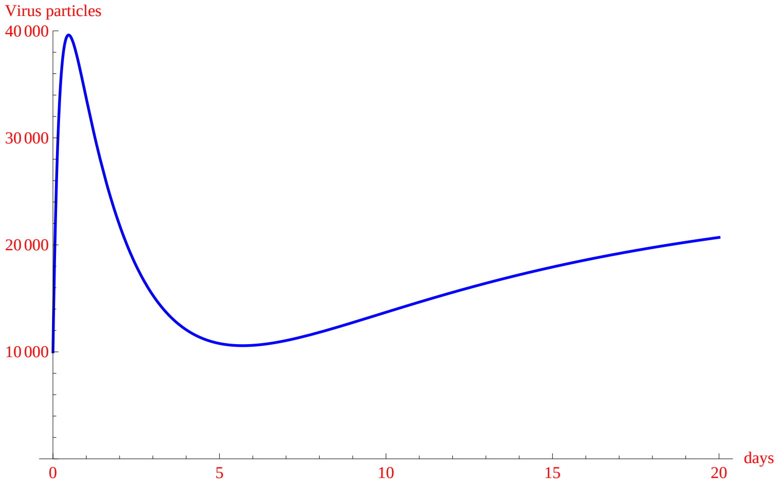

FIGURE 3. Simulated behavior of particles cells with parameter values given in Table 2, $R_0 = 0.49$

[5] Vergu, E., Mallet, A., Golmard, J. L., *A modeling approach to the impact of HIV mutations on the immune system*, Comp. Bio. Med., **35**(2005), 1-24.

[6] White, E., Comiskey, C., *Heroin epidemics, treatment and ODE modelling*, Mathematical Biosciences, **208**(2007), 312-324.

[7] Evans, L. C., *Partial Differential Equations*, AMS, Providence, RI, 1998.

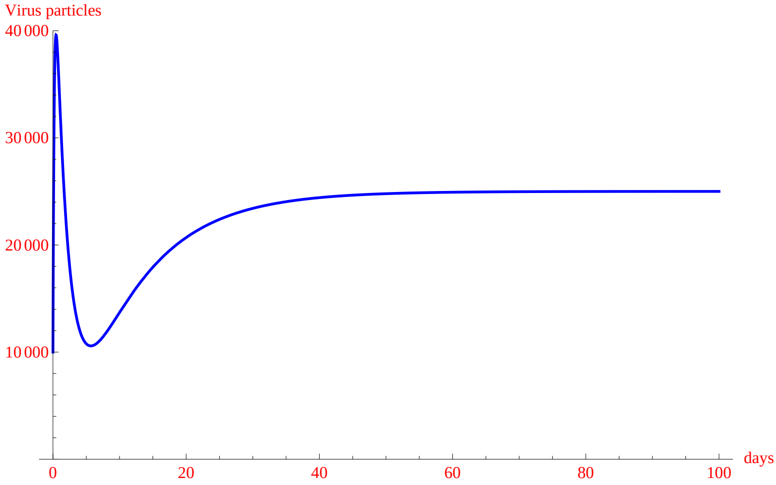

FIGURE 4. Simulated behavior of particles cells with parameter values given in Table 2, $R_0 = 0.49$, and $0 < t < 100$

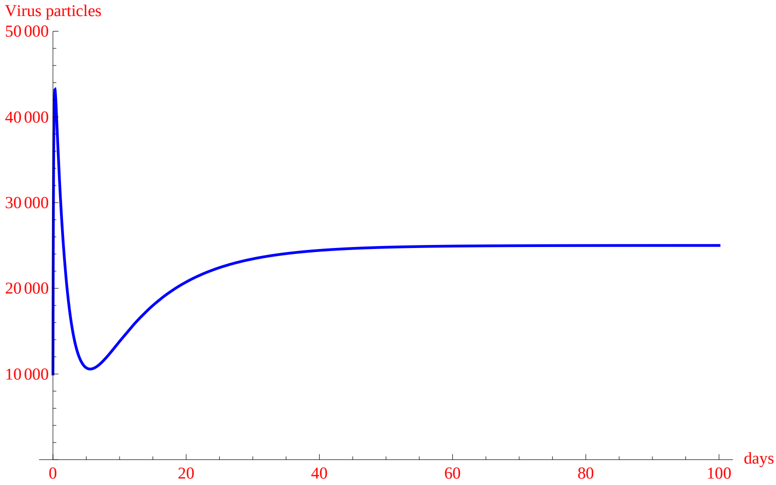

FIGURE 5. Simulated behavior for virus particles, the values of the parameters are the same as those in Table 2 except $N = 200, u = 10$

[8] Inokuti, M., Sekine, H., Mura, T., *General use of the Lagrange multiplier in nonlinear mathematics*, Nemat-Nassers (Ed.) Variational Method in the Mechanics of Solids. Pergamon Press, Oxford, 1978, 156-162.

[9] Ji-Huan He, *Variational iteration method – a kind of non-linear analytical technique: Some examples*, International J. of Nonlinear Mechanics, **34**(1999), no. 4, 699-708.

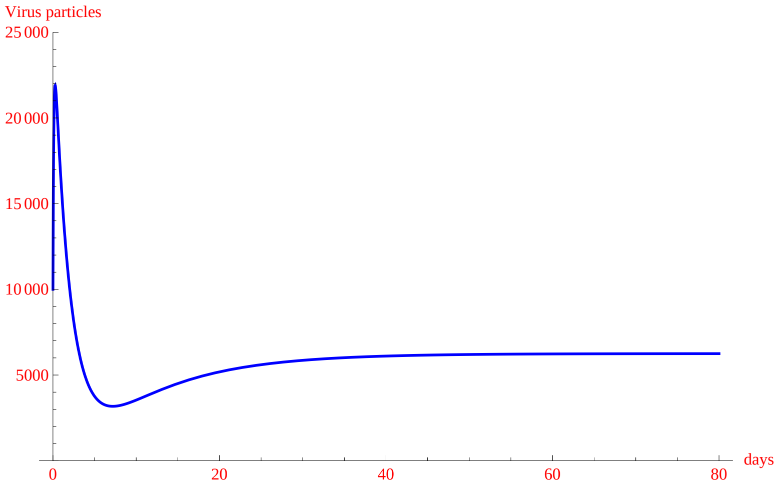

FIGURE 6. Simulated behavior for virus particles, the values of the parameters are the same as those in Table 2 except $u = 10$

[10] Ji-Huan He, *Variational iteration method for autonomous ordinary differential systems*, Comput. Appl. Math., **114**(2000), no. 2-3, 115-123.

[11] Ji-Huan He, *Some asymptotic methods for strongly nonlinear equations*, Intern J. of Modern Physics, **20**(2006), 1141-1199.

[12] Ji-Huan He, *Variational iteration method – Some recent results and new interpretations*, Comput. Appl. Math., **207**(2007), no. 1, 3-17.

[13] Dehghan, M., Tatari, M., *Identifying an unknown function in a parabolic equation with overspecified data via He's variational iteration method*, Chaos, Solitons and Fractals, **36**(2008), no. 1, 157-166.

[14] Dalal, N., Greenhalgh, D., Xuerong Mao, *A stochastic model for internal HIV dynamics*, Math. Analy. Appl. (in press).

[15] Naresh, R., Tripathi, A., Omar, S., *Modelling the spread of AIDS epidemic with vertical transmision*, Appl. Math. Comput., **178**(2006), 262-272.

[16] Khalili, S., Armaou, A., *Sensitivity analysis of HIV infection response to treatment via stochastic modeling*, Chem. Eng. Sci. (in press).

[17] Verhulst, F., *Nonlinear Differential Equations and Dynamical Systems*, Springer, London, 1990.

[18] Wu, H., Tan, W. Y., *Deterministic and Stochastic Models of AIDS Epidemics and HIV Infections with Intervention*, World Scientific Publishing Co., 2005.

[19] Xiaohua Xia, *Modelling of HIV infection, Vaccine readiness, drug effectiveness and therapeutical failures*, Journal of Process Control, **17**(2007), 253-260.

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Boundary integral equations for the problem of 2D Brinkman flow past several voids

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Abstract. In this paper we obtain the existence and uniqueness result for the classical solution of the boundary value problem which describes the $2D$ flow of an incompressible Newtonian fluid in a porous medium and in the presence of $N \geq 2$ voids.

Mathematics Subject Classification (2010): 76D07, 76D03, 31A10.

Keywords: Brinkman flow, potential theory, boundary integral representation, existence and uniqueness result.

1. Introduction

The problem of viscous incompressible fluid flow through porous media has various chemical, biotechnology, and geological applications, concerning: the treatment of transport and chemical reaction within catalyst particles in fixed and fluidized beds, the modeling of polymer molecules as porous particles, immobilization of cells or enzymes and perfusion chromatography for purifying proteins and other bio-molecules, the flow of various kinds of fluids past porous rocks embedded in porous soil. In [2] Kohr and Sekhar have used the potential theory, as well as the Brinkman model, in order to obtain the existence and uniqueness result of the classical solution to a boundary value problem which describes the flow of an unbounded viscous incompressible fluid in the presence of a porous body embedded in another porous medium. Also, in [3] the authors obtained an indirect boundary integral formulation for the three-dimensional viscous flow problem in a granular material with one void. The method of matched asymptotic expansions and the method of boundary integral equations have been used in [4] in order to study the two-dimensional steady flow of a viscous incompressible fluid at low Reynolds number past a porous body of arbitrary shape. In this paper we show the existence and uniqueness result for the classical solution of a boundary value problem that describes the two-dimensional flow of an incompressible Newtonian fluid in a porous medium and in the presence of $N \geq 2$ voids by using the Brinkman model for the external flow, as well as the Stokes model for the internal flow. We use a boundary integral method that reduces the flow problem to a system of Fredholm integral equations of the second kind that has a unique solution in some Banach spaces.

2. The mathematical formulation of the problem

Let us consider an otherwise unbounded homogeneous granular material in which $N \geq 2$ fluid obstacles (voids) are given. The k-th void occupies the bounded domain $D_k \subset \mathbb{R}^2$ whose boundary Γ_k is a closed Lypaunov curve in the class $C^{1,\alpha}$, $\alpha \in (0,1]$, $k = 1, \ldots, N$. Let us denote by D_0 the set given by $D_0 = \cup_{k=1}^{N} D_k$. We denote by D_e the unbounded domain with the boundary $\Gamma = \cup_{k=1}^{N} \Gamma_k$, and assume that at great distances, i.e., far from the voids, the fluid flow is uniform with velocity and pressure fields $\mathbf{U}_\infty$ and p_∞, respectively.

Let us now assume that the flow in the unbounded domain D_e is described by the Brinkman model, i.e., the Brinkman and continuity equations. Thus, the non-dimensional volume averaged velocity and pressure fields $\mathbf{v}^e$ and p^e satisfy in D_e the following equations:

$$-\nabla p^e + (\nabla^2 - \chi^2)\mathbf{v}^e = \mathbf{0} \ \text{ in } D_e, \tag{2.1}$$

$$\nabla \cdot \mathbf{v}^e = 0 \ \text{ in } D_e, \tag{2.2}$$

where $\chi > 0$ is the constant having the expression $\chi = \frac{a}{\sqrt{\kappa}}\sqrt{\frac{\mu_f}{\mu_{eff}}}$, a is a characteristic length (connected to the sizes of the curves Γ_k, $k = 1, \ldots, N$) and k is the permeability of the porous medium. Note that if $\mu_f = \mu_{eff}$, then χ becomes $\chi = a/\sqrt{\kappa}$.

The flow inside each void is assumed to be described by the Stokes system, i.e., by the Stokes and continuity equations:

$$-\nabla p^i + \nabla^2 \mathbf{v}^i = \mathbf{0} \ \text{ in } D_0, \tag{2.3}$$

$$\nabla \cdot \mathbf{v}^i = 0 \ \text{ in } D_0. \tag{2.4}$$

Also, we assume that the velocity and boundary traction fields are continuous across each curve Γ_k, $k = 1, \ldots, N$, i.e.,

$$\mathbf{v}^i = \mathbf{v}^e, \quad \mathbf{t}^i = \mathbf{t}^e \text{ on } \Gamma_k. \tag{2.5}$$

Note that $\mathbf{t}^e$ is the boundary traction corresponding to the external fields $\mathbf{v}^e$ and p^e, and $\mathbf{t}^i$ is the boundary traction due to the internal fields $\mathbf{v}^i$ and p^i.

At large distances, the fields $\mathbf{v}^p = \mathbf{v}^e - \mathbf{U}^\infty$ and $p^p = p^e - P^\infty$ vanish such that

$$(|\mathbf{v}^p||\nabla \mathbf{v}^p|)(\mathbf{x}) = o(|\mathbf{x}|^{-1}), \quad (|\mathbf{v}^p||p^p|)(\mathbf{x}) = o(|\mathbf{x}|^{-1}) \ \text{ as } |\mathbf{x}| \to \infty, \tag{2.6}$$

where $\mathbf{U}^\infty$ and P^∞ are the non-dimensional undisturbed velocity and pressure fields.

Therefore, the considered flow problem reduces to the boundary value problem consisting of the system of equations (2.1)-(2.4) subject to the transmission and far field conditions (2.5)-(2.6) and having as unknowns the fields $\mathbf{v}^e$, p^e, $\mathbf{v}^i$ and p^i. We show that this problem has a unique classical solution $((\mathbf{v}^e, p^e), (\mathbf{v}^i, p^i)) \in ((C^2(D_e) \cap C^0(\overline{D}_e)) \times C^1(D_e)) \times ((C^2(D_0) \cap C^0(\overline{D}_0)) \times C^1(D_0))$, where $D_0 = \cup_{k=1}^N D_k$.

3. Uniqueness of the solution

First, we show the following uniqueness result:

Theorem 3.1. *The boundary value problem* (2.1)-(2.6) *has at most one classical solution* $((\mathbf{v}^e, p^e), (\mathbf{v}^i, p^i)) \in ((C^2(D_e) \cap C^0(\overline{D}_e)) \times C^1(D_e)) \times ((C^2(D_0) \cap C^0(\overline{D}_0)) \times C^1(D_0))$.

Proof. Let us assume that the boundary value problem (2.1)-(2.6) has two classical solutions and let $((\mathbf{v}_0^e, p_0^e), (\mathbf{v}_0^i, p_0^i))$ be their difference. Therefore, the pairs $(\mathbf{v}_0^e, p_0^e)$ and $(\mathbf{v}_0^i, p_0^i)$ satisfy the following equations, boundary and far field conditions:

$$-\nabla p_0^i + \nabla^2 \mathbf{v}_0^i = \mathbf{0} \ \text{ and } \ \nabla \cdot \mathbf{v}_0^i = 0 \ \text{ in } D_0, \tag{3.1}$$

$$-\nabla p_0^e + (\nabla^2 - \chi^2)\mathbf{v}_0^e = \mathbf{0} \ \text{ and } \ \nabla \cdot \mathbf{v}_0^e = 0 \ \text{ in } D_e, \tag{3.2}$$

$$\mathbf{v}_0^i = \mathbf{v}_0^e \ \text{ and } \ \mathbf{t}_0^i = \mathbf{t}_0^e \ \text{ on } \Gamma_k, \ k = 1, \ldots, N, \tag{3.3}$$

$$(|\mathbf{v}_0^e||\nabla \mathbf{v}_0^e|)(\mathbf{x}) = o(|\mathbf{x}|^{-1}), \quad (|\mathbf{v}_0^e||p_0^e|)(\mathbf{x}) = o(|\mathbf{x}|^{-1}) \ \text{ as } |\mathbf{x}| \to \infty. \tag{3.4}$$

In addition, the fields $\mathbf{v}_0^e$ and p_0^e satisfy the energy identity (see e.g. [1], p.24)

$$2\int_{D_e} E_{kj}(\mathbf{v}_0^e) E_{kj}(\mathbf{v}_0^e) d\mathbf{x} = -\sum_{k=1}^{N} \int_{\Gamma_k} \mathbf{v}_0^e \cdot \mathbf{t}_0^e d\Gamma_k, \tag{3.5}$$

where

$$E_{kj}(\mathbf{v}_0^e) = \frac{1}{2}\left(\frac{\partial v_{0,k}^e}{\partial x_j} + \frac{\partial v_{0,j}^e}{\partial x_k}\right)$$

and $\mathbf{t}_0^e = (t_{0,1}^e, t_{0,2}^e)$ is the boundary traction due to the fields $\mathbf{v}_0^e = (v_{0,1}^e, v_{0,2}^e)$ and p_0^e, i.e.,

$$t_{0,j}^e = T_{jk}(\mathbf{v}_0^e) n_k = (-p_0^e \delta_{jk} + 2E_{jk}(\mathbf{v}_0^e)) n_k. \tag{3.6}$$

In the relations (3.5) and (3.6) and in what follows we use Einstein's repeated-index summation convention. Also we denote by $\mathbf{n} = (n_1, n_2)$ the outward unit normal to Γ.

Now, making use of the fact that the fields $\mathbf{v}_0^i$ and p_0^i satisfy the equations (3.2), we get the identity (see e.g. [1], p.15):

$$\int_{D_k} (\chi^2 |\mathbf{v}_0^i|^2 + 2E_{kj}(\mathbf{v}_0^i) E_{kj}(\mathbf{v}_0^i)) d\mathbf{x} = \int_{\Gamma_k} \mathbf{v}_0^i \cdot \mathbf{t}_0^i d\Gamma_k, \quad k = 1, \ldots, N, \tag{3.7}$$

where

$$E_{jk}(\mathbf{v}_0^i) = \frac{1}{2}\left(\frac{\partial v_{0,j}^i}{\partial x_k} + \frac{\partial v_{0,k}^i}{\partial x_j}\right)$$

and $\mathbf{t}_0^i = (t_{0,1}^i, t_{0,2}^i)$ is the boundary traction due to the fields $\mathbf{v}_0^i = (v_{0,1}^i, v_{0,2}^i)$ and p_0^i, defined as in (3.6).

Taking into account the boundary conditions (3.3), as well as the identities (3.5) and (3.7), we obtain the equality

$$2\int_{D_e} E_{jk}(\mathbf{v}_0^e)E_{jk}(\mathbf{v}_0^e)d\mathbf{x} = -\sum_{k=1}^{N}\int_{D_k}\big(\chi^2|\mathbf{v}_0^i|^2 + 2E_{jk}(\mathbf{v}_0^i)E_{jk}(\mathbf{v}_0^i)\big)d\mathbf{x}, \quad (3.8)$$

where the left-hand side is non-negative and the right-hand side is less than or equal to zero. Thus, we obtain that

$$\int_{D_e} E_{jk}(\mathbf{v}_0^e)E_{jk}(\mathbf{v}_0^e)d\mathbf{x} = 0,$$

$$\int_{D_k} (\chi^2|\mathbf{v}_0^i|^2 + 2E_{jk}(\mathbf{v}_0^i)E_{jk}(\mathbf{v}_0^i))d\mathbf{x} = 0, \ k = 1, \dots, N.$$

Therefore, we find that

$$\mathbf{v}_0^i = \mathbf{0} \ \text{ in } D_k, \quad k = 1, \dots, N \qquad (3.9)$$

and, due to (3.4),

$$\mathbf{v}_0^e = \mathbf{0} \ \text{ in } D_e. \qquad (3.10)$$

In view of (3.1) and (3.10) it follows that $p_0^e = c_e \in \mathbb{R}$ in D_e. The decay condition of p_0^e at infinity yields that $c_e = 0$, i.e., $p_0^e = 0$ in D_e. Hence we have

$$\mathbf{v}_0^e = \mathbf{0} \ \text{ and } \ p_0^e = 0 \text{ in } D_e. \qquad (3.11)$$

Using similar arguments, we obtain

$$\mathbf{v}_0^i = \mathbf{0} \ \text{ and } \ p_0^i = c_k \in \mathbb{R} \ \text{ in } D_k, \quad k = 1, \dots, N. \qquad (3.12)$$

On the other hand, the properties (3.11) yield that

$$\mathbf{t}_0^e = \mathbf{0} \text{ on } \Gamma_k, \quad k = 1, \dots, N, \qquad (3.13)$$

and, in view of the second of the conditions (3.3), it follows that $\mathbf{t}_0^i = -c_k\mathbf{n} = \mathbf{0}$ on Γ_k, $k = 1, \dots, N$. Therefore, we get $c_k = 0$, $k = 1, \dots, N$. Consequently, we have

$$\mathbf{v}_0^i = \mathbf{0}, \quad p_0^i = 0 \text{ in } D_0. \qquad (3.14)$$

The relations (3.11) and (3.14) yield the desired uniqueness result. This completes the proof of Theorem 3.1. □

4. Potential theory for the Brinkman and Stokes equations

In this section we will present the fundemental solution for the Brimkman and Stokes equations and the main properties of the potential theory for the Brinkman system of equations (2.1)-(2.2) and respectively for the Stokes system (2.3)-(2.4).

4.1. The fundamental solutions of the Brinkman and Stokes equations

The components of the fundamental Brinkman tensor $\mathcal{G}^{\chi^2}$ and those of its associated pressure vector $\mathbf{\Pi}^{\chi^2}$, which determine the fundamental solution $(\mathcal{G}^{\chi^2}, \mathbf{\Pi}^{\chi^2})$ of the Brinkman system in $\mathbb{R}^2$, are given by (see e.g. [1, p. 81]):

$$\mathcal{G}_{jk}^{\chi^2}(\mathbf{x}-\mathbf{y}) = \delta_{jk} A_1(\chi|\mathbf{x}-\mathbf{y}|) + \frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^2} A_2(\chi|\mathbf{x}-\mathbf{y}|) \quad \text{and}$$

$$\Pi_j^{\chi^2}(\mathbf{x}-\mathbf{y}) = 2\frac{x_j-y_j}{|\mathbf{x}-\mathbf{y}|^2}, \tag{4.1}$$

where

$$\begin{aligned} A_1(z) &= 2\{K_0(z) + z^{-1}K_1(z) - z^{-2}\}, \\ A_2(z) &= 2\{-K_0(z) - 2z^{-1}K_1(z) + 2z^{-2}\}, \end{aligned} \tag{4.2}$$

and and K_ν is the modified Bessel function of the second kind and order ν.

The corresponding stress and pressure tensors $\mathbf{S}^{\chi^2}$ and $\mathbf{\Lambda}^{\chi^2}$ have the following components (see e.g. [1, p. 82, 196]):

$$\begin{aligned} S_{ijk}^{\chi^2}(\mathbf{x}-\mathbf{y}) &= -\Pi_j^{\chi^2}(\mathbf{x}-\mathbf{y})\delta_{ik} + \frac{\partial \mathcal{G}_{ij}^{\chi^2}(\mathbf{x}-\mathbf{y})}{\partial x_k} + \frac{\partial \mathcal{G}_{kj}^{\chi^2}(\mathbf{x}-\mathbf{y})}{\partial x_i} \\ = -2\Bigg\{ \delta_{ik}\frac{x_j-y_j}{|\mathbf{x}-\mathbf{y}|^2} D_1(\chi|\mathbf{x}-\mathbf{y}|) &+ \left(\delta_{kj}\frac{x_i-y_i}{|\mathbf{x}-\mathbf{y}|^2} + \delta_{ij}\frac{x_k-y_k}{|\mathbf{x}-\mathbf{y}|^2}\right) D_2(\chi|\mathbf{x}-\mathbf{y}|) \\ &+ \frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^4} D_3(\chi|\mathbf{x}-\mathbf{y}|)\Bigg\}, \end{aligned} \tag{4.3}$$

$$\Lambda_{ik}^{\chi^2}(\mathbf{x}-\mathbf{y}) = 2\frac{\delta_{ik}}{|\mathbf{x}-\mathbf{y}|^2}\big(-\chi^2|\mathbf{x}-\mathbf{y}|^2 \ln|\mathbf{x}-\mathbf{y}| - 2\big) + 8\frac{(x_i-y_i)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^4}, \tag{4.4}$$

where

$$\begin{aligned} D_1(z) &= 2K_2(z) + 1 - 4z^{-2}, \\ D_2(z) &= 2K_2(z) + zK_1(z) - 4z^{-2}, \\ D_3(z) &= -8K_2(z) - 2zK_1(z) + 16z^{-2}. \end{aligned} \tag{4.5}$$

The components of the fundamental tensor $\mathcal{G}$ and those of its associated pressure vector $\mathbf{\Pi}$, which determine the fundamental solution $(\mathcal{G}, \mathbf{\Pi})$ of the Stokes system in $\mathbb{R}^2$, are given by (see e.g. [1, p. 38])

$$\mathcal{G}_{jk}(\mathbf{x}-\mathbf{y}) = -\delta_{jk}\ln|\mathbf{x}-\mathbf{y}| + \frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^2}, \quad \Pi_j(\mathbf{x}-\mathbf{y}) = 2\frac{x_j-y_j}{|\mathbf{x}-\mathbf{y}|^2}, \tag{4.6}$$

and the stress and pressure tensors $\mathbf{S}$ and $\mathbf{\Lambda}$ have the components (see e.g. [1, p. 39, 132])

$$\begin{aligned} S_{ijk}(\mathbf{x}-\mathbf{y}) &= -4\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^4}, \\ \Lambda_{ik}(\mathbf{x}-\mathbf{y}) &= 4\left(-\frac{\delta_{ik}}{|\mathbf{x}-\mathbf{y}|^2} + 2\frac{(x_i-y_i)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^4}\right). \end{aligned} \tag{4.7}$$

4.2. Boundary potentials for the Brinkman and Stokes equations

Let $\mathcal{C} \in \mathbb{R}^2$ be a closed Lypaunov curve in the class $C^{1,\alpha}$, $\alpha \in (0,1]$. The *single-* and *double-layer potentials*, $\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})$ and $\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})$, associated with the Brinkman system and having the densities $\mathbf{g}$ and $\mathbf{h}$, respectively, are given by

$$\mathbf{V}_{\chi^2}(\mathbf{x},\mathbf{g}) = \frac{1}{4\pi}\int_{\mathcal{C}} \mathcal{G}^{\chi^2}(\mathbf{x}-\mathbf{y})\cdot\mathbf{g}(\mathbf{y})d\mathcal{C}(\mathbf{y}) \quad \text{for } \mathbf{x}\in\mathbb{R}^2\setminus\mathcal{C} \tag{4.8}$$

$$\left(\mathbf{W}_{\chi^2}\right)_k(\mathbf{x},\mathbf{h}) = \frac{1}{4\pi}\int_{\mathcal{C}} S^{\chi^2}_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y})h_j(\mathbf{y})d\mathcal{C}(\mathbf{y}) \quad \text{for } \mathbf{x}\in\mathbb{R}^2\setminus\mathcal{C}, \tag{4.9}$$

and the corresponding pressure functions $P^s_{\chi^2}(\cdot,\mathbf{g})$ and $P^d_{\chi^2}(\cdot,\mathbf{h})$ have the expressions

$$P^s_{\chi^2}(\mathbf{x},\mathbf{g}) = \frac{1}{4\pi}\int_{\mathcal{C}} \Pi^{\chi^2}_j(\mathbf{x}-\mathbf{y})g_j(\mathbf{y})d\mathcal{C}(\mathbf{y}) \quad \text{for } \mathbf{x}\in\mathbb{R}^2\setminus\mathcal{C} \tag{4.10}$$

$$P^d_{\chi^2}(\mathbf{x},\mathbf{h}) = \frac{1}{4\pi}\int_{\mathcal{C}} \Lambda^{\chi^2}_{j\ell}(\mathbf{x}-\mathbf{y})n_\ell(\mathbf{y})h_j(\mathbf{y})d\mathcal{C}(\mathbf{y}) \quad \text{for } \mathbf{x}\in\mathbb{R}^2\setminus\mathcal{C}. \tag{4.11}$$

The pairs $(\mathbf{V}_{\chi^2}(\cdot,\mathbf{g}), P^s_{\chi^2}(\cdot,\mathbf{g}))$ and $(\mathbf{W}_{\chi^2}(\cdot,\mathbf{h}), P^d_{\chi^2}(\cdot,\mathbf{h}))$ satisfy the Brinkman system in both domains D_0 and D_e, respectively.

The *single-* and *double-layer potentials*, $\mathbf{V}(\cdot,\mathbf{g})$ and $\mathbf{W}(\cdot,\mathbf{h})$, for the Stokes system and with the densities $\mathbf{g}$ and $\mathbf{h}$, respectively, can be obtained as in (4.8) and (4.9), but with $\mathcal{G}$ and $S_{jk\ell}$ instead of $\mathcal{G}^{\chi^2}$ and $S^{\chi^2}_{jk\ell}$. Similarly, the pressure terms $P^s(\cdot,\mathbf{g})$ and $P^d(\cdot,\mathbf{h})$ can be obtained as in (4.10) and (4.11), but with Π_j and $\Lambda_{j\ell}$ instead of ${\Pi_j}^{\chi^2}$ and $\Lambda^{\chi^2}_{j\ell}$.

Let us denote by $\mathbf{H}_{\chi^2}(\cdot,\mathbf{g})$ the normal stress due to the single-layer potential $\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})$ and defined in a neighborhood $U\subset\mathbb{R}^2$ of $\mathcal{C}$ by the relation

$$\left(\mathbf{H}_{\chi^2}\right)_k(\mathbf{x},\mathbf{g}) = T_{k\ell}(\mathbf{V}_{\chi^2}(\mathbf{g}))(\mathbf{x})n_\ell(\tilde{\mathbf{x}}), \quad \mathbf{x}\in\widetilde{\mathcal{U}}\setminus\mathcal{C},$$

where $\tilde{\mathbf{x}}$ is the orthogonal projection of $\mathbf{x}\in U$ onto $\mathcal{C}$. On the components, we have

$$\left(\mathbf{H}_{\chi^2}\right)_k(\mathbf{x},\mathbf{g}) = \frac{1}{4\pi}\int_{S} S^{\chi^2}_{kj\ell}(\mathbf{x}-\mathbf{y})n_\ell(\bar{\mathbf{x}})g_j(\mathbf{y})d\mathcal{C}(\mathbf{y}), \quad \mathbf{x}\in U\setminus\mathcal{C},\ k=1,2. \tag{4.12}$$

The stress field due to the single-layer potential $\mathbf{V}(\cdot,\mathbf{g})$ is defined in U by the relation:

$$t_j(\mathbf{V}(\mathbf{g}))(\mathbf{x}) = T_{j\ell}(\mathbf{V}(\mathbf{g}))(\mathbf{x})n_\ell(\tilde{\mathbf{x}}), \quad \mathbf{x}\in U\setminus\mathcal{C},\ j=1,2. \tag{4.13}$$

Let $\mathbf{K}^{\chi^2}(\mathbf{y},\mathbf{x})$ be the kernel of the double-layer potential $\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})$, whose components are given by $K^{\chi^2}_{jk}(\mathbf{y},\mathbf{x}) = S^{\chi^2}_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y})$. Similarly, the components of the kernel of the double-layer potential $\mathbf{W}(\cdot,\mathbf{h})$ are denoted by $K_{jk}(\mathbf{y},\mathbf{x})$, and are given by the relation $K_{jk}(\mathbf{y},\mathbf{x}) = S_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y})$.

Let us now consider the following decomposition of the tensors $\mathcal{G}^{\chi^2}$ and $\mathbf{S}^{\chi^2}$:

$$\begin{aligned}\mathcal{G}^{\chi^2}_{kj}(\mathbf{x}-\mathbf{y}) &= \mathcal{G}_{kj}(\mathbf{x}-\mathbf{y}) + \mathcal{G}^{c}_{kj}(\mathbf{x}-\mathbf{y}),\\ S^{\chi^2}_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y}) &= S_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y}) + S^{c}_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y}),\end{aligned} \tag{4.14}$$

where the matrix kernel $\mathcal{G}^c$ with the components $\mathcal{G}^c_{kj}$ and the matrix kernel $\mathbf{S}^c\mathbf{n}$ with the components $S^c_{jk\ell}n_l$ are continuous. Thus, one obtains the following result which shows the continuity behaviour and the jump formulas for the single- and double-layer potentials associated to the Brinkman system (for e.g. [4]):

Theorem 4.1. *a) Let $\mathcal{C}$ be a closed Lyapunov curve in $\mathbb{R}^2$, i.e., $\mathcal{C} \in C^{1,\alpha}$, $\alpha \in (0,1]$, and let densities $\mathbf{g} \in C^0(\mathcal{C})$ and $\mathbf{h} \in C^0(\mathcal{C})$ be given. Also let $\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})$, $\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})$ and $\mathbf{H}_{\chi^2}(\cdot,\mathbf{g})$ be the boundary potentials given by* (4.8), (4.9) *and* (4.12). *Then on $\mathcal{C}$ we have:*

$$\left(\mathbf{V}_{\chi^2}\right)^+(\cdot,\mathbf{g}) = \left(\mathbf{V}_{\chi^2}\right)^-(\cdot,\mathbf{g}) = \mathbf{V}_{\chi^2}(\cdot,\mathbf{g}), \tag{4.15}$$

$$\left(\mathbf{W}_{\chi^2}\right)^+(\cdot,\mathbf{h}) - \left(\mathbf{W}_{\chi^2}\right)^*(\cdot,\mathbf{h}) = \frac{1}{2}\mathbf{h} = \left(\mathbf{W}_{\chi^2}\right)^*(\cdot,\mathbf{h}) - \left(\mathbf{W}_{\chi^2}\right)^-(\cdot,\mathbf{h}), \tag{4.16}$$

$$\left(\mathbf{H}_{\chi^2}\right)^+(\cdot,\mathbf{g}) - \left(\mathbf{H}_{\chi^2}\right)^*(\cdot,\mathbf{g}) = -\frac{1}{2}\mathbf{g} = \left(\mathbf{H}_{\chi^2}\right)^*(\cdot,\mathbf{g}) - \left(\mathbf{H}_{\chi^2}\right)^-(\cdot,\mathbf{g}). \tag{4.17}$$

In addition, if $\mathbf{h} \in C^{1,\beta}(\mathcal{C})$, $\beta \in (0,\alpha)$, then there exist the limiting values of the boundary traction due to the double-layer potential $\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})$ on both sides of $\mathcal{C}$, $\mathbf{T}^+(\mathbf{W}_{\chi^2}(\mathbf{h}))$ and $\mathbf{T}^-(\mathbf{W}_{\chi^2}(\mathbf{h}))$, and they are equal, i.e.,

$$\mathbf{T}^+(\mathbf{W}_{\chi^2}(\mathbf{h})) = \mathbf{T}^-(\mathbf{W}_{\chi^2}(\mathbf{h})) \equiv \mathbf{T}(\mathbf{W}_{\chi^2}(\mathbf{h})) \quad \text{on } \mathcal{C}. \tag{4.18}$$

The superscript $+$ $(-)$ is used for the limiting value of a field evaluated from the external side (the internal side) of $\mathcal{C}$, and the symbol $*$ refers to the principal value of a double-layer integral on $\mathcal{C}$. The relations (4.15)-(4.18) also hold for the boundary potentials associated with the Stokes system.

The functions $\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})$, $\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})$, $P^s_{\chi^2}(\cdot,\mathbf{g})$, $P^d_{\chi^2}(\cdot,\mathbf{h})$ satisfy the relations

$$\mathbf{V}_{\chi^2}(\mathbf{x},\mathbf{g}) = O(|\mathbf{x}|^{-2}), \quad \mathbf{W}_{\chi^2}(\mathbf{x},\mathbf{h}) = O(|\mathbf{x}|^{-1}) \ \text{ as } |\mathbf{x}| \to \infty, \tag{4.19}$$

$$P^s_{\chi^2}(\mathbf{x},\mathbf{g}) = O(|\mathbf{x}|^{-1}), \quad P^d_{\chi^2}(\mathbf{x},\mathbf{h}) = O(\ln|\mathbf{x}|) \ \text{ as } |\mathbf{x}| \to \infty, \tag{4.20}$$

and in the case $\chi = 0$, we have:

$$\mathbf{V}(\mathbf{x},\mathbf{g}) = O(\ln|\mathbf{x}|), \quad P^s(\mathbf{x},\mathbf{h}) = O(|\mathbf{x}|^{-1}) \ \text{ as } |\mathbf{x}| \to \infty, \tag{4.21}$$

$$\mathbf{W}(\mathbf{x},\mathbf{h}) = O(|\mathbf{x}|^{-1}), \quad P^d(\mathbf{x},\mathbf{h}) = O(|\mathbf{x}|^{-2}) \ \text{ as } |\mathbf{x}| \to \infty. \tag{4.22}$$

4.3. Complementary integral operators

For $\lambda \in (0,\alpha)$, let $\mathcal{V}_{\chi^2} : C^{\lambda}(\mathcal{C}) \to C^{1,\lambda}(\mathcal{C})$ and $\mathbf{K}_{\chi^2} : C^{1,\lambda}(\mathcal{C}) \to C^{1,\lambda}(\mathcal{C})$ be the single- and double-layer integral operators for the Brinkman system, i.e.,

$$\mathcal{V}_{\chi^2}\mathbf{g} = \mathbf{V}_{\chi^2}(\cdot,\mathbf{g}), \quad \mathbf{K}_{\chi^2}\mathbf{h} = \mathbf{W}^*_{\chi^2}(\cdot,\mathbf{h}), \quad \forall\ \mathbf{g} \in C^{\lambda}(\mathcal{C}),\ \mathbf{h} \in C^{1,\lambda}(\mathcal{C}),$$

Similarly, $\mathcal{V} : C^{\lambda}(\mathcal{C}) \to C^{1,\lambda}(\mathcal{C})$ and $\mathbf{K} : C^{1,\lambda}(\mathcal{C}) \to C^{1,\lambda}(\mathcal{C})$ are the corresponding integral operators for the Stokes system.

Also, let $\mathbf{D}_{\chi^2} : C^{1,\lambda}(\mathcal{C}) \to C^{\lambda}(\mathcal{C})$ be the operator given in (4.18), i.e.,

$$\left(\mathbf{D}_{\chi^2}\mathbf{h}\right)_j(\mathbf{x}) = \text{p.f.} \int_{\mathcal{C}} D^{\chi^2}_{j\ell}(\mathbf{x},\mathbf{y})h_\ell(\mathbf{y})d\mathcal{C}(\mathbf{y}), \tag{4.23}$$

where

$$D^{\chi^2}_{j\ell}(\mathbf{x},\mathbf{y}) = -\Lambda^{\chi^2}_{\ell k}(\mathbf{x}-\mathbf{y})n_k(\mathbf{y})n_j(\mathbf{x})$$
$$+\Big(\frac{\partial}{\partial x_j}S^{\chi^2}_{\ell ik}(\mathbf{y}-\mathbf{x}) + \frac{\partial}{\partial x_i}S^{\chi^2}_{\ell jk}(\mathbf{y}-\mathbf{x})\Big)n_i(\mathbf{x})n_k(\mathbf{y}).$$

The corresponding operator for the Stokes system is denoted by $\mathbf{D}_0$. The operators $\mathbf{D}_{\chi^2}$ and $\mathbf{D}_0$ belong to the class of hypersingular operators.

Let us introduce the notations

$$\Lambda^c_{\ell k}(\mathbf{x}-\mathbf{y}) = \Lambda^{\chi^2}_{\ell k}(\mathbf{x}-\mathbf{y}) - \Lambda_{\ell k}(\mathbf{x}-\mathbf{y}), \quad K^c_{jk}(\mathbf{y},\mathbf{x}) = K^{\chi^2}_{jk}(\mathbf{y},\mathbf{x}) - K_{jk}(\mathbf{y},\mathbf{x}), \tag{4.24}$$

in view of which we are now able to define the complementary integral operators for the Stokes-Brinkman-coupled system.

The complementary single- and double-layer operators $\mathcal{V}_{\chi^2,0} : C^{\lambda}(\mathcal{C}) \to C^{1,\lambda}(\mathcal{C})$ and $\mathbf{K}_{\chi^2,0} : C^{1,\lambda}(\mathcal{C}) \to C^{1,\lambda}(\mathcal{C})$ are given by

$$\mathcal{V}_{\chi^2,0} = \mathcal{V}_{\chi^2} - \mathcal{V}, \quad \mathbf{K}_{\chi^2,0} = \mathbf{K}_{\chi^2} - \mathbf{K}, \tag{4.25}$$

and the adjoint of the complementary double-layer operator $\mathbf{K}^{'}_{\chi^2,0} : C^{\lambda}(\mathcal{C}) \to C^{\lambda}(\mathcal{C})$ has the expression $\mathbf{K}^{'}_{\chi^2,0} = \mathbf{K}^{'}_{\chi^2} - \mathbf{K}^{'}$, where $\mathbf{K}^{'}_{\chi^2}$ is the adjoint operators of $\mathbf{K}_{\chi^2}$.

In addition, the complementary hypersingular operator

$$\mathbf{D}_{\chi^2,0} : C^{1,\lambda}(\mathcal{C}) \to C^{\lambda}(\mathcal{C})$$

is given by $\mathbf{D}_{\chi^2,0} = \mathbf{D}_{\chi^2} - \mathbf{D}_0$.

We have following compactness result whose proof can be consulted in [4]:

Theorem 4.2. *If $\mathcal{C}$ is a closed Lyapunov curve in $\mathbb{R}^2$, i.e., $\mathcal{C} \in C^{1,\alpha}$, $\alpha \in (0,1]$, and $\lambda \in (0,\alpha)$, then the complementary boundary integral operators*

$$\mathcal{V}_{\chi^2,0} : C^{\lambda}(\mathcal{C}) \to C^{1,\lambda}(\mathcal{C}), \quad \mathbf{K}_{\chi^2,0} : C^{1,\lambda}(\mathcal{C}) \to C^{1,\lambda}(\mathcal{C}),$$
$$\mathbf{K}'_{\chi^2,0} : C^{\lambda}(\mathcal{C}) \to C^{\lambda}(\mathcal{C}), \quad \mathbf{D}_{\chi^2,0} : C^{1,\lambda}(\mathcal{C}) \to C^{\lambda}(\mathcal{C})$$

are compact.

In addition, if $\mathbf{h} \in C^{1,\lambda}(\mathcal{C})$, then $\mathbf{T}^+(\mathbf{W}_{\chi^2,0}(\mathbf{h}))$ and $\mathbf{T}^-(\mathbf{W}_{\chi^2,0}(\mathbf{h}))$ exist everywhere on $\mathcal{C}$ and they are equal.

5. The boundary integral formulation of the problem

In order to prove that the boundary value problem (2.1)-(2.6) has a unique classical solution, we consider the following boundary integral representations:

$$\begin{aligned} v_k^e(\mathbf{x}) &= U_k^\infty + \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} K_{jk}^{\chi^2}(\mathbf{y},\mathbf{x})\phi_j(\mathbf{y})d\Gamma_l(\mathbf{y}) \\ &+ \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \mathcal{G}_{kj}^{\chi^2}(\mathbf{x}-\mathbf{y})\Big(h_j(\mathbf{y}) \\ &- \frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}), \quad \mathbf{x}\in D_e \end{aligned} \tag{5.1}$$

$$\begin{aligned} p^e(\mathbf{x}) &= P^\infty(\mathbf{x}) + \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \Lambda_{jk}^{\chi^2}(\mathbf{x}-\mathbf{y})n_k(\mathbf{y})\phi_j(\mathbf{y})d\Gamma_l(\mathbf{y}) \\ &+ \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \Pi_j^{\chi^2}(\mathbf{x}-\mathbf{y})\Big(h_j(\mathbf{y}) \\ &- \frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}),\ \mathbf{x}\in D_e, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} v_k^i(\mathbf{x}) &= \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} K_{jk}(\mathbf{y},\mathbf{x})\phi_j(\mathbf{y})d\Gamma_l(\mathbf{y}) \\ &+ \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \mathcal{G}_{kj}(\mathbf{x}-\mathbf{y})\Big(h_j(\mathbf{y}) \\ &- \frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}) \\ &- \int_{\Gamma_m} h_j(\mathbf{y})d\Gamma_m(\mathbf{y}), \quad \mathbf{x}\in D_m, m=1,\ldots,N \end{aligned} \tag{5.3}$$

$$\begin{aligned} p^i(\mathbf{x}) &= \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \Lambda_{jk}(\mathbf{x}-\mathbf{y})n_k(\mathbf{y})\phi_j(\mathbf{y})d\Gamma_l(\mathbf{y}) \\ &+ \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \Pi_j(\mathbf{x}-\mathbf{y})\Big(h_j(\mathbf{y}) \\ &- \frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}), \quad \mathbf{x}\in D_0, \end{aligned} \tag{5.4}$$

where $\mathbf{\Phi} = (\phi_1,\phi_2) \in C^{1,\lambda}(\Gamma)$ and $\mathbf{h} = (h_1,h_2) \in C^{\lambda}(\Gamma)$ are unknown densities, $\lambda \in (1,\alpha)$, and $|\Gamma_k| = \int_{\Gamma_k} d\Gamma_k$ is the length of Γ_k, $k = 1,\ldots,N$.

Let us observe that the boundary integral representations (5.1)-(5.4) satisfy the system of equations (2.1)-(2.4), as well as far field conditions (2.5)-(2.6).

Now, imposing the transmission condition (2.5), we obtain the equations

$$\phi_k(\mathbf{x}_0) + \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} K^c_{jk}(\mathbf{y},\mathbf{x}_0)\phi_j(\mathbf{y})d\Gamma_l(\mathbf{y}) + \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} \mathcal{G}^c_{kj}(\mathbf{x}_0-\mathbf{y})\Big(h_j(\mathbf{y}) - \frac{1}{|\Gamma_m|}\int_{\Gamma_m} h_j(\mathbf{y})d\Gamma_m(\mathbf{y})\Big)\, d\Gamma_l(\mathbf{y}) + \int_{\Gamma_m} h_j(\mathbf{y})d\Gamma_m(\mathbf{y}) = -U^\infty_k,\ \mathbf{x}_0\in\Gamma_m,\ m=1,\dots,N. \tag{5.5}$$

Taking into account the second of the boundary conditions (2.5), we obtain the boundary integral equations

$$-h_k(\mathbf{x}_0) + \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} K^c_{kj}(\mathbf{x}_0,\mathbf{y})\Big(h_j(\mathbf{y}) - \frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}) + T_{kj}(\mathbf{W}^c(\mathbf{\Phi}))(\mathbf{x}_0)n_j(\mathbf{x}_0) = -t^\infty_k(\mathbf{x}_0),\ \mathbf{x}_0\in\Gamma_m,\ m=1,\dots,N, \tag{5.6}$$

where t^∞_k are the components of the stress field associated to the velocity field $\mathbf{U}^\infty$, i.e.,

$$t^\infty_k(\mathbf{x}) = p^\infty n_k(\mathbf{x}).$$

We mention that the integrals on Γ_m who appears in (5.5) and (5.6) are understood in the sense of principal value.

Therefore, the boundary value problem (2.1)-(2.6) reduces to the system of boundary integral equations (5.5) and (5.6). In view of Theorem 4.2 it follows that all operators that appear in the boundary integral equations (5.5) and (5.6) are compact, as mappings into one of the spaces $C^{1,\lambda}(\Gamma)$, $C^\lambda(\Gamma)$. Thus, these equations are Fredholm integral equations of the second kind with the unknowns $(\mathbf{\Phi},\mathbf{h})\in C^{1,\lambda}(\Gamma)\times C^\lambda(\Gamma)$.

We have the following existence and uniqueness result (see also [2]):

Theorem 5.1. *Let Γ_k be closed Lyapunov curves of class $C^{1,\alpha}$ in $\mathbb{R}^2$, $\alpha\in(0,1]$, $k=1,\dots,N$, $\Gamma=\cup_{k=1}^N\Gamma_k$, and let $\lambda\in(0,\alpha)$. Then the system of Fredholm integral equations of the second kind* (5.5) *and* (5.6) *has a unique solution $(\mathbf{\Phi},\mathbf{h})\in C^{1,\lambda}(\Gamma)\times C^\lambda(\Gamma)$. In addition, the boundary integral representations* (5.1)-(5.4)*, obtained with the densities $\mathbf{\Phi}$ and $\mathbf{h}$, determine the unique classical solution $((\mathbf{v}^e,p^e),(\mathbf{v}^i,p^i))\in((C^2(D_e)\cap C^1(\overline{D}_e))\times(C^1(D_e)\cap C^0(\overline{D}_e)))\times((C^2(D_0)\cap C^1(\overline{D}_0))\times(C^1(D_0)\cap C^0(\overline{D}_0)))$ to the boundary value problem consisting of the equations* (2.1)-(2.4) *and the boundary and far field conditions* (2.5)-(2.6).

Proof. Let us consider the following homogeneous system of integral equations

$$\begin{aligned}\phi_k^0(\mathbf{x}_0) &+ \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} K_{jk}^c(\mathbf{y},\mathbf{x}_0)\phi_j^0(\mathbf{y})d\Gamma_l(\mathbf{y}) \\ &+ \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_k} \mathcal{G}_{kj}^c(\mathbf{x}_0-\mathbf{y})\Big(h_j^0(\mathbf{y}) - \frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j^0(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}) \\ &+ \int_{\Gamma_m} h_j^0(\mathbf{y})d\Gamma_m(\mathbf{y}) = 0,\ \mathbf{x}_0\in\Gamma_m,\ m=1,\dots,N, \end{aligned} \tag{5.7}$$

$$\begin{aligned}-h_k^0(\mathbf{x}_0) &+ \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} K_{kj}^c(\mathbf{x}_0,\mathbf{y})\Big(h_j^0(\mathbf{y}) - \frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j^0(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}) \\ &+ T_{kj}(\mathbf{W}^c(\mathbf{\Phi^0}))(\mathbf{x}_0)n_j(\mathbf{x}_0) = 0,\ \mathbf{x}_0\in\Gamma_m,\ m=1,\dots,N. \end{aligned} \tag{5.8}$$

Also let $(\mathbf{\Phi}^0,\mathbf{h}^0)\in C^{1,\lambda}(\Gamma)\times C^{\lambda}(\Gamma)$ be an arbitrary solution to this system, and let $(\mathbf{u}^e,q^e)$ and $(\mathbf{u}^i,q^i)$ be the fields given by the following boundary integral representations:

$$\begin{aligned}u_k^e(\mathbf{x}) =& \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l} K_{jk}^{\chi^2}(\mathbf{y},\mathbf{x})\phi_j^0(\mathbf{y})d\Gamma_l(\mathbf{y}) + \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l}\mathcal{G}_{kj}^{\chi^2}(\mathbf{x}-\mathbf{y})\Big(h_j^0(\mathbf{y}) \\ &- \frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j^0(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}),\quad \mathbf{x}\in\mathbb{R}^2\setminus\Gamma_m, m=1,\dots,N \end{aligned} \tag{5.9}$$

$$\begin{aligned}q^e(\mathbf{x}) =& \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l}\Lambda_{jk}^{\chi^2}(\mathbf{x}-\mathbf{y})\phi_j^0(\mathbf{y})d\Gamma_l(\mathbf{y}) \\ &+ \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l}\Pi_j^{\chi^2}(\mathbf{x}-\mathbf{y})\Big(h_j^0(\mathbf{y}) \\ &- \frac{1}{|\Gamma_l|}\int_{\Gamma_l} h_j^0(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}),\quad \mathbf{x}\in\mathbb{R}^2\setminus\Gamma_m, m=1,\dots,N \end{aligned} \tag{5.10}$$

and

$$\begin{aligned}u_k^i(\mathbf{x}) =& \frac{1}{4\pi}\int_{\Gamma_m} K_{jk}(\mathbf{y},\mathbf{x})\phi_j^0(\mathbf{y})d\Gamma_m(\mathbf{y}) + \frac{1}{4\pi}\int_{\Gamma_m}\mathcal{G}_{kj}(\mathbf{x}-\mathbf{y})\Big(h_j^0(\mathbf{y}) \\ &- \frac{1}{|\Gamma_m|}\int_{\Gamma_m} h_j^0(\mathbf{z})d\Gamma_m(\mathbf{z})\Big)d\Gamma_m(\mathbf{y}) \\ &- \int_{\Gamma_m} h_j^0(\mathbf{y})d\Gamma_m(\mathbf{y}),\quad \mathbf{x}\in\mathbb{R}^2\setminus\Gamma_m, m=1,\dots,N \end{aligned} \tag{5.11}$$

$$q^i(\mathbf{x}) = \frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l}\Lambda_{jk}(\mathbf{x}-\mathbf{y})n_k(\mathbf{y})\phi_j^0(\mathbf{y})d\Gamma_l(\mathbf{y})$$
$$+\frac{1}{4\pi}\sum_{l=1}^{N}\int_{\Gamma_l}\Pi_j(\mathbf{x}-\mathbf{y})\Big(h_j^0(\mathbf{y})$$
$$-\frac{1}{|\Gamma_l|}\int_{\Gamma_l}h_j^0(\mathbf{z})d\Gamma_l(\mathbf{z})\Big)d\Gamma_l(\mathbf{y}),\quad \mathbf{x}\in\mathbb{R}^2\setminus\Gamma_m, m=1,\ldots,N. \quad (5.12)$$

Because the pairs $(\mathbf{V}_{\chi^2}(\cdot,\mathbf{g}), P^s_{\chi^2}(\cdot,\mathbf{g}))$ and $(\mathbf{W}_{\chi^2}(\cdot,\mathbf{h}), P^d_{\chi^2}(\cdot,\mathbf{h}))$ satisfy the Brinkman system in both domains D_0 and D_e, respectively, and the pairs $(\mathbf{V}(\cdot,\mathbf{g}), P^s(\cdot,\mathbf{g}))$ and $(\mathbf{W}(\cdot,\mathbf{h}), P^d(\cdot,\mathbf{h}))$ satisfy the Stokes system in both domains D_0 and D_e, respectively, we obtain that:

$$\nabla\cdot\mathbf{u}^e = 0,\quad -\nabla q^e + (\nabla^2-\chi^2)\mathbf{u}^e = \mathbf{0} \text{ in } \mathbb{R}^2\setminus\Gamma, \quad (5.13)$$
$$\nabla\cdot\mathbf{u}^i = 0,\quad -\nabla q^i + \nabla^2\mathbf{u}^e = \mathbf{0} \text{ in } \mathbb{R}^2\setminus\Gamma. \quad (5.14)$$

Taking into account the relations (4.21) we have that:

$$(|\mathbf{u}^e||\nabla\mathbf{u}^e|)(\mathbf{x}) = o(|\mathbf{x}|^{-1}),\quad (|\mathbf{u}^e||q^e|)(\mathbf{x}) = o(|\mathbf{x}|^{-1}) \text{ as } |\mathbf{x}|\to\infty. \quad (5.15)$$

Therefore, the fields $\mathbf{u}^e$ and q^e satisfy the identity

$$\int_{D_e}(2E_{jk}(\mathbf{u}^e)E_{jk}(\mathbf{u}^e)+\chi^2|\mathbf{u}^e|^2)d\mathbf{x} = -\sum_{l=1}^{N}\int_{\Gamma_l}u_k^{e+}(\mathbf{x})t_k^+(\mathbf{u}^e)(\mathbf{x})d\Gamma_l(\mathbf{x}), \quad (5.16)$$

where $t_k^{\pm}(\mathbf{u}^e) = T_{kj}^{\pm}(\mathbf{u}^e)n_j$, and

$$T_{kj}(\mathbf{u}^e) = -q^e\delta_{kj} + 2E_{kj}(\mathbf{u}^e),\quad E_{jk}(\mathbf{u}^e) = \frac{1}{2}\left(\frac{\partial u_j^e}{\partial x_k}+\frac{\partial u_k^e}{\partial x_j}\right).$$

Similarly, the fields $\mathbf{u}^i$ and q^i satisfy the identity (see e.g. [1], p. 15)

$$2\int_{D_0}E_{jk}(\mathbf{u}^i)E_{jk}(\mathbf{u}^i)d\mathbf{x} = \sum_{l=1}^{N}\int_{\Gamma_l}u_k^{i-}(\mathbf{x})t_k^-(\mathbf{u}^i)(\mathbf{x})d\Gamma_l(\mathbf{x}), \quad (5.17)$$

where $t_k^{\pm}(\mathbf{u}^i) = T_{kj}^{\pm}(\mathbf{u}^i)n_j$, and

$$T_{kj}(\mathbf{u}^i) = -q^i\delta_{kj} + 2E_{kj}(\mathbf{u}^i),\quad E_{jk}(\mathbf{u}^i) = \frac{1}{2}\left(\frac{\partial u_j^i}{\partial x_k}+\frac{\partial u_k^i}{\partial x_j}\right).$$

Now, taking into account the formulas (4.15)-(4.17), we obtain the properties

$$u_k^{e+} = u_k^{i-},\quad t_k^+(\mathbf{u}^e) = t_k^-(\mathbf{u}^i) \text{ on } \Gamma_l, l=1\ldots,N \quad (5.18)$$

which yield the equality

$$\sum_{l=1}^{N}\int_{\Gamma_l}u_k^{e+}(\mathbf{x})t_k^+(\mathbf{u}^e)(\mathbf{x})d\Gamma_l = \sum_{k=1}^{N}\int_{\Gamma_l}u_k^{i-}(\mathbf{x})t_k^-(\mathbf{u}^i)(\mathbf{x})d\Gamma_l. \quad (5.19)$$

From the properties (5.16), (5.17) and (5.19) we deduce that

$$\int_{D_e}(2E_{jk}(\mathbf{u}^e)E_{jk}(\mathbf{u}^e)+\chi^2|\mathbf{u}^e|^2)d\mathbf{x} = -2\int_{D_0}E_{jk}(\mathbf{u}^i)E_{jk}(\mathbf{u}^i)d\mathbf{x}, \quad (5.20)$$

and hence

$$\mathbf{u}^e = \mathbf{0} \text{ in } D_e, \tag{5.21}$$

$$E_{jk}(\mathbf{u}^i) = \mathbf{0} \text{ in } D_m, j, k = 1, 2,\ m = 1, \ldots, N. \tag{5.22}$$

Using Killing's theorem, we deduce that there exists some real constants a_0^k and b_0^k such that

$$\mathbf{u}^i = a_0^m + b_0^m \times \mathbf{x} \text{ in } \ D_m, \quad m = 1, \ldots, N. \tag{5.23}$$

But,

$$0 = u^{e+} = u^{i-} \text{ on } \Gamma_m\, m = 1, \ldots, N$$

thus we obtain that

$$a_0^m = b_0^m = 0.$$

Then , we have:

$$\mathbf{u}^i = \mathbf{0} \text{ in } \ D_m, m = 1, \ldots, N. \tag{5.24}$$

In addition, in view of the second of equations (5.13) and from the fact that the pressure field q^e vanishes at infinity, we obtain

$$q^e = 0 \text{ in } D_e. \tag{5.25}$$

Similarly, we deduce that $q^i = c_m^0 \in \mathbb{R}$ in D_m. On the other hand, from the relations (5.18), (5.21) and (5.25) we get

$$t_k^-(\mathbf{u}^i) = t_k^+(\mathbf{u}^e) = 0 \text{ on } \ \Gamma_m, \quad m = 1, \ldots, N \tag{5.26}$$

and hence the constant c_m^0 must be equal to zero, i.e.,

$$\mathbf{u}^i = \mathbf{0}, \quad q^i = 0 \text{ in } D_0. \tag{5.27}$$

Now, using the jump formula

$$\mathbf{u}^{e+} - \mathbf{u}^{e-} = \mathbf{\Phi^0} \text{ on } \Gamma_m, \quad m = 1, \ldots, N$$

(see the properties (4.15) and (4.16)) as well as the result (5.21), we deduce that

$$\mathbf{u}^{e-} = -\mathbf{\Phi}^0 \text{ on } \Gamma_m, \quad m = 1, \ldots, N. \tag{5.28}$$

Similarly, from the jump formula

$$\mathbf{u}^{i+} - \mathbf{u}^{i-} = \mathbf{\Phi}^0 \text{ on } \Gamma_m, \quad m = 1, \ldots, N$$

as well as the result (5.27), we find that

$$\mathbf{u}^{i+} = \mathbf{\Phi}^0 \text{ on } \Gamma_m, \quad m = 1, \ldots, N. \tag{5.29}$$

On the other hand, from the relations (4.17) we deduce that the boundary traction due to the fields $\mathbf{u}^e$ and q^e has a jump across every curve Γ_k given by the formula

$$\mathbf{t}^+(\mathbf{u}^e) - \mathbf{t}^-(\mathbf{u}^e) = -\Big(\mathbf{h}^0 - \frac{1}{|\Gamma_m|}\int_{\Gamma_m} \mathbf{h^0} d\Gamma_m\Big) \text{ on } \Gamma_m, \quad m = 1, \ldots, N. \tag{5.30}$$

But $\mathbf{t}^+(\mathbf{u}^e) = \mathbf{0}$ on Γ_k and hence

$$\mathbf{t}^-(\mathbf{u}^e) = \mathbf{h}^0 - \frac{1}{|\Gamma_m|}\int_{\Gamma_m} \mathbf{h^0} d\Gamma_m \text{ on } \Gamma_m, \quad m = 1, \ldots, N. \tag{5.31}$$

With similar kind of arguments as before, we get the relation

$$\mathbf{t}^{+}(\mathbf{u}^{i}) = -\Big(\mathbf{h}^{0} - \int_{\Gamma_m} \mathbf{h}^{0} d\Gamma_m\Big) \text{ on } \Gamma_m, \quad m = 1, \dots, N. \tag{5.32}$$

In addition, the fields $(\mathbf{u}^{e}, q^{e})$ satisfy the identity

$$\int_{D_0} (2E_{jk}(\mathbf{u}^{e})E_{jk}(\mathbf{u}^{e}) + \chi^2|\mathbf{u}^{e}|^2)d\mathbf{x} = \sum_{l=1}^{N} \int_{\Gamma_l} u_k^{e-}(\mathbf{x})t_k^{-}(\mathbf{u}^{e})(\mathbf{x})d\Gamma_l(\mathbf{x}) \tag{5.33}$$

and, in view of the properties (5.28) and (5.31), this identity takes the form

$$\int_{D_0} (2E_{jk}(\mathbf{u}^{e})E_{jk}(\mathbf{u}^{e}) + \chi^2|\mathbf{u}^{e}|^2)d\mathbf{x} = -\sum_{l=1}^{N} \int_{\Gamma_k} \mathbf{\Phi}^{0} \cdot \Big(\mathbf{h}^{0} - \frac{1}{|\Gamma_l|} \int_{\Gamma_l} \mathbf{h}^{\mathbf{0}} d\Gamma_l\Big) d\Gamma_l. \tag{5.34}$$

Since

$$\int_{\Gamma_l} \Big(\mathbf{h}^{0} - \frac{1}{\Gamma_l} \int_{\Gamma_l} \mathbf{h}^{0} d\Gamma_l\Big) d\Gamma_l = 0, \tag{5.35}$$

we deduce that

$$\mathbf{V}\Big(\mathbf{x}, \mathbf{h}^{0} - \frac{1}{|\Gamma_l|} \int_{\Gamma_l} \mathbf{h}^{\mathbf{0}} d\Gamma_l\Big) = \mathcal{O}(|\mathbf{x}|^{-1}) \text{ as } |\mathbf{x}| \to \infty, \tag{5.36}$$

the fields $\mathbf{u}^{i}$ and q^{i} behave at infinity as follows (see also the relations (4.21)):

$$\nabla^{s}\mathbf{u}^{i}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1-s}), \quad q^{i}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1}) \text{ as } |\mathbf{x}| \to \infty,\ s = 0, 1, \tag{5.37}$$

and hence they satisfy the far field conditions (2.6). Consequently, we get the following identity:

$$2\int_{D_e} E_{jk}(\mathbf{u}^{i})E_{jk}(\mathbf{u}^{i})d\mathbf{x} = -\sum_{l=1}^{N} \int_{\Gamma_l} u_k^{i+}(\mathbf{x})t_k^{+}(\mathbf{u}^{i})(\mathbf{x})d\Gamma_l(\mathbf{x}), \tag{5.38}$$

which, in view of the properties (5.29) and (5.32), becomes

$$2\int_{D_e} E_{jk}(\mathbf{u}^{i})E_{jk}(\mathbf{u}^{i})d\mathbf{x} = \sum_{l=1}^{N} \int_{\Gamma_l} \mathbf{\Phi}^{0} \cdot \Big(\mathbf{h}^{0} - \frac{1}{|\Gamma_l|} \int_{\Gamma_l} \mathbf{h}^{0} d\Gamma_l\Big) d\Gamma_l. \tag{5.39}$$

Therefore, from the identities (5.34) and (5.39) we obtain that

$$\mathbf{u}^{e} = \mathbf{0} \text{ in } D_0 \tag{5.40}$$

and

$$\mathbf{u}^{i} = \mathbf{0} \text{ in } D_e. \tag{5.41}$$

The property (5.41), the equation $-\nabla q^{i} + (\nabla^2 - \chi^2)\mathbf{u}^{i} = \mathbf{0}$ in D_e, and the fact that the pressure field q^{i} vanishes at infinity lead to the additional result

$$q^{i} = 0 \text{ in } D_e. \tag{5.42}$$

From the relation (5.40), we get that

$$\mathbf{u}^{e-} = \mathbf{0} \text{ on } \Gamma_l, \quad l = 1, \dots, N.$$

Using the above relation and (5.28), we obtain that:

$$\mathbf{\Phi}^{0} = \mathbf{0} \text{ on } \Gamma_l, \quad l = 1, \dots, N. \tag{5.43}$$

In addition, according to the relations (5.32), (5.41) and (5.42) we find that

$$\mathbf{t}^{+}(\mathbf{u}^{i}) = \mathbf{0} \quad \text{on} \quad \Gamma_l, \quad l = 1, \ldots, N, \tag{5.44}$$

i.e.,

$$\mathbf{h} = \frac{1}{|\Gamma_l|}\int_{\Gamma_l} \mathbf{h} d\Gamma_l := \mathbf{c}_l \in \mathbb{R}^2 \quad \text{on} \quad \Gamma_l, \quad l = 1, \ldots, N. \tag{5.45}$$

So, we obtain that

$$\mathbf{0} = \mathbf{u}^{i} = -\int_{\Gamma_l} \mathbf{h} d\Gamma_l in D_l, \quad l = 1, \ldots, N \tag{5.46}$$

and hence

$$\int_{\Gamma_l} \mathbf{h} d\Gamma_l = 0, \quad l = 1, \ldots, N. \tag{5.47}$$

Finally, from the relations(5.45) and (5.46) we find that

$$\mathbf{h}^{0} = \mathbf{0} \quad \text{on} \quad \Gamma_l, \quad l = 1, \ldots, N. \tag{5.48}$$

The relations (5.43) and (5.48) shows that the homogeneous system of equations (5.7) and (5.8) has only the trivial solution in the space $C^{1,\lambda}(\Gamma) \times C^{\lambda}(\Gamma)$. Consequently, in view of Fedholm's alternative [5] we deduce that the non-homogeneous system of Fredholm integral equations of the second kind (5.5) and (5.6) has a unique solution$(\mathbf{\Phi}, \mathbf{h}) \in C^{1,\lambda}(\Gamma) \times C^{\lambda}(\Gamma)$, as desired. □

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References

[1] Kohr, M., Pop I., *Viscous Incompressible Flow for Low Reynolds Numbers*, WIT Press, Southampton, UK, 2004.

[2] Kohr, M., Raja Sekhar, G. P., *Existence and uniqueness result for two-dimensional porous media flows with porous inclusions based on Brinkman equation*, Engineering Analysis with Boundary Elements, **31**(2007), 604-613.

[3] Kohr, M., Raja Sekhar, G. P., *Existence and uniqueness result for the problem of viscous flow in a granular material with a void*, Quarterly of Applied Mathematics, **65**(2007), no. 4, 683-704.

[4] Kohr, M., Wendland, W. L., Raja Sekhar, G. P., *Boundary integral equations for two-dimensional low Reynolds number flow past a porous body*, Mathematical Methods in the Applied Sciences, Published Online, Sep 26 2008. DOI: 10.1002/mma.1074.

[5] Hsiao, G., Wendland, W.L., *Boundary Integral equations*, Springer, Heidelberg, 2008.

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