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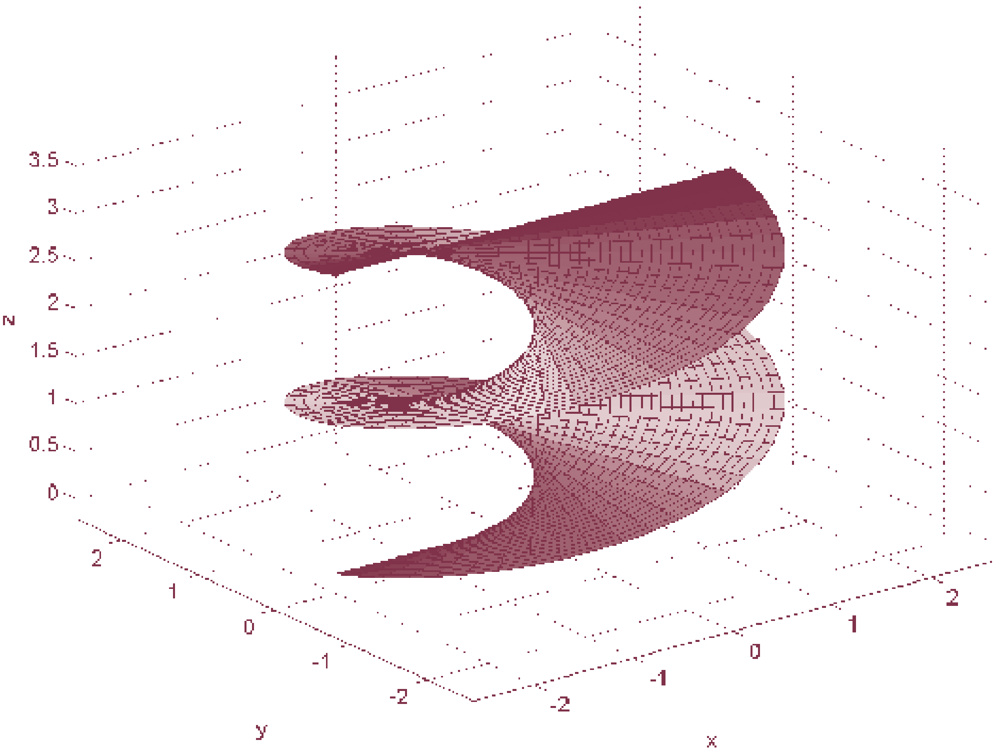

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ON THE BEHAVIOR NEAR 0 AND NEAR ∞ OF FUNCTIONALS ON $W_0^{1,p}(\Omega)$ INVOLVING NONLINEAR OSCILLATING TERMS

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Abstract. The behavior near 0 and near ∞ of energy functionals on $W_0^{1,p}(\Omega)$ associated to boundary value problems for quasilinear elliptic equations is studied. As a consequence, some results concerning the existence of infinitely many solutions for the Dirichlet problem are established.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with $C^{1,1}$ boundary $\partial\Omega$.

Let $f : \Omega\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory function. In recent years, some authors have investigated the problem of finding infinitely many solutions for the problem

$$\begin{cases} -\Delta_p u = f(x,u) \ \text{ in } \ \Omega \\ B(u) = 0 \end{cases} \tag{P}$$

in the case in which the nonlinearity $f(x,\cdot)$ has an oscillatory behavior near 0 or near ∞. Here, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-laplacian operator, with $p > 1$, and B is a given boundary operator. The reader is referred, for instance, to [1] [2], [6] and [11] for problem (P) with Neumann boundary condition, that is with $Bu = \frac{\partial u}{\partial \nu}_{|\partial\Omega}$, and to [3], [4], [7], [9] and [10] for problem (P) with Dirichlet boundary condition, that is with $Bu = u_{|\partial\Omega}$ (see also reference of [9], [10] for a wider overview on the subject). In all of these papers, the existence of infinitely many solutions is obtained by showing that the energy functional associated to problem (P)

$$\Psi(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx - \int_\Omega \left(\int_0^{u(x)} f(x,t)dt\right) dx, \tag{1.1}$$

defined on $W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$ according to whether the Neumann or the Dirichlet boundary condition is considered, possesses infinitely many critical points. Therefore,

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in particular, solutions are always understood in weak sense. In practice, the previous circumstance is realized by showing that if $f(x,\cdot)$ has a suitable oscillatory behavior near ∞ or near 0 then there exists a sequence of local minima $\{u_n\}$ for the functional Ψ which is unbounded with respect to some norm or satisfying $\lim_{n\to+\infty}\Psi(u_n)=0$ and $\Psi(u_n)<0$ for all $n\in\mathbb{N}$. In order to find the local minima u_n, a direct variational method is used in [1] where it is proved that the for a certain sequence on spheres $\{S_n\}$ of $W^{1,p}(\Omega)$, the infimum of Ψ on each S_n is strictly greater of the global minimum of Ψ on the closed ball B_n having S_n as boundary; truncation methods are, instead, used in [2], [3], [7], [9], [10] where it is proved that the global minima of certain truncations Ψ_n of Ψ are, actually, local minima of this latter; finally in [4], [6], [11], taking advantage of the compact embedding of $W^{1,p}(\Omega)$ in $C^0(\Omega)$ when $p>N$, a variational result of [12] on the multiplicity of critical points is applied. Once a sequence of local minima is found out, the successive step, as said before, is to show that this sequence contains infinitely many pairwise distinct elements. It is quite simple to realize this when the Neumann problem is considered. Indeed, due to the fact that the constant functions belong to $W^{1,p}(\Omega)$, it is suffice to require that there exists a sequence $\{\xi_n\}$ of real numbers such that

$$a)\qquad \lim_{n\to+\infty}\xi_n\to\begin{cases}0,\\ \pm\infty\end{cases};$$

$$b)\qquad \limsup_{n\to+\infty}\frac{\int_\Omega\int_0^{\xi_n}f(x,t)dt}{|\xi_n|^p}=+\infty$$

(see, for instance, [2], [6], [11] where, however, a slight weaker condition is required). The above question becomes more delicate when the Dirichlet problem is considered, especially if $f(x,\cdot)$ is sign-changing. In this case, the validity of $a)$ and $b)$ is no more sufficient to realize that the sequence of local minima u_n contains infinitely many pairwise distinct elements. To achieve this goal, more sophisticated conditions must be imposed. In particular, in [3] it is showed that under the following assumptions:

there exist a nonempty open set D in Ω, a positive number $\sigma>0$ and a sequence $\{\xi_n\}$ in $]0,+\infty[$ such that $\lim_{n\to+\infty}\xi_n=0$ and

$$a_1)\qquad \limsup_{n\to+\infty}\frac{\operatorname*{ess\,inf}_{x\in D}\int_0^{\xi_n}f(x,t)dt}{\xi_n^p}=+\infty;$$

$$b_1)\qquad \operatorname*{ess\,inf}_{x\in D}\inf_{\xi\in[0,\xi_n]}\int_0^{\xi}f(x,t)dt\geq-\sigma\operatorname*{ess\,inf}_{x\in D}\int_0^{\xi_n}f(x,t)dt\qquad\text{for all } n\in\mathbb{N},$$

the sequence of local minima $\{u_n\}$ can be chosen having the following property:

$$\Psi(u_n) < 0 \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n\to+\infty} \Psi(u_n) = 0.$$

Note that, as showed in [3], assumptions $a_1)$ and $b_1)$, when f is independent of x, are weaker than the following ones assumed in [10]

$$a_2) \quad \limsup_{\xi\to 0^+} \frac{\int_0^\xi f(t)dt}{\xi^p} = +\infty \ ;$$

$$b_2) \quad \liminf_{\xi\to 0^+} \frac{\int_0^\xi f(t)dt}{\xi^p} = 0,$$

where similar conclusions are obtained. Finally, in [9] it is showed that the sequence of local minima $\{u_n\}$ satisfies the following property $\lim_{n\to+\infty} \frac{u_n(x)}{\mathrm{dist}(x,\partial\Omega)} = +\infty$ uniformly in Ω assuming that $a_2)$, $b_2)$ hold with $\xi \to 0^+$ replaced by $\xi \to +\infty$. It is worth of noticing the fact that in [9] the authors obtained, besides $\{u_n\}$, a sequence of saddle points $\{v_n\}$ having the same property of $\{u_n\}$. Moreover, note that this property proves that u_n and v_n turn out to be positive in Ω rather than simply nonnegative (as in [3] and [10]).

The aim of this paper is to give a new contribution on this topic. In particular, we will establish the existence of a sequence of pairwise distinct local minima for the functional Ψ keeping assumption $a_1)$ but replacing assumption $b_1)$ with the following one

$\tilde{b}_1)$ there exists a positive number $\sigma > 0$ and a nonempty open set D in Ω such that

$$\int_0^{\xi_n} F_+(\xi)d\xi \geq -\sigma \int_0^{\xi_n} F_-(\xi)d\xi$$

for all $n \in \mathbb{N}$,

where $F : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$F(\xi) \leq \operatorname*{ess\,inf}_{x\in D} \int_0^\xi f(x,t)dt \quad \text{and}$$

$$F_+(t) = \max\{F(t), 0\}, \quad F_-(\xi) = \min\{F(\xi), 0\},$$

for all $\xi \in \mathbb{R}$.

To motivate our main result, we promptly exhibit an example of function f (for simplicity independent of x) which satisfy $a_1)$ and $\tilde{b}_1)$ but not $b_1)$:

let α, β, η be three positive numbers such that $1 < \beta < \alpha < p$ and $\beta > \alpha - \eta$ and let $g \in C^1(]0, \infty[)$ be a bounded nonnegative function such that

$$\int_0^1 \frac{g(t)}{t^\eta} < +\infty \tag{1.2}$$

and satisfying

$$\lim_{n\to+\infty} g(t_n) > 0 \quad \text{and} \quad g(s_n) = 0 \quad \text{for every} \quad n \in \mathbb{N}, \tag{1.3}$$

where $\{t_n\}$ and $\{s_n\}$ are two decreasing sequences in $]0, +\infty[$ such that

$$\sup_{n\in\mathbb{N}} \frac{t_n}{t_{n+1}} < +\infty \quad \text{and} \quad \lim_{n\to+\infty} t_n = \lim_{n\to+\infty} s_n = 0.$$

Put

$$f(t) = t^{\alpha-1} - t^{\beta-1}(\beta^{-1}tg'(t) + g(t)) \;\text{ for }\; t > 0 \;\text{ and }\; f(t) = 0 \;\text{ otherwise.}$$

Let us to show that f is the functions we are looking for. At first, note that

$$F(\xi) = \int_0^\xi f(t)dt = \frac{1}{\alpha}\xi^\alpha - \frac{1}{\beta}\xi^\beta g(\xi) \quad \text{for every} \quad \xi \geq 0.$$

Now, fix any sequence $\{\xi_n\}$ in $]0, +\infty[$ such that $\lim_{n\to+\infty} \xi_n = 0$ and $\xi_n < t_1$ for every $n \in \mathbb{N}$ and denote by k_n the smallest integer such that $t_{k_n} \leq \xi_n$. It follows $\lim_{n\to+\infty} t_{k_n} = 0$ and, for every $n \in \mathbb{N}$,

$$\inf_{\xi\in[0,\xi_n]} F(\xi) \leq F(t_{k_n}) = \frac{1}{\alpha}t_{k_n}^\alpha - \frac{1}{\beta}t_{k_n}^\beta g(t_{n_k})$$

$$F(\xi_n) \leq \sup_{\xi\in[0,\xi_n]} F(\xi) \leq \frac{1}{\alpha}t_{k_n-1}^\alpha.$$

Hence, for every $\sigma > 0$ and $n \in \mathbb{N}$, one has

$$\inf_{\xi\in[0,\xi_n]} F(\xi) + \sigma F(\xi_n) \leq \frac{1}{\alpha}t_{k_n}^\alpha - \frac{1}{\beta}t_{k_n}^\beta g(t_{n_k}) + \sigma\frac{1}{\alpha}t_{k_n-1}^\alpha =$$

$$\frac{1}{\alpha}t_{k_n}^\alpha \left(1 + \sigma\frac{t_{n_k-1}^\alpha}{t_{k_n}^\alpha} - \frac{\alpha}{\beta}t_{k_n}^{\beta-\alpha}g(t_{k_n})\right).$$

Consequently, in view of (1.3), one has

$$\inf_{\xi\in[0,\xi_n]} F(\xi) + \sigma F(\xi_n) < 0 \quad \text{for every} \quad n \in \mathbb{N}, \quad \text{with } n \text{ large enough.}$$

This means that condition b_1) does not hold.

Now, note that for every $\xi \in]0, 1[$ with

$$\xi < \left(\frac{\beta}{2\alpha}\int_0^1 \frac{g(\tau)}{\tau^\eta}d\tau\right)^{\frac{1}{\beta+\eta-\alpha}}$$

one has:

$$\int_{F_\xi^+} F(\tau)d\tau \geq \int_0^\xi F(\tau)d\tau = \frac{\xi^\alpha}{\alpha} - \frac{1}{\beta}\int_0^\xi \tau^\beta d\tau \geq \frac{\xi^\alpha}{\alpha} - \frac{\xi^{\beta+\eta}}{\beta}\int_0^1 \frac{g(\tau)}{\tau^\eta}d\tau \quad \text{and}$$

$$\int_{F_\xi^-} F(\tau)d\tau \geq -\frac{1}{\beta}\int_0^\xi \tau^\beta d\tau \geq -\frac{\xi^{\beta+\eta}}{\beta}\int_0^1 \frac{g(\tau)}{\tau^\eta}d\tau.$$

Therefore,

$$\int_{F_\xi^+} F(\tau)d\tau + \int_{F_\xi^-} F(\tau)d\tau \geq \frac{\xi^\alpha}{\alpha} - 2\frac{\xi^{\beta+\eta}}{\beta}\int_0^1 \frac{g(\tau)}{\tau^\eta}d\tau > 0.$$

Thus, in view of (1.2), condition $\tilde{b}_1$) holds with $\sigma = 1$. Finally, note that, thanks to the properties of the sequence $\{s_n\}$, condition a_1) holds as well.

To exhibit a concrete example of function g satisfying (1.2) and (1.3), an easy calculation shows that it is enough to take

$$g(t) = e^{-t^{-\rho}\cos^2(t^{-1})}\sin^2(t^{-1}) \qquad \text{with} \qquad \rho > 2(\eta - 1) \qquad \text{if} \quad \eta \geq 1$$

$$g(t) = \sin^2(t^{-1}) \qquad\qquad \text{if} \quad \eta < 1.$$

2. **The results**

In what follows, the following notations will be used:

- for every $\xi \in \mathbb{R}$ and every nonempty open set D in Ω, the set $\{u \in W_0^{1,p}(\Omega) : \; 0 \leq u(x) \leq \xi \;$ a.e. in $D\}$ is denoted by $X_{\xi,D}$;
- given a nonempty set A in $\mathbb{R}$ and a positive number ε, the symbol $(A)_\varepsilon$ denotes the (closed) ε-dilatation of A, that is the set $\{t \in \mathbb{R} : \; \inf_{\tau\in A}|t - \tau| \leq \varepsilon\}$. Moreover, we will use the notation $int(A)$ to denote the interior of A;
- given a positive real number τ and a function $h : [0,\tau] \to \mathbb{R}$, the symbols h_τ^+ and h_τ^- denote, respectively, the sets $\{t \in [0,\tau] : \; h(t) > 0\}$ and $\{t \in [0,\tau] : \; h(t) \leq 0\}$;
- for any Lebesgue measurable set A in $\mathbb{R}^N$, the symbol $|A|$ denotes its Lebesgue measure.

Moreover, we equip the space $W_0^{1,p}(\Omega)$ with its standard norm

$$\|\cdot\| = \left(\int_\Omega |\nabla(\cdot)|^p dx\right)^{\frac{1}{p}}$$

and for every continuous function $F : \mathbb{R} \to \mathbb{R}$ we put

$$\Psi_F(u) = \frac{1}{p}\|u\|^p - \int_\Omega F(u(x))dx$$

for all $u \in W_0^{1,p}(\Omega)$ such that $F(u(\cdot)) \in L^1(\Omega)$. Our main result is Theorem 2.1 below. It allow us to determinate an upper estimate of the number $\inf_{X_\xi} \Psi_F$ in terms of constants which depend, besides Ω, only on the ratio of the areas of the regions delimited by the x-axis and the graphs of F_+ and F_- in $[0, \xi]$.

Theorem 2.1. *Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exist two positive numbers σ, ξ such that*

i) $F(\xi) > 0$;

ii) *the set* $\Lambda_F \overset{def}{=} \left\{ \tau \in [0, \xi] : \int_{F_\tau^+} F(t)dt > -\sigma \int_{F_\tau^-} F(t)dt \right\}$ *is nonempty and*

$\sup \Lambda_F = \xi$.

Then, for every nonempty open set D in Ω, there exist two positive constants C_1, C_2 depending only on D and σ such that

$$\inf_{X_{\xi,D}} \Psi_F \leq C_1 \xi^p - C_2 F(\xi) + |D| \sup_{t \in [0, \inf \Lambda_F]} |F(t)| \tag{2.1}$$

Proof. We consider the case in which $\inf_{[0,\xi]} F < 0$ as, otherwise, the proof is similar and simpler. Moreover, note that, without loss of generality, the number σ can be chosen in $]0, 1[$. Choose $x_0 \in D$ and fix $r, R > 0$, with $R > r$, such that $\overline{B}(x_0, R) \subset D$. From $i)$ one has $int(F_\xi^+) \neq \emptyset$. Now, let $\varepsilon \in]0, 1[$. Note that the connected components of the compact set $(F_\xi^-)_\varepsilon$ are intervals of length at least 2ε. Therefore, $(F_\xi^-)_\varepsilon$ is union of a finite number of pairwise disjoint compact intervals I_l with $l = 1, .., m$, namely

$$(F_\xi^-)_\varepsilon = \cup_{l=1}^m I_l.$$

As a consequence, one has

$$int(F_\xi^+) \supset]0, \xi[\setminus \cup_{l=1}^m I_l = \cup_{\alpha=0}^L]a_{2\alpha+1}, a_{2\alpha}[= int(F_\xi^+) \setminus (F_\xi^-)_\varepsilon \tag{2.2}$$

where a_α is a finite decreasing sequence of positive real numbers. This sequence, of course, depends on ε. However, for brevity reasons, we do not explicitly indicate this dependence. Now, observe that a_{2L+1} is nonincreasing with respect to ε and so we can consider the following limit

$$\lim_{\varepsilon \to 0^+} a_{2L+1} \overset{def}{=} \rho.$$

Let us to show that,

$$\rho \leq \inf \Lambda_F. \tag{2.3}$$

Indeed, arguing by contradiction, assume that

$$\rho > \inf \Lambda_F.$$

Then, it should exist $\tau \in \Lambda_F$ such that $\tau \leq \rho$. In particular $F(t) \leq 0$ for all $t \in [0, \tau]$ and, consequently, $\int_{F_\tau^+} F(t)dt = 0$ which is absurd if $\tau \in \Lambda_F$. Thus, if we fix any $\eta > 0$, we can choose ε small enough in order that

$$a_{2L+1} < \inf \Lambda_F + \eta. \tag{2.4}$$

Moreover, since

$$\lim_{\varepsilon \to 0^+} \int_{(F_\xi^-)_\varepsilon \cap F_\xi^+} F(t)dt = 0,$$

we can also assume, choosing ε smaller if necessary, that

$$\frac{\int_{int(F_\xi^+)\setminus(F_\xi^-)_\varepsilon}}{\int_{(F_\xi^-)_\varepsilon \cap F_\xi^+} F(t)dt} > 1, \tag{2.5}$$

and, thanks to condition i),

$$\xi \notin (F_\xi^-)_\varepsilon. \tag{2.6}$$

In particular, from (2.6) we have $\xi = a_0$. Moreover, from (2.2) and (2.5) one has

$$\frac{\sum_{\alpha=0}^{L} \int_{a_{2\alpha+1}}^{a_{2\alpha}} F(t)dt}{\int_{(F_\xi^-)_\varepsilon \cap F_\xi^+} F(t)dt} > 1. \tag{2.7}$$

Let us put

$$\eta_1 = \frac{R-r}{a_0} \quad \text{and} \quad \eta_2 = \frac{\eta_1}{4} \cdot \min\left\{ \left(\frac{r}{R}\right)^{N-1}, \frac{r^{N-1}\sigma}{R^{N-1} - \sigma r^{N-1}} \right\}. \tag{2.8}$$

Define the following function

$$u_\xi(x) = \begin{cases} a_0 & \text{if } x \in B(x_0, r) \\ \dfrac{a_\alpha - a_{\alpha+1}}{R_{\alpha+1} - R_\alpha}(R_{\alpha+1} - |x - x_0|) + a_{\alpha+1} & \text{if } x \in B(x_0, R_{\alpha+1}) \setminus \overline{B}(x_0, R_\alpha) \\ & \text{and } \alpha = 0, .., 2L \\ \dfrac{a_{2L+1}}{R - R_{2L+1}}(R - |x - x_0|) & \text{if } a_{2L+1} \neq 0 \text{ and} \\ & x \in \overline{B}(x_0, R) \setminus \overline{B}(x_0, R_{2L+1}) \\ 0 \quad \text{otherwise} & \end{cases}$$

where

$$\begin{aligned} R_{2\alpha+1} &= R_{2\alpha} + \eta_1(a_{2\alpha} - a_{2\alpha+1}) \text{ for all } \alpha = 0, .., L \text{ and} \\ R_{2\alpha} &= R_{2\alpha-1} + \eta_2(a_{2\alpha-1} - a_{2\alpha}) \text{ for all } \alpha = 1, .., L \end{aligned} \tag{2.9}$$

with $R_0 = r$. Note that u_ξ is a Lipschitz function in Ω with $u_{\xi|\partial\Omega} = 0$, hence $u_\xi \in W_0^{1,p}(\Omega)$. Moreover, if we put $\mu = 0$ if $a_{2L+1} = 0$ and

$$\mu = \omega_N \left(\frac{a_{2L+1}}{R - R_{2L+1}} \right)^p (R^N - R_{2L+1}^N)$$

if $a_{2L+1} \neq 0$, taking in mind that $\eta_2 \leq \eta_1$ and in view of (2.8), one has

$$\int_\Omega |\nabla u_\xi|^p dx = \sum_{\alpha=0}^{2L} \omega_N \left(\frac{a_\alpha - a_{\alpha+1}}{R_{\alpha+1} - R_\alpha} \right)^p (R_{\alpha+1}^N - R_\alpha^N) + \mu$$

$$= \frac{\omega_N}{\eta_2^{p-1}} \sum_{\alpha=0}^{2L} (a_\alpha - a_{\alpha+1}) \sum_{i=0}^{N-1} R_{\alpha+1}^{N-1-i} R_\alpha^i + \mu$$

$$\leq N\omega_N R^{N-1} \left[\frac{4}{\min\left\{ \left(\frac{r}{R}\right)^{N-1}, \frac{r^{N-1}\sigma}{R^{N-1} - \sigma r^{N-1}} \right\}} \right]^{p-1} \frac{a_0^{p-1}}{(R-r)^{p-1}} (a_0 - a_{2L+1}) + \mu. \quad (2.10)$$

Since from (2.9) it follows $R_{2L+1} \leq R_0 + \eta_1(a_0 - a_{2L+1})$, the following estimate holds

$$\mu \leq N\omega_N R^{N-1} \frac{a_{2L+1}}{(R-r)^{p-1}} a_0^{p-1}. \quad (2.11)$$

Therefore, from (2.10) and (2.11), there exists a constant C depending only on Ω and ρ such that (recall that $a_0 = \xi$)

$$\int_\Omega |\nabla u_\xi|^p dx \leq C\xi^p \quad (2.12)$$

Moreover, since $0 \leq u_\xi(x) \leq \xi$ for all $x \in D$, we finally infer $u_\xi \in X_{\xi,D}$.

Put, for simplicity,

$$D_\alpha = B(x_0, R_{\alpha+1}) \setminus \overline{B}(x_0, R_\alpha)$$

for all $\alpha = 0, .., 2L$, as well as

$$\gamma_\alpha(t) = \frac{a_\alpha - a_{\alpha+1}}{R_{\alpha+1} - R_\alpha}(R_{\alpha+1} - t) + a_{\alpha+1}$$

for all $\alpha = 0, .., 2L$ and $t \in \mathbb{R}$. Finally, put $\delta = 0$ if $a_{2L+1} = 0$ and

$$\delta = \int_{B(x_0,R)\setminus B(x_0,R_{2L+1})} F\left(\frac{a_{2L+1}}{R - R_{2L+1}} (R - |x - x_0|) \right) dx$$

if $a_{2L+1} > 0$.

An upper estimate for $|\delta|$ in the case $a_{2L+1} > 0$ can be obtained noticing that, by (2.4), one gets

$$|\delta| \leq \omega_N \int_0^{a_{2L+1}} \left(R - \frac{R - R_{2L+1}}{a_{2L+1}}\right)^{N-1} \frac{R - R_{2L+1}}{a_{2L+1}} |F(t)| dt$$
$$\leq \frac{\omega_N R^N}{a_{2L+1}} \int_0^{a_{2L+1}} |F(t)| dt \leq |D| \sup_{t \in [0, \inf \Lambda_F + \eta]} |F(t)|. \quad (2.13)$$

Now, observe that (2.2) and (2.6) imply

$$\bigcup_{\alpha=1}^{L}]a_{2\alpha}, a_{2\alpha-1}[= \left((F_\xi^-)_\varepsilon \cap F_\xi^+\right) \cup F_\xi^- \quad (2.14)$$

and

$$\{a_0\} \cup ((F_\xi^-)_\varepsilon \cap F_\xi^+) \cup \bigcup_{\alpha=0}^{L}]a_{2\alpha+1}, a_{2\alpha}[= F_\xi^+. \quad (2.15)$$

Thus, if we put

$$I_1 = \sum_{\alpha=0}^{L} \int_{a_{2\alpha+1}}^{a_{2\alpha}} F(t) dt, \quad I_2 = \int_{(F_\xi^-)_\varepsilon \cap F_\xi^+} F(t) dt$$

and

$$I_+ = \int_{F_\xi^+} F(t) dt, \quad I_- = \int_{F_\xi^-} F(t) dt$$

we have

$$I_+ = I_1 + I_2.$$

Moreover, by assumption ii), we also have

$$I_- > -\frac{1}{\sigma} I_+.$$

With this in mind and in view of (2.12), (2.14), (2.15), we get

$$
\begin{aligned}
&\Psi_F(u_\xi) \leq \\
&\frac{C}{p}\xi^p - \int_\Omega F(u_\xi) = \frac{C}{p}\xi^p - \sum_{\alpha=0}^{2L}\int_{D_\alpha} F(\gamma_\alpha(|x-x_0|))dx - \int_{B(x_0,r)} F(\xi)dx - \delta \\
&= \frac{C}{p}\xi^p - \omega_N r^N F(\xi) - \delta - \\
&\omega_N \sum_{\alpha=0}^{2L}\int_{a_{\alpha+1}}^{a_\alpha}\left[R_{\alpha+1} - \frac{R_{\alpha+1}-R_\alpha}{a_\alpha - a_{\alpha+1}}(t-a_{\alpha+1})\right]^{N-1}\frac{R_{\alpha+1}-R_\alpha}{a_\alpha - a_{\alpha+1}}F(t)dt = \\
&\frac{C}{p}\xi^p - \omega_N r^N F(\xi) - \delta - \omega_N \sum_{\alpha=0}^{L}\int_{a_{2\alpha+1}}^{a_{2\alpha}}[R_{2\alpha+1} - \eta_1(t-a_{2\alpha+1})]^{N-1}\eta_1 F(t)dt + \\
&\sum_{\alpha=1}^{L}\int_{a_{2\alpha}}^{a_{2\alpha-1}}[R_{2\alpha} - \eta_2(t-a_{2\alpha})]^{N-1}\eta_2 F(t)dt \leq \\
&\frac{C}{p}\xi^p - \omega_N r^N F(\xi) + |\delta| - \omega_N\left(\eta_1 r^{N-1}I_1 + \eta_2 r^{N-1}I_2 + \eta_2 R^{N-1}I_-\right) \leq \\
&\frac{C}{p}\xi^p - \omega_N r^N F(\xi) + |\delta| - \omega_N\left(\eta_1 r^{N-1}I_1 + \eta_2 r^{N-1}I_2 - \eta_2\frac{R^{N-1}}{\sigma}(I_1+I_2)\right) = \\
&\frac{C}{p}\xi^p - \omega_N r^N F(\xi) + |\delta| - \\
&\omega_N\left(\left(\eta_1 r^{N-1} - \eta_2\frac{R^{N-1}}{\sigma}\right)I_1 - \eta_2\left(r^{N-1} - \frac{R^{N-1}}{\sigma}\right)I_2\right)
\end{aligned}
$$

At this point, thanks to the choice of η_2 (see (2.8)) and inequality (2.7), one can easily check that

$$\left(\eta_1 r^{N-1} - \eta_2\frac{R^{N-1}}{\sigma}\right)I_1 - \eta_2\left(r^{N-1} - \frac{R^{N-1}}{\sigma}\right)I_2 > 0.$$

Therefore, by the previous inequalities and (2.13), it follows

$$\Psi_F(u_\xi) \leq \frac{C}{p}\xi^p - \frac{\omega_N r^N}{2}F(\xi) + |D|\sup_{t\in[0,\inf\Lambda_F+\eta]}|F(t)|$$

Then, since $u_\xi \in X_\xi$, by the arbitrariness of η conclusion follows. □

Theorem 2.1 can now be applied to study the behavior of Ψ_F near 0 and near ∞. To this end, we have the following two Corollaries

Corollary 2.2. *Let $F:\mathbb{R}\to\mathbb{R}$ be a continuous function with $F(0)=0$. Assume that there exists a sequence $\{\xi_n\}$ in $]0,+\infty[$ with $\lim_{n\to+\infty}\xi_n=0$ and a positive number σ such that*

$$j)\quad \limsup_{n\to+\infty}\frac{F(\xi_n)}{\xi_n^p}=+\infty;$$

$$jj)\quad \int_{F^+_{\xi_n}}F(t)dt\geq-\sigma\int_{F^-_{\xi_n}}F(t)dt \quad \textit{for all } n\in\mathbb{N}.$$

Then, for every nonempty open set D in Ω, up to subsequence of $\{\xi_n\}$, one has

$$\inf\nolimits_{X_{\xi_n,D}}\Psi_F<0 \textit{ for all } n\in\mathbb{N} \quad \textit{and} \quad \lim_{n\to+\infty}\inf\nolimits_{X_{\xi_n,D}}\Psi_F=0.$$

Proof. By assumption $j)$, up to a subsequence, we can suppose $F(\xi_n)>0$ for all $n\in\mathbb{N}$. From this, replacing σ with $\frac{\sigma}{2}$ if necessary, by assumption $jj)$ we have that, for all $n\in\mathbb{N}$, the sets

$$\Lambda^n_F=\left\{\tau\in[0,\xi_n]:\ \int_{F^+_\tau}F(t)dt>-\sigma\int_{F^-_\tau}F(t)dt\right\}$$

are nonempty with $\sup\Lambda^n_F=\xi_n$. This fact jointly to $\lim_{n\to+\infty}\xi_n=0$ imply $\inf\Lambda^n_F=0$ for all $n\in\mathbb{N}$. Therefore, thanks to Theorem 2.1, there exist two positive constants C_1,C_2 depending only on σ and D such that

$$\inf_{X_{\xi_n,D}}\Psi_F\leq C_1\xi_n^p-C_2F(\xi_n)$$

for all $n\in\mathbb{N}$. Then, in view of $j)$, conclusion follows. □

Corollary 2.3. *Let $F:\mathbb{R}\to\mathbb{R}$ be a continuous function with $F(0)=0$. Assume that there exists a sequence $\{\xi_n\}$ in $]0,+\infty[$ with $\lim_{n\to+\infty}\xi_n=+\infty$ and a positive number σ such that*

$$k)\quad \limsup_{n\to+\infty}\frac{F(\xi_n)}{\xi_n^p}=+\infty;$$

$$kk)\quad \int_{F^+_{\xi_n}}F(t)dt\geq-\sigma\int_{F^-_{\xi_n}}F(t)dt \quad \textit{for all } n\in\mathbb{N}.$$

Then, for every nonempty open set D in Ω, up to subsequence of $\{\xi_n\}$, one has

$$\lim_{n\to+\infty}\inf\nolimits_{X_{\xi_n,D}}\Psi_F=-\infty.$$

Proof. As in Corollary 2.2 we are able to apply Theorem 2.1. In this case, there exist positive constants C_1,C_2 depending only on σ and D such that, up to a subsequence of $\{\xi_n\}$, one has

$$\inf_{X_{\xi_n,D}}\Psi_F\leq C_1\xi_n^p-C_2F(\xi_n)+|D|\sup_{t\in[0,\inf\Lambda^n_F]}|F(t)|.$$

for all $n\in\mathbb{N}$. Since the sequence $\{\inf\Lambda^n_F\}$ is bounded, conclusion follows by assumption $k)$. □

We are now in position to state and prove two results concerning the existence of infinitely many weak solutions for some elliptic boundary value problems. The first one is a variant of Theorem 2.1 of [2].

Theorem 2.4. *Let* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function such that* $f(x, 0) = 0$ *for a.e.* $x \in \Omega$ *and let* D *be a nonempty open bounded set in* Ω. *Let* $F : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $F(0) = 0$ *and*

$$F(\xi) \leq \operatorname*{ess\,inf}_{x \in D} \int_0^\xi f(x,t)dt \quad for \quad every \quad \xi \in \mathbb{R}.$$

Assume that there exist a nonempty open set D *in* Ω, *two positive numbers* σ, s *and two sequences* $\{\xi_n\}, \{t_n\}$ *in* $]0, +\infty[$ *with* $\lim_{n\to+\infty} \xi_n = \lim_{n\to+\infty} t_n = 0$ *such that*

a) $\sup_{t\in[0,s]} |f(\cdot, s)| \in L^m(\Omega)$ *where* $m \geq 1$ *with* $m > \frac{Np}{Np-N+p}$ *if* $p \leq N$;
b) $f(x, t_n) \leq 0$ *for all* $n \in \mathbb{N}$ *and for almost all* $x \in \Omega$;
c) $\limsup_{n\to+\infty} \dfrac{F(\xi_n)}{\xi_n^p} = +\infty$;
d) $\int_{F^+_{\xi_n}} F(t)dt \geq -\sigma \int_{F^-_{\xi_n}} F(t)dt \quad$ *for all* $n \in \mathbb{N}$;

Then, the following Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in} \quad \Omega \\ u_{|\Omega} = 0 \end{cases} \tag{P_f}$$

admits a sequence of pairwise distinct nonnegative weak solutions $\{u_n\}$ *in* $W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ *satisfying*

$$\lim_{n\to+\infty} \|u_n\| = \lim_{n\to+\infty} \max_{x\in\overline{\Omega}} |u_n(x)| = 0.$$

Proof. Up to subsequences, we can suppose $t_n, \xi_n \in]0, s]$ and $\xi_n \leq t_n$ for all $n \in \mathbb{N}$. After that, fix $n \in \mathbb{N}$ and put

$$\Psi_n(u) = \frac{1}{p}\|u\|^p - \int_\Omega \left(\int_0^{u(x)} f_n(x,t)dt \right) dx$$

for all $u \in W_0^{1,p}(\Omega)$, where

$$f_n(x,t) = \begin{cases} f(x,t) & \text{if} \quad (x,t) \in \Omega \times [0, t_n] \\ f(x,t_n) & \text{if} \quad (x,t) \in \Omega \times]t_n, +\infty[\\ 0 & \text{if} \quad (x,t) \in \Omega \times]-\infty, 0[\end{cases}.$$

By standard results, condition $a)$ implies that functional Ψ_n is sequentially weakly lower semicontinuous and Gateâux differentiable in $W_0^{1,p}(\Omega)$. Moreover, one has

$$\lim_{\|u\|\to+\infty} \Psi_n(u) = +\infty.$$

Therefore, Ψ_n achieves its global minimum at a point $u_n \in W_0^{1,p}(\Omega)$. Thus, u_n is a weak solution of the problem

$$\begin{cases} -\Delta_p u = f_n(x,u) \quad \text{in} \quad \Omega \\ u_{|\partial\Omega} = 0 \end{cases}.$$

By regularity results ([5]), one has $u \in C^1(\overline{\Omega})$. Moreover, by the Maximum Principle we easily infer that $0 \le u_n(x) \le t_n$ (see for instance Lemma 4.1 of [8]). Consequently, u_n is actually a weak solution of problem (P_f). Finally, note that

$$\Psi_n(u_n) = \inf_{W_0^{1,p}(\Omega)} \Psi_n \le \inf_{X_{\xi_n,D}} \Psi_n \le \inf_{X_{\xi_n,D}} \Psi_F \tag{2.16}$$

where

$$\Psi_F(u) = \frac{1}{p}\|u\|^p - \int_\Omega F(u(x))dx, \qquad u \in W_0^{1,p}(\Omega).$$

At this point, assumptions c), d) allows us to apply Corollary 2.2 and this yields, thanks to (2.16),

$$\Psi_n(u_n) < 0 \qquad \text{for all } n \in \mathbb{N} \qquad \text{and} \qquad \lim_{n\to+\infty} \Psi_n(u_n) = 0.$$

Then, conclusion easily follows. □

Our second application is a result analogous to Theorem 2.4 which states the existence of a norm-unbounded sequence of pairwise distinct weak solutions for problem (P_f). The proof is practically the same of Theorem 2.4 and so it is omitted.

Theorem 2.5. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(x,0) = 0$ for a.e. $x \in \Omega$ and let D be a nonempty open bounded set in Ω. Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $F(0) = 0$ and*

$$F(\xi) \le \operatorname*{ess\,inf}_{x\in D} \int_0^\xi f(x,t)dt \quad \textit{for every} \quad \xi \in \mathbb{R}.$$

Assume that there exist a nonempty open set D in Ω, a positive number σ and two sequences $\{\xi_n\}, \{t_n\}$ in $]0,+\infty[$ with $\lim_{n\to+\infty} \xi_n = \lim_{n\to+\infty} t_n = +\infty$ such that

a) $\sup_{t\in[0,s]} |f(\cdot,s)| \in L^m(\Omega)$ *for every $s > 0$, where $m \ge 1$ with $m > \frac{Np}{Np-N+p}$ if $p \le N$;*

b) $f(x,t_n) \le 0$ *for all $n \in \mathbb{N}$ and for almost all $x \in \Omega$;*

c) $\limsup_{n\to+\infty} \dfrac{F(\xi_n)}{\xi_n^p} = +\infty$;

d) $\int_{F_{\xi_n}^+} F(t)dt \ge -\sigma \int_{F_{\xi_n}^-} F(t)dt$ *for all $n \in \mathbb{N}$;*

Then, the following Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in} \quad \Omega \\ u_{|\Omega} = 0 \end{cases} \qquad (P_f)$$

admits a sequence of pairwise distinct nonnegative weak solutions $\{u_n\}$ *in* $W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ *satisfying*

$$\lim_{n\to+\infty} \|u_n\| = +\infty.$$

References

[1] Anello, G., *Existence of infinitely many weak solutions for a Neumann problem*, Nonlinear Anal. **57**(2) (2004), 199–209.

[2] Anello, G., Cordaro, G., *Infinitely many positive solutions for the Neumann problem involving the p-Laplacian*, Colloq. Math. **97**(2) (2003), 221–231.

[3] Anello, G., Cordaro, G., *Infinitely many arbitrarily small positive solutions for the Dirichlet problem involving the p-Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A, **132**(3) (2002) 511–519.

[4] Cammaroto, F., Chinni, A., Di Bella, B., *Infinitely many solutions for the Dirichlet problem involving the p-Laplacian*, Nonlinear Anal., **61**(1-2) (2005), 41–49.

[5] Di Benedetto, E., $C^{1+\alpha}$ *Local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal., **7**(8) (1983), 827-850.

[6] Faraci, F., Kristaly, A., *On an open question of Ricceri concerning a Neumann problem*, Glasg. Math. J., **49** (2007)(2), 189–195.

[7] Saint-Raymond, J., *On the multiplicity of the solutions of the equation* $-\Delta u = \lambda \cdot f(u)$, J. Differential Equations, **180**(1) (2002), 65–88.

[8] Chang, K. C., *Infinite dimensional Morse theory and multiple solution problem*, Birkhäuser, Boston, 1993.

[9] Obersnel, F., Omari, P., *Positive solutions of elliptic problems with locally oscillating nonlinearities*, J. Math. Anal. Appl., **323**(2) (2006), 913–929.

[10] Omari, P., Zanolin, F., *An elliptic problem with arbitrarily small positive solutions*, Electron. J. Diff. Eqns Conf., **5** (2000), 301–308.

[11] Ricceri, B., *Infinitely many solutions of the Neumann problem for elliptic equations involving the p-Laplacian*, Bull. London Math. Soc., **33**(3) (2001), 331–340.

[12] Ricceri, B., *A general variational principle and some of its applications*, J. Comput. Appl. Math., **113** (2000), 401-410.

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A REMARK ON PERTURBED ELLIPTIC NEUMANN PROBLEMS

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Abstract. The aim of this paper is to establish the existence of infinitely many solutions for perturbed eigenvalue elliptic Neumann problems involving the p–Laplacian. To be precise, we show that an appropriate oscillating behaviour of the nonlinear term, even under small perturbations, ensures again the existence of infinitely many solutions.

1. **Introduction**

Very recently in [6], presenting a version of the infinitely many critical points theorem of B. Ricceri (see [12, Theorem 2.5]), the existence of an unbounded sequence of weak solutions for a Sturm-Liouville problem, having discontinuous nonlinearities, has been established. In a such approach, an appropriate oscillating behavior of the nonlinear term either at infinity or at zero is required. This type of methodology has been used then in several works in order to obtain existence results for different kinds of problems (see, for instance, [2, 3, 5, 7, 8, 9, 10, 11]).

In particular, in [3, Theorem 3], by following this approach, it was proved the existence of infinitely many solutions for the following elliptic Neumann problem

$$\begin{cases} -\Delta_p u + q(x)|u|^{p-2}u = \lambda h(x,u) & \text{in } \Omega \\ \partial u/\partial\nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbf{R}^N$ be a bounded open set with smooth boundary $\partial\Omega$, ν is the outer unit normal to $\partial\Omega$, $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p–Laplacian, $p > N$, $q \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega q > 0$, $h : \Omega \times \mathbf{R} \to \mathbf{R}$ is a continuous function and λ is a positive real parameter. For reader's convenience, Theorem 3 of [3] is here recalled.

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Theorem 1.1. *Let* $h : \Omega \times \mathbf{R} \to \mathbf{R}$ *be an* L^1*–Carathéodory function. Put* $H(x, \xi) := \int_0^\xi h(x, t)\, dt$ *for all* $(x, \xi) \in \Omega \times \mathbf{R}$ *and assume that*

$$\liminf_{\xi\to+\infty} \frac{\int_\Omega \max_{|t|\le\xi} H(x,t)dx}{\xi^p} < \frac{1}{c^p\|q\|_1} \limsup_{\xi\to+\infty} \frac{\int_\Omega H(x,\xi)dx}{\xi^p}, \tag{1.1}$$

where

$$c = \sup_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\sup_{x\in\overline{\Omega}} |u(x)|}{\left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega q(x)|u(x)|^p dx\right)^{\frac{1}{p}}}, \tag{1.2}$$

and $\|q\|_1 := \int_\Omega q(x)dx$.

Then, for each

$$\lambda \in \left] \frac{\|q\|_1}{p \limsup_{\xi\to+\infty} \frac{\int_\Omega H(x,\xi)dx}{\xi^p}}, \frac{1}{pc^p \liminf_{\xi\to+\infty} \frac{\int_\Omega \max_{|t|\le\xi} H(x,t)dx}{\xi^p}} \right[,$$

problem (P_λ) *possesses an unbounded sequence of weak solutions in* $W^{1,p}(\Omega)$.

In recent years, multiplicity results for Neumann problems have widely been investigated (see [13] and [14]) as well as the existence of three solutions for perturbed Neumann problems has been obtained (see [15] and [4]). The aim of this note is to point out, as a consequence of Theorem 1.1, existence results of infinitely many solutions for perturbed Neumann problems. To be precise, we prove the existence of infinitely many weak solutions for the following perturbed Neumann problem

$$\begin{cases} -\Delta_p u + q(x)|u|^{p-2}u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega \\ \partial u/\partial\nu = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{$N^{f,g}_{\lambda,\mu}$}$$

where $f, g : \Omega \times \mathbf{R} \to R$ are two L^1–Carathéodory functions and λ, μ are real parameters.

Precisely, requiring that the nonlinear term f has a suitable oscillating behavior at infinity, in Theorem 2.1, we establish the existence of a precise interval Λ such that for every $\lambda \in \Lambda$ and every L^1–Carathéodory function g which satisfies a certain growth at infinity, choosing μ sufficiently small, the perturbed problem $(N^{f,g}_{\lambda,\mu})$ admits an unbounded sequence of weak solutions in $W^{1,p}(\Omega)$.

As an example, we present here a special case of our main result.

Theorem 1.2. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous nonnegative function. Put $F(\xi) := \int_0^\xi f(t)\,dt$ for all $\xi \in \mathbf{R}$, and assume that*

$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = 0, \quad \text{and} \quad \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = +\infty.$$

Then, for every nonnegative continuous function $g : \Omega \times \mathbf{R} \to \mathbf{R}$ such that

$$g'_\infty) \;\; G^\star_\infty := \lim_{\xi \to +\infty} \frac{\int_0^1 \left(\int_0^\xi g(x,t)dt \right) dx}{\xi^2} < +\infty,$$

and for every $\mu \in \left[0, \frac{1}{4G^\star_\infty}\right[$, the following problem

$$\begin{cases} -u'' + u = f(u) + \mu g(x,u) & \text{in} \quad]0,1[\\ u'(0) = u'(1) = 0, \end{cases} \qquad (N^{f,g}_\mu)$$

admits a sequence of pairwise distinct positive classical solutions.

An analogous result (see Theorem 2.6) can be obtained if we replace the oscillating behavior condition at infinity, by a similar one at zero. In this setting, a sequence of pairwise distinct non-zero solutions which converges to zero is achieved.

The note is arranged as follows. In Section 2 we show our abstract results, while in Section 3 a concrete example of application is given.

2. Main results

We recall here some basic definitions and notations. A function $h : \Omega \times \mathbf{R} \to \mathbf{R}$ is called an L^1-Carathéodory function if $x \mapsto h(x,t)$ is measurable for all $t \in \mathbf{R}$, $t \mapsto h(x,t)$ is continuous for almost every $x \in \Omega$ and for all $M > 0$ one has $\sup_{|t| \le M} |h(x,t)| \in L^1(\Omega)$. Clearly, if h is continuous in $\overline{\Omega} \times \mathbf{R}$, then it is L^1-Carathéodory.

Let $W^{1,p}(\Omega)$ be the usual Sobolev space endowed with the norm

$$\|u\| = \Big(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega q(x)|u(x)|^p dx \Big)^{\frac{1}{p}},$$

that is equivalent to the usual one.

A *weak solution* of the problem $(N^{f,g}_{\lambda,\mu})$ is any $u \in W^{1,p}(\Omega)$, such that

$$\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) dx + \int_\Omega q(x)|u(x)|^{p-2}u(x)v(x)dx +$$

$$-\lambda \int_\Omega f(x,u(x))v(x)dx - \mu \int_\Omega g(x,u(x))v(x)dx = 0, \quad \forall v \in W^{1,p}(\Omega).$$

Set $F(x,\xi) := \int_0^\xi f(x,t)dt$ for every $(x,\xi) \in \Omega \times \mathbf{R}$, and

$$\lambda_1 := \frac{\|q\|_1}{p \limsup_{\xi\to+\infty} \dfrac{\int_\Omega F(x,\xi)dx}{\xi^p}}, \quad \lambda_2 := \frac{1}{pc^p \liminf_{\xi\to+\infty} \dfrac{\int_\Omega \max_{|t|\le\xi} F(x,t)dx}{\xi^p}}. \tag{2.1}$$

Our result reads as follows

Theorem 2.1. *Assume that*

$$\liminf_{\xi\to+\infty} \frac{\int_\Omega \max_{|t|\le\xi} F(x,t)dx}{\xi^p} < \frac{1}{c^p\|q\|_1} \limsup_{\xi\to+\infty} \frac{\int_\Omega F(x,\xi)dx}{\xi^p}, \tag{2.2}$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, for every L^1-Carathédory function $g : \Omega\times\mathbf{R} \to \mathbf{R}$ whose potential $G(x,\xi) := \int_0^\xi g(x,t)\ dt$, $\forall\ (x,\xi) \in \Omega \times \mathbf{R}$, is a nonnegative function satisfying

$$g_\infty)\ \ G_\infty := \lim_{\xi\to+\infty} \frac{\int_\Omega \max_{|t|\le\xi} G(x,t)dx}{\xi^p} < +\infty,$$

and for every $\mu \in [0, \mu_{g,\lambda}[$, where

$$\mu_{g,\lambda} := \frac{1}{pc^pG_\infty}\left(1 - \lambda pc^p \liminf_{\xi\to+\infty} \frac{\int_\Omega \max_{|t|\le\xi} F(x,t)dx}{\xi^p}\right),$$

the problem $(N^{f,g}_{\lambda,\mu})$ admits a sequence of weak solutions which is unbounded in $W^{1,p}(\Omega)$.

Proof. Our aim is to apply Theorem 1.1. To this end, fix $\overline{\lambda} \in]\lambda_1, \lambda_2[$ and let g be a function satisfies hypothesis $g_\infty)$. In the non-perturbed case, i.e. $\mu = 0$, the thesis is trivial. Owing to $\overline{\lambda} < \lambda_2$, one has

$$\mu_{g,\overline{\lambda}} := \frac{1}{pc^pG_\infty}\left(1 - \overline{\lambda} pc^p \liminf_{\xi\to+\infty} \frac{\int_\Omega \max_{|t|\le\xi} F(x,t)dx}{\xi^p}\right) > 0.$$

Take $0 < \overline{\mu} < \mu_{g,\overline{\lambda}}$ and put

$$\eta_1 := \lambda_1 = \frac{\|q\|_1}{p \limsup_{\xi\to+\infty} \dfrac{\int_\Omega F(x,\xi)dx}{\xi^p}}, \quad \eta_2 := \frac{1}{pc^p \liminf_{\xi\to+\infty} \dfrac{\int_\Omega \max_{|t|\le\xi} F(x,t)dx}{\xi^p} + pc^p\dfrac{\overline{\mu}}{\overline{\lambda}}G_\infty}.$$

If $G_\infty = 0$ clearly one has $\eta_1 = \lambda_1$, $\eta_2 = \lambda_2$ and

$$\overline{\lambda} \in \Lambda^\star :=]\eta_1, \eta_2[.$$

If $G_\infty \neq 0$, from $\overline{\mu} < \mu_{g,\overline{\lambda}}$, it follows that

$$\overline{\lambda} p c^p \liminf_{\xi \to +\infty} \frac{\displaystyle\int_\Omega \max_{|t| \leq \xi} F(x,t)dx}{\xi^p} + p c^p G_\infty \overline{\mu} < 1,$$

that means

$$\overline{\lambda} < \frac{1}{p c^p \liminf_{\xi \to +\infty} \frac{\displaystyle\int_\Omega \max_{|t| \leq \xi} F(x,t)dx}{\xi^p} + p c^p \frac{\overline{\mu}}{\overline{\lambda}} G_\infty} = \eta_2.$$

On the other hand, by our hypothesis, $\overline{\lambda} > \eta_1$.
Hence, one has

$$\overline{\lambda} \in \Lambda^\star :=]\eta_1, \eta_2[.$$

Now, put

$$H(x,\xi) := F(x,\xi) + \frac{\overline{\mu}}{\overline{\lambda}} G(x,\xi),$$

for every $x \in \Omega$ and $\xi \in \mathbf{R}$.
Then

$$\frac{\displaystyle\int_\Omega \max_{|t| \leq \xi} H(x,t)dx}{\xi^p} \leq \frac{\displaystyle\int_\Omega \max_{|t| \leq \xi} F(x,t)dx}{\xi^p} + \frac{\overline{\mu}}{\overline{\lambda}} \frac{\displaystyle\int_\Omega \max_{|t| \leq \xi} G(x,t)dx}{\xi^p},$$

and taking into account hypothesis g_∞), it follows that

$$\liminf_{\xi \to +\infty} \frac{\displaystyle\int_\Omega \max_{|t| \leq \xi} H(x,t)dx}{\xi^p} \leq \liminf_{\xi \to +\infty} \frac{\displaystyle\int_\Omega \max_{|t| \leq \xi} F(x,t)dx}{\xi^p} + \frac{\overline{\mu}}{\overline{\lambda}} G_\infty. \tag{2.3}$$

Moreover, taking into account that the potential G is a nonnegative function, we obtain

$$\limsup_{\xi \to +\infty} \frac{\displaystyle\int_\Omega H(x,\xi)dx}{\xi^p} \geq \limsup_{\xi \to +\infty} \frac{\displaystyle\int_\Omega F(x,\xi)dx}{\xi^p}. \tag{2.4}$$

Conditions (2.3) and (2.4) yield

$$\overline{\lambda} \in \Lambda^\star \subseteq \left] \frac{\|q\|_1}{p \limsup_{\xi \to +\infty} \frac{\displaystyle\int_\Omega H(x,\xi)dx}{\xi^p}}, \frac{1}{p c^p \liminf_{\xi \to +\infty} \frac{\displaystyle\int_\Omega \max_{|t| \leq \xi} H(x,t)dx}{\xi^p}} \right[. \tag{2.5}$$

So the conclusion follows at once from Theorem 1.1 observing that, from (2.5), condition (1.1) is clearly verified. The proof is complete. □

Remark 2.2. If Ω is convex, an explicit upper bound for the constant c is

$$c \leq 2^{\frac{p-1}{p}} \max \left\{ \frac{1}{\|q\|_1^{\frac{1}{p}}}, \frac{d}{N^{\frac{1}{p}}} \left(\frac{p-1}{p-N} \operatorname{meas}(\Omega) \right)^{\frac{p-1}{p}} \frac{\|q\|_\infty}{\|q\|_1} \right\},$$

where "meas(Ω)" denotes the Lebesgue measure of the set Ω, $d := \operatorname{diam}(\Omega)$ and $\|q\|_\infty := \max_{x\in\overline{\Omega}} |u(x)|$. See, for instance, [1, Remark 1].

Remark 2.3. If

$$\liminf_{\xi\to+\infty} \frac{\int_\Omega \max_{|t|\leq\xi} F(x,t)dx}{\xi^p} = 0, \quad \text{and} \quad \limsup_{\xi\to+\infty} \frac{\int_\Omega F(x,\xi)dx}{\xi^p} = +\infty,$$

clearly, hypothesis (2.2) is verified and Theorem 2.1 guarantees the existence of infinitely many weak solutions for problem $(N^{f,g}_{\lambda,\mu})$, for every pair $(\lambda,\mu) \in D$, where

$$D :=]0,+\infty[\times \left[0, \frac{1}{pc^p G_\infty}\right[.$$

Moreover, under the assumption $G_\infty = 0$, the main result ensures the existence of infinitely many weak solutions for the problem $(N^{f,g}_{\lambda,\mu})$, for every $\mu > 0$.

Remark 2.4. Assuming that, in Theorem 2.1, the L^1–Carathéodory function f is nonnegative, hypothesis (2.2) can be written as follows

$$\liminf_{\xi\to+\infty} \frac{\int_\Omega F(x,\xi)dx}{\xi^p} < \frac{1}{c^p\|q\|_1} \limsup_{\xi\to+\infty} \frac{\int_\Omega F(x,\xi)dx}{\xi^p},$$

as well as

$$\mu_{g,\lambda} := \frac{1}{pc^p G_\infty} \left(1 - \lambda p c^p \liminf_{\xi\to+\infty} \frac{\int_\Omega F(x,\xi)dx}{\xi^p} \right).$$

Moreover if in addition, we consider the autonomous case, condition (2.2) assumes the following form

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} < \frac{1}{c^p\|q\|_1} \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p}.$$

Further, in this setting, one has

$$\lambda_1 := \frac{\|q\|_1}{p \operatorname{meas}(\Omega) \limsup\limits_{\xi\to+\infty} \frac{F(\xi)}{\xi^p}}, \quad \lambda_2 := \frac{1}{pc^p \operatorname{meas}(\Omega) \liminf\limits_{\xi\to+\infty} \frac{F(\xi)}{\xi^p}},$$

and

$$\mu_{g,\lambda} := \frac{1}{pc^p G_\infty}\left(1 - \lambda p \operatorname{meas}(\Omega) c^p \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^p}\right).$$

Remark 2.5. We point out that Theorem 1.2 in Introduction is a particular case of Theorem 2.1 taking into account Remarks 2.3 and 2.4.

Replacing the conditions at infinity of the potential by a similar at zero, the same result holds and, in addition, the sequence of pairwise distinct solutions uniformly converges to zero. Precisely, set

$$\lambda_1^* := \frac{\|q\|_1}{p \limsup\limits_{\xi\to 0^+} \frac{\int_\Omega F(x,\xi)dx}{\xi^p}}, \quad \lambda_2^* := \frac{1}{pc^p \liminf\limits_{\xi\to 0^+} \frac{\int_\Omega \max_{|t|\le\xi} F(x,t)dx}{\xi^p}}. \tag{2.6}$$

From Theorem 4 of [3], arguing as in the proof of Theorem 2.1, we obtain the following result.

Theorem 2.6. *Assume that*

$$\liminf_{\xi\to 0^+} \frac{\int_\Omega \max_{|t|\le\xi} F(x,t)dx}{\xi^p} < \frac{1}{c^p\|q\|_1} \limsup_{\xi\to 0^+} \frac{\int_\Omega F(x,\xi)dx}{\xi^p}. \tag{2.7}$$

Then, for each $\lambda \in]\lambda_1^*, \lambda_2^*[$*, for every* L^1-*Carathédory function* $g : \Omega\times\mathbf{R} \to \mathbf{R}$ *whose potential* $G(x,\xi) := \int_0^\xi g(x,t)\,dt$, $\forall\,(x,\xi) \in \Omega\times\mathbf{R}$*, is a nonnegative function satisfying*

$$g_0)\ \ G_0 := \lim_{\xi\to 0^+} \frac{\int_\Omega \max_{|t|\le\xi} G(x,t)dx}{\xi^p} < +\infty,$$

and for every $\mu \in [0, \mu_{g,\lambda}[$*, where*

$$\mu_{g,\lambda} := \frac{1}{pc^p G_0}\left(1 - \lambda p c^p \liminf_{\xi\to 0^+} \frac{\int_\Omega \max_{|t|\le\xi} F(x,t)dx}{\xi^p}\right),$$

the problem $(N_{\lambda,\mu}^{f,g})$ *possesses a sequence of non-zero weak solutions which strongly converges to* 0 *in* $W^{1,p}(\Omega)$*.*

3. **Application**

Let $g : \Omega \times \mathbf{R} \to \mathbf{R}$ be a continuous and nonnegative function such that

$$\lim_{\xi \to +\infty} \frac{\displaystyle\int_0^1 \left(\int_0^\xi g(x,t)dt \right) dx}{\xi^2} < +\infty.$$

Consider the following Neumann problem

$$\begin{cases} -u'' + u = \lambda f(u) + \mu g(x,u) & \text{in} \quad]0,1[\\ u'(0) = u'(1) = 0, \end{cases} \tag{$N_{\lambda,\mu}^{f,g}$}$$

where $f : \mathbf{R} \to \mathbf{R}$ is defined as follows

$$f(t) := \begin{cases} t\cos^2(\ln(t)) & \text{if } t > 0 \\ \\ 0 & \text{if } t \leq 0. \end{cases}$$

A direct computation ensures that

$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = \frac{2-\sqrt{2}}{8},$$

and

$$\limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = \frac{2+\sqrt{2}}{8}.$$

Moreover,

$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} < \frac{1}{2} \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}.$$

From Theorem 2.1, for each $\lambda \in \Lambda := \left] \frac{4}{2+\sqrt{2}}, \frac{2}{(2-\sqrt{2})} \right[$, and for every

$$0 \leq \mu < \frac{1}{4}\left(1 - \lambda \frac{(2-\sqrt{2})}{2}\right) \left(\lim_{\xi \to +\infty} \frac{\displaystyle\int_0^1 \left(\int_0^\xi g(x,t)dt \right) dx}{\xi^2} \right)^{-1},$$

problem $(N_{\lambda,\mu}^{f,g})$ possesses a sequence of pairwise distinct classical solutions.
For instance, for every $(\lambda, \mu) \in \Lambda \times [0, +\infty[$, the problem

$$\begin{cases} -u'' + u = \lambda f(u) + \mu \dfrac{\sqrt{|u|}}{1+\sqrt{x}} & \text{in} \quad]0,1[\\ u'(0) = u'(1) = 0, \end{cases} \tag{$N_{\lambda,\mu}^{f,g}$}$$

possesses a sequence of pairwise distinct classical solutions.

References

[1] Bonanno, G., Candito, P., *Three solutions to a Neumann problem for elliptic equations involving the p–Laplacian*, Arch. Math. (Basel), **80** (2003), 424–429.

[2] Bonanno, G., Candito, P., *Infinitely many solutions for a class of discrete non-linear boundary value problems,* Appl. Anal., **88** (2009), 605–616.

[3] Bonanno, G., D'Aguì, G., *On the Neumann problem for elliptic equations involving the p–Laplacian*, J. Math. Anal. Appl., **358** (2009), 223–228.

[4] Bonanno, G., D'Aguì, G., *Multiplicity results for a perturbed elliptic Neumann problem*, Abstr. Appl. Anal., **2010** (2010), 1–10.

[5] Bonanno, G., Livrea, R., *Multiple periodic solutions for Hamiltonian systems with not coercive potential*, J. Math. Anal. Appl., **363** (2010), 627–638.

[6] Bonanno, G., Molica Bisci, G., *Infinitely many solutions for a boundary value problem with discontinuous nonlinearities*, Bound. Value Probl., **2009** (2009), 1–20.

[7] Bonanno, G., Molica Bisci, G., *Infinitely many solutions for a Dirichlet problem involving the p–Laplacian*, Proc. Roy. Soc. Edinburgh, Sect. A, **140** (2010), 737–752.

[8] Bonanno, G., Molica Bisci, G., O'Regan, D., *Infinitely many weak solutions for a class of quasilinear elliptic systems*, Math. Comput. Modelling, **52** (2010), 152–160.

[9] Bonanno, G., Molica Bisci, G., Rădulescu, V., *Arbitrarily small weak solutions for a nonlinear eigenvalue problem in Orlicz-Sobolev spaces*, preprint.

[10] Bonanno, G., Tornatore, E., *Infinitely many solutions for a mixed boundary value problem*, Ann. Polon. Math., to appear.

[11] Bonanno, G., Di Bella, B., *Infinitely many solutions for a fourth-order elastic beam equation*, preprint.

[12] Ricceri, B., *A general variational principle and some of its applications*, J. Comput. Appl. Math., **113** (2000), 401–410.

[13] Ricceri, B., *Infinitely many solutions of the Neumann problem for elliptic equations involving the p–Laplacian,* Bull. London Math. Soc., **33** (2001), 331–340.

[14] Ricceri, B., *A multiplicity theorem for the Neumann problem,* Proc. Amer. Math. Soc., **134** (2006), 1117–1124.

[15] Ricceri, B., *A note on the Neumann problem,* Complex Var. Elliptic Equ., **55** (2010), 593–599.

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CRITICAL POINT METHODS IN DEGENERATE ANISOTROPIC PROBLEMS WITH VARIABLE EXPONENT

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Abstract. We work on the anisotropic variable exponent Sobolev spaces and we consider the problem: $-\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{P_+^+ - 2} u = f(x, u)$ in Ω, $u \geq 0$ in Ω and $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary and $\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u)$ is a $\overrightarrow{p}(\cdot)$ - Laplace type operator. Relying on the critical point theory and using the mountain-pass theorem, we prove the existence of a unique nontrivial weak solution for our problem.

1. Introduction

We are interested in discussing the problem:

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{P_+^+ - 2} u = f(x, u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, the operator $\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u)$ is a $\overrightarrow{p}(\cdot)$ - Laplace type operator, $\overrightarrow{p}(x) = (p_1(x), p_2(x), ...p_N(x))$, $b \in L^\infty(\overline{\Omega})$, $P_+^+ = \max_{i \in \{1,...,N\}} \{\sup_{x \in \Omega} p_i(x)\}$ and a_i, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions fulfilling some adequate hypotheses. In order to detail the conditions imposed on the functions involved in our problem we make the following notation. We denote by $A_i : \Omega \times \mathbb{R} \to \mathbb{R}$

$$A_i(x, s) = \int_0^s a_i(x, t)dt \quad \text{for all } i \in \{1, ..., N\},$$

and by $F : \Omega \times \mathbb{R} \to \mathbb{R}$

$$F(x, s) = \int_0^s f(x, t)dt.$$

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We set $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) \,:\, \min_{x\in\overline{\Omega}} p(x) > 1\}$ and we denote, for any $p \in C_+(\overline{\Omega})$,

$$p^+ = \sup_{x\in\Omega} p(x) \qquad \text{and} \qquad p^- = \inf_{x\in\Omega} p(x).$$

For $\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N$, $\overrightarrow{p}(x) = (p_1(x), p_2(x), ...p_N(x))$ with $p_i \in C_+(\overline{\Omega})$, $i \in \{1, ..., N\}$ we denote by $\overrightarrow{P}_+$, $\overrightarrow{P}_- \in \mathbb{R}^N$ the vectors

$$\overrightarrow{P}_+ = (p_1^+, ..., p_N^+), \quad \overrightarrow{P}_- = (p_1^-, ..., p_N^-),$$

and by P_+^+, P_-^+, $P_-^- \in \mathbb{R}^+$ the following:

$$P_+^+ = \max\{p_1^+, ..., p_N^+\}, \quad P_-^+ = \max\{p_1^-, ..., p_N^-\}, \quad P_-^- = \min\{p_1^-, ..., p_N^-\}.$$

We define $P_-^\star \in \mathbb{R}^+$ and $P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^\star = \frac{N}{\sum_{i=1}^N 1/p_i^- - 1}\,, \qquad P_{-,\infty} = \max\{P_-^+, P_-^\star\}\,.$$

Now we can state the conditions satisfied by the functions b, p_i, A_i, a_i, f, for all $i \in \{1, ..., N\}$:

(b) there exists $b_0 > 0$ such that $b(x) \geq b_0$ for all $x \in \Omega$;

(p) $p_i \in C_+(\overline{\Omega})$ is logarithmic Hölder continuous (that is, there exists $M > 0$ such that $|p_i(x) - p_i(y)| \leq -M/\log(|x-y|)$ for all x, $y \in \Omega$ with $|x - y| \leq 1/2$), $p_i(x) < N$ for all $x \in \Omega$ and $\sum_{i=1}^N 1/p_i^- > 1$;

(A1) there exists a positive constant $c_{1,i}$ such that a_i satisfies the growth condition

$$|a_i(x, s)| \leq c_{1,i}(1 + |s|^{p_i(x)-1}),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$;

(A2) the following inequalities hold:

$$|s|^{p_i(x)} \leq a_i(x, s)s \leq p_i(x)\ A_i(x, s),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$;

(A3) a_i is fulfilling

$$(a_i(x, s) - a_i(x, t))(s - t) > 0,$$

for all $x \in \Omega$ and s, $t \in \mathbb{R}$ with $s \neq t$;

(f1) there exist a positive constant c_3 and $q \in C(\overline{\Omega})$ with $1 < P_-^- < P_+^+ < q^- < q^+ < P_-^\star$, such that f satisfies the growth condition

$$|f(x, s)| \leq c_3|s|^{q(x)-1},$$

for all $x \in \Omega$ and $s \in \mathbb{R}$;

(f2) f verifies the Ambrosetti-Rabinowitz type condition: there exists a constant $\mu > P_+^+$ such that for every $x \in \Omega$

$$0 < \mu F(x,s) \leq s f(x,s), \quad \forall s > 0;$$

(f3) f is fulfilling

$$(f(x,s) - f(x,t))(s-t) < 0,$$

for all $x \in \overline{\Omega}$ and $s,\, t \in \mathbb{R}$ with $s \neq t$.

Our main result is stated by the following theorem.

Theorem 1.1. *Suppose that conditions* (b)*,* (p)*,* (A1)-(A3)*,* (f1)-(f3) *are fulfilled, where* $b \in L^{\infty}\left(\overline{\Omega}\right)$ *and* a_i*,* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *are Carathéodory functions. Then there is a unique nontrivial weak solution to problem (1.1).*

Notice that $\sum_{i=1}^{N} \partial_{x_i} a_i\left(x, \partial_{x_i} u\right)$ is a $\overrightarrow{p}(\cdot)$ - Laplace type operator, since by choosing $a_i(x,s) = |s|^{p_i(x)-2}s$ for all $i \in \{1,...,N\}$, we have $A_i(x,s) = \frac{1}{p_i(x)}\,|s|^{p_i(x)}$ for all $i \in \{1,...,N\}$, and we arrive at the anisotropic variable exponent Laplace operator

$$\Delta_{\overrightarrow{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i} u|^{p_i(x)-2}\,\partial_{x_i} u\right). \tag{1.2}$$

We bring to your attention that when choosing $p_1,...,p_N$ to be constant functions and $\overrightarrow{p} = (p_1,...,p_N)$, we obtain the anisotropic $\overrightarrow{p}$ - Laplace operator

$$\Delta_{\overrightarrow{p}}(u) = \sum_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i} u|^{p_i-2}\,\partial_{x_i} u\right),$$

while when choosing $p_1 = ... = p_N = p$ in (1.2) we obtain an operator similar to the variable exponent $p(\cdot)$ - Laplace operator

$$\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u),$$

where p is a continuous function. Therefore it is not only a study of boundary value problems, it is also a "boundary" study, if we take into consideration the fact that the theory of anisotropic variable exponent Lebesgue-Sobolev spaces is situated at the boundary between the the anisotropic Sobolev spaces theory developed by [25, 27, 28, 30, 31] and the variable exponent Sobolev spaces theory developed by [5, 6, 7, 8, 9, 11, 12, 15, 22, 23, 24, 29]. In this newly formed direction of PDEs, various articles appeared and continue to appear. For the proof of our main result we need to explore techniques similar to those used by [2, 3, 4, 14, 20, 21]. In order to not repeat the same arguments we import some of the results presented in these papers, indicating the place where all the calculus details may be found. Generally speaking, we are relying on the critical point theory, that is, we associate to (1.1) a functional energy whose critical points represent the weak solutions of the problem. Among the previously enumerated papers, our work is more closely related to [2, 14, 19], where are also used general $\overrightarrow{p}(\cdot)$ - Laplace type operators under conditions resembling to (A1)-(A3). In [14] the authors prove that the functional energy is proper, weakly lower semi-continuous and coercive, thus it has a minimizer which is a weak solution to their problem. In [2, 19] the authors establish the multiplicity of the solution in addition to its existence. The first paper utilize the symmetric mountain-pass theorem of Ambrosetti and Rabinowitz, while the second one is combining the mountain-pass theorem of Ambrosetti and Rabinowitz with the Ekeland's variational principle. For more variational methods that could prove useful we send the reader to [10, 17, 18].

An interesting remark is that operators fulfilling conditions like (A1)-(A3) were not just considered when working on anisotropic variable exponent Lebesgue-Sobolev spaces. To give some examples, we refer to [16, 22], where it was discussed the following type of problem

$$\begin{cases} -\text{div}(a(x,\nabla u)) = f(x,u) & \text{for} \quad x \in \Omega, \\ u = 0 & \text{for} \quad x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary. The preference for conditions (A1)-(A3) comes from the fact that, as said before, the anisotropic variable exponent Laplace operator satisfies them. But the association with the $\overrightarrow{p}(\cdot)$ - Laplace is not the only reason, since there are other well known operators that satisfy these conditions. Indeed, when choosing $a_i(x,s) = (1+|s|^2)^{(p_i(x)-2)/2}s$ for all $i \in \{1,...,N\}$, we have $A_i(x,s) = \frac{1}{p_i(x)}[(1+|s|^2)^{p_i(x)/2} - 1]$ for all $i \in \{1,...,N\}$, and we obtain the anisotropic variable mean curvature operator

$$\sum_{i=1}^{N} \partial_{x_i} \left[\left(1 + |\partial_{x_i} u|^2\right)^{(p_i(x)-2)/2} \partial_{x_i} u \right].$$

Now that we have examples of appropriate operators we can pass to shortly describing the structure of the rest of the paper. In the second section we recall the definition and some important properties of the variable exponent spaces, anisotropic spaces and anisotropic variable exponent spaces. In the third section we define the notion of weak solution to problem (1.1) and we introduce the functional energy associated to this problem. Then we present several auxiliary results and we use them to prove the main theorem.

2. **Abstract framework**

In what follows we consider p, $p_i \in C_+(\overline{\Omega})$, $i \in \{1, \ldots, N\}$ to be logarithmic Hölder continuous. We define the variable exponent Lebesgue space by

$$L^{p(\cdot)}(\Omega)=\{u : u \text{ is a measurable real–valued function such that } \int_\Omega |u(x)|^{p(x)}dx<\infty\}$$

endowed with the so-called Luxemburg norm

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0 \ : \quad \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

We mention that for p constant this norm becomes the norm of the classical Lebesgue space L^p, that is,

$$|u|_p = \left(\int_\Omega |u|^p\right)^{1/p}.$$

The space $\left(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)}\right)$ has some important properties. It is a separable and reflexive Banach space ([15, Theorem 2.5, Corollary 2.7]). The inclusion between spaces generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents in $C_+(\overline{\Omega})$ such that $p_1 \le p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous ([15, Theorem 2.8]). The following Hölder-type inequality

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(\cdot)}|v|_{p'(\cdot)} \le 2|u|_{p(\cdot)}|v|_{p'(\cdot)} \tag{2.1}$$

holds true for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ ([15, Theorem 2.1]), where we denoted by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise i.e. $1/p(x) + 1/p'(x) = 1$ ([15, Corollary 2.7]).
The $p(\cdot)$-modular of the $L^{p(\cdot)}(\Omega)$ space, that is, the function $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$,

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\, dx,$$

plays a key role in handling this space. We remind some of its properties (see again [15]): if $u \in L^{p(\cdot)}(\Omega)$, $(u_n) \subset L^{p(\cdot)}(\Omega)$ and $p^+ < \infty$, then,

$$\begin{aligned}
|u|_{p(\cdot)} < 1\ (= 1; > 1) \quad &\Leftrightarrow \quad \rho_{p(\cdot)}(u) < 1\ (= 1; > 1)\\
|u|_{p(\cdot)} > 1 \quad &\Rightarrow \quad |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+}\\
|u|_{p(\cdot)} < 1 \quad &\Rightarrow \quad |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-}\\
|u|_{p(\cdot)} \to 0\ (\to \infty) \quad &\Leftrightarrow \quad \rho_{p(\cdot)}(u) \to 0\ (\to \infty) \qquad (2.2)\\
\lim_{n\to\infty} |u_n - u|_{p(\cdot)} = 0 \quad &\Leftrightarrow \quad \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0.
\end{aligned}$$

We define the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$,

$$W^{1,p(\cdot)}(\Omega) = \Big\{ u \in L^{p(\cdot)}(\Omega) : \partial_{x_i} u \in L^{p(\cdot)}(\Omega),\ i \in \{1, 2, ...N\} \Big\}$$

endowed with the norm

$$\|u\| = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}. \tag{2.3}$$

The space $\big(W^{1,p(\cdot)}(\Omega), \|\cdot\|\big)$ is a separable and reflexive Banach space. In what concerns the Sobolev space with zero boundary values, we denote it by $W_0^{1,p(\cdot)}(\Omega)$ and we define it as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|$. We consider the norms

$$\|u\|_{1,p(\cdot)} = |\nabla u|_{p(\cdot)},$$

and

$$\|u\|_{p(\cdot)} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p(\cdot)}$$

which are equivalent to (2.3) on $W_0^{1,p(\cdot)}(\Omega)$. The space $W_0^{1,p(\cdot)}(\Omega)$ is also a separable and reflexive Banach space. Furthermore, if $s \in C_+(\overline{\Omega})$ and $s(x) < p^\star(x)$ for all $x \in \overline{\Omega}$, where $p^\star(x) = Np(x)/[N - p(x)]$ if $p(x) < N$ and $p^\star(x) = \infty$ if $p(x) \geq N$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact and continuous.

We define now the anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{\overrightarrow{p}(\cdot)} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}.$$

The space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ can be considered a natural generalization of both the variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ and the classical anisotropic Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega)$, where $\overrightarrow{p}$ is the constant vector $(p_1, ..., p_N)$. The space $W_0^{1,\overrightarrow{p}}(\Omega)$ endowed with the norm

$$\|u\|_{1,\overrightarrow{p}} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i}$$

is a reflexive Banach space for all $\overrightarrow{p} \in \mathbb{R}^N$ with $p_i > 1$, $i \in \{1, ..., N\}$. This result can be easily extended to $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, see [21]. Another extension can be made in what concerns the embedding between $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $L^{q(\cdot)}(\Omega)$ [21, Theorem 1]: if $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, the components of $\overrightarrow{p}$ verify (p) and $q \in C(\overline{\Omega})$ verifies $1 < q(x) < P_{-,\infty}$ for all $x \in \overline{\Omega}$, then the embedding

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

is continuous and compact.

3. Proof of the main result

Working under the hypotheses of Theorem 1.1, we denote $W_0^{1,\vec{p}(\cdot)}(\Omega)$ by E and we start by giving the definition of the weak solution for problem (1.1).

Definition 3.1. By a weak solution to problem (1.1) we understand a function $u \in E$ such that

$$\int_\Omega \left[\sum_{i=1}^N a_i\left(x, \partial_{x_i} u\right) \partial_{x_i}\varphi + b(x)|u|^{P_+^+-2}u\varphi - f(x,u)\varphi \right] dx = 0, \tag{3.1}$$

for all $\varphi \in E$.

As said in the introductory section, we base our proof on the critical point theory, thus we associate to problem (1.1) the energy functional $I : E \to \mathbb{R}$ defined by

$$I(u) = \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right) dx + \frac{1}{P_+^+} \int_\Omega b(x)|u|^{P_+^+} dx - \int_\Omega F(x, u_+)\, dx,$$

where $u_+(x) = \max\{u(x), 0\}$.

For all $i \in \{1, 2, ...N\}$, we denote by J_i, $K : E \to \mathbb{R}$ the functionals

$$J_i(u) = \int_\Omega A_i\left(x, \partial_{x_i} u\right) dx \quad \text{and} \quad K(u) = \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right) dx - \int_\Omega F(x, u_+)\, dx.$$

In what follows we present several results concerning the functionals J_i, K or other terms of I.

Lemma 3.2. *(*[14, Lemma 3.4]*) For $i \in \{1, 2, ...N\}$,*

(i) *the functional J_i is well-defined on E;*
(ii) *the functional J_i is of class $C^1(E, \mathbb{R})$ and*

$$\langle J_i'(u), \varphi \rangle = \int_\Omega a_i(x, \partial_{x_i} u) \partial_{x_i}\varphi\, dx,$$

for all $u, \varphi \in E$.

Lemma 3.3. *(*[2, Section 4, Claim 2]*) There exist ρ, $r > 0$ such that $K(u) \geq r > 0$, for any $u \in E$ with $\|u\|_{\vec{p}(\cdot)} = \rho$.*

Remark 3.4. In the proof of [2, Section 4, Claim 2] appeared the fact that f was considered odd in its second variable, thus F was even in its second variable. In our case, this hypothesis is not necessary, since we are interested in $F(x, u_+)$, which has its second variable nonnegative.

Lemma 3.5. *(see* [2, Section 4, Claim 1]*)* *(i) For all $u \in E$,*

$$\sum_{i=1}^{N} \int_\Omega \left[A_i(x, \partial_{x_i} u) - \frac{1}{\mu} a_i(x, \partial_{x_i} u) \partial_{x_i} u \right] dx \geq \left(\frac{1}{P_+^+} - \frac{1}{\mu} \right) \left(\frac{\|u\|_{\vec{p}(\cdot)}^{P_-^-}}{N^{P_-^- -1}} - N \right),$$

where μ is the constant from (f2).

(ii) For any sequence $(u_n)_n \subset E$ weakly convergent to $u \in E$,

$$\lim_{n\to\infty} \int_\Omega f(x, u_n)(u_n - u) dx = 0.$$

Lemma 3.6. *(*[2, Section 3, Lemma 2]*) Let $(u_n)_n \subset E$ be a sequence which is weakly convergent to $u \in E$ and*

$$\limsup_{n\to\infty} \int_\Omega \sum_{i=1}^{N} a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u) dx \leq 0.$$

Then $(u_n)_n$ converges strongly to u in E.

By Lemma 3.2 and by a standard calculus, I is well-defined on E and $I \in C^1(E, \mathbb{R})$ with the derivative given by

$$\langle I'(u), \varphi \rangle = \int_\Omega \sum_{i=1}^{N} a_i(x, \partial_{x_i} u) \partial_{x_i} \varphi \, dx + \int_\Omega b(x) |u|^{P_+^+ - 2} u \varphi \, dx - \int_\Omega f(x, u) \varphi \, dx,$$

for all $u, \varphi \in E$, therefore the critical points of I are weak solutions to (1.1). Being concerned with the existence of critical points, we search for help in the mini-max principles theory (see for example [1, 13, 26]) and we find it in the mountain-pass theorem of Ambrosetti and Rabinowitz without the Palais-Smale condition. Following the steps described by the statement of this theorem, we first prove two auxiliary results.

Lemma 3.7. *There exist ρ, $r > 0$ such that $I(u) \geq r > 0$, for any $u \in E$ with $\|u\|_{\vec{p}(\cdot)} = \rho$.*

Proof. By hypothesis (b),

$$\frac{1}{P_+^+} \int_\Omega b(x) |u|^{P_+^+} dx \geq \frac{b_0}{P_+^+} |u|_{P_+^+}^{P_+^+} \geq 0,$$

for all $u \in E$. Hence

$$I(u) \geq \int_\Omega \sum_{i=1}^{N} A_i \left(x, \partial_{x_i} u \right) \, dx - \int_\Omega F(x, u_+) \, dx,$$

for all $u \in E$. Using Lemma 3.3, we deduce that we can find ρ, $r > 0$ such that $I(u) \geq r > 0$, for all $u \in E$ with $\|u\|_{\vec{p}(\cdot)} = \rho$. ☐

Lemma 3.8. *There exists $e \in E$ with $\|e\|_{\vec{p}(\cdot)} > \rho$ (ρ given by Lemma 3.7) such that $I(e) < 0$.*

Proof. Since for all $i \in \{1, ..., N\}$, $A_i(x,s) = \int_0^s a_i(x,t)\,dt$, by a simple change of variables and by condition (A1) we get

$$A_i(x,s) \le c_{1,i} \int_0^1 |a_i(x,ts)s|\ dt \le c_{1,i}\left(|s| + \frac{|s|^{p_i(x)}}{p_i(x)}\right) \quad \text{for all } x \in \Omega,\ s \in \mathbb{R}.$$

Setting $C_1 = \max\{c_{1,i} : \ i \in \{1, 2, ...N\}\}$, by the above relation we obtain that

$$\int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i}(tu)\right)\, dx \le C_1 \int_\Omega \sum_{i=1}^N |\partial_{x_i}(tu)|\, dx + \frac{C_1}{P_-^-}\int_\Omega \sum_{i=1}^N |\partial_{x_i}(tu)|^{p_i(x)}\, dx$$

for all $u \in E$.

On the other hand, by rewriting condition (f2), we deduce that there exists a positive constant c_4 such that

$$F(x,s) \ge c_4\,|s|^\mu, \quad \forall x \in \Omega,\ \forall s \ge 0,$$

therefore

$$I(tu) \le C_1\, t \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|\, dx + \frac{C_1\, t^{P_+^+}}{P_-^-} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)}\, dx$$

$$+ \frac{t^{P_+^+}}{P_+^+}\int_\Omega b(x)|u|^{P_+^+}\, dx - c_4\, t^\mu \int_\Omega |u|^\mu\, dx,$$

for all $u \in E$ and $t > 1$. Then, due to the fact that $\mu > P_+^+ > 1$, for $u \not\equiv 0$ we have $I(tu) \to -\infty$ when $t \to \infty$ and we can choose t large enough and $e = tu \in E$ with $\|e\|_{\vec{p}(\cdot)} > \rho$ such that

$$I(e) < 0.$$

□

Proof of Theorem 1.1. Proof of existence. By Lemma 3.7, Lemma 3.8 and the mountain-pass theorem of Ambrosetti and Rabinowitz, there exist a sequence $(u_n)_n \subset E$ such that

$$I(u_n) \to \alpha > 0 \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

Notice that from the definition of the functional I we can consider $(u_n)_n$ to be a sequence of nonnegative functions. We will prove that $(u_n)_n$ is bounded in E by arguing by contradiction, more exactly by assuming that, passing eventually to a subsequence still denoted by $(u_n)_n$,

$$\|u_n\|_{\vec{p}(\cdot)} \to \infty \quad \text{when } n \to \infty. \tag{3.3}$$

Combining relations (3.2) and(3.3) we infer

$$\begin{aligned}
1+\alpha+\|u_n\|_{\vec{p}(\cdot)} &\geq I(u_n)-\frac{1}{\mu}\langle I'(u_n),u_n\rangle \\
&\geq \sum_{i=1}^{N}\int_\Omega\left[A_i(x,\partial_{x_i}u_n)-\frac{1}{\mu}a_i(x,\partial_{x_i}u_n)\partial_{x_i}u_n\right]dx+ \\
&\quad +\left(\frac{1}{P_+^+}-\frac{1}{\mu}\right)\int_\Omega b(x)|u|^{P_+^+}\,dx- \\
&\quad -\int_\Omega\left[F(x,u_n)-\frac{1}{\mu}u_nf(x,u_n)\right]dx \\
&\geq \sum_{i=1}^{N}\int_\Omega\left[A_i(x,\partial_{x_i}u_n)-\frac{1}{\mu}a_i(x,\partial_{x_i}u_n)\partial_{x_i}u_n\right]dx,
\end{aligned}$$

for sufficiently large n, since μ is the constant from (f2). By Lemma 3.5(i) we come to

$$1+\alpha+\|u_n\|_{\vec{p}(\cdot)}\geq\left(\frac{1}{P_+^+}-\frac{1}{\mu}\right)\left(\frac{\|u_n\|_{\vec{p}(\cdot)}^{P_-^-}}{N^{P_-^--1}}-N\right),$$

hence by dividing by $\|u_n\|_{\vec{p}(\cdot)}^{P_-^-}$ and passing to the limit as $n\to\infty$ we obtain the desired contradiction and we conclude that $(u_n)_n$ is bounded in E. We know that the space E is reflexive, thus there is $u_0\in E$ such that, up to a subsequence, $(u_n)_n$ converges weakly to u_0 in E. We need to show that $(u_n)_n$ converges strongly to u_0 in E.

The fact that $P_+^+<P_{-,\infty}$ implies that the embedding $E\hookrightarrow L^{P_+^+}(\Omega)$ is compact. Thus $(u_n)_n$ converges strongly to u_0 in $L^{P_+^+}(\Omega)$. By the Hölder-type inequality (2.1),

$$\left|\int_\Omega b(x)|u_n|^{P_+^+-2}u_n(u_n-u_0)dx\right|\leq 2\|b\|_{L^\infty(\Omega)}\left||u_n|^{P_+^+-2}u_n\right|_{\frac{P_+^+}{P_+^+-1}}|u_n-u_0|_{P_+^+}.$$

Using the strong convergence of $(u_n)_n$ to u_0 in $L^{P_+^+}(\Omega)$, the above relation and (2.2) we come to

$$\lim_{n\to\infty}\int_\Omega b(x)|u_n|^{P_+^+-2}u_n(u_n-u_0)dx=0. \tag{3.4}$$

Let us consider the relations

$$\begin{aligned}
<I'(u_n),u_n-u_0>=\int_\Omega\Big[\sum_{i=1}^{N}a_i(x,\partial_{x_i}u_n)(\partial_{x_i}u_n-\partial_{x_i}u_0)+b(x)|u_n|^{P_+^+-2}u_n(u_n-u_0) \\
-f(x,u_n)(u_n-u_0)\Big]dx
\end{aligned}$$

and, from (3.2),

$$\lim_{n\to\infty} < I'(u_n), u_n - u_0 >= 0.$$

Combining these relations with (3.4) and Lemma 3.5(ii) we obtain

$$\lim_{n\to\infty} \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u_0) dx = 0.$$

Using Lemma 3.6 we deduce that $(u_n)_n$ converges strongly to u_0 in E. By (3.2) u_0 is a critical point to I and $I(u_0) = \alpha > 0$. Since $I(0) = 0$ it follows that u_0 is a nontrivial weak solution to (1.1). □

Proof of uniqueness. Let us assume that there exist two nontrivial solutions to problem (1.1), that is, u_1 and u_2. We replace the solution u by u_1 in (3.1) and we choose $\varphi = u_1 - u_2$. We obtain

$$\int_\Omega \Big[\sum_{i=1}^N a_i\left(x, \partial_{x_i} u_1\right)\, \partial_{x_i}(u_1 - u_2) + b(x)|u_1|^{P_+^+ - 2} u_1(u_1 - u_2) \\ - f(x, u_1)(u_1 - u_2) \Big] dx = 0. \tag{3.5}$$

Next, we replace the solution u by u_2 in (3.1) and we choose $\varphi = u_2 - u_1$. We infer

$$\int_\Omega \Big[\sum_{i=1}^N a_i\left(x, \partial_{x_i} u_2\right)\, \partial_{x_i}(u_2 - u_1) + b(x)|u_2|^{P_+^+ - 2} u_2(u_2 - u_1) \\ - f(x, u_2)(u_2 - u_1) \Big] dx = 0. \tag{3.6}$$

Putting together (3.5) and (3.6) we arrive at

$$\int_\Omega \left\{ \sum_{i=1}^N \left[a_i\left(x, \partial_{x_i} u_1\right) - a_i\left(x, \partial_{x_i} u_2\right)\right] \left(\partial_{x_i} u_1 - \partial_{x_i} u_2\right) \right\} dx + \\ + \int_\Omega b(x) \left[|u_1|^{P_+^+ - 2} u_1 - |u_2|^{P_+^+ - 2} u_2 \right] (u_1 - u_2)\, dx - \\ - \int_\Omega \left[f(x, u_1) - f(x, u_2)\right](u_1 - u_2)\, dx = 0.$$

By hypotheses (A3) and (f3), all the terms in the above equality are positive unless $u_1 = u_2$, and this yields the uniqueness of the solution. □

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References

[1] Ambrosetti, A., Rabinowitz, P. H., *Dual variational methods in critical point theory*, J. Funct. Anal., **14** (1973), 349–381.

[2] Boureanu, M.-M., *Infinitely many solutions for a class of degenerate anisotropic elliptic problems with variable exponent*, Taiwanese Journal of Mathematics, to appear.

[3] Boureanu, M.-M., *Existence of solutions for anisotropic quasilinear elliptic equations with variable exponent*, Advances in Pure and Applied Mathematics, **1** (2010), 387–411.

[4] Boureanu, M.-M., Pucci, P., Rădulescu, V., *Multiplicity of solutions for a class of anisotropic elliptic equations with variable exponent*, Complex Variables and Elliptic Equations, to appear.

[5] Edmunds, D.E., Lang, J., Nekvinda, A., *On $L^{p(x)}$ norms*, Proc. Roy. Soc. London Ser. A, **455** (1999), 219–225.

[6] Edmunds, D.E., Rákosník, J., *Density of smooth functions in $W^{k,p(x)}(\Omega)$*, Proc. Roy. Soc. London Ser. A, **437** (1992), 229–236.

[7] Edmunds, D.E., Rákosník, J., *Sobolev embedding with variable exponent*, Studia Math., **143** (2000), 267–293.

[8] Fan, X., *Solutions for $p(x)$–Laplacian Dirichlet problems with singular coefficients*, J. Math. Anal. Appl., **312** (2005), 464–477.

[9] Fan, X., Zhang, Q., Zhao, D., *Eigenvalues of $p(x)$–Laplacian Dirichlet problem*, J. Math. Anal. Appl., **302** (2005), 306–317.

[10] Ghergu, M., Rădulescu, V., *Singular Elliptic Problems. Bifurcation and Asymptotic Analysis*, Oxford Lecture Series in Mathematics and Its Applications, **37**, Oxford University Press, 2008.

[11] Harjulehto, P., *Variable exponent Sobolev spaces with zero boundary values*, Math. Bohem., **132** (2007), no. 2, 125–136.

[12] Hästö, P., *On the density of continuous functions in variable exponent Sobolev spaces*, Rev. Mat. Iberoamericana, **23** (2007), 74–82.

[13] Jabri, Y., *The Mountain Pass Theorem. Variants, Generalizations and some Applications*, Cambridge University Press, 2003.

[14] Kone, B., Ouaro, S., Traore, S., *Weak solutions for anisotropic nonlinear elliptic equations with variable exponents*, Electronic Journal of Differential Equations, **2009** (2009), No. 144, 1–11.

[15] Kováčik, O., Rákosník, J., *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. J., **41** (1991), 592–618.

[16] Kristály, A., Lisei, H., Varga, C., *Multiple solutions for p-Laplacian type equations*, Nonlinear Anal. TMA, **68** (2008), 1375–1381.

[17] Kristály, A., Rădulescu, V., Varga, C., *Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics and its Applications, **136**, Cambridge University Press, Cambridge, 2010.

[18] Kristály, A., Varga, C., *An Introduction to Critical Point Theory for Nonsmooth Functions*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2004.

[19] Mihăilescu, M., Moroşanu, G., *Existence and multiplicity of solutions for an anisotropic elliptic problem involving variable exponent growth conditions*, Applicable Analysis, **89** (2010), 257–271.

[20] Mihăilescu, M., Pucci, P., Rădulescu, V., *Nonhomogeneous boundary value problems in anisotropic Sobolev spaces*, C. R. Acad. Sci. Paris, Ser. I, **345** (2007), 561–566.

[21] Mihăilescu, M., Pucci, P., Rădulescu, V., *Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent*, J. Math. Anal. Appl., **340** (2008), 687–698.

[22] Mihăilescu, M., Rădulescu, V., *A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids*, Proc. Roy. Soc. London Ser. A, **462** (2006), 2625–2641.

[23] Mihăilescu, M., Rădulescu, V., *On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent*, Proc. Amer. Math. Soc., **135** (2007), 2929–2937.

[24] Musielak, J., *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Vol. **1034**, Springer, Berlin, 1983.

[25] Nikol'skii, S. M.., *On imbedding, continuation and approximation theorems for differentiable functions of several variables*, Russian Math. Surv., **16** (1961), 55–104.

[26] Rabinowitz, P. H., *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conf. Ser. in Math., **65**, Amer. Math. Soc., 1986.

[27] Rákosník, J., *Some remarks to anisotropic Sobolev spaces I*, Beiträge zur Analysis, **13** (1979), 55–68.

[28] Rákosník, J., *Some remarks to anisotropic Sobolev spaces II*, Beiträge zur Analysis, **15** (1981), 127–140.

[29] Samko, S., Vakulov, B., *Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators*, J. Math. Anal. Appl., **310** (2005), 229–246.

[30] Troisi, M., *Teoremi di inclusione per spazi di Sobolev non isotropi*, Ricerche Mat., **18** (1969), 3–24.

[31] Ven'–tuan, L., *On embedding theorems for spaces of functions with partial derivatives of various degree of summability*, Vestnik Leningrad Univ., **16** (1961), 23–37.

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INFINITELY MANY SOLUTION FOR A NONLINEAR NAVIER BOUNDARY VALUE PROBLEM INVOLVING THE p-BIHARMONIC

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Abstract. The existence of infinitely many solutions is established for a class of nonlinear elliptic equations involving the p-biharmonic operator and under Navier boundary value conditions. The approach adopted is fully based on critical point theory.

1. **Introduction**

In this paper, we are interested in studying the existence of infinitely many solutions for the following nonlinear elliptic Navier boundary value problem involving the p-biharmonic

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x,u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbf{R}^N$ with a smooth enough boundary $\partial\Omega$, $(N \geq 1)$, $p > \max\{1, N/2\}$, Δ is the usual Laplace operator, λ is a positive parameter and $f \in C^0(\bar{\Omega} \times \mathbf{R})$.

In these latest years, many authors looked for multiple solutions of boundary value problems involving biharmonic and p-biharmonic type operators, see for instance [5], [11], [12], [14] and the references cited therein.

More precisely, in [10], assuming that $f(x,\cdot)$ is odd and by using the Symmetric Mountain Pass Theorem of Ambrosetti-Rabinowitz, the existence of infinitely many solutions for nonlinear elliptic equations with a general p-biharmonic type operator and under either Navier or Dirichlet boundary conditions has been obtained. In [9], see also [13], requiring that the nonlinearity f is the sum of an odd term and a non-odd perturbation, via perturbation theory, the existence of infinitely many sign-changing solutions for problem (1.1) for $p = 2$ and $N \geq 5$ has been established.

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Moreover, in such frameworks, some additional suitable growth conditions, for example that f is p-sublinear at zero and p-superlinear at infinity, are supposed.

Here, we achieve our goal under different assumptions on f which turn out to be mutually independent with respect to those adopted on the above mentioned papers, see Examples 3.3 and 3.7. In particular, we obtain well precise intervals of parameters such that problem (1.1) admits either an unbounded sequence of solutions (Theorem 3.1) provided that f has a suitable behaviour at infinity or a sequence of non-zero solutions (Theorem 3.8) strongly converging to zero if a similar behaviour occurs at zero. Moreover, we explicitly observe that in the autonomous case (Theorem 3.4) our conclusions are sharped (Remark 3.5).

On the other hand, it is worth noticing that the results contained in [4], where the authors required that the nonlinearity changes sign in a suitable way, are included in the case $\alpha = 0$ and $\beta = \infty$ treated here (Remark 3.2), where the nonlinearity can also be nonnegative (Corollary 3.6). This is due to the fact that we use a more precise version of Ricceri's variational principle [7], given by Bonanno and Molica Bisci in [1]. Very recently, the same approach adopted here has also been followed in [3] to look for infinitely many solutions for a fourth order equation in the one dimensional case which, as particular case, contains problem (1.1) with $p = 2$.

For general references and for a complete and exhaustive overview on variational methods we refer the reader to the excellent monographs [6] and [8].

2. **Preliminaries**

Here and in the sequel Ω is an open bounded subset of $\mathbf{R}^N$ $(N \geq 1)$, $p > \max\{1, N/2\}$, while X denotes the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega |\Delta u(x)|^p dx \right)^{1/p} \qquad \forall u \in X. \tag{2.1}$$

The Rellich Kondrachov Theorem assures that X is compactly imbedded in $C^0(\bar{\Omega})$, being

$$k := \sup_{u \in X \setminus \{0\}} \frac{\|u\|_{C^0(\bar{\Omega})}}{\|u\|} < +\infty. \tag{2.2}$$

Let $f \in C^0(\bar{\Omega} \times \mathbf{R})$ and let us put

$$F(x,t) := \int_0^t f(x,\xi) d\xi \quad \forall (x,t) \in \bar{\Omega} \times \mathbf{R}.$$

For our approach we will use the functionals $\Phi, \Psi : X \to \mathbf{R}$ defined by putting

$$\Phi(u) := \frac{1}{p}\|u\|^p, \quad \Psi(u) := \int_\Omega F(x, u(x)) dx$$

for every $u \in X$. It is simple to verify that Φ and Ψ are well defined, as well as Gâteaux differentiable. Moreover, in view of the fact that Φ is continuous and convex, it turns out sequentially weakly lower semicontinuous, while, since Ψ has compact derivative, it results sequentially weakly continuous. In particular, one has

$$\Phi'(u)(v) = \int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)dx, \quad \Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)dx$$

for every $u, v \in X$.

We explicitly observe that, in view of (2.2), one has that, for every $r > 0$

$$\Phi^{-1}(]-\infty, r[) := \{u \in X : \ \Phi(u) < r\} \subseteq \{u \in C^0(\bar{\Omega}) : \ \|u\|_{C^0} < k(pr)^{1/p}\}. \quad (2.3)$$

Finally, if we recall that a weak solution of problem (1.1) is a function $u \in X$ such that

$$\int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)dx - \lambda \int_\Omega f(x,u(x))v(x)dx = 0 \quad \forall v \in X,$$

it is obvious that our goal is to find critical points of the functional $\Phi - \lambda\Psi$. For this aim, our main tool is a general critical points theorem due to Bonanno and Molica Bisci (see [1]) that is a generalization of a previous result of Ricceri [7] and that here we state in a smooth version for the reader's convenience.

Theorem 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbf{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{v \in \Phi^{-1}(]-\infty,r[)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(a) *for every $r > \inf_X \Phi$ and every $\lambda \in \left]0, \frac{1}{\varphi(r)}\right[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(b) *If $\gamma < +\infty$ then, for each $\lambda \in \left]0, \frac{1}{\gamma}\right[$, the following alternative holds: either*

(b_1) I_λ possesses a global minimum,

or

(b_2) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.

(c) *If* $\delta < +\infty$ *then, for each* $\lambda \in \left]0, \frac{1}{\delta}\right[$*, the following alternative holds: either*

(c_1) *there is a global minimum of* Φ *which is a local minimum of* I_λ*, or*

(c_2) *there is a sequence of pairwise distinct critical points (local minima) of* I_λ *which weakly converges to a global minimum of* Φ*.*

3. Main results

Fixed $x^0 \in \Omega$, let us pick $0 < s_1 < s_2$ such that $B(x^0, s_2) \subseteq \Omega$ and put

$$L := \frac{\Gamma\left(1 + \frac{N}{2}\right)}{\pi^{N/2}} \left(\frac{s_2^2 - s_1^2}{2Nk}\right)^p \frac{1}{s_2^N - s_1^N}, \tag{3.1}$$

where Γ denotes the Gamma function and k is defined in (2.2).

Theorem 3.1. *Assume that*

(i_1) $F(x,t) \geq 0$ *for every* $(x,t) \in \Omega \times [0, +\infty[$*;*

(i_2) *There exist* $x^0 \in X$*,* $0 < s_1 < s_2$ *as considered in (3.1) such that, if we put*

$$\alpha := \liminf_{t \to +\infty} \frac{\int_\Omega \max_{|\xi| \leq t} F(x,\xi)dx}{t^p}, \qquad \beta := \limsup_{t \to +\infty} \frac{\int_{B(x^0,s_1)} F(x,t)dx}{t^p},$$

one has

$$\alpha < L\beta. \tag{3.2}$$

Then, for every $\lambda \in \Lambda := \dfrac{1}{pk^p}\left]\dfrac{1}{L\beta}, \dfrac{1}{\alpha}\right[$ *problem (1.1) admits an unbounded sequence of weak solutions.*

Proof. With the purpose of applying Theorem 2.1, we begin observing that, for every $r > 0$, taking in mind (2.3), one has

$$\varphi(r) \leq \frac{\sup_{\Phi^{-1}(]-\infty, r[)} \Psi}{r} \leq \frac{\int_\Omega \max_{|\xi| \leq k(pr)^{1/p}} F(x,\xi)dx}{r}. \tag{3.3}$$

At this point, we consider a sequence $\{t_n\}$ of positive numbers such that $t_n \to +\infty$ and

$$\lim_{n \to +\infty} \frac{\int_\Omega \max_{|\xi| \leq t_n} F(x,\xi)dx}{t_n^p} = \alpha. \tag{3.4}$$

For every $n \in \mathbf{N}$ let us consider $r_n = \frac{1}{p}\left(\frac{t_n}{k}\right)^p$. Putting together (3.3), (3.4) and (3.2) one has

$$\gamma \leq \liminf_{n \to +\infty} \varphi(r_n) \leq pk^p \lim_{n \to +\infty} \frac{\int_\Omega \max_{|\xi| \leq t_n} F(x,\xi)dx}{t_n^p} < +\infty. \tag{3.5}$$

Moreover, we can also observe that, owing to (3.4) and (3.5),

$$\Lambda \subseteq \left]0, \frac{1}{\gamma}\right[.$$

Fix $\lambda \in \Lambda$ and claim that

$$\Phi - \lambda\Psi \text{ is unbounded from below.} \tag{3.6}$$

Indeed, since $\frac{1}{\lambda} < pk^p L\beta$, we can consider a sequence $\{\tau_n\}$ of positive numbers and $\eta > 0$ such that $\tau_n \to +\infty$ and

$$\frac{1}{\lambda} < \eta < pk^p L \frac{\int_{B(x^0,s_1)} F(x,\tau_n)dx}{\tau_n^p} \tag{3.7}$$

for every $n \in \mathbf{N}$ large enough. Let $\{w_n\}$ be a sequence in X defined by putting

$$w_n(x) = \begin{cases} \tau_n & \text{if } x \in B(x^0, s_1) \\ \frac{\tau_n}{s_2^2 - s_1^2}[s_2^2 - \sum_{i=1}^N (x_i - x_i^0)^2] & \text{if } x \in B(x^0, s_2) \setminus B(x^0, s_1) \\ 0 & \text{if } x \in \Omega \setminus B(x^0, s_2). \end{cases} \tag{3.8}$$

Fixed $n \in \mathbf{N}$, a simple computation shows that

$$\Phi(w_n) = \frac{1}{p}\left(\frac{2N\tau_n}{s_2^2 - s_1^2}\right)^p \frac{\pi^{N/2}}{\Gamma(1 + N/2)}(s_2^N - s_1^N) = \frac{\tau_n^p}{pk^p L}. \tag{3.9}$$

On the other hand, thanks to assumption (i_1), one has

$$\Psi(w_n) = \int_\Omega F(x, w_n(x))dx \geq \int_{B(x^0,s_1)} F(x,\tau_n)dx. \tag{3.10}$$

According to (3.9), (3.10) and (3.7) we achieve

$$\Phi(w_n) - \lambda\Psi(w_n) \leq \frac{\tau_n^p}{pk^p L} - \lambda \int_{B(x^0,s_1)} F(x,\tau_n)dx < \frac{\tau_n^p}{pk^p L}(1 - \lambda\eta)$$

for every $n \in \mathbf{N}$ large enough. Hence, (3.6) holds.
The alternative of Theorem 2.1 (case (b)) assures the existence of an unbounded sequence $\{u_n\}$ of critical points of the functional $\Phi - \lambda\Psi$ and the proof is complete in view of the considerations made in the previous section. $\square$

Remark 3.2. We explicitly observe that it is easier to verify assumption (3.2) provided that $\alpha = 0$ and $\beta = +\infty$ and of course in this case the interval Λ becomes $]0, +\infty[$. This situations occurs, for instance, in [4].

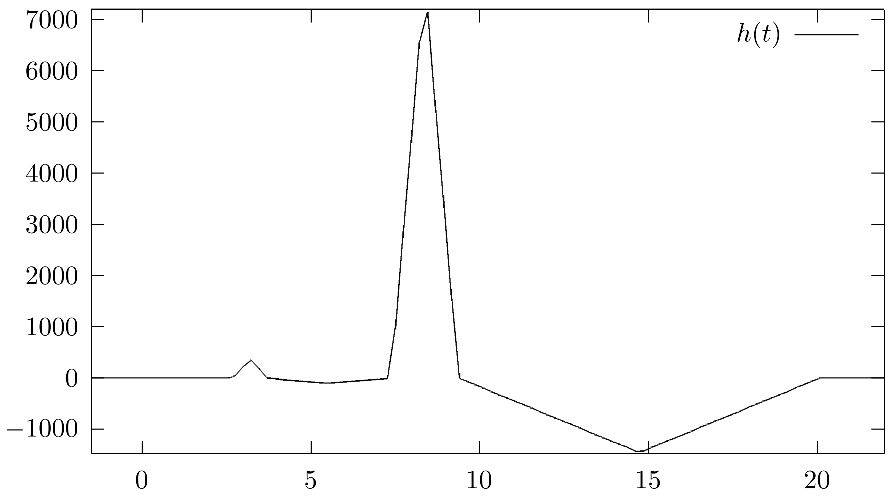
Example 3.3. Let Ω be an open, bounded subset of $\mathbf{R}^2$ and $g \in C^0(\bar{\Omega}) \setminus \{0\}$ a nonnegative function. Put

$$a_n := e^{n!}, \qquad b_n := e^{n!} + n, \qquad c_n := (a_n + b_n)/2, \qquad d_n := (b_n + a_{n+1})/2$$

for every $n \in \mathbf{N}^* := \mathbf{N} \setminus \{0\}$ and define the following function

$$h(t) := \begin{cases} \sum_{n\in\mathbf{N}^*} \mathbf{1}_{[a_n,b_n[} \frac{b_n^4}{c_n-a_n}\left(1-\frac{|t-c_n|}{c_n-a_n}\right) & \text{if } t \in \cup_{n\in\mathbf{N}^*}[a_n,b_n[, \\ \sum_{n\in\mathbf{N}^*} \mathbf{1}_{[b_n,a_{n+1}[} \frac{-b_n^4}{d_n-b_n}\left(1-\frac{|t-d_n|}{d_n-b_n}\right) & \text{if } t \in \cup_{n\in\mathbf{N}^*}[b_n,a_{n+1}[, \\ 0 & \text{otherwise}, \end{cases}$$

where the symbol $\mathbf{1}_{[r,s[}$ denotes the characteristic function of the interval $[r,s[$. A qualitative graph of h is shown in the figure below.



Moreover, let us put

$$f(x,t) := g(x)h(t), \tag{3.11}$$

for every $(x,t) \in \bar{\Omega} \times \mathbf{R}$. Hence, one has that

$$F(x,t) = \int_0^t f(x,\xi)d\xi = g(x)H(t)$$

for every $(x,t) \in \bar{\Omega} \times \mathbf{R}$, where

$$H(t) = \int_0^t h(\tau)d\tau \qquad \forall t \in \mathbf{R}.$$

It is easy to verify that, for every $n \in \mathbf{N}^*$,

$$\int_{a_n}^{b_n} h(\tau)d\tau = b_n^4 \quad \text{and} \quad \int_{b_n}^{a_{n+1}} h(\tau)d\tau = -b_n^4.$$

From this, a simple computation gives

$$H(a_n) = 0, \qquad \int_\Omega \max_{|\xi|\le a_{n+1}} F(x,\xi)dx = H(b_n)\int_\Omega g(x)dx = b_n^4 \int_\Omega g(x)dx.$$

Hence,

$$\alpha \le \int_\Omega g(x)dx \lim_{n\to+\infty} \frac{b_n^4}{a_{n+1}^3} = 0.$$

Moreover, let $x^0 \in \Omega$ such that $g(x^0) > 0$, fix $s_1 > 0$ such that $B(x^0, s_1) \subset \Omega$ and $g(x) > 0$ for every $x \in B(x^0, s_1)$, one has

$$\beta \ge \int_{B(x^0,s_1)} g(x)dx \lim_{n\to+\infty} \frac{H(b_n)}{b_n^3} = +\infty.$$

Applying Theorem 3.1, we can conclude that, for every $\lambda > 0$ the following problem

$$\begin{cases} \Delta(|\Delta u|\Delta u) = \lambda g(x)h(u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits an unbounded sequence of weak solutions.

In order to give the best formulation of the previous Theorem 3.1 in the autonomous case, let us observe that the function $s : \bar\Omega \to \mathbf{R}_0^+$ defined by

$$s(x) = d(x, \partial\Omega) \qquad \forall x \in \bar\Omega$$

is Lipschitz continuous. Hence, there exists $y^0 \in \Omega$ such that

$$\bar s = s(y^0) = \max_{x\in\Omega} s(x),$$

that is $\bar s$ is the biggest possible radius among all the balls contained in Ω.

Moreover, let $\bar\mu \in]0,1[$ be the point where the function $\frac{\mu^N(1-\mu^2)^p}{1-\mu^N}$ attains its maximum in $]0,1[$.

Theorem 3.4. *Let $h : \mathbf{R} \to \mathbf{R}$ be a continuous function such that:*

$(\mathrm{i}_1)'$ $H(t) = \int_0^t h(\xi)dx \ge 0$ *for every* $t \in [0,+\infty[$;

$(\mathrm{i}_2)'$ *Put*

$$\alpha' := \liminf_{t\to+\infty} \frac{\max_{|\xi|\le t} H(t)}{t^p}, \qquad \beta' := \limsup_{t\to+\infty} \frac{H(t)}{t^p},$$

one has

$$\alpha' < L'\beta',$$

where

$$L' = \frac{\bar s^{2p}}{(2Nk)^p|\Omega|} \frac{\bar\mu^N(1-\bar\mu^2)^p}{1-\bar\mu^N}. \tag{3.12}$$

Then, for every $\lambda \in \frac{1}{pk^p|\Omega|}\left]\frac{1}{L'\beta'}, \frac{1}{\alpha'}\right[$ *the following problem*

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda h(u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.13}$$

admits an unbounded sequence of weak solutions.

Proof. Put $x^0 = y^0$, $s_2 = \bar{s}$, $s_1 = \bar{\mu}\bar{s}$ and $f(x,t) = h(t)$ for every $(t,x) \in \bar{\Omega} \times \mathbf{R}$. Obviously $(\mathrm{i}_1)'$ implies (i_1). Moreover,

$$\alpha = |\Omega|\alpha', \quad \beta = \frac{\pi^{N/2}}{\Gamma(1+N/2)}(\bar{s}\bar{\mu})^N\beta', \quad L = \frac{|\Omega|\Gamma(1+N/2)}{\pi^{N/2}(\bar{\mu}\bar{s})^N}L'.$$

Hence, in view of $(\mathrm{i}_2)'$, one has

$$\alpha < |\Omega|L'\beta' = L\beta,$$

that is (i_2) holds and the conclusion follows directly from Theorem 3.1. □

Remark 3.5. It is worth noticing that in our framework, whenever $\beta' < +\infty$, taking in mind the properties of $\bar{s}$ and $\bar{\mu}$, the choice of such a L' is the best possible.

An immediate consequence of Theorem 3.4 is the following

Corollary 3.6. *Let* $h : \mathbf{R} \to \mathbf{R}$ *be a continuous and nonnegative function such that*

$$\liminf_{t\to+\infty} \frac{H(t)}{t^p} < L' \limsup_{t\to+\infty} \frac{H(t)}{t^p}, \tag{3.14}$$

being L' *defined in (3.12). Then, for every*

$$\lambda \in \Lambda' := \frac{1}{pk^p|\Omega|}\left]\frac{1}{L'\limsup_{t\to+\infty}\frac{H(t)}{t^p}}, \frac{1}{\liminf_{t\to+\infty}\frac{H(t)}{t^p}}\right[,$$

problem (3.13) admits an unbounded sequence of weak solutions.

Proof. It follows from Theorem 3.4 observing that, in view of the nonnegativity of h, $(\mathrm{i}_1)'$ holds and $\alpha' = \liminf_{t\to+\infty}\frac{H(t)}{t^p}$. □

Example 3.7. Let $\Omega =]0,4[$, $p = 2$ and $h : \mathbf{R} \to \mathbf{R}$ be a function defined by putting

$$h(t) := \begin{cases} 2t\left(1 + 2\sin^2(\ln t) + 2\sin(\ln t)\cos(\ln t)\right) & \text{if } t \in]0,+\infty[, \\ 0 & \text{if } t \in]-\infty, 0]. \end{cases}$$

Obviously, h is continuous and nonnegative. Moreover,

$$H(t) = \int_0^t h(\xi)d\xi = \begin{cases} t^2\left(1 + 2\sin^2(\ln t)\right) & \text{if } t \in]0,+\infty[, \\ 0 & \text{if } t \in]-\infty, 0]. \end{cases}$$

Hence, putted $a_n = e^{n\pi}$ and $b_n = e^{\frac{2n+1}{2}\pi}$ for every $n \in \mathbf{N}$, one has that

$$\liminf_{t\to+\infty} \frac{H(t)}{t^p} \le \lim_{n\to+\infty} \frac{H(a_n)}{a_n^2} = 1 \tag{3.15}$$

and

$$\limsup_{t\to+\infty} \frac{H(t)}{t^p} \ge \lim_{n\to+\infty} \frac{H(b_n)}{b_n^2} = 3. \tag{3.16}$$

In view of Proposition 2.1 of [2], one has that

$$k \le \frac{1}{2\pi}. \tag{3.17}$$

Hence, from (3.12), (3.17) and the definition of $\bar{\mu}$, it follows that

$$L' = \frac{1}{k^2}\bar{\mu}(1-\bar{\mu})(1+\bar{\mu})^2 > \frac{9}{4}\pi^2. \tag{3.18}$$

Putting together (3.15), (3.16) and the definition of L', it is simple to verify that condition (3.14) holds, as well as

$$\left]\frac{2}{27}, \frac{\pi^2}{2}\right[\subset \Lambda'.$$

Finally, applying Corollary 3.6, one has that for every $\lambda \in \left]\frac{2}{27}, \frac{\pi^2}{2}\right[$ the following problem

$$\begin{cases} u^{iv} = \lambda h(u) & \text{in }]0,4[, \\ u(0) = u(4) = 0, \\ u''(0) = u''(4) = 0, \end{cases}$$

admits an unbounded sequence of weak solutions.

Similar reasonings assure the existence of infinitely many weak solutions to problem (1.1) converging at zero. More precisely, the following result holds.

Theorem 3.8. *Assume that* (i$_1$) *is satisfied. Suppose that*

(j$_2$) *There exist $x^0 \in X$, $0 < s_1 < s_2$ as considered in (3.1) such that, if we put*

$$\alpha^0 := \liminf_{t\to 0^+} \frac{\int_\Omega \max_{|\xi|\le t} F(x,\xi)dx}{t^p}, \qquad \beta^0 := \limsup_{t\to 0^+} \frac{\int_{B(x^0,s_1)} F(x,t)dx}{t^p},$$

one has

$$\alpha^0 < L\beta^0. \tag{3.19}$$

Then, for every $\lambda \in \frac{1}{pk^p}\left]\frac{1}{L\beta^0}, \frac{1}{\alpha^0}\right[$ problem (1.1) admits a sequence $\{u_n\}$ of weak solutions such that $u_n \to 0$.

Proof. Once observed that $\min_X \Phi = \Phi(0) = 0$, let $\{t_n\}$ be a sequence of positive numbers such that $t_n \to 0^+$ and

$$\lim_{n\to+\infty} \frac{\int_\Omega \max_{|\xi|\le t_n} F(x,\xi)dx}{t_n^p} = \alpha^0 < +\infty. \tag{3.20}$$

Putting $r_n = \frac{1}{p}\left(\frac{t_n}{k}\right)^p$ for every $n \in \mathbf{N}$ and working as in the proof of Theorem 3.1, it follows that $\delta < +\infty$.

Fix now $\lambda \in \frac{1}{pk^p}\left]\frac{1}{L\beta^0}, \frac{1}{\alpha^0}\right[$ and claim that

$$\Phi - \lambda\Psi \text{ has not a local minimum at zero.} \tag{3.21}$$

Let $\{\tau_n\}$ be a sequence of positive numbers and $\eta > 0$ such that $\tau_n \to 0^+$ and

$$\frac{1}{\lambda} < \eta < pk^pL\frac{\int_{B(x^0,s_1)} F(x,\tau_n)dx}{\tau_n^p} \tag{3.22}$$

for every $n \in \mathbf{N}$ large enough. Let $\{w_n\}$ be the sequence in X defined in (3.8). Putting together (3.9), (3.10) and (3.22) we achieve

$$\Phi(w_n) - \lambda\Psi(w_n) < \frac{\tau_n^p}{pk^pL}(1-\lambda\eta) < 0 = \Phi(0) - \lambda\Psi(0)$$

for every $n \in \mathbf{N}$ large enough, that implies claim (3.21) in view of the fact that $\|w_n\| \to 0$.

The alternative of Theorem 2.1 (case (c)) completes the proof. □

Remark 3.9. In the same spirit of the previous Theorem 3.8, it could be possible to obtain suitable versions of Theorem 3.4, as well as Corollary 3.6, when the '$\liminf$' and the '$\limsup$' are considered for $t \to 0^+$, in order to assure the existence of arbitrarily small weak solutions of problem (3.13).

References

[1] Bonanno, G., Molica Bisci, G., *Infinitely many solutions for a boundary value problem with discontinuous nonlinearities*, Bound. Value Probl., **2009** (2009), 1–20.

[2] Bonanno, G., Di Bella, B., *A boundary value problem for a fourth-order elastic beam equation*, J. Math. Anal. Appl., **343** (2008), 1166-1176.

[3] Bonanno, G., Di Bella, B., *Infinitely many solutions for a fourth-order elastic beam equation*, submitted.

[4] Guo, H. M., Geng, D., *Infinitely many solutions for the Dirichlet problem involving the p- biharmonic-like equation*, J. South China Normal Univ. Natur. Sci. Ed., **28** (2009), 18–21.

[5] Micheletti, A. M., Pistoia, A., *Multiplicity results for a fourth-order semilinear elliptic problems*, Nonlinear Anal. **60**, (1998), 895-908.

[6] Rabinowitz, P. H., *Minimax methods in critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conferences in Math., vol.**65** , Amer. Math. Soc. Providence, RI, 1985.

[7] Ricceri, B., *A general variational principle and some of its applications*, J. Comput. Appl. Math., **133** (2000), 401-410.

[8] Struwe, M., *Variational Methods*, Springer, Berlin, 1996.

[9] Wang, Y., Shen, Y., Infinitely many sign-changing solutions for a class of biarmonic equations without symmetry, Nonlinear Anal., **71** (2009), 967-977.

[10] Wang, W., Zhao, P., *Nonuniformly nonlinear elliptic equations of p-biarmonic type*, J. Math. Anal. Appl. **348** (2008), 730-738.

[11] Li, C., Tang, C. L., *Three solutions for a Navier boundary value problem involving the p-biharmonic*, Nonlinear Anal., **72** (2010) 1339-1347.

[12] Yang, Z., Geng, D., Yan, H., *Existence of multiple solutions for a semilinear biharmonic equations with critical exponent*, Acta Math. Sci. Ser. A, **27** (1) (2006), 129-142.

[13] Tehrani, H. T., *Infinitely many solutions for indefinite semilinear elliptic equations without symmetry*, Comm. Partial Differential Equations, **21** (1996), 541-557.

[14] Zhang, J., Li, S., *Multiple non trivial solutions for some fourth-order semilinear elliptic problems*, Nonlinear Anal., **60** (2005), 221-230.

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COMBINED VARIATIONAL AND SUB-SUPERSOLUTION APPROACH FOR MULTI-VALUED ELLIPTIC VARIATIONAL INEQUALITIES

SIEGFRIED CARL

Abstract. This paper provides a variational approach for a class of multi-valued elliptic variational inequalities governed by the p-Laplacian and Clarke's generalized gradient of some locally Lipschitz function including a number of (multi-valued) elliptic boundary value problems as special cases. Since only local growth conditions are imposed on the multi-valued term, the problem under consideration is neither coercive nor of variational structure beforehand meaning that it cannot be related to the derivative of some associated (nonsmooth) potential. By combining a recently developed sub-supersolution method for multi-valued elliptic variational inequalities and a suitable modification of the given locally Lipschitz function the main goal of this paper is to construct a (nonsmooth) functional whose critical points turn out to be solutions of the problem under consideration lying in an ordered interval of sub-supersolution.

1. **Introduction**

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and let $V = W^{1,p}(\Omega)$ and $V_0 = W_0^{1,p}(\Omega)$, $1 < p < +\infty$, denote the usual Sobolev spaces with their dual spaces V^* and V_0^*, respectively. Let K be a closed, convex subset of V, and let $j : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function $(x, s) \mapsto j(x, s)$ that is only supposed to be measurable in $x \in \Omega$ and locally Lipschitz continuous in $s \in \mathbb{R}$. Let q denote the Hölder conjugate to p, i.e., q satisfies $1/p + 1/q = 1$. In this paper we are dealing with the following multi-valued variational inequality: Find $u \in K$, $\eta \in L^q(\Omega)$ such that

$$\langle -\Delta_p u, v - u\rangle + \int_\Omega \eta\,(v - u)\,dx \geq 0, \quad \forall\, v \in K, \tag{1.1}$$

$$\eta(x) \in \partial j(x, u(x)) \ \text{ for a.a. } x \in \Omega, \tag{1.2}$$

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where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $s \mapsto \partial j(x,s)$ denotes Clarke's generalized gradient of some locally Lipschitz function $s \mapsto j(x,s)$, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing. The operator $-\Delta_p$ is defined by

$$\langle -\Delta_p u, \varphi \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \, \nabla \varphi \, dx, \quad \forall \, \varphi \in V,$$

which implies that $-\Delta_p : V \to V^*$ is continuous, bounded, monotone, and thus pseudomonotone, see [1, Theorem 2.109, Lemma 2.111].

Only for the sake of simplifying our presentation and in order to emphasize the key ideas we have confined our consideration to problem (1.1)-(1.2). Making use of the arguments developed in this paper, more general multi-valued problems can be considered as well such as, for example, the following one: Find $u \in K$, $\eta \in L^q(\Omega)$, and $\xi \in L^q(\partial\Omega)$ such that

$$\begin{cases} \eta(x) \in \partial j_1(x, u(x)), \text{ a.e. } x \in \Omega, \quad \xi(x) \in \partial j_2(x, \gamma u(x)), \text{ a.e. } x \in \partial\Omega, \\ \langle -\Delta_p u - h, v - u \rangle + \displaystyle\int_\Omega \eta \, (v - u) \, dx + \int_{\partial\Omega} \xi \, (\gamma v - \gamma u) \, d\sigma \geq 0, \ \forall \, v \in K, \end{cases} \tag{1.3}$$

where $\gamma : V \to L^p(\partial\Omega)$ denotes the trace operator, and $h \in V^*$.

The main goal of this paper is to develop a variational approach to the multi-valued variational inequality (1.1)-(1.2). Since only a local growth condition is imposed on the multi-valued term, the problem under consideration is neither coercive nor of variational structure beforehand meaning that it cannot be related to the derivative of some associated (nonsmooth) potential. Therefore, the main difficulty one is faced with is to associate to (1.1)-(1.2) a corresponding potential that can be studied by (nonsmooth) variational methods. By combining a recently developed sub-supersolution method for elliptic variational inequalities (see [1]) with a suitable modification of the given locally Lipschitz function, the aim of this paper is to construct a (nonsmooth) functional whose critical points turn out to be solutions of problem (1.1)-(1.2).

2. **Special Cases**

Let us consider a few special cases that are included in (1.1)-(1.2).

Example 2.1. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Consider its primitive given by

$$j(x,s) := \int_0^s f(x,t) \, dt.$$

Then the function $s \mapsto j(x,s)$ is continuously differentiable, and thus Clarke's gradient reduces to a singleton, i.e.,

$$\partial j(x,s) = \{\partial j(x,s)/\partial s\} = \{f(x,s)\}.$$

If $K = V$, then (1.1)-(1.2) becomes the following quasilinear elliptic boundary value problem (BVP)

$$\langle -\Delta_p u, v\rangle + \int_\Omega f(x,u)\, v\, dx = 0, \quad \forall\, v \in V, \tag{2.1}$$

which is the formulation for the weak solution of the quasilinear Neumann BVP

$$-\Delta_p u + f(x,u) = 0 \;\text{ in }\; \Omega, \;\; \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \;\; \partial\Omega, \tag{2.2}$$

where $\partial/\partial\nu$ denotes the outward pointing conormal derivative associated with $-\Delta_p$.

Example 2.2. If $K = V_0$, and j as in Example 2.1, then (1.1)-(1.2) is equivalent to

$$u \in V_0: \quad \langle -\Delta_p u, v\rangle + \int_\Omega f(x,u)\, v\, dx = 0, \quad \forall\, v \in V_0, \tag{2.3}$$

which is nothing but the weak formulation of the homogeneous Dirichlet problem

$$-\Delta_p u + f(x,u) = 0 \;\text{ in }\; \Omega, \;\; u = 0 \quad \text{on} \;\; \partial\Omega. \tag{2.4}$$

Example 2.3. If $K = V_0$ or $K = V$, then (1.1)-(1.2) reduces to elliptic inclusion problems, which for $K = V_0$ yields the following multi-valued Dirichlet problem

$$-\Delta_p u + \partial j_1(x,u) \ni 0 \;\text{ in }\; \Omega, \;\; u = 0 \quad \text{on} \;\; \partial\Omega, \tag{2.5}$$

and for $K = V$ the multi-valued Neumann BVP

$$-\Delta_p u + \partial j(x,u) \ni 0 \;\text{ in }\; \Omega, \;\; \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \;\; \partial\Omega. \tag{2.6}$$

Example 2.4. Let Γ_1 and Γ_2 be relatively open subsets of $\partial\Omega$ satisfying $\overline{\Gamma_1} \cup \overline{\Gamma_2} = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. If $K \subseteq V$ is the closed subspace given by

$$K = \{v \in V : \gamma v = 0 \;\text{ on }\; \Gamma_1\},$$

then we obtain the following special case of (1.1)-(1.2):

$$-\Delta_p u + \partial j(x,u) \ni 0 \;\text{ in }\; \Omega, \;\; \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \;\; \Gamma_2, \;\; u = 0 \;\text{ on }\; \Gamma_1. \tag{2.7}$$

Example 2.5. If $K \subseteq V$, and $j = 0$, then (1.1)-(1.2) is equivalent to the usual variational inequality of the form

$$u \in K: \quad \langle -\Delta_p u, v - u\rangle \geq 0, \quad \forall\, v \in K.$$

3. Definitions, Assumptions and Preliminaries

Based on comparison principles for nonsmooth variational problems developed in [1] we first provide a natural extension of the notion of sub-supersolution to the multi-valued variational problem (1.1)-(1.2). To this end we introduce the following notations for functions w, z and sets W and Z of functions defined on Ω: $w \wedge z = \min\{w,z\}$, $w \vee z = \max\{w,z\}$, $W \wedge Z = \{w \wedge z : w \in W,\ z \in Z\}$, $W \vee Z = \{w \vee z : w \in W,\ z \in Z\}$, and $w \wedge Z = \{w\} \wedge Z$, $w \vee Z = \{w\} \vee Z$.

Definition 3.1. A function $\underline{u} \in V$ is called a **subsolution** of (1.1)-(1.2) if there is an $\underline{\eta} \in L^q(\Omega)$ satisfying

$$
\begin{aligned}
&(i)\ \underline{u} \vee K \subseteq K,\\
&(ii)\ \underline{\eta}(x) \in \partial j(x, \underline{u}(x)), \text{for a.e. } x \in \Omega,\\
&(iii)\ \langle -\Delta_p \underline{u}, v - \underline{u}\rangle + \int_\Omega \underline{\eta}\,(v - \underline{u})\,dx \ge 0, \quad \text{for all} \quad v \in \underline{u} \wedge K.
\end{aligned}
$$

Definition 3.2. A function $\overline{u} \in V$ is called a **supersolution** of (1.1)-(1.2) if there is an $\overline{\eta} \in L^q(\Omega)$ satisfying

$$
\begin{aligned}
&(i)\ \overline{u} \wedge K \subseteq K,\\
&(ii)\ \overline{\eta}(x) \in \partial j(x, \overline{u}(x)),\ \text{for a.e. } x \in \Omega,\\
&(iii)\ \langle -\Delta_p \overline{u}, v - \overline{u}\rangle + \int_\Omega \overline{\eta}\,(v - \overline{u})\,dx \ge 0, \quad \text{for all } v \in \overline{u} \vee K.
\end{aligned}
$$

Remark 3.3. Note that the notions for sub- and supersolution defined in Definition 3.1 and Definition 3.2 have a symmetric structure, i.e., one obtains the definition for the supersolution $\overline{u}$ from the definition of the subsolution by replacing $\underline{u}$ in Definition 3.1 by $\overline{u}$, and interchanging $\vee$ by $\wedge$.

To see that Definitions 3.1 and 3.2 are in fact natural extensions of the usual notions of sub-supersolutions for elliptic BVP let us consider the following special cases.

Example 3.4. Consider Example 2.1, i.e., $K = V$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and j is the primitive of f as given above. Then Clarke's generalized gradient ∂j reduces to a singleton, i.e.,

$$\partial j(x,s) = \{f(x,s)\},$$

and (1.1)-(1.2) becomes the quasilinear elliptic BVP (2.1). If $\underline{u} \in V$ is a subsolution according to Definition 3.1, then the first condition (i) is trivially satisfied. The second condition (ii) of Definition 3.1 means that

$$\underline{\eta}(x) = f(x, \underline{u}(x)), \text{ for a.e. } x \in \Omega.$$

Since $K = V$, any $v \in \underline{u} \wedge V$ has the form $v = \underline{u} \wedge \varphi = \underline{u} - (\underline{u} - \varphi)^+$ with $\varphi \in V$, where $w^+ = \max\{w, 0\}$, condition (iii) becomes

$$\langle -\Delta_p \underline{u}, -(\underline{u} - \varphi)^+ \rangle + \int_\Omega f(\cdot, \underline{u}) \, (-(\underline{u} - \varphi)^+) \, dx \geq 0, \quad \forall \, \varphi \in V. \tag{3.1}$$

Since $\underline{u} \in V$, we have

$$M = \{(\underline{u} - \varphi)^+ : \varphi \in V\} = V \cap L^p_+(\Omega),$$

where $L^p_+(\Omega)$ is the positive cone of $L^p(\Omega)$, and thus we obtain from inequality (3.1)

$$\langle -\Delta_p \underline{u}, \chi \rangle + \int_\Omega f(x, \underline{u}) \, \chi \, dx \leq 0, \quad \forall \, \chi \in V \cap L^p_+(\Omega, \tag{3.2}$$

which is nothing but the usual notion of a (weak) subsolution for the BVP (2.1). Similarly, one verifies that $\overline{u} \in V$ which is a supersolution according to Definition 3.2 is equivalent with the usual supersolution of the BVP (2.1).

Example 3.5. In case that $K = V_0$, and j as in Example 3.4, then (1.1)-(1.2) is equivalent to the BVP (2.3) (resp. (2.4)). Let us consider the notion of subsolution in this case given via Definition 3.1. For $\underline{u} \in V$ condition (i) means $\underline{u} \vee V_0 \subseteq V_0$. This last condition is satisfied if and only if

$$\gamma \underline{u} \leq 0 \quad \text{i.e.,} \quad \underline{u} \leq 0 \ \text{ on } \ \partial\Omega, \tag{3.3}$$

and condition (ii) means, as above,

$$\underline{\eta}(x) = f(x, \underline{u}(x)), \text{ a.e. } x \in \Omega.$$

Since any $v \in \underline{u} \wedge V_0$ can be represented in the form $v = \underline{u} - (\underline{u} - \varphi)^+$ with $\varphi \in V_0$, from (iii) of Definition 3.1 we obtain

$$\langle -\Delta_p \underline{u}, -(\underline{u} - \varphi)^+ \rangle + \int_\Omega f(\cdot, \underline{u}) \, (-(\underline{u} - \varphi)^+) \, dx \geq 0, \ \forall \, \varphi \in V_0. \tag{3.4}$$

Set $\chi = (\underline{u} - \varphi)^+$, then (3.4) results in

$$\langle -\Delta_p \underline{u}, \chi \rangle + \int_\Omega f(\cdot, \underline{u}) \, \chi \, dx \leq 0, \ \forall \, \chi \in M_0, \tag{3.5}$$

where $M_0 := \{\chi \in V : \chi = (\underline{u} - \varphi)^+, \ \varphi \in V_0\} \subseteq V_0 \cap L^p_+(\Omega)$. In [1] it has been proved that the set M_0 is a dense subset of $V_0 \cap L^p_+(\Omega)$, which shows that (3.5) together with (3.3) is nothing but the weak formulation for the subsolution of the Dirichlet problem (2.3). Similarly, $\overline{u} \in V$ given by Definition 3.6 is shown to be a supersolution of the Dirichlet problem (2.3).

Assumption on j.

(H1) The function $j : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies: $x \mapsto j(x,s)$ is measurable in Ω for all $s \in \mathbb{R}$, and $s \mapsto j(x,s)$ is locally Lipschitz continuous in $\mathbb{R}$ for a.e. $x \in \Omega$.

We next introduce a certain local L^q-boundedness condition for Clarke's generalized gradient $s \mapsto \partial j(x,s)$.

Definition 3.6. Let $[v,w] \subset L^p(\Omega)$ be an ordered interval. Clarke's gradient $\partial j : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ is called **$\mathbf{L^q}$-bounded with respect to the ordered interval $[\mathbf{v},\mathbf{w}]$** provided that there exists $k_\Omega \in L^q_+(\Omega)$ such that for a.e. $x \in \Omega$ and for all $s \in [v(x), w(x)]$ the inequality

$$|\eta| \le k_\Omega(x), \quad \forall\, \eta \in \partial j(x,s),$$

is fulfilled.

Remark 3.7. (i) We note that $\partial j : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ is trivially L^q-bounded with respect to any ordered interval $[v,w] \subset L^p(\Omega)$ if we suppose the following natural growth condition on ∂j: There exist $c > 0$, $k_\Omega \in L^q_+(\Omega)$ such that

$$|\eta| \le k_\Omega(x) + c|s|^{p-1}, \quad \forall\, \eta \in \partial j(x,s),$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$.

(ii) If ∂j is a singleton, i.e., $\partial j(x,s) = \{f(x,s)\}$ then accordingly we call the function $(x,s) \mapsto f(x,s)$ L^q-bounded with respect to the ordered interval $[v,w] \subset L^p(\Omega)$ provided that there exists $k_\Omega \in L^q_+(\Omega)$ such that for a.e. $x \in \Omega$ and for all $s \in [v(x), w(x)]$ the inequalty

$$|f(x,s)| \le k_\Omega(x)$$

is fulfilled.

The construction of an appropriate functional related to (1.1)-(1.2) relies amongst others on a suitable modification of the function j outside the interval $[\underline{u}, \overline{u}]$ formed by a given pair of sub- and supersolutions. Let $(\underline{u}, \underline{\eta}) \in V \times L^q(\Omega)$ and $(\overline{u}, \overline{\eta}) \in V \times L^q(\Omega)$ satisfy the conditions of Definition 3.1 and Definition 3.2, respectively, with $\underline{u} \le \overline{u}$. Then we define the following modification $\tilde{j}$ of the given j:

$$\tilde{j}(x,s) = \begin{cases} j(x,\underline{u}(x)) + \underline{\eta}(x)(s - \underline{u}(x)) & \text{if} \quad s < \underline{u}(x), \\ j(x,s) & \text{if} \quad \underline{u}(x) \le s \le \overline{u}(x), \\ j(x,\overline{u}(x)) + \overline{\eta}(x)(s - \overline{u}(x)) & \text{if} \quad s > \overline{u}(x). \end{cases} \tag{3.6}$$

Assumption on j.

(H2) Let $\underline{u}$ and $\overline{u}$ be sub-and supersolution of (1.1)-(1.2) such that $\underline{u} \le \overline{u}$. We assume that $\partial j : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ is L^q-bounded with respect to the ordered interval $[\underline{u}, \overline{u}]$.

Lemma 3.8. *Let hypotheses (H1)–(H2) be satisfied. Then the function $\tilde{j} : \Omega\times\mathbb{R} \to \mathbb{R}$ has the following properties:*

(i) *$x \mapsto \tilde{j}(x,s)$ is measurable in Ω for all $s \in \mathbb{R}$, and $s \mapsto \tilde{j}(x,s)$ is Lipschitz continuous in $\mathbb{R}$ for a.e. $x \in \Omega$.*

(ii) *Let $\partial\tilde{j}$ denote Clarke's generalized gradient of $s \mapsto \tilde{j}(x,s)$, then for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$ the growth*

$$|\eta| \le k_\Omega(x), \quad \forall\, \eta \in \partial\tilde{j}(x,s)$$

is fulfilled.

(iii) *Clarke's generalized gradient of $s \mapsto \tilde{j}(x,s)$ is given by*

$$\partial\tilde{j}(x,s) = \begin{cases} \underline{\eta}(x) & \text{if} \quad s < \underline{u}(x), \\ \partial\tilde{j}(x,\underline{u}(x)) & \text{if} \quad s = \underline{u}(x), \\ \partial j(x,s) & \text{if} \quad \underline{u}(x) < s < \overline{u}(x), \\ \partial\tilde{j}(x,\overline{u}(x)) & \text{if} \quad s = \overline{u}(x), \\ \overline{\eta}(x) & \text{if} \quad s > \overline{u}(x), \end{cases} \tag{3.7}$$

and the inclusions $\partial\tilde{j}(x,\underline{u}(x)) \subseteq \partial j(x,\underline{u}(x))$ and $\partial\tilde{j}(x,\overline{u}(x)) \subseteq \partial j(x,\overline{u}(x))$ hold true.

Proof. The proof follows immediately from the definition (3.6) of $\tilde{j}$, and using the assumptions (H1)–(H2) on j as well as from the fact that Clarke's generalized gradient $\partial j_1(x,s)$ is a convex set. ☐

Using $\tilde{j}$ we define an integral functional $\tilde{J}$ on $L^p(\Omega)$ given by

$$\tilde{J}(u) = \int_\Omega \tilde{j}(x,u(x))\,dx, \quad u \in L^p(\Omega). \tag{3.8}$$

Due to (ii) of Lemma 3.8, and applying Lebourg's mean value theorem (see [1, Theorem 2.177]) the functional $\tilde{J} : L^p(\Omega) \to \mathbb{R}$ is well-defined and Lipschitz continuous, so that Clarke's generalized gradients $\partial\tilde{J} : L^p(\Omega) \to 2^{(L^p(\Omega))^*}$ is well-defined too. Moreover, Aubin–Clarke theorem (cf. [8, p. 83]) provides the following characterization of the generalized gradient. For $u \in L^p(\Omega)$ we have

$$\tilde{\eta} \in \partial\tilde{J}(u) \Longrightarrow \tilde{\eta} \in L^q(\Omega) \text{ with } \tilde{\eta}(x) \in \partial\tilde{j}(x,u(x)) \text{ for a.e. } x \in \Omega. \tag{3.9}$$

Lemma 3.9. *Let $i : V \hookrightarrow L^p(\Omega)$ denote the embedding operator and let $i^* : L^q(\Omega) \hookrightarrow V^*$ be its adjoint operator. Then Clarke's generalized gradient of $\tilde{J}$ at $u \in V$ is given* *by*

$$\partial\tilde{J}(u) = \partial(\tilde{J}\circ i)(u) = (i^*\circ\partial\tilde{J}\circ i)(u) = i^*\partial\tilde{J}(u), \quad \forall\, u\in V.$$

Proof. Apply the chain rule, cf. [1, Corollary 2.180]. □

Finally, let b be the cut-off function related to an ordered pair $(\underline{u},\overline{u})$ of sub-supersolution and defined as follows:

$$b(x,s) = \begin{cases} (s-\overline{u}(x))^{p-1} & \text{if} \quad s>\overline{u}(x) \\ 0 & \text{if} \quad \underline{u}(x)\le s\le\overline{u}(x) \\ -(\underline{u}(x)-s)^{p-1} & \text{if} \quad s<\underline{u}(x). \end{cases}$$

Apparently, $b:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying the growth condition

$$|b(x,s)| \le k(x) + c_1|s|^{p-1} \tag{3.10}$$

for a.e. $x\in\Omega$ and for all $s\in\mathbb{R}$, where $c_1>0$ is a constant and $k\in L^q_+(\Omega)$. Moreover, one has the following estimate

$$\int_\Omega b(x,u(x))\,u(x)\,dx \ge c_2\|u\|^p_{L^p(\Omega)} - c_3, \quad \forall u\in L^p(\Omega), \tag{3.11}$$

for some constants $c_2>0$ and $c_3>0$. Due to (3.10) the functional $\mathbb{B}$ given by

$$\mathbb{B}(u) = \int_\Omega\int_0^{u(x)} b(x,s)\,dsdx, \quad \forall\, u\in L^p(\Omega) \tag{3.12}$$

is well defined, and $\mathbb{B}\in C^1(V,\mathbb{R})$ with

$$\langle\mathbb{B}'(u),\varphi\rangle = \int_\Omega b(x,u(x))\,\varphi(x)\,dx, \quad \forall u\in V. \tag{3.13}$$

Lemma 3.10. *There exist constants $c_4>0$, $c_5>0$ such that*

$$\mathbb{B}(u) \ge c_4\|u\|^p_{L^p(\Omega)} - c_5, \quad \forall u\in L^p(\Omega). \tag{3.14}$$

Proof. From the definition of the cut-off function b we readily see that β given by

$$\beta(x,s) = \begin{cases} \frac{1}{p}(s-\overline{u}(x))^{p} & \text{if} \quad s>\overline{u}(x) \\ 0 & \text{if} \quad \underline{u}(x)\le s\le\overline{u}(x) \\ \frac{1}{p}(\underline{u}(x)-s)^{p} & \text{if} \quad s<\underline{u}(x). \end{cases} \tag{3.15}$$

is a primitive of $s\mapsto b(x,s)$, i.e., $\partial\beta(x,s)/\partial s = b(x,s)$, which yields

$$\int_0^{u(x)} b(x,s)\,ds = \beta(x,u(x)) - \beta(x,0). \tag{3.16}$$

By using (3.15) we get the estimate

$$|\beta(x,0)| \le \frac{1}{p}(|\underline{u}|^p + |\overline{u}|^p). \tag{3.17}$$

For functions v, $w \in L^p(\Omega)$ we denote

$$\{v < (\leq) w\} = \{x \in \Omega : v(x) < (\leq) w(x)\}.$$

We next estimate the first term on the right-hand side of (3.16). To this end we make use of the following inequality:

$$|u(x)|^p \leq c\left(|u(x) - \overline{u}(x)|^p + |\overline{u}(x)^p\right)$$

for some generic positive constant c, which yields

$$\frac{1}{p}|u(x) - \overline{u}(x)|^p \geq \frac{1}{pc}|u(x)|^p - \frac{1}{p}|\overline{u}(x)|^p, \tag{3.18}$$

and similarly

$$\frac{1}{p}|\underline{u}(x) - u(x)|^p \geq \frac{1}{pc}|u(x)|^p - \frac{1}{p}|\underline{u}(x)|^p. \tag{3.19}$$

With the help of (3.15) and (3.18)–(3.19) we obtain

$$\begin{aligned} \int_\Omega \beta(x, u(x))\, dx &= \int_{\{u>\overline{u}\}} \beta(x, u(x))\, dx + \int_{\{u<\underline{u}\}} \beta(x, u(x))\, dx \\ &\geq \frac{1}{pc}\int_\Omega |u(x)|^p\, dx - \frac{1}{pc}\int_{\{\underline{u}\leq u\leq\overline{u}\}} |u(x)|^p\, dx \\ &\quad -\frac{1}{p}\int_\Omega \left(|\underline{u}(x)|^p + |\overline{u}(x)|^p\right) dx \\ &\geq \frac{1}{pc}\|u\|^p_{L^p(\Omega)} - \frac{2}{p}\left(\|\underline{u}\|^p_{L^p(\Omega)} + \|\overline{u}\|^p_{L^p(\Omega)}\right). \end{aligned} \tag{3.20}$$

Finally, (3.16), (3.17) and (3.20) imply the assertion of the lemma, i.e.,

$$\mathbb{B}(u) \geq \frac{1}{pc}\|u\|^p_{L^p(\Omega)} - \frac{3}{p}\left(\|\underline{u}\|^p_{L^p(\Omega)} + \|\overline{u}\|^p_{L^p(\Omega)}\right)$$

with

$$c_4 = \frac{1}{pc},\ c_5 = \frac{3}{p}\left(\|\underline{u}\|^p_{L^p(\Omega)} + \|\overline{u}\|^p_{L^p(\Omega)}\right).$$

□

4. **Combined Variational and Sub-Supersolution Approach**

In this section we formulate and prove our main result. A crucial role in our approach plays the following functional

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx + \mathbb{B}(u) + \tilde{J}(u), \quad u \in V, \tag{4.1}$$

where $\mathbb{B}$ and $\tilde{J}(u)$ are defined by (3.8) and (3.12), respectively.

Lemma 4.1. *Let hypotheses (H1) and (H2) be satisfied. Then the functional $\Phi : V \to \mathbb{R}$ is locally Lipschitz continuous, bounded below, coercive, and weakly lower semicontinuous.*

Proof. By the definition of $\tilde{J}$ and due to Lemma 3.8 (ii) we readily see that $\tilde{J} : L^p(\Omega) \to \mathbb{R}$ is Lipschitz continuous, which in view of the compact embedding $V \hookrightarrow L^p(\Omega)$ shows that $\tilde{J} : V \to \mathbb{R}$ is weakly lower semicontinuous. The functionals

$$u \mapsto P(u) := \frac{1}{p} \int_\Omega |\nabla u|^p \, dx \text{ and } u \mapsto \mathbb{B}(u)$$

are $C^1(V, \mathbb{R})$, and thus, in particular, locally Lipschitz continuous as well. The derivative $P' + \mathbb{B}' : V \to V^*$ results in

$$\langle P'(u) + \mathbb{B}'(u), \varphi \rangle = \langle -\Delta_p u + \mathbb{B}'(u), \varphi \rangle = \int_\Omega \Big(|\nabla u|^{p-2} \nabla u \nabla \varphi + b(\cdot, u)\varphi \Big) \, dx, \quad \forall \, \varphi \in V.$$

Taking into account (3.10) and applying [1, Theorem 2.109, Lemma 2.111] we see that the operator $P' + \mathbb{B}' : V \to V^*$ is bounded, and pseudomonotone, which in view of [10, Proposition 41.8] implies that the functional $P + \mathbb{B} : V \to \mathbb{R}$ is weakly lower semicontinuous. Due to Lemma 3.8 (ii), (iii) the functional $\tilde{J} : L^p(\Omega) \to \mathbb{R}$ is (globally) Lipschitz continuous with Lipschitz constant L. Thus by means of Lemma 3.10 we obtain the following estimate (for some constant $c_6 > 0$)

$$\Phi(u) = P(u) + \mathbb{B}(u) + \tilde{J}(u) \geq \frac{1}{p} \|\nabla u\|^p_{L^p(\Omega)} + c_4 \|u\|^p_{L^p(\Omega)} - L\|u\|_{L^p(\Omega)} - c_6, \quad (4.2)$$

which shows that $\Phi : V \to \mathbb{R}$ is bounded below and coercive. □

Let $I_K : V \to \mathbb{R} \cup \{+\infty\}$ be the indicator function related to the given closed convex set $K \neq \emptyset$, i.e.,

$$I_K(u) = \begin{cases} 0 & \text{if} \quad u \in K, \\ +\infty & \text{if} \quad u \notin K, \end{cases}$$

which is known to be proper, convex, and lower semicontinuous, and thus weakly lower semicontinuous as well (cf. [10, Proposition 38.7]). The following functional $\mathbb{E} : V \to \mathbb{R} \cup \{+\infty\}$ will allow us to study the multi-valued variational inequality (1.1)–(1.2) via variational methods for nonsmooth and nonconvex functionals:

$$\mathbb{E}(u) = \Phi(u) + I_K(u), \quad u \in V, \quad (4.3)$$

i.e., $\mathbb{E}$ is the sum of a locally Lipschitz functional and a convex, proper and lower semicontinuous functional. This type of functional has been studied, e.g., in [9].

Definition 4.2. The function $u \in V$ is called a **critical point** of $\mathbb{E} : V \to \mathbb{R} \cup \{+\infty\}$ if the following holds:

$$\Phi^o(u; v-u) + I_K(v) - I_K(u) \ge 0, \quad \forall\, v \in V,$$

where $\Phi^o(u;v)$ denotes Clarke's generalized directional derivative of Φ at u in the direction v.

The following definition is equivalent to Definition 4.2, see [9, p.46].

Definition 4.3. The function $u \in V$ is called a **critical point** of $\mathbb{E} : V \to \mathbb{R} \cup \{+\infty\}$ if and only if

$$0 \in \partial\Phi(u) + \partial I_K(u),$$

where $\partial\Phi(u)$ denotes Clarke's generalized gradient of Φ at u, and $\partial I_K(u)$ is the subdifferential of I_K at u in the sense of convex analysis.

Our main result is given by the following theorem.

Theorem 4.4. *Let hypotheses (H1)–(H2) be satisfied. Then the functional* $\mathbb{E} = \Phi + I_K : V \to \mathbb{R} \cup \{+\infty\}$ *possesses critical points. Moreover, any critical point* u *of* $\mathbb{E}$ *is a solution of the multi-valued variational inequality (1.1)–(1.2) which belongs to the ordered interval* $[\underline{u}, \overline{u}]$ *formed by the given ordered sub- and supersolution.*

Proof. (a) *Existence of critical points*

By Lemma 4.1 in conjunction with the properties of the indicator function I_K, the functional $\mathbb{E} : V \to \mathbb{R} \cup \{+\infty\}$ defined in (4.3) is weakly lower semicontinuous, bounded below and coercive. Applying the basic minimization principle (cf. e.g. [10, Proposition 38.15]) there exists a global minimizer u of $\mathbb{E}$ which necessarily is a critical point of $\mathbb{E}$ (see [9]), i.e., $u \in K$ and $0 \in \partial\Phi(u) + \partial I_K(u)$.

(b) *Critical points are solutions of (1.1)–(1.2) in* $[\underline{u}, \overline{u}]$

Let $u \in K$ be a critical point of $\mathbb{E}$, which implies the existence of an $\xi \in \partial\Phi(u)$ satisfying $-\xi \in \partial I_K(u)$. The latter is equivalent to

$$\langle \xi, v-u \rangle \ge 0, \quad \forall\, v \in K. \tag{4.4}$$

Since Φ is the sum of a differentiable functional and Lipschitz continuous functional, we have

$$\partial\Phi(u) = P'(u) + \mathbb{B}'(u) + \partial\tilde{J}(u),$$

and $\xi \in \partial\Phi(u)$ leads to

$$\xi = P'(u) + \mathbb{B}'(u) + i^*\tilde{\eta}, \tag{4.5}$$

where $\tilde{\eta} \in \partial\tilde{J}(u)$, which in turn implies (see (3.9)) that $\tilde{\eta} \in L^q(\Omega) \hookrightarrow V^*$ and $\tilde{\eta}(x) \in \partial\tilde{j}(x, u(x))$. Hence by (4.4), (4.5) it follows that to any critical point u of

$\mathbb{E}$ there is an $\tilde\eta \in L^q(\Omega)$ such that the following multi-valued variational inequality holds:

$$u \in K,\ \tilde\eta(x) \in \partial\tilde{j}(x, u(x)) : \langle -\Delta_p u + \mathbb{B}'(u) + i^*\tilde\eta, v - u\rangle \ge 0, \quad \forall\, v \in K. \tag{4.6}$$

By comparison we are going to prove next that any solution of (4.6) belongs to the interval $[\underline{u}, \overline{u}]$. We first note that (4.6) is equivalent to

$$u \in K,\ \tilde\eta(x) \in \partial\tilde{j}(x, u(x)) : \langle -\Delta_p u, v - u\rangle + \int_\Omega \big(b(\cdot, u) + \tilde\eta\big)\,(v - u)\,dx \ge 0, \quad \forall\, v \in K. \tag{4.7}$$

Let us show first that for any solution u of (4.7) the inequality $u \le \overline{u}$ holds, where $\overline{u}$ is the given supersolution of (1.1)–(1.2). To this end we recall the definition of $\overline{u}$ according to Definition 3.2: $\overline{u} \in V$ satisfies

$$(i)\ \overline{u} \wedge K \subseteq K,$$
$$(ii)\ \overline{\eta}(x) \in \partial j(x, \overline{u}(x)),\ \text{ for a.e. } x \in \Omega,$$
$$(iii)\ \langle -\Delta_p \overline{u}, v - \overline{u}\rangle + \int_\Omega \overline{\eta}\,(v - \overline{u})\,dx \ge 0, \quad \text{for all } v \in \overline{u} \vee K.$$

We apply the special test function $v = \overline{u} \vee u = \overline{u} + (u - \overline{u})^+$ in (iii), and $v = \overline{u} \wedge u = u - (u - \overline{u})^+ \in K$ in (4.7), and get by adding the resulting inequalities (with $A := -\Delta_p$ for short)

$$\langle A\overline{u} - Au, (u - \overline{u})^+\rangle - \int_\Omega b(\cdot, u)\,(u - \overline{u})^+\,dx + \int_\Omega (\overline{\eta} - \tilde\eta)\,(u - \overline{u})^+\,dx \ge 0. \tag{4.8}$$

Applying Lemma 3.8 (iii) we have

$$\int_\Omega (\overline{\eta} - \tilde\eta)\,(u - \overline{u})^+\,dx = \int_{\{u > \overline{u}\}} (\overline{\eta} - \tilde\eta)\,(u - \overline{u})\,dx = 0, \tag{4.9}$$

because $\tilde\eta(x) = \overline{\eta}(x)$ for $x \in \{u > \overline{u}\}$. Taking the definition of the cut-off function b into account we get

$$\int_\Omega b(\cdot, u)\,(u - \overline{u})^+\,dx = \int_\Omega \Big((u - \overline{u})^+\Big)^p\,dx. \tag{4.10}$$

The first trem on the left-hand side of (4.8)yields the estimate

$$\langle A\overline{u} - Au, (u - \overline{u})^+\rangle = -\langle Au - A\overline{u}, (u - \overline{u})^+\rangle \le 0. \tag{4.11}$$

Applying the results (4.9)–(4.11) to (4.8) we finally obtain

$$\int_\Omega \Big((u - \overline{u})^+\Big)^p\,dx = 0,$$

which implies $(u - \overline{u})^+ = 0$, and thus $u \le \overline{u}$. The proof for $\underline{u} \le u$ can be done in a similar way.

So far we have shown that any solution u of the multi-valued variational inequality (4.7) belongs to the interval $[\underline{u},\overline{u}]$, and thus satisfies: $u \in K$, $b(x,u(x)) = 0$, $\tilde{\eta} \in L^q(\Omega)$ and

$$\tilde{\eta}(x) \in \partial\tilde{j}(x,u(x)), \text{ a.e. } x \in \Omega, \tag{4.12}$$

$$\langle Au, v-u\rangle + \int_\Omega \tilde{\eta}\,(v-u)\,dx \geq 0, \ \forall\, v \in K. \tag{4.13}$$

From Lemma 3.8 (iii) we see that $\partial\tilde{j}(x,u(x)) \subseteq \partial j(x,u(x))$ for any $u \in [\underline{u},\overline{u}]$, and therefore we also have

$$\tilde{\eta}(x) \in \partial j(x,u(x)), \text{ a.e. } x \in \Omega,$$

which shows that the solution $u \in [\underline{u},\overline{u}]$ of the problem (4.7) is in fact a solution of the original multi-valued variational inequality (1.1)–(1.2). This completes the proof. □

Remark 4.5. (i) Theorem 4.4 yields the desired variational tool in form of the nonsmooth functional $\mathbb{E}$ given by (4.1) and (4.3), which not only allows to get existence results for the multi-valued variational inequality (1.1)–(1.2), but also to localize the critical points of $\mathbb{E}$, i.e., any critical point of $\mathbb{E}$ belongs automatically to the ordered interval $[\underline{u},\overline{u}]$. Under the assumptions (H1)–(H2) we were able to verify the existence of critical points by showing that $\mathbb{E}$ has a global minimizer. Under more specific assumptions on j other types of critical points may occur which allows the study of multiple solutions for (1.1)–(1.2).

(ii) By inspection of the notion of sub- and supersolution according to Definition 3.1 and Definition 3.2, respectively, one readily observes that any solution of the multi-valued variational inequality (1.1)–(1.2) is both a subsolution and supersolution provided the closed, convex set $K \subseteq V$ satisfies the following lattice condition:

$$K \wedge K \subseteq K, \quad K \vee K \subseteq K.$$

(iii) In specific cases for K the general potential $\mathbb{E}$ may be replaced by a more simpler potential having the same critical points which according to Theorem 4.4 are solutions of (1.1)–(1.2) within the order interval $[\underline{u},\overline{u}]$ of sub-supersolution. In regard with the latter two examples will be considered in the next section.

5. **Applications**

In this section we apply Theorem 4.4 to the special cases given in Example 2.1 and Example 2.2. In particular, the corresponding functional $\mathbb{E}$ is studied in more detail. In the result we may replace $\mathbb{E}$ by functionals $\hat{\mathbb{E}}$ and $\hat{\mathbb{E}}_0$ that are simpler to handle and that have the same critical points as $\mathbb{E}$.

Example 5.1. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, and let j be its primitive given by

$$j(x, s) := \int_0^s f(x, t)\, dt. \tag{5.1}$$

Then the function $s \mapsto j(x, s)$ is continuously differentiable, and thus Clarke's gradient reduces to a singleton, i.e.,

$$\partial j(x, s) = \{\partial j(x, s)/\partial s\} = \{f(x, s)\}.$$

If $K = V$, then (1.1)-(1.2) reduces to the following quasilinear elliptic BVP

$$\langle -\Delta_p u, v\rangle + \int_\Omega f(x, u)\, v\, dx = 0, \quad \forall\, v \in V, \tag{5.2}$$

which is Example 2.1 and which is equivalent to a homogeneous Neumann problem. Let $\underline{u}$ and $\overline{u}$ be sub- and supersolution with $\underline{u} \le \overline{u}$. The hypothesis (H1) on j given by (5.1) is trivially satisfied. To fulfill hypothesis (H2) we need to impose an L^q-boundedness with respect to $[\underline{u}, \overline{u}]$ on f, i.e., for some $k_\Omega \in L^q_+(\Omega)$ the following inequality is assumed to be satisfied:

$$|f(x, s)| \le k_\Omega(x), \quad \text{for a.e. } x \in \Omega,\ \forall\, s \in [\underline{u}(x), \overline{u}(x)]. \tag{5.3}$$

The associated potential $\mathbb{E}$ whose critical points are solutions of (5.2) within $[\underline{u}, \overline{u}]$ is now given by (note that $I_V = 0$)

$$\mathbb{E}(u) = \frac{1}{p} \int_\Omega |\nabla u|^p\, dx + \mathbb{B}(u) + \tilde{J}(u),$$

where $\mathbb{B}$ and $\tilde{J}$ are given by (3.12) and (3.8), respectively. Our aim is to replace the functional $\tilde{J}$ by a functional $\hat{J}$ that can easier be handled and that satisfies

$$\tilde{J}(u) - \hat{J}(u) = C, \quad \forall\, u \in V,$$

where C is a constant (not depending on u). Let $\hat{\mathbb{E}}$ be defined by

$$\hat{\mathbb{E}}(u) := \frac{1}{p} \int_\Omega |\nabla u|^p\, dx + \mathbb{B}(u) + \hat{J}(u), \quad u \in V. \tag{5.4}$$

Since $\mathbb{E}$ and $\hat{\mathbb{E}}$ differ only by some constant C, we have that u is a critical point of $\mathbb{E}$ if and only if u is a critical point of $\hat{\mathbb{E}}$. Therefore, Theorem 4.4 holds true if $\mathbb{E}$ is replaced by $\hat{\mathbb{E}}$.

For the construction of the new functional $\hat{J}$ let us first recall $\tilde{j}$ where j is given by (5.1):

$$\tilde{j}(x, s) = \begin{cases} j(x, \underline{u}(x)) + f(x, \underline{u}(x))(s - \underline{u}(x)) & \text{if} \quad s < \underline{u}(x), \\ j(x, s) & \text{if} \quad \underline{u}(x) \le s \le \overline{u}(x), \\ j(x, \overline{u}(x)) + f(x, \overline{u}(x))(s - \overline{u}(x)) & \text{if} \quad s > \overline{u}(x). \end{cases} \tag{5.5}$$

Since $s \mapsto \tilde{j}(x,s)$ given by (5.5) is differentiable, Clarke's gradient $\partial\tilde{j}(x,s)$ is single-valued, i.e., $\partial\tilde{j}(x,s) = \{\frac{\partial}{\partial s}\tilde{j}(x,s)\}$ which apparently is given by

$$\frac{\partial}{\partial s}\tilde{j}(x,s) = \begin{cases} f(x,\underline{u}(x)) & \text{if} \quad s < \underline{u}(x), \\ f(x,s) & \text{if} \quad \underline{u}(x) \le s \le \overline{u}(x), \\ f(x,\overline{u}(x) & \text{if} \quad s > \overline{u}(x). \end{cases} \tag{5.6}$$

By means of the truncation τ related to the given sub- and supersolution and defined by

$$\tau(x,s) = \begin{cases} \underline{u}(x) & \text{if } s < \underline{u}(x), \\ s & \text{if } \underline{u}(x) \le s \le \overline{u}(x), \\ \overline{u}(x) & \text{if } s > \overline{u}(x), \end{cases} \tag{5.7}$$

we may rewrite (5.6) in the following compact form

$$\frac{\partial}{\partial s}\tilde{j}(x,s) = (f\circ\tau)(x,s) := f(x,\tau(x,s)).$$

Define the function $(x,s) \mapsto \hat{j}(x,s)$ by

$$\hat{j}(x,s) := \int_0^s (f\circ\tau)(x,t)\,dt. \tag{5.8}$$

Then we have

$$\frac{\partial}{\partial s}\tilde{j}(x,s) = \frac{\partial}{\partial s}\hat{j}(x,s),$$

and thus

$$\hat{j}(x,s) = \tilde{j}(x,s) - \tilde{j}(x,0),$$

which yields

$$\tilde{J}(u) - \hat{J}(u) = \tilde{J}(0) =: C \quad \forall\, u \in V,$$

where

$$\hat{J}(u) := \int_\Omega \hat{j}(x,u(x))\,dx = \int_\Omega \int_0^{u(x)} (f\circ\tau)(x,t)\,dt\,dx, \quad \forall\, u \in V. \tag{5.9}$$

Applying Theorem 4.4 to the elliptic problem (5.2) we get the following result.

Corollary 5.2. *Let $\underline{u}$, $\overline{u}$ be sub-and supersolution of (5.2) with $\underline{u} \le \overline{u}$, and assume the local L^q-boundedness with respect to $[\underline{u},\overline{u}]$ for f. Then the (smooth) functional*

$$\hat{\mathbb{E}}(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \mathbb{B}(u) + \hat{J}(u), \quad u \in V,$$

with $\hat{J}$ given by (5.9) possesses critical points. Any critical point u of $\hat{\mathbb{E}}$ is a solution of the elliptic problem (5.2) which belongs to the interval $[\underline{u},\overline{u}]$.

Example 5.3. Let j and f satisfy the same condition as in Example 2.1, and assume $K = V_0$. Then (1.1)–(1.2) is equivalent to the homogeneous Dirichlet boundary value problem (2.3) which is:

$$u \in V_0 : \quad \langle -\Delta_p u, v \rangle + \int_\Omega f(x,u)\, v\, dx = 0, \quad \forall\, v \in V_0. \tag{5.10}$$

Similarly as in the previous subsection the potential $\mathbb{E} : V \to \mathbb{R} \cup \{+\infty\}$ provided by Theorem 4.4 and whose critical points are the solutions of (5.10)in $[\underline{u}, \overline{u}]$ may be replaced first by the following simpler functional $\hat{\mathbb{E}} : V_0 \to \mathbb{R}$ (note V_0 is a closed subspace of V):

$$\hat{\mathbb{E}}(u) := \frac{1}{p} \int_\Omega |\nabla u|^p\, dx + \mathbb{B}(u) + \int_\Omega \int_0^{u(x)} (f \circ \tau)(x,t)\, dt\, dx, \quad u \in V_0. \tag{5.11}$$

The specific properties of problem (5.10) allow even further to simplify the associated potential $\hat{\mathbb{E}}$ in that the term $\mathbb{B}(u)$ which is required in the general situation and in the previous subsection may now be dropped, i.e., we have the following result.

Corollary 5.4. *Let $\hat{\mathbb{E}}_0 : V_0 \to \mathbb{R}$ be defined by*

$$\hat{\mathbb{E}}_0(u) := \frac{1}{p} \int_\Omega |\nabla u|^p\, dx + \int_\Omega \int_0^{u(x)} (f \circ \tau)(x,t)\, dt\, dx, \quad u \in V_0.$$

Then $\hat{\mathbb{E}}_0 \in C^1(V_0, \mathbb{R})$ possesses critical points in V_0, and any critical point $u \in V_0$ of $\hat{\mathbb{E}}_0$ is a solution of (5.10) satisfying $\underline{u} \le u \le \overline{u}$. Moreover, $\hat{\mathbb{E}}$ and $\hat{\mathbb{E}}_0$ have the same critical points.

Proof. As $\|u\|_{V_0}^p := \int_\Omega |\nabla u|^p\, dx$ defines an equivalent norm in V_0, and since

$$\Big| \int_\Omega \int_0^{u(x)} (f \circ \tau)(x,t)\, dt\, dx \Big| \le \|k_\Omega\|_{L^q(\Omega)}\, \|u\|_{L^p(\Omega)},$$

we readily see that $\hat{\mathbb{E}}_0 : V_0 \to \mathbb{R}$ is bounded below, coercive, and weakly lower semicontinous. Thus there is a global minimizer $u \in V_0$ of $\hat{\mathbb{E}}_0$ which is a critical point, i.e, we have

$$0 = \langle \hat{\mathbb{E}}_0'(u), \varphi \rangle = \int_\Omega \Big(|\nabla u|^{p-2} \nabla u \nabla \varphi + (f \circ \tau)(\cdot, u)\, \varphi \Big) dx, \quad \forall\, \varphi \in V_0. \tag{5.12}$$

The supersolution $\overline{u} \in V$ of (5.10) satisfies: $\overline{u}|_{\partial\Omega} \ge 0$ and the inequality

$$\int_\Omega \Big(|\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla \varphi + f(\cdot, \overline{u})\, \varphi \Big) dx \ge 0, \quad \forall\, \varphi \in V_0 \cap L^p_+(\Omega). \tag{5.13}$$

Subtracting (5.13) from (5.12) and using $\varphi = (u - \overline{u})^+ \in V_0$ we get

$$\int_\Omega \Big(|\nabla u|^{p-2} \nabla u - |\nabla \overline{u}|^{p-2} \nabla \overline{u} \Big) \nabla (u - \overline{u})^+\, dx + \int_\Omega \big((f \circ \tau)(\cdot, u) - f(\cdot, \overline{u}) \big) (u - \overline{u})^+\, dx \le 0. \tag{5.14}$$

Applying the definition of τ we readily see that

$$\int_\Omega \left((f\circ\tau)(\cdot,u) - f(\cdot,\overline{u})\right)(u-\overline{u})^+\,dx = 0,$$

which by means of (5.14) implies

$$\begin{aligned} 0 &\geq \int_\Omega \Big(|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla\overline{u}\Big)\nabla(u-\overline{u})^+\,dx \\ &= \int_{\{u>\overline{u}\}} \Big(|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla\overline{u}\Big)\nabla(u-\overline{u})\,dx \geq 0. \end{aligned}$$

Hence it follows that $\nabla(u-\overline{u}) = 0$ a.e. in $\{u>\overline{u}\}$, which means $\nabla(u-\overline{u})^+ = 0$ a.e. in Ω, and thus $\|(u-\overline{u})^+\|_{V_0} = 0$, i.e., $(u-\overline{u})^+ = 0$ a.e. in Ω, that is $u \leq \overline{u}$. In a similar way one shows that $\underline{u} \leq u$ holds true which proves that any critical point u of $\hat{\mathbb{E}}_0$ is a solution of (5.10) because $\underline{u} \leq u \leq \overline{u}$ and therefore $(f\circ\tau)(x,u(x)) = f(x,u(x))$. So far we know that critical points of both $\hat{\mathbb{E}}$ and $\hat{\mathbb{E}}_0$ are necessarily solutions of (5.10) in $[\underline{u},\overline{u}]$. By (3.12) and (3.16) we see that $\mathbb{B}(u) = c$ for $u\in[\underline{u},\overline{u}]$, and therefore

$$\hat{\mathbb{E}}(u) = \hat{\mathbb{E}}_0 + c, \quad \forall\, u \in [\underline{u},\overline{u}],$$

which shows that $\hat{\mathbb{E}}$ and $\hat{\mathbb{E}}_0$ have the same critical points. This completes the proof. □

Example 5.5. For illustration let us consider the Dirichlet problem depending on a parameter $\lambda\in\mathbb{R}$:

$$-\Delta_p u = \lambda\,|u|^{p-2}u - g(u), \quad \text{in} \quad V_0^*, \tag{5.15}$$

where we assume the following assumptions on g:

(g1) $g:\mathbb{R}\to\mathbb{R}$ is continuous;
(g2) $\lim_{|s|\to\infty} \frac{g(s)}{|s|^{p-2}s} = +\infty$;
(g3) $\lim_{s\to 0} \frac{g(s)}{|s|^{p-2}s} = 0$.

The following specific g which is of exponential growth satisfies (g1)–(g3):

$$g(s) = \begin{cases} |s|^{p-2}se^{-s-1} & \text{if } s<-1 \\ \frac{|s|^p}{2}((s-1)\cos(s+1)+s+1) & \text{if } -1\leq s\leq 1 \\ (1+(s-1))s^{p-1}e^{s-1} & \text{if } s>1, \end{cases}$$

see [4]. By means of (g2) one readily verifies that $\overline{u} = M > 0$ with M sufficiently large is a supersolution of (5.15), and $\underline{u} = -M$ with $M>0$ sufficiently large is a subsolution of (5.15). Since $s\mapsto \lambda\,|s|^{p-2}s - g(s)$ is continuous, and thus bounded in

$[\underline{u}, \overline{u}] = [-M, M]$ we may apply Corollary 5.4 with τ given by

$$\tau(s) = \begin{cases} -M & \text{if } s < -M, \\ s & \text{if } -M \le s \le M, \\ M & \text{if } s > M, \end{cases}$$

and $\hat{\mathbb{E}}_0 : V_0 \to \mathbb{R}$ given by

$$\hat{\mathbb{E}}_0(u) = \frac{1}{p}\,\|\nabla u\|_p^p - \int_\Omega \int_0^{u(x)} (\lambda|\tau(s)|^{p-2}\tau(s) - g(\tau(s)))\,ds\,dx.$$

According to Corollary 5.4 the functional $\hat{\mathbb{E}}_0$ has critical points, and any critical point $u \in V_0$ is a solution of (5.15) satisfying $-M \le u \le M$ for $M > 0$ sufficiently large. Due to (g3) problem (5.15) always has the trivial solution. How to decide the existence of nontrivial, and moreover, multiple nontrivial solutions? In a first step we can show that the global minimizer of $\hat{\mathbb{E}}_0$ is a nontrivial solution of (5.15) in $[-M, M]$ provided $\lambda > \lambda_1$, where λ_1 is the first eigenvalue of $(-\Delta_p, V_0)$. Let φ_1 be the (normalized, positive) eigenfunction corresponding to λ_1 ($\|\varphi_1\|_p = 1$), then it is known that $\varphi_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. By using (g3) we have for $\varepsilon > 0$ small the estimate:

$$\begin{aligned}
\hat{\mathbb{E}}_0(\varepsilon\varphi_1) &= \frac{\lambda_1}{p}\,\varepsilon^p - \int_\Omega \int_0^{\varepsilon\varphi_1(x)} (\lambda s^{p-1} - g(s))\,ds\,dx \\
&= \frac{\lambda_1 - \lambda}{p}\,\varepsilon^p + \int_\Omega \int_0^{\varepsilon\varphi_1(x)} g(s)\,ds\,dx \\
&\qquad (g3) \Longrightarrow \frac{|g(s)|}{|s|^{p-1}} < \lambda - \lambda_1,\ \forall\ s:\ |s| < \delta_\lambda \\
&\le \frac{\lambda_1 - \lambda}{p}\,\varepsilon^p + \int_\Omega \int_0^{\varepsilon\varphi_1(x)} \frac{|g(s)|}{s^{p-1}}\,s^{p-1}\,ds\,dx \\
&\qquad \text{choose } \varepsilon : \varepsilon\|\varphi_1\|_\infty < \delta_\lambda \\
&< \frac{\lambda_1 - \lambda}{p}\,\varepsilon^p + \frac{\lambda - \lambda_1}{p}\,\varepsilon^p = 0.
\end{aligned}$$

Therefore, the global minimizer u of $\hat{\mathbb{E}}_0$ satisfies $\hat{\mathbb{E}}_0(u) \le \hat{\mathbb{E}}_0(\varepsilon\varphi_1) < 0$, and thus $u \ne 0$ is a nontrivial solution.

Remark 5.6. Multiple solution results for (1.1)–(1.2) in case that $K = V_0$ which refers to the Dirichlet problem for elliptic equations with smooth or nonsmooth functions j have been obtained by the author jointly with D. Motreanu and K. Perera in [2, 3, 4, 5, 6, 7]. In particular, in [4] multiple solutions have been obtained for (5.15) by applying a combined approach of variational and comparison principles in the smooth case. The approach developed in this paper allows to extend the study of multiple solutions to a wide range of (nonsmooth) multi-valued variational inequality in the form (1.1)–(1.2) or, more general, in the form (1.3).

References

[1] Carl, S., Le, Vy K., Motreanu, D., *Nonsmooth Variational Problems and Their Inequalities. Comparison Principles and Applications*, Springer Monogr. Math., Springer, New York, 2007.

[2] Carl, S., Motreanu, D., *Multiple and sign-changing solutions for the multivalued p-Laplacian equation*, Math. Nachr., **283** (2010), 965–981.

[3] Carl, S., Motreanu, D., *Sign-changing solutions for nonlinear elliptic problems depending on parameters*, Int. J. Differential Equations, **2010**, Article ID 536236, DOI: 10.1155/2010/ (2010)/536236.

[4] Carl, S., Motreanu, D., *Constant-sign and sign-changing solutions for nonlinear eigenvalue problems*, Nonlinear Anal., **68** (2008), 2668–2676.

[5] Carl, S., Motreanu, D., *Constant-sign and sign-changing solutions of a nonlinear eigenvalue problem involving the p-Laplacian*, Differential Integral Equations, **20** (3) (2007), 309–324.

[6] Carl, S., Motreanu, D., *Sign-changing and extremal constant-sign solutions of nonlinear elliptic problems with supercritical nonlinearities*, Commun. Appl. Nonlinear Anal., **14** (2007), 85–100.

[7] Carl, S., Perera, K., *Sign-changing and multiple solutions for the p-Laplacian*, Abstract Appl. Anal., **7** (2002), 613–625.

[8] Clarke, F. H., *Optimization and Nonsmooth Analysis*, Classics Appl. Math. **5**, SIAM, Philadelphia, 1990.

[9] Motreanu, D., Radulescu, V., *Variational and Non-variational Methods in Nonlinear Analysis and Boundary Value Problems*, Kluwer Academic Publishers, Boston, Dordrecht, London (2003).

[10] Zeidler, E., *Nonlinear Functional Analysis and Its Applications*, Vol. III. Springer-Verlag, New York (1985).

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INFINITELY MANY SOLUTIONS FOR A CLASS OF ELLIPTIC VARIATIONAL-HEMIVARIATIONAL INEQUALITY PROBLEMS

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Abstract. The aim of the present paper is to give some results on the existence of infinitely many solutions for a class of nonlinear elliptic variational-hemivariational inequalities. The approach is based on a result of infinitely many critical points.

1. **Introduction**

In mechanics and physics there is a variety of variational inequality formulations which arise when the material laws or the boundary conditions are derived by a convex, generally not everywhere differentiable and finite superpotential ([12]).The variational inequalities have a precise physical meaning: they express the principle of virtual work (or power) in its inequality form. Moreover, there exists a variety of nonmonotone laws which manifests the need for the derivation of variational formulations for nonconvex and not everywhere differentiable and finite energy functions (nonconvex superpotentials). Such variational formulations have been called by P.D. Panagiotopoulos ([10], [11]) hemivariational inequalities and describe large families of important problems in physics and engineering. It should also be noted that the hemivariational inequalities are closely connected to the notion of the generalized gradient of Clarke, which in the case of lack of convexity plays the same role as the subdifferential in the case of convexity (at least for static mechanical problems). Roughly speaking, variational-hemivariational inequalities may be regarded as hemivariational inequalities subject to variational constraints. Consequently, a further term, namely the subdifferential of some proper, convex, and lower semicontinuous function, appears inside the equation.

Several authors have been interested in the study of variational-hemivariational inequalities, for example, S. A. Marano and D. Motreanu, in the very nice paper

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[8], studied the existence of infinitely many solutions for a class of elliptic variational hemivariational inequality with $p-$Laplacian.

Let Ω be a non-empty, bounded, open subset of the Euclidian space $\mathbb{R}^N$, $N \geq 3$, with a boundary of class C^1, let $p \in]N, +\infty[$, and let $q \in L^\infty(\Omega)$ satisfy $\operatorname{ess\,inf}_{x\in\Omega} q(x) > 0$. Given a closed convex subset K of $W^{1,p}(\Omega)$ containing the constant functions, they consider the following variational-hemivariational inequality problem

Find $u \in K$ fulfilling

$$-\int_\Omega \left[|\nabla u(x)|^{p-2}\nabla u(x)\nabla(v(x)-u(x)) + q(x)|u(x)|^{p-2}u(x)(v(x)-u(x))\right] dx$$

$$\leq \int_\Omega [\alpha(x)F^\circ(u(x);(v(x)-u(x))) + \beta(x)G^\circ(u(x);(v(x)-u(x)))]dx, \quad \forall v \in W^{1,p}(\Omega)$$

where $F(\xi) = \displaystyle\int_0^\xi f(t)dt$, $G(\xi) = \displaystyle\int_0^\xi g(t)dt$ for all $\xi \in \mathbb{R}$, with $f, g : \mathbb{R} \to \mathbb{R}$ locally essentially bounded, α, $\beta \in L^1(\Omega)$ such that $min\{\alpha(x), \beta(x)\} \geq 0$ a.e. in Ω.

In the study of this problem, they apply a result obtained by the same authors ([8, Theorem 1.1]), on the existence of infinitely many critical points.

The main purpose of the present paper is to establish the existence of infinitely many solutions for an elliptic variational-hemivariational inequality with $p-$Laplacian type: *Find $u \in K$ fulfilling*

$$-\int_\Omega \left[|\nabla u(x)|^{p-2}\nabla u(x)\nabla(v(x)-u(x)) + q(x)|u(x)|^{p-2}u(x)(v(x)-u(x))\right] dx$$

$$\leq \lambda \int_\Omega F^\circ(x, u(x);(v(x)-u(x)))dx$$

for all $v \in K$, with λ positive real parameter.

The approach is based on a result of infinitely many critical points due to G. Bonanno and G. Molica Bisci [4] which is a more precise version of [8, Theorem 1.1].

It is worth noticing that our results allow us to consider also the case when the sign of the nonlinear term is constant, see for instance Theorem 3.2 and Example 3.3, in which the nonlinear term h is nonpositive. We observe that this case cannot be investigated by applying [8, Theorem 2.1], (see Remark 3.5).

2. **Preliminaries**

Let $(X, \|\cdot\|)$ be a real Banach space. We denote by X^* the dual space of X, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X^* and X. A function $h : X \to \mathbb{R}$ is called locally Lipschitz continuous when to every $x \in X$ there correspond a neighborhood V_x of x and a constant $L_x \geq 0$ such that

$$|h(z) - h(w)| \leq L_x \|z - w\| \quad \forall\, z, w \in V_x\,.$$

If $x, z \in X$, we write $h^\circ(x; z)$ for the generalized directional derivative of h at the point x along the direction z, i.e.,

$$h^\circ(x; z) := \limsup_{w \to x,\, t \to 0^+} \frac{h(w + tz) - h(w)}{t}\,.$$

For locally Lipschitz h_1, $h_2 : X \to \mathbb{R}$, we have

$$(h_1 + h_2)^\circ(x, z) \leq h_1^\circ(x, z) + h_2^\circ(x, z), \qquad \forall x, z \in X. \tag{2.1}$$

The generalized gradient of the function h in x, denoted by $\partial h(x)$, is the set

$$\partial h(x) := \{x^* \in X^* : \langle x^*, z\rangle \leq h^\circ(x; z)\ \forall\, z \in X\}\,.$$

We say that $x \in X$ is a (generalized) critical point of h when

$$h^\circ(x; z) \geq 0 \quad \forall\, z \in X,$$

that clearly signifies $0 \in \partial h(x)$.

When a non-smooth functional, $g : X \to]-\infty, +\infty]$, is expressed as a sum of a locally Lipschitz function, $h : X \to \mathbb{R}$, and a convex, proper, and lower semicontinuous function, $j : X \to]-\infty, +\infty]$, that is $g := h + j$, a (generalized) critical point of g is every $u \in X$ such that

$$h^\circ(u; v - u) + j(v) - j(u) \geq 0,$$

for all $v \in X$ (see [9, Chapter 3]).

Here and in the sequel X is a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous functional, $\Upsilon : X \to \mathbb{R}$ is a sequentially weakly upper semicontinuous functional, λ is a positive real parameter, $j : X \to]-\infty, +\infty]$ is a convex, proper and lower semicontinuous functional and $D\,(j)$ is the effective domain of j.
Write

$$\Psi := \Upsilon - j \quad \text{and} \quad I_\lambda := \Phi - \lambda\Psi = (\Phi - \lambda\Upsilon) + \lambda j.$$

We also assume that Φ is coercive and

$$D(j) \cap \Phi^{-1}(]-\infty, r[) \neq \emptyset \tag{2.2}$$

for all $r > \inf_X \Phi$. Moreover, from (2.2) and provided $r > \inf_X \Phi$, we can define

$$\varphi(r) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\left(\displaystyle\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)\right) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Assuming also that Φ and Υ are locally Lipschitz functionals, in [4] the authors obtained the following result, which is a more precise version of [8, Theorem 1.1].

Theorem 2.1. *Under the above assumptions on X, Φ and Ψ, one has*

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(] - \infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(b) *If $\gamma < +\infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds: either*

(b_1) *I_λ possesses a global minimum,*

or

(b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.*

(c) *If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds: either*

(c_1) *there is a global minimum of Φ which is a local minimum of I_λ,*

or

(c_2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ, with $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$, which weakly converges to a global minimum of Φ.*

3. Existence Results

In this section, we present an applications of Theorem 2.1 to a Neumann-type problem for a variational-hemivariational inequality involving the p-Laplacian.

Let Ω be a non-empty, bounded, open subset of the Euclidian space $\mathbb{R}^N$, $N \geq 3$, with a boundary of class C^1, let $p \in]N, +\infty[$, and let $q \in L^\infty(\Omega)$ satisfy $\operatorname{ess\,inf}_{x\in\Omega} q(x) > 0$. On the space $W^{1,p}(\Omega)$, we consider the norm

$$\|u\| := \left(\int_\Omega (|\nabla u(x)|^p + q(x)|u(x)|^p) dx \right)^{\frac{1}{p}},$$

which is equivalent to the usual one.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be locally essentially bounded. Put

$$F(x, \xi) = \int_0^\xi f(x, t) dt.$$

The function F is locally Lipschitz. So, it makes sense to consider its generalized directional derivative F°.

Given a closed convex subset K of $W^{1,p}(\Omega)$ containing the constant functions, denote by (P) the following variational-hemivariational inequality problem:

Find $u \in K$ fulfilling

$$-\int_\Omega \left[|\nabla u(x)|^{p-2}\nabla u(x)\nabla(v(x)-u(x)) + q(x)|u(x)|^{p-2}u(x)(v(x)-u(x))\right] dx$$

$$\leq \lambda \int_\Omega F^\circ(x,u(x);(v(x)-u(x)))dx$$

for all $v \in K$, with λ positive real parameter.

Put

$$c = \sup_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\sup_{x\in\Omega}|u(x)|}{\left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega q(x)|u(x)|^p dx\right)^{\frac{1}{p}}}. \tag{3.1}$$

From (3.1), we infer at once that

$$c^p\|q\|_1 \geq 1. \tag{3.2}$$

Let

$$A = \liminf_{\xi\to+\infty} \frac{\displaystyle\int_\Omega \max_{|t|\leq\xi} -F(x,t)dx}{\xi^p}, \quad B = \limsup_{\xi\to+\infty} \frac{\displaystyle\int_\Omega -F(x,\xi)dx}{\xi^p},$$

and

$$\lambda_1 = \frac{\|q\|_1}{pB}, \quad \lambda_2 = \frac{1}{pc^pA}. \tag{3.3}$$

Our main result is the following.

Theorem 3.1. *Assume that*

$$\liminf_{\xi\to+\infty} \frac{\displaystyle\int_\Omega \max_{|t|\leq\xi}(-F(x,t))\,dx}{\xi^p} < \frac{1}{c^p\|q\|_1}\limsup_{\xi\to+\infty} \frac{\displaystyle\int_\Omega (-F(x,\xi))\,dx}{\xi^p}. \tag{3.4}$$

Then, for each $\lambda \in]\lambda_1,\lambda_2[$, where λ_1, λ_2 are given in (3.3), problem (P) possesses an unbounded sequence of solutions.

Proof. Our aim is to apply part (b) of Theorem 2.1. Take as X the Sobolev space $W^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \Big(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega q(x)|u(x)|^p dx\Big)^{\frac{1}{p}}.$$

For each $u \in X$, put

$$\Phi(u) := \frac{1}{p}\|u\|^p, \qquad \Upsilon(u) := \int_\Omega -F(x,u(x))dx.$$

and

$$j(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

Since, $\Psi := \Upsilon - j$,

$$I_\lambda := \frac{1}{p}\|u\|^p - \lambda\left(\int_\Omega -F(x,u(x))dx - j(u)\right) = \left(\frac{1}{p}\|u\|^p - \lambda\int_\Omega -F(x,u(x))dx\right) + \lambda j.$$

Pick $\lambda \in]\lambda_1, \lambda_2[$. Let $\{\rho_n\}$ be a real sequence such that $\lim_{n\to\infty} \rho_n = +\infty$ and

$$\lim_{n\to\infty} \frac{\int_\Omega \max_{|t|\le\rho_n}(-F(x,t))dx}{\rho_n^p} = A.$$

Put $r_n = \frac{1}{p}\left(\frac{\rho_n}{c}\right)^p$ for all $n \in \mathbb{N}$. Taking into account $\|v\|^p < pr_n$ and $\|v\|_\infty \le c\|v\|$, one has $|v(x)| \le \rho_n$, for every $x \in \Omega$. Therefore,

$$\varphi(r_n) = \inf_{\|u\|^p < pr_n} \frac{\sup_{\|v\|^p<pr_n}\left(\int_\Omega -F(x,v(x))dx - j(v)\right) - \left(\int_\Omega -F(x,u(x))dx - j(u)\right)}{r_n - \frac{\|u\|^p}{p}}$$

$$\le \frac{\sup_{\|v\|^p<pr_n}\left(\int_\Omega -F(x,v(x))dx - j(v)\right)}{r_n} \le \frac{\sup_{\|v\|^p<pr_n}\int_\Omega -F(x,v(x))dx}{r_n}$$

$$\le \frac{\int_\Omega \max_{|t|\le\rho_n} -F(x,t)dx}{r_n}$$

Hence,

$$\varphi(r_n) \le pc^p \frac{\int_\Omega \max_{|t|\le\rho_n}(-F(x,t))dx}{\rho_n^p} \qquad \forall n \in \mathbb{N}.$$

Then,

$$\gamma \le \liminf_{n\to+\infty} \varphi(r_n) \le pc^p A < +\infty.$$

Now, we claim that the functional $\Phi - \lambda\Psi$ is unbounded from below.

Let $\{d_n\}$be a real sequence such that $\lim_{n\to\infty} d_n = +\infty$ and

$$\lim_{n\to\infty} \int_\Omega \frac{-F(x,d_n)dx}{d_n^p} = B. \tag{3.5}$$

For each $n \in \mathbb{N}$, put $w_n(x) = d_n$, for all $x \in \Omega$. Clearly $w_n \in W^{1,p}(\Omega)$ for each $n \in \mathbb{N}$. Hence,

$$\|w_n\|^p = d_n^p\|q\|_1$$

and

$$\Phi(w_n) - \lambda\Psi(w_n) = \frac{\|w_n\|^p}{p} - \lambda \int_\Omega -F(x, w_n(x))dx + \lambda j(w_n)$$
$$= \frac{d_n^p \|q\|_1}{p} - \lambda \int_\Omega -F(x, d_n)dx.$$

Now, if $B < +\infty$, let $\epsilon \in \left]0, B - \frac{\|q\|_1}{p\lambda}\right[$. From (3.5) there exists ν_ϵ such that

$$\int_\Omega -F(x, d_n)dx > (B - \epsilon)d_n^p, \qquad \forall n > \nu_\epsilon.$$

Therefore,

$$\Phi(w_n) - \lambda\Psi(w_n) = \frac{d_n^p \|q\|_1}{p} - \lambda \int_\Omega -F(x, d_n)dx < \frac{d_n^p \|q\|_1}{p} - \lambda d_n^p (B - \epsilon)$$
$$= d_n^p \left(\frac{\|q\|_1}{p} - \lambda(B - \epsilon) \right).$$

From the choice of ϵ, one has

$$\lim_{n\to+\infty} [\Phi(w_n) - \lambda\Psi(w_n)] = -\infty.$$

If $B = +\infty$, fix $M > \frac{\|q\|_1}{p\lambda}$. From (3.5) there exists ν_M such that

$$\int_\Omega -F(x, d_n)dx > M d_n^p, \quad \forall n > \nu_M.$$

Moreover,

$$\Phi(w_n) - \lambda\Psi(w_n) = \frac{d_n^p \|q\|_1}{p} - \lambda \int_\Omega -F(x, d_n)dx < \frac{d_n^p \|q\|_1}{p} - \lambda M d_n^p = d_n^p \left(\frac{\|q\|_1}{p} - \lambda M \right).$$

Taking into account the choice of M, also in this case, one has

$$\lim_{n\to+\infty} [\Phi(w_n) - \lambda\Psi(w_n)] = -\infty.$$

From part (b) of Theorem the functional $\Phi - \lambda\Psi$ admits a sequence of critical points $u_n \subseteq W^{1,p}(\Omega)$ such that $\lim_{n\to\infty} \Phi(u_n) = +\infty$, that means for each point u_n

$$(\Phi - \lambda\Upsilon)^\circ(u_n, v - u_n) + j(v) - j(u_n) \geq 0 \qquad \forall v \in X. \quad (H)$$

Since Φ is bounded on bounded sets and taking into account that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$, then $\{u_n\}$ has to be unbounded. Moreover, from (H) we obtain $u_n \in K, \ \forall n \in \mathbb{N}$, so

$$(\Phi - \lambda\Upsilon)^\circ(u_n, v - u_n) \geq 0 \qquad \forall v \in K.$$

From (2.1) and the regularity of Φ, it follows

$$\Phi^{'}(u_n, v - u_n) + \lambda[-\Upsilon(u_n, v - u_n)]^\circ \geq 0 \qquad \forall v \in K.$$

Therefore,

$$\int_\Omega \left[|\nabla u_n(x)|^{p-2}\nabla u_n(x)\nabla(v(x)-u_n(x)) + q(x)|u_n(x)|^{p-2}u_n(x)(v(x)-u_n(x))\right] dx +$$

$$+\lambda \left[\int_\Omega F(x,u_n(x);(v(x)-u_n(x)))dx\right]^\circ \geq 0 \qquad \forall v \in K.$$

From an inequality concerning the integral functionals ([7]), we have

$$[-\Upsilon(u_n, v-u_n)]^\circ = \left[\int_\Omega F(x,u_n(x);(v(x)-u_n(x)))dx\right]^\circ \leq$$

$$\leq \int_\Omega F^\circ(x,u_n(x);(v(x)-u_n(x)))dx.$$

Then,

$$\int_\Omega \left[|\nabla u_n(x)|^{p-2}\nabla u_n(x)\nabla(v(x)-u_n(x)) + q(x)|u_n(x)|^{p-2}u_n(x)(v(x)-u_n(x))\right] dx +$$

$$+\lambda \int_\Omega F^\circ(x,u_n(x);(v(x)-u_n(x)))dx \geq 0;$$

that is

$$-\int_\Omega \left[|\nabla u_n(x)|^{p-2}\nabla u_n(x)\nabla(v(x)-u_n(x)) + q(x)|u_n(x)|^{p-2}u_n(x)(v(x)-u_n(x))\right] dx$$

$$\leq \lambda \int_\Omega F^\circ(x,u_n(x);(v(x)-u_n(x)))dx.$$

□

Given $\alpha \in L^1(\Omega)$, such that $\alpha(x) \geq 0$ a.e. in Ω, let $h : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded, such that $h(x) \leq 0$ a.e. in $\mathbb{R}$. Consider the following problem:

(P_H) *Find $u \in K$ fulfilling*

$$-\int_\Omega \left[|\nabla u(x)|^{p-2}\nabla u(x)\nabla(v(x)-u(x)) + q(x)|u(x)|^{p-2}u(x)(v(x)-u(x))\right] dx$$

$$\leq \lambda \int_\Omega \alpha(x)H^\circ(u(x);(v(x)-u(x)))dx$$

for all $v \in K$, with λ positive real parameter.

An immediate consequence of Theorem 3.1 is the following

Theorem 3.2. *Assume that*

$$\liminf_{\xi\to+\infty}\frac{(-H(\xi))}{\xi^p} < \frac{1}{c^p\|q\|_1}\limsup_{\xi\to+\infty}\frac{(-H(\xi))}{\xi^p}.$$

Then for every $\lambda \in \left]\frac{\|q\|_1}{p\|\alpha\|_1 \limsup_{\xi\to+\infty}\frac{(-H(\xi))}{\xi^p}}, \frac{1}{pc^p\|\alpha\|_1\liminf_{\xi\to+\infty}\frac{(-H(\xi))}{\xi^p}}\right[$, *problem* (P_H) *possesses an unbounded sequence of solutions.*

Example 3.3. Put

$$a_n := \frac{2n!(n+2)!-1}{4(n+1)!}, \quad b_n := \frac{2n!(n+2)!+1}{4(n+1)!}$$

for every $n \in \mathbb{N}$, and define the non-positive (and discontinuous) function $h : \mathbb{R} \to \mathbb{R}$ as follows

$$h(\xi) := \begin{cases} -2(n+1)![n^{p-1}(n+1)!^p - (n-1)^{p-1}n!^p] & \text{if } \xi \in \bigcup_{n\geq 0}]a_n, b_n[\\ 0 & \text{otherwise.} \end{cases}$$

Direct computations ensure that

$$\limsup_{\xi\to+\infty} \frac{-H(\xi)}{\xi^p} = +\infty \quad \text{and} \quad \liminf_{\xi\to+\infty} \frac{-H(\xi)}{\xi^p} = 0.$$

Owing to Theorem 3.2 for each $\lambda > 0$ the problem (P_H) possesses a sequence of solutions.

Remark 3.4. We explicitly observe that we cannot apply [8, Theorem 2.1] to the problem of Example 3.3, since hypotheses (3.6), (3.7), recalled below, do not hold, namely, supposed that there exist two sequences $\{\xi_n\} \subseteq \mathbb{R}$, $\{r_n\} \subseteq \mathbb{R}^+$ such that $\lim_{n\to+\infty} r_n = +\infty$,

$$H(\xi_n) = \inf_{|\xi|\leq c(pr_n)^{1/p}} H(\xi), \quad \forall n \in \mathbb{N}, \tag{3.6}$$

$$\frac{1}{p}\|q\|_1|\xi_n|^p < r_n \quad \forall n \in \mathbb{N}, \tag{3.7}$$

and taking into account that H is nonincreasing, we obtain that $c^p\|q\|_1 < 1$, which contradicts (3.2).

Remark 3.5. When f is an L^1-Carathéodory while $K = W^{1,p}(\Omega)$ the above inequality takes the form

$$-\int_\Omega \left[|\nabla u(x)|^{p-2}\nabla u(x)\nabla(w(x)) + q(x)|u(x)|^{p-2}u(x)(w(x))\right] dx$$

$$= \lambda \int_\Omega f(x, u(x))w(x)dx, \qquad \forall w \in W^{1,p}(\Omega).$$

Therefore, in such a case, a function $u \in W^{1,p}(\Omega)$ solves (P) if and only if it is a weak solution to the Neumann problem

$$\begin{cases} \Delta_p u - q(x)|u|^{p-2}u = \lambda f(x,u) & \textit{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \textit{in } \partial\Omega, \end{cases}$$

with ν being the outer unit normal to $\partial\Omega$.

We observe that this problem has been addressed recently in [2], by applying directly Theorem 2.1 to smooth functionals.

The results can be applied to study the above problem with discontinuous nonlinear term (see, for instance [1], [3]).

References

[1] Bonanno, G., Candito, P., *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Diff. Eqs., **244** (2008) 3031-3059.

[2] Bonanno, G., D'Aguì, G., *On the Neumann Problem for elliptic equations involving the p−Laplacian*, J. Math. Anal. Appl., **358** (2009), 223-228.

[3] Bonanno, G., Giovannelli, N., *An eigenvalue Dirichlet problem involving the p-Laplacian with discontinuous nonlinearities*, J. Math. Anal. Appl., **308** (2005). 596-604.

[4] Bonanno, G., Molica Bisci, G., *Infinitely many solutions for a boundary value problem with discontinuous nonlinearities*, Boundary Value Problems, (2009), 1-20.

[5] Brézis, H., *Analyse Fonctionelle - Théorie et Applications*, Masson, Paris, 1983.

[6] Chang, K. C.., *Variational methods for nondifferentiable functionals and their applications to partial differential equations,* J. Math. Anal. Appl., **80** (1981), 102–129.

[7] Clarke, F. H., *Optimization and Nonsmooth Analysis,* Classics Appl. Math., **5**, SIAM, Philadelphia, 1990.

[8] Marano, S. A., Motreanu, D., *Infinitely many critical points of non-differentiable functions and applications to a Neumann type problem involving the p-Laplacian*, J. Diff. Eqs., **182** (2002), 108-120.

[9] Motreanu, D., Panagiotopoulos, P. D., *Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities*, Nonconvex Optim. Appl., **29** Kluwer, Dordrecht, 1998.

[10] Naniewicz, Z., Panagiotopoulos, P. D., *Mathematical Theory of Hemivariational Inequalities and Applications,* Marcel Dekker, NewYork, 1995.

[11] Panagiotopoulos, P. D., *Hemivariational Inequalites. Applications to Mechanics and Engineering*, Springer, New York, 1993.

[12] Panagiotopoulos, P. D., *Inequality Problems in Mechanics. Convex and Nonconvex Energy Functions,* Birkhauser Verlag, Basel/Boston 1985.

[13] Ricceri, B., *A general variational principle and some of its applications*, J. Comput. Appl. Math., **133** (2000), 401-410.

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A NOTE ON THE EXISTENCE OF INFINITELY MANY SOLUTIONS FOR THE ONE DIMENSIONAL PRESCRIBED CURVATURE EQUATION

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Abstract. In the present paper we deal with the one dimensional prescribed curvature equation. We prove, under a suitable oscillatory behaviour at zero of the nonlinearity, the existence of infinitely many solutions. Our approach combines variational techniques with classical regularity results.

1. **Introduction**

In the present paper we deal with the one dimensional prescribed curvature problem

$$(P) \qquad \begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' = h(t)f(u) & \text{in }]0,1[\\ \\ u(0) = u(1) = 0 \end{cases}$$

where $h : [0,1] \to \mathbb{R}$ is a positive bounded function with $\operatorname{ess\,inf}_{[0,1]} h > 0$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. The problem of existence and multiplicity results for such problem is one of the most investigated issue in calculus of variations and differential geometry. We focus here on the existence of infinitely many solutions in the same spirit of some recent papers of Obersnel and Omari who studied the problem under different sets of assumptions on the nonlinearity f.

A sequence of weak solutions (tending in the C^1 norm to zero) has been obtained in [3] in any space dimension N via the Lusternik-Schnirelmann theory, provided the nonlinearity is odd and its primitive is subquadratic at zero. The same thesis for the one dimensional autonomous equation has been achieved in [2] via the analysis of some generalized Fučík spectrum under different behaviour of the

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nonlinearity. Namely if f is superlinear or sublinear at zero and satisfies, with its primitive, suitable conditions at $+\infty$ and $-\infty$, the authors proved the existence of infinitely many solutions which are possibly discontinuous at the points where they attain the value zero. Finally we mention the paper [4], where the method of sub and super solutions guarantees, in any space dimension N, the existence of a sequence of weak solutions tending in the C^1 norm to zero.

We will prove the existence of infinitely many solutions for the one dimensional prescribed curvature problem under suitable oscillatory assumptions at zero on the nonlinearity f. We propose a new approach without requiring symmetry or conditions at $+\infty$.

Following the variational approach of [3], we will apply a variational principle by Ricceri [5] to an elliptic regularized problem to obtain a sequence of pairwise distinct critical points for the energy functional associated and subsequently, by the means of classical regularity results, we will achieve the existence of infinitely many solutions for the original problem.

Throughout the sequel by a solution of (P) we mean a *weak solution*, that is a function $u \in W_0^{1,2}(]0,1[)$ such that

$$\int_0^1 \frac{u'(t)}{\sqrt{1+u(t)'^2}} v'(t)dt - \int_0^1 h(t)f(u(t))v(t)dt = 0$$

for every $v \in W_0^{1,2}(]0,1[)$.

Our main result is

Theorem 1.1. *Assume that*

i) there exist two sequences $\{a_k\}$ *and* $\{b_k\}$ *in* $]0,\infty[$ *with* $b_{k+1} < a_k < b_k$, $\lim_{k\to\infty} b_k = 0$ *and* $\lim\limits_{k\to\infty} \frac{a_k}{b_k} = 0$ *such that* $f(s) \leq 0$ *for every* $s \in [a_k, b_k]$*;*

ii) if $F(s) = \int_0^s f(t)dt$*, then*

$$-\infty < \liminf_{s\to 0^+} \frac{F(s)}{s^2} \leq \limsup_{s\to 0^+} \frac{F(s)}{s^2} = +\infty;$$

iii) $\limsup\limits_{k\to\infty} \frac{\max_{[0,a_k]} F}{b_k^2} < \frac{7\sqrt{2}}{32} \frac{1}{\|h\|_{L^1(]0,1[)}}.$

Then, problem (P) *admits a sequence of non negative weak solutions* $\{u_k\} \subseteq C^1([0,1])$ *which satisfy* $\lim_{k\to\infty} \|u_k\|_{C^1([0,1])} = 0$.

2. Proof of Theorem 1.1

2.1. **Preliminaries.** Our main tool is the following variational principle by Ricceri which is a consequence of a more general result.

Theorem 2.1. *([5], Theorem 2.5) Let X be a Hilbert space, $\Phi, \Psi : X \to \mathbb{R}$ two sequentially weakly lower semicontinuous, continuously Gâteaux differentiable functionals. Assume that Ψ is strongly continuous and coercive. For each $\rho > \inf_X \Psi$, set*

$$\varphi(\rho) := \inf_{\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(u) - \inf_{\overline{\Psi^{-1}(]-\infty,\rho[)}^w} \Phi}{\rho - \Psi(u)}, \tag{2.1}$$

where $\Psi^{-1}(]-\infty,\rho[) := \{u \in X : \Psi(u) < \rho\}$ and $\overline{\Psi^{-1}(]-\infty,\rho[)}^w$ is its closure in the weak topology of X. Furthermore, set

$$\delta := \liminf_{\rho \to (\inf_X \Psi)^+} \varphi(\rho). \tag{2.2}$$

If $\delta < +\infty$ then, for every $\lambda > \delta$, either $\Phi + \lambda\Psi$ possesses a local minimum, which is also a global minimum of Ψ, or there is a sequence $\{u_k\}$ of pairwise distinct critical points of $\Phi + \lambda\Psi$, with $\lim_{k\to\infty} \Psi(u_k) = \inf_X \Psi$, weakly converging to a global minimum of Ψ.

2.2. **Proof.** In the present section we will give the proof of Theorem 1.1. Following an idea of Obersnel and Omari in [3], we apply Theorem 2.1 to a modified problem and then, by the means of a regularity result by Lieberman (see [1]) we prove that the critical points of the energy are actually solutions of the original problem.

We split the proof in several steps.

Step 1. A modified problem.

Notice first that assumptions $i)$ and $ii)$ imply that $f(0) = 0$. We truncate f as follows:

$$g(s) = \begin{cases} 0 & s < 0 \\ f(s) & 0 \leq s < b_1 \\ f(b_1) & s \geq b_1 \end{cases}$$

where b_1 is from assumption (i). The function g is continuous and if $G : \mathbb{R} \to \mathbb{R}$ denotes its primitive, that is $G(s) = \int_0^s g(t)dt$, g and G satisfy the assumptions $(i) - (iii)$ of Theorem 1.1. Define also $a : [0, +\infty[\to]0, +\infty[$ by

$$a(s) = \begin{cases} \dfrac{1}{\sqrt{1+s}} & 0 \leq s < 1 \\ \dfrac{\sqrt{2}}{16}(s-2)^2 + \dfrac{7\sqrt{2}}{16} & 1 \leq s < 2 \\ \dfrac{7\sqrt{2}}{16} & s \geq 2. \end{cases}$$

The function a is of class $C^{1,1}([0,+\infty[)$ and, for every $s \geq 0$, satisfies $\frac{7\sqrt{2}}{16} \leq a(s) \leq 1$. Denote by A its primitive, that is $A(s) = \int_0^s a(t)dt$, verifying then

$$\frac{7\sqrt{2}}{16} s \leq A(s) \leq s. \tag{2.3}$$

We introduce now the auxiliary problem

$$(P') \qquad \begin{cases} -\left(a(|u'|^2)u'\right)' = h(t)g(u) & \text{in }]0,1[\\ \\ u(0) = u(1) = 0 \end{cases}$$

Denote by X the space $W_0^{1,2}(]0,1[)$, endowed with the norm $\|u\| = \left(\int_0^1 |u'(t)|^2 dt\right)^{1/2}$. It is well known that the space X is compactly embedded into $C^0([0,1])$ and $\|u\|_\infty \leq \|u\|$ where $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$. Let Ψ and $\Phi : X \to \mathbb{R}$ be the functionals defined by

$$\Psi(u) = \frac{1}{2}\int_0^1 A(|u'(t)|^2)dt, \quad \Phi(u) = -\int_0^1 h(t)G(u(t))dt, \quad u \in X.$$

Due to (2.3), Ψ is well defined on X, continuous and coercive. Moreover, by the convexity of the function $s \to A(s^2)$ in $\mathbb{R}$, Ψ is convex and then sequentially weakly lower semicontinuous. The functional Φ is well defined and sequentially weakly continuous. Moreover Ψ and Φ are continuously Gâteaux differentiable with derivative given by

$$\Psi'(u)(v) = \int_0^1 a(|u'(t)|^2)u'(t)v'(t)dt, \qquad \Phi'(u)(v) = -\int_0^1 h(t)g(u(t))v(t)dt,$$

for every $u, v \in X$. With these assumptions, the function φ from (2.1) reads as follows:

$$\varphi(\rho) = \inf_{\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(u) - \inf_{\Psi^{-1}(]-\infty,\rho])} \Phi}{\rho - \Psi(u)},$$

where $\Psi^{-1}(]-\infty,\rho]) = \{u \in X \ : \ \Psi(u) \leq \rho\}$.

Step 2. We claim that $\delta < 1$.

Recall that $\delta := \liminf_{\rho\to 0^+} \varphi(\rho)$ and clearly $\delta \geq 0$.

Notice that from $G(s) = 0$ for every $s \leq 0$ and *i)* it follows that

$$\max_{[-b_k,b_k]} G = \max_{[0,b_k]} G = \max_{[0,a_k]} G.$$

Let $\bar{s}_k \in [0, a_k]$ such that $G(\bar{s}_k) = \max_{[-b_k,b_k]} G$ and denote by $s_k = \dfrac{7\sqrt{2}b_k^2}{32}$.

We have that

$$\Psi^{-1}(]-\infty, s_k]) \subseteq \{v \in X : \|v\|_\infty \leq b_k\}.$$

Indeed, if $v \in X$ is such that $\Psi(v) \le s_k$, then by (2.3) we have

$$\frac{7\sqrt{2}}{32}\|v\|^2 \le \Psi(v) \le s_k,$$

and clearly

$$\|v\|_\infty^2 \le b_k^2,$$

which is our claim. Hence,

$$\sup_{\Psi^{-1}(]-\infty,s_k])}(-\Phi(v)) \le \max_{[-b_k,b_k]} G\int_0^1 h(t)dt = G(\bar{s}_k)\|h\|_{L^1(]0,1[)}. \tag{2.4}$$

By assumption (ii), $\liminf_{s\to 0^+}\dfrac{G(s)}{s^2} > -\infty$. It follows the existence of $\underline{M} > 0$ and $\tau \in]0,b_1[$ such that

$$G(s) > -\underline{M}s^2 \quad \text{for every } s \in]0,\tau[. \tag{2.5}$$

Choose now l such that

$$\limsup_{k\to\infty}\frac{\max_{[0,a_k]}G}{b_k^2} < l < \frac{7\sqrt{2}}{32}\frac{1}{\|h\|_{L^1(]0,1[)}}.$$

By $i)$ and $iii)$, for k big enough,

$$\frac{\max_{[0,a_k]}G}{b_k^2}\|h\|_{L^1(]0,1[)} + \left(\frac{1}{2}\underline{M}\|h\|_\infty + \frac{64\sqrt{2}}{7}l\|h\|_{L^1(]0,1[)}\right)\frac{a_k^2}{b_k^2} < l\|h\|_{L^1(]0,1[)}$$

which implies, as $\bar{s}_k \le a_k$

$$\frac{G(\bar{s}_k)}{s_k}\|h\|_{L^1(]0,1[)} + \left(\frac{1}{2}\underline{M}\|h\|_\infty + \frac{64\sqrt{2}}{7}l\|h\|_{L^1(]0,1[)}\right)\frac{\bar{s}_k^2}{s_k} < l\|h\|_{L^1(]0,1[)}. \tag{2.6}$$

Define

$$w_{\bar{s}_k}(t) = \begin{cases} 4\bar{s}_k t, & \text{if} \quad 0 \le t < \frac{1}{4} \\ \bar{s}_k, & \text{if} \quad \frac{1}{4} \le t < \frac{3}{4} \\ 4\bar{s}_k(1-t), & \text{if} \quad \frac{3}{4} \le t \le 1. \end{cases}$$

Clearly $w_{\bar{s}_k} \in X$ and

$$\Psi(w_{\bar{s}_k}) \le \frac{1}{2}\|w_{\bar{s}_k}\|^2 = 4\bar{s}_k^2. \tag{2.7}$$

So, using the definition of $w_{\bar{s}_k}$,

$$\begin{aligned} -\Phi(w_{\bar{s}_k}) &= \int_0^{1/4} h(t)G(w_{\bar{s}_k})dt + \int_{1/4}^{3/4} h(t)G(\bar{s}_k)dt + \int_{3/4}^1 h(t)G(w_{\bar{s}_k})dt \\ &> -\frac{\underline{M}\|h\|_\infty}{4}\bar{s}_k^2 + \frac{h_0}{2}G(\bar{s}_k) - \frac{\underline{M}\|h\|_\infty}{4}\bar{s}_k^2 \\ &> -\frac{\underline{M}\|h\|_\infty}{2}\bar{s}_k^2. \end{aligned} \tag{2.8}$$

where $h_0 = \operatorname{ess\,inf}_{[0,1]} h$.

Putting together (2.4), (2.8), (2.6) and (2.7) we obtain

$$\begin{aligned}\sup_{\Psi-1(]-\infty,s_k])}(-\Phi(v))+\Phi(w_{\bar{s}_k}) &\leq G(\bar{s}_k)\|h\|_{L^1([0,1])}+\frac{\underline{M}\|h\|_\infty}{2}\bar{s}_k^2\\ &< \frac{16\sqrt{2}}{7}l\|h\|_{L^1([0,1])}(s_k-4\bar{s}_k^2)\\ &\leq \frac{16\sqrt{2}}{7}l\|h\|_{L^1([0,1])}(s_k-\Psi(w_{\bar{s}_k})).\end{aligned}$$

Since $s_k \to 0$ as $k \to \infty$, we get

$$\delta \leq \liminf_k \frac{\Phi(w_{\bar{s}_k})-\inf_{\Psi^{-1}(]-\infty,s_k])}\Phi(v)}{s_k-\Psi(w_{\bar{s}_k})} \leq \frac{16\sqrt{2}}{7}l\|h\|_{L^1([0,1])}<1.$$

Step 3. 0 is not a local minimum of $\Psi+\Phi$.

We will construct a sequence of functions in X tending in norm to zero where the energy attains negative value. By assumption (ii), $\limsup_{s\to 0^+}\frac{G(s)}{s^2}=+\infty$ and so if $\overline{M}>0$ is such that

$$\overline{M} > \frac{8+\|h\|_\infty \underline{M}}{h_0}, \tag{2.9}$$

(where $\underline{M}$ is as in Step 2), there exists a sequence $\{\tilde{s}_k\} \subset]0,\tau[$ converging to zero such that

$$G(\tilde{s}_k) > \overline{M}\tilde{s}_k^2. \tag{2.10}$$

Let $w_{\tilde{s}_k}$ defined as

$$w_{\tilde{s}_k}(t)=\begin{cases}4\tilde{s}_k t, & \text{if}\quad 0\leq t<\frac{1}{4}\\ \tilde{s}_k, & \text{if}\quad \frac{1}{4}\leq t<\frac{3}{4}\\ 4\tilde{s}_k(1-t), & \text{if}\quad \frac{3}{4}\leq t\leq 1.\end{cases}$$

It is clear that $\|w_{\tilde{s}_k}\| \to 0$ as $k\to\infty$. Let us prove that $\Psi(w_{\tilde{s}_k})+\Phi(w_{\tilde{s}_k})<0$. Indeed, by (2.5) and (2.10) we have

$$\begin{aligned}\Psi(w_{\tilde{s}_k})+\Phi(w_{\tilde{s}_k}) &\leq 4\tilde{s}_k^2-\int_0^{1/4}h(t)G(w_{\tilde{s}_k})dt-\int_{1/4}^{3/4}h(t)G(\tilde{s}_k)dt-\int_{3/4}^1 h(t)G(w_{\tilde{s}_k})dt\\ &\leq 4\tilde{s}_k^2+\frac{\underline{M}\|h\|_\infty}{4}\tilde{s}_k^2-\frac{\overline{M}h_0}{2}\tilde{s}_k^2+\frac{\underline{M}\|h\|_\infty}{4}\tilde{s}_k^2\\ &=\tilde{s}_k^2\left(4+\frac{\underline{M}\|h\|_\infty}{2}-\frac{\overline{M}h_0}{2}\right)\\ &<0=\Psi(0)+\Phi(0).\end{aligned}$$

Our claim is achieved.

Step 4. Existence of a sequence of critical points for $\Psi+\Phi$.

We apply Theorem 2.1 to the functionals Ψ and Φ with $\lambda = 1$. One has that 0 is the global minimum of Ψ and by Step 3 is not a local minimum of $\Psi + \Phi$, hence there exists a sequence $\{u_k\}$ of pairwise distinct critical points of the energy such that $\lim_{k\to\infty} \Psi(u_k) = 0$. In particular,

$$\lim_{k\to\infty} \|u_k\|_\infty = 0. \tag{2.11}$$

Let us prove that the critical points of the energy are non negative. Assume that u is a critical point of $\Psi + \Phi$ and that the set $C = \{t \in [0, 1] : u(t) < 0\}$ is non empty, i.e. has a positive measure. Then, the function $v = \min\{0, u\}$ still belongs to X and

$$\begin{aligned} 0 &= \Psi'(u)(v) + \Phi'(u)(v) = \int_0^1 a(|u'(t)|^2)u'(t)v'(t)dt - \int_0^1 h(t)g(u(t))v(t)dt \\ &= \int_0^1 a(|u'(t)|^2)u'(t)^2 dt \end{aligned}$$

which implies $u = 0$, a contradiction. Hence, by using (2.11), for k big enough, $0 \le u_k(t) \le b_1$ for every $t \in [0, 1]$.

Step 5. Proof concluded.

If u_k is a critical point of $\Psi + \Phi$, then it is a weak solution of the auxiliary problem (P'), it is non negative and bounded from above by b_1, as proven in Step 4.

We are going to prove now that for k big enough, $\|u_k'\|_\infty \le 1$.

From [1], there exists $\alpha \in]0, 1]$ and $c > 0$ such that $u_k \in C^{1,\alpha}([0, 1])$ and

$$\|u_k\|_{C^{1,\alpha}([0,1])} \le c \qquad \text{for every } k \in \mathbb{N}. \tag{2.12}$$

Let us prove now that

$$\lim_{k\to\infty} \|u_k\|_{C^1([0,1])} = 0.$$

Indeed assume by contradiction that there exists a sequence $\{u_{k_h}\}$ such that $\lim_{h\to\infty} \|u_{k_h}\|_{C^1([0,1])} > 0$. Then, since $\lim_{k\to\infty} \|u_k\|_\infty = 0$ it must be

$$\lim_{h\to\infty} \|u'_{k_h}\|_\infty > 0. \tag{2.13}$$

From Ascoli Arzela' Theorem, there exists a subsequence still denoted by $\{u_{k_h}\}$ such that $\{u'_{k_h}\}$ is uniformly convergent to zero, in contradiction with (2.13).

In particular, for k big enough, we have that $\|u_k\|_{C^1([0,1])} \le 1$ and this implies at once that u_k is a weak solution of the original problem.

Remark 2.2. It is still an open question whether Theorem 1.1 is valid without assumption (iii).

Remark 2.3. We point out that our method works when the space dimension N is equal to 1. Indeed in our application of Theorem 2.1 it is crucial to embed the space X in $C^0([0,1])$. Notice however that the variational principle by Ricceri is valid for every space dimension, but it is still an open question how to apply it when there is no embedding into the space of continuous functions.

We conclude this note with an example of application of Theorem 1.1.

Example 2.4. Let $a_k = \frac{1}{k!k}$ and $b_k = \frac{1}{k!}$. Choose a constant $l \in]0, \frac{7\sqrt{2}}{32}[$.

Define $f : \mathbb{R} \to \mathbb{R}$ as follows

$$f(s) = \begin{cases} 4l(b_k^2 - b_{k+1}^2)\dfrac{(s - b_{k+1})}{(a_k - b_{k+1})^2}, & \text{if} \quad b_{k+1} \leq s \leq \frac{a_k + b_{k+1}}{2} \\ 4l(b_k^2 - b_{k+1}^2)\dfrac{(a_k - s)}{(a_k - b_{k+1})^2}, & \text{if} \quad \frac{a_k + b_{k+1}}{2} \leq s \leq a_k \\ 0, & \text{otherwise} \end{cases}$$

The function f is continuous and satisfies all the assumptions of Theorem 1.1. In particular, $\liminf_{s \to 0^+} \dfrac{F(s)}{s^2} = l$. Then, problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1 + u'^2}}\right)' = f(u) & \text{in }]0, 1[\\ \\ u(0) = u(1) = 0 \end{cases}$$

admits a sequence of non-negative weak solutions tending to zero in the C^1 norm.

References

[1] Lieberman, G. M., *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal., **12** (1988), 1203–1219.

[2] Obersnel, F., *Classical and non-classical sign-changing solutions of a one-dimensional autonomous prescribed curvature equation*, Adv. Nonlinear Stud., **7**, (2007) 671–682.

[3] Obersnel, F., Omari, P., *Multiple non-trivial solutions of the Dirichlet problem for the prescribed mean curvature equation*, to appear in Contemporary Mathematics.

[4] Obersnel, F., Omari, P., *Positive solutions of the Dirichlet problem for the prescribed mean curvature equation*, J. Diff. Eqs., **249** (2010), 1674–1725.

[5] Ricceri, B., *A general variational principle and some of its applications*, J. Comput. Appl. Math., **113** (2000), 401–410.

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ON A DEGENERATE AND SINGULAR ELLIPTIC EQUATION WITH CRITICAL EXPONENT AND NON-STANDARD GROWTH CONDITIONS

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Abstract. In this paper we study a class of degenerate and singular elliptic equations involving critical exponents and non-standard growth conditions in the whole space $\mathbb{R}^N$. We show the existence of at least one nontrivial solution using as main argument Ekeland's variational principle.

1. **Introduction**

In this paper we are concerned with the study of the following problem

$$-\text{div}(|x|^\alpha \nabla u) = \lambda g(x)|u|^{q(x)-2}u + |u|^{2^\star_\alpha - 2}u \qquad \text{in } \mathbb{R}^N \tag{1.1}$$

where $N \geq 2$, $0 < \alpha < 2$, $2^\star_\alpha = 2N/(N-2+\alpha)$ is the critical exponent, $q : \mathbb{R}^N \to (1, 2^\star_\alpha)$ is a function satisfying $q \in L^\infty(\mathbb{R}^N)$, $g : \mathbb{R}^N \to (0,\infty)$ is a measurable function satisfying certain properties that will be described later in the paper and $\lambda > 0$ is a constant.

The main interest in studying problem (1.1) is due to the presence of the degenerate and singular potential $|x|^\alpha$ in the divergence operator. This potential leads to a differential operator

$$\text{div}(|x|^\alpha \nabla u(x))$$

which is degenerate and singular in the sense that

$$\lim_{|x|\to 0} |x|^\alpha = 0 \text{ and } \lim_{|x|\to\infty} |x|^\alpha = \infty,$$

provided that $\alpha \in (0, 2)$. Consequently, we will analyze equation (1.1) in the case when the operator $\text{div}(|x|^\alpha \nabla u(x))$ is not strictly elliptic in the sense pointed out in D. Gilbarg & N. S. Trudinger [6] (see, page 31 in [6] for the definition of strictly

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elliptic operators). It follows that some of the techniques that can be applied in solving equations involving strictly elliptic operators fail in this new context. For instance some concentration phenomena may occur in the degenerate and singular case which lead to a lack of compactness. On the other hand, such kind of problems are exacerbated by the presence of the critical exponent $2^\star_\alpha$ in the right-hand side of equation (1.1).

2. **Preliminary results**

In this paper the convenient (and natural) functional space where we are seeking solutions for problem (1.1) is $\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$, which is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the inner product

$$\langle u,v\rangle_\alpha := \int_{\mathbb{R}^N} |x|^\alpha \nabla u \nabla v dx.$$

Recall that $\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ is a Hilbert space with respect to the norm

$$\|u\|^2_\alpha := \int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^2 dx$$

We say that $u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ is a *weak solution* of (1.1) if

$$\int_{\mathbb{R}^N} |x|^\alpha \nabla u \nabla v dx - \lambda \int_{\mathbb{R}^N} g(x)|u|^{q(x)-2}uv\, dx - \int_{\mathbb{R}^N} |u|^{2^\star_\alpha - 2}uv\, dx = 0$$

for all $v \in C_0^\infty(\mathbb{R}^N)$.

Remark 2.1. Actually, it can be proved that $\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N\setminus\{0\})}^{\|\cdot\|_\alpha}$ (see [2]).

The starting point of the variational approach to problems of this type is the following inequality which can be obtained essentially "interpolating" between Sobolev's and Hardy's inequalities [1] (see also [3] and [4]).

Lemma 2.2. *(Caffarelli-Kohn-Nirenberg) Let* $N \geq 2$, $\alpha \in (0,2)$ *and denote* $2^\star_\alpha = \frac{2N}{N-2+\alpha}$. *Then there exists* $C_\alpha > 0$ *such that*

$$\left(\int_{\mathbb{R}^N} |\varphi|^{2^\star_\alpha} dx\right)^{2/2^\star_\alpha} \leq C_\alpha \int_{\mathbb{R}^N} |x|^\alpha |\nabla\varphi|^2 dx \qquad (2.1)$$

for every $\varphi \in C_0^\infty(\mathbb{R}^N)$.

Remark 2.3. By Lemma 2.2 we deduce that $\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ is continuously embedded in $L^{2^\star_\alpha}(\mathbb{R}^N)$.

On the other hand, in order to study problem (1.1), we will appeal to the variable exponent Lebesgue spaces $L^{q(\cdot)}(\mathbb{R}^N)$. We point out certain properties of that spaces according to the papers of Kováčik and Rákosník [7] and Mihăilescu and Rădulescu [8, 9].

For any function $p : \mathbb{R}^N \to (1, \infty)$ with $p \in L^\infty(\mathbb{R}^N)$ define

$$p^- := \operatorname{ess}\inf_{x \in \mathbb{R}^N} p(x) \quad \text{and} \quad p^+ := \operatorname{ess}\sup_{x \in \mathbb{R}^N} p(x)\,.$$

It is usually assumed that $p^+ < +\infty$, since this condition implies many useful properties for the associated variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^N)$. This function space is defined by

$$L^{p(\cdot)}(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable } : \int_{\mathbb{R}^N} |u(x)|^{p(x)}\, dx < \infty \right\}.$$

$L^{p(\cdot)}(\mathbb{R}^N)$ is a Banach space when endowed with the so-called *Luxemburg norm*, defined by

$$|u|_{p(\cdot)} := \inf \left\{ \mu > 0 \ : \int_{\mathbb{R}^N} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \le 1 \right\}.$$

The variable exponent Lebesgue space is a special case of an Orlicz-Musielak space. For constant functions p the space $L^{p(\cdot)}(\mathbb{R}^N)$ reduces to the classical Lebesgue space $L^p(\mathbb{R}^N)$, endowed with the standard norm

$$\|u\|_{L^p(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |u(x)|^p dx \right)^{1/p}.$$

We recall that if $1 < p^- \le p^+ < +\infty$ the variable exponent Lebesgue spaces are separable and reflexive.

We denote by $L^{p'(\cdot)}(\mathbb{R}^N)$ the conjugate space of $L^{p(\cdot)}(\mathbb{R}^N)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(\cdot)}(\mathbb{R}^N)$ and $v \in L^{p'(\cdot)}(\mathbb{R}^N)$ the Hölder type inequality

$$\left| \int_{\mathbb{R}^N} uv\, dx \right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(\cdot)} |v|_{p'(\cdot)} \tag{2.2}$$

holds.

A key role in the theory of variable exponent Lebesgue and Sobolev (defined below) spaces is played by the *modular* of the space $L^{p(\cdot)}(\mathbb{R}^N)$, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) := \int_{\mathbb{R}^N} |u(x)|^{p(x)}\, dx.$$

If $u \in L^{p(\cdot)}(\mathbb{R}^N)$ then the following relations hold:

$$|u|_{p(\cdot)} > 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+}; \tag{2.3}$$

$$|u|_{p(\cdot)} < 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-}; \tag{2.4}$$

$$|u|_{p(\cdot)} = 1 \Leftrightarrow \rho_{p(\cdot)}(u) = 1. \tag{2.5}$$

3. **The main result**

In this paper we study the existence of nontrivial weak solutions for problem (1.1) in the case when $q : \mathbb{R}^N \to (1, 2_\alpha^\star)$, $q \in L^\infty(\mathbb{R}^N)$ satisfies the property that there exists $x_0 \in \overline{\Omega}$ and $s > 0$ such that q is continuous on the ball centered in x_0 of radius s, that is $B_s(x_0)$, and

$$1 < q(x_0) < 2\,. \tag{3.1}$$

Our main result is given by the following theorem.

Theorem 3.1. *Assume $q : \mathbb{R}^N \to (1, 2_\alpha^\star)$, $q \in L^\infty(\mathbb{R}^N)$ satisfies the property that there exists $x_0 \in \mathbb{R}^N$ and $s > 0$ such that q is continuous in $B_s(x_0)$ and relation (3.1) is fulfilled. Assume that $g : \mathbb{R}^N \to (0, \infty)$ satisfies $g \in L^\infty(\mathbb{R}^N) \cap L^{r(\cdot)}(\mathbb{R}^N)$, where $r(x) = \frac{2_\alpha^\star}{2_\alpha^\star - q(x)}$ for each $x \in \mathbb{R}^N$. Then, there exists $\lambda^\star > 0$ such that problem (1.1) has a nontrivial weak solution for any $\lambda \in (0, \lambda^\star)$.*

4. **Proof of the main result**

In order to prove Theorem 3.1 we define the functional $J : \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^2 \, dx - \lambda \int_{\mathbb{R}^N} \frac{g(x)}{q(x)} |u|^{q(x)} \, dx - \frac{1}{2_\alpha^\star} \int_{\mathbb{R}^N} |u|^{2_\alpha^\star} \, dx\,.$$

Standard arguments show that $J \in C^1(\mathcal{D}_\alpha^{1,2}(\mathbb{R}^N), \mathbb{R})$ and

$$\langle J'(u), v \rangle = \int_{\mathbb{R}^N} |x|^\alpha \nabla u \nabla v dx - \lambda \int_{\mathbb{R}^N} g(x) |u|^{q(x)-2} uv \, dx - \int_{\mathbb{R}^N} |u|^{2_\alpha^\star - 2} uv \, dx\,,$$

for all u, $v \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$. Thus, we remark that in order to find weak solutions of equation (1.1) it is enough to find critical points for the functional J.

Lemma 4.1. *There exists $\lambda^\star > 0$ such that for any $\lambda \in (0, \lambda^\star)$ there exist $\xi > 0$ and $\theta > 0$ such that*

$$J(u) \ge \theta, \quad \forall\, u \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N) \text{ with } \|u\|_\alpha = \xi\,.$$

Proof. By Lemma 2.2 and Remark 2.3 it follows taht

$$|u|_{2_\alpha^\star} \le C_\alpha^{1/2} \|u\|_\alpha, \quad \forall\, u \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)\,. \tag{4.1}$$

Consider $\xi \in (0,1)$ with $\xi < 1/\sqrt{C_\alpha}$. Then the above relation implies

$$|u|_{2_\alpha^\star} < 1, \quad \forall\, u \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N), \text{ with } \|u\|_\alpha = \xi\,. \tag{4.2}$$

On the other hand, by relation (2.4) we have

$$| \, |u|^{q(\cdot)} \, |_{\frac{2^\star_\alpha}{q(\cdot)}}^{2^\star_\alpha/q^-} \le |u|_{2^\star_\alpha}^{2^\star_\alpha}, \quad \forall \, u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N), \text{ with } \|u\|_\alpha = \xi \, . \tag{4.3}$$

Since $g \in L^{r(\cdot)}(\mathbb{R}^N)$, with $r(x) = \frac{2^\star_\alpha}{2^\star_\alpha - q(x)}$ we deduce by Hölder's inequality that there exists a constant $c_1 > 0$ such that

$$\int_{\mathbb{R}^N} g(x)|u|^{q(x)} \, dx \le c_1 |g|_{r(\cdot)} | \, |u|^{q(\cdot)} \, |_{\frac{2^\star_\alpha}{q(\cdot)}}, \quad \forall \, u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N) \tag{4.4}$$

Relations (4.1), (4.2), (4.3) and (4.4) imply that

$$\int_{\mathbb{R}^N} g(x)|u|^{q(x)} \, dx \le c_1 |g|_{r(\cdot)} C_\alpha^{q^-/2} \|u\|_\alpha^{q^-}, \quad \forall \, u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N), \text{ with } \|u\|_\alpha = \xi \, . \tag{4.5}$$

Relations (2.1) and (4.5) yield that for any $u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ with $\|u\|_\alpha = \xi$ the following inequalities hold true

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_\alpha^2 - \lambda \int_{\mathbb{R}^N} \frac{g(x)}{q(x)} |u|^{q(x)} \, dx - \frac{1}{2^\star_\alpha} \cdot |u|_{2^\star_\alpha}^{2^\star_\alpha} \\ &\ge \frac{1}{2}\|u\|_\alpha^2 - \lambda \int_{\mathbb{R}^N} \frac{g(x)}{q(x)} |u|^{q(x)} \, dx - \frac{C_\alpha^{2^\star_\alpha/2}}{2^\star_\alpha} \cdot \|u\|_\alpha^{2^\star_\alpha} \\ &\ge \tfrac{1}{2}\|u\|_\alpha^2 - \tfrac{\lambda}{q^-} \cdot c_2^{q^-} \|u\|_\alpha^{q^-} - c_3 \|u\|_\alpha^{2^\star_\alpha}, \end{aligned} \tag{4.6}$$

where c_2 and c_3 are two positive constants. In other words, for any $u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ with $\|u\|_\alpha = \xi$ we have

$$J(u) \ge \|u\|_\alpha^{q^-} \cdot \left[\frac{1}{2}\|u\|_\alpha^{2-q^-} - \frac{\lambda}{q^-} \, c_2^{q^-} - c_3 \|u\|_\alpha^{2^\star_\alpha - q^-} \right] .$$

Define $Q : [0, \infty) \to \mathbb{R}$ by

$$Q(t) = \frac{1}{2} \, t^{2-q^-} - c_3 \, t^{2^\star_\alpha - q^-} .$$

Since relation (3.1) holds true we deduce that $q^- < 2 < 2^\star_\alpha$ and thus, it is clear that there exists $\beta > 0$ such that $\max_{t \ge 0} Q(t) = Q(\beta) > 0$. We take $\lambda^\star = \frac{q^-}{c_2^{q^-}} \, Q(\beta)$ and we remark that there exists $\theta > 0$ such that for any $\lambda \in (0, \lambda^\star)$ we have

$$J(u) \ge \theta, \quad \forall \, u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N) \text{ with } \|u\|_\alpha = \xi \, .$$

Lemma 4.1 is verified. □

Lemma 4.2. *There exists $\phi \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ such that $\phi \ge 0$, $\varphi \ne 0$ and $J(t\phi) < 0$, for $t > 0$ small enough.*

Proof. Since there exists $x_0 \in \mathbb{R}^N$ and $s > 0$ such that q is continuous in $B_s(x_0)$ and relation (3.1) is satisfied we deduce that there exists $\theta \in (1, 2)$ such that the open set $\Omega_0 := \{x \in \Omega; \, q(x) < \theta\}$ is nonempty and bounded.

Let $\phi \in C_0^\infty(\mathbb{R}^N)$ be such that $\text{supp}(\phi) \supset \overline{\Omega}_0$, $\phi(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \le \phi \le 1$ in $\mathbb{R}^N$. For any $t \in (0,1)$ we have

$$\begin{aligned} J(t\phi) &= \frac{t^2}{2}\int_{\mathbb{R}^N} |x|^\alpha |\nabla\phi|^2\,dx - \lambda\int_{\mathbb{R}^N} \frac{t^{q(x)}g(x)}{q(x)}|\phi|^{q(x)}\,dx - \frac{t^{2^\star_\alpha}}{2^\star_\alpha}\int_{\mathbb{R}^N}|\phi|^{2^\star_\alpha}\,dx \\ &\le \frac{t^2}{2}\int_{\mathbb{R}^N} |x|^\alpha |\nabla\phi|^2\,dx - \frac{\lambda}{q^+}\int_{\Omega_0} g(x)t^{q(x)}|\phi|^{q(x)}\,dx \\ &\le \frac{t^2}{2}\int_{\mathbb{R}^N} |x|^\alpha |\nabla\phi|^2\,dx - \frac{\lambda\cdot t^\theta}{q^+}\int_{\Omega_0} g(x)|\phi|^{q(x)}\,dx\,. \end{aligned}$$

It is clear that

$$J(t\phi) < 0\,,$$

providing that

$$0 < t < \min\left\{1, \frac{\lambda\cdot 2}{q^+}\cdot\frac{\int_{\Omega_0} g(x)|\phi|^{q(x)}\,dx}{\int_{\mathbb{R}^N}|x|^\alpha|\nabla\phi|^2\,dx}\right\}.$$

Lemma 4.2 is verified. □

Proof of Theorem 3.1. By inequality (4.6) we obtain that J is bounded from below on $\overline{B_\xi(0)}$. Thus, usinging Ekeland's variational principle (see [5] or [10]) to the functional $J : \overline{B_\xi(0)} \to \mathbb{R}$, it follows that there exists $u_\epsilon \in \overline{B_\xi(0)}$ such that

$$\begin{aligned} J(u_\epsilon) &< \inf_{\overline{B_\xi(0)}} J + \epsilon \\ J(u_\epsilon) &< J(u) + \epsilon\cdot\|u-u_\epsilon\|_\alpha, \quad u \ne u_\epsilon. \end{aligned}$$

Using Lemmas 4.1 and 4.2 we find

$$\inf_{\partial B_\xi(0)} J \ge \theta > 0 \quad\text{and}\quad \inf_{\overline{B_\xi(0)}} J < 0\,.$$

We choose $\epsilon > 0$ such that

$$0 < \epsilon \le \inf_{\partial B_\xi(0)} J - \inf_{\overline{B_\xi(0)}} J\,.$$

Therefore, $J(u_\epsilon) < \inf_{\partial B_\xi(0)} J$ and thus, $u_\epsilon \in B_\xi(0)$.

We define $I : \overline{B_\xi(0)} \to \mathbb{R}$ by $I(u) = J(u) + \epsilon\cdot\|u-u_\epsilon\|_\alpha$. It is clear that u_ϵ is a minimum point of I and thus

$$\frac{I(u_\epsilon + \delta\cdot v) - I(u_\epsilon)}{\delta} \ge 0$$

for small $\delta > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{J(u_\epsilon + \delta\cdot v) - J(u_\epsilon)}{\delta} + \epsilon\cdot\|v\|_\alpha \ge 0.$$

Letting $\delta \to 0$ it follows that $\langle J^{'}(u_\epsilon), v\rangle + \epsilon\cdot\|v\|_\alpha > 0$ and we infer that $\|J^{'}(u_\epsilon)\| \le \epsilon$.

We deduce that there exists a sequence $\{u_n\} \subset B_\xi(0)$ such that

$$J(u_n) \to c = \inf_{B_\xi(0)} J < 0 \quad \text{and} \quad J^{'}(u_n) \to 0. \tag{4.7}$$

It is clear that $\{u_n\}$ is bounded in $\mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$. Thus, there exists $w \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$ such that, up to a subsequence, $\{u_n\}$ converges weakly to u in $\mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$. Then Lemma 2.2 (actually, Remark 2.3) implies that $\{u_n\}$ converges weakly to u in $L^{2_\alpha^\star}(\Omega)$. Using these information and the fact that $g \in L^{2_\alpha^\star/q(\cdot)}(\mathbb{R}^N)$, we get that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g(x)|u_n|^{q(x)-2}u_n v\, dx = \int_{\mathbb{R}^N} g(x)|u|^{q(x)-2}uv\, dx\,,$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2_\alpha^\star-2}u_n v\, dx = \int_{\mathbb{R}^N} |u|^{2_\alpha^\star-2}uv\, dx\,,$$

for any $v \in C_0^\infty(\mathbb{R}^N)$.

On the other hand, relation (4.7) implies

$$\lim_{n\to\infty} \langle J'(u_n), v\rangle = 0\,,$$

for all $v \in C_0^\infty(\mathbb{R}^N)$ and actually, (by density) for all $v \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$.

The above information implies

$$J'(u) = 0\,,$$

and thus, u is a weak solution of equation (1.1).

We prove now that $u \neq 0$. Assume by contradiction that $u \equiv 0$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^\alpha |\nabla u_n|^2\, dx = l \geq 0\,.$$

Since by relation (4.7) we have $\lim_{n\to\infty} \langle J'(u_n), u_n\rangle = 0$ and $\{u_n\}$ converges weakly to 0 in $L^{2_\alpha^\star}(\mathbb{R}^N)$ and $g \in L^{2_\alpha^\star/q(\cdot)}(\mathbb{R}^N)$ we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g(x)|u_n|^{q(x)}\, dx = 0\,,$$

or

$$\int_{\mathbb{R}^N} |x|^\alpha |\nabla u_n|^2\, dx - \int_{\mathbb{R}^N} |u_n|^{2_\alpha^\star}\, dx = o(1)$$

or

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2_\alpha^\star}\, dx = l\,.$$

Using again (4.7) we deduce

$$0 > c + o(1) = \frac{1}{2}\int_{\mathbb{R}^N} |x|^\alpha |\nabla u_n|^2\, dx - \lambda \int_{\mathbb{R}^N} \frac{g(x)}{q(x)}|u_n|^{q(x)}\, dx - \frac{1}{2_\alpha^\star}\int_{\mathbb{R}^N} |u_n|^{2_\alpha^\star}\, dx$$

$$\to \left(\frac{1}{2} - \frac{1}{2_\alpha^\star}\right) l \geq 0$$

and that is a contradiction. We conclude that $u \neq 0$.

Thus, Theorem 3.1 is proved. □

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References

[1] Caffarelli, L., Kohn, R., Nirenberg, L., *First order interpolation inequalities with weights*, Compositio Math., **53** (1984), 259-275.

[2] Catrina, F., Wang, Z.-Q., *On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence) and symmetry of extremal function*, Commun. Pure Appl. Math., **54** (2001), 229-258.

[3] Caldiroli, P., Musina, R., *On the existence of extremal functions for a weighted Sobolev embedding with critical exponent*, Calc. Var. Partial Differential Equations, **8** (1999), 365-387.

[4] aldiroli, P., Musina, R., *On a variational degenerate elliptic problem*, NoDEA Nonlinear Differential Equations Appl., **7** (2000), 187-199.

[5] Ekeland, I., *On the variational principle*, J. Math. Anal. App., **47** (1974), 324-353.

[6] Gilbarg, D., Trudinger, N. S., *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1998.

[7] Kováčik, O., Rákosník, J., *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. J., **41** (1991), 592-618.

[8] Mihăilescu, M., Rădulescu, V., *A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids*, Proc. Roy. Soc. London Ser. A, **462** (2006), 2625-2641.

[9] Mihăilescu, M., Rădulescu, V., *On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent*, Proceedings Amer. Math. Soc., **135** (2007), 2929-2937.

[10] Willem, M., *Minimax Theorems*, Birkhäuser, Boston, 1996.

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REMARKS ON A LIMITING CASE IN THE TREATMENT OF NONLINEAR PROBLEMS WITH MOUNTAIN PASS GEOMETRY

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Abstract. We study a class of nonlinear elliptic problem with linear growth and Dirichlet boundary condition. By means of the mountain pass theorem, we establish the existence of a positive solution. The proof of the Palais-Smale condition differs with respect to the standard case that corresponds to nonlinearities with a superlinear behaviour.

1. **Introduction**

The mountain pass theorem of A. Ambrosetti and P. Rabinowitz [2] is a result of great importance in the determination of critical points to energy functionals that occur in the theory of partial differential equations. The original version of A. Ambrosetti and P. Rabinowitz corresponds to the case of mountains of positive altitude. Their proof relies on some deep deformation techniques developed by R. Palais and S. Smale [11, 12], who put the main ideas of the Morse theory into the framework of differential topology on infinite dimensional manifolds. H. Brezis and L. Nirenberg provided in [4] a simpler proof which combines two major tools: Ekeland's variational principle and the pseudogradient lemma. Ekeland's variational principle is the nonlinear version of the Bishop–Phelps theorem and it may be also viewed as a generalization of Fermat's theorem. As pointed out by H. Brezis and F. Browder [3], the mountain pass theorem "extends ideas already present in Poincaré and Birkhoff". An important contribution is due to P. Pucci and J. Serrin [14, 15, 16], who studied he case of mountains of zero altitude.

In its simplest form, the mountain pass theorem considers functions $J : X \to \mathbb{R}$ of class C^1, where X is a real Banach space. It is assumed that J satisfies the following geometric conditions:

(H1) *there exist two numbers* $R > 0$ *and* $c_0 \in \mathbb{R}$ *such that* $J(u) \geq c_0$ *for every*

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$u \in S_R := \{v \in X;\ \|v\| = R\}$;
(H2) *$J(0) < c_0$ and $J(e) < c_0$ for some $e \in X$ with $\|e\| > R$.*
With an additional compactness condition of Palais-Smale type it then follows that the function J has a critical point $u_0 \in X \setminus \{0, e\}$, with corresponding critical value $c \geq c_0$. In essence, this critical value occurs because 0 and e are low points on either side of the "mountain" S_R, so that between 0 and e there must be a lowest critical point, that is, a mountain pass. Condition (H2) signifies that the mountain should have positive altitude. P. Pucci and J. Serrin [14, 15] proved that the mountain pass theorem continues to hold for a mountain of zero altitude, provided it also has nonzero thickness. In addition, if $c = c_0$, then the "pass" itself occurs precisely on the mountain. Roughly speaking, P. Pucci and J. Serrin showed that the mountain pass theorem still remains true if (H1) is strengthened a little, to the form
(H1)$'$ *there exist real numbers c_0, R, r such that $0 < r < R$ and $J(u) \geq c_0$ for every $u \in A := \{v \in X;\ r < \|v\| < R\}$,*
while hypothesis (H2) is weakened and replaced with
(H2)$'$ *$J(0) \leq c_0$ and $J(e) \leq c_0$ for some $e \in X$ with $\|e\| > R$.*

As stated above, the Palais-Smale compactness condition is also crucial for the mountain pass theorem. Let X be a real Banach space and $J \in C^1(X, \mathbb{R})$. We recall that J satisfies the Palais-Smale condition if any sequence $(u_n)_n$ in X such that

$$(J(u_n))_n \text{ is bounded and } J'(u_n) \to 0,$$

admits a convergent subsequence. As pointed out by M. Struwe [19, p. 169], "recent advances in the calculus of variations have shown that the Palais-Smale condition holds for problems in a broad range of energies. Moreover, the failure of the Palais-Smale condition at certain levels reflects highly interesting phenomena related to internal symmetries of the systems under study, which geometrically can be described as *separation of spheres*, or mathematically as *singularities*, respectively as *change in topology*. Speaking in physical terms, we might observe *phase transitions* or *particle creation* at the energy levels where the Palais-Smale condition fails".

The mountain pass theorem has the following simple geometric interpretation. Consider two valleys A and B such that A is surrounded by a mountain ridge that separates it from B. To go from A to B, we must cross the mountain chain. If we want to climb as little as possible, we would have to consider the maximal elevation of each path. The path with the minimal one of these elevations will cross a mountain pass.

We refer to the books by A. Ambrosetti and A. Malchiodi [1], M. Ghergu and V. Rădulescu [7], Y. Jabri [8], A. Kristály, V. Rădulescu, and Cs. Varga [9], J. Mawhin and M. Willem [10], P. Rabinowitz [17], M. Schechter [18], M. Struwe [19], M. Willem [20], and W. Zou [21] for relevant applications of the mountain pass theory. We also refer to the recent paper by P. Pucci and V. Rădulescu [13] for a history of this result and several applications, including in the nonsmooth setting.

2. **Main Result**

The main application of the mountain pass theorem provided by A. Ambrosetti and P. Rabinowitz [2] concerns the Emden–Fowler equation (see R. Emden [5] and R. H. Fowler [6])

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. The standard example here is given by $f(u) = |u|^{p-1}u$, where $1 < p < (N+2)/(N-2)$ for $N \geq 3$ ($1 < p < \infty$ if $N = 1$ or $N = 2$). More generally, the nonlinear term in problem (2.1) can be assumed to satisfy the following assumptions:

(1) $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that

$$|f(u)| \leq C\,(1 + u^p) \qquad \text{for all } u \geq 0,$$

for some $C > 0$, where $1 < p < (N+2)/(N-2)$;

(2) $f(0) = f'(0) = 0$;

(3) there exists $\mu > 2$ such that

$$0 < \mu F(u) \leq u f(u) \qquad \text{for all } u \text{ large enough,}$$

where $F(u) := \int_0^u f(t)\,dt$.

The purpose of the present paper is to study problem (2.1) in the case of f has a linear behaviour. This means that f fulfills the same growth condition as in hypothesis (1) above, but provided that $p = 1$. Simple examples show that we cannot expect that a solution exists in all cases. Indeed, if we consider the simplest case corresponding to $f(u) = \lambda u$, where λ is a positive parameter, then problem (2.1) has a solution if and only if $\lambda = \lambda_1$. Here, λ_1 stands for the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$. Even if we drop the restriction that the solution is positive, a nontrivial solution of (2.1) does not exist if λ is not an eigenvalue of the Laplace operator. These simple remarks show that the lowest eigenvalue λ_1 of $(-\Delta)$ must play a central role in the existence of a solution to problem (2.1), provided that the nonlinear term f has a linear growth.

Our main result is stated in what follows. We point out that the nonlinear term is not assumed to satisfy the above Ambrosetti-Rabinowitz technical assumption (3).

Theorem 2.1. *Assume $f:\mathbb{R}\to\mathbb{R}$ is a differentiable function such that $f(0)=0$,*

$$f'(0)<\lambda_1 \tag{2.2}$$

and

$$\lambda_1<\lim_{u\to\infty}\frac{f(u)}{u}<\infty\,. \tag{2.3}$$

Then problem (2.1) has at least a solution.

Proof. Since we are looking for positive solutions, it is natural to consider the continuous function

$$f_0(u):=\begin{cases} f(u) & \text{if } u\geq 0\\ 0 & \text{if } u<0.\end{cases}$$

By our hypotheses, there is some $C>0$ such that for all $u\in\mathbb{R}$,

$$|f_0(u)|\leq C(1+|u|). \tag{2.4}$$

Assumption (2.3) implies that $C>\lambda_1$. Relation (2.3) also shows that

$$f_0(u)\geq C_1u-C_2\qquad \text{for all } u\geq 0,$$

for some C_1, $C_2>0$ with $C_1>\lambda_1$.

Set $F_0(u):=\int_0^u f(t)\,dt$. Therefore

$$|F_0(u)|\leq C\left(u+\frac{u^2}{2}\right)$$

and

$$F_0(u)\geq C_1\frac{u^2}{2}-C_3\qquad \text{for all } u\geq 0,$$

where $C_3>0$.

We associate to (2.1) the energy functional $J:H_0^1(\Omega)\to\mathbb{R}$ defined by

$$J(u):=\int_\Omega|\nabla u|^2\,dx-\int_\Omega F_0(u)\,dx\qquad \text{for all } u\in H_0^1(\Omega).$$

It follows from (2.4) and the Sobolev embedding theorem that J is well defined on $H_0^1(\Omega)$. Moreover, J is of class C^1 and for all $v\in H_0^1(\Omega)$,

$$J'(u)(v)=\int_\Omega[\nabla u\nabla v-f_0(u)v]\,dx.$$

We first argue that the geometric assumptions of the mountain pass theorem are fulfilled. Fix $1<p<(N+2)/(N-2)$. In order to check condition (H1) we observe that hypothesis (2.3) implies that there are some $0 < \alpha < \lambda_1$ and $C_4 > 0$ such that for all $u \in H_0^1(\Omega)$

$$|f_0(u)| \leq \alpha\,|u| + C_4|u|^p.$$

Thus, $F_0(u) \leq \alpha u^2/2 + C_5\,|u|^{p+1}$, where $C_5 > 0$. It follows that

$$J(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\alpha}{2}\int_\Omega u^2\,dx - C_5\int_\Omega |u|^{p+1}\,dx.$$

By Poincaré's inequality we have

$$\int_\Omega u^2\,dx \leq \frac{1}{\lambda_1}\int_\Omega |\nabla u|^2\,dx \qquad \text{for all } u \in H_0^1(\Omega).$$

We deduce that

$$J(u) \geq \frac{\lambda_1 - \alpha}{2\lambda_1}\,\|u\|^2_{H_0^1(\Omega)} - C_5\int_\Omega |u|^{p+1}\,dx,$$

for all $u \in H_0^1(\Omega)$. Using Sobolev embeddings we conclude that $J(u) > 0$ for all u with $\|u\| = R$, provided that $R > 0$ is small enough. This shows that condition (H1) is fulfilled.

Now, we prove that assumption (H2) in the mountain pass theorem holds true. Let $e_1 > 0$ be the first eigenfunction of $(-\Delta)$ in $H_0^1(\Omega)$, hence $\int_\Omega |\nabla e_1|^2\,dx = \lambda_1 \int_\Omega e_1^2\,dx$. Thus, for all $t > 0$,

$$J(te_1) \leq \frac{t^2\lambda_1^2}{2}\int_\Omega e_1^2\,dx - \frac{t^2C_1^2}{2}\int_\Omega e_1^2\,dx + C_3|\Omega| < 0,$$

provided that t is sufficiently large. This follows from the fact that $C_1 > \lambda_1$, which is a direct consequence of our assumption (2.3).

To complete the proof, it remains to show that the energy functional J satisfies the Palais-Smale condition. For this purpose, let (u_n) be a sequence in $H_0^1(\Omega)$ such that

$$\sup_n |J(u_n)| < \infty \tag{2.5}$$

and

$$\lim_{n\to\infty} \|J'(u_n)\|_{H^{-1}(\Omega)} = 0. \tag{2.6}$$

We now claim that

$$(u_n) \text{ is bounded in } H_0^1(\Omega). \tag{2.7}$$

We first observe that relation (2.6) implies

$$-\Delta u_n = f_0(u_n) + \psi_n\,, \tag{2.8}$$

where $\|\psi_n\|_{H^{-1}(\Omega)} \to 0$. Taking into account the linear growth of f_0 and using (2.8), we deduce that our claim (2.7) follows after proving that (u_n) is bounded in

$L^2(\Omega)$. Arguing by contradiction, we assume that $\|u_n\|_{L^2(\Omega)} \to \infty$. It follows that $v_n := u_n/\|u_n\|_{L^2(\Omega)}$ satisfies

$$-\Delta v_n = \frac{f_0(u_n)}{\|u_n\|_{L^2(\Omega)}} + \frac{\psi_n}{\|u_n\|_{L^2(\Omega)}}\,. \tag{2.9}$$

We now observe that $f_0(u_n)/\|u_n\|_{L^2(\Omega)}$ is bounded in $L^2(\Omega)$. Thus, after multiplication with v_n in (2.9) and integration, we deduce that (v_n) is bounded in $H^1_0(\Omega)$. So, up to a subsequence,

$$v_n \rightharpoonup v \qquad \text{in } H^1_0(\Omega).$$

Moreover, since $\|v_n\|_{L^2(\Omega)} = 1$, we also have

$$\|v\|_{L^2(\Omega)} = 1. \tag{2.10}$$

Next, we observe that the definition of f_0 implies that there is some $C > 0$ such that $f_0(u) \leq -C$ for all $u \in \mathbb{R}$. Thus, by (2.9),

$$-\Delta u_n \geq -C + \psi_n \qquad \text{in } \Omega\,.$$

So, by the maximum principle, $u_n \geq -C + \rho_n$, where $\rho_n \to 0$ in $H^1_0(\Omega)$ and hence $\rho_n \to 0$ a.e. in Ω (at a subsequence). This implies that

$$v \geq 0 \text{ a.e. in } \Omega. \tag{2.11}$$

On the other hand, our assumption (2.3) implies

$$f_0(u) \geq Au - B \qquad \text{for all } u \in \mathbb{R},$$

where $A > \lambda_1$ and $B > 0$. This implies that

$$-\Delta u_n \geq Au_n - B + \psi_n \qquad \text{in } \Omega. \tag{2.12}$$

Let $e_1 > 0$ be the first eigenfunction of the Laplace operator in $H^1_0(\Omega)$. After multiplication with e_1 in relation (2.12) we obtain

$$\lambda_1 \int_\Omega u_n \psi_n \, dx \geq A \int_\Omega u_n e_1 \, dx - C + \langle \psi_n, e_n \rangle_{H^{-1}, H^1_0}\,.$$

Dividing by $\|u_n\|_{L^2(\Omega)}$ and taking $n \to \infty$ we obtain $\lambda_1 \int_\Omega v e_1 \, dx \geq A \int_\Omega v e_1 \, dx$, which contradicts relations (2.10) and (2.11). This concludes the proof of our claim (2.7). From now on, using the same ideas as in the proof of Theorem 9 in [13], we can deduce that any bounded sequence (u_n) in $H^1_0(\Omega)$ satisfying relation (2.6) has the property that contains a strongly convergent subsequence in $H^1_0(\Omega)$. These arguments do not depend on the linear growth of the nonlinear term f. Our proof is now complete. □

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References

[1] Ambrosetti, A., Malchiodi, A., *Nonlinear Analysis and Semilinear Elliptic Problems*, Cambridge Studies in Advanced Mathematics, vol. 104, Cambridge University Press, Cambridge, 2007.

[2] Ambrosetti, A., Rabinowitz, P., *Dual variational methods in critical point theory and applications*, J. Funct. Anal., **14** (1973), 349-381.

[3] Brezis, H., Browder, F., *Partial differential equations in the 20th century*, Adv. Math., **135** (1998), 76-144.

[4] Brezis, H., Nirenberg, L., *Remarks on finding critical points*, Comm. Pure Appl. Math., **44** (1991), 939-964.

[5] Emden, R., *Die Gaskügeln*, Teubner, Leipzig, 1907.

[6] Fowler, R. H., *Further studies of Emden and similar differential equations*, Quart. J. Math., **2** (1931), 259-288.

[7] Ghergu, M., Rădulescu, V., *Singular Elliptic Problems. Bifurcation and Asymptotic Analysis*, Oxford Lecture Series in Mathematics and Its Applications, vol. 37, Oxford University Press, 2008.

[8] Jabri, Y., *The Mountain Pass Theorem. Variants, Generalizations and Some Applications*, Encyclopedia of Mathematics and its Applications, vol. 95, Cambridge University Press, Cambridge, 2003.

[9] Kristály, A., Rădulescu, V., Varga, Cs., *Variational Principles in Mathematical Physics, Geometry, and Economics. Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics and its Applications, vol. 136, Cambridge University Press, Cambridge, 2010.

[10] Mawhin, J., Willem, M., *Critical Point Theory and Hamiltonian Systems*, Applied Mathematical Sciences, vol. 74, Springer, New York, 1989.

[11] Palais, R., *Ljusternik-Schnirelmann theory on Banach manifolds*, Topology, **5** (1966), 115-132.

[12] Palais, R., Smale, S., *A generalized Morse theory*, Bull. Amer. Math. Soc., **70** (1964), 165-171.

[13] Pucci, P., Rădulescu, V., *The impact of the mountain pass theory in nonlinear analysis: a mathematical survey*, Boll. Unione Mat. Ital., in press.

[14] Pucci, P., Serrin, J., *Extensions of the mountain pass theorem*, J. Funct. Anal., **59** (1984), 185-210.

[15] Pucci, P., Serrin, J., *A mountain pass theorem*, J. Differential Equations, **60** (1985), 142-149.

[16] Pucci, P., Serrin, J., *The structure of the critical set in the mountain pass theorem*, Trans. Amer. Math. Soc., **299** (1987), 115-132.

[17] Rabinowitz, P., *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conference Series in Mathematics, **65**, 1986 American Mathematical Society, Providence, RI.

[18] Schechter, M., *Minimax Systems and Critical Point Theory*, Birkhäuser, Boston, 2009.

[19] Struwe, M., *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer-Verlag, Berlin, 1990.

[20] Willem, M., *Minimax Theorems*, Progress in Nonlinear Differential Equations and their Applications, vol. 24, Birkhäuser, Boston, 1996.
[21] Zou, W., *Sign-changing Critical Point Theory*, Springer, New York, 2008.

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A MULTIPLICITY RESULT FOR NONLOCAL PROBLEMS INVOLVING NONLINEARITIES WITH BOUNDED PRIMITIVE

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Abstract. The aim of this paper is to provide the first application of Theorem 3 of [2] in a case where the dependence of the underlying equation from the real parameter is not of affine type. The simplest particular case of our result reads as follows:
Let $f : \mathbf{R} \to \mathbf{R}$ be a non-zero continuous function such that

$$\sup_{\xi\in\mathbf{R}} |F(\xi)| < +\infty$$

where $F(\xi) = \int_0^\xi f(s)ds$.
Moreover, let $k : [0, +\infty[\to \mathbf{R}$ and $h :] - \mathrm{osc}_{\mathbf{R}}F, \mathrm{osc}_{\mathbf{R}}F[\to \mathbf{R}$ be two continuous and non-decreasing functions, with $k(t) > 0$ for all $t > 0$ and $h^{-1}(0) = \{0\}$. Then, for each μ large enough, there exist an open interval $A \subseteq]\inf_{\mathbf{R}} F, \sup_{\mathbf{R}} F[$ and a number $\rho > 0$ such that, for every $\lambda \in A$, the problem

$$\begin{cases} -k\left(\int_0^1 |u'(t)|^2 dt\right) u'' = \mu h\left(\int_0^1 F(u(t))dt - \lambda\right) f(u) \text{ in } [0,1] \\ u(0) = u(1) = 0 \end{cases}$$

has at least three solutions whose norms in $H^1_0(0,1)$ are less than ρ.

In [2], we established the following result:

Theorem 1.1. *Let X be a separable and reflexive real Banach space, $I \subseteq \mathbf{R}$ an interval, and $\Psi : X \times I \to \mathbf{R}$ a continuous function satisfying the following conditions:*
(a_1) for each $x \in X$, the function $\Psi(x, \cdot)$ is concave;
(a_2) for each $\lambda \in I$, the function $\Psi(\cdot, \lambda)$ is C^1, sequentially weakly lower semicontinuous, coercive, and satisfies the Palais-Smale condition;
(a_3) there exists a continuous concave function $h : I \to \mathbf{R}$ such that

$$\sup_{\lambda\in I} \inf_{x\in X} (\Psi(x, \lambda) + h(\lambda)) < \inf_{x\in X} \sup_{\lambda\in I} (\Psi(x, \lambda) + h(\lambda)) .$$

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Then, there exist an open interval $A \subseteq I$ and a positive real number ρ, such that, for each $\lambda \in J$, the equation

$$\Psi'_x(x, \lambda) = 0$$

has at least three solutions in X whose norms are less than ρ. A consequence of Theorem 1.1 is as follows:

Theorem 1.2. *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbf{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbf{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact; $I \subseteq \mathbf{R}$ an interval. Assume that*

$$\lim_{\|x\| \to +\infty} (\Phi(x) + \lambda\Psi(x)) = +\infty$$

for all $\lambda \in I$, and that there exists a continuous concave function $h : I \to \mathbf{R}$ such that

$$\sup_{\lambda \in I} \inf_{x \in X} (\Phi(x) + \lambda\Psi(x) + h(\lambda)) < \inf_{x \in X} \sup_{\lambda \in I} (\Phi(x) + \lambda\Psi(x) + h(\lambda)) \ .$$

Then, there exist an open interval $A \subseteq I$ and a positive real number ρ such that, for each $\lambda \in A$, the equation

$$\Phi'(x) + \lambda\Psi'(x) = 0$$

has at least three solutions in X whose norms are less than ρ.

In appraising the literature, it is quite surprising to realize that, while Theorem 1.2 has been proved itself to be one of the most frequently used abstract multiplicity results in the last decade, it seems that there is no article where Theorem 1.1 has been applied to some Ψ which does not depend on λ in an affine way. For an up-dated bibliographical account related to Theorem 1.2, we refer to [3].

The aim of this paper is to offer a first contribution to fill this gap.

To state our results, let us fix some notation.

For a generic function $\psi : X \to \mathbf{R}$, we denote by $\operatorname{osc}_X \psi$ the (possibly infinite) number $\sup_X \psi - \inf_X \psi$.

In the sequel, $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary. We consider the space $H^1_0(\Omega)$ equipped with the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^2 dx \right)^{\frac{1}{2}} \ .$$

If $I \subseteq \mathbf{R}$ is an interval, with $0 \in I$, and $g : \Omega \times I \to \mathbf{R}$ is a function such that $g(x, \cdot)$ is continuous in I for all $x \in \Omega$, we set

$$G(x,\xi) = \int_0^\xi g(x,t)dt$$

for all $(x, \xi) \in \Omega \times I$.

When $n \geq 2$, we denote by $\mathcal{A}$ the class of all Carathéodory functions $f : \Omega \times \mathbf{R} \to \mathbf{R}$ such that

$$\sup_{(x,\xi)\in\Omega\times\mathbf{R}} \frac{|f(x,\xi)|}{1+|\xi|^q} < +\infty \ ,$$

for some q with $0 < q < \frac{n+2}{n-2}$ if $n \geq 3$ and $0 < q < +\infty$ if $n = 2$. When $n = 1$, we denote by $\mathcal{A}$ the class of all Carathéodory functions $f : \Omega \times \mathbf{R} \to \mathbf{R}$ such that, for each $r > 0$, the function $x \to \sup_{|t|\leq r} |f(x,t)|$ belongs to $L^1(\Omega)$.

If $f \in \mathcal{A}$, for each $u \in H_0^1(\Omega)$, we set

$$J_f(u) = \int_\Omega F(x, u(x))dx \ .$$

The functional J_f is C^1 and its derivative is compact. Moreover, we set

$$\alpha_f = \inf_{H_0^1(\Omega)} J_f \ ,$$

$$\beta_f = \sup_{H_0^1(\Omega)} J_f$$

and

$$\omega_f = \beta_f - \alpha_f \ .$$

Clearly, when f does not depend on x, we have

$$\alpha_f = \operatorname{meas}(\Omega) \inf_{\mathbf{R}} F$$

and

$$\beta_f = \operatorname{meas}(\Omega) \sup_{\mathbf{R}} F \ .$$

Our main result reads as follows:

Theorem 1.3. *Let $f, g \in \mathcal{A}$ be such that*

$$\sup_{(x,\xi)\in\Omega\times\mathbf{R}} \max\{|F(x,\xi)|, G(x,\xi)\} < +\infty$$

and

$$\sup_{u\in H_0^1(\Omega)} \left| \int_\Omega F(x,u(x))dx \right| > 0 \ .$$

Then, for every pair of continuous and non-decreasing functions $k : [0, +\infty[\to \mathbf{R}$ *and* $h :]-\omega_f, \omega_f[\to \mathbf{R}$*, with* $k(t) > 0$ *for all* $t > 0$ *and* $h^{-1}(0) = \{0\}$*, for which the number*

$$\theta^* = \inf \left\{ \frac{\frac{1}{2}K\left(\int_\Omega |\nabla u(x)|^2 dx\right) - \int_\Omega G(x, u(x))dx}{H\left(\int_\Omega F(x, u(x))dx\right)} : u \in H_0^1(\Omega), \int_\Omega F(x, u(x))dx \neq 0 \right\}$$

is non-negative, and for every $\mu > \theta^*$*, there exist an open interval* $A \subseteq]\alpha_f, \beta_f[$ *and a number* $\rho > 0$ *such that, for every* $\lambda \in A$*, the problem*

$$\begin{cases} -k\left(\int_\Omega |\nabla u(x)|^2 dx\right)\Delta u = \mu h\left(\int_\Omega F(x, u(x))dx - \lambda\right) f(x, u) + g(x, u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

has at least three weak solutions whose norms in $H_0^1(\Omega)$ *are less than* ρ*.*

Clearly, a weak solution of the above problem problem is any $u \in H_0^1(\Omega)$ such that

$$k\left(\int_\Omega |\nabla u(x)|^2 dx\right) \int_\Omega \nabla u(x)\nabla v(x)dx =$$

$$= \mu h\left(\int_\Omega F(x, u(x))dx - \lambda\right) \int_\Omega f(x, u(x))v(x)dx + \int_\Omega g(x, u(x))v(x)dx$$

for all $v \in H_0^1(\Omega)$.

So, the weak solutions of the problem are exactly the critical points in $H_0^1(\Omega)$ of the functional

$$u \to \frac{1}{2}K(\|u\|^2) - \int_\Omega G(x, u(x))dx - \mu H\left(\int_\Omega F(x, u(x))dx - \lambda\right) .$$

The problem that we are considering is a nonlocal one. We refer to the very recent paper [1] for a relevant discussion and an up-dated bibliography as well.

From what we said above, it is clear that our proof of Theorem 1.3 is based on the use of Theorem 1.1. This is made possible by the following proposition:

Proposition 1.4. *Let* X *be a non-empty set and let* $\gamma : X \to [0, +\infty[$*,* $J : X \to \mathbf{R}$ *be two functions such that* $\gamma(x_0) = J(x_0) = 0$ *for some* $x_0 \in X$*. Moreover, assume that* J *is bounded and takes at least four values. Finally, let* $\varphi :]-\operatorname{osc}_X J, \operatorname{osc}_X J[\to [0, +\infty[$ *be a continuous function such that*

$$\varphi^{-1}(0) = \{0\} \tag{1.1}$$

and

$$\min\left\{ \liminf_{t \to (-\operatorname{osc}_X J)^+} \varphi(t), \liminf_{t \to (\operatorname{osc}_X J)^-} \varphi(t) \right\} > 0 . \tag{1.2}$$

Put

$$\theta = \inf_{x \in J^{-1}(]\inf_X J, \sup_X J[\setminus\{0\})} \frac{\gamma(x)}{\varphi(J(x))} .$$

Then, for each $\mu > \theta$, we have

$$\sup_{\lambda\in]\inf_X J,\sup_X J[}\inf_{x\in X}(\gamma(x)-\mu\varphi(J(x)-\lambda)) < \inf_{x\in X}\sup_{\lambda\in]\inf_X J,\sup_X J[}(\gamma(x)-\mu\varphi(J(x)-\lambda)) .$$

Proof. First, we make some remarks on the definition of θ. Since J takes at least four values, the set $J^{-1}(]\inf_X J,\sup_X J[\setminus\{0\}$ is non-empty. So, if $x \in J^{-1}(]\inf_X J,\sup_X J[\setminus\{0\}$, we have $J(x)\in]-\operatorname{osc}_X J,\operatorname{osc}_X J[\setminus\{0\}$ (recall that $\inf_X J \leq 0 \leq \sup_X J$), and so $\varphi(J(x)) > 0$. Hence, θ is a well-defined non-negative real number. Now, fix $\mu > \theta$. Since φ is continuous, we have

$$\inf_{\lambda\in]\inf_X J,\sup_X J[}\varphi(J(x)-\lambda) = 0$$

for all $x \in X$. Hence

$$\inf_{x\in X}\sup_{\lambda\in]\inf_X J,\sup_X J[}(\gamma(x)-\mu\varphi(J(x)-\lambda)) = \inf_{x\in X}\left(\gamma(x)-\mu\inf_{\lambda\in]\inf_X J,\sup_X J[}\varphi(J(x)-\lambda)\right)$$
$$= \inf_X \gamma = 0 . \tag{1.3}$$

Now, since $\mu > \theta$, there is $x_1 \in X$ such that

$$\gamma(x_1) - \mu\varphi(J(x_1)) < 0 .$$

So, again by the continuity of φ, for $\epsilon, \delta > 0$ small enough, we have

$$\gamma(x_1) - \mu\varphi(J(x_1) - \lambda) < -\epsilon \tag{1.4}$$

for all $\lambda \in [-\delta, \delta]$. On the other hand, (1.1) and (1.2) imply that

$$\nu := \inf_{\lambda\in]\inf_X J,\sup_X J[\setminus[-\delta,\delta]}\varphi(-\lambda) > 0 . \tag{1.5}$$

From (1.4) and (1.5), recalling that $\gamma(x_0) = J(x_0) = 0$, it clearly follows

$$\sup_{\lambda\in]\inf_X J,\sup_X J[}\inf_{x\in X}(\gamma(x)-\mu\varphi(J(x)-\lambda)) \leq \max\{-\epsilon,-\mu\nu\} < 0$$

and so the conclusion follows in view of (1.3). □

Remark 1.5. It is clear that if a $\varphi :]-\operatorname{osc}_X J,\operatorname{osc}_X J[\to [0,+\infty[$ satisfies (1.1) and is convex, then it is continuous and satisfies (1.2) too.

A joint application of Theorem 1.3 and Proposition 1.4 gives

Theorem 1.6. *Let X be a separable and reflexive real Banach space and let $\eta, J : X \to \mathbf{R}$ be two C^1 functionals with compact derivative and $\eta(0) = J(0) = 0$. Assume also that J is bounded and non-constant, and that η is bounded above.*

Then, for every sequentially weakly lower semicontinuous and coercive C^1 functional $\psi : X \to \mathbf{R}$ whose derivative admits a continuous inverse on X^* and *with* $\psi(0) = 0$*, for every convex* C^1 *function* $\varphi :]-\mathrm{osc}_X J, \mathrm{osc}_X J[\to [0, +\infty[$*, with* $\varphi^{-1}(0) = \{0\}$*, for which the number*

$$\hat{\theta} = \inf_{x \in J^{-1}(\mathbf{R} \setminus \{0\})} \frac{\psi(x) - \eta(x)}{\varphi(J(x))}$$

is non-negative, and for every $\mu > \hat{\theta}$ *there exist an open interval* $A \subseteq]\inf_X J, \sup_X J[$ *and a number* $\rho > 0$ *such that, for each* $\lambda \in A$*, the equation*

$$\psi'(x) = \mu\varphi'(J(x) - \lambda)J'(x) + \eta'(x)$$

has at least three solutions whose norms are less than ρ*.*

Proof. We apply Theorem 1.1 taking $I =]\inf_X J, \sup_X J[$ and

$$\Psi(x, \lambda) = \psi(x) - \eta(x) - \mu\varphi(J(x) - \lambda)$$

for all $(x, \lambda) \in X \times I$.

Clearly, Ψ is C^1 in X, continuous in $X \times I$ and concave in I. By Corollary 41.9 of [4], the functionals η, J are sequentially weakly continuous. Hence, for each $\lambda \in I$, the functional $\Psi(\cdot, \lambda)$ is sequentially weakly lower semicontinuous. Moreover, it is coercive, since ψ is so and $\sup_{x \in X} \max\{|J(x)|, \eta(x)\} < +\infty$. Moreover, it is clear that, for each $\lambda \in I$, the derivative of the functional $\eta(\cdot) + \varphi(J(\cdot) - \lambda)$ is compact (due to the assumptions on η and J and to the fact that φ' is bounded on the compact interval $[\inf_X J, \sup_X J] - \lambda)$, and so, by Example 38.25 of [4], the functional $\Psi(\cdot, \lambda)$ satisfies the Palais-Smale condition. Now, to realize that condition (a_3) is satisfied, we use Remark 1.5 and Proposition 1.4 with $\gamma = \psi - \eta$, observing that $\hat{\theta} = \theta$ since the range of J is an interval. Then, we see that all the assumptions of Theorem 1.1 are satisfied, and the conclusion follows in view of the chain rule. □

It is worth noticing the following consequence of Theorem 1.6:

Theorem 1.7. *Let* X *be a separable and reflexive real Banach space, let* $J : X \to \mathbf{R}$ *be a non-constant bounded* C^1 *functional with compact derivative and* $J(0) = 0$*, and let* $\psi : X \to \mathbf{R}$ *be a sequentially weakly lower semicontinuous and coercive* C^1 *functional whose derivative admits a continuous inverse on* X^* *and with* $\psi(0) = 0$*. Assume that there exists* $\mu > 0$ *such that*

$$\inf_{x \in X}(\psi(x) - \mu(e^{J(x)} - 1)) < 0 \leq \inf_{x \in X}(\psi(x) - \mu J(x)) \ . \tag{1.6}$$

Then, there exist an open interval $A \subseteq]\mu e^{-\sup_X J}, \mu e^{-\inf_X J}[$ *and a number* $\rho > 0$ *such that, for each* $\lambda \in A$*, the equation*

$$\psi'(x) = \lambda e^{J(x)} J'(x)$$

has at least three solutions whose norms are less than ρ*.*

Proof. From (1.6), it clearly follows that

$$0 \leq \inf_{x \in J^{-1}(\mathbf{R}\setminus\{0\})} \frac{\psi(x) - \mu J(x)}{e^{J(x)} - J(x) - 1} < \mu \ .$$

Consequently, we can apply Theorem 1.6 with $\eta = \mu J$ and $\varphi(t) = e^t - t - 1$, so that $\mu > \hat{\theta}$. Then, there exist an open interval $B \subseteq] \inf_X J, \sup_X J[$ and a number ρ such that, for each $\nu \in B$ the equation

$$\psi'(x) = \mu(e^{J(x)-\nu} - 1)J'(x) + \mu J'(x) = \mu e^{-\nu} e^{J(x)} J'(x)$$

has at least three solutions whose norms are less that ρ. Therefore, the conclusion follows taking

$$A = \{\mu e^{-\nu} : \nu \in B\} \ ,$$

and the proof is complete. □

Proof of Theorem 1.3. Let us apply Theorem 1.6 taking

$$X = H^1_0(\Omega) \ ,$$

$$J = J_f \ ,$$

$$\eta = J_g \ ,$$

$$\varphi = H$$

and

$$\psi(u) = \frac{1}{2} K(\|u\|^2)$$

for all $u \in X$.

Since $f, g \in \mathcal{A}$, the functionals J_f, J_g are C^1, with compact derivative. Since K is C^1, increasing and coercive, the functional ψ is sequentially weakly lower semicontinuous, C^1 and coercive. Let us show that ψ' has a continuous inverse on X^* (identified to X, since X is a real Hilbert space). To this end, note that the continuous function $t \to tk(t^2)$ is increasing in $[0, +\infty[$ and onto $[0, +\infty[$. Denote by σ its inverse and consider the operator $T : X \to X$ defined by

$$T(v) = \begin{cases} \frac{\sigma(\|v\|)}{\|v\|} v & \text{if} v \neq 0 \\ 0 & \text{if} \qquad v = 0. \end{cases}$$

Since σ is continuous and $\sigma(0) = 0$, the operator T is continuous in X. For each $u \in X \setminus \{0\}$, since $k(\|u\|^2) > 0$, we have

$$T(\psi'(u)) = T(k(\|u\|^2)u) = \frac{\sigma(k(\|u\|^2)\|u\|)}{k(\|u\|^2)\|u\|} k(\|u\|^2)u = \frac{\|u\|}{k(\|u\|^2)\|u\|} k(\|u\|^2)u = u \ ,$$

as desired. Clearly, the assumptions on h imply that φ is non-negative, convex, with $\varphi^{-1}(0) = \{0\}$. So, all the assumptions of Theorem 1.6 are satisfied, and the conclusion follows. $\square$

We conclude pointing out the following sample of application of Theorem 1.3 which is made possible by the fact that h is assumed to have the required properties on $]-\omega_f, \omega_f[$ only.

Example 1.8. Let $f : \mathbf{R} \to \mathbf{R}$ be a non-zero function belonging to $\mathcal{A}$, with $\sup_{\mathbf{R}} |F| < +\infty$ and let $k : [0, +\infty[\to \mathbf{R}$ be a continuous and non-decreasing function, with $k(t) > 0$ for all $t > 0$.

Then, for each μ large enough, there exist an open interval

$$A \subseteq]\text{meas}(\Omega) \inf_{\mathbf{R}} F, \text{meas}(\Omega) \sup_{\mathbf{R}} F[$$

and a number $\rho > 0$ such that, for every $\lambda \in A$, the problem

$$\begin{cases} -k\left(\int_\Omega |\nabla u(x)|^2 dx\right) \Delta u = \mu \dfrac{\int_\Omega F(u(x))dx - \lambda}{(\text{meas}(\Omega)\text{osc}_{\mathbf{R}} F)^2 - \left(\int_\Omega F(u(x))dx - \lambda\right)^2} f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions whose norms in $H_0^1(\Omega)$ are less than ρ.

References

[1] Fan, X. L., *On nonlocal $p(x)$-Laplacian Dirichlet problems*, Nonlinear Anal., **72** (2010), 3314-3323.

[2] Ricceri, B., *On a three critical points theorem*, Arch. Math. (Basel), **75** (2000), 220-226.

[3] Ricceri, B., *Nonlinear eigenvalue problems*, in Handbook of Nonconvex Analysis and Applications, D. Y. Gao and D. Motreanu eds., International Press, to appear.

[4] Zeidler, E., *Nonlinear functional analysis and its applications*, vol. III, Springer-Verlag, 1985.

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ON A CLASS OF ANALYTIC AND MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY AL-OBOUDI DIFFERENTIAL OPERATOR

SERAP BULUT

Abstract. In this paper, we introduce a new class of analytic functions defined by Al-Oboudi differential operator. For the functions belonging to this class, we obtain coefficient inequalities, Hadamard product, radii of close-to convexity, starlikeness and convexity, extreme points, the integral means inequalities for the fractional derivatives, and further we give distortion theorems using fractional calculus techniques.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$.

For $f \in \mathcal{A}$, Al-Oboudi [1] introduced the following operator:

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = (1-\delta) f(z) + \delta z f'(z) = D_\delta f(z), \quad \delta \geq 0 \tag{1.3}$$

$$D^n f(z) = D_\delta (D^{n-1} f(z)), \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{1.4}$$

If f is given by (1.1), then from (1.3) and (1.4) we see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k, \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{1.5}$$

with $D^n f(0) = 0$.

When $\delta = 1$, we get Sălăgean's differential operator [4].

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Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p = 1, 2, \ldots) \tag{1.6}$$

which are analytic and p-valent in the open unit disk $\mathbb{U}$.

We can write the following equalities for the functions $f \in \mathcal{A}_p$:

$$D^0_{\delta,p} f(z) = f(z), \tag{1.7}$$

$$D^1_{\delta,p} f(z) = (1-\delta) f(z) + \frac{\delta}{p} z f'(z) = D_{\delta,p} f(z), \quad \delta \geq 0 \tag{1.8}$$

$$D^n_{\delta,p} f(z) = D_{\delta,p}(D^{n-1}_{\delta,p} f(z)), \quad (n \in \mathbb{N}). \tag{1.9}$$

If f is given by (1.6), then from (1.8) and (1.9) we see that

$$D^n_{\delta,p} f(z) = z^p + \sum_{k=p+1}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n a_k z^k, \quad (n \in \mathbb{N}_0). \tag{1.10}$$

Let $\mathcal{T}_p$ denote the subclass of $\mathcal{A}_p$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \qquad (a_k \geq 0). \tag{1.11}$$

If f is given by (1.11), then from (1.8) and (1.9) we see that

$$D^n_{\delta,p} f(z) = z^p - \sum_{k=p+1}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n a_k z^k, \quad (n \in \mathbb{N}_0). \tag{1.12}$$

Definition 1.1. A function $f \in \mathcal{T}_p$ is in $\mathcal{R}^n_p(\alpha, \beta, \gamma, \mu)$ if and only if

$$\left| \frac{\left(D^n_{\delta,p} f(z)\right)' - p z^{p-1}}{\alpha \left(D^n_{\delta,p} f(z)\right)' + (\beta - \gamma)} \right| < \mu, \quad (z \in \mathbb{U},\ n \in \mathbb{N}_0), \tag{1.13}$$

for $0 \leq \alpha < 1$, $0 \leq \gamma < 1$, $0 < \beta \leq 1$, $0 < \mu < 1$. Here $D^n_{\delta,p} f(z)$ is defined as in (1.12).

In this paper, basic properties of the class $\mathcal{R}^n_p(\alpha, \beta, \gamma, \mu)$ are studied, such as coefficient bounds, Hadamard product, radii of close-to convexity, starlikeness and convexity, extreme points, the integral means inequalities for the fractional derivatives, and further distortion theorems are given using fractional calculus techniques.

2. Coefficient inequalities

Theorem 2.1. *A function $f \in \mathcal{T}_p$ is in the class $\mathcal{R}_p^n(\alpha, \beta, \gamma, \mu)$ if and only if*

$$\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)a_k \leq \mu(\alpha p+\beta-\gamma). \tag{2.1}$$

The result is sharp for the function f given by

$$f(z) = z^p - \frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)} z^k \quad (k \geq p+1).$$

Proof. Suppose that $f \in \mathcal{R}_p^n(\alpha, \beta, \gamma, \mu)$. Then we have from (1.13)

$$\left| \frac{\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n a_k z^{k-1}}{\alpha\left(pz^{p-1} - \sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n a_k z^{k-1}\right) + (\beta-\gamma)} \right| < \mu.$$

So, we obtain

$$\Re\left\{ \frac{\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n a_k z^{k-1}}{\alpha\left(pz^{p-1} - \sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n a_k z^{k-1}\right) + (\beta-\gamma)} \right\} < \mu.$$

If we choose z real and let $z \to 1^-$, then we get

$$\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)a_k \leq \mu(\alpha p+\beta-\gamma).$$

Conversely, suppose that the inequality (2.1) holds true and let

$$z \in \partial\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}.$$

Then we find from (1.13) that

$$\begin{aligned}
& \left|\left(D_{\delta,p}^{n}f(z)\right)' - pz^{p-1}\right| - \mu\left|\alpha\left(D_{\delta,p}^{n}f(z)\right)' + (\beta-\gamma)\right| \\
= & \left|\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n} a_k z^{k-1}\right| \\
& -\mu\left|\alpha\left(pz^{p-1} - \sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n} a_k z^{k-1}\right) + (\beta-\gamma)\right| \\
\leq & \sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n} a_k |z|^{k-1} - \mu(\alpha p+\beta-\gamma) \\
& +\mu\alpha\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n} a_k |z|^{k-1} \\
= & \sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n} (1+\mu\alpha)a_k - \mu(\alpha p+\beta-\gamma) \leq 0.
\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f \in \mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$. □

Corollary 2.2. *If $f \in \mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$, then*

$$a_{p+1} \leq \frac{\mu(\alpha p+\beta-\gamma)p^n}{(p+1)(p+\delta)^n(1+\mu\alpha)}.$$

Theorem 2.3. *Let the functions*

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \qquad (a_k \geq 0), \tag{2.2}$$

$$g(z) = z^p - \sum_{k=p+1}^{\infty} b_k z^k \qquad (b_k \geq 0) \tag{2.3}$$

be in the class $\mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$. Then for $0 \leq \lambda \leq 1$, the function h defined by

$$h(z) = (1-\lambda)f(z) + \lambda g(z) = z^p - \sum_{k=p+1}^{\infty} c_k z^k,$$

$$c_k := (1-\lambda)a_k + \lambda b_k \geq 0$$

is also in the class $\mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$.

Proof. Suppose that each of the functions f and g is in the class $\mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$. Then, making use of (2.1), we see that

$$\begin{aligned}\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)c_k &= (1-\lambda)\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)a_k\\ &\quad+\lambda\sum_{k=p+1}^{\infty} k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)b_k\\ &\le(1-\lambda)\mu(\alpha p+\beta-\gamma)+\lambda\mu(\alpha p+\beta-\gamma)\\ &=\mu(\alpha p+\beta-\gamma)\end{aligned}$$

which completes the proof of Theorem 2.3. □

3. Hadamard product

Next we define the modified Hadamard product of functions f and g, which are defined by (2.2) and (2.3), respectively, by

$$f*g(z)=z^p-\sum_{k=p+1}^{\infty} a_k b_k z^k \qquad (a_k\ge 0, b_k\ge 0).$$

Theorem 3.1. *If each of the functions f and g is in the class $\mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$, then*

$$f*g(z)\in\mathcal{R}_p^n(\alpha,\beta,\gamma,\eta),$$

where

$$\eta\ge\frac{\mu^2(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)^2-\mu^2\alpha(\alpha p+\beta-\gamma)}.$$

Proof. From Theorem 2.1, we have

$$\sum_{k=p+1}^{\infty}\frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}a_k\le 1 \tag{3.1}$$

and

$$\sum_{k=p+1}^{\infty}\frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}b_k\le 1. \tag{3.2}$$

We need to find the smallest η such that

$$\sum_{k=p+1}^{\infty}\frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\eta\alpha)}{\eta(\alpha p+\beta-\gamma)}a_k b_k\le 1. \tag{3.3}$$

From (3.1) and (3.2) we find, by means of the Cauchy-Schwarz inequality, that

$$\sum_{k=p+1}^{\infty} \frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n}(1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}\sqrt{a_k b_k} \leq 1. \tag{3.4}$$

Thus it is enough to show that

$$\frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n}(1+\eta\alpha)}{\eta(\alpha p+\beta-\gamma)}a_k b_k \leq \frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n}(1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}\sqrt{a_k b_k},$$

that is

$$\sqrt{a_k b_k} \leq \frac{\eta(1+\mu\alpha)}{\mu(1+\eta\alpha)}. \tag{3.5}$$

On the other hand, from (3.4) we have

$$\sqrt{a_k b_k} \leq \frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n}(1+\mu\alpha)}. \tag{3.6}$$

Therefore in view of (3.5) and (3.6) it is enough to show that

$$\frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n}(1+\mu\alpha)} \leq \frac{\eta(1+\mu\alpha)}{\mu(1+\eta\alpha)}$$

which simplifies to

$$\eta \geq \frac{\mu^2(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n}(1+\mu\alpha)^2-\mu^2\alpha(\alpha p+\beta-\gamma)}.$$

□

4. **Close-to convexity, starlikeness and convexity**

A function $f \in \mathcal{T}_p$ is said to be p-valently close-to convex of order ρ if it satisfies

$$\Re\left\{f'(z)\right\} > \rho$$

for some ρ $(0 \leq \rho < p)$ and for all $z \in \mathbb{U}$.

Also a function $f \in \mathcal{T}_p$ is said to be p-valently starlike of order ρ if it satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho$$

for some ρ $(0 \leq \rho < p)$ and for all $z \in \mathbb{U}$.

Further a function $f \in \mathcal{T}_p$ is said to be p-valently convex of order ρ if

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \rho$$

for some ρ $(0 \leq \rho < p)$ and for all $z \in \mathbb{U}$.

Theorem 4.1. *If $f \in \mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$, then f is p-valently close-to convex of order ρ in $|z| < r_1(\alpha,\beta,\gamma,\mu,\rho)$, where*

$$r_1(\alpha,\beta,\gamma,\mu,\rho) = \inf_k \left\{ \frac{\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)(p-\rho)}{\mu(\alpha p+\beta-\gamma)} \right\}^{\frac{1}{k-p}}, \quad k \geq p+1.$$

Proof. It is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p-\rho.$$

We have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{k=p+1}^{\infty} k a_k |z|^{k-p} < p-\rho \tag{4.1}$$

and

$$\sum_{k=p+1}^{\infty} k \left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha) a_k \leq \mu(\alpha p+\beta-\gamma). \tag{4.2}$$

Hence (4.1) is true if

$$\frac{k|z|^{k-p}}{p-\rho} \leq \frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}. \tag{4.3}$$

Solving (4.3) for $|z|$, we obtain

$$|z| \leq \left\{ \frac{\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)(p-\rho)}{\mu(\alpha p+\beta-\gamma)} \right\}^{\frac{1}{k-p}}.$$

□

Theorem 4.2. *If $f \in \mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$, then f is p-valently starlike of order ρ in $|z| < r_2(\alpha,\beta,\gamma,\mu,\rho)$, where*

$$r_2(\alpha,\beta,\gamma,\mu,\rho) = \inf_k \left\{ \frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)(p-\rho)}{\mu(\alpha p+\beta-\gamma)(k-\rho)} \right\}^{\frac{1}{k-p}}, \quad k \geq p+1.$$

Proof. We need to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p-\rho.$$

The inequality

$$\left|\frac{zf'(z)}{f(z)} - p\right| = \left|\frac{-\sum_{k=p+1}^{\infty}(k-p)a_k z^{k-p}}{1-\sum_{k=p+1}^{\infty} a_k z^{k-p}}\right| \leq \frac{\sum_{k=p+1}^{\infty}(k-p)a_k |z|^{k-p}}{1-\sum_{k=p+1}^{\infty} a_k |z|^{k-p}} < p-\rho$$

holds true if

$$\frac{(k-\rho)\,|z|^{k-p}}{p-\rho} \leq \frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n}(1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}.$$

Then f is starlike of order ρ. □

Theorem 4.3. *If $f \in \mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$, then f is p-valently convex of order ρ in $|z| < r_3(\alpha,\beta,\gamma,\mu,\rho)$, where*

$$r_3(\alpha,\beta,\gamma,\mu,\rho) = \inf_k \left\{ \frac{\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n}(1+\mu\alpha)p(p-\rho)}{\mu(\alpha p+\beta-\gamma)(k-\rho)} \right\}^{\frac{1}{k-p}}, \quad k \geq p+1.$$

Proof. We must show that

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| < p-\rho.$$

Since

$$\begin{aligned}\left|1+\frac{zf''(z)}{f'(z)}-p\right| &= \left|\frac{-\sum_{k=p+1}^{\infty}k(k-p)a_k z^{k-p}}{p-\sum_{k=p+1}^{\infty}ka_k z^{k-p}}\right| \\ &\leq \frac{\sum_{k=p+1}^{\infty}k(k-p)a_k\,|z|^{k-p}}{p-\sum_{k=p+1}^{\infty}ka_k\,|z|^{k-p}} < p-\rho,\end{aligned}$$

if

$$\frac{k(k-\rho)\,|z|^{k-p}}{p(p-\rho)} \leq \frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n}(1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}$$

then f is convex of order ρ. □

5. **Extreme points**

Theorem 5.1. *Let $f_p(z) = z^p$ and*

$$f_k(z) = z^p - \frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^{n}(1+\mu\alpha)}z^k \quad (k \geq p+1).$$

Then $f \in \mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$ if and only if it can be expresses in the form

$$f(z) = \lambda_p f_p(z) + \sum_{k=p+1}^{\infty}\lambda_k f_k(z),$$

where $\lambda_k \geq 0$ and $\lambda_p = 1-\sum_{k=p+1}^{\infty}\lambda_k$.

Proof. Assume that

$$f(z)=\lambda_p f_p(z)+\sum_{k=p+1}^{\infty}\lambda_k f_k(z).$$

Then

$$\begin{aligned} f(z) &= \left(1-\sum_{k=p+1}^{\infty}\lambda_k\right)z^p+\sum_{k=p+1}^{\infty}\lambda_k\left(z^p-\frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}z^k\right)\\ &= z^p-\sum_{k=p+1}^{\infty}\lambda_k\frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}z^k. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{k=p+1}^{\infty}\left[k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)\right]\lambda_k\frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}\\ &= \mu(\alpha p+\beta-\gamma)\sum_{k=p+1}^{\infty}\lambda_k\\ &= \mu(\alpha p+\beta-\gamma)(1-\lambda_p)\\ &\le \mu(\alpha p+\beta-\gamma). \end{aligned}$$

Therefore, we have $f\in\mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$.

Conversely, suppose that $f\in\mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$. Since

$$a_k\le\frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}\quad(k\ge p+1),$$

we can set

$$\lambda_k=\frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}{\mu(\alpha p+\beta-\gamma)}a_k\quad(k\ge p+1),$$

$$\lambda_p=1-\sum_{k=p+1}^{\infty}\lambda_k.$$

Then

$$\begin{aligned} f(z) &= z^p-\sum_{k=p+1}^{\infty}a_k z^k\\ &= \lambda_p z^p+\sum_{k=p+1}^{\infty}\lambda_k\left(z^p-\frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}z^k\right)\\ &= \lambda_p f_p(z)+\sum_{k=p+1}^{\infty}\lambda_k f_k(z). \end{aligned}$$

This completes the proof of Theorem 5.1. □

Corollary 5.2. *The extreme points of* $\mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$ *are given by*

$$f_p(z)=z^p, \qquad f_k(z)=z^p-\frac{\mu(\alpha p+\beta-\gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n(1+\mu\alpha)}z^k \quad (k\geq p+1).$$

6. The main integral means inequalities for the fractional derivative

We discuss the integral means inequalities for functions $f\in\mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$.

The following definitions of fractional derivatives by Owa [3] (also by Srivastava and Owa [5]) will be required in our investigation.

Definition 6.1. The fractional integral of order λ is defined by

$$D_z^{-\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z\frac{f(t)}{(z-t)^{1-\lambda}}dt \quad (\lambda>0),$$

where the function f is analytic in a simply connected region of the complex z-plane containing the origin and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)>0$.

Definition 6.2. The fractional derivative of order λ is defined, for a function f, by

$$D_z^{\lambda}f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z\frac{f(t)}{(z-t)^{\lambda}}dt \quad (0\leq\lambda<1), \tag{6.1}$$

where the function f is analytic in a simply connected region of the complex z-plane containing the origin and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)>0$.

Definition 6.3. Under the hypothesis of Definition 6.2, the fractional derivative of order $p+\lambda$ is defined, for a function f, by

$$D_z^{p+\lambda}f(z)=\frac{d^p}{dz^p}D_z^{\lambda}f(z), \tag{6.2}$$

where $0\leq\lambda<1$ and $p\in\mathbb{N}_0$.

It readily follows from Definitions 6.1 and 6.2 that

$$D_z^{-\lambda}z^k=\frac{\Gamma(k+1)}{\Gamma(k+\lambda+1)}z^{k+\lambda} \quad (\lambda>0,\ k\in\mathbb{N}) \tag{6.3}$$

and

$$D_z^{\lambda}z^k=\frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)}z^{k-\lambda} \quad (0\leq\lambda<1,\ k\in\mathbb{N}), \tag{6.4}$$

respectively.

We will also need the concept of subordination between analytic functions and a subordination theorem of Littlewood [2] in our investigation.

Definition 6.4. Given two functions f and g, which are analytic in $\mathbb{U}$, the function f is said to be subordinate to g in $\mathbb{U}$ if there exists a function w analytic in $\mathbb{U}$ with

$$w(0) = 0, \qquad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

We denote this subordination by

$$f(z) \prec g(z).$$

Lemma 6.5. *If the functions f and g are analytic in $\mathbb{U}$ with*

$$f(z) \prec g(z),$$

then, for $\sigma > 0$ and $z = re^{i\theta}$ $\quad (0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^\sigma \, d\theta \leq \int_0^{2\pi} |g(z)|^\sigma \, d\theta.$$

Theorem 6.6. *Let $f \in \mathcal{R}_p^n(\alpha, \beta, \gamma, \mu)$ and suppose that*

$$\sum_{j=p+1}^{\infty} (j-q)_{q+1} a_j \leq \frac{\mu(\alpha p + \beta - \gamma)\Gamma(k+1)\Gamma(2+p-\lambda-q)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)\Gamma(k+1-\lambda-q)\Gamma(p+1-q)} \tag{6.5}$$

for $0 \leq q \leq j$, $0 \leq \lambda < 1$, where $(j-q)_{q+1}$ denotes the Pochhammer symbol defined by

$$(j-q)_{q+1} = (j-q)(j-q+1)\cdots j. \tag{6.6}$$

Also let the function f_k be defined by

$$f_k(z) = z^p - \frac{\mu(\alpha p + \beta - \gamma)}{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)} z^k \quad (k \geq p+1). \tag{6.7}$$

If there exists an analytic function w defined by

$$(w(z))^{k-p} = \frac{k\left[1+\left(\frac{k}{p}-1\right)\delta\right]^n (1+\mu\alpha)}{\mu(\alpha p + \beta - \gamma)} \frac{\Gamma(k+1-\lambda-q)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} (j-q)_{q+1} \Psi(j) a_j z^{j-p} \tag{6.8}$$

$(k \geq q)$, with

$$\Psi(j) = \frac{\Gamma(j-q)}{\Gamma(j+1-\lambda-q)}, \quad (0 \leq \lambda < 1,\ j \geq p+1), \tag{6.9}$$

then, for $\sigma > 0$ and $z = re^{i\theta}$ $\quad (0 < r < 1)$,

$$\int_0^{2\pi} \left|D_z^{q+\lambda} f(z)\right|^\sigma d\theta \leq \int_0^{2\pi} \left|D_z^{q+\lambda} f_k(z)\right|^\sigma d\theta, \quad (0 \leq \lambda < 1). \tag{6.10}$$

Proof. Let $f(z) = z^p - \sum_{j=p+1}^{\infty} a_j z^j$. By means of (6.4) and Definition 6.3, we have

$$\begin{aligned} D_z^{q+\lambda} f(z) &= \frac{\Gamma(p+1) z^{p-\lambda-q}}{\Gamma(p+1-\lambda-q)} \left[1 - \sum_{j=p+1}^{\infty} \frac{\Gamma(j+1)\Gamma(p+1-\lambda-q)}{\Gamma(p+1)\Gamma(j+1-\lambda-q)} a_j z^{j-p} \right] \\ &= \frac{\Gamma(p+1) z^{p-\lambda-q}}{\Gamma(p+1-\lambda-q)} \left[1 - \sum_{j=p+1}^{\infty} \frac{\Gamma(p+1-\lambda-q)}{\Gamma(p+1)} (j-q)_{q+1} \Psi(j) a_j z^{j-p} \right], \end{aligned}$$

where

$$\Psi(j) = \frac{\Gamma(j-q)}{\Gamma(j+1-\lambda-q)}, \quad (0 \leq \lambda < 1, \ j \geq p+1).$$

Since Ψ is a decreasing function of j, we get

$$0 < \Psi(j) \leq \Psi(p+1) = \frac{\Gamma(p+1-q)}{\Gamma(2+p-\lambda-q)}.$$

Similarly, from (6.7), (6.4), and Definition 6.3, we have

$$D_z^{q+\lambda} f_k(z)$$

$$= \frac{\Gamma(p+1) z^{p-\lambda-q}}{\Gamma(p+1-\lambda-q)} \left[1 - \frac{\mu(\alpha p + \beta - \gamma)\Gamma(k+1)\Gamma(p+1-\lambda-q)}{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)\Gamma(p+1)\Gamma(k+1-\lambda-q)} z^{k-p} \right].$$

For $\sigma > 0$ and $z = re^{i\theta}$ $\quad (0 < r < 1)$, we must show that

$$\begin{aligned} &\int_0^{2\pi} \left| 1 - \sum_{j=p+1}^{\infty} \frac{\Gamma(p+1-\lambda-q)}{\Gamma(p+1)} (j-q)_{q+1} \Psi(j) a_j z^{j-p} \right|^\sigma d\theta \\ \leq &\int_0^{2\pi} \left| 1 - \frac{\mu(\alpha p + \beta - \gamma)}{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)} \frac{\Gamma(k+1)\Gamma(p+1-\lambda-q)}{\Gamma(p+1)\Gamma(k+1-\lambda-q)} z^{k-p} \right|^\sigma d\theta. \end{aligned}$$

So, by applying Lemma 6.5, it is enough to show that

$$\begin{aligned} &1 - \sum_{j=p+1}^{\infty} \frac{\Gamma(p+1-\lambda-q)}{\Gamma(p+1)} (j-q)_{q+1} \Psi(j) a_j z^{j-p} \\ \prec \quad &1 - \frac{\mu(\alpha p + \beta - \gamma)}{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)} \frac{\Gamma(k+1)\Gamma(p+1-\lambda-q)}{\Gamma(p+1)\Gamma(k+1-\lambda-q)} z^{k-p}. \end{aligned}$$

If the above subordination holds true, then we have an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ such that

$$\begin{aligned} & 1 - \sum_{j=p+1}^{\infty} \frac{\Gamma(p+1-\lambda-q)}{\Gamma(p+1)} (j-q)_{q+1} \Psi(j) a_j z^{j-p} \\ = \quad & 1 - \frac{\mu(\alpha p + \beta - \gamma)}{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)} \frac{\Gamma(k+1)\Gamma(p+1-\lambda-q)}{\Gamma(p+1)\Gamma(k+1-\lambda-q)} (w(z))^{k-p}. \end{aligned}$$

By the condition of the theorem, we define the function w by

$$(w(z))^{k-p} = \frac{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)}{\mu(\alpha p + \beta - \gamma)} \frac{\Gamma(k+1-\lambda-q)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} (j-q)_{q+1} \Psi(j) a_j z^{j-p}$$

which readily yields $w(0) = 0$. For such a function w, we have

$$\begin{aligned} |w(z)|^{k-p} &\leq \frac{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)}{\mu(\alpha p + \beta - \gamma)} \frac{\Gamma(k+1-\lambda-q)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} (j-q)_{q+1} \Psi(j) a_j |z|^{j-p} \\ &\leq |z| \frac{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)}{\mu(\alpha p + \beta - \gamma)} \frac{\Gamma(k+1-\lambda-q)}{\Gamma(k+1)} \Psi(p+1) \sum_{j=p+1}^{\infty} (j-q)_{q+1} a_j \\ &= |z| \frac{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)\Gamma(k+1-\lambda-q)\Gamma(p+1-q)}{\mu(\alpha p + \beta - \gamma)\Gamma(k+1)\Gamma(2+p-\lambda-q)} \sum_{j=p+1}^{\infty} (j-q)_{q+1} a_j \\ &\leq |z| < 1 \end{aligned}$$

by means of the hypothesis of the theorem.

Thus theorem is proved. □

As a special case $q = 0$, we have following result from Theorem 6.6.

Corollary 6.7. *Let $f \in \mathcal{R}_p^n(\alpha, \beta, \gamma, \mu)$ and suppose that*

$$\sum_{j=p+1}^{\infty} j a_j \leq \frac{\mu(\alpha p + \beta - \gamma)\Gamma(k+1)\Gamma(2+p-\lambda)}{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)\Gamma(k+1-\lambda)\Gamma(p+1)} \qquad (k \geq p+1).$$

If there exists an analytic function w defined by

$$(w(z))^{k-p} = \frac{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)}{\mu(\alpha p + \beta - \gamma)} \frac{\Gamma(k+1-\lambda)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} j \Psi(j) a_j z^{j-p},$$

with

$$\Psi(j) = \frac{\Gamma(j)}{\Gamma(j+1-\lambda)}, \quad (0 \leq \lambda < 1,\ j \geq p+1),$$

then, for $\sigma > 0$ *and* $z = re^{i\theta} \quad (0 < r < 1)$,

$$\int_0^{2\pi} \left| D_z^{\lambda} f(z) \right|^{\sigma} d\theta \leq \int_0^{2\pi} \left| D_z^{\lambda} f_k(z) \right|^{\sigma} d\theta, \quad (0 \leq \lambda < 1).$$

Letting $q = 1$ in Theorem 6.6, we have the following.

Corollary 6.8. *Let* $f \in \mathcal{R}_p^n(\alpha, \beta, \gamma, \mu)$ *and suppose that*

$$\sum_{j=p+1}^{\infty} j(j-1)a_j \leq \frac{\mu(\alpha p + \beta - \gamma)\Gamma(k+1)\Gamma(1+p-\lambda)}{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)\Gamma(k-\lambda)\Gamma(p)}.$$

If there exists an analytic function w *defined by*

$$(w(z))^{k-p} = \frac{k\left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^n (1+\mu\alpha)}{\mu(\alpha p + \beta - \gamma)} \frac{\Gamma(k-\lambda)}{\Gamma(k+1)} \sum_{j=p+1}^{\infty} j(j-1)\Psi(j) a_j z^{j-p}$$

with

$$\Psi(j) = \frac{\Gamma(j-1)}{\Gamma(j-\lambda)}, \quad (0 \leq \lambda < 1,\ j \geq p+1),$$

then, for $\sigma > 0$ *and* $z = re^{i\theta} \quad (0 < r < 1)$,

$$\int_0^{2\pi} \left| D_z^{1+\lambda} f(z) \right|^{\sigma} d\theta \leq \int_0^{2\pi} \left| D_z^{1+\lambda} f_k(z) \right|^{\sigma} d\theta, \quad (0 \leq \lambda < 1).$$

7. Distortion theorems involving operators of fractional calculus

Theorem 7.1. *If* $f \in \mathcal{R}_p^n(\alpha, \beta, \gamma, \mu)$, *then we have*

$$\left| D_z^{-\lambda} f(z) \right| \leq \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)} |z|^{p+\lambda} \left[1 + \frac{\mu(\alpha p + \beta - \gamma)p^n}{(p+\delta)^n(1+\mu\alpha)(p+\lambda+1)} |z|\right] \tag{7.1}$$

and

$$\left| D_z^{-\lambda} f(z) \right| \geq \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)} |z|^{p+\lambda} \left[1 - \frac{\mu(\alpha p + \beta - \gamma)p^n}{(p+\delta)^n(1+\mu\alpha)(p+\lambda+1)} |z|\right], \tag{7.2}$$

for $\lambda > 0$.

Proof. Suppose that $f \in \mathcal{R}_p^n(\alpha, \beta, \gamma, \mu)$. Using Theorem 2.1, we find that

$$(p+1)\left[1 + \frac{\delta}{p}\right]^n (1+\mu\alpha) \sum_{k=p+1}^{\infty} a_k \leq \mu(\alpha p + \beta - \gamma) \tag{7.3}$$

or

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{\mu(\alpha p + \beta - \gamma)p^n}{(p+1)(p+\delta)^n(1+\mu\alpha)}. \tag{7.4}$$

From (6.3), we have

$$\begin{aligned}\frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)}z^{-\lambda}D_z^{-\lambda}f(z) &= z^p - \sum_{k=p+1}^{\infty}\frac{\Gamma(k+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(k+\lambda+1)}a_k z^k \\ &= z^p - \sum_{k=p+1}^{\infty}\Psi(k)a_k z^k, \qquad (7.5)\end{aligned}$$

where

$$\Psi(k) = \frac{\Gamma(k+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(k+\lambda+1)}. \qquad (7.6)$$

Clearly, Ψ is a decreasing function of k and we get

$$0 < \Psi(k) \leq \Psi(p+1) = \frac{p+1}{p+\lambda+1}.$$

Using (7.4) – (7.6), we obtain

$$\begin{aligned}\left|\frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)})z^{-\lambda}D_z^{-\lambda}f(z)\right| &\leq |z|^p + \Psi(p+1)\,|z|^{p+1}\sum_{k=p+1}^{\infty}a_k \\ &\leq |z|^p + \frac{\mu(\alpha p+\beta-\gamma)p^n}{(p+1)\,(p+\delta)^n\,(1+\mu\alpha)}\frac{p+1}{p+\lambda+1}\,|z|^{p+1}\end{aligned}$$

which is equivalent to (7.1) and

$$\begin{aligned}\left|\frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)}z^{-\lambda}D_z^{-\lambda}f(z)\right| &\geq |z|^p - \Psi(p+1)\,|z|^{p+1}\sum_{k=p+1}^{\infty}a_k \\ &\geq |z|^p - \frac{\mu(\alpha p+\beta-\gamma)p^n}{(p+1)\,(p+\delta)^n\,(1+\mu\alpha)}\frac{p+1}{p+\lambda+1}\,|z|^{p+1}\end{aligned}$$

which is precisely the assertion (7.2). □

The proof of Theorem 7.2 below is similar to that of Theorem 7.1, which we have detailed above fairly fully. Indeed, instead of (6.3), we make use of (6.4) to prove Theorem 7.2.

Theorem 7.2. *If $f \in \mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$, then we have*

$$\left|D_z^{\lambda}f(z)\right| \leq \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\,|z|^{p-\lambda}\left[1+\frac{\mu(\alpha p+\beta-\gamma)p^n}{(p+\delta)^n(1+\mu\alpha)(p-\lambda+1)}\,|z|\right] \qquad (7.7)$$

and

$$\left|D_z^{\lambda}f(z)\right| \geq \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\,|z|^{p-\lambda}\left[1-\frac{\mu(\alpha p+\beta-\gamma)p^n}{(p+\delta)^n(1+\mu\alpha)(p-\lambda+1)}\,|z|\right]. \qquad (7.8)$$

Corollary 7.3. *If $f \in \mathcal{R}_p^n(\alpha,\beta,\gamma,\mu)$, then we have*

$$|z|^p - \frac{\mu(\alpha p+\beta-\gamma)p^n}{(p+\delta)^n(1+\mu\alpha)(p+\lambda+1)}\,|z|^{p+1} \leq |f(z)|$$

$$\leq |z|^p + \frac{\mu(\alpha p + \beta - \gamma)p^n}{(p+\delta)^n(1+\mu\alpha)(p+\lambda+1)} |z|^{p+1} . \tag{7.9}$$

Proof. From Definition 6.1, we have

$$\lim_{\lambda \to 0} D_z^{-\lambda} f(z) = f(z).$$

Therefore, letting $\lambda = 0$ in (7.1) and (7.2), we obtain (7.9). □

Corollary 7.4. *If $f \in \mathcal{R}_p^n(\alpha, \beta, \gamma, \mu)$, then we have*

$$p |z|^{p-1} - \frac{\mu(\alpha p + \beta - \gamma)p^n}{(p+\delta)^n(1+\mu\alpha)(p-\lambda+1)} |z|^p \leq |f'(z)|$$

$$\leq p |z|^{p-1} + \frac{\mu(\alpha p + \beta - \gamma)p^n}{(p+\delta)^n(1+\mu\alpha)(p-\lambda+1)} |z|^p . \tag{7.10}$$

Proof. From Definition 6.2, we have

$$\lim_{\lambda \to 1} D_z^{\lambda} f(z) = f'(z).$$

Therefore, letting $\lambda = 1$ in (7.7) and (7.8), we obtain (7.10). □

References

[1] Al-Oboudi, F. M., *On univalent functions defined by a generalized Sălăgean operator*, Int. J. Math. Math. Sci., **25-28** (2004), 1429-1436.

[2] Littlewood, J. E., *On inequalities in the theory of functions*, Proc. London Math. Soc., **23** (1925), no. 1, 481-519.

[3] Owa, S., *On the distortion theorems I*, Kyungpook Math. J., **18** (1978), no. 1, 53-59.

[4] Sălăgean, G., *Subclasses of univalent functions*, Complex Analysis-Fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., Springer, Berlin, **1013** (1983), 362-372.

[5] Srivastava, H. M., Owa, S., *Univalent Functions, Fractional Calculus, and Their Applications*, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester, UK; JohnWiley & Sons, New York, NY, USA, 1989.

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SUFFICIENT CONDITIONS FOR UNIVALENCE AND QUASICONFORMAL EXTENSIONS IN SEVERAL COMPLEX VARIABLES

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Abstract. The method of subordination chains is used to established an univalence criterion which contains as particular cases some univalence criteria for holomorphic mappings in the unit ball B of $\mathbb{C}^n$. We also obtain a sufficient condition for a normalized mapping $f \in \mathcal{H}(B)$ to be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

1. **Introduction and preliminaries**

Pfaltzgraff [16] was the first who obtained an univalence criterion in the n-variable case. He [17] also initiated the study of quasiconformal extensions for quasiregular holomorphic mappings defined on the unit ball of $\mathbb{C}^n$.

The problems of univalence criteria and quasiconformal extensions for holomorphic mappings on the unit ball in $\mathbb{C}^n$ have been studied by P. Curt [3], [4], [5], [7], H. Hamada and G. Kohr [14], [15], P. Curt and G. Kohr [9], [10], [11], D. Răducanu [18].

In this work we generalize the results due to J.A. Pfaltzgraff [16], [17], P. Curt [3], [5], [7], D. Răducanu [18].

Let $\mathbb{C}^n$ denote the space of n-complex variables $z = (z_1, \ldots, z_n)$ with the usual inner product $\langle z, w\rangle = \sum_{i=1}^{n} z_i\overline{w}_i$ and Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. Let B denote the open unit ball in $\mathbb{C}^n$.

Let $\mathcal{H}(B)$ be the set of holomorphic mappings from B into $\mathbb{C}^n$. Also, let $\mathcal{L}(\mathbb{C}^n)$ be the space of continuous linear mappings from $\mathbb{C}^n$ into $\mathbb{C}^n$ with the standard operator norm

$$\|A\| = \sup\{\|Az\| : \ \|z\| = 1\}.$$

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By I we denote the identity in $\mathcal{L}(\mathbb{C}^n)$. A mapping $f \in \mathcal{H}(B)$ is said to be normalized if $f(0) = 0$ and $Df(0) = I$.

We say that a mapping $f \in \mathcal{H}(B)$ is K-quasiregular, $K \geq 1$, if

$$\|Df(z)\|^n \leq K|\det Df(z)|, \quad z \in B.$$

A mapping $f \in \mathcal{H}(B)$ is called quasiregular if is K-quasiregular for some $K \geq 1$. Every quasiregular holomorphic mapping is locally biholomorphic.

Let G and G' be domains in $\mathbb{R}^m$. A homeomorphism $f : G \to G'$ is said to be K-quasiconformal if it is differentiable a.e., ACL (absolutely continuous on lines) and

$$\|Df(x)\|^m \leq K|\det Df(x)| \text{ a.e. } , x \in G$$

where $Df(x)$ denotes the real Jacobian matrix of f and K is a constant.

If $f, g \in \mathcal{H}(B)$, we say that f is subordinate to g (and write $f \prec g$) if there exists a Schwarz mapping v (i.e. $v \in \mathcal{H}(B)$ and $\|v(z)\| \leq \|z\|$, $z \in B$) such that $f(z) = g(v(z))$, $z \in B$.

A mapping $L : B \times [0, \infty) \to \mathbb{C}^n$ is called a subordination chain if the following conditions hold:

(i) $L(0, t) = 0$ and $L(\cdot, t) \in \mathcal{H}(B)$ for $t \geq 0$;

(ii) $L(\cdot, s) \prec L(\cdot, t)$ for $0 \leq s \leq t < \infty$.

An important role in our discussion is played by the n-dimensional version of the class of holomorphic functions on the unit disc with positive real part

$$\mathcal{N} = \{h \in \mathcal{H}(B) : \ h(0) = 0, \ \text{Re } \langle h(z), z\rangle > 0, \ z \in B \setminus \{0\}\}$$

$$\mathcal{M} = \{h \in \mathcal{N}; \ Dh(0) = I\}.$$

It is known that normalized univalent subordination chains satisfy the generalized Loewner differential equation ([12], [8]).

By using an elementary change of variable, it is not difficult to reformulate the mentioned result in the case of nonnormalized subordination chains $L(z, t) = a(t)z + \dots$, where $a : [0, \infty) \to \mathbb{C}$, $a(\cdot) \in C^1([0, \infty))$, $a(0) = 1$ and $\lim\limits_{t\to\infty} |a(t)| = \infty$.

Theorem 1.1. *Let $L(z, t) : B \times [0, \infty) \to \mathbb{C}^n$ be a Loewner chain such that $L(z, t) = a(t)z + \dots$, where $a \in C^1([0, \infty))$, $a(0) = 1$, and $\lim\limits_{t\to\infty} |a(t)| = \infty$. Then there exists a mapping $h = h(z, t) : B \times [0, \infty) \to \mathbb{C}^n$ such that $h(\cdot, t) \in \mathcal{N}$ for $t \geq 0$, $h(z, \cdot)$ is measurable on $[0, \infty)$ for $z \in B$ and*

$$\frac{\partial L}{\partial z}(z, t) = DL(z, t)h(z, t), \ \textit{a.e. } t \geq 0, \ \forall \ z \in B. \tag{1.1}$$

We shall use the following theorem to prove our results [6]. We mention that this result is a simplified version of Theorem 3 [3] due to Theorem 1.2 [12].

Theorem 1.2. *Let $L(z,t) = a(t)z + \dots,$ be a function from $B \times [0,\infty)$ into $\mathbb{C}^n$ such that*

(i) $L(\cdot,t) \in \mathcal{H}(B)$, for each $t \geq 0$

(ii) $L(z,t)$ is absolutely continuous of t, locally uniformly with respect to B.

Let $h(z,t)$ be a function from $B \times [0,\infty)$ into $\mathbb{C}^n$ such that

(iii) $h(\cdot,t) \in \mathcal{N}$ for each $t \geq 0$

(iv) $h(z,\cdot)$ is measurable on $[0,\infty)$ for each $z \in B$.

Suppose $h(z,t)$ satisfies:

$$\frac{\partial L}{\partial t}(z,t) = DL(z,t)h(z,t) \ \text{a.e.}\ t \geq 0,\ \forall\ z \in B.$$

Further, suppose

(a) $a(0) = 1$, $\lim\limits_{t\to\infty} |a(t)| = \infty$, $a(\cdot) \in C^1([0,\infty))$.

(b) There is a sequence $\{t_m\}_m$, $t_m > 0$, $t_m \to \infty$ such that

$$\lim_{m\to\infty} \frac{L(z,t)}{a(t_m)} = F(z) \tag{1.2}$$

locally uniformly in B, where $F \in \mathcal{H}(B)$.

Then for each $t \geq 0$, $L(\cdot,t)$ is univalent on B.

Also, we shall use the following result that was recently proved by P. Curt and G. Kohr [11].

Theorem 1.3. *Let $L(z,t) : B \times [0,\infty) \to \mathbb{C}^n$, $L(z,t) = a(t)z + \dots,$ be a Loewner chain such that $a(\cdot) \in C^1[0,\infty)$, $a(0) = 1$ and $\lim\limits_{t\to\infty} |a(t)| = \infty$. Assume that the following conditions hold:*

(i) There exists $K > 0$ such that $L(\cdot,t)$ is K-quasiregular for each $t \geq 0$.

(ii) There exist some constants $M > 0$ and $\beta \in [0,1)$ such that

$$\|DL(z,t)\| \leq \frac{M|a(t)|}{(1-\|z\|)^\beta}, \quad z \in B,\ t \in [0,\infty) \tag{1.3}$$

(ii) There exists a sequence $\{t_m\}_{m\in\mathbb{N}}$, $t_m > 0$, $\lim\limits_{m\to\infty} t_m = \infty$, and a mapping $F \in \mathcal{H}(B)$ such that

$$\lim_{m\to\infty} \frac{L(z,t_m)}{a(t_m)} = F(z) \ \text{locally uniformly on}\ B.$$

Further, assume that the mapping $h(z,t)$ defined by Theorem 1.1 satisfies the following conditions

(iv) There exists a constant $C > 0$ such that

$$C\|z\|^2 \leq \operatorname{Re}\,\langle h(z,t), z\rangle, \quad z \in B,\ t \in [0,\infty) \tag{1.4}$$

(v) There exists a constant $C_1 > 0$ such that

$$\|h(z,t)\| \leq C_1, \quad z \in B,\ t \in [0,\infty). \tag{1.5}$$

Then the function $f = L(\cdot, 0)$ *extends to a quasiconformal homeomorphism of* $\mathbb{R}^{2n}$ *onto itself.*

2. **Univalence criteria**

In this section, by using the Loewner chains method, we obtain some univalence criteria involving the first and second derivative of an holomorphic mapping in the unit ball B.

Theorem 2.1. *Let* $f \in \mathcal{H}(B)$ *be a normalized mapping (i.e.* $f(0) = 0$ *and* $Df(0) = I$*). Let* $\beta \in \mathbb{R}$, $\beta \geq 2$ *and* α, c *be complex numbers such that*

$$c \neq -1, \quad \alpha \neq 1 \quad \text{and} \quad \left|\frac{c+\alpha}{1-\alpha}\right| \leq 1.$$

If the function $f(z) - \alpha z$, $z \in B$ *is locally biholomorphic on* B *and if the following conditions hold*

$$\left\|(Df(z) - \alpha I)^{-1}(cDf(z) + \alpha I) - \left(\frac{\beta}{2} - 1\right) I\right\| < \frac{\beta}{2}, \quad z \in B \tag{2.1}$$

and

$$\Big\| \|z\|^{\beta}(Df(z) - \alpha I)^{-1}(cDf(z) + \alpha I) \tag{2.2}$$

$$+(1 - \|z\|^{\beta})(Df(z) - \alpha I)^{-1}D^2 f(z)(z, \cdot) + \left(1 - \frac{\beta}{2}\right) I\Big\| < \frac{\beta}{2}, \quad z \in U$$

then the function f *is univalent on* B.

Proof. we will show that the relations (2.1) and (2.2) allow us to embed f as the initial element $f(z) = L(z, 0)$ of an appropriate subordination chain.

We define

$$L(z,t) = f(e^{-t}z) + \frac{1}{1+c}(e^{\beta t} - 1)e^{-t}[Df(e^{-t}z) - \alpha I](z), \ t \geq 0, \ z \in B. \tag{2.3}$$

Since

$$a(t) = e^{(\beta-1)t}\frac{1-\alpha}{1+c}\left(1 + e^{-t}\frac{c+\alpha}{1-\alpha}\right) \quad \text{and} \quad \left|\frac{c+\alpha}{1-\alpha}\right| \leq 1$$

we deduce that $a(t) \neq 0$, $a(0) = 1$, $\lim_{t\to\infty} |a(t)| = \infty$ and $a(\cdot) \in C^1([0, \infty))$.

It can be easily verified that:

$$L(z,t) = a(t)z + (\text{holormohic term}) \quad \text{so} \quad \lim_{t\to\infty} \frac{L(z,t)}{a(t)} = z$$

locally uniformly with respect to $z \in B$, and thus (1.2) holds with $F(z) = z$.
It is obvious that L satisfies the absolute continuity requirements of Theorem 1.2.

From (2.3) we obtain:

$$DL(z,t) = \frac{1}{1+c}\frac{\beta}{2}e^{(\beta-1)t}[Df(e^{-t}z) - \alpha I]\Big[I - I \tag{2.4}$$

$$\frac{2}{\beta}e^{-\beta t}(c+1)(Df(e^{-t}z)-\alpha I)^{-1}Df(e^{-t}z)+\frac{2}{\beta}(1-e^{-\beta t})I$$
$$+\frac{2}{\beta}(1-e^{-\beta t})(Df(e^{-t}z)-\alpha I)^{-1}D^2f(e^{-t}z)(e^{-t}z,\cdot)\Big].$$

By using the obvious equality:

$$(c+1)[Df(e^{-t}z)-\alpha I]^{-1}Df(e^{-t}z)=Df(e^{-t}z)-\alpha I$$

the relation (2.4) becomes

$$DL(z,t)=\frac{1}{1+c}\frac{\beta}{2}e^{(\beta-1)t}[Df(e^{-t}z)-\alpha I]\Big\{I+\left(\frac{2}{\beta}-1\right)I \tag{2.5}$$
$$+\frac{2}{\beta}e^{-\beta t}[Df(e^{-t}z)-\alpha I]^{-1}[cDf(e^{-t}z)+\alpha I]$$
$$+\frac{2}{\beta}(1-e^{-\beta t})[Df(e^{-t}z)-\alpha I]^{-1}D^2f(e^{-t}z)(e^{-t}z,\cdot)\Big\}.$$

If we denote, for each fixed $(z,t)\in B\times[0,\infty)$, by $E(z,t)$ the linear operator

$$E(z,t)=-\frac{2}{\beta}e^{-\beta t}(Df(e^{-t}z)-\alpha I)^{-1}(cDf(e^{-t}z)+\alpha I) \tag{2.6}$$
$$-\frac{2}{\beta}(1-e^{-\beta t})(Df(e^{-t}z)-\alpha I)^{-1}D^2f(e^{-t}z)(e^{-t}z,\cdot)+\left(1-\frac{2}{\beta}\right)I,$$

then (2.5) becomes:

$$DL(z,t)=\frac{1}{1+c}\frac{\beta}{2}e^{(\beta-1)t}[Df(e^{-t}z)-\alpha I][I-E(z,t)]. \tag{2.7}$$

We will prove next that for each $z\in B$ and $t\in[0,\infty)$, $I-E(z,t)$ is an invertible operator.

For $t=0$,

$$E(z,0)=-\frac{2}{\beta}\left[(Df(z)-\alpha I)^{-1}(cDf(z)+\alpha I)-\frac{\beta}{2}I+I\right].$$

By using the condition (2.1) we obtain that $\|E(z,0)\|<1$ and in consequence $I-E(z,0)$ is an invertible operator.

For $t>0$ since $E(,t):\overline{B}\to\mathcal{L}(\mathbb{C}^n)$ is holomorphic, by using the weak maximum modulus theorem we obtain that $\|E(z,t)\|$ can have no maximum in B unless $\|E(z,t)\|$ is of constant value throughout $\overline{B}$.

If $z=0$ and $t>0$, since $\beta\geq 2$, we have

$$\|E(0,t)\|=\frac{2}{\beta}\left|1+\frac{c+\alpha}{1-\alpha}e^{-\beta t}-\frac{\beta}{2}\right|<1. \tag{2.8}$$

Also, we have

$$\|E(z,t)\|\leq\max_{\|w\|=1}\|E(w,t)\|. \tag{2.9}$$

If we let now $u = e^{-t}w$, where $\|w\| = 1$, then $\|u\| = e^{-t}$ and so

$$E(w,t) = -\frac{2}{\beta}\|u\|^{\beta}[Df(u) - \alpha I]^{-1}(cDf(u) + \alpha I)$$

$$-\frac{2}{\beta}(1 - \|u\|^{\beta})(Df(u) - \alpha I)^{-1}D^2 f(u)(u,\cdot) - \frac{2}{\beta}\left(1 - \frac{\beta}{2}\right) I.$$

By using (2.2), (2.8) and the previous equality we obtain

$$\|E(z,t)\| < 1, \quad t > 0.$$

Hence for $t > 0$, $I - E(z,t)$ is an invertible operator, too.

Further computations show that:

$$\frac{\partial L}{\partial t}(z,t) = \frac{1}{1+c}e^{(\beta-1)t}\frac{\beta}{2}[Df(e^{-t} - z) - \alpha I]\Big[I + \left(1 - \frac{2}{\beta}\right) I$$

$$-\frac{2}{\beta}e^{-\beta t}(Df(e^{-t}z) - \alpha I)^{-1}(cDf(e^{-t}z) + \alpha I)$$

$$-\frac{2}{\beta}(1 - e^{-\beta t})[Df(e^{-t}z) - \alpha I]^{-1}D^2 f(e^{-t}z)(e^{-t}z,\cdot)\Big](z)$$

and

$$\frac{\partial L}{\partial z}(z,t) = \frac{1}{c}e^{(\beta-1)t}\frac{\beta}{2}[Df(e^{-t}z) - \alpha I][I + E(z,t)](z). \tag{2.10}$$

In conclusion, by using (2.7) and (2.10) we obtain

$$\frac{\partial L}{\partial t}(z,t) = DL(z,t)[I - E(z,t)]^{-1}[I + E(z,t)](z), \quad z \in B.$$

Hence $L(z,t)$ satisfies the differential equation (1.1) for all $z \in B$ and $t \geq 0$ where

$$h(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z), \quad z \in B. \tag{2.11}$$

It remains to show that the function defined by (2.11) satisfies the conditions of Theorem 1.2. Clearly $h(z,t)$ satisfies the holomorphy and measurability requirements and $h(0,t) = 0$.

Furthermore, the inequality:

$$\|g(z,t) - z\| = \|E(z,t)(h(z,t) + z)\| \leq \|E(z,t)\| \cdot \|h(z,t) + z\| < \|h(z,t) + z\|$$

implies that Re $\langle h(z,t), z\rangle > 0$, $\forall\ z \in B \setminus \{0\}$, $t \geq 0$.

Since all the assumptions of Theorem 1.2 are satisfied, it follows that the functions $L(\cdot,t)$ $(t \geq 0)$ are univalent in B.

In particular $f = L(\cdot,0)$ is univalent in B.

Remark 2.2. If $\beta = 2$, $\alpha = 0$ and $c = 0$, then Theorem 2.1 becomes the n-dimensional version of Becker's univalence criterion [17].

If $\beta = 2$, $f = g$, then Theorem 2.1 becomes the n-dimensional version of Ahlfors and Becker's univalence criterion [3].

If $c = 0$ then Theorem 2.1 becomes Theorem 2 [4].
If $\alpha = 0$ and $c = 0$ then Theorem 2.1 becomes Theorem 2 [5].
If $\beta = 2$ then Theorem 2.1 becomes Theorem 2.1 [18].

3. **Quasiconformal extensions**

In this section we present a sufficient condition for a normalized holomorphic mapping on B to be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Theorem 3.1. *Let $f \in \mathcal{H}(B)$ be a normalized mapping (i.e. $f(0) = 0$, $Df(0) = I$) such that the mapping $f(z) - \alpha z$, $z \in B$, is quasiregular. Also let $\beta \geq 2$, and α, c be complex numbers such that*

$$c \neq -1, \quad \alpha \neq 1 \quad \textit{and} \quad \left|\frac{c+\alpha}{1-\alpha}\right| \leq 1.$$

If there is $q \in [0,1)$ such that $1 - \frac{2}{\beta} \leq q < \frac{2}{\beta}$,

$$\frac{2}{\beta}\left\|(Df(z) - \alpha I)^{-1}(cDf(z) + \alpha I) - \left(\frac{\beta}{2} - 1\right) I\right\| \leq q < 1, \ z \in B \tag{3.1}$$

and

$$\frac{2}{q}\Big\|\|z\|^{\beta}(Df(z) - \alpha I)^{-1}(cDf(z) + \alpha I) \tag{3.2}$$
$$+(1 - \|z\|^{\beta})(Df(z) - \alpha I)^{-1}D^2f(z)(z, \cdot) + \left(1 - \frac{\beta}{2}\right) I\Big\| \leq q < 1, \quad z \in B$$

then f extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Proof. The conditions (3.1) and (3.2) enable us to embed f as the initial element $f(z) = L(z, 0)$ of the subordination chain defined by (2.3). In Theorem 2.1 we proved that L (defined by (2.3)) is a subordination chain which satisfies the generalized Loewner equation (1.1) where the mapping h is defined by (2.11) and the mapping $E : B \times [0, \infty) \to \mathcal{L}(\mathbb{C}^n)$ is defined by (2.6).

Next, we will show that $\|E(z,t)\| \leq q$ for all $(z,t) \in B \times [0, \infty)$. We have

$$\|E(z,0)\| = \frac{2}{\beta}\left\|(Df(z) - \alpha I)^{-1}(cDf(z) + \alpha I) - \left(\frac{\beta}{2} - 1\right) I\right\| \leq q < 1,$$

$z \in B$, according to condition (3.1). Next, let $t \in (0, \infty)$.

In view of the maximum principle for holomorphic mappings into complex Banach spaces, by using the condition (3.2), we obtain:

$$\|E(z,t)\| \leq \max_{\|w\|=1} \|E(z,t)\|$$

$$= \frac{2}{\beta} \max_{\|w\|=1} \Big\|\|we^{-t}\|^{\beta}[Df(we^{-t}) - \alpha I]^{-1}[cDf(we^{-t}) + \alpha I]$$

$$+(1-\|we^{-t}\|)^{\beta}[Df(we^{-t})-\alpha I]^{-1}[D^2f(we^{-t})(we^{-t},\cdot)+I\left(1-\frac{\beta}{2}\right)\Big\| \le q<1,$$

$z \in B$.

Therefore $\|E(z,t)\| \le q < 1$, $z \in B$, $t \in [0,\infty)$.

From now on, for the simplicity of the notations, we will denote by g the function defined by $g(z) = f(z) - \alpha z$, $z \in B$. By taking into account the conditions (3.1) and (3.2) from the hypothesis, we deduce that

$$\begin{aligned}
&(1-\|z\|^{\beta})\|[Dg(z)]^{-1}D^2g(z)(z,\cdot)\| \\
&= (1-\|z\|^{\beta})\|[Df(z)-\alpha I]^{-1}D^2f(z)(z,\cdot)\| \\
&\le q\frac{\beta}{2} + \left\|\|z\|^{\beta}(Df(z)-\alpha I)^{-1}(cDf(z)-\alpha I) + \left(1-\frac{\beta}{2}\right)I\right\| \\
&\le q\frac{\beta}{2}\Big\|\|z\|^{\beta}\left\{(Df(z)-\alpha I)^{-1}(cDf(z)-\alpha I) - \left(\frac{\beta}{2}-1\right)I\right\} \\
&\qquad + \left(1-\frac{\beta}{2}\right)(1-\|z\|^{\beta})I\Big\| \\
&\le q\frac{\beta}{2} + \|z\|^{\beta}\cdot\frac{\beta}{2}\cdot q + \left(\frac{\beta}{2}-1\right)(1-\|z\|^{\beta}) \\
&= \|z\|^{\beta}\left(q\frac{\beta}{2}-\frac{\beta}{2}+1\right) + q\frac{\beta}{2}+\frac{\beta}{2}-1 \\
&\le \max_{x\in[0,1]}\left\{x\left(q\frac{\beta}{2}-\frac{\beta}{2}+1\right)+q\frac{\beta}{2}+\frac{\beta}{2}-1\right\} \\
&= \max\left\{q\frac{\beta}{q}+\frac{\beta}{2}-1, q\beta\right\} = q\beta = 2\gamma
\end{aligned}$$

where $\gamma = \dfrac{q\beta}{2} < 1$.

Since $\beta \ge 2$, we deduce from the above relation that:

$$(1-\|z\|^2)\|[Dg(z)]^{-1}D^2g(z)(z,\cdot)\| \le 2\gamma, \quad z \in U. \tag{3.3}$$

From the previous inequality, by using a similar argument with that used in the proof of Theorem 2.1 [17] we obtain that there exists $M > 0$ such that

$$|\det Dg(z)| \le \frac{M}{(1-\|z\|)^{n\gamma}}, \quad z \in B \tag{3.4}$$

and hence

$$\|Dg(z)\| \le \frac{L}{(1-\|z\|)^{\gamma}} \quad \text{where} \quad L = \sqrt[n]{MK}. \tag{3.5}$$

We prove now that the mappings $L(\cdot,t)$ are quasiregular. Since g is a quasiregular holomorphic mapping and the following inequality holds

$$1-q \le \|I-E(z,t)\| \le 1+q, \quad z \in B,\ t \ge 0,$$

by using (2.7) we easily obtain

$$\|DL(z,t)\| \le \frac{\beta}{2}e^{(\beta-1)t}\frac{1}{|1+c|}\|Dg(ze^{-t})\|\|I-E(z,t)\| \tag{3.6}$$

$$\le |a(t)|\cdot\frac{1+q}{1-q}\cdot\frac{1}{|1-\alpha|}\cdot\frac{L}{(1-\|z\|)^{\gamma}} = \frac{|a(t)|L^*}{(1-\|z\|)^{\gamma}}.$$

On the other hand, we have

$$\|DL(z,t)\|^n \le \left(\frac{\beta}{2}\right)^n e^{n(\beta-1)t}\frac{1}{|1+c|^n}\|Dg(ze^{-t})\|^n(1+q)^n \tag{3.7}$$

$$\le \left(\frac{\beta}{2}\right)^n e^{n(\beta-1)t}\frac{1}{|1+c|^n}|\det Dg(ze^{-t})|(1+q^n)$$

$$\le \left(\frac{1+q}{1-q}\right)^n K|\det DL(z,t)|, \quad z\in B,\ t\ge 0.$$

By using Remark 2.2 from [9] we have that the function $h(z,t)$ satisfies the conditions (iv) and (v) of Theorem 1.3.

Since all the conditions of Theorem 1.3 are satisfied, it results that the function f admits a quasiconformal extension of $\mathbb{R}^{2n}$ onto itself.

Remark 3.2. If $\beta = 2$, $c = 0$ and $\alpha = 0$ in Theorem 3.1, we obtain the n-dimensional version of the quasiconformal extension result due to Becker [17].

If $\beta = 2$ and $\alpha = 0$ in Theorem 3.1 we obtain the n-dimensional version of the quasiconformal extension result due to Ahlfors and Becker [3].

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References

[1] Becker, J., *Löewnersche Differentialgleichung und quasikonform fortsetzbare schlichte Functionen*, J. Reine Angew. Math., **255** (1972), 23-43.

[2] Brodksii, A. A., *Quasiconformal extension of biholomorphic mappings*, In: Theory of Mappings and Approximation of Functions, 3-34, Naukova Durka, Kiew, 1983.

[3] Curt, P., *A generalization in n-dimensional complex space of Ahlfors and Becker's criterion for univalence*, Studia Univ. Babeş-Bolyai, Mathematica, **39** (1994), no. 1, 31-38.

[4] Curt, P., *A univalence criterion for holomorphic mappings in* $\mathbb{C}^n$, Mathematica (Cluj), **37(60)** (1995), no. 1-2, 67-71.

[5] Curt, P., *A sufficient condition for univalence of holomorphic mappings in* $\mathbb{C}^n$, Mem. Sect. Stiinţ. Iasi, seria IV, XIX, 1996, 49-53.

[6] Curt, P., *Special Chapters of Geometric Function Theory of Several Complex Variables*, Editura Albastră, Cluj-Napoca, 2001 (in Romanian).

[7] Curt, P., *Quasiconformal extensions of holomorphic maps in* $\mathbb{C}^n$, Mathematica, **46(69)** (2004), no. 1, 55-60.

[8] Curt, P., Kohr, G., *Subordination chains and Loewner differential equations in several complex variables*, Ann. Univ. M. Curie-Sklodowska, Sect. A, **57**(2003), 35-43.

[9] Curt, P., Kohr, G., *Quasiconformal extensions and q-subordination chains in* $\mathbb{C}^n$, Mathematica (Cluj), **49(72)**(2007), 149-159.

[10] Curt, P., Kohr, G., *The asymptotical case of certain quasiconformal extensions results for holomorphic mappings in* $\mathbb{C}^n$, Bull. Belgian Math. Soc. Simon Stevin, **14**(2007), 653-667.

[11] Curt, P., Kohr, G., *Some remarks concerning quasiconformal extensions in several complex variables*, J. Inequalities Appl., 2008, 16 pages, Article ID690932.

[12] Graham, I., Hamada, H., Kohr, G., *Parametric representation of univalent mappings in several complex variables*, Canadian Journal of Mathematics, **54** (2002), no. 2, 324-351.

[13] Graham, I., Kohr, G., *Geometric Function Theory in One and Higher Dimensions*, Marcel Dekker Inc., New York, 2003.

[14] Hamada, H., Kohr, G., *Loewner chains and quasiconformal extension of holomorphic mappings*, Ann. Polon. Math., **81** (2003), 85-100.

[15] Hamada, H., Kohr, G., *Quasiconformal extension of biholomorphic mappings in several complex variables*, J. Anal. Math., **96**(2005), 269-282.

[16] Pfaltzgraff, J. A., *Subordination chains and univalence of holomorphic mappings in* $\mathbb{C}^n$, Math. Ann., **210** (1974), 55-68.

[17] Pfaltzgraff, J. A., *Subordination chains and quasiconformal extension of holomorphic maps in* $\mathbb{C}^n$, Ann. Acad. Sci. Fenn. Ser. A, Math., **1**(1975), 13-25.

[18] Răducanu, D., *Sufficient conditions for univalence in* $\mathbb{C}^n$, IJMMS, 32:12(2002), 701-706.

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ON S-DISCONNECTED SPACES

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Abstract. The structure of the class of S-disconnected spaces is studied. Two types of S-disconnectedness of topological spaces are introduced. Properties of these spaces in the context of connectedness of spaces are investigated.

1. Introduction

A certain class of non-Hausdorff spaces, called irreducible spaces, was introduced by MacDonald [18]. Pipitone and Russo [27] have defined S-connected spaces. In [34] Thompson proved that these two notions are equivalent. It should be also noticed that Levine has defined the so-called D-spaces [16], which are irreducible spaces, in fact. On the other hand, the notion of hyperconnected spaces, due to Steen and Seebach [32] is equivalent to the notion of D-spaces (Sharma [29]). Some properties of hyperconnected spaces were investigated by Noiri [22].

2. Preliminaries

Throughout the present paper (X, τ) and (Y, σ) denote topological spaces on which no separation axioms are assumed. The closure (resp. interior) in (X, τ) of a subset S of (X, τ) will be denoted by $\mathrm{cl}(S)$ (resp. $\mathrm{int}(S)$). The set S is said to be regular open (resp. regular closed) in (X, τ), if $S = \mathrm{int}(\mathrm{cl}(S))$ (resp. $S = \mathrm{cl}(\mathrm{int}(S))$). A subset S of X is said to be semi-open [15] (resp. α-open [21]) if $S \subset \mathrm{cl}(\mathrm{int}(S))$ (resp. $S \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(S)))$). Levine defined [15] S as semi-open if there exists an open subset G of (X, τ) such that $G \subset S \subset \mathrm{cl}(G)$. The complement of a semi-open set is said to be semi-closed [4]. The semi-closure of a subset S of (X, τ) [4], denoted by $\mathrm{scl}(S)$, is defined as an intersection of all semi-closed sets of (X, τ) containing S. The set $\mathrm{scl}(S)$ is semi-closed. The semi-interior of S in (X, τ) [4], denoted by $\mathrm{sint}(S)$, is defined as a union of all semi-open subsets A of (X, τ) such that $A \subset S$. It is well known that

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$X \setminus \mathrm{sint}(A) = \mathrm{scl}(X \setminus A)$ and $X \setminus \mathrm{scl}(A) = \mathrm{sint}(X \setminus A)$ [4, Theorem 1.6]. The family of all semi-open (resp. semi-closed; α-open; closed; regular open) subsets of (X, τ) we denote by $\mathrm{SO}(X, \tau)$ (resp. $\mathrm{SC}(X, \tau)$; τ^α; $\mathrm{c}(\tau)$; $\mathrm{RO}(X, \tau)$). The family τ^α forms a topology on X, different from τ, in general. The following inclusions hold in each (X, τ): $\tau \subset \tau^\alpha \subset \mathrm{SO}(X, \tau)$. The inclusion $\tau \subset \mathrm{SO}(X, \tau)$ implies $\mathrm{c}(\tau) \subset \mathrm{SC}(X, \tau)$. The reverses of these inclusions are not necessarily true, in general. A topological space (X, τ) is said to be semi-connected (briefly: S-connected), if X is not the union of two disjoint nonempty semi-open subsets of (X, τ). In the opposite case (X, τ) is called semi-disconnected (briefly: S-disconnected). Pipitone and Russo [27, Esempio 3.3, 11, p. 30] showed that connectedness does not imply S-connectedness, in general. A topological space (X, τ) is said to be extremally disconnected (briefly: e.d.), if $\mathrm{cl}(G) \in \tau$ for each $G \in \tau$.

3. p. S-disconnectedness and s.p. S-disconnectedness

In 1983 Janković proved the following characterization of e.d. spaces: *an (X, τ) is e.d. if and only if* $\mathrm{SO}(X, \tau) = \tau^\alpha$ [13, Theorem 2.9(f)]. Later (in 1984), Reilly and Vamanamurthy showed that (X, τ) is disconnected if and only if (X, τ^α) is disconnected [28, Theorem 2]. These two theorems give a motivation to investigate S-disconnectedness of not e.d. spaces from the connectedness point of view. For e.d. spaces we have what follows: *an (X, τ) is disconnected if and only if it is S-disconnected* [12, Theorem 3.2(2)].

Definition 3.1. A not e.d. topological space (X, τ) is called to be ***properly S-disconnected*** (briefly: ***p. S-disc.***), if there exist $A, U \subset X$ such that $A \in \mathrm{SO}(X, \tau) \setminus \tau^\alpha$, $U \in \tau^\alpha$, $U \cup A = X$, and $U \cap A = \emptyset$.

Theorem 3.2. *Let (X, τ) be a topological space. The following are equivalent:*

1. *(X, τ) is p. S-disc.*
2. *There exist $A, U \subset X$ such that $A \in \mathrm{SO}(X, \tau) \setminus \tau^\alpha$, $U \in \mathrm{RO}(X, \tau)$, $U \cup A = X$, and $U \cap A = \emptyset$.*
3. *There exist $A, U \subset X$ such that $A \in \mathrm{SO}(X, \tau) \setminus \tau$, $U \in \mathrm{RO}(X, \tau)$, $U \cup A = X$, and $U \cap A = \emptyset$.*
4. *There exist $A, U \subset X$ such that $A \in \mathrm{SO}(X, \tau) \setminus \tau^\alpha$, $U \in \tau$, $U \cup A = X$, and $U \cap A = \emptyset$.*
5. *There exist $A, U \subset X$ such that $A \in \mathrm{SO}(X, \tau) \setminus \mathrm{RO}(X, \tau)$, $U \in \mathrm{RO}(X, \tau)$, $U \cup A = X$, and $U \cap A = \emptyset$.*
6. *There exist $A, U \subset X$ such that $A \in \mathrm{SO}(X, \tau) \setminus \tau$, $U \in \tau$, $U \cup A = X$, and $U \cap A = \emptyset$.*

Proof. Implications: (2)⇒(3), (4)⇒(1), (3)⇒(5), (3)⇒(6) are obvious.
(1)⇒(2). By hypothesis there exist sets $A \in \mathrm{SO}(X,\tau) \setminus \tau^\alpha$, $U \in \tau^\alpha$ such that $U \cup A = X$ and $U \cap A = \emptyset$. (2) follows from [9, Lemma 2.2], because $U \in \tau^\alpha \cap \mathrm{SC}(X,\tau)$.
(3)⇒(4). Let $A, U \subset X$ be such that $A \in \mathrm{SO}(X,\tau) \setminus \tau$, $U \in \mathrm{RO}(X,\tau)$, $U \cup A = X$, and $U \cap A = \emptyset$. Suppose $A \in \tau^\alpha \setminus \tau$. Hence $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$ and A is regular closed. Therefore $A \in \tau$. A contradiction.
(5)⇒(3). Suppose $A \in \tau \setminus \mathrm{RO}(X,\tau)$. Then $A = \mathrm{int}(A) = \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$, because A is regular closed. Hence A is regular open. A contradiction.
(6)⇒(3). Use [9, Lemma 2.2(2)]. □

Let us remark that in Definition 3.1 and in conditions (2)–(6) of Theorem 3.2 we have $\emptyset \neq U \neq X$.

Example 3.3. (a). Consider $X = \{a,b,c\}$ with the topology

$$\tau = \big\{\emptyset, X, \{a\}, \{b\}, \{a,b\}\big\}.$$

Since $\mathrm{SO}(X,\tau) = \big\{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\big\}$ and $\tau^\alpha = \tau$, then the equality $X = \{a\} \cup \{b,c\}$ implies p. S-disconnectedness of (X,τ).
(b). Take the space of reals $\mathbb{R}$ with the usual topology. Then $\mathbb{R}$ is p. S-disc., since $\mathbb{R} = (-\infty, a] \cup (a, +\infty)$.

Definition 3.4. A not e.d. topological space (X,τ) is called to be ***super-properly S-disconnected*** (briefly: ***s.p. S-disc.***), if there exist $A, B \subset X$ such that $A, B \in \mathrm{SO}(X,\tau) \setminus \tau^\alpha$, $A \cup B = X$, and $A \cap B = \emptyset$.

Example 3.5. Let $X = \{a,b,c,d\}$ and $\tau = \big\{\emptyset, X, \{a\}, \{d\}, \{a,d\}\big\}$. Since $\{a,b\}$, $\{c,d\} \in \mathrm{SO}(X,\tau) \setminus \tau^\alpha$ and $X = \{a,b\} \cup \{c,d\}$, then (X,τ) is s.p. S-disc.

It should be noticed that the space from Example 3.3 is not s.p. S-disc.

The following remark is obvious.

Remark 3.6. A topological space (X,τ) is S-disconnected if and only if (X,τ) is s.p. S-disc. or p. S-disc. or disconnected.

If (X,τ) is p. S-disc. or s.p. S-disc., then there exists $A \in \mathrm{SO}(X,\tau) \setminus \tau$. The reverse implication is not true, in general, as the following example shows.

Example 3.7. Let $X = \{a,b,c\}$, $\tau = \big\{\emptyset, X, \{a\}\big\}$. For this space we have $\mathrm{SO}(X,\tau) \setminus \tau = \big\{\{a,b\}, \{a,c\}\big\}$.

Observe that the spaces in Examples 3.3 and 3.5 are connected.

Remark 3.8. Example 3.7 shows that there exists a connected space, which is not p. S-disc.

Example 3.9. Let $X=\mathbb{R}^2\setminus D$, where $D=\{(x,y):\ x=0\}$. In X consider the subset topology τ of the Euclidean topology of the plane. If $U=\{(x,y)\in X:\ x<0\}$ and $V=\{(x,y)\in X:\ x>0\}$, then it is clear that (X,τ) is not connected. Let now

$$A=\{(x,y)\in X:\ y<0\}\cup\{(x,y)\in X:\ y=0,x\in\mathbb{Q}\},$$
$$B=\{(x,y)\in X:\ y>0\}\cup\{(x,y)\in X:\ y=0,x\in\mathbb{R}\setminus\mathbb{Q}\},$$

where $\mathbb{Q}$ stands for the set of rationals. One easily checks that $A,B\in \mathrm{SO}\,(X,\tau)\setminus\tau^\alpha$. This shows that (X,τ) is s.p. S-disc. Note that if $a<b$ and $ab\neq 0$, then we can put also

$$A=\{(x,y)\in X:\ y<0\}\cup\{(x,y)\in X:\ y=0,\ x=a \text{ or } x>b\},$$
$$B=\{(x,y)\in X:\ y>0\}\cup\{(x,y)\in X:\ y=0,\ x<a \text{ or } a<x\leq b\}.$$

Example 3.10. Let $X=\{a,b,c,d\}$ and

$$\tau=\big\{\emptyset,X,\{a\},\{b,c,d\},\{b\},\{d\},\{b,d\},\{a,b\},\{a,d\},\{a,b,d\}\big\}.$$

For this space we have $\tau=\tau^\alpha$ and $\mathrm{SO}\,(X,\tau)=\tau\cup\big\{\{b,c\},\{c,d\},\{a,b,c\},\{a,c,d\}\big\}$. Partitions $X=\{a\}\cup\{b,c,d\}=\{a,d\}\cup\{b,c\}$ show respectively that (X,τ) is disconnected and p. S-disc. One observes that this space is not s.p. S-disc.

Example 3.11. Let $X=\{a,b,c\}$, $\tau=\big\{\emptyset,X,\{a\},\{b,c\}\big\}$. The space (X,τ) is disconnected and not p. S-disc.

Theorem 3.12. *A topological space (X,τ) is s.p. S-disc. if and only if there exists a set $A\in\mathrm{SO}\,(X,\tau)\setminus\tau^\alpha$ with $\mathrm{scl}\,(A)\in\big(\mathrm{SO}\,(X,\tau)\setminus\tau^\alpha\big)\cap\big(\mathrm{SC}\,(X,\tau)\setminus \mathrm{c}\,(\tau^\alpha)\big)$.*

Proof. Necessity. Let (X,τ) be s.p. S-disc., i.e., for certain $A,B\in\mathrm{SO}\,(X,\tau)\setminus\tau^\alpha$ we have $X=A\cup B$ and $A\cap B=\emptyset$. Clearly $A,B\in\mathrm{SC}\,(X,\tau)\setminus\mathrm{c}\,(\tau^\alpha)$. Thus for A we obtain $\mathrm{scl}\,(A)=A\in\big(\mathrm{SO}\,(X,\tau)\setminus\tau^\alpha\big)\cap\big(\mathrm{SC}\,(X,\tau)\setminus\mathrm{c}\,(\tau^\alpha)\big)$ (analogously for B).
Sufficiency. Let (X,τ) be such a space that for a certain $U\in\mathrm{SO}\,(X,\tau)\setminus\tau^\alpha$ we have $\mathrm{scl}\,(U)\in\big(\mathrm{SO}\,(X,\tau)\setminus\tau^\alpha\big)\cap\big(\mathrm{SC}\,(X,\tau)\setminus\mathrm{c}\,(\tau^\alpha)\big)$. Put $A=\mathrm{scl}\,(U)$. So, for $B=X\setminus\mathrm{scl}\,(U)$ we infer without difficulties that $B\in\mathrm{SO}\,(X,\tau)\setminus\tau^\alpha$. Therefore (X,τ) is s.p. S-disc. and the proof is complete. □

Lemma 3.13. *Assume that for a (X,τ) the two conditions below hold.*

($\star$) *There exist disjoint subsets $A\in\mathrm{SO}\,(X,\tau)\setminus\tau^\alpha$, $B\in\mathrm{SO}\,(X,\tau)\setminus\{\emptyset\}$ with $X=A\cup B$.*

($\star\star$) *There exists a point $x\in\big(A\setminus\mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(A)))\big)\setminus(\mathrm{cl}\,(B)\setminus B)$, where $\mathrm{cl}\,(B)\neq X$.*

Then (X,τ) is disconnected.

Proof. Suppose (X,τ) is connected. We have

$$X = \operatorname{int}\big(\operatorname{cl}(\operatorname{int}(A)) \cup \operatorname{cl}(\operatorname{int}(B))\big) \subset \operatorname{int}(\operatorname{cl}(\operatorname{int}(A))) \cup \operatorname{cl}(\operatorname{int}(B)) \subset X$$

(see [1, Lemma 1.1]) and $\operatorname{int}(A) \neq \emptyset \neq \operatorname{int}(B)$. Thus, $X = \operatorname{int}(\operatorname{cl}(\operatorname{int}(A))) \cup \operatorname{cl}(\operatorname{int}(B))$. One easily checks that

$$\operatorname{int}(\operatorname{cl}(\operatorname{int}(A))) \cap \operatorname{cl}(\operatorname{int}(B)) = \emptyset \tag{3.1}$$

and similarly

$$\operatorname{int}(\operatorname{cl}(\operatorname{int}(B))) \cap \operatorname{cl}(\operatorname{int}(A)) = \emptyset. \tag{3.2}$$

Since $\operatorname{int}(\operatorname{cl}(\operatorname{int}(A))) \cap \operatorname{int}(\operatorname{cl}(\operatorname{int}(B))) = \emptyset$, $\operatorname{int}(\operatorname{cl}(\operatorname{int}(A))) \neq \emptyset \neq \operatorname{int}(\operatorname{cl}(\operatorname{int}(B)))$, we infer from the supposition that $X \setminus \big(\operatorname{int}(\operatorname{cl}(\operatorname{int}(A))) \cup \operatorname{int}(\operatorname{cl}(\operatorname{int}(B)))\big) \neq \emptyset$. So, we obtain $X = \operatorname{int}(\operatorname{cl}(\operatorname{int}(A))) \cup \operatorname{int}(\operatorname{cl}(\operatorname{int}(B))) \cup (\operatorname{cl}(B) \cap \operatorname{cl}(A))$, because $\operatorname{cl}(\operatorname{int}(\operatorname{cl}(S))) = \operatorname{cl}(S)$ for any semi-open subset of every topological space. Let $\operatorname{cl}(A) \neq X$ (the case $\operatorname{cl}(A) = X$ we leave to the reader). It is easy to see that we have $\operatorname{cl}(A) \setminus A = X \setminus (A \cup \operatorname{int}(B))$, $\operatorname{cl}(B) \setminus B = X \setminus (B \cup \operatorname{int}(A))$, and consequently $(\operatorname{cl}(A) \setminus A) \cap (\operatorname{cl}(B) \setminus B) = \emptyset$. So, we get what follows: $X = \operatorname{int}(\operatorname{cl}(\operatorname{int}(A))) \cup \operatorname{int}(\operatorname{cl}(\operatorname{int}(B))) \cup \big((A \cup (\operatorname{cl}(A) \setminus A)) \cap (B \cup (\operatorname{cl}(B) \setminus B))\big) = \operatorname{int}(\operatorname{cl}(\operatorname{int}(A))) \cup \operatorname{int}(\operatorname{cl}(\operatorname{int}(B))) \cup \big(A \cap (\operatorname{cl}(B) \setminus B)\big) \cup \big(B \cap (\operatorname{cl}(A) \setminus A)\big)$. Let x be a point fulfilling the condition $(\star\star)$. We shall show that $x \notin \operatorname{int}(\operatorname{cl}(\operatorname{int}(B)))$. Suppose not. By (3.2) we get $\operatorname{int}(\operatorname{cl}(\operatorname{int}(B))) \cap \operatorname{int}(A) = \emptyset$; hence $x \notin \operatorname{cl}(\operatorname{int}(A))$ what contradicts $x \in A \in \mathrm{SO}(X,\tau)$. Therefore $x \in A \cap (\operatorname{cl}(B) \setminus B)$. But, $x \notin \operatorname{cl}(B) \setminus B$ by $(\star\star)$. This shows that (X,τ) is disconnected. □

Theorem 3.14. *Each s.p. S-disc. space fulfilling the condition $(\star\star)$ is disconnected.*

Proof. It follows directly from Definition 3.4 and Lemma 3.13. □

Here, from the connectedness and e.d. points of view, the following is worth noticing.

Example 3.15. A space (X,τ) may be disconnected and not e.d. Consider $X = \{a,b,c,d\}$ and $\tau = \big\{\emptyset, X, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,d\}, \{b,d\}, \{b,c,d\}, \{a,b,d\}\big\}$. We have $X = \{a\} \cup \{b,c,d\}$ and $\operatorname{cl}(\{a,b\}) = \{a,b,c\} \notin \tau$.

Example 3.3(b) guarantees the existence of a not e.d. space which is connected.

Example 3.16. (a). Let $X = \{a,b,c\}$ and $\tau = \big\{\emptyset, X, \{a\}, \{a,b\}\big\}$. This space is e.d. and connected. See also Example 3.7.
(b). The space from Example 3.11 is e.d. and disconnected.

4. **Some properties**

Lemma 4.1. *Let* (X,τ) *be any space. If* $S \in \mathrm{SO}(X,\tau) \setminus \tau^{\alpha}$ *then* $\mathrm{cl}(\mathrm{int}(S)) \in \mathrm{SO}(X,\tau) \setminus \tau^{\alpha}$.

Proof. It is clear that $\mathrm{cl}(\mathrm{int}(S)) \in \mathrm{SO}(X,\tau)$. If we suppose $\mathrm{cl}(\mathrm{int}(S)) \in \tau^{\alpha}$, then $S \subset \mathrm{cl}(\mathrm{int}(S)) \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathrm{int}(S))))) = \mathrm{int}(\mathrm{cl}(\mathrm{int}(S)))$. A contradiction. □

Theorem 4.2. *If* (X,τ) *is s.p. S-disc., then* (X,τ) *is p. S-disc.*

Proof. Assume that (X,τ) is s.p. S-disc. Then, for certain $A, B \in \mathrm{SO}(X,\tau) \setminus \tau^{\alpha} \subset \mathrm{SO}(X,\tau) \setminus \tau$ we have $X = A \cup B$ and $A \cap B = \emptyset$. Clearly $A \cup \mathrm{cl}(\mathrm{int}(B)) = X$ and hence $\mathrm{int}(A) \cup \mathrm{cl}(\mathrm{int}(B)) \subset X$. But, with [1, Lemma 1.1(b)] we obtain $X = \mathrm{int}\big(A \cup \mathrm{cl}(\mathrm{int}(B))\big) \subset \mathrm{int}(A) \cup \mathrm{cl}(\mathrm{int}(B))$. So, consequently

$$X = \mathrm{int}(A) \cup \mathrm{cl}(\mathrm{int}(B)).$$

It is easy to check that $\mathrm{int}(A) \cap \mathrm{cl}(\mathrm{int}(B)) = \emptyset$. Observe that $\mathrm{int}(A) \neq \emptyset$ and $\mathrm{cl}(\mathrm{int}(B))$ is a nonempty semi-open subset of (X,τ), which is not open (by Lemma 4.1). Thus, by Theorem 3.2(6), (X,τ) is p. S-disc. □

Theorem 4.2 implies the following obvious corollary.

Corollary 4.3. *If* (X,τ) *is s.p. S-disc., then there exists an* $A \subset X$ *such that* $A \in \mathrm{c}(\tau) \cap \big(\mathrm{SO}(X,\tau) \setminus \tau\big)$.

Theorem 4.4. *A connected topological space* (X,τ) *is p. S-disc. if and only if there exists* $A \in \mathrm{SO}(X,\tau) \setminus \tau$ *with* $\mathrm{cl}(A) \notin \tau$.

Proof. We apply Theorem 3.2(6). Necessity is obvious. For a strong sufficiency, i.e., with *any* (X,τ), suppose that $A \in \mathrm{SO}(X,\tau) \setminus \tau$ and $\mathrm{cl}(A) \notin \tau$. Then, since $\mathrm{cl}(A) \in \mathrm{SO}(X,\tau)$, from $X = (X \setminus \mathrm{cl}(A)) \cup \mathrm{cl}(A)$ it follows that (X,τ) is p. S-disc. □

Remark 4.5. If a space (X,τ) is not e.d. then there exists an $A \in \mathrm{SO}(X,\tau) \setminus \tau^{\alpha}$ with $\mathrm{scl}(A) \notin \tau$.

Proof. Suppose for each $A \in \mathrm{SO}(X,\tau) \setminus \tau^{\alpha}$, $\mathrm{scl}(A) \in \tau$. Since (X,τ) is not e.d., there is an $A' \in \tau^{\alpha}$ such that $\mathrm{scl}(A') \notin \tau$ [31, Theorem 2.1(iii)]. But with [14, Proposition 2.7(a)] we have $\mathrm{scl}(A') = \mathrm{int}(\mathrm{cl}(A'))$. A contradiction. □

Corollary 4.6. *If a space* (X,τ) *is connected and not p. S-disc., then for each* $A \in \mathrm{SO}(X,\tau) \setminus \tau$ *we have* $\mathrm{cl}(A) = X$.

Proof. By Theorem 4.4 we get that either $X = \mathrm{cl}(A)$ or $X \neq \mathrm{cl}(A) \in \tau$, but obviously the second relation is not possible. □

Theorem 4.7. *Let* (X,τ) *be a connected topological space. Then, the following are equivalent:*

(a) (X,τ) *is s.p. S-disc. or p. S-disc.*

(b) *There exists an* $A \in \mathrm{SO}(X,\tau) \setminus \tau$ *with* $\mathrm{scl}(A) \neq X$.

Proof. Strong (a)⇒(b). Let (X,τ) be p. S-disc. Then $X = U \cup A$ for such sets $U \in \tau \setminus \{X, \emptyset\}$, $A \in \mathrm{SO}(X,\tau) \setminus \tau$ that $U \cap A = \emptyset$. Consider the set $\mathrm{scl}(A)$. Since A is closed, then $\mathrm{scl}(A) = A \neq X$.

(b)⇒(a). Assume that for a certain $A' \in \mathrm{SO}(X,\tau) \setminus \tau$ we have $\mathrm{scl}(A') \neq X$. Put $A = \mathrm{scl}(A')$. Hence $\emptyset \neq B = X \setminus A \neq X$ and by [33, Corollary 2.2] we have $A, B \in \mathrm{SO}(X,\tau)$. The sets A and B cannot be both α-open in (X,τ), since (X,τ) is connected by hypothesis. Thus our space is s.p. S-disc. or p. S-disc. □

Lemma 4.8. *If a connected space* (X,τ) *is p. S-disc., then there exist sets* $U, V \in \mathrm{RO}(X,\tau) \setminus \{\emptyset\}$ *such that* $X = \mathrm{cl}(U) \cup V$, $\mathrm{cl}(U) \cap V = \emptyset$ *and* $\mathrm{cl}(U) \cap \mathrm{cl}(V) \neq \emptyset$.

Proof. Let (X,τ) be p. S-disc. and connected. By Theorem 3.2(5) there exist sets $A \in \mathrm{SO}(X,\tau) \setminus \mathrm{RO}(X,\tau)$, $V \in \mathrm{RO}(X,\tau)$ such that $X = A \cup V$ and $A \cap V = \emptyset$ (obviously $V \neq \emptyset$). Then $A \in \big(\mathrm{SO}(X,\tau) \cap \mathrm{SC}(X,\tau)\big) \setminus \{\emptyset, X\}$ and by [6, Proposition 2.1(c)] there exists a set $U \in \mathrm{RO}(X,\tau) \setminus \{\emptyset\}$ such that $U \subset A \subset \mathrm{cl}(U)$. Hence $A = \mathrm{cl}(A) = \mathrm{cl}(U)$ and $\mathrm{cl}(U) \cap V = \emptyset$. Observe that if $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$, then (X,τ) is disconnected and this contradicts connectedness of (X,τ). Therefore, $\mathrm{cl}(U) \cap \mathrm{cl}(V) \neq \emptyset$. □

By the proof of Lemma 4.8 it can be easily deduced what follows.

Theorem 4.9. *If a connected space* (X,τ) *is p. S-disc., then there exists an open but not regular open, disconnected subset of* (X,τ).

Proof. Our consideration relies on the proof of Lemma 4.8 (including the notation). We shall show only that the set $U \cup V$ is not regular open. Suppose that $U \cup V \in \mathrm{RO}(X,\tau)$. Hence $\mathrm{int}\big(\mathrm{cl}(U) \cup \mathrm{cl}(V)\big) = U \cup V \subsetneq X$. But, $\mathrm{int}\big(\mathrm{cl}(U) \cup \mathrm{cl}(V)\big) = X$, a contradiction. □

Corollary 4.10. *If* (X,τ) *is S-disconnected and connected, then there exists an open disconnected subset of* (X,τ).

Proof. See Remark 3.6 and Theorem 4.9. □

Lemma 4.11. *If a space* (X,τ) *is connected and if there exist sets* $U, V \in \mathrm{RO}(X,\tau) \setminus \{\emptyset\}$ *such that* $X = \mathrm{cl}(U) \cup V$ *and* $\mathrm{cl}(U) \cap V = \emptyset$, *then* (X,τ) *is p. S-disc.*

Proof. The set $\mathrm{cl}(U) \in \mathrm{SO}(X,\tau) \setminus \tau$, because (X,τ) is connected. So, by Theorem 3.2(3), (X,τ) is p. S-disc. □

Theorem 4.12. *Let a space* (X,τ) *be connected. Then the following are equivalent:*

1. (X,τ) *is p. S-disc.*
2. *There exist* $U, V \in \mathrm{RO}(X,\tau) \setminus \{\emptyset\}$ *such that* $X = \mathrm{cl}(U) \cup V$, $\mathrm{cl}(U) \cap V = \emptyset$.
3. *There exist* $U, V \in \tau \setminus \{\emptyset\}$ *such that* $X = \mathrm{cl}(U) \cup V$, $\mathrm{cl}(U) \cap V = \emptyset$.

4. *There exist* $U, V \in \tau^\alpha \setminus \{\emptyset\}$ *such that* $X = \mathrm{cl}\,(U) \cup V$, $\mathrm{cl}\,(U) \cap V = \emptyset$.
5. *There exist* $U, V \in \mathrm{RO}\,(X, \tau^\alpha) \setminus \{\emptyset\}$ *such that* $X = \alpha\text{-}\mathrm{cl}\,(U) \cup V$, $\alpha\text{-}\mathrm{cl}\,(U) \cap V = \emptyset$, *where* $\alpha\text{-}\mathrm{cl}\,(.)$ *denotes the closure operator with respect to* τ^α*-topology on* X.
6. *There exist* $U, V \in \tau^\alpha \setminus \{\emptyset\}$ *such that* $X = \alpha\text{-}\mathrm{cl}\,(U) \cup V$, $\alpha\text{-}\mathrm{cl}\,(U) \cap V = \emptyset$.

Proof. (1)⇔(2). Follows by Lemmas 4.8 and 4.11.
(2)⇒(3). Obvious.
(2)⇐(3). It can be easily seen that (3)⇒(1): by Theorem 3.2(6) and connectedness of (X, τ).
(3)⇒(4). Obvious.
(3)⇐(4). We shall show only that (4)⇒(1). By hypothesis we have $U \subset \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(U)))$ and $U \neq \emptyset$. Hence $\mathrm{cl}\,(U) \in \mathrm{SO}\,(X, \tau)$ and $\mathrm{cl}\,(U) \neq \emptyset$. Also, $\mathrm{cl}\,(U) \notin \tau^\alpha$ up to connectedness of (X, τ) [28, Theorem 2]. Therefore (X, τ) is p. S-disc.
(5)⇔(2) and (6)⇔(4) follow by the proof of [14, Corollary 2.3] and [14, Proposition 2.2]. □

Remark 4.13. In Theorem 3.2, the class $\mathrm{SO}\,(X, \tau)$ can be replaced also by $\mathrm{SO}\,(X, \tau^\alpha)$ [21, Proposition 3] and the class $\mathrm{RO}\,(X, \tau)$ by $\mathrm{RO}\,(X, \tau^\alpha)$.

Theorem 4.14. *Let* (X, τ) *be a connected space. The following are equivalent:*

1. (X, τ) *is p.* S*-disc.*
2. *There exists a set* $B \in \mathrm{SC}\,(X, \tau)$ *such that* $B \neq X$ *and* $\mathrm{int}\,(B) \neq \emptyset$.
3. *There exists a set* $B \in \mathrm{SC}\,(X, \tau)$ *such that* $B \neq X$ *and* $\mathrm{sint}\,(B) \neq \emptyset$.

Proof. (1)⇒(2). Let (X, τ) be p. S-disc. By hypothesis the space (X, τ) is connected. On the other hand, from Theorem 3.2(5) we infer that there exists a set $B \in \mathrm{RO}\,(X, \tau) \subset \mathrm{SC}\,(X, \tau)$ with $B \neq X$ and $\mathrm{int}\,(B) \neq \emptyset$.
(2)⇒(3). Obvious.
(3)⇒(1). Suppose there exists a set $B \in \mathrm{SC}\,(X, \tau)$ with $B \neq X$ and $\mathrm{sint}\,(B) \neq \emptyset$. From [4, Theorems 1.4(2) and 1.12] we get that B is semi-closed if and only if $\mathrm{sint}\,(\mathrm{scl}\,(B)) \subset B$. Hence $\emptyset \neq \mathrm{sint}\,(\mathrm{scl}\,(B)) \neq X$. By [33, Lemma 2.7], $\mathrm{sint}\,(\mathrm{scl}\,(B)) \in \mathrm{SO}\,(X, \tau) \cap \mathrm{SC}\,(X, \tau)$. Put $U = \mathrm{int}\,(\mathrm{sint}\,(\mathrm{scl}\,(B)))$. Clearly $U \neq \emptyset$ and $U \neq X$. We have $X \setminus U = \mathrm{cl}\,(\mathrm{scl}\,(\mathrm{sint}\,(X \setminus B)))$. and $A = X \setminus U \in \mathrm{SO}\,(X, \tau)$, since by [33, Lemma 2.2(iii)], the set $\mathrm{scl}\,(\mathrm{sint}\,(X \setminus B))$ belongs to $\mathrm{SO}\,(X, \tau)$. Also $\emptyset \neq A \neq X$. The set A cannot be a member of τ, because (X, τ) is connected. So, by Theorem 3.2(6) the space (X, τ) is p. S-disc. □

5. Mappings and p. S-disconnectedness

A function $f\colon (X,\tau) \to (Y,\sigma)$ is called *contra-continuous* [8] if the preimage $f^{-1}(V) \in \mathrm{c}\,(\tau)$ for each $V \in \sigma$.

Remark 5.1. (**a**). From [9, Theorem 5.1] and Example 3.3 we infer that there exists a subclass of not e.d. spaces (X,τ) such that any contra-continuous mapping $f\colon (X,\tau) \to (Y,\sigma)$, where (Y,σ) is T_1, is constant.
(**b**). Also with [9, Theorem 5.1] we get that if a bijection $f\colon (X,\tau) \to (\mathbb{R},\tau_e)$, τ_e the usual topology, is open and contra-continuous then (X,τ) is not p. S-disc. Therefore, there is no open and contra-continuous bijection $f\colon (\mathbb{R},\tau_e) \to (\mathbb{R},\tau_e)$ (compare Example 3.3(b)).

A metric space X is connected if and only if each continuous mapping f : $X \to \mathbb{R}$ is Darboux. This implies

Remark 5.2. From Example 3.9 we infer that there exist an s.p. S-disc. metric space X and a continuous mapping $f\colon X \to \mathbb{R}$ which is not Darboux.

A function $f\colon (X,\tau) \to (Y,\sigma)$ is called *almost continuous* (in the sense S& S) [30, Theorem 2.2] (resp. *α-continuous* [20]; *irresolute* [5]) if the preimage $f^{-1}(V) \in \tau$ (resp. $f^{-1}(V) \in \tau^{\alpha}$; $f^{-1}(V) \in \mathrm{SO}\,(X,\tau)$) for every $V \in \mathrm{RO}\,(Y,\sigma)$ (resp. $V \in \sigma$; $V \in \mathrm{SO}\,(Y,\sigma)$). α-continuous mappings are called *strongly semi-continuous* in [24]. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called *pre-semi-open* [5] if $f(A) \in \mathrm{SO}\,(Y,\sigma)$ for each $A \in \mathrm{SO}\,(X,\tau)$. A bijection $f\colon (X,\tau) \to (Y,\sigma)$ is said to be a *semi-homeomorphism* (in the sense of Crossley and Hildebrand) [5], if it is pre-semi-open and irresolute. It is well known that connected spaces are preserved under semi-homeomorphims [5, Theorem 2.12] or almost continuous surjections [17, Theorem 4] or α-continuous surjections [24, Theorem 3.1]. Thus, the following is clear.

Remark 5.3. Let (X,τ) be p. S-disc. and connected, and let $f : (X,\tau) \to (Y,\sigma)$ be a semi-homeomorphism or an almost continuous surjection, or α-continuous surjection. Then (Y,σ) is connected.

For the case of semi-homeomorphism we shall show a stronger result in the sequel.

Theorem 5.4. *Let $f : (X,\tau) \to (Y,\sigma)$ be a continuous surjection and (Y,σ) be a p. S-disc. connected space. Then, there is a proper subset of X which is open and disconnected (in (X,τ)).*

Proof. From Theorem 4.9 we infer that there exists an open and disconnected proper subset S of (Y,σ). So, $f^{-1}(S)$ is an open and disconnected proper subset of (X,τ). □

Corollary 5.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous surjection and (Y, σ) be a connected and S-disconnected space. Then, there is an open disconnected subset of (X, τ).*

Proof. Remark 3.6 and Theorem 5.4. □

Theorem 5.6. *Let $f : (X, \tau) \to (Y, \sigma)$ be a homeomorphism and (Y, σ) be p. S-disc. and connected. Then $X = A \cup B$, where $A \cap B = \emptyset$, A is an open disconnected subset of (X, τ), and $B \in \mathrm{c}(\tau) \setminus \tau$.*

Proof. We apply Theorems 5.4 and an obvious fact that there is no open bijection $f : (X, \tau) \to (Y, \sigma)$, where (X, τ) is disconnected and (Y, σ) is connected. □

Theorem 3.2 is followed by the series of results given below, concerning preimages and images of p. S-disc. spaces under some well known types of functions. Straightforward proofs are omitted.

Theorem 5.7. *Let (X, τ) be connected, (Y, σ) be s.p. S-disc., and let $f : (X, \tau) \to (Y, \sigma)$ be an irresolute surjection. Then (X, τ) is p. S-disc.*

A function $f\colon (X, \tau) \to (Y, \sigma)$ is called *completely continuous* [2] (resp. *an R-map* [3]; *α-irresolute* [19]) if the preimage $f^{-1}(V) \in \mathrm{RO}\,(X, \tau)$ (resp. $f^{-1}(V) \in \mathrm{RO}\,(X, \tau)$; $f^{-1}(V) \in \tau^{\alpha}$) for every $V \in \sigma$ (resp. $V \in \mathrm{RO}\,(Y, \sigma)$; $V \in \sigma^{\alpha}$).

Theorem 5.8. *Let (X, τ) be not e.d. and connected, (Y, σ) be p. S-disc., and let a surjection $f : (X, \tau) \to (Y, \sigma)$ fulfil one of the following conditions:*

1. *f is irresolute and almost continuous;*
2. *f is irresolute and it is an R-map;*
3. *f is irresolute and α-continuous.*

Then (X, τ) is p. S-disc.

Remark 5.9. If (X, τ) is e.d. and connected, if (Y, σ) is p. S-disc., then it is clear by [13, Theorem 2.9(b)] and [11, Lemma 1(i)] (for the case (3)) that there is no surjection $f : (X, \tau) \to (Y, \sigma)$ fulfilling (1) or (2) or (3) of Theorem 5.8.

Obviously, (2) is a particular case of (1) in Theorem 5.8. Since each continuous function is almost continuous, each completely continuous function is an R-map and each α-irresolute function is α-continuous, therefore the next corollary is obvious. None of these three implications is reversible, see respectively: [30, Example 2.1], [26, Example 4.6], and [19, Example 1].

Corollary 5.10. *Let (X, τ) be not e.d. and connected, (Y, σ) be p. S-disc., and a surjection $f : (X, \tau) \to (Y, \sigma)$ fulfils one of the following conditions:*

(1') *f is irresolute and continuous;*
(2') *f is irresolute and completely continuous;*

(3') *f is irresolute and α-irresolute.*

Then (X, τ) is p. S-disc.

Remark 5.11. (a). [7, Example 7.1] shows that there exists an irresolute mapping, which is not almost continuous and hence: not an R-map, not continuous, and not completely continuous.
(b). [7, Example 7.2] guarantees the existence of not irresolute mapping, which is continuous (hence almost continuous).
(c). Notions of irresolutness and α-continuity are independent of each other, see [25, Example 3.11 and Theorem 3.12]. In [10] the author has shown that concepts of irresolutness and α-irresolutness are independent of each other.

Example 5.12. Let $X = \{a, b\} = Y$, $\tau = \{\emptyset, X, \{a\}\}$, and $\sigma = \{\emptyset, Y, \{b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. then f is an R-map, but it is not irresolute.

Example 5.13. Let $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, X, \{b\}, \{a, c\}\}$, and $\sigma = \{\emptyset, Y, \{b\}\}$. Then, the identity function $f : (X, \tau) \to (Y, \sigma)$ is completely continuous and not irresolute.

The result from Theorem 5.8 for the case (2) may be strengthened (see Theorem 5.20 below).

A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be *almost open* [30] (resp. *R-open*; *α-open* [20]) if the image $f(U) \in \sigma$ (resp. $f(U) \in \mathrm{RO}\,(Y, \sigma)$; $f(U) \in \sigma^{\alpha}$) for every $U \in \mathrm{RO}\,(X, \tau)$ (resp. $U \in \mathrm{RO}\,(X, \tau)$; $U \in \tau$).

Theorem 5.14. *Let (X, τ) be p. S-disc., (Y, σ) be not e.d. and connected, and let a bijection $f : (X, \tau) \to (Y, \sigma)$ fulfil one of the following conditions:*

(a) *f is pre-semi-open and almost open;*
(b) *f is pre-semi-open and R-open;*
(c) *f is pre-semi-open and α-open;*

Then (Y, σ) is p. S-disc.

Proof. Apply respective parts of Theorem 3.2 (obviously: (b)⇒(a)). □

Remark 5.15. By the same reasoning as mentioned in Remark 5.9, there is no bijection between a p. S-disc. space (X, τ) and an e.d. connected space (Y, σ) fulfilling (a) or (b) or (c) of Theorem 5.14.

Remark 5.16. (a). [23, Example 1.8] shows that there exists an almost open function (in fact, R-open), which is not pre-semi-open.

(b). Let $X=\{a,b,c\}=Y$, $\tau=\{\emptyset,X\}$, and $\sigma=\{\emptyset,Y,\{a\},\{b\},\{a,b\}\}$. The mapping $f:(X,\tau)\to(Y,\sigma)$ defined as follows: $f(a)=a$, $f(b)=f(c)=b$, is almost open, but it is not R-open.

Example 5.17. Let $X=\{a,b,c\}=Y$, $\tau=\{\emptyset,X,\{b\},\{a,c\}\}$, and $\sigma=\{\emptyset,Y,\{b\}\}$. Define $f:(X,\tau)\to(Y,\sigma)$ as in Remark 5.16(b). Then, f is pre-semi-open and not almost open (hence not R-open).

Example 5.18. Let $X=\{a,b,c\}=Y$, $\tau=\{\emptyset,X,\{b\},\{b,c\}\}$, and $\sigma=\{\emptyset,Y,\{a\},\{b\},\{a,b\}\}$. The identity function $f:(X,\tau)\to(Y,\sigma)$ is pre-semi-open and not α-open.

Example 5.19. Let $X=\{a,b,c\}=Y$, $\tau=\{\emptyset,X,\{a\}\}$, and $\sigma=\{\emptyset,Y,\{a\},\{b,c\}\}$. We define a mapping $f:(X,\tau)\to(Y,\sigma)$ as follows $f(a)=f(b)=a$, $f(c)=b$. Then, f is α-open and not pre-semi-open.

Theorem 5.20. *Let (X,τ) be connected, (Y,σ) be p. S-disc. and connected, and a surjection $f:(X,\tau)\to(Y,\sigma)$ be an R-map. Then (X,τ) is p. S-disc.*

Proof. By Theorem 4.12(2) there exist $U_1,V_1\in\mathrm{RO}\,(Y,\sigma)\setminus\{\emptyset\}$ such that $Y=\mathrm{cl}_Y(U_1)\cup V_1$ and $\mathrm{cl}_Y(U_1)\cap V_1=\emptyset$. Clearly $\mathrm{cl}_Y(U_1)$ is regular closed in (Y,σ). It is obvious that the set $f^{-1}\big(\mathrm{cl}_Y(U_1)\big)$ is regular closed in (X,τ). So, we have $X=f^{-1}(Y)=\mathrm{cl}_X\big(\mathrm{int}_X\big(f^{-1}\big(\mathrm{cl}_Y(U_1)\big)\big)\big)\cup f^{-1}(V_1)$, where $U=\mathrm{int}_X\big(f^{-1}\big(\mathrm{cl}_Y(U_1)\big)\big)\in\tau\setminus\{\emptyset\}$, $V=f^{-1}(V_1)\in\tau\setminus\{\emptyset\}$, and $\mathrm{cl}_X(U)\cap V=\emptyset$. This proves that (X,τ) is p. S-disc., since, by hypothesis, it is connected (Theorem 4.12(3)). □

Theorem 5.21. *Let (X,τ) be a connected p. S-disc. space and $f:(X,\tau)\to(Y,\sigma)$ be a semi-homeomorphism. Then (Y,σ) is connected and p. S-disc.*

Proof. Since (X,τ) is connected and p. S-disc., by [5, Theorem 2.12] and Theorem 4.14(3) respectively, (X,τ) is connected and there exists a set $B\in\mathrm{SC}\,(X,\tau)$ with $B\neq X$ and $\mathrm{sint}_X(B)\neq\emptyset$. By [5, Theorem 2.12] the space (Y,σ) is connected. Obviously, $f(B)\neq Y$. Recall that for every semi-homeomorphism $f:X\to Y$ and any $B\subset X$ we have $f\big(\mathrm{sint}_X(B)\big)=\mathrm{sint}_Y(f(B))$ [5, Corollary 1.2]. So, $\mathrm{sint}_Y(f(B))\neq\emptyset$. It is not difficult to see that each bijective pre-semi-open map preserves semi-closed sets. Therefore $f(B)\in\mathrm{SC}\,(Y,\sigma)$ and applying once more Theorem 4.14(3) we finish the proof. □

Corollary 5.22. *Let $f:(X,\tau)\to(Y,\sigma)$ be a homeomorphism, (X,τ) be connected and p. S-disc. Then, (Y,σ) is connected and p. S-disc.*

Proof. [5, Theorem 1.9] and Theorem 5.21. □

References

[1] Andrijević, D., *Semi-preopen sets*, Mat. Vesnik, **38** (1986), 24–32.

[2] Arya, S. P., Gupta, R., *On strongly continuous mappings*, Kyungpook Math. J., **14** (1974), 131–143.

[3] Carnahan, D. A., *Some properties related to compactness in topological spaces*, Ph.D. thesis, University of Arkansas, 1973.

[4] Crossley, C. G., Hildebrand, S. K., *Semi-closure*, Texas J. Sc., **22** (1971), no. 2-3, 99–112.

[5] Crossley, C. G., Hildebrand, S. K., *Semi-topological properties*, Fund. Math., **74** (1972), 233–254.

[6] Di Maio, G., Noiri, T., *On s-closed spaces*, Indian J. Pure Appl. Math., **18** (1987), no. 3, 226–233.

[7] Di Maio, G., Noiri, T., *Weak and strong forms of irresolute functions*, Third National Conference in Topology (Italian), Trieste 1986, Rend. Circ. Mat. Palermo, Suppl., **18** (1988), no. 2, 255-273.

[8] Dontchev, J., *Contra-continuous functions and strongly S-closed spaces*, Internat. J. Math. Math. Sci., **19** (1996), no. 2, 303–310.

[9] Dontchev, J., Noiri, T., *Contra-semicontinuous functions*, Math. Pannonica, **10** (1999), no. 2, 159–168.

[10] Duszyński, Z., *On pre-semi-open mappings*, submitted to Periodica Math. Hungar.

[11] Garg, G. L., Sivaraj, D., *Semitopological properties*, Mat. Vesnik, **36** (1984), 137–142.

[12] Jafari, S., Noiri, T., *Properties of β-connected spaces*, Acta Math. Hungar., **101** (2003), no. 3, 227–236.

[13] Janković, D. S., *On locally irreducible spaces*, Ann. Soc. Sci. Bruxelles, **97** (1983), no. 2, 59–72.

[14] Janković, D. S., *A note on mappings of extremally disconnected spaces*, Acta Math. Hung., **46** (1985), no. 1-2, 83–92.

[15] Levine, N., *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70** (1963), 36–41.

[16] Levine, N., *Dense topologies*, Amer. Math. Monthly, **75** (1968), 847–852.

[17] Long, P. E., Carnahan, D. A., *Comparing almost continuous functions*, Proc. Amer. Math. Soc., **38** (1973), 413–418.

[18] MacDonald, I. G., *Algebraic geometry*, W. A. Benjamin Inc., New York 1968, 13–22.

[19] Maheshwari, S. N., Thakur, S. S., *On α-irresolute mappings*, Tamkang J. Math., **11** (1980), 205–214.

[20] Mashhour, A. S., Hasanein, I. A., El-Deeb, S. N., *α-continuous and α-open mappings*, Acta Math. Hungar., **41** (1983), 213–218.

[21] Njåstad, O., *On some classes of nearly open sets*, Pacific J. Math., **15** (1965), 961–970.

[22] Noiri, T., *A note on hyperconnected sets*, Mat. Vesnik, **3(16)(31)** (1979), no. 1, 53–60.

[23] Noiri, T., *Semi-continuity and weak-continuity*, Czechoslovak Mathematical Journal, **31**(**106**) (1981), 314–321.

[24] Noiri, T., *A function which preserves connected spaces*, Čas. Pěst. Mat., **107** (1982), 393–396.

[25] Noiri, T., *On α-continuous functions*, Čas. Pěst. Mat., **109** (1984), 118–126.

[26] Noiri, T., *Super-continuity and some strong forms of continuity*, Indian J. Pure Appl. Math., **15** (1984), no. 3, 241–250.

[27] Pipitone, V., Russo, G., *Spazi semiconnessi a spazi semiaperti*, Rend. Circ. Mat. Palermo, **24** (1975), no. 2, 273–285.

[28] Reilly, I. L., Vamanamurthy, M. K., *Connectedness and strong semi-continuity*, Čas. pěst. mat., **109** (1984), 261–265.

[29] Sharma, A. K., *On some properties of hyperconnected spaces*, Mat. Vesnik, **1(14)(29)** (1977), no. 1, 25–27.

[30] Singal, M. K., Singal, A. R., *Almost-continuous mappings*, Yokohama Math. J., **16** (1968), 63–73.

[31] Sivaraj, D., *A note on extremally disconnected spaces*, Indian J. Pure Appl. Math., **17**(12) (1986), 1373–1375.

[32] Steen, L. A., Seebach, Jr., J. A., *Counterexamples in Topology*, Holt, Reinhart and Winston, Inc., New York 1970.

[33] Tadros, S. F., Khalaf, A. B., *On regular semi-open sets and s^*-closed spaces*, Tamkang J. Math., **23** (1992), no. 4, 337–348.

[34] Thompson, T., *Characterizations of irreducible spaces*, Kyungpook Math. J., **21** (1981), no. 2, 191–194.

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SUBORDINATION RESULTS AND INTEGRAL MEANS INEQUALITIES FOR $K-$UNIFORMLY STARLIKE FUNCTIONS DEFINED BY CONVOLUTION INVOLVING THE HURWITZ-LERCH ZETA FUNCTION

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Abstract. In this paper, we introduce a generalized class of $k-$uniformly starlike functions and obtain the subordination results and integral means inequalities. Some interesting consequences of our results are also pointed out.

1. **Introduction**

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C};\ |z| < 1\}$. For functions $f \in A$ given by (1.1) and $g \in A$ given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U. \tag{1.2}$$

In terms of the Hadamard product (or convolution), we choose g as a fixed function in A such that $(f * g)(z)$ exists for any $f \in\ A$, and for various choices of g we get different linear operators which have been studied in recent past. To illustrate some of these cases which arise from the convolution structure (1.2), we consider the following examples.

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The following we recall a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (cf., e.g., [28], p. 121 et sep.])

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.3}$$

$$(a \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; s \in \mathbb{C}, \Re(s) > 1 \text{ and } |z| = 1)$$

where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \{\mathbb{N}\}$, $(\mathbb{Z} := \{\pm 1, \pm 2, \pm 3, ...\}); \mathbb{N} := \{1, 2, 3, ...\}$.

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [4], Ferreira and Lopez [5], Garg et al. [8], Lin and Srivastava [15], Lin et al. [16], and others. In 2007, Srivastava and Attiya [27] (see also Raducanu and Srivastava [20], and Prajapat and Goyal [19]) introduced and investigated the linear operator:

$$\mathcal{J}_{\mu,b} : \mathcal{A} \to \mathcal{A}$$

defined, in terms of the Hadamard product (or convolution), by

$$\mathcal{J}_{\mu,b} f(z) = \mathcal{G}_{\mu,b} * f(z) \tag{1.4}$$

$(z \in U; b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}; f \in \mathcal{A})$, where, for convenience,

$$G_{\mu,b}(z) := (1+b)^\mu [\Phi(z, \mu, b) - b^{-\mu}] \quad (z \in U). \tag{1.5}$$

It is easy to observe from (1.4) and (1.5) that, for $f(z)$ of the form(1.1),we have

$$\mathcal{J}_{\mu,b} f(z) = z + \sum_{n=2}^{\infty} C_n(b, \mu) a_n z^n \tag{1.6}$$

$$C_n(b, \mu) = \left(\frac{1+b}{n+b}\right)^\mu \tag{1.7}$$

where (and throughout this paper unless otherwise mentioned) the parameters μ, b and $C_n(b, \mu)$ are constrained as follows:

$$b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C} \quad \text{and} \quad C_n(b, \mu) = \left(\frac{1+b}{n+b}\right)^\mu.$$

For $f(z) \in \mathcal{A}$ and $z \in \mathcal{U}$

$$\mathcal{J}_{\mu,b} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^\mu a_n z^n \tag{1.8}$$

For various choices of μ we get different operators and are listed below.

$$\mathcal{J}_{0,b}(f)(z) := f(z), \tag{1.9}$$

$$\mathcal{J}_{1,b}(f)(z) := \int_0^z \frac{f(t)}{t}dt := A(f)(z), \tag{1.10}$$

$$\mathcal{J}_{1,\nu}(f)(z) := \frac{1+\nu}{z^\nu}\int_0^z t^{1-\nu}f(t)dt := \mathcal{F}_\nu(f)(z), (\nu > -1), \tag{1.11}$$

$$\mathcal{J}_{\sigma,1}(f)(z) := z + \sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^\sigma a_n z^n = \mathcal{I}^\sigma(f)(z)(\sigma > 0), \tag{1.12}$$

where $\mathcal{A}(f)$ and $\mathcal{F}_\gamma$ are the integral operators introduced by Alexandor [1] and Bernardi [3], respectively, and $\mathcal{I}^\sigma(f)$ is the Jung-Kim-Srivastava integral operator [11] closely related to some multiplier transformation studied by Fleet [6].

In this paper, by making use of the operator$\mathcal{J}_{\mu,b}$ we introduced a new subclass of analytic functions with negative coefficients and discuss some interesting properties of this generalized function class.

For $0 \leq \gamma < 1$ and $k \geq 0$, we let $\mathcal{J}_b^\mu(\gamma, k)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\text{Re}\,\left\{\frac{z(\mathcal{J}_b^\mu f(z))'}{\mathcal{J}_b^\mu f(z)} - \gamma\right\} > k\left|\frac{z(\mathcal{J}_b^\mu f(z))'}{\mathcal{J}_b^\mu f(z)} - 1\right|,\ z \in U, \tag{1.13}$$

where $\mathcal{J}_b^\mu f(z)$ is given by (1.4). We further let $T\mathcal{J}_b^\mu(\gamma, k) = \mathcal{J}_b^\mu(\gamma, k) \cap T$, where

$$T \ := \ \left\{f \in A : f(z) = z - \sum_{n=2}^{\infty}|a_n|z^n, \quad z \in U\right\} \tag{1.14}$$

is a subclass of A introduced and studied by Silverman [23].

By suitably specializing the values of μ, γ and k in the class $\mathcal{J}_b^\mu(\gamma, k)$, we obtain the various subclasses, we present some examples.

Example 1.1. If $\mu = 0$ then

$$\mathcal{J}_b^0(\gamma, k) \equiv \mathbb{S}(\gamma, k) := \left\{f \in A : \text{Re}\,\left\{\tfrac{zf'(z)}{f(z)} - \gamma\right\} > k\left|\tfrac{zf'(z)}{f(z)} - 1\right|,\ z \in U\right\}.$$

Further $T\mathbb{S}(\gamma, k) = \mathbb{S}(\gamma, k) \cap T$, where T is given by (1.14). The class $T\mathbb{S}(\gamma, k) \equiv UST(\gamma, k)$. A function in $UST(\gamma, k)$ is called $k-$uniformly starlike of order γ, $0 \leq \gamma < 1$ and Note that the classes $UST(\gamma, 0)$ and $UST(0, 0)$ were first introduced in [23]. We also observe that $UST(\gamma, 0) \equiv T^*(\gamma)$ is well-known subclass of starlike functions of order γ.

Example 1.2. If $\mu = 1$ and $b = \nu$ with $\nu > -1$ then

$$\mathcal{J}_\nu^1(\gamma, k) \equiv B_\nu(\gamma, k) = \left\{f \in A : \text{Re}\,\left(\frac{z(J_\nu f(z))'}{J_\nu f(z)} - \gamma\right) > k\left|\frac{z(J_\nu f(z))'}{J_\nu f(z)} - 1\right|,\ z \in U\right\},$$

where J_ν is a Bernardi operator [3] defined by

$$J_\nu f(z) := \frac{\nu+1}{z^\nu}\int_0^z t^{\nu-1}f(t)dt.$$

Note that the operator J_1 was studied earlier by Libera [13] and Livingston [17]. Further, $TB_\nu(\gamma,k) = B_\nu(\gamma,k) \cap T$, where T is given by (1.14).

Example 1.3. If $\mu = \sigma$ and $b = 1$ with $\sigma > 0$ then

$$\mathcal{J}_1^\sigma(\gamma,k) \equiv \mathcal{I}^\sigma(\gamma,k) = \left\{ f \in A : \text{Re}\ \left(\frac{z(\mathcal{I}^\sigma f(z))'}{\mathcal{I}^\sigma f(z)} - \gamma \right) > k \left| \frac{z(\mathcal{I}^\sigma f(z))'}{\mathcal{I}^\sigma f(z)} - 1 \right|,\ z \in U \right\},$$

where $\mathcal{I}^\sigma$ is the Jung-Kim-Srivastava integral operator [11] defined by

$$\mathcal{I}^\sigma f(z) := z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1} \right)^\sigma a_n z^n.$$

Further, $T\mathcal{I}^\sigma(\gamma,k) = \mathcal{I}^\sigma(\gamma,k) \cap T$, where T is given by (1.14).

Remark 1.4. Observe that, specializing the parameters μ γ and k in the class $\mathcal{J}_b^\mu(\gamma,k)$, we obtain various classes introduced and studied by Goodman [9, 10], Kanas et.al., [12], Ma and Minda [18], Rønning [21, 22] and others.

The object of the present paper is to investigate the coefficient estimates,extremepoint.Further, we obtain the subordination results and integral means inequalities for the generalized class k- uniformly starlike functions. Some interesting consequences of our results are also pointed out.

2. **Coefficient Estimates**

We first mention a sufficient condition for function $f(z)$ of the form (1.1) to belong to the class $\mathcal{J}_b^\mu(\gamma,k)$,given by the following theorem which can be established easily on lines similar to Aouf and Murugusundaramoorthy [2] hence we omit the details.

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in $\mathcal{J}_b^\mu(\gamma,k)$ if*

$$\sum_{n=2}^{\infty} [n(1+k) - (\gamma+k)] C_n(b,\mu)\ |a_n| \le 1-\gamma, \tag{2.1}$$

where $0 \le \gamma < 1$, $k \ge 0$, and $C_n(b,\mu)$ is given by (1.7).

Theorem 2.2. *Let $0 \le \gamma < 1$, $k \ge 0$ and a function f of the form (1.14) to be in the class $T\mathcal{J}_b^\mu(\gamma,k)$ if and only if*

$$\sum_{n=2}^{\infty} [n(1+k) - (\gamma+k)] C_n(b,\mu)\ |a_n| \le 1-\gamma, \tag{2.2}$$

where $C_n(b,\mu)$ is given by (1.7).

Corollary 2.3. *If $f \in \mathcal{TJ}_b^\mu(\gamma, k)$, then*

$$|a_n| \leq \frac{1-\gamma}{[n(1+k)-(\gamma+k)]C_n(b,\mu)}, \quad , \; 0 \leq \gamma < 1, k \geq 0, \tag{2.3}$$

where $C_n(b,\mu)$ is given by (1.7).
Equality holds for the function $f(z) = z - \frac{1-\gamma}{[n(1+k)-(\gamma+k)]C_n(b,\mu)} z^n$.

Theorem 2.4. (Extreme Points) *Let*

$$f_1(z) = z \quad \text{and} \quad f_n(z) = z - \frac{1-\gamma}{[n(1+k)-(\gamma+k)]C_n(b,\mu)} z^n, \; \; n \geq 2,$$

for $0 \leq \gamma < 1$, $k \geq 0$, and $C_n(b,\mu)$ is given by (1.7). Then $f(z)$ is in the class $T\mathcal{J}_b^\mu(\gamma, k)$ if and only if it can be expressed in the form $f(z) = \sum\limits_{n=1}^{\infty} \omega_n f_n(z)$,

where $\omega_n \geq 0$ and $\sum\limits_{n=1}^{\infty} \omega_n = 1$.

3. **Subordination Results**

Before stating and proving our subordination theorem for the class $\mathcal{TJ}_b^\mu(\gamma, k)$, we need the following definitions and lemmas.

Definition 3.1. For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h, denoted by $g \prec h$, if there exists an analytic function w such that $w(0) = 0$, $|w(z)| < 1$ and $g(z) = h(w(z))$, for all $z \in U$.

Definition 3.2. A sequence $\{b_n\}_{n=1}^\infty$ of complex numbers is said to be a subordinating sequence if, whenever $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$, $a_1 = 1$ is regular, univalent and convex in U, we have

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z), \quad z \in U. \tag{3.1}$$

In 1961, Wilf [29] proved the following subordinating factor sequence.

Lemma 3.3. *The sequence $\{b_n\}_{n=1}^\infty$ is a subordinating factor sequence if and only if*

$$Re \left\{ 1 + 2\sum_{n=1}^{\infty} b_n z^n \right\} > 0, \quad z \in U. \tag{3.2}$$

Theorem 3.4. *Let $f \in \mathcal{TJ}_b^\mu(\gamma, k)$ and $g(z)$ be any function in the usual class of convex functions C, then*

$$\frac{(2+k-\gamma))C_2}{2[1-\gamma+(2+k-\gamma)C_2]}(f * g)(z) \prec g(z) \tag{3.3}$$

where $0 \leq \gamma < 1$; $k \geq 0$ with

$$C_2 = C_2(b,\mu) = \left(\frac{1+b}{2+b}\right)^\mu \tag{3.4}$$

and

$$Re\ \{f(z)\} > -\frac{[1-\gamma+(2+k-\gamma)C_2]}{(2+k-\gamma)C_2}, \quad z \in U. \tag{3.5}$$

The constant factor $\frac{(2+k-\gamma)C_2}{2[1-\gamma+(2+k-\gamma)C_2]}$ *in (3.3) cannot be replaced by a larger number.*

Proof. Let $f \in \mathcal{T}\mathcal{J}_b^{\mu}(\gamma, k)$ and suppose that $g(z) = z + \sum\limits_{n=2}^{\infty} c_n z^n \in C$. Then

$$\begin{aligned}&\frac{(2+k-\gamma)C_2}{2[1-\gamma+(2+k-\gamma)C_2]}(f*g)(z)\\ &\qquad = \frac{(2+k-\gamma)C_2}{2[1-\gamma+(2+k-\gamma)C_2]}\left(z+\sum_{n=2}^{\infty} c_n a_n z^n\right).\end{aligned} \tag{3.6}$$

Thus, by Definition 3.2, the subordination result holds true if

$$\left\{\frac{(2+k-\gamma)C_2}{2[1-\gamma+(2+k-\gamma)C_2]}\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 3.3, this is equivalent to the following inequality

$$\text{Re}\left\{1+\sum_{n=1}^{\infty}\frac{(2+k-\gamma)C_2}{[1-\gamma+(2+k-\gamma)C_2]}a_n z^n\right\} > 0, \quad z \in U. \tag{3.7}$$

Since $\frac{(n(1+k)-(\gamma+k))C_n(b,\mu)}{(1-\gamma)} \geq \frac{(2+k-\gamma)C_2}{(1-\gamma)} > 0$, for $n \geq 2$ we have

$$\begin{aligned}&\text{Re}\left\{1+\frac{(2+k-\gamma)C_2}{[1-\gamma+(2+k-\gamma)C_2]}\sum_{n=1}^{\infty}a_n z^n\right\}\\ &= \text{Re}\left\{1+\frac{(2+k-\gamma)C_2}{[1-\gamma+(2+k-\gamma)C_2]}z+\frac{\sum\limits_{n=2}^{\infty}(2+k-\gamma)C_2 a_n z^n}{[1-\gamma+(2+k-\gamma)C_2]}\right\}\\ &\geq 1-\frac{(2+k-\gamma)C_2}{[1-\gamma+(2+k-\gamma)C_2]}r\\ &\qquad -\frac{1}{[1-\gamma+(2+k-\gamma)C_2]}\sum_{n=2}^{\infty}|[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]C_n(b,\mu)a_n|\,r^n\\ &\geq 1-\frac{(2+k-\gamma)C_2}{[1-\gamma+(2+k-\gamma)C_2]}r-\frac{1-\gamma}{[1-\gamma+(2+k-\gamma)C_2]}r\\ &> 0, \quad |z| = r < 1,\end{aligned}$$

where we have also made use of the assertion (2.1) of Theorem 2.1. This evidently proves the inequality (3.7) and hence the subordination result (3.3) asserted by Theorem 3.4. The inequality (3.5) follows from (3.3) by taking

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in C.$$

Next we consider the function

$$F(z) := z - \frac{1-\gamma}{(2+k-\gamma)C_2} z^2$$

where $0 \le \gamma < 1,\ k \ge 0$, and C_2 is given by (3.4). Clearly $F \in \mathcal{T}\mathcal{J}_b^{\mu}(\gamma, k)$. For this function ,(3.3)becomes

$$\frac{(2+k-\gamma)C_2}{2[1-\gamma+(2+k-\gamma)C_2]} F(z) \prec \frac{z}{1-z}.$$

It is easily verified that

$$\min\left\{\text{Re}\left(\frac{(2+k-\gamma)C_2}{2[1-\gamma+(2+k-\gamma)C_2]} F(z)\right)\right\} = -\frac{1}{2}, \quad z \in U.$$

This shows that the constant $\frac{(2+k-\gamma)C_2}{2[1-\gamma+(2+k-\gamma)C_2]}$ cannot be replaced by any larger one. □

By taking different choices of μ, γ and k in the above theorem and in view of Examples 1 and 2 in Section 1, we state the following corollaries for the subclasses defined in those examples.

Corollary 3.5. *If $f \in \mathbb{S}^*(\gamma, k)$, then*

$$\frac{2+k-\gamma}{2[3+k-\gamma]}(f * g)(z) \prec g(z), \tag{3.8}$$

where $0 \le \gamma < 1,\ ,\ k \ge 0,\ g \in C$ and

$$Re\{f(z)\} > -\frac{3+k-2\gamma}{2+k-\gamma}, \quad z \in U.$$

The constant factor

$$\frac{2+k-\gamma}{2[3+k-2\gamma]}$$

in (3.8) cannot be replaced by a larger one.

Remark 3.6. Corollary 3.5, yields the result obtained by Singh [26] when $\gamma = k = 0$.

Remark 3.7. Corollary 3.5 yields the results obtained by Frasin [7] for the special values of γ and k.

Corollary 3.8. *If* $f \in B_\nu(\gamma, k)$, *then*

$$\frac{(\nu+1)(2+k-\gamma)}{2[(\nu+2)(1-\gamma)+(\nu+1)(2+k-\gamma)]}(f*g)(z) \prec g(z), \tag{3.9}$$

where $0 \le \gamma < 1,\ ,\ k \ge 0,\ \nu > -1,\ g \in C$ *and*

$$Re\{f(z)\} > -\frac{[(\nu+2)(1-\gamma)+(\nu+1)(2+k-\gamma)]}{(\nu+1)(2+k-\gamma)},\ \ z \in U.$$

The constant factor

$$\frac{(\nu+1)(2+k-\gamma)}{2[(\nu+2)(1-\gamma)+(\nu+1)(2+k-\gamma)]}$$

in (3.9) cannot be replaced by a larger one.

4. **Integral Means Inequalities**

Due to , Littlewood [14] we obtain integral means inequalities for the functions in the family$\mathcal{TJ}_b^\mu(\gamma, k)$. We also state the integral means inequalities for several known as well as new subclasses.

Lemma 4.1. *If the functions* f *and* g *are analytic in* U *with* $g \prec f$, *then for* $\eta > 0$, *and* $0 < r < 1$,

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\eta d\theta \le \int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta. \tag{4.1}$$

In [23], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured in [24] and settled in [25], that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta \le \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\eta d\theta,$$

for all $f \in T$, $\eta > 0$ and $0 < r < 1$. In [25], he also proved his conjecture for the subclasses $T^*(\gamma)$ and $C(\gamma)$ of T.

Applying Lemma 4.1, Theorem 2.2 and Theorem 2.4, we obtain the following integral means inequalities for the functions in the family $\mathcal{TJ}_b^\mu(\gamma, k)$.

Theorem 4.2. *Suppose* $f \in \mathcal{TJ}_b^\mu(\gamma, k)$, $\eta > 0$, $0 \le \gamma < 1$, $k \ge 0$ *and* $f_2(z)$ *is defined by*

$$f_2(z) = z - \frac{1-\gamma}{(2+k-\gamma)C_2}z^2,$$

where C_2 is given by (3.4). Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\eta \, d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta \, d\theta. \tag{4.2}$$

Proof. For $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$, (4.2) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \right|^\eta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{(1-\gamma)}{(2+k-\gamma)C_2} z \right|^\eta d\theta.$$

By Lemma 4.1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{1-\gamma}{(2+k-\gamma)C_2} z.$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{1-\gamma}{(2+k-\gamma)C_2} w(z), \tag{4.3}$$

and using (2.2), we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{n=2}^{\infty} \frac{[n(1+k)-(\gamma+k)]C_n(b,\mu)}{1-\gamma} |a_n| z^{n-1} \right| \\ &\leq |z| \sum_{n=2}^{\infty} \frac{[n(1+k)-(\gamma+k)]C_n(b,\mu)}{1-\gamma} |a_n| \\ &\leq |z|, \end{aligned}$$

where $C_n(b,\mu)$ is given by (1.7). This completes the proof by Theorem 2.2. ☐

In view of the Examples 1 and 2 in Section 1 and Theorem 4.2, we can state the following corollaries without proof for the classes defined in those examples.

Corollary 4.3. *If $f \in T\mathbb{S}(\gamma, k)$, $0 \leq \gamma < 1$, $k \geq 0$ and $\eta > 0$, then the assertion (4.2) holds true where*

$$f_2(z) = z - \frac{1-\gamma}{[2+k-\gamma)]} z^2.$$

Remark 4.4. Fixing $k = 0$, Corollary 4.3 lead the integral means inequality for the class $T^*(\gamma)$ obtained in [25].

Corollary 4.5. *If $f \in TB_\nu(\gamma, k)$, $\nu > -1$, $0 \leq \gamma < 1$, $k \geq 0$ and $\eta > 0$, then the assertion (4.2) holds true where*

$$f_2(z) = z - \frac{(1-\gamma)(\nu+2)}{(\nu+1)[2+k-\gamma]} z^2 \ .$$

Concluding Remarks. The various results presented in this paper would provide interesting extensions and generalizations of those considered earlier for simpler analytic function classes. The details involved in the derivations of such specializations of the results presented in this paper are fairly straight- forward.

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References

[1] Alexander, J. W., *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math., **17** (1915), 1222.

[2] Aouf, M. K., Murugusundaramoorthy, G., *On a subclass of uniformly convex functions defined by the Dziok-Srivastava Operator,* Austral. J. Math. Anal. Appl., **3** (2007).

[3] Bernardi, S.D., *Convex and starlike univalent functions,* Trans. Amer. Math. Soc., **135** (1969), 429–446.

[4] Choi, J., Srivastava, H. M., *Certain families of series associated with the Hurwitz-Lerch Zeta function*, Appl. Math. Comput., **170** (2005), 399-409.

[5] Ferreira, C., Lopez, J. L., *Asymptotic expansions of the Hurwitz-Lerch Zeta function*, J. Math. Anal. Appl., **298** (2004), 210-224.

[6] Flett, T. M., *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl., **38**(1972), 746765.

[7] Frasin, B. A., *Subordination results for a class of analytic functions defined by a linear operator,* J. Ineq. Pure Appl. Math., **7 (134)** (2006), no. 4, 1–7.

[8] Garg, M., Jain, K., Srivastava, H. M., *Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch Zeta functions*, Integral Transform. Spec. Funct., **17** (2006), 803-815.

[9] Goodman, A. W., *On uniformly convex functions,* Ann. Polon. Math., **56** (1991), 87–92.

[10] W. Goodman, A. W. *On uniformly starlike functions,* J. Math. Anal. Appl., **155** (1991), 364–370.

[11] Jung, I. B., Kim, Y. C., Srivastava, H. M., *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl., **176** (1993), 138147.

[12] Kanas, S., Srivastava, H. M., *Linear operators associated with $k-$uniformly convex functions,* Integral Transform Spec. Funct., **9** (2000), 121–132.

[13] Libera, R. J., *Some classes of regular univalent functions,* Proc. Amer. Math. Soc., **16** (1965), 755–758.

[14] Littlewood, J. E., *On inequalities in theory of functions,* Proc. London Math. Soc., **23** (1925), 481–519.

[15] Lin, S. D., Srivastava, H. M., *Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations*, Appl. Math. Comput., **154** (2004), 725-733.

[16] Lin, S. D., Srivastava, H. M., Wang, P. Y., *Some espansion formulas for a class of generalized Hurwitz-Lerch Zeta functions*, Integral Transform. Spec. Funct., **17** (2006), 817-827.

[17] Livingston, A. E., *On the radius of univalence of certain analytic functions,* Proc. Amer. Math. Soc., **17** (1966), 352–357.

[18] Ma, W. C., Minda, D., *Uniformly convex functions,* Annal. Polon. Math., **57** (1992), no. 2, 165–175.

[19] Prajapat, J. K., Goyal, S. P., *Applications of Srivastava-Attiya operator to the classes of strongly starlike and strongly convex functions*, J. Math. Inequal., **3** (2009), 129-137.

[20] Raducanu, D., Srivastava, H. M., *A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch Zeta function*, Integral Transform. Spec. Funct., **18** (2007), 933-943.

[21] Rønning, F., *Uniformly convex functions and a corresponding class of starlike functions,* Proc. Amer. Math. Soc. **118** (1993), 189–196.

[22] Rønning, F., *On starlike functions associated with parabolic regions,* Ann. Univ. Mariae Curie-Sklodowska Sect. A, **45** (1991), 117–122.

[23] Silverman, H., *Univalent functions with negative coefficients,* Proc. Amer. Math. Soc., **51** (1975), 109–116.

[24] Silverman, H., *A survey with open problems on univalent functions whose coefficients are negative,* Rocky Mt. J. Math., **21** (1991), 1099–1125.

[25] Silverman, H., *Integral means for univalent functions with negative coefficients,* Houston J. Math., **23** (1997), 169–174.

[26] Singh, S., *A subordination theorem for spirallike functions,* Internat. J. Math. and Math. Sci., **24** (2000), no. 7, 433–435.

[27] Srivastava, H. M., Attiya, A. A., *An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination*, Integral Transform. Spec. Funct., **18** (2007), 207–216.

[28] Srivastava, H. M., Choi, J., *Series associated with the Zeta and related functions*, Dordrecht, Boston, London: Kluwer Academic Publishers, 2001.

[29] Wilf, H. S., *Subordinating factor sequence for convex maps of the unit circle,* Proc. Amer. Math. Soc., **12** (1961), 689–693.

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ON INVERSE-CONVEX MEROMORPHIC FUNCTIONS

ALINA TOTOI

Abstract. We introduce a new class K_i of meromorphic functions, called the class of inverse-convex functions, and we study some properties and we prove some theorems for this class. We also study some properties for the integral operator

$$I_\gamma(g)(z) = \frac{\gamma}{z^{\gamma+1}} \int_0^z t^\gamma g(t)dt,\ \gamma \in \mathbb{C},$$

with respect to this class K_i.

1. **Introduction and preliminaries**

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane, $\dot{U} = U \setminus \{0\}$ and $H(U) = \{f : U \to \mathbb{C} : f \text{ is holomorphic in } U\}$.
Let $A_n = \{f \in H(U) : f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots\}$, $n \in \mathbb{N}^*$, and for $n = 1$ we denote A_1 by A and this set is called *the class of analytic functions normalized at the origin*.
Let K be the class of normalized convex functions on the unit disc U, i.e.

$$K = \left\{ f \in A : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U \right\}$$

and let S^* be the class of normalized starlike functions on U, i.e.

$$S^* = \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\}.$$

We denote by M_0 the class of meromorphic functions in $\dot{U}$ of the form

$$g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots,\ z \in \dot{U}.$$

Let

$$M_0^* = \left\{ g \in M_0 : \text{g is univalent in } \dot{U} \text{ and } \operatorname{Re} \left[-\frac{zg'(z)}{g(z)} \right] > 0,\ z \in \dot{U} \right\}$$

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be called the class of meromorphic starlike functions in $\dot{U}$.
We note that if f is a normalized starlike function on U, then the function $g = \dfrac{1}{f}$ belongs to the class M_0^*.
For $\alpha \in [0, 1)$ and $\beta > 1$ let

$$M_0^*(\alpha) = \left\{ g \in M_0 : \text{g is univalent in } \dot{U} \text{ and } \operatorname{Re}\left[-\frac{zg'(z)}{g(z)}\right] > \alpha,\, z \in \dot{U} \right\}$$

and

$$M_0^*(\alpha, \beta) = \left\{ g \in M_0 : \text{g is univalent in } \dot{U} \text{ and } \alpha < \operatorname{Re}\left[-\frac{zg'(z)}{g(z)}\right] < \beta,\, z \in \dot{U} \right\}.$$

Let

$$M_0^c = \left\{ g \in M_0 : \text{g is univalent in } \dot{U} \text{ and } \operatorname{Re}\left[-\left(\frac{zg''(z)}{g'(z)} + 1\right)\right] > 0,\, z \in \dot{U} \right\}$$

be called the class of meromorphic convex functions in $\dot{U}$.
It's easy to see that $M_0^c \subset M_0^*$.

Theorem 1.1. [1, Theorem 2.4f.], [2, p.212] *Let* $p \in H[a, n]$ *with* $\operatorname{Re} a > 0$ *and let* $P : U \to \mathbb{C}$ *be a function with* $\operatorname{Re} P(z) > 0$, $z \in U$. *If*

$$\operatorname{Re}\,[p(z) + P(z)zp'(z)] > 0,\, z \in U,$$

then $\operatorname{Re} p(z) > 0$, $z \in U$.

Lemma 1.2. [1, Exemple 2.4e.], [2, p.211] *Let* $p \in H[a, n]$ *with* $\operatorname{Re} a > 0$ *and let* $\alpha : U \to \mathbb{R}$. *If*

$$\operatorname{Re}\left[p(z) + \alpha(z)\frac{zp'(z)}{p(z)}\right] > 0,\, z \in U,$$

then $\operatorname{Re} p(z) > 0$, $z \in U$.

Definition 1.3. Let $g : \dot{U} \to \mathbb{C}$ be a meromorphic function in $\dot{U}$ of the form

$$g(z) = \frac{\alpha_{-1}}{z} + \alpha_0 + \alpha_1 z + \cdots,\, z \in \dot{U}.$$

We say that the function g is inverse-convex in $\dot{U}$ if there exists a convex function f defined on U with $f(0) = 0$ such that $f(z)g(z) = 1$ for each $z \in \dot{U}$.

Remark 1.4. 1. From the above definition we notice that if g is inverse-convex, then $g(z) \neq 0$, $z \in \dot{U}$ and g is univalent in $\dot{U}$.

2. If $\alpha_{-1} = 1$, i.e. $g \in M_0$, we can easily see that the function f from the above definition is also normalized, hence a function $g \in M_0$ is inverse-convex in $\dot{U}$ if there exists a function $f \in K$ such that $f(z)g(z) = 1$ for each $z \in \dot{U}$. We will denote the class of these functions by K_i(the class of normalized inverse-convex functions on $\dot{U}$).

3. If g is inverse-convex in $\dot{U}$ and $\lambda \in \mathbb{C}^*$, then the meromorphic function λg is also inverse-convex in $\dot{U}$.

4. If $g \in K_i$, then $g \in M_0^*\left(\frac{1}{2}\right)$.

Definition 1.5. Let $g : \dot{U} \to \mathbb{C}$ be a meromorphic function in $\dot{U}$ of the form

$$g(z) = \frac{\alpha_{-1}}{z} + \alpha_0 + \alpha_1 z + \cdots .$$

We say that the function g is close-to-inverse-convex in $\dot{U}$ if there exists an inverse-convex function ψ on $\dot{U}$ such that

$$\operatorname{Re} \frac{g'(z)}{\psi'(z)} > 0,\ z \in \dot{U}.$$

We denote by C_i the class of normalized close-to-inverse-convex functions on $\dot{U}$.

For $\beta > 1$ we say that a close-to-inverse-convex function g is in the class $C_{i;\beta}$ if the function $\psi \in K_i \cap M_0^*(0, \beta)$.

2. **Main results**

Theorem 2.1. (Theorem of analytical characterization of the inverse-convexity for meromorphic functions) *Let $g : \dot{U} \to \mathbb{C}$ be a meromorphic function in $\dot{U}$ of the form*

$$g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots ,$$

such that $g(z) \neq 0$, $z \in \dot{U}$. Then the function g is inverse-convex on $\dot{U}$ if and only if g is univalent on $\dot{U}$ and

$$\operatorname{Re} \left[\frac{zg''(z)}{g'(z)} - 2\frac{zg'(z)}{g(z)} + 1 \right] > 0,\ z \in \dot{U}.$$

Proof. Suppose that $g \in K_i$. Then there exists $f \in K$ such that $f(z)g(z) = 1$, $z \in \dot{U}$, so

$$g(z) = \frac{1}{f(z)},\ z \in \dot{U},\ f \in K. \tag{2.1}$$

Because f is univalent also is g, and if we consider the second differential for the equality $f(z)g(z) = 1$, $z \in \dot{U}$ we obtain

$$f''(z)g(z) + 2f'(z)g'(z) + f(z)g''(z) = 0. \tag{2.2}$$

Dividing (2.2) by $f(z)g'(z) \neq 0$, $z \in \dot{U}$ and multiplying the result with z we will have

$$\frac{zf''(z)}{f'(z)} \frac{f'(z)g(z)}{f(z)g'(z)} + 2\frac{zf'(z)}{f(z)} + \frac{zg''(z)}{g'(z)} = 0. \tag{2.3}$$

Using the derivative for $f(z)g(z) = 1$ we obtain

$$\frac{f'(z)g(z)}{f(z)g'(z)} = -1. \tag{2.4}$$

From (2.4) and (2.3) we have

$$\frac{zf''(z)}{f'(z)} + 1 = \frac{zg''(z)}{g'(z)} - 2\frac{zg'(z)}{g(z)} + 1,\ z \in \dot{U},$$

and, since we know that $\mathrm{Re}\,\frac{zf''(z)}{f'(z)} + 1 > 0$, $z \in U$, we obtain

$$\mathrm{Re}\left[\frac{zg''(z)}{g'(z)} - 2\frac{zg'(z)}{g(z)} + 1\right] > 0,\ z \in \dot{U}.$$

To prove the sufficiency we consider the function $f(z) = \frac{1}{g(z)}$, $z \in \dot{U}$, with $f(0) = 0$ and we prove that $f \in K$. $\square$

Remark 2.2. 1. An easy computation shows that the function

$$f(z) = \log(1+z), z \in U \left(\text{with} \quad \log(1+z)\Big|_{z=0} = 0\right)$$

is convex on U and normalized, so the function $g(z) = \frac{1}{f(z)}$, $z \in \dot{U}$ belongs to the class K_i.

On the other hand we have

$$\frac{zg''(z)}{g'(z)} + 1 = \frac{\log(1+z) + 2z}{(1+z)\log(1+z)},$$

and it's easy to see that the inequality

$$\mathrm{Re}\left[-\left(\frac{zg''(z)}{g'(z)} + 1\right)\right] > 0$$

doesn't hold for each $z \in \dot{U}$ $\left(\text{for exemple we can take } z = \frac{1}{2}\right)$, so $g \notin M_0^c$. In other words, $K_i \neq M_0^c$.

2. We know that the function $f(z) = \frac{z}{1 + e^{i\tau}z} \in K$, so

$$g(z) = \frac{1}{f(z)} = \frac{1}{z} + e^{i\tau} \in K_i.$$

But on the other hand, it's easy to show that $g \in M_0^c$, hence $K_i \cap M_0^c \neq \emptyset$.

3. If $g \in K_i$, then $f = \frac{1}{g} \in K \subset S^*$, so $g \in M_0^*$. Therefore, we have $K_i \subset M_0^*$.

Theorem 2.3. (Duality theorem between the classes M_0^* and K_i) *Let $g : \dot{U} \to \mathbb{C}$ be a function in M_0. Then $g \in K_i$ if and only if the function*

$$G(z) = -\frac{g^2(z)}{zg'(z)} \in M_0^*.$$

Proof. Using the definition we have $g \in K_i$ if and only if $f = \dfrac{1}{g} \in K$.

On the other hand, in view of Alexander's duality theorem (see [2], [3]) we deduce that

$$f \in K \quad \text{is equivalent to} \quad F(z) = zf'(z) = -\frac{zg'(z)}{g^2(z)} \in S^*.$$

But, we know that $F \in S^*$ is equivalent to $G = \dfrac{1}{F} \in M_0^*$. So, we obtained

$$g \in K_i \quad \text{if and only if} \quad G(z) = -\frac{1}{z}\frac{g^2(z)}{g'(z)} \in M_0^*.$$

□

Theorem 2.4. (Distortion theorem for the class K_i) *If the function g belongs to the class K_i, then we have:*

$$\frac{1}{r} - 1 \le |g(z)| \le \frac{1}{r} + 1,\ |z| = r \in (0,1) \quad \left(\textit{equivalent to } \left||g(z)| - \frac{1}{|z|}\right| \le 1,\ z \in \dot{U}\right),$$

$$\left(\frac{1-r}{r+r^2}\right)^2 \le |g'(z)| \le \left(\frac{1+r}{r-r^2}\right)^2,\ |z| = r \in (0,1).$$

For $|g(z)|$ these estimates are sharp and we have equality for $g(z) = \dfrac{1}{z} + e^{i\tau}$, $\tau \in \mathbb{R}$.

Proof. If $g \in K_i$, then $f = \dfrac{1}{g} \in K$ and in view of the distortion theorem for the class K we have

$$\frac{r}{1+r} \le |f(z)| \le \frac{r}{1-r} \tag{2.5}$$

$$\frac{1}{(1+r)^2} \le |f'(z)| \le \frac{1}{(1-r)^2},\ |z| = r < 1. \tag{2.6}$$

From (2.5) taking $f = \dfrac{1}{g}$ we obtain the bounds for $|g(z)|$ and since $r = |z|$ we have

$$\frac{1}{|z|} - 1 \le |g(z)| \le \frac{1}{|z|} + 1 \Leftrightarrow$$

$$\left||g(z)| - \frac{1}{|z|}\right| \le 1.$$

For the bounds of $|g'(z)|$ we use: $g' = -g^2 f'$, the bounds for $|g(z)|$ and (2.6). □

Remark 2.5. If $f : U \to \mathbb{C}$ is a function of the form $f(z) = z + a_1 z^2 + a_2 z^3 + \cdots$, then the function $g : \dot{U} \to \mathbb{C}$ defined as $g(z) = \dfrac{1}{f(z)}$, $z \in \dot{U}$ has the form

$$g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n + \cdots ,$$

where

$$\begin{cases} \alpha_0 = & -a_1 \\ \alpha_1 = & -a_2 - \alpha_0 a_1 \\ \vdots & \\ \alpha_n = & -a_{n+1} - \alpha_0 a_n - \alpha_1 a_{n-1} - \cdots - \alpha_{n-1} a_1 \\ \vdots & \end{cases}$$

We know that if a function f belongs to the class K and it is of the form presented above then we have $|a_n| \leq 1$ for each $n \in \mathbb{N}^*$ and therefore, after a short computation we obtain that

$$|\alpha_n| \leq 2^n, \forall n \in \mathbb{N}.$$

So, if $g \in K_i$, $g(z) = \dfrac{1}{z} + \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n + \cdots$, then $|\alpha_n| \leq 2^n$, $\forall n \in \mathbb{N}$.

Theorem 2.6. *Let be $g \in K_i$, $\lambda \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Re} \lambda > 2|\lambda|^2$, $\beta = \dfrac{\operatorname{Re} \lambda}{2|\lambda|^2}$ and $\operatorname{Re} \left[-\dfrac{zg'(z)}{g(z)} \right] < \beta$ (i.e. $g \in K_i \cap M_0^*(0, \beta)$), then the function*

$$h_\lambda(z) = g(z) + \lambda z g'(z),\ z \in \dot{U},$$

is close-to-inverse-convex.

Proof. From $h_\lambda(z) = g(z) + \lambda z g'(z)$ we obtain $h'_\lambda(z) = g'(z) + \lambda g'(z) + \lambda z g''(z)$ which is equivalent to

$$\frac{h'_\lambda(z)}{\lambda g'(z)} = 1 + \frac{1}{\lambda} + \frac{zg''(z)}{g'(z)} = \frac{1}{\lambda} + 2\frac{zg'(z)}{g(z)} + \left[\frac{zg''(z)}{g'(z)} - 2\frac{zg'(z)}{g(z)} + 1 \right]$$

so

$$\operatorname{Re} \frac{h'_\lambda(z)}{\lambda g'(z)} = \operatorname{Re} \frac{1}{\lambda} + 2\operatorname{Re} \frac{zg'(z)}{g(z)} + \operatorname{Re} \left[\frac{zg''(z)}{g'(z)} - 2\frac{zg'(z)}{g(z)} + 1 \right] > 0,\ z \in \dot{U}.$$

For the last inequality we have used the fact that $g \in K_i$ implies

$$\operatorname{Re} \left[\frac{zg''(z)}{g'(z)} - 2\frac{zg'(z)}{g(z)} + 1 \right] > 0,\ z \in \dot{U},$$

and we have also used the condition

$$\operatorname{Re} \left[-\frac{zg'(z)}{g(z)} \right] < \frac{\operatorname{Re} \lambda}{2|\lambda|^2} \quad \text{equivalent to} \quad \operatorname{Re} \frac{1}{\lambda} + 2\operatorname{Re} \frac{zg'(z)}{g(z)} > 0.$$

Therefore, we have

$$\operatorname{Re}\frac{h'_\lambda(z)}{\lambda g'(z)} > 0,\ z \in \dot{U}$$

meaning that the function h_λ is close-to-inverse-convex with respect to the inverse-convex function λg.

We note that we need $\operatorname{Re}\lambda > 2|\lambda|^2$ because $\beta > 1$ and that implies $|\lambda| < 1/2$. □

For $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$ we consider the integral operator $I_\gamma : M_0 \to M_0$ given by

$$I_\gamma(g)(z) = \frac{\gamma}{z^{\gamma+1}}\int_0^z t^\gamma g(t)dt \tag{2.7}$$

and we have the following result.

Theorem 2.7. *Let be $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 1$ and $\beta = \frac{\operatorname{Re}\gamma + 1}{2}$.*
If $I_\gamma[K_i] \subset K_i$, then $I_\gamma[C_{i;\beta}] \subset C_i$.

Proof. Let $G = I_\gamma(g)$. If we take the second derivative for the relation

$$G(z) = I_\gamma(g)(z) = \frac{\gamma}{z^{\gamma+1}}\int_0^z t^\gamma g(t)dt$$

we obtain

$$(\gamma+2)G'(z) + zG''(z) = \gamma g'(z). \tag{2.8}$$

If $g \in C_{i;\beta}$, then there exists a function $\psi \in K_i \cap M_0^*(0,\beta)$ such that

$$\operatorname{Re}\frac{g'(z)}{\psi'(z)} > 0,\ z \in U. \tag{2.9}$$

Let's denote $\phi = I_\gamma(\psi)$. From $I_\gamma[K_i] \subset K_i$ we obtain that $\phi \in K_i$.
We also have the relation

$$(\gamma+2)\phi'(z) + z\phi''(z) = \gamma\psi'(z). \tag{2.10}$$

If we denote

$$p(z) = \frac{G'(z)}{\phi'(z)},$$

then $p(0) = 1$ and the relation (2.8) can be rewritten in the following form

$$(\gamma+2)p(z)\phi'(z) + z[p'(z)\phi'(z) + p(z)\phi''(z)] = \gamma g'(z). \tag{2.11}$$

Using (2.11) and (2.10) we obtain

$$p(z) + \frac{zp'(z)}{(\gamma+2) + \frac{z\phi''(z)}{\phi'(z)}} = \frac{g'(z)}{\psi'(z)}$$

which is equivalent to

$$p(z) + \frac{zp'(z)}{P(z)} = \frac{g'(z)}{\psi'(z)}, \quad \text{where} \quad P(z) = (\gamma+2) + \frac{z\phi''(z)}{\phi'(z)}.$$

Using (2.9) we deduce that

$$\operatorname{Re}\left[p(z)+\frac{zp'(z)}{P(z)}\right]>0,\ z\in U. \tag{2.12}$$

The relation (2.10) is equivalent to $\phi'(z)P(z)=\gamma\psi'(z)$ and using the logarithmic derivative for this equality we obtain

$$P(z)+\frac{zP'(z)}{P(z)}=\gamma+2+\frac{z\psi''(z)}{\psi'(z)}=\left[\frac{z\psi''(z)}{\psi'(z)}-2\frac{z\psi'(z)}{\psi(z)}+1\right]+2\frac{z\psi'(z)}{\psi(z)}+\gamma+1.$$

Since we know that

1. $\psi\in K_i$, i.e.

$$\operatorname{Re}\left[\frac{z\psi''(z)}{\psi'(z)}-2\frac{z\psi'(z)}{\psi(z)}+1\right]>0,\ z\in U.$$

2. $\psi\in M_0^*(0,\beta)$, i.e.

$$\operatorname{Re}\left[-\frac{z\psi'(z)}{\psi(z)}\right]<\beta=\frac{\operatorname{Re}\gamma+1}{2}$$

we have

$$\operatorname{Re}\left[P(z)+\frac{zP'(z)}{P(z)}\right]>0,\ z\in U.$$

It is easy to see that $P(0)=\gamma$, so $\operatorname{Re}P(0)>0$ and using Lemma 1.2 we obtain $\operatorname{Re}P(z)>0,\ z\in U$.

Using (2.12), $\operatorname{Re}P(z)>0,\ z\in U$ and Theorem 1.1 we have

$$\operatorname{Re}p(z)>0,\ z\in U$$

which is the same with

$$\operatorname{Re}\frac{G'(z)}{\phi'(z)}>0,\ z\in U,\quad \text{hence}\quad G\in C_i.$$

□

References

[1] Miller, S. S., Mocanu, P. T., *Differential subordinations. Theory and applications*, Marcel Dekker Inc. New York, Basel, 2000.

[2] Mocanu, P. T., Bulboacă, T., Sălăgean, G., *The geometric theory of univalent functions*, (Romanian) Casa Cărţii de Ştiinţă, Cluj-Napoca, 2006.

[3] Pommerenke, Ch., *Univalent Functions*, Vanderhoeck & Ruprecht, Göttingen, 1975.

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VORONOVSKAYA TYPE THEOREMS FOR SMOOTH PICARD AND GAUSS-WEIERSTRASS SINGULAR OPERATORS

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Abstract. In this article we continue with the study of approximation properties of smooth Picard singular integral operators and smooth Gauss-Weierstrass singular integral operators over the real line. We produce some Voronovskaya type theorems and give some quantitative results regarding the rate of convergence of the above mentioned singular integral operators.

1. **Introduction**

We are motivated by the approximation properties of *Picard* and *Gauss-Weierstrass singular integrals* of a function f defined by the following

$$P_{\xi}(f;x) := \frac{1}{2\xi}\int_{-\infty}^{\infty} f(x+y)e^{-|y|/\xi}dy, \tag{1.1}$$

$$W_{\xi}(f;x) := \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty} f(x+y)e^{-y^2/\xi}dy, \tag{1.2}$$

for all $x \in \mathbb{R}$, $\xi > 0$, see [5], chp. 16, 17, and [4], chp. 21, and [6].

Next we mention the *smooth Picard singular integral operators* $P_{r,\xi}(f;x)$ and the *smooth Gauss-Weierstrass singular integral operators* $W_{r,\xi}(f;x)$ defined next, basic approximation properties of them were studied in [1], [2], [3], [7], [8] and [9].

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we set

$$\alpha_j = \begin{cases} (-1)^{r-j}\binom{r}{j}j^{-n}, & j = 1,\dots,r, \\ 1 - \sum\limits_{j=1}^{r}(-1)^{r-j}\binom{r}{j}j^{-n}, & j = 0, \end{cases} \tag{1.3}$$

that is $\sum\limits_{j=0}^{r}\alpha_j = 1$.

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Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f^{(n)}$ exists and it is bounded and Lebesgue measurable. We define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue singular integrals

$$P_{r,\xi}(f;x) = \frac{1}{2\xi} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt) \right) e^{-|t|/\xi} dt, \tag{1.4}$$

and

$$W_{r,\xi}(f;x) = \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt) \right) e^{-t^2/\xi} dt. \tag{1.5}$$

Note 1.1. The operators $P_{r,\xi}$ and $W_{r,\xi}$ are not, in general, positive, see [1] and [7] respectively.

Note 1.2. In particular we have $P_{1,\xi} = P_\xi$ and $W_{1,\xi} = W_\xi$.

In Section 2 we will give some elementary properties of the integrals defined in (1.4) and (1.5). Then, in Section 3, we will prove some Voronovskaya type asymptotic theorems, see also [11].

2. **Auxiliary result**

From (1.4) and (1.5) we also obtain

$$P_{r,\xi}(f;x) = \sum_{j=0}^{r} \frac{1}{2\xi} \alpha_j \int_{-\infty}^{\infty} f(x+jt) e^{-|t|/\xi} dt, \tag{2.1}$$

and

$$W_{r,\xi}(f;x) = \sum_{j=0}^{r} \frac{1}{\sqrt{\pi\xi}} \alpha_j \int_{-\infty}^{\infty} f(x+jt) e^{-t^2/\xi} dt. \tag{2.2}$$

By means of elementary calculations, we obtain

Lemma 2.1. For every $n \in \mathbb{N}_0$, and $\xi > 0$, we have

$$I_n := \int_0^{\infty} t^n e^{-t/\xi} dt = n! \xi^{n+1}, \tag{2.3}$$

and

$$I_n^* := \int_0^{\infty} t^n e^{-t^2/\xi} dt = \begin{cases} \xi^{\frac{n+1}{2}} \cdot \frac{1}{2} \cdot \left(\frac{n-1}{2}\right)!, & n-odd \\ \xi^{\frac{n+1}{2}} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{1}{2^n} \cdot \frac{n!}{\left(\frac{n}{2}\right)!}, & n-even. \end{cases} \tag{2.4}$$

Proof. Easy. ∎

3. **Results**

We present first our main result.

Theorem 3.1. Let $f:\mathbb{R}\to\mathbb{R}$ be such that $f^{(n)}$exists, $n\in\mathbb{N}$, and is bounded and Lebesgue measurable on $\mathbb{R}$, and let $\xi\to 0+$, $0<\alpha\leq 1$. Then

$$P_{r,\xi}(f;x)-f(x)=\sum_{m=1}^{\lfloor\frac{n-1}{2}\rfloor} f^{(2m)}(x)\left(\sum_{j=1}^{r}\alpha_j j^{2m}\right)\xi^{2m}+o(\xi^{n-\alpha}), \tag{3.1}$$

and

$$W_{r,\xi}(f;x)-f(x)=\sum_{m=1}^{\lfloor\frac{n-1}{2}\rfloor} \frac{f^{(2m)}(x)}{m!2^{2m}}\left(\sum_{j=1}^{r}\alpha_j j^{2m}\right)\xi^{m}+o(\xi^{\frac{n-\alpha}{2}}). \tag{3.2}$$

Proof. We notice by $\frac{1}{2\xi}\int_{-\infty}^{\infty}e^{-|t|/\xi}dt=1$ and $\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}e^{-t^2/\xi}dt=1$, that $P_{r,\xi}(c,x)=c, W_{r,\xi}(c,x)=c$, for any c constant, and therefore (see also [7] and [2], formula (3.3) there) we have

$$P_{r,\xi}(f;x)-f(x)=\frac{1}{2\xi}\left(\sum_{j=0}^{r}\alpha_j\int_{-\infty}^{\infty}(f(x+jt)-f(x))e^{-|t|/\xi}dt\right), \tag{3.3}$$

and

$$W_{r,\xi}(f;x)-f(x)=\frac{1}{\sqrt{\pi\xi}}\left(\sum_{j=0}^{r}\alpha_j\int_{-\infty}^{\infty}(f(x+jt)-f(x))e^{-t^2/\xi}dt\right). \tag{3.4}$$

Using Taylor's formula for f, we have

$$f(x+jt)=\sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}(jt)^k+\frac{f^{(n)}(\gamma)}{n!}(jt)^n, \tag{3.5}$$

with γ between x and $x+jt$.

We obtain

$$P_{r,\xi}(f;x)-f(x) \overset{(3.3)}{=} \frac{1}{2\xi}\left(\sum_{j=1}^{r}\alpha_j\int_{-\infty}^{\infty}\left(\left[\sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}(jt)^k+\frac{f^{(n)}(\gamma)}{n!}(jt)^n\right]\right.\right.$$
$$\left.\left.-f(x)\right)e^{-|t|/\xi}dt\right)$$
$$= \frac{1}{2\xi}\left(\sum_{j=1}^{r}\alpha_j\int_{-\infty}^{\infty}\left[\sum_{k=1}^{n-1}\frac{f^{(k)}(x)}{k!}j^kt^k+\frac{f^{(n)}(\gamma)}{n!}j^nt^n\right]e^{-|t|/\xi}dt\right)$$
$$= \frac{1}{2\xi}\left(\sum_{j=1}^{r}\alpha_j\left[\sum_{k=1}^{n-1}\frac{f^{(k)}(x)}{k!}j^k\left(\int_{-\infty}^{\infty}t^ke^{-|t|/\xi}dt\right)\right.\right.$$
$$\left.\left.+\frac{j^n}{n!}\int_{-\infty}^{\infty}f^{(n)}(\gamma)t^ne^{-|t|/\xi}dt\right]\right)$$
$$\overset{(2.3)}{=} \frac{1}{2\xi}\left(\sum_{j=1}^{r}\alpha_j\left[\sum_{m=1}^{\lfloor\frac{n-1}{2}\rfloor}f^{(2m)}(x)j^{2m}2\xi^{2m+1}\right.\right.$$
$$\left.\left.+\frac{j^n}{n!}\int_{-\infty}^{\infty}f^{(n)}(\gamma)t^ne^{-|t|/\xi}dt\right]\right)$$
$$= \sum_{j=1}^{r}\alpha_j\left[\left(\sum_{m=1}^{\lfloor\frac{n-1}{2}\rfloor}f^{(2m)}(x)j^{2m}\xi^{2m}\right)+\frac{j^n}{2\xi n!}\int_{-\infty}^{\infty}f^{(n)}(\gamma)t^ne^{-|t|/\xi}dt\right],$$

and

$$W_{r,\xi}(f;x)-f(x) \overset{(3.4)}{=} \frac{1}{\sqrt{\pi\xi}}\left(\sum_{j=1}^{r}\alpha_j\int_{-\infty}^{\infty}\left(\left[\sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}(jt)^k+\frac{f^{(n)}(\gamma)}{n!}(jt)^n\right]\right.\right.$$
$$\left.\left.-f(x)\right)e^{-t^2/\xi}dt\right)$$
$$= \frac{1}{\sqrt{\pi\xi}}\left(\sum_{j=1}^{r}\alpha_j\int_{-\infty}^{\infty}\left[\sum_{k=1}^{n-1}\frac{f^{(k)}(x)}{k!}(jt)^k+\frac{f^{(n)}(\gamma)}{n!}(jt)^n\right]e^{-t^2/\xi}dt\right)$$
$$= \frac{1}{\sqrt{\pi\xi}}\left(\sum_{j=1}^{r}\alpha_j\left[\sum_{k=1}^{n-1}\frac{f^{(k)}(x)}{k!}j^k\left(\int_{-\infty}^{\infty}t^ke^{-t^2/\xi}dt\right)\right.\right.$$
$$\left.\left.+\frac{j^n}{n!}\int_{-\infty}^{\infty}f^{(n)}(\gamma)t^ne^{-t^2/\xi}dt\right]\right)$$

$$\overset{(2.4)}{=} \frac{1}{\sqrt{\pi\xi}} \left(\sum_{j=1}^{r} \alpha_j \left[\sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{f^{(2m)}(x)}{m!} j^{2m} \xi^{\frac{2m+1}{2}} \frac{\sqrt{\pi}}{2^{2m}} + \frac{j^n}{n!} \int_{-\infty}^{\infty} f^{(n)}(\gamma) t^n e^{-t^2/\xi} dt \right] \right)$$

$$= \sum_{j=1}^{r} \alpha_j \left[\sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} f^{(2m)}(x) \frac{j^{2m}\xi^m}{m! 2^{2m}} + \frac{j^n}{\sqrt{\pi\xi} n!} \int_{-\infty}^{\infty} f^{(n)}(\gamma) t^n e^{-t^2/\xi} dt \right].$$

Therefore we have obtained

$$P_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} f^{(2m)}(x) \left(\sum_{j=1}^{r} \alpha_j j^{2m} \right) \xi^{2m} = \sum_{j=1}^{r} \alpha_j \frac{j^n}{2\xi n!} \int_{-\infty}^{\infty} f^{(n)}(\gamma) t^n e^{-|t|/\xi} dt, \tag{3.6}$$

and

$$W_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} f^{(2m)}(x) \frac{1}{m! 2^{2m}} \left(\sum_{j=1}^{r} \alpha_j j^{2m} \right) \xi^m = \sum_{j=1}^{r} \alpha_j \frac{j^n}{\sqrt{\pi\xi} n!} \int_{-\infty}^{\infty} f^{(n)}(\gamma) t^n e^{-t^2/\xi} dt. \tag{3.7}$$

Hence

$$\Delta_\xi := \frac{1}{\xi^n} \left[(P_{r,\xi}(f;x) - f(x)) - \sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} \xi^{2m} f^{(2m)}(x) \left(\sum_{j=1}^{r} \alpha_j j^{2m} \right) \right]$$

$$= \sum_{j=1}^{r} \alpha_j \frac{j^n}{2\xi^{n+1} n!} \int_{-\infty}^{\infty} f^{(n)}(\gamma) t^n e^{-|t|/\xi} dt$$

$$= \frac{1}{2n!\xi^{n+1}} \int_{-\infty}^{\infty} \left(\sum_{j=1}^{r} \alpha_j j^n f^{(n)}(\gamma) \right) t^n e^{-|t|/\xi} dt$$

$$= \frac{1}{2n!\xi^{n+1}} \left[\int_{-\infty}^{\infty} \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} f^{(n)}(\gamma) \right) t^n e^{-|t|/\xi} dt \right], \tag{3.8}$$

and

$$
\begin{aligned}
\Delta_\xi^* \quad : \quad &= \frac{1}{\xi^{\frac{n-1}{2}}} \left[(W_{r,\xi}(f;x) - f(x)) - \sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} \xi^m f^{(2m)}(x) \frac{1}{m! 2^{2m}} \left(\sum_{j=1}^{r} \alpha_j j^{2m} \right) \right] \\
&= \sum_{j=1}^{r} \alpha_j \frac{j^n}{\sqrt{\pi\xi}\xi^{\frac{n-1}{2}} n!} \int_{-\infty}^{\infty} f^{(n)}(\gamma) t^n e^{-t^2/\xi} dt \\
&= \frac{1}{\sqrt{\pi\xi}\xi^{\frac{n-1}{2}} n!} \int_{-\infty}^{\infty} \left(\sum_{j=1}^{r} \alpha_j j^n f^{(n)}(\gamma) \right) t^n e^{-t^2/\xi} dt \\
&= \frac{1}{\sqrt{\pi\xi}\xi^{\frac{n-1}{2}} n!} \left[\int_{-\infty}^{\infty} \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} f^{(n)}(\gamma) \right) t^n e^{-t^2/\xi} dt \right]. \qquad (3.9)
\end{aligned}
$$

Call

$$
\Phi_n(x,t) := \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} f^{(n)}(\gamma). \qquad (3.10)
$$

Thus

$$
\Delta_\xi = \frac{1}{2n!\xi^{n+1}} \left[\int_{-\infty}^{\infty} \Phi_n(x,t) t^n e^{-|t|/\xi} dt \right], \qquad (3.11)
$$

and

$$
\Delta_\xi^* = \frac{1}{\sqrt{\pi\xi}\xi^{\frac{n-1}{2}} n!} \left[\int_{-\infty}^{\infty} \Phi_n(x,t) t^n e^{-t^2/\xi} dt \right]. \qquad (3.12)
$$

Using Hölder's inequality we obtain

$$
\begin{aligned}
|\Delta_\xi| &\leq \frac{1}{2n!\xi^{n+1}} \left[\int_{-\infty}^{\infty} \left| \Phi_n(x,t) t^n e^{-|t|/(2\xi)} e^{-|t|/(2\xi)} \right| dt \right] \\
&\leq \frac{1}{2n!\xi^{n+1}} \left(\int_{-\infty}^{\infty} \left(\Phi_n(x,t) e^{-|t|/(2\xi)} \right)^2 dt \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \left(t^n e^{-|t|/(2\xi)} \right)^2 dt \right)^{\frac{1}{2}} \\
&= \frac{1}{2n!\xi^{n+1}} \left(\int_{-\infty}^{\infty} \Phi_n^2(x,t) e^{-|t|/\xi} dt \right)^{\frac{1}{2}} \left(2(2n)!\xi^{2n+1} \right)^{\frac{1}{2}} \\
&= \frac{\sqrt{(2n)!}}{n!} \left(\frac{1}{2\xi} \int_{-\infty}^{\infty} \Phi_n^2(x,t) e^{-|t|/\xi} dt \right)^{\frac{1}{2}} \\
&= \sqrt{\binom{2n}{n}} \left(\frac{1}{2\xi} \int_{-\infty}^{\infty} \Phi_n^2(x,t) e^{-|t|/\xi} dt \right)^{\frac{1}{2}}, \qquad (3.13)
\end{aligned}
$$

and

$$\begin{aligned}
\left|\Delta_{\xi}^{*}\right| &\leq \frac{1}{\sqrt{\pi\xi}\xi^{\frac{n-1}{2}}n!}\left[\int_{-\infty}^{\infty}\left|\Phi_n(x,t)t^n e^{-t^2/(2\xi)}e^{-t^2/(2\xi)}\right|dt\right] \\
&\leq \frac{1}{\sqrt{\pi\xi}\xi^{\frac{n-1}{2}}n!}\left(\int_{-\infty}^{\infty}\left(\Phi_n(x,t)e^{-t^2/(2\xi)}\right)^2 dt\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\left(t^n e^{-t^2/(2\xi)}\right)^2 dt\right)^{\frac{1}{2}} \\
&= \frac{1}{\xi^{\frac{n-1}{2}}n!}\left(\frac{\xi^n}{2^{2n}}\frac{(2n)!}{(n)!}\right)^{\frac{1}{2}}\left(\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}\Phi_n^2(x,t)e^{-t^2/\xi}dt\right)^{\frac{1}{2}} \\
&= \frac{\sqrt{\xi}}{n!2^n}\left(\frac{(2n)!}{n!}\right)^{\frac{1}{2}}\left(\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}\Phi_n^2(x,t)e^{-t^2/\xi}dt\right)^{\frac{1}{2}}. \qquad (3.14)
\end{aligned}$$

So far we have obtained

$$|\Delta_\xi| \leq \sqrt{\binom{2n}{n}}\left(\frac{1}{2\xi}\int_{-\infty}^{\infty}\Phi_n^2(x,t)e^{-|t|/\xi}dt\right)^{\frac{1}{2}}, \qquad (3.15)$$

and

$$\left|\Delta_{\xi}^{*}\right| \leq \frac{\sqrt{\xi}}{n!2^n}\left(\frac{(2n)!}{n!}\right)^{\frac{1}{2}}\left(\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}\Phi_n^2(x,t)e^{-t^2/\xi}dt\right)^{\frac{1}{2}}. \qquad (3.16)$$

Since we assumed that $f^{(n)}$ exists and it is bounded and it is Lebesgue measurable, we obtain

$$\left\|f^{(n)}\right\|_{\infty} < M, \text{ for some } M \geq 0.$$

Therefore

$$\begin{aligned}
|\Phi_n(x,t)| &\leq \left(\sum_{j=1}^{r}\binom{r}{j}\right)M \\
&= (2^r-1)M. \qquad (3.17)
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\frac{1}{2\xi}\int_{-\infty}^{\infty}\Phi_n^2(x,t)e^{-|t|/\xi}dt\right)^{\frac{1}{2}} &\leq (2^r-1)M\left(\frac{1}{2\xi}\int_{-\infty}^{\infty}e^{-|t|/\xi}dt\right)^{\frac{1}{2}} \\
&= (2^r-1)M, \qquad (3.18)
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}\Phi_n^2(x,t)e^{-t^2/\xi}dt\right)^{\frac{1}{2}} &\leq (2^r-1)M\left(\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}e^{-t^2/\xi}dt\right)^{\frac{1}{2}} \\
&= (2^r-1)M. \qquad (3.19)
\end{aligned}$$

Therefore

$$|\Delta_\xi| \leq (2^r-1)M\sqrt{\binom{2n}{n}} =: \lambda, \qquad (3.20)$$

and

$$\left|\Delta_{\xi}^{*}\right| \leq (2^{r}-1)\, M \frac{\sqrt{\xi}}{n!2^{n}} \left(\frac{(2n)!}{n!}\right)^{\frac{1}{2}} =: \sqrt{\xi}\lambda^{*}. \tag{3.21}$$

Consequently we get $(0 < \alpha \leq 1)$

$$\frac{1}{\xi^{n-\alpha}} \left| (P_{r,\xi}(f;x) - f(x)) - \sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} \xi^{2m} f^{(2m)}(x) \left(\sum_{j=1}^{r} \alpha_j j^{2m}\right) \right| \leq \lambda \xi^{\alpha} \to 0, \tag{3.22}$$

as $\xi \to 0+$, and

$$\frac{1}{\xi^{\frac{n-\alpha}{2}}} \left| (W_{r,\xi}(f;x) - f(x)) - \sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} \xi^{m} \frac{f^{(2m)}(x)}{m!2^{2m}} \left(\sum_{j=1}^{r} \alpha_j j^{2m}\right) \right| \leq \lambda^{*} \xi^{\frac{\alpha}{2}} \to 0, \tag{3.23}$$

as $\xi \to 0+$.

Notice that $n-1-2m \geq 0$ and also $\frac{n-1}{2} - m \geq 0$.

From the last we conclude the claims of the theorem. □

Corollary 3.2. **($n = 1$ case)** Let f such that f'exists and it is bounded and Lebesgue measurable on $\mathbb{R}$. Let $\xi \to 0+$, $0 < \alpha \leq 1$. Then

$$P_{r,\xi}(f;x) - f(x) = o\left(\xi^{1-\alpha}\right), \tag{3.24}$$

and

$$W_{r,\xi}(f;x) - f(x) = o\left(\xi^{\frac{1-\alpha}{2}}\right). \tag{3.25}$$

Proof. In Theorem 3.1, we place $n = 1$. □

Corollary 3.3. **($n = 2$ case)** Let f such that f''exists and it is bounded and Lebesgue measurable on $\mathbb{R}$. Let $\xi \to 0+$, $0 < \alpha \leq 1$. Then

$$P_{r,\xi}(f;x) - f(x) = o\left(\xi^{2-\alpha}\right), \tag{3.26}$$

and

$$W_{r,\xi}(f;x) - f(x) = o\left(\xi^{1-\frac{\alpha}{2}}\right). \tag{3.27}$$

Proof. In Theorem 3.1, we place $n = 2$. □

Corollary 3.4. **($n = 3$ case)** Let f such that $f^{(3)}$ exists and it is bounded and Lebesgue measurable on $\mathbb{R}$. Let $\xi \to 0+$, $0 < \alpha \leq 1$. Then

$$P_{r,\xi}(f;x) - f(x) = \xi^{2} f''(x) \left(\sum_{j=1}^{r} \alpha_j j^{2}\right) + o(\xi^{3-\alpha}), \tag{3.28}$$

and

$$W_{r,\xi}(f;x) - f(x) = \frac{\xi f''(x)}{4}\left(\sum_{j=1}^{r}\alpha_j j^2\right) + o(\xi^{\frac{3-\alpha}{2}}). \tag{3.29}$$

Proof. In Theorem 3.1, we place $n = 3$. □

Corollary 3.5. **($n = 4$ case)** Let f such that $f^{(14)}$ exists and it is bounded and Lebesgue measurable on $\mathbb{R}$. Let $\xi \to 0+$, $0 < \alpha \leq 1$. Then

$$P_{r,\xi}(f;x) - f(x) = \xi^2 f''(x)\left(\sum_{j=1}^{r}\alpha_j j^2\right) + o(\xi^{4-\alpha}), \tag{3.30}$$

and

$$W_{r,\xi}(f;x) - f(x) = \frac{\xi f''(x)}{4}\left(\sum_{j=1}^{r}\alpha_j j^2\right) + o(\xi^{2-\frac{\alpha}{2}}). \tag{3.31}$$

Proof. In Theorem 3.1, we place $n = 4$. □

References

[1] Anastassiou, G. A., *Basic Convergence with Rates of Smooth Picard Singular Operators,* J. Comput. Anal. Appl., **8** (2006), no. 4, 313–334.

[2] Anastassiou, G. A., *L_p Convergence with Rates of Smooth Picard Singular Operators,* J. of Computational Analysis and Applications, **8** (2006), no. 4, 313-334.

[3] Anastassiou, G. A., *Global smoothness and uniform convergence of smooth Picard singular operators,* Comput. Math. Appl., **50** (2005), no. 10-12, 1755–1766.

[4] Anastassiou, G. A., *Quantitative Approximation*, Quantitative Approximations, Chapman & Hall/CRC, Boca Raton, New York, 2001.

[5] Anastassiou, G. A., Gal, S. G., *Approximation Theory*, Birkhäuser, Boston, Berlin, 2000.

[6] Anastassiou, G. A., Gal, S., *Convergence of generalized singular integrals to the unit, multivariate case*, Applied Math. Rev., Vol. 1, World Sci. Publ. Co., Singapore, 2000, pp. 1–8.

[7] Anastassiou, G. A., Mezei, R. A., *L_p Convergence with Rates of Smooth Gauss-Weierstrass Singular Operators,* submitted 2008.

[8] Anastassiou, G. A., Mezei, R. A., *Uniform Convergence with Rates of Smooth Gauss-Weierstrass Singular Integral Operators,* submitted 2009.

[9] Anastassiou, G. A., Mezei, R. A., *Global Smoothness and Uniform Convergence of Smooth Gauss-Weierstrass Singular Operators,* submitted 2009.

[10] DeVore, R. A., Lorentz, G. G., *Constructive Approximation*, Springer-Verlag, Vol. 303, Berlin, New York, 1993.

[11] Rempulska, L., Walczak, Z., *On modified Picard and Gauss-Weierstrass singular integrals*, Ukraïn. Mat. Zh. 57 (2005), no. 11, 1577–1584; translation in Ukrainian Math. J. 57 (2005), no. 11, 1844–1852.

[12] Schumaker, L., *Spline Functions. Basic Theory*, J. Wiley & Sons, New York, 1981.

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BLENDING SURFACES ON CIRCULAR DOMAINS GENERATED BY HERMITE INTERPOLATION

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Abstract. In this article we use the univariate Hermite interpolation to construct the surfaces on circular domains. The surfaces match given circles. We study the parabolical points of these surfaces. Some examples and graphs are given. These surfaces can be used in civil engineering or in Computer Aided Geometric Design (CAGD).

1. **Introduction**

The blending surfaces have been created by Coons S.A. [7]. These surfaces match a given curve. They can be used in civil engineering (roof-surfaces for large halls) or in Computer Aided Geometric Design (CAGD).

In some previous papers there were constructed the blending surfaces on the rectangular or triangular domains (see [2]-[6]). In [1] we constructed the blending surfaces on circular domain using Lagrange interpolation. In this paper we use Hermite interpolation to get the surfaces which match the given circles. We give the explicit and the parametrical representations for these surfaces. We study the position of the parabolical points because the maximal stress holds in these points (see [3], [5], [6], [9]).

2. **Construction of the surfaces**

Let $0 = y_0 < y_1 < ... < y_{l-1} < y_l = a$, $s_j \in \mathbb{N}$, $j = \overline{0,l}$, $\alpha_{jq} \in \mathbb{R}$, $j = \overline{0,l}$, $q = \overline{1,s_j}$ and $f : [0,a] \to \mathbb{R}$ a function with the properties

$$\begin{aligned} &f(0) = h > 0, f(a) = 0,\\ &f(y_j) = h_j > 0, j = \overline{1,l-1},\\ &f^{(q)}(y_j) = \alpha_{jq}, j = \overline{0,l}, q = \overline{1,s_j}. \end{aligned} \tag{2.1}$$

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Let the univariate Hermite function

$$(H_n f)(y) = \sum_{j=0}^{l} \sum_{q=0}^{s_j} h_{jq}(y) f^{(q)}(y_j)$$

with $n = l + s_0 + ... + s_l$. The cardinal functions are given by (see [10])

$$h_{jq}(y) = \frac{(y-y_j)^q}{q!} v_j(y) \sum_{\sigma=0}^{s_j-q} \frac{(y-y_j)^\sigma}{\sigma!} \left[\frac{1}{v_j(y)}\right]_{y=y_j}^{(\sigma)}, j = \overline{0,l}, q = \overline{0,s_j}$$

where

$$v_j(y) = v(y)/(y-y_j)^{s_j+1}$$

with

$$v(y) = \prod_{j=0}^{l} (y-y_j)^{s_j+1}.$$

Taking into account (2.1) we obtain

$$(H_n f)(y) = \sum_{j=0}^{l} \sum_{q=1}^{s_j} h_{jq}(y) \alpha_{jq} + \sum_{j=1}^{l-1} h_{j0}(y) h_j + h_{00}(y) h. \tag{2.2}$$

The function (2.2) has the properties

$$\begin{aligned} &(H_n f)(0) = h, (H_n f)(a) = 0, \\ &(H_n f)(y_j) = h_j, j = \overline{1, l-1}, \\ &(H_n f)^{(q)}(y_j) = \alpha_{jq}, j = \overline{0,l}, q = \overline{1, s_j}. \end{aligned}$$

Let $D = \{(X,Y) \in \mathbb{R}^2 | X^2 + Y^2 \le a^2\}$ and $C_j = \{(X,Y) \in \mathbb{R}^2 | X^2 + Y^2 = y_j^2\}$, $j = \overline{1, n-1}$ in the XOY plane.

If we make the substitution

$$y = \sqrt{X^2 + Y^2} \tag{2.3}$$

in (2.2), we obtain the surfaces

$$\widetilde{F}(X,Y) = \sum_{j=0}^{l} \sum_{q=1}^{s_j} h_{jq}(\sqrt{X^2+Y^2}) \alpha_{jq} + \sum_{j=1}^{l-1} h_{j0}(\sqrt{X^2+Y^2}) h_j + \tag{2.4}$$

$$+ h_{00}(\sqrt{X^2+Y^2}) h, \quad X^2 + Y^2 \le a^2.$$

The surfaces (2.4) have the properties

$$\widetilde{F}|_{\partial D} = 0,$$

$$\widetilde{F}|_{C_j} = h_j, j = \overline{1, n-1},$$

$$\widetilde{F}(0,0) = h.$$

It follows that the surfaces $\widetilde{F}$ match the circle $X^2+Y^2=a^2, Z=0$ (the surfaces are staying on the border of domain D), the circles $X^2+Y^2=y_j^2, Z=h_j$ for $j=\overline{1,n-1}$ and the height of the surfaces in the center of domain D is h.

We can give a parametrical representation for the surfaces

$$\begin{cases} X = u\cos v \\ Y = u\sin v \\ Z = \sum\limits_{j=0}^{l}\sum\limits_{q=1}^{s_j} h_{jq}(u)\alpha_{jq} + \sum\limits_{j=1}^{l-1} h_{j0}(u)h_j + h_{00}(u)h \end{cases} \quad u\in[0,a], v\in[0,2\pi].$$

3. **Parabolical points**

If we take $\alpha_{00}=h, \alpha_{j0}=h_j$, $j=\overline{1,l-1}, \alpha_{l0}=0$ we obtain

$$\widetilde{F}(X,Y)=\sum_{j=0}^{l}\sum_{q=0}^{s_j} h_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}.$$

The parabolical points of the surfaces $\widetilde{F}$ satisfy the condition

$$\widetilde{F}_{XX}(X,Y)\widetilde{F}_{YY}(X,Y)-(\widetilde{F}_{XY}(X,Y))^2=0. \tag{3.1}$$

The second partial derivatives of the function $\widetilde{F}$ can be expressed by

$$\widetilde{F}_{XX}(X,Y)=\frac{X^2}{X^2+Y^2}\sum_{j=0}^{l}\sum_{q=0}^{s_j} h''_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}+ \tag{3.2}$$

$$+\frac{Y^2}{(X^2+Y^2)^{\frac{3}{2}}}\sum_{j=0}^{l}\sum_{q=0}^{s_j} h'_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}$$

$$\widetilde{F}_{XY}(X,Y)=\frac{XY}{X^2+Y^2}\sum_{j=0}^{l}\sum_{q=0}^{s_j} h''_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}- \tag{3.3}$$

$$-\frac{XY}{(X^2+Y^2)^{\frac{3}{2}}}\sum_{j=0}^{l}\sum_{q=0}^{s_j} h'_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}$$

$$\widetilde{F}_{YY}(X,Y)=\frac{Y^2}{X^2+Y^2}\sum_{j=0}^{l}\sum_{q=0}^{s_j} h''_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}+ \tag{3.4}$$

$$+\frac{X^2}{(X^2+Y^2)^{\frac{3}{2}}}\sum_{j=0}^{l}\sum_{q=0}^{s_j} h'_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}$$

if $(X,Y)\neq(0,0)$. From (3.1) and (3.2)-(3.4) we obtain

$$\frac{1}{\sqrt{X^2+Y^2}}\left(\sum_{j=0}^{l}\sum_{q=0}^{s_j} h'_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}\right)\left(\sum_{j=0}^{l}\sum_{q=0}^{s_j} h''_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}\right)=0.$$

We consider the following two polynomial equations with unknown y

$$\sum_{j=0}^{l}\sum_{q=0}^{s_j} h'_{jq}(y)\alpha_{jq} = 0, \tag{3.5}$$

$$\sum_{j=0}^{l}\sum_{q=0}^{s_j} h''_{jq}(y)\alpha_{jq} = 0. \tag{3.6}$$

If the equations have no roots in $[0, a]$ then the surfaces generated by function $\widetilde{F}$ have no parabolical points. If $y = \overline{y} \in (0, a]$ is a solution for one of the equations (3.5) or (3.6), the surfaces have parabolical points of which projections on XOY plane are the circle $X^2 + Y^2 = \overline{y}^2$. If $\overline{y} = 0$ then the point $(0, 0, h)$ is the parabolical point of the surfaces if

$$\lim_{X,Y\to 0} \frac{1}{\sqrt{X^2+Y^2}} \left(\sum_{j=0}^{l}\sum_{q=0}^{s_j} h'_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}\right)\left(\sum_{j=0}^{l}\sum_{q=0}^{s_j} h''_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}\right) \tag{3.7}$$

is zero. It is important to study the position of the parabolical points. The maximal stress holds in these points. The surfaces are more resistent if the projection of parabolical points are situated as closed as possible to the border, if possible just on the border or outside of domain (see [6], [9]). We can give conditions on parameters α_{ij} to control the position of the parabolical points.

4. **Examples**

We study the parabolical points in two particular cases.

First, we take $l = 1$, $s_0 = 1$, $s_1 = 0$. We have

$$h_{00}(y) = -\frac{y^2}{a^2} + 1, h_{01}(y) = -\frac{y^2}{a} + y.$$

From (2.4), it follows that the surface is given by

$$\widetilde{F}_1(X,Y) = \left(-\frac{X^2+Y^2}{a} + \sqrt{X^2+Y^2}\right)\alpha_{01} + \left(-\frac{X^2+Y^2}{a^2} + 1\right)h, X^2+Y^2 \le a^2$$

It has the properties

$$\widetilde{F}_1|_{\partial D} = 0, \widetilde{F}_1(0,0) = h.$$

The equations (3.5) and (3.6) become

$$\left(-\frac{2}{a^2}h - \frac{2}{a}\alpha_{01}\right)y + \alpha_{01} = 0 \tag{4.1}$$

$$-\frac{2}{a^2}h - \frac{2}{a}\alpha_{01} = 0 \tag{4.2}$$

Let $\overline{y} = \frac{a^2\alpha_{01}}{2(h+a\alpha_{01})}$. We have the following cases:

i) If $\alpha_{01} = -\frac{h}{a}$ then the relation (4.2) holds. It follows that all the points of the surface are parabolical points.

ii) If $\alpha_{01} < -\frac{2h}{a}$ or $\alpha_{01} > 0$ the equation (4.1) has the solution $y = \overline{y}$ in interval $(0, a)$. It follows that the surface has parabolical points of which projections are situated on a circle, inside of domain D.

iii) If $\alpha_{01} = -\frac{2h}{a}$ then $y = a$ is the solution of the equation (4.1). The parabolical points are on the border of domain D.

iv) If $\alpha_{01} \in (-2\frac{h}{a}, -\frac{h}{a})$ then the equation (4.1) has the solution $y = \overline{y} > a$. The surface has no parabolical points.

v) If $\alpha_{01} \in (-\frac{h}{a}, 0)$ the equation (4.1) has the solution $y = \overline{y} < 0$. The surface has no parabolical points.

vi) If $\alpha_{01} = 0$ then $y = 0$ is the solution of the equation (4.1). The limit (3.7) is different to zero. It follows that the surface has no parabolical points.

In Figure 1 we plot the surface $\widetilde{F}_1$ for $a = 3$, $h = 5$ and some values of α_{01}.

In second case, we take $l = 1$, $s_0 = 0$, $s_1 = 1$. We have

$$h_{00}(y) = \frac{y^2}{a^2} - 2\frac{y}{a} + 1, h_{11}(y) = \frac{y^2}{a} - y.$$

From (2.4) we get the surface

$$\widetilde{F}_2(X,Y) = \left(\frac{X^2+Y^2}{a} - \sqrt{X^2+Y^2}\right)\alpha_{11} + \left(\frac{X^2+Y^2}{a^2} - 2\frac{\sqrt{X^2+Y^2}}{a} + 1\right)h,$$

$$X^2 + Y^2 \leq a^2$$

The surface has the properties

$$\widetilde{F}_2|_{\partial D} = 0, \widetilde{F}_2(0,0) = h.$$

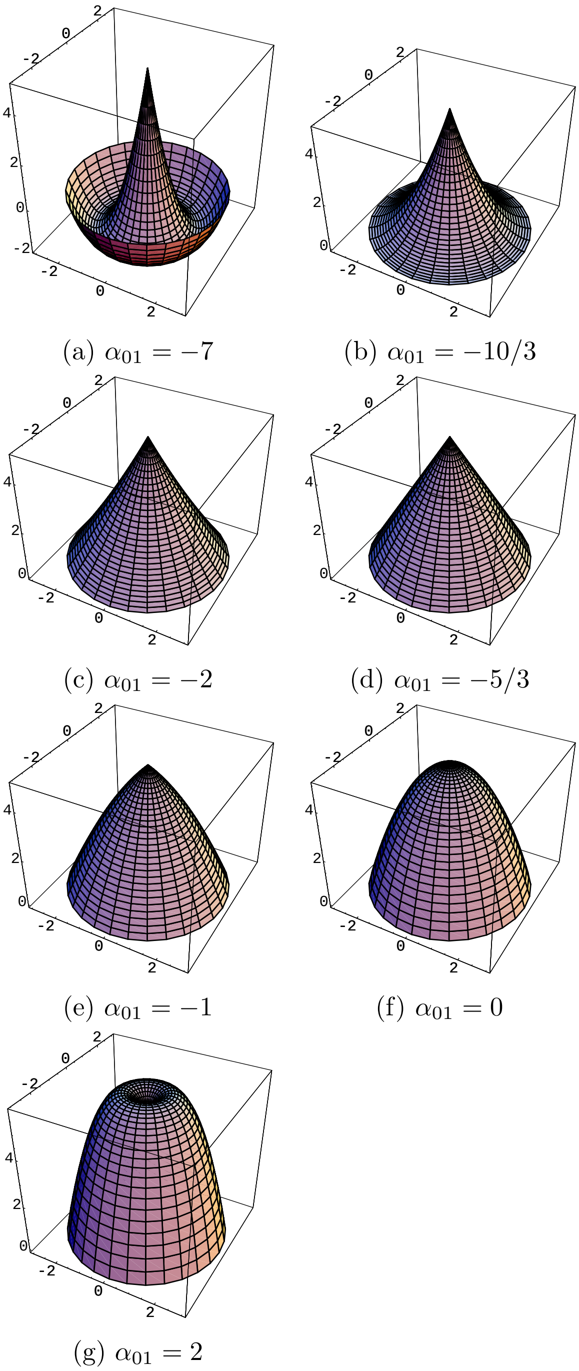
The equations (3.5) and (3.6) become

$$\left(\frac{2}{a^2}h + \frac{2}{a}\alpha_{11}\right)y - \frac{2}{a}h - \alpha_{11} = 0 \tag{4.3}$$

$$\frac{2}{a^2}h + \frac{2}{a}\alpha_{11} = 0 \tag{4.4}$$

Let $\overline{y} = \frac{a(2h+a\alpha_{11})}{2(h+a\alpha_{11})}$. We have the following cases:

i) If $\alpha_{11} = -\frac{h}{a}$ then the relation (4.4) holds. It follows that all the points of the surface are parabolical points.



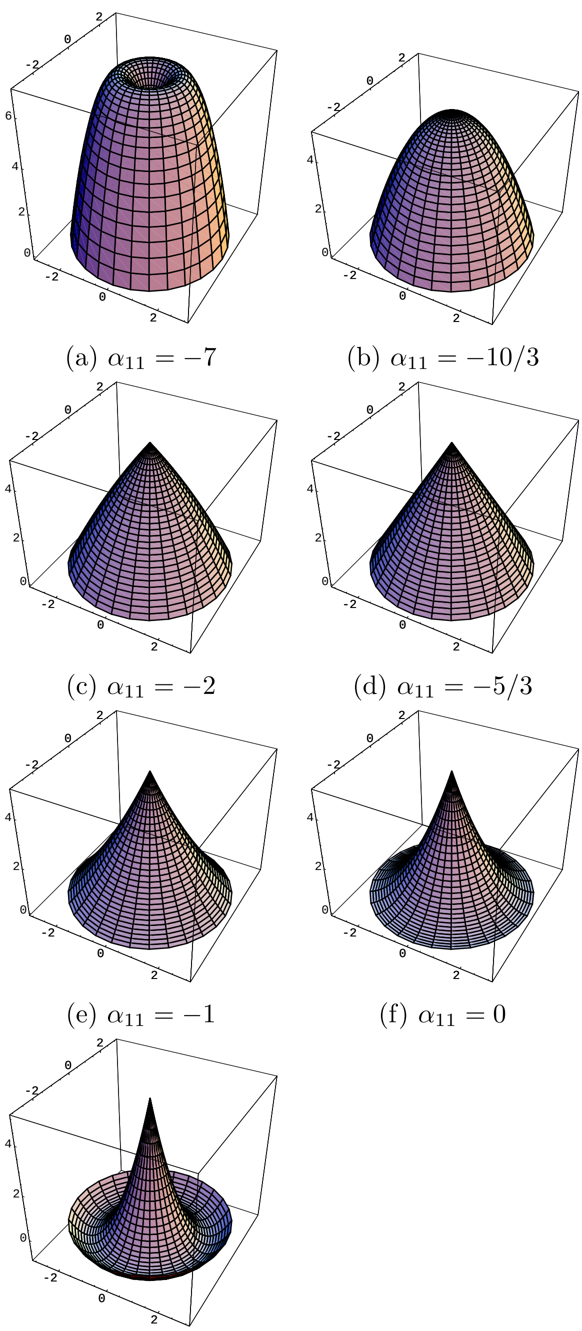
(a) $\alpha_{01} = -7$ (b) $\alpha_{01} = -10/3$

(c) $\alpha_{01} = -2$ (d) $\alpha_{01} = -5/3$

(e) $\alpha_{01} = -1$ (f) $\alpha_{01} = 0$

(g) $\alpha_{01} = 2$

FIGURE 1. The surface $\widetilde{F}_1$ for $a = 3$ and $h = 5$.



(a) $\alpha_{11} = -7$ (b) $\alpha_{11} = -10/3$

(c) $\alpha_{11} = -2$ (d) $\alpha_{11} = -5/3$

(e) $\alpha_{11} = -1$ (f) $\alpha_{11} = 0$

(g) $\alpha_{11} = 2$

FIGURE 2. The surface $\widetilde{F}_2$ for $a = 3$ and $h = 5$.

ii) If $\alpha_{11} < -\frac{2h}{a}$ or $\alpha_{01} > 0$ the equation (4.3) has the solution $y = \overline{y}$ in interval $(0, a)$. The surface has parabolical points of which projections are situated on a circle, inside of domain D.

iii) If $\alpha_{11} = -\frac{2h}{a}$ then $y = 0$ is solution of the equation (4.3). The limit (3.7) is different to zero. It follows that the surface has no parabolical points.

iv) If $\alpha_{11} \in (-2\frac{h}{a}, -\frac{h}{a})$ then the equation (4.3) has the solution $y = \overline{y} < 0$. It follows that the surface has no parabolical points.

v) If $\alpha_{11} \in (-\frac{h}{a}, 0)$ the equation (4.3) has the solution $y = \overline{y} > a$. The surface has no parabolical points.

vi) If $\alpha_{11} = 0$ then $y = a$ is the solution of the equation (4.3). The parabolical points of the surface are on the border of domain D.

In Figure 2 we plot the surface $\widetilde{F}_2$ for $a = 3$, $h = 5$ and some values of α_{11}.

References

[1] Birou, M., *Some surfaces generated by blending interpolation*, submitted.

[2] Coman, Gh., Gânscă, I., Tâmbulea, L., *Some practical application of blending interpolation*, Itinerant Seminar on Functional Equations, Approximation and Convexity, Univ. Babeş-Bolyai, Cluj-Napoca, 1989, Preprint 89-6, 5-22.

[3] Coman, Gh., Gânscă, I., Tâmbulea, L., *Some new roof-surfaces generated by blending interpolation technique*, Studia Univ. Babeş-Bolyai Math., **36** (1991), No. 1, 119-130.

[4] Coman, Gh., Gânscă, I, Tâmbulea, L., *Surfaces generated by blending interpolation*, Studia Univ. Babeş-Bolyai Math., **38** (1993), No. 3, 39-48.

[5] Coman, Gh., Tâmbulea, L., Gânscă, I., *Multivariate approximation*, Seminar on Numerical and Statistical Calculus, Babeş-Bolyai Univ., Cluj-Napoca, 1996, Preprint 96-1, 29-60.

[6] Gânscă, I., Coman, Gh., *Aproximare blending cu aplicaţii in construcţii*, Bul. Stiint. Inst. Politehn. Cluj-Napoca Ser. Electrotehn. Energet. Inform., **24** (1981), 35-40.

[7] Coons, S. A., *Surface for computer aided design of space forms*, Project MAC, Design Div., Dep. of Mech. Engineering, MIT, 1964.

[8] Farin, G., *Curves and surfaces for Computer Aided Geometric Design*, Academic Press, Inc Harcourt Brace Jovanovich, Publishers, Boston, San Diego, New York, London, Sidney, Tokyo, Toronto, 1990.

[9] Mihailescu, M., Horvath, I., *Velaroidal shells for covering universal industrial halls*, Acta Tech. Acad. Hungaricae, **85** (1977), 35-145.

[10] Stancu, D. D., *Asupra formulei de interpolare a lui Hermite si a unor aplicatii ale acesteia*, Stud. si Cerc. Matematice, Cluj-Napoca (1957), No. 8, 329-335.

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APPROXIMATION AND SHAPE PRESERVING PROPERTIES OF THE NONLINEAR BASKAKOV OPERATOR OF MAX-PRODUCT KIND

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Abstract. Starting from the study of the *Shepard nonlinear operator of max-prod type* in [2], [3], in the recent monograph [5], Open Problem 5.5.4, pp. 324-326, the *Baskakov max-prod type operator* is introduced and the question of the approximation order by this operator is raised. The aim of this note is to obtain for the discussed operator an upper pointwise estimate of the approximation error of the form $C\omega_1(f;\sqrt{\frac{x(1+x)}{n}})$ (with the explicit constant $C = 12$) and to prove by a counterexample that in some sense, for arbitrary f this type of order of approximation with respect to $\omega_1(f;)$ cannot be improved. However, for some subclasses of functions including for example the nondecreasing concave functions, the essentially better order of approximation $\omega_1(f;\frac{x+1}{n})$ is obtained. Finally, some shape preserving properties are proved.

1. **Introduction**

Starting from the study of the *Shepard nonlinear operator of max-prod type* in [2], [3], by the Open Problem 5.5.4, pp. 324-326 in the recent monograph [5], the following *nonlinear Baskakov operator of max-prod type* is introduced (here $\bigvee$ means maximum)

$$V_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{\infty} b_{n,k}(x)},$$

where $b_{n,k}(x) = \binom{n+k-1}{k} x^k/(1+x)^{n+k}$.

The aim of this note is to obtain for the discussed operator an upper pointwise estimate of the approximation error of the form $C\omega_1(f;\sqrt{\frac{x(1+x)}{n}})$ (with the explicit

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constant $C = 12$) Also, one proves by a counterexample that in some sense, in general this type of order of approximation with respect to $\omega_1(f;\cdot)$ cannot be improved. However, for some subclasses of functions, including for example the bounded, nondecreasing concave functions, the essentially better order $\omega_1(f;(x+1)/n)$ is obtained. This allows us to put in evidence large classes of functions (e.g. bounded, nondecreasing concave polygonal lines on $[0,\infty)$) for which the order of approximation given by the max-product Baskakov operator, is essentially better than the order given by the linear Baskakov operator. Finally, some shape preserving properties are presented.

Section 2 presents some general results on nonlinear operators, in Section 3 we prove several auxiliary lemmas, Section 4 contains the approximation results, while in Section 5 we present some shape preserving properties.

2. **Preliminaries**

For the proof of the main result we need some general considerations on the so-called nonlinear operators of max-prod kind. Over the set of positive reals, $\mathbb{R}_+$, we consider the operations $\vee$ (maximum) and $\cdot$, product. Then $(\mathbb{R}_+,\vee,\cdot)$ has a semiring structure and we call it as Max-Product algebra.

Let $I\subset\mathbb{R}$ be a bounded or unbounded interval, and

$$CB_+(I)=\{f:I\to\mathbb{R}_+; f \text{ continuous and bounded on } I\}.$$

The general form of $L_n: CB_+(I)\to CB_+(I)$, (called here a discrete max-product type approximation operator) studied in the paper will be

$$L_n(f)(x)=\bigvee_{i=0}^{n}K_{n,i}(x)\cdot f(x_{n,i}),$$

or

$$L_n(f)(x)=\bigvee_{i=0}^{\infty}K_{n,i}(x)\cdot f(x_{n,i}),$$

where $n\in\mathbb{N}$, $f\in CB_+(I)$, $K_{n,i}\in CB_+(I)$ and $x_i\in I$, for all i. These operators are nonlinear, positive operators and moreover they satisfy a pseudo-linearity condition of the form

$$L_n(\alpha\cdot f\vee\beta\cdot g)(x)=\alpha\cdot L_n(f)(x)\vee\beta\cdot L_n(g)(x),\forall\alpha,\beta\in\mathbb{R}_+, f,g\in CB_+(I).$$

In this section we present some general results on these kinds of operators which will be useful later in the study of the Baskakov max-product kind operator considered in Introduction.

Lemma 2.1. ([1]) *Let $I\subset\mathbb{R}$ be a bounded or unbounded interval,*

$$CB_+(I)=\{f:I\to\mathbb{R}_+; f \text{ continuous and bounded on } I\},$$

and $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$ *be a sequence of operators satisfying the following properties :*

(i) If $f, g \in CB_+(I)$ *satisfy* $f \le g$ *then* $L_n(f) \le L_n(g)$ *for all* $n \in N$ *;*

(ii) $L_n(f+g) \le L_n(f) + L_n(g)$ *for all* $f, g \in CB_+(I)$.

Then for all $f, g \in CB_+(I)$, $n \in N$ *and* $x \in I$ *we have*

$$|L_n(f)(x) - L_n(g)(x)| \le L_n(|f-g|)(x).$$

Proof. Since is very simple, we reproduce here the proof in [1]. Let $f, g \in CB_+(I)$. We have $f = f - g + g \le |f-g| + g$, which by the conditions $(i)-(ii)$ successively implies $L_n(f)(x) \le L_n(|f-g|)(x) + L_n(g)(x)$, that is $L_n(f)(x) - L_n(g)(x) \le L_n(|f-g|)(x)$.

Writing now $g = g - f + f \le |f-g| + f$ and applying the above reasonings, it follows $L_n(g)(x) - L_n(f)(x) \le L_n(|f-g|)(x)$, which combined with the above inequality gives $|L_n(f)(x) - L_n(g)(x)| \le L_n(|f-g|)(x)$. □

Remarks. 1) It is easy to see that the Baskakov max-product operator satisfy the conditions (i) and (ii) in Lemma 2.1. In fact, instead of (i) it satisfies the stronger condition

$$L_n(f \vee g)(x) = L_n(f)(x) \vee L_n(g)(x),\ f, g \in CB_+(I).$$

Indeed, taking in the above equality $f \le g$, $f, g \in CB_+(I)$, it easily follows $L_n(f)(x) \le L_n(g)(x)$.

2) In addition, it is immediate that the Baskakov max-product operator is positive homogeneous, that is $L_n(\lambda f) = \lambda L_n(f)$ for all $\lambda \ge 0$.

Corollary 2.2. ([1]) *Let* $L_n : CB_+(I) \to CB_+(I)$, $n \in N$ *be a sequence of operators satisfying the conditions (i)-(ii) in Lemma 2.1 and in addition being positive homogenous. Then for all* $f \in CB_+(I)$, $n \in N$ *and* $x \in I$ *we have*

$$|f(x) - L_n(f)(x)| \le \left[\frac{1}{\delta} L_n(\varphi_x)(x) + L_n(e_0)(x)\right] \omega_1(f;\delta)_I + f(x) \cdot |L_n(e_0)(x) - 1|,$$

where $\delta > 0$, $e_0(t) = 1$ *for all* $t \in I$, $\varphi_x(t) = |t - x|$ *for all* $t \in I$, $x \in I$, $\omega_1(f;\delta)_I = \max\{|f(x) - f(y)|; x, y \in I, |x - y| \le \delta\}$ *and if* I *is unbounded then we suppose that there exists* $L_n(\varphi_x)(x) \in \mathbb{R}_+ \bigcup \{+\infty\}$, *for any* $x \in I$, $n \in \mathbb{N}$.

Proof. The proof is identical with that for positive linear operators and because of its simplicity we reproduce it below. Indeed, from the identity

$$L_n(f)(x) - f(x) = [L_n(f)(x) - f(x) \cdot L_n(e_0)(x)] + f(x)[L_n(e_0)(x) - 1],$$

it follows (by the positive homogeneity and by Lemma 2.1)

$$|f(x) - L_n(f)(x)| \le |L_n(f(x))(x) - L_n(f(t))(x)| + |f(x)| \cdot |L_n(e_0)(x) - 1| \le$$

$$L_n(|f(t) - f(x)|)(x) + |f(x)| \cdot |L_n(e_0)(x) - 1|.$$

Now, since for all $t, x \in I$ we have

$$|f(t) - f(x)| \leq \omega_1(f; |t-x|)_I \leq \left[\frac{1}{\delta}|t-x| + 1\right] \omega_1(f;\delta)_I,$$

replacing above we immediately obtain the estimate in the statement. □

An immediate consequence of Corollary 2.2 is the following.

Corollary 2.3. ([1]) *Suppose that in addition to the conditions in Corollary 2.2, the sequence* $(L_n)_n$ *satisfies* $L_n(e_0) = e_0$*, for all* $n \in N$*. Then for all* $f \in CB_+(I)$*,* $n \in N$ *and* $x \in I$ *we have*

$$|f(x) - L_n(f)(x)| \leq \left[1 + \frac{1}{\delta} L_n(\varphi_x)(x)\right] \omega_1(f;\delta)_I.$$

The nonlinear max-product Baskakov operator satisfies for all $n \in \mathbb{N}$, $n \geq 2$ all the hypothesis in the Lemma 2.1, Corollaries 2.2 and 2.3 as can be seen from the following considerations.

Lemma 2.4. *Let* $n \in \mathbb{N}$*,* $n \geq 2$*. We have*

$$\bigvee_{k=0}^{\infty} b_{n,k}(x) = b_{n,j}(x), \text{ for all } x \in \left[\frac{j}{n-1}, \frac{j+1}{n-1}\right], \ j = 0, 1, 2, ...$$

Proof. First we show that for fixed $n \in \mathbb{N}, n \geq 2$ and $0 \leq k < k+1$ we have

$$0 \leq b_{n,k+1}(x) \leq b_{n,k}(x), \text{ if and only if } x \in [0, (k+1)/(n-1)].$$

Indeed, the inequality one reduces to

$$0 \leq \binom{n+k}{k+1} \frac{x^{k+1}}{(1+x)^{n+k+1}} \leq \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}},$$

which after simple calculus is obviously equivalent to

$$0 \leq x \leq \frac{k+1}{n-1}.$$

By taking $k = 0, 1, 2, ..$ in the inequality just proved above, we get

$$b_{n,1}(x) \leq b_{n,0}(x), \text{ if and only if } x \in [0, 1/(n-1)],$$

$$b_{n,2}(x) \leq b_{n,1}(x), \text{ if and only if } x \in [0, 2/(n-1)],$$

$$b_{n,3}(x) \leq b_{n,2}(x), \text{ if and only if } x \in [0, 3/(n-1)],$$

so on,

$$b_{n,k+1}(x) \leq b_{n,k}(x), \text{ if and only if } x \in [0, (k+1)/(n-1)],$$

and so on.

From all these inequalities, reasoning by recurrence we easily obtain :

$$\text{if } x \in [0, 1/(n-1)] \text{ then } b_{n,k}(x) \leq b_{n,0}(x), \text{ for all } k = 0, 1, 2, ...$$

$$\text{if } x \in [1/(n-1), 2/(n-1)] \text{ then } b_{n,k}(x) \le b_{n,1}(x), \text{ for all } k = 0, 1, 2, ...$$

$$\text{if } x \in [2/(n-1), 3/(n-1)] \text{ then } b_{n,k}(x) \le b_{n,2}(x), \text{ for all } k = 0, 1, 2, ...$$

and so on, in general

$$\text{if } x \in [j/(n-1), (j+1)/(n-1)] \text{ then } b_{n,k}(x) \le b_{n,j}(x), \text{ for all } k = 0, 1, 2, ...$$

which proves the lemma. □

In what follows we need some notations.

For each $n \in \mathbb{N}$, $n \ge 2$, $k, j \in \{0, 1, 2, ..., \}$, and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$, $x > 0$ let us denote

$$m_{k,n,j}(x) = \frac{b_{n,k}(x)}{b_{n,j}(x)} = \frac{\binom{n+k-1}{k}}{\binom{n+j-1}{j}} \left(\frac{x}{1+x}\right)^{k-j}.$$

and for $x = 0$ let us denote $m_{0,n,0}(x) = 1$ and $m_{k,n,0}(x) = 0$ for all $k \in \{1, 2, ..., \}$.

Also, for any $n \in \mathbb{N}$, $n \ge 2$, $k \in \{0, 1, .., \}$ and $j \in \{0, 1, .., \}$, let us define the functions $f_{k,n,j} : [\frac{j}{n-1}, \frac{j+1}{n-1}] \to \mathbb{R}$,

$$f_{k,n,j}(x) = m_{k,n,j}(x) f\left(\frac{k}{n}\right) = \frac{\binom{n+k-1}{k}}{\binom{n+j-1}{j}} \left(\frac{x}{1+x}\right)^{k-j} f\left(\frac{k}{n}\right).$$

From Lemma 2.4, it follows that for each $j \in \{0, 1, ..., \}$ and for all $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ we can write

$$V_n^{(M)}(f)(x) = \bigvee_{k=0}^{\infty} f_{k,n,j}(x).$$

Lemma 2.5. *Let $n \in \mathbb{N}$, $n \ge 2$. For all $k, j \in \{0, 1, 2, ...\}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ we have*

$$m_{k,n,j}(x) \le 1.$$

Proof. Let $j \in \{0, 1, ..., \}$ and let $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. By Lemma 2.4, it immediately follows that

$$m_{k,n,j}(x) \le m_{j,n,j}(x).$$

Since $m_{j,n,j}(x) = 1$, the conclusion of the lemma is immediate. □

Lemma 2.6. *For any arbitrary bounded function $f : [0, \infty) \to \mathbb{R}_+$, $V_n^{(M)}(f)$ is positive, bounded, continuous and satisfies $V_n^{(M)}(f)(0) = f(0)$, for all $n \in \mathbb{N}$, $n \ge 3$.*

Proof. The positivity of $V_n^{(M)}(f)$ is immediate. Also, taking into account that $b_{n,0}(0) = 1$ and $b_{n,k}(0) = 0$ for all $k \in \{1, 2, ..., \}$ we immediately obtain that $V_n^{(M)}(f)(0) = f(0)$.

If f is bounded then let $M \in \mathbb{R}_+$ be such that $f(x) \leq M$ for all $x \in [0, \infty)$. Let $x \in [0, \infty)$ and let $j \in \{0, 1, ...\}$ be such that $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. Then

$$V_n^{(M)}(f)(x) = \bigvee_{k=0}^{\infty} f_{k,n,j}(x) = \bigvee_{k=0}^{\infty} m_{k,n,j}(x) f\left(\frac{k}{n}\right).$$

Since by Lemma 2.5. we have $m_{k,n,j}(x) \leq 1$ for all $k \in \{0, 1, ..., \}$ and since $f\left(\frac{k}{n}\right) \leq M$ for all $k \in \{0, 1, ..., \}$, it is immediate that $V_n^{(M)}(f)(x) \leq M$.

With respect to continuity, it suffices to prove that on each subinterval of the form $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, with $j \in \{0, 1, ...\}$, $V_n^{(M)}(f)$ is continuous. For this purpose, for $j \in \{0, 1, ...\}$ fixed and for any $l \in \mathbb{N}$ let us define the function $g_{l,j} : [\frac{j}{n-1}, \frac{j+1}{n-1}] \to \mathbb{R}_+$, $g_{l,j}(x) = \bigvee_{k=0}^{l} f_{k,n,j}(x)$. It is clear that for each $l \in \mathbb{N}$ the function $g_{l,j}$ is continuous on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, as a maximum of finite number of continuous functions. Since, for all $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ we have

$$0 \leq V_n^{(M)}(f)(x) = \max\left\{\bigvee_{k=0}^{l} f_{k,n,j}(x), \bigvee_{k=l+1}^{\infty} f_{k,n,j}(x)\right\}$$

$$\leq \bigvee_{k=0}^{l} f_{k,n,j}(x) + \bigvee_{k=l+1}^{\infty} f_{k,n,j}(x),$$

it follows that for all $l \in \mathbb{N}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ we have

$$0 \leq V_n^{(M)}(f)(x) - g_{l,j}(x) \leq \bigvee_{k=l+1}^{\infty} f_{k,n,j}(x) = \bigvee_{k=l+1}^{\infty} m_{k,n,j}(x) f\left(\frac{k}{n}\right)$$

$$\leq M \bigvee_{k=l+1}^{\infty} m_{k,n,j}(x).$$

For $l \geq j$, by the proof Lemma 2.4 it follows that

$$m_{l,n,j}(x) \geq m_{l+1,n,j}(x) \geq m_{l+2,n,j}(x) \geq ....$$

Also, for $l \geq j$ it is easy to prove that $m_{l,n,j}(x) \leq m_{l,n,j}(\frac{j+1}{n-1})$ for all $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. From all these reasonings it follows that

$$0 \leq V_n^{(M)}(f)(x) - g_{l,j}(x) \leq M m_{l+1,n,j}(\frac{j+1}{n-1})$$

for all $l \geq j$. Let us consider the sequence $(a_l)_{l \geq j}$, $a_l = M m_{l+1,n,j}(\frac{j+1}{n})$. By simple calculus we get $\lim_{l \to \infty} \frac{a_{l+1}}{a_l} = \lim_{l \to \infty} \left(\frac{(n+l+1)(j+1)}{(l+2)(n+j)}\right) = \frac{j+1}{n+j} < 1$, which immediately implies that $\lim_{l \to \infty} a_l = 0$. This implies that $V_n^{(M)}(f)$ is the uniform limit of a sequence of continuous functions on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, $g_{l,j}$, $l \in \mathbb{N}$, which implies the continuity of $V_n^{(M)}(f)$ on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$. □

Remark. From Lemmas 2.4-2.6, it is clear that $V_n^{(M)}(f)$ satisfies all the conditions in Lemma 2.1, Corollary 2.2 and Corollary 2.3 for $I = [0, \infty)$.

3. **Auxiliary Results**

Remark. Note since by Lemma 2.6 we have $V_n^{(M)}(f)(0) = f(0)$ for all $n \geq 3$, notice that in the notations, proofs and statements of the all approximation results, that is in Lemmas 3.1, 3.2, Theorem 4.1, Lemma 4.2, Corollaries 4.4, 4.5, in fact we always may suppose that $x > 0$.

For each $n \in \mathbb{N}$, $n \geq 3$, $k, j \in \{0, 1, 2, ...,\}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$, let us denote

$$M_{k,n,j}(x) = m_{k,n,j}(x) \left| \frac{k}{n} - x \right|.$$

It is clear that if $k \geq \frac{n}{n-1}(j+1)$ then

$$M_{k,n,j}(x) = m_{k,n,j}(x)(\frac{k}{n} - x)$$

and if $k \leq \frac{n}{n-1}j$ then

$$M_{k,n,j}(x) = m_{k,n,j}(x)(x - \frac{k}{n}).$$

Also, for each $n \in \mathbb{N}$, $n \geq 3$, $k, j \in \mathbb{N}$, $k \geq \frac{n}{n-1}(j+1)$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ let us denote

$$\overline{M}_{k,n,j}(x) = m_{k,n,j}(x)(\frac{k}{n-1} - x)$$

and for each $n \in \mathbb{N}$, $n \geq 3$, $k, j \in \mathbb{N}$, $k \leq \frac{n}{n+1}j$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ let us denote

$$\underline{M}_{k,n,j}(x) = m_{k,n,j}(x)(x - \frac{k}{n-1}).$$

Lemma 3.1. *Let $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ and $n \in \mathbb{N}$, $n \geq 3$.*

(i) For all $k, j \in \{0, 1, 2, ...,\}$ with $k \geq \frac{n}{n-1}(j+1)$ we have

$$M_{k,n,j}(x) \leq \overline{M}_{k,n,j}(x).$$

(ii) For all $k, j \in \mathbb{N}$, $k \geq \frac{n}{n-2}(j+1)$ we have

$$\overline{M}_{k,n,j}(x) \leq 2M_{k,n,j}(x).$$

(iii) For all $k, j \in \mathbb{N}$, $k \leq \frac{n}{n+1}j$ we have

$$\underline{M}_{k,n,j}(x) \leq M_{k,n,j}(x) \leq 2\underline{M}_{k,n,j}(x).$$

Proof. (i) The inequality $M_{k,n,j}(x) \leq \overline{M}_{k,n,j}(x)$ is immediate.

(ii) Since the function $h(x) = \frac{\frac{k}{n-1}-x}{\frac{k}{n}-x}$ is nondecreasing on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$ we get

$$\frac{\overline{M}_{k,n,j}(x)}{M_{k,n,j}(x)} = \frac{\frac{k}{n-1} - x}{\frac{k}{n} - x} \leq \frac{\frac{k}{n-1} - \frac{j+1}{n-1}}{\frac{k}{n} - \frac{j+1}{n-1}} = \frac{n(k-j-1)}{n(k-j-1)-k}.$$

We have

$$\begin{aligned} \frac{n(k-j-1)}{n(k-j-1)-k} &\leq 2 \Leftrightarrow n(k-j-1) \leq 2n(k-j-1) - 2k \\ &\Leftrightarrow 2k \leq n(k-j-1) \Leftrightarrow n(j+1) \leq k(n-2) \\ &\Leftrightarrow k \geq \frac{n}{n-2}(j+1). \end{aligned}$$

which proves (ii).

(iii) The inequality $\underline{M}_{k,n,j}(x) \leq M_{k,n,j}(x)$ is immediate.

On the other hand, tacking account of the fact that the function $h(x) = \frac{x-\frac{k}{n}}{x-\frac{k}{n-1}}$ is nonincreasing on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$ we get

$$\frac{M_{k,n,j}(x)}{\underline{M}_{k,n,j}(x)} = \frac{x - \frac{k}{n}}{x - \frac{k}{n-1}} \leq \frac{\frac{j}{n-1} - \frac{k}{n}}{\frac{j}{n-1} - \frac{k}{n-1}} = \frac{n(j-k)+k}{n(j-k)}.$$

We have

$$\begin{aligned} \frac{n(j-k)+k}{n(j-k)} &\leq 2 \Leftrightarrow n(j-k) + k \leq 2n(j-k) \\ &\Leftrightarrow k \leq n(j-k) \Leftrightarrow k(n+1) \leq nj \Leftrightarrow k \leq \frac{n}{n+1}j. \end{aligned}$$

which proves (iii). □

Lemma 3.2. *Let $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ and $n \in \mathbb{N}$, $n \geq 3$.*

(i) If $j \in \{0, 1, 2, ...\}$ is such that $k \geq \frac{n}{n-1}(j+1)$ and

$$n[(k-j)^2 - (k+1)] + kj - j^2 - k^2 - j \geq 0,$$

then $\overline{M}_{k,n,j}(x) \geq \overline{M}_{k+1,n,j}(x)$.

(ii) If $k \in \{1, 2, ...j\}$ is such that $k \leq \frac{n}{n+1}j$ and

$$n[(k-j)^2 - k] + kj - j^2 - k^2 + k \geq 0,$$

then $\underline{M}_{k,n,j}(x) \geq \underline{M}_{k-1,n,j}(x)$.

Proof. (i) We observe that

$$\frac{\overline{M}_{k,n,j}(x)}{\overline{M}_{k+1,n,j}(x)} = \frac{k+1}{n+k} \cdot \frac{x+1}{x} \cdot \frac{\frac{k}{n-1} - x}{\frac{k+1}{n-1} - x}.$$

Since the function $g(x)=\frac{x+1}{x}\cdot\frac{\frac{k}{n-1}-x}{\frac{k+1}{n-1}-x}$ clearly is nonincreasing, it follows that $g(x)\geq g(\frac{j+1}{n-1})=\frac{n+j}{j+1}\cdot\frac{k-j-1}{k-j}$ for all $x\in[\frac{j}{n-1},\frac{j+1}{n-1}]$.

Then

$$\frac{\overline{M}_{k,n,j}(x)}{\overline{M}_{k+1,n,j}(x)}\geq\frac{k+1}{n+k}\cdot\frac{n+j}{j+1}\cdot\frac{k-j-1}{k-j}.$$

Through simple calculus we obtain

$$\begin{aligned}&(k+1)(n+j)(k-j-1)-(n+k)(j+1)(k-j)\\=\;&n[(k-j)^2-(k+1)]+kj-j^2-k^2-j\end{aligned}$$

which proves (i).

(ii) We observe that

$$\frac{\underline{M}_{k,n,j}(x)}{\underline{M}_{k-1,n,j}(x)}=\frac{n+k-1}{k}\cdot\frac{x}{1+x}\cdot\frac{x-\frac{k}{n-1}}{x-\frac{k-1}{n}}.$$

Since the function $h(x)=\frac{x}{1+x}\cdot\frac{x-\frac{k}{n-1}}{x-\frac{k-1}{n-1}}$ is nondecreasing, it follows that $h(x)\geq h(\frac{j}{n-1})=\frac{j}{n+j-1}\cdot\frac{j-k}{j-k+1}$ for all $x\in[\frac{j}{n-1},\frac{j+1}{n-1}]$.

Then

$$\frac{\underline{M}_{k,n,j}(x)}{\underline{M}_{k-1,n,j}(x)}\geq\frac{n+k-1}{k}\cdot\frac{j}{n+j-1}\cdot\frac{j-k}{j-k+1}.$$

Through simple calculus we obtain

$$\begin{aligned}&j(n+k-1)(j-k)-k(n+j-1)(j-k+1)\\=\;&n[(j-k)^2-k]+kj-j^2-k^2+k\end{aligned}$$

which proves (ii) and the lemma. □

4. **Approximation Results**

If $V_n^{(M)}(f)(x)$ represents the *Baskakov operator of max-product kind* defined in Introduction, then the first main result of this section is the following.

Theorem 4.1. *Let $f:[0,\infty)\to\mathbb{R}_+$ be bounded and continuous on $[0,\infty)$. Then we have the estimate*

$$|V_n^{(M)}(f)(x)-f(x)|\leq 12\omega_1\left(f,\sqrt{\frac{x(x+1)}{n-1}}\right),n\in\mathbb{N},n\geq 4,x\in[0,\infty),$$

where

$$\omega_1(f,\delta)=\sup\{|f(x)-f(y)|;x,y\in[0,\infty),|x-y|\leq\delta\}.$$

Proof. It is easy to check that the max-product Baskakov operator fulfills the conditions in Corollary 2.3 and we have

$$|V_n^{(M)}(f)(x) - f(x)| \leq \left(1 + \frac{1}{\delta_n} V_n^{(M)}(\varphi_x)(x)\right) \omega_1(f, \delta_n), \quad (4.1)$$

where $\varphi_x(t) = |t - x|$. So, it is enough to estimate

$$E_n(x) := V_n^{(M)}(\varphi_x)(x) = \frac{\bigvee_{k=0}^{\infty} b_{n,k}(x) \left|\frac{k}{n} - x\right|}{\bigvee_{k=0}^{\infty} b_{n,k}(x)}, x \in [0, \infty).$$

Let $x \in [j/n - 1, (j+1)/n - 1]$ where $j \in \{0, 1, ...,\}$ is fixed, arbitrary. By Lemma 2.4 we easily obtain

$$E_n(x) = \max_{k=0,1,...,} \{M_{k,n,j}(x)\}, x \in [j/n - 1, (j+1)/n - 1].$$

In all what follows we may suppose that $j \in \{1, 2, ...,\}$, because for $j = 0$ we get $E_n(x) < 5\sqrt{\frac{x(x+1)}{n-1}}$, for all $x \in [0, 1/n - 1]$. Indeed, in this case we have

$$M_{k,n,0}(x) = \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k \left|\frac{k}{n} - x\right|,$$

which for $k = 0$ gives

$$M_{k,n,0}(x) = x = \sqrt{x} \cdot \sqrt{x} \leq \sqrt{x} \cdot \frac{1}{\sqrt{n-1}} \leq \sqrt{\frac{x(x+1)}{n-1}}.$$

Also, for $k = 1$ we have $x \leq \frac{2}{n}$ which implies $\left|\frac{1}{n} - x\right| \leq \frac{1}{n}$ and further one

$$M_{1,n,0}(x) \leq \binom{n}{1} \left(\frac{x}{1+x}\right) \cdot \frac{1}{n} = \frac{x}{1+x} \leq x \leq \sqrt{\frac{x(x+1)}{n-1}}.$$

Suppose now that $k \geq 2$. We observe that in this case all the hypothesis of the Lemma 3.1 (i) are fulfilled, therefore in this case we have $M_{k,n,0}(x) \leq \overline{M}_{k,n,0}(x)$. Also by Lemma 3.2 (i), for $j = 0$ it follows that $\overline{M}_{k,n,0}(x) \geq \overline{M}_{k+1,n,0}(x)$ for every $k \geq 2$ such that $(n-1)k^2 - nk - n \geq 0$. Because the function $f(x) = (n-1)x^2 - nx - n$, $x \geq 1$ is nondecreasing and because $f(\sqrt{n}) \geq 0$, it follows that $\overline{M}_{k,n,0}(x) \geq \overline{M}_{k+1,n,0}(x)$ for every $k \in \mathbb{N}$, $k \geq \sqrt{n}$. Let us denote $A = \{k \in \mathbb{N}, 2 \leq k \leq \sqrt{n} + 1\}$ and let $k \in A$.

We have by Lemma 2.5

$$\begin{aligned}
&\overline{M}_{k,n,0}(x) = \\
= &\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k\left(\frac{k}{n-1}-x\right) \\
\leq &\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k\frac{k}{n-1} \leq \binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k \cdot \frac{3k}{2n} \\
= &\frac{(n+k-1)!}{k!(n-1)!}\cdot\left(\frac{x}{1+x}\right)^k\cdot\frac{3k}{2n} = \binom{n+k-1}{k-1}\left(\frac{x}{1+x}\right)^{k-1}\left(\frac{3x}{2(1+x)}\right) \\
= &\binom{n+k-1}{k-1}\left(\frac{1/n}{1+1/n}\right)^{k-1}\left(\frac{x}{1+x}\cdot\frac{1+1/n}{1/n}\right)^{k-1}\cdot\left(\frac{3x}{2(1+x)}\right) \\
= &\binom{n+k-1}{k-1}\left(\frac{1/n}{1+1/n}\right)^{k-1}\left(\frac{(n+1)x}{1+x}\right)^{k-1}\cdot\left(\frac{3x}{2(1+x)}\right) \\
= &m_{k-1,n+1,0}\left(\frac{1}{n}\right)\left(\frac{(n+1)x}{1+x}\right)^{k-1}\cdot\left(\frac{3x}{2(1+x)}\right) \\
\leq &\left(\frac{(n+1)x}{1+x}\right)^{k-1}\cdot\left(\frac{3x}{2(1+x)}\right).
\end{aligned}$$

Since the function $g(x) = \left(\frac{(n+1)x}{1+x}\right)^{k-1}$ is nondecreasing on the interval $[0, \frac{1}{n-1}]$, it follows that

$$g(x) \leq g\left(\frac{1}{n-1}\right) = \left(\frac{n+1}{n}\right)^{k-1}$$

for all $x \in [0, \frac{1}{n-1}]$. Then

$$\begin{aligned}
\overline{M}_{k,n,0}(x) &\leq \frac{3}{2}\cdot\left(\frac{n+1}{n}\right)^{k-1}\cdot\frac{x}{1+x} \leq \frac{3}{2}\cdot\left(\frac{n+1}{n}\right)^{\sqrt{n}}\cdot\frac{x}{1+x} \\
&< \frac{3}{2}\cdot\left(\frac{n+1}{n}\right)^{n}\cdot\frac{x}{1+x} < \frac{3e}{2}\cdot x \leq \frac{3e}{2}\cdot\sqrt{\frac{x(x+1)}{n-1}} < 5\sqrt{\frac{x(x+1)}{n-1}}.
\end{aligned}$$

Based on the above results, we obtain

$$\begin{aligned}
&E_n(x) = \\
= &\max_{k=0,1,\ldots,}\{M_{k,n,0}(x)\} \leq \max\{M_{0,,n,0}(x), M_{1,n,0}(x), \max_{k=2,3,\ldots,}\{\overline{M}_{k,n,0}(x)\}\} \\
= &\max\{M_{0,,n,0}(x), M_{1,n,0}(x), \max_{k\in A}\{\overline{M}_{k,n,0}(x)\}\} < 5\sqrt{\frac{x(x+1)}{n-1}}.
\end{aligned}$$

So it remains to obtain an upper estimate for each $M_{k,n,j}(x)$ when $j = 1, 2, ...,$ is fixed, $x \in [j/(n-1), (j+1)/(n-1)]$ and $k = 0, 1, ...,$. In fact we will prove that

$$M_{k,n,j}(x) < 6\sqrt{\frac{x(x+1)}{n}}, \text{ for all } x \in [j/(n-1), (j+1)/(n-1)], k = 0, 1, ..., \quad (4.2)$$

which immediately will imply that

$$E_n(x) \leq 6\sqrt{\frac{x(x+1)}{n}}, \text{ for all } x \in [0, \infty), n \in \mathbb{N},$$

and taking $\delta_n = 6\sqrt{\frac{x(x+1)}{n}}$ in (4.1), we immediately obtain the estimate in the statement.

In order to prove (4.2) we distinguish the following cases:
1) $\frac{n}{n+1} \cdot j \leq k \leq \frac{n}{n-1} \cdot (j+1)$; 2) $k > \frac{n}{n-1} \cdot (j+1)$ and 3) $k < \frac{n}{n+1} \cdot j$.
Case 1) We have

$$\frac{k}{n} - x \leq \frac{\frac{n}{n-1} \cdot (j+1)}{n} - \frac{j}{n-1} = \frac{j+1}{n-1} - \frac{j}{n-1} = \frac{1}{n-1}.$$

On the other hand

$$\begin{aligned}\frac{k}{n} - x &\geq \frac{\frac{n}{n+1} \cdot j}{n} - \frac{j+1}{n-1} = \frac{j}{n+1} - \frac{j+1}{n-1} = \frac{-2j}{(n-1)(n+1)} - \frac{1}{n-1} \\ &\geq \frac{-2x}{n+1} - \frac{1}{n-1}.\end{aligned}$$

Therefore $\left|\frac{k}{n} - x\right| \leq \frac{2x}{n-1} + \frac{1}{n-1}$. It is immediate that $\frac{x}{n-1} \leq \sqrt{\frac{x(x+1)}{n-1}}$ for all $x \geq 0$. On the other hand, $\frac{1}{n-1} = \sqrt{\frac{1}{n-1}} \cdot \sqrt{\frac{1}{n-1}} \leq \sqrt{\frac{1}{n-1}} \cdot \sqrt{\frac{j}{n-1}} \leq \sqrt{\frac{1}{n-1}} \cdot \sqrt{x} \leq \sqrt{\frac{x(x+1)}{n-1}}$. It follows that

$$M_{k,n,j}(x) = m_{k,n,j}(x)\left|\frac{k}{n} - x\right| \leq \left|\frac{k}{n} - x\right| \leq 3\sqrt{\frac{x(x+1)}{n-1}}.$$

Case 2). Subcase a). Suppose first that $n[(k-j)^2 - (k+1)] + kj - j^2 - k^2 - j < 0$. Denoting $k = j + \alpha$, the previous inequality becomes $\alpha^2(n-1) - \alpha(n+j) - (j+1)(n+j) < 0$ where evidently $\alpha \geq 1$.Let us define the function $f(t) = t^2(n-1) - t(n+j) - (j+1)(n+j)$, $t \in \mathbb{R}$. We claim that $f\left(\sqrt{\frac{3(j+1)(n+j)}{n-1}}\right) > 0$ which will imply $\alpha < \sqrt{\frac{3(j+1)(n+j)}{n-1}}$ and further one $k - j < \sqrt{\frac{3(j+1)(n+j)}{n-1}}$. Indeed, after simple calculation we get

$$\begin{aligned} f\left(\sqrt{\frac{3(j+1)(n+j)}{n-1}}\right) &= (n+j)\sqrt{j+1}\left(2\sqrt{j+1}-\sqrt{\frac{3(n+j)}{n-1}}\right) \\ &= (n+j)\sqrt{j+1}\left(\sqrt{4j+4}-\sqrt{3+\frac{3j+3}{n-1}}\right) \\ &\geq (n+j)\sqrt{j+1}\left(\sqrt{4j+4}-\sqrt{3+\frac{3j+3}{2}}\right) \\ &= (n+j)\sqrt{j+1}\left(\sqrt{4j+4}-\sqrt{4+\frac{3j+1}{2}}\right) > 0 \end{aligned}$$

where we used the obvious inequality $4j > \frac{3j+1}{2}$ for all $j \geq 1$.

Based on the above results we have

$$\begin{aligned} & \overline{M}_{k,n,j}(x) = \\ = \; & m_{k,n,j}(x)(\frac{k}{n-1}-x) \leq \frac{k}{n-1}-x \leq \frac{k}{n-1}-\frac{j}{n-1} \\ = \; & \frac{k-j}{n-1} < \frac{\sqrt{\frac{3(j+1)(n+j)}{n-1}}}{n-1} = \frac{1}{\sqrt{n-1}}\cdot\sqrt{\frac{3(j+1)(n+j)}{(n-1)^2}} \\ \leq \; & \frac{1}{\sqrt{n-1}}\cdot\sqrt{\frac{6j(n+j)}{(n-1)^2}} \\ = \; & \frac{1}{\sqrt{n-1}}\cdot\sqrt{\frac{6j}{n-1}}\cdot\sqrt{\frac{n+j}{n-1}} \\ = \; & \frac{1}{\sqrt{n-1}}\cdot\sqrt{\frac{6j}{n-1}}\cdot\sqrt{\frac{n+j-1}{n-1}}\cdot\sqrt{\frac{n+j}{n+j-1}} \\ = \; & \frac{1}{\sqrt{n-1}}\cdot\sqrt{\frac{6j}{n-1}}\cdot\sqrt{1+\frac{j}{n-1}}\cdot\sqrt{\frac{n+j}{n+j-1}} \leq \sqrt{\frac{6x(x+1)}{n-1}}\cdot\frac{2}{\sqrt{3}} \\ = \; & 2\sqrt{2}\sqrt{\frac{x(x+1)}{n-1}}. \end{aligned}$$

Subcase b). Suppose now that $n[(k-j)^2-(k+1)]+kj-j^2-k^2-j \geq 0$. Because n and j are fixed, we can define the real function

$$\begin{aligned} g(x) &= n[(x-j)^2-(x+1)]+xj-j^2-x^2-j \\ &= (n-1)x^2-x(2nj-j+n)+nj^2-n-j^2-j, \end{aligned}$$

for all $x \in \mathbb{R}$. For $x \geq \frac{n}{n-1}(j+1)$ we get

$$\begin{aligned} g'(x) &= 2(n-1)x - 2nj + j - n \geq 2(n-1) \cdot \frac{n(j+1)}{n-1} - 2nj + j - n \\ &= n + j > 0. \end{aligned}$$

Therefore, g is nondecreasing on the interval $[\frac{n}{n-1}(j+1), \infty)$. Since

$$g\left(\frac{n}{n-1}(j+1)\right) = -nj - n - j^2 - j < 0$$

and because $\lim_{x\to\infty} g(x) = \infty$, by the monotonicity of g too, it follows that there exists $\overline{k} \in \mathbb{N}$, $\overline{k} > \frac{n}{n-1}(j+1)$ of minimum value, such that $g(\overline{k}) = n[(\overline{k} - j)^2 - (\overline{k}+1)] + \overline{k}j - j^2 - \overline{k}^2 - j \geq 0$. Denote $k_1 = \overline{k} - 1$ where evidently $k_1 \geq j + 1$. If $k_1 \geq \frac{n}{n-1}(j+1)$, then from the properties of g and by the way we choose $\overline{k}$ it results that $g(k_1) < 0$. If $k_1 < \frac{n}{n-1}(j+1)$, then $j < k_1 < \frac{n}{n-1}(j+1)$. Since g is a quadratic function and because $g(j) < 0$ and $g\left(\frac{n}{n-1}(j+1)\right) < 0$, it is immediate that we get to the same conclusion as in the other case, that is $g(k_1) < 0$ or equivalently $\alpha^2(n-1) - \alpha(n+j) - (j+1)(n+j) < 0$, where $k_1 = j + \alpha$. Using the same technique as in subcase a) we get $k_1 - j < \sqrt{\frac{3(j+1)(n+j)}{n-1}}$. Then

$$\begin{aligned} \overline{M}_{\overline{k},n,j}(x) &= m_{\overline{k},n,j}(x)(\frac{\overline{k}}{n-1} - x) \leq \frac{\overline{k}}{n-1} - x \\ &\leq \frac{\overline{k}}{n-1} - \frac{j}{n-1} = \frac{k_1 - j}{n-1} + \frac{1}{n-1} \\ &< 2\sqrt{2}\sqrt{\frac{x(x+1)}{n-1}} + \frac{1}{n-1} \\ &\leq 2\sqrt{2}\sqrt{\frac{x(x+1)}{n-1}} + \sqrt{\frac{x(x+1)}{n-1}} < 4\sqrt{\frac{x(x+1)}{n-1}}. \end{aligned}$$

By Lemma 3.2., (i) it follows that $\overline{M}_{\overline{k},n,j}(x) \geq \overline{M}_{\overline{k}+1,n,j}(x) \geq ....$ We thus obtain $\overline{M}_{k,n,j}(x) < 4\sqrt{\frac{x(x+1)}{n-1}}$ for any $k \in \{\overline{k}, \overline{k}+1, ..., \}$.

Therefore, in both subcases, by Lemma 3.1, (i) too, we get

$$M_{k,n,j}(x) < 4\sqrt{\frac{x(x+1)}{n-1}}.$$

Case 3). Subcase a). Suppose first that $n[(k-j)^2 - k] + kj - j^2 - k^2 + k < 0$. Denoting $k = j - \alpha$ the previous inequality becomes $\alpha^2(n-1) + \alpha(n+j-1) - nj - j^2 + j < 0$ where evidently $\alpha \geq 1$. Let us define the function $f(t) = t^2(n-1) + t(n+j-1) - nj - j^2 + j$, $t \in \mathbb{R}$. Because $f\left(\sqrt{\frac{j(n+j-1)}{n-1}}\right) = (n+j-1) \cdot \sqrt{\frac{j(n+j-1)}{n-1}} > 0$ it follows that $\alpha < \sqrt{\frac{j(n+j-1)}{n-1}}$ and further one $j - k < \sqrt{\frac{j(n+j-1)}{n-1}}$. Then we obtain

$$\begin{aligned}
&\underline{M}_{k,n,j}(x) = \\
= \quad & m_{k,n,j}(x)(x - \frac{k}{n-1}) \leq \frac{j+1}{n-1} - \frac{k}{n-1} = \frac{j-k}{n-1} + \frac{1}{n-1} \\
< \quad & \frac{\sqrt{\frac{j(n+j-1)}{n-1}}}{n-1} + \sqrt{\frac{x(x+1)}{n-1}} = \sqrt{\frac{1}{n-1}} \cdot \sqrt{\frac{j(n+j-1)}{(n-1)^2}} + \sqrt{\frac{x(x+1)}{n-1}} \\
= \quad & \sqrt{\frac{1}{n-1}} \cdot \sqrt{\frac{j}{(n-1)}} \cdot \sqrt{1 + \frac{j}{(n-1)}} + \sqrt{\frac{x(x+1)}{n-1}} \leq 2\sqrt{\frac{x(x+1)}{n-1}}.
\end{aligned}$$

Subcase b). Suppose now that $n[(k-j)^2 - k] + kj - j^2 - k^2 + k \geq 0$. Because n and j are fixed we can define the real function

$$\begin{aligned}
g(x) &= n[(x-j)^2 - x] + xj - j^2 - x^2 + x \\
&= (n-1)x^2 - x(2nj + j + n + 1) + nj^2 - j^2,
\end{aligned}$$

for all $x \in \mathbb{R}$. For $x \leq \frac{n}{n+1} \cdot j$ we get

$$\begin{aligned}
g'(x) &= 2(n-1)x - (2nj + j + n + 1) \leq \frac{2(n-1)nj}{n+1} - (2nj + j + n + 1) \\
&\leq 2nj - (2nj + j + n + 1) = -j - n - 1 < 0.
\end{aligned}$$

Therefore, g in nonincreasing on the interval $[0, \frac{nj}{n+1}]$. We have

$$\begin{aligned}
g(\frac{nj}{n+1}) &= \frac{(n-1)n^2j^2}{(n+1)^2} - \frac{nj}{n+1} \cdot (2nj + j + n + 1) + nj^2 - j^2 \\
&\leq \frac{n^2j^2}{n+1} - \frac{nj}{n+1} \cdot (2nj + j + n + 1) + nj^2 - j^2 \\
&= \frac{-n^2j^2 - nj^2 - n^2j - nj}{n+1} + nj^2 - j^2 = \frac{-n^2j - nj}{n+1} - j^2 \\
&= -nj - j^2 < 0.
\end{aligned}$$

Based on the above result and because $g(0) > 0$, by the monotonicity of g too, it follows that there exists $\widetilde{k} \in \mathbb{N}$, $\widetilde{k} < \frac{nj}{n+1}$ of maximum value, such that $g(\widetilde{k}) = n[(\widetilde{k} - j)^2 - \widetilde{k}] + \widetilde{k}j - j^2 - \widetilde{k}^2 + \widetilde{k} \geq 0$. Denoting $k_2 = \widetilde{k} + 1$ and reasoning as in case (ii), subcase b) we obtain $g(k_2) < 0$. Further, reasoning as in case (iii), subcase a) we obtain $j - k_2 < \sqrt{\frac{j(n+j-1)}{n-1}}$. It follows

$$\begin{aligned}
\underline{M}_{\widetilde{k},n,j}(x) &= m_{\widetilde{k},n,j}(x)(x - \frac{\widetilde{k}}{n-1}) \leq \frac{j+1}{n-1} - \frac{\widetilde{k}}{n-1} \\
&= \frac{j-k_2}{n-1} + \frac{2}{n-1} < 3\sqrt{\frac{x(x+1)}{n-1}}.
\end{aligned}$$

By Lemma 3.2, (ii) it follows that $\underline{M}_{\widetilde{k},n,j}(x) \geq \underline{M}_{\widetilde{k}-1,n,j}(x) \geq ... \geq \underline{M}_{0,n,j}(x)$. We thus obtain $\underline{M}_{k,n,j}(x) < 3\sqrt{\frac{x(x+1)}{n-1}}$ for any $k \in \{0, 1, ..., \widetilde{k}\}$

In both subcases, by Lemma 3.1, (iii) too, we get $M_{k,n,j}(x) < 6\sqrt{\frac{x(x+1)}{n-1}}$.

In conclusion, collecting all the estimates in the above cases and subcases we easily get the relationship (4.2), which completes the proof. □

Remark. In what follows we prove that the order of approximation in Theorem 4.1 cannot be improved. Indeed, for $n \in \mathbb{N}$, $n \geq 3$ let us take $j_n = (n-1)^2 - 1$, $k_n = j_n + [\sqrt{\frac{(j_n+1)(n+j_n)}{n-1}}] + 1 = j_n + [(n-1)\sqrt{n}] + 1$, $x_n = \frac{j_n+1}{n-1} = n-1$. Because $\lim_{n\to\infty}\left(k_n - \frac{n}{n-2}\cdot(j+1)\right) = \infty$, it follows that there exists $n_0 \in \mathbb{N}$, $n_0 \geq 3$ such that $k_n \geq \frac{n}{n-2}\cdot(j_n+1)$ for each $n \in \mathbb{N}$, $n \geq n_0$.Then, according to Lemma 3.1 (ii) for each $n \in \mathbb{N}$, $n \geq n_0$ we can write

$$\overline{M}_{k_n,n,j_n}(x_n) = \frac{\binom{n+k_n-1}{k_n}x_n^{k_n}/(1+x_n)^{n+k_n}}{\binom{n+j_n-1}{j_n}x_n^{j_n}/(1+x_n)^{n+j_n}}\left(\frac{k_n}{n-1} - x_n\right)$$

$$= \frac{(n+k_n-1)!}{(n+j_n-1)!}\cdot\frac{j_n!}{k_n!}\left(\frac{x_n}{1+x_n}\right)^{k_n-j_n}\left(\frac{k_n}{n-1} - x_n\right)$$

$$\geq \frac{(n+j_n)(n+j_n+1)...(n+k_n-1)}{(j_n+1)(j_n+2)...k_n}\left(\frac{n-1}{n}\right)^{[(n-1)\sqrt{n}]+1}\cdot$$

$$\cdot\left(\frac{k_n}{n-1} - \frac{j_n+1}{n-1}\right)$$

$$= \frac{(n+j_n)(n+j_n+1)...(n+k_n-1)}{(j_n+1)(j_n+2)...k_n}\left(\frac{n-1}{n}\right)^{[(n-1)\sqrt{n}]+1}\cdot$$

$$\cdot\ \frac{k_n-j_n}{n-1}\cdot\frac{k_n-j_n-1}{k_n-j_n}$$

$$\geq \frac{(n+j_n)(n+j_n+1)...(n+k_n-1)}{(j_n+1)(j_n+2)...k_n}\left(\frac{n-1}{n}\right)^{[(n-1)\sqrt{n}]+1}\cdot\frac{k_n-j_n}{2(n-1)}.$$

Since

$$\frac{k_n-j_n}{n-1} = \frac{[\sqrt{\frac{(j_n+1)(n+j_n)}{n-1}}]+1}{n-1} \geq \frac{\sqrt{\frac{(j_n+1)(n+j_n)}{n-1}}}{n-1} = \frac{\sqrt{\frac{(j_n+1)}{n-1}(1+\frac{j_n+1}{n-1})}}{\sqrt{n-1}}$$

$$= \frac{\sqrt{x_n(x_n+1)}}{\sqrt{n-1}},$$

it follows that

$$\overline{M}_{k_n,n,j_n}(x_n) \geq$$

$$\frac{(n+j_n)(n+j_n+1)...(n+k_n-1)}{(j_n+1)(j_n+2)...k_n}\left(\frac{n-1}{n}\right)^{[(n-1)\sqrt{n}]+1}\cdot\frac{\sqrt{x_n(x_n+1)}}{2\sqrt{n-1}}.$$

It is easy to prove that if $0 < a \le b$ then $\frac{b}{a} \ge \frac{b+1}{a+1}$. Using this result, we get

$$\frac{n+j_n}{j_n+1} \ge \frac{n+j_n+1}{j_n+2} \ge ... \ge \frac{n+k_n-1}{k_n},$$

which implies

$$\overline{M}_{k_n,n,j_n}(x_n)$$
$$\ge \left(\frac{n+k_n-1}{k_n}\right)^{[(n-1)\sqrt{n}]+1} \left(\frac{n-1}{n}\right)^{[(n-1)\sqrt{n}]+1} \cdot \frac{\sqrt{x_n(x_n+1)}}{2\sqrt{n-1}}.$$

We have

$$\lim_{n\to\infty} \left(\frac{n+k_n-1}{k_n}\right)^{[(n-1)\sqrt{n}]+1} \left(\frac{n-1}{n}\right)^{[(n-1)\sqrt{n}]+1}$$
$$= \lim_{n\to\infty} \left(\frac{(n-1)^2+(n-1)+[(n-1)\sqrt{n}]}{(n-1)^2+[(n-1)\sqrt{n}]}\right)^{[(n-1)\sqrt{n}]+1} \cdot$$
$$\cdot \left(\frac{n-1}{n}\right)^{[(n-1)\sqrt{n}]+1}$$
$$= \lim_{n\to\infty} \left(\frac{(n-1)^2+(n-1)+(n-1)\sqrt{n}}{(n-1)^2+(n-1)\sqrt{n}}\right)^{(n-1)\sqrt{n}+1} \cdot$$
$$\cdot \left(\frac{n-1}{n}\right)^{(n-1)\sqrt{n}+1}$$
$$= \lim_{n\to\infty} \left(\frac{n+\sqrt{n}}{(n-1)+\sqrt{n}}\right)^{(n-1)\sqrt{n}+1} \left(\frac{n-1}{n}\right)^{(n-1)\sqrt{n}+1}$$
$$= \lim_{n\to\infty} \left(1+\frac{1}{n-1+\sqrt{n}}\right)^{(n-1)\sqrt{n}+1} \left(1-\frac{1}{n}\right)^{(n-1)\sqrt{n}+1}$$
$$= \lim_{n\to\infty} \left[1+\left(\frac{1-\sqrt{n}}{n(n-1+\sqrt{n})} - \frac{1}{n(n-1+\sqrt{n})}\right)\right]^{(n-1)\sqrt{n}+1} = e^{-1}.$$

It follows that there exists $n_1 \in \mathbb{N}$, such that

$$\left(\frac{n+k_n-1}{k_n}\right)^{[(n-1)\sqrt{n}]+1} \left(\frac{n-1}{n}\right)^{[(n-1)\sqrt{n}]+1} \ge e^{-2}$$

for any $n \ge n_1$. Then we get

$$\overline{M}_{k_n,n,j_n}(x_n) \ge \frac{\sqrt{x_n(x_n+1)}}{2e^2\sqrt{n-1}}$$

for all $n \ge \max\{n_0, n_1\}$. Taking into account Lemma 3.1, (ii) too, it follows that for all $n \ge \max\{n_0, n_1\}$ we have $M_{k_n,n,j_n}(x_n) \ge \frac{\sqrt{x_n(x_n+1)}}{4e^2\sqrt{n-1}}$ for all $n \ge \max\{n_0, n_1\}$, which combined with the fact $\lim\limits_{n\to\infty} x_n = \infty$ will imply the desired conclusion.

In what follows we will prove that for large subclasses of functions f, the order of approximation $\omega_1(f;\sqrt{x(x+1)/(n-1)})$ in Theorem 4.1 can essentially be improved to $\omega_1(f;(x+1)/(n-1))$.

For this purpose, for any $n\in\mathbb{N}$, $n\geq 3$, $k\in\{0,1,..,\}$ and $j\in\{0,1,..,\}$, let us denote $A_j=\{k\in\mathbb{N}: j\leq k\leq \frac{n}{n-1}(j+1)+1\}$.

We need the following auxiliary lemmas.

Lemma 4.2. *Let $f:[0,\infty)\to[0,\infty)$ be bounded and suppose that there exists $j\in\{0,1,...,\}$ and $x\in[j/(n-1),(j+1)/(n-1)]$ such that*

$$V_n^{(M)}(f)(x)=\bigvee_{k\in A_j} f_{k,n,j}(x),$$

Then

$$\left|V_n^{(M)}(f)(x)-f(x)\right|\leq 2\omega_1\left(f;\frac{x+1}{n-1}\right), n\geq 3.$$

Proof. We distinguish two cases:

Case (i) Suppose that $V_n^{(M)}(f)(x)\leq f(x)$. Because $V_n^{(M)}(f)(x)\geq f_{j,n,j}(x)=f(\frac{j}{n})$ it follows that $f(\frac{j}{n})\leq V_n^{(M)}(f)(x)\leq f(x)$, which implies

$$\left|V_n^{(M)}(f)(x)-f(x)\right|=f(x)-V_n^{(M)}(f)(x)\leq f(x)-f(\frac{j}{n}).$$

By simple calculation we have $0\leq x-\frac{j}{n}\leq\frac{j+1}{n-1}-\frac{j}{n}=\frac{j}{(n-1)n}+\frac{1}{n-1}\leq\frac{x+1}{n-1}$. Therefore, in this case we obtain

$$\left|V_n^{(M)}(f)(x)-f(x)\right|\leq\omega_1\left(f;\frac{x+1}{n-1}\right).$$

Case (ii) Suppose that $V_n^{(M)}(f)(x)>f(x)$. From the hypothesis we get that there exists $\overline{k}\in A_j$ such that $V_n^{(M)}(f)(x)=f_{\overline{k},n,j}(x)$, which implies

$$\begin{aligned}\left|V_n^{(M)}(f)(x)-f(x)\right| &= V_n^{(M)}(f)(x)-f(x)=f_{\overline{k},n,j}(x)-f(x)\\ &= m_{\overline{k},n,j}(x)f(\frac{\overline{k}}{n})-f(x)\leq f(\frac{\overline{k}}{n})-f(x).\end{aligned}$$

We have $\frac{\overline{k}}{n}-x\leq\frac{\frac{n}{n-1}(j+1)+1}{n}-\frac{j}{n-1}=\frac{1}{n-1}+\frac{1}{n}\leq\frac{2}{n-1}\leq\frac{2(x+1)}{n-1}$. On the other hand we have

$$\begin{aligned}\frac{\overline{k}}{n}-x &\geq \frac{j}{n}-\frac{j+1}{n-1}=\frac{-j}{n(n-1)}-\frac{1}{n-1}\geq\frac{-x}{n}-\frac{1}{n-1}\\ &\geq \frac{-x}{n-1}-\frac{1}{n-1}=\frac{-(x+1)}{n-1}.\end{aligned}$$

Therefore, we obtain $\left|\frac{\overline{k}}{n}-x\right|\leq\frac{2(x+1)}{n-1}$ and it follows that $\left|V_n^{(M)}(f)(x)-f(x)\right|\leq 2\omega_1\left(f;\frac{x+1}{n-1}\right)$ which proves the lemma. □

Lemma 4.3. *If the function* $f:[0,\infty)\to[0,\infty)$ *is concave, then the function* $g:(0,\infty)\to[0,\infty), g(x)=\frac{f(x)}{x}$ *is nonincreasing.*

Proof. Let $x, y\in(0,\infty)$ be with $x\le y$. Then

$$f(x)=f\left(\frac{x}{y}y+\frac{y-x}{y}0\right)\ge\frac{x}{y}f(y)+\frac{y-x}{y}f(0)\ge\frac{x}{y}f(y),$$

which implies $\frac{f(x)}{x}\ge\frac{f(y)}{y}$. □

Corollary 4.4. *If* $f:[0,\infty)\to[0,\infty)$ *is bounded, nondecreasing and such that the function* $g:(0,\infty)\to[0,\infty), g(x)=\frac{f(x)}{x}$ *is nonincreasing, then*

$$\left|V_n^{(M)}(f)(x)-f(x)\right|\le 2\omega_1\left(f;\frac{x+1}{n-1}\right),\ \textit{for all } x\in[0,\infty), n\ge 3.$$

Proof. Since f is nondecreasing it follows (see the proof of Theorem 5.3 in the next section)

$$V_n^{(M)}(f)(x)=\bigvee_{k\ge j}^{\infty}f_{k,n,j}(x),\ \text{for all } x\in[j/(n-1),(j+1)/(n-1)].$$

Let $x\in[0,\infty)$ and $j\in\{0,1,...,\}$ such that $x\in[\frac{j}{n-1},\frac{j+1}{n-1}]$. Let $k\in\{1,2...,\}$ be with $k\ge j$. Then

$$\begin{aligned} f_{k+1,n,j}(x) &= \frac{\binom{n+k}{k+1}}{\binom{n+j-1}{j}}(\frac{x}{1+x})^{k+1-j}f(\frac{k+1}{n}) \\ &= \frac{\binom{n+k-1}{k}}{\binom{n+j-1}{j}}\cdot\frac{n+k}{k+1}(\frac{x}{1+x})^{k-j}\frac{x}{1+x}f(\frac{k+1}{n}). \end{aligned}$$

Since $g(x)$ is nonincreasing we get $\frac{f(\frac{k+1}{n})}{\frac{k+1}{n}}\le\frac{f(\frac{k}{n})}{\frac{k}{n}}$ that is $f(\frac{k+1}{n})\le\frac{k+1}{k}f(\frac{k}{n})$. From $x\le\frac{j+1}{n-1}$ it follows

$$\begin{aligned} f_{k+1,n,j}(x) &\le \frac{\binom{n+k-1}{k}}{\binom{n+j-1}{j}}(\frac{x}{1+x})^{k-j}\frac{j+1}{n+j}\cdot\frac{n+k}{k+1}\cdot\frac{k+1}{k}f(\frac{k}{n}) \\ &= f_{k,n,j}(x)\frac{j+1}{n+j}\cdot\frac{n+k}{k}=\frac{(n+j)k+n(j+1-k)+k}{(n+j)k}\cdot f_{k,n,j}(x). \end{aligned}$$

Since for each $k\ge\frac{n}{n-1}(j+1)$ we get $n(j+1-k)+k\le 0$, it follows that $f_{k,n,j}(x)\ge f_{k+1,n,j}(x)$ for any $k\ge\frac{n}{n-1}(j+1)$ which will immediately imply that $V_n^{(M)}(f)(x)=\bigvee_{k\in A_j}f_{k,n,j}(x)$. By Lemma 4.2 we immediately obtain the desired conclusion. □

Corollary 4.5. *Let $f : [0,\infty) \to [0,\infty)$ be a bounded, nondecreasing, concave function. Then*

$$\left|V_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{x+1}{n-1}\right), \text{ for all } x \in [0,\infty), n \geq 3.$$

Proof. The proof is immediate by Lemma 4.3 and Corollary 4.4. □

Remarks. 1) If we suppose, for example, that in addition to the hypothesis in Corollary 4.4, $f : [0,\infty) \to [0,\infty)$ is a Lipschitz function, that is there exists $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$, for all $x, y \in [0,\infty)$, then it follows that the order of pointwise approximation on $[0,\infty)$ by $V_n^{(M)}(f)(x)$ is $\frac{x+1}{n-1}$, which is essentially better than the order $\frac{a}{\sqrt{n}}$ obtained from Theorem 4.1 on each compact subinterval $[0, a]$ for f Lipschitz function on $[0,\infty)$.

2) It is known that for the linear Baskakov operator given by

$$V_n(f)(x) = (1+x)^{-n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k f(k/n),$$

the following pointwise approximation result is known (see [4])

$$|V_n(f)(x) - f(x)| \leq C\omega_2^{\varphi}(f; \sqrt{x(1+x)/n}), x \in [0,\infty), n \in \mathbb{N}, \tag{4.3}$$

where $\varphi(x) = \sqrt{x(1+x)}$ and $\omega_2^{\varphi}(f;\delta)$ is the Ditzian-Totik second order modulus of smoothness on $[0,\infty)$ defined by

$$\omega_2^{\varphi}(f;\delta)$$
$$= \sup\{\sup\{|f(x+h\varphi(x)) - 2f(x) + f(x - h\varphi(x))|; x \geq h^2/(1-h^2)\}, h \in [0,\delta]\},$$

with $\delta < 1$.

Now, for example, if f has the second derivative bounded by the constant K on $[0,\infty)$, because in this case we have $\omega_2^{\varphi}(f;\delta) \leq K\delta^2$, then by (4.3) we obtain the estimate

$$|V_n(f)(x) - f(x)| \leq CK\frac{x(1+x)}{n}, x \in [0,\infty), n \in \mathbb{N},$$

while by Corollary 4.5 it follows the much better estimate (on large subintervals of $[0,\infty)$)

$$|V_n^{(M)}(f)(x) - f(x)| \leq \frac{4\|f'\|(1+x)}{n}, x \in [0,\infty), n \in \mathbb{N}, n \geq 3.$$

Also, if f is, for example a nondecreasing concave polygonal line on $[0,\infty)$, constant on an interval $[a,\infty)$, then by simple reasonings we get that $\omega_2^{\varphi}(f;\delta) \sim \delta$ for $\delta \leq 1$ and by (4.3) it easily follows the estimate

$$|V_n(f)(x) - f(x)| \leq C\frac{\sqrt{x(1+x)}}{\sqrt{n}}, x \in [0,\infty), n \in \mathbb{N}, \tag{4.4}$$

while because such of function f obviously is a Lipschitz function on $[0, \infty)$ (as having bounded all the derivative numbers) by Corollary 4.5 we get the essentially better estimate than in (4.4)

$$|V_n^{(M)}(f)(x) - f(x)| \leq \frac{C(1+x)}{n}, x \in [0, \infty), n \in \mathbb{N}, n \geq 3.$$

In a similar manner, by Corollary 4.4 we can produce many subclasses of functions for which the order of approximation given by the max-product Baskakov operator is essentially better than the order of approximation given by the linear Baskakov operator. Intuitively, the max-product Baskakov operator has better approximation properties than its linear counterpart, for non-differentiable functions in a finite number of points (with the graphs having some "corners"), as for example for functions defined as a maximum of a finite number of continuous functions on $[0, \infty)$.

3) Since it is clear that a bounded nonincreasing concave function on $[0, \infty)$ necessarily one reduces to a constant function, the approximation of such functions is not of interest.

5. **Shape Preserving Properties**

In this section we will present some shape preserving properties.

Remark. Note that because of the continuity of $V_n^{(M)}(f)(x)$ on $[0, \infty)$ in Lemma 2.6, it will suffice to prove the shape properties of $V_n^{(M)}(f)(x)$ on $(0, \infty)$ only. As a consequence, in the notations and proofs below we always may suppose that $x > 0$.

Lemma 5.1. *Let $n \in \mathbb{N}$, $n \geq 3$. If $f : [0, \infty) \to \mathbb{R}_+$ is a nondecreasing function then for any $k \in \{0, 1, ..., \}$, $j \in \{0, 1, ..., \}$ with $k \leq j$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ we have $f_{k,n,j}(x) \geq f_{k-1,n,j}(x)$.*

Proof. Because $k \leq j$, by direct computation it follows that $m_{k,n,j}(x) \geq m_{k-1,n,j}(x)$. From the monotonicity of f we get $f\left(\frac{k}{n}\right) \geq f\left(\frac{k-1}{n}\right)$. Thus we obtain

$$m_{k,n,j}(x) f\left(\frac{k}{n}\right) \geq m_{k-1,n,j}(x) f\left(\frac{k-1}{n}\right),$$

which proves the lemma. □

Corollary 5.2. *Let $n \in \mathbb{N}$, $n \geq 3$. If $f : [0, \infty) \to \mathbb{R}_+$ is nonincreasing then $f_{k,n,j}(x) \geq f_{k+1,n,j}(x)$ for any $k \in \{0, 1, ..., \}$, $j \in \{0, 1, .., \}$ with $k \geq j$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$.*

Proof. Because $k \geq j$, by direct computation it follows that $m_{k,n,j}(x) \geq m_{k+1,n,j}(x)$. From the monotonicity of f we get $f\left(\frac{k}{n}\right) \geq f\left(\frac{k+1}{n}\right)$. Thus we obtain

$$m_{k,n,j}(x) f\left(\frac{k}{n}\right) \geq m_{k+1,n,j}(x) f\left(\frac{k+1}{n}\right),$$

which proves the corollary. □

Theorem 5.3. *If $f : [0, \infty) \to \mathbb{R}_+$ is nondecreasing and bounded (on $[0, \infty)$), then $V_n^{(M)}(f)$ is nondecreasing and bounded, for any $n \in \mathbb{N}$ with $n \geq 3$.*

Proof. Because $V_n^{(M)}(f)$ is continuous on $[0, \infty)$, it suffice to prove that on each subinterval of the form $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, with $j \in \{0, 1, ..., \}$, $V_n^{(M)}(f)$ is nondecreasing.

So let $j \in \{0, 1, ..., \}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. Because f is nondecreasing, from Lemma 5.1 it follows that

$$f_{j,n,j}(x) \geq f_{j-1,n,j}(x) \geq f_{j-2,n,j}(x) \geq ... \geq f_{0,n,j}(x).$$

But then it is immediate that

$$V_n^{(M)}(f)(x) = \bigvee_{k=j}^{\infty} f_{k,n,j}(x),$$

for all $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. Clearly that for $k \geq j$ the function $f_{k,n,j}$ is nondecreasing and since $V_n^{(M)}(f)$ is defined as the supremum of nondecreasing functions, it follows that it is nondecreasing. □

Corollary 5.4. *If $f : [0, \infty) \to \mathbb{R}_+$ is nonincreasing then $V_n^{(M)}(f)$ is nonincreasing, for any $n \in \mathbb{N}$ with $n \geq 3$.*

Proof. Because $V_n^{(M)}(f)$ is continuous on $[0, \infty)$, it suffice to prove that on each subinterval of the form $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, with $j \in \{0, 1, ..., \}$, $V_n^{(M)}(f)$ is nonincreasing.

So let $j \in \{0, 1, ..., \}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. Because f is nonincreasing, from Corollary 5.2 it follows that

$$f_{j,n,j}(x) \geq f_{j+1,n,j}(x) \geq f_{j+2,n,j}(x) \geq ... \geq f_{n,n,j}(x).$$

But then it is immediate that

$$V_n^{(M)}(f)(x) = \bigvee_{k=0}^{j} f_{k,n,j}(x),$$

for all $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. Clearly that for $k \leq j$ the function $f_{k,n,j}$ is nonincreasing and since $V_n^{(M)}(f)$ is defined as the maximum of nonincreasing functions, it follows that it is nonincreasing. □

In what follows, let us consider the following concept generalizing the monotonicity and convexity.

Definition 5.5. Let $f : [0, \infty) \to \mathbb{R}$ be continuous on $[0, \infty)$. One says that f is quasi-convex on $[0, \infty)$ if it satisfies the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}, \text{ for all } x, y \in [0, \infty) \text{ and } \lambda \in [0, 1].$$

(see e.g. the book [5], p. 4, (iv)).

Remark. By [6], the continuous function f is quasi-convex on the bounded interval $[0, a]$, equivalently means that there exists a point $c \in [0, a]$ such that f is nonincreasing on $[0, c]$ and nondecreasing on $[c, a]$. But this property easily can be extended to continuous quasiconvex functions on $[0, \infty)$, in the sense that there exists $c \in [0, \infty]$ ($c = \infty$ by convention for nonincreasing functions on $[0, \infty)$) such that f is nonincreasing on $[0, c]$ and nondecreasing on $[c, \infty)$. This easily follows from the fact that the quasiconvexity of f on $[0, \infty)$ means the quasiconvexity of f on any bounded interval $[0, a]$, with arbitrary large $a > 0$.

The class of quasi-convex functions includes the both classes of nondecreasing functions and of nonincreasing functions (obtained from the class of quasi-convex functions by taking $c = 0$ and $c = \infty$, respectively). Also, it obviously includes the class of convex functions on $[0, \infty)$.

Corollary 5.6. *If $f : [0, \infty) \to \mathbb{R}_+$ is continuous, bounded and quasi-convex on $[0, \infty)$ then $V_n^{(M)}(f)$ is quasi-convex on $[0, \infty)$ for any $n \in \mathbb{N}$ with $n \geq 3$.*

Proof. If f is nonincreasing (or nondecreasing) on $[0, \infty)$ (that is the point $c = \infty$ (or $c = 0$) in the above Remark) then by the Corollary 5.4 (or Theorem 5.3, respectively) it follows that for all $n \in \mathbb{N}$, $V_n^{(M)}(f)$ is nonincreasing (or nondecreasing) on $[0, \infty)$.

Suppose now that there exists $c \in (0, \infty)$, such that f is nonincreasing on $[0, c]$ and nondecreasing on $[c, \infty)$. Define the functions $F, G : [0, \infty) \to \mathbb{R}_+$ by $F(x) = f(x)$ for all $x \in [0, c]$, $F(x) = f(c)$ for all $x \in [c, \infty)$ and $G(x) = f(c)$ for all $x \in [0, c]$, $G(x) = f(x)$ for all $x \in [c, \infty)$.

It is clear that F is nonincreasing and continuous on $[0, \infty)$, G is nondecreasing and continuous on $[0, \infty)$ and that $f(x) = \max\{F(x), G(x)\}$, for all $x \in [0, \infty)$.

But it is easy to show (see also Remark 1 after the proof of Lemma 2.1) that

$$V_n^{(M)}(f)(x) = \max\{V_n^{(M)}(F)(x), V_n^{(M)}(G)(x)\}, \text{ for all } x \in [0, \infty),$$

where by the Corollary 5.4 and Theorem 5.3 , $V_n^{(M)}(F)(x)$ is nonincreasing and continuous on $[0, \infty)$ and $V_n^{(M)}(G)(x)$ is nondecreasing and continuous on $[0, \infty)$. We have two cases : 1) $V_n^{(M)}(F)(x)$ and $V_n^{(M)}(G)(x)$ do not intersect each other ; 2) $V_n^{(M)}(F)(x)$ and $V_n^{(M)}(G)(x)$ intersect each other.

Case 1). We have $\max\{V_n^{(M)}(F)(x), V_n^{(M)}(G)(x)\} = V_n^{(M)}(F)(x)$ for all $x \in [0, \infty)$ or $\max\{V_n^{(M)}(F)(x), V_n^{(M)}(G)(x)\} = V_n^{(M)}(G)(x)$ for all $x \in [0, \infty)$, which obviously proves that $V_n^{(M)}(f)(x)$ is quasi-convex on $[0, \infty)$.

Case 2). In this case it is clear that there exists a point $c' \in [0, \infty)$ such that $V_n^{(M)}(f)(x)$ is nonincreasing on $[0, c']$ and nondecreasing on $[c', \infty)$, which by the considerations in the above Remark implies that $V_n^{(M)}(f)(x)$ is quasiconvex on $[0, \infty)$ and proves the corollary. □

It is of interest to exactly calculate $V_n^{(M)}(f)$ for $f(x) = e_0(x) = 1$ and for $f(x) = e_1(x) = x$. In this sense we can state the following.

Lemma 5.7. *For all* $x \in [0, \infty)$ *and* $n \in \mathbb{N}$, $n \geq 3$ *we have* $V_n^{(M)}(e_0)(x) = 1$ *and*

$$V_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,0}(x)}{b_{n,0}(x)} = \frac{x}{1+x}, \text{ if } x \in [0, 1/n],$$

$$V_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,1}(x)}{b_{n,0}(x)} = \frac{(n+1)x^2}{(1+x)^2}, \text{ if } x \in [1/n, 1/(n-1)],$$

$$V_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,1}(x)}{b_{n,1}(x)} = \frac{x}{1+x} \cdot \frac{n+1}{n}, \text{ if } x \in [1/(n-1), 2/n],$$

$$V_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,2}(x)}{b_{n,1}(x)} = \frac{x^2}{(1+x)^2} \cdot \frac{(n+1)(n+2)}{2n}, \text{ if } x \in [2/n, 2/(n-1)],$$

$$V_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,2}(x)}{b_{n,2}(x)} = \frac{x}{1+x} \cdot \frac{n+2}{n}, \text{ if } x \in [2/(n-1), 3/n],$$

$$V_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,3}(x)}{b_{n,2}(x)} = \frac{x^2}{(1+x)^2} \cdot \frac{(n+2)(n+3)}{3n}, \text{ if } x \in [3/n, 3/(n-1)],$$

and so on, in general we have

$$V_n^{(M)}(e_1)(x) = \frac{x}{1+x} \cdot \frac{n+j}{n}, \text{ if } x \in [j/(n-1), (j+1)/n],$$

$$V_n^{(M)}(e_1)(x) = \frac{x^2}{(1+x)^2} \cdot \frac{(n+j)(n+j+1)}{n(j+1)}, \text{ if } x \in [(j+1)/n, (j+1)/(n-1)],$$

for $j \in \{0, 1, ..., \}$.

Proof. The formula $V_n^{(M)}(e_0)(x) = 1$ is immediate by the definition of $V_n^{(M)}(f)(x)$.

To find the formula for $V_n^{(M)}(e_1)(x)$ we will use the explicit formula in Lemma 2.4 which says that

$$\bigvee_{k=0}^{\infty} b_{n,k}(x) = b_{n,j}(x), \text{ for all } x \in \left[\frac{j}{n-1}, \frac{j+1}{n-1}\right], j = 0, 1, ...,$$

where $b_{n,k}(x) = \binom{n+k-1}{k} x^k / (1+x)^{n+k}$.

Since

$$\max_{k=0,1...,} \left\{ b_{n,k}(x) \frac{k}{n} \right\} = \max_{k=1,...,n} \left\{ b_{n,k}(x) \frac{k}{n} \right\} = x \cdot \max_{k=0,1...,} \{ b_{n+1,k}(x \},$$

we obtain

$$V_n^{(M)}(e_1)(x) = x \cdot \frac{\bigvee_{k=0}^{\infty} b_{n+1,k}(x)}{\bigvee_{k=0}^{\infty} b_{n,k}(x)}$$

Now the conclusion of the lemma is immediate by applying Lemma 2.4 to both expressions $\bigvee_{k=0}^{\infty} b_{n+1,k}(x)$, $\bigvee_{k=0}^{\infty} b_{n,k}(x)$, taking into account that we get the following division of the interval $[0, \infty)$

$$0 < \frac{1}{n} \leq \frac{1}{n-1} \leq \frac{2}{n} \leq \frac{2}{n-1} \leq \frac{3}{n} \leq \frac{3}{n-1} \leq \frac{4}{n} \leq \frac{4}{n-1} ...., .$$

□

Remarks. 1) The convexity of f on $[0, \infty)$ is not preserved by $V_n^{(M)}(f)$ as can be seen from Lemma 5.8. Indeed, while $f(x) = e_1(x) = x$ is obviously convex on $[0, \infty)$, it is easy to see that $V_n^{(M)}(e_1)$ is not convex on $[0, 1]$.

2) Also, if f is supposed to be starshaped on $[0, \infty)$ (that is $f(\lambda x) \leq \lambda f(x)$ for all $x, \lambda \in [0, \infty)$), then again by Lemma 5.8 it follows that $V_n^{(M)}(f)$ for $f(x) = e_1(x)$ is not starshaped on $[0, \infty)$, although $e_1(x)$ obviously is starshaped on $[0, \infty)$.

Despite of the absence of the preservation of the convexity, we can prove the interesting property that for any arbitrary nonincreasing function f, the max-product Baskakov operator $V_n^{(M)}(f)$ is piecewise convex on $[0, \infty)$. We present the following.

Theorem 5.8. *Let $n \in \mathbb{N}$ be with $n \geq 3$. For any nonincreasing function $f : [0, \infty) \to [0, \infty)$, $V_n^{(M)}(f)$ is convex on any interval of the form $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, $j = 0, 1, ...,$.*

Proof. From the proof of Corollary 5.4 we have

$$V_n^{(M)}(f)(x) = \bigvee_{k=0}^{j} f_{k,n,j}(x),$$

for any $j \in \{0, 1, ...,\}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$.

We will prove that for any fixed j and $k \leq j$, each function $f_{k,n,j}(x)$ is convex on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, which will imply that $V_n^{(M)}(f)$ can be written as a maximum of some convex functions on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$.

Since $f \geq 0$ it suffices to prove that the functions $g_{k,j} : [0, 1] \to \mathbb{R}_+$, $g_{k,j}(x) = \left(\frac{x}{1+x}\right)^{k-j}$ are convex on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$.

For $k = j$, $g_{j,j}$ is constant so is convex.

For $k = j - 1$ it follows $g_{j-1,j}(x) = \frac{x+1}{x}$ for any $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. Then $g''_{j-1,j}(x) = \frac{2}{x^3} > 0$ for any $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$.

If $k \leq j - 2$ then $g''_{k,j}(x) = (k - j) \left(\frac{x}{1+x}\right)^{k-j-2} \cdot \frac{1}{(x+1)^4} \cdot (k - j - 1 - 2x) > 0$, for any $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$.

Since all the functions $g_{k,j}$ are convex on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, we get that $V_n^{(M)}(f)$ is convex on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$ as maximum of these functions, which proves the theorem. □

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References

[1] Bede, B., Gal, S.G., *Approximation by nonlinear Bernstein and Favard-Szász-Mirakjan operators of max-product kind*, Journal of Concrete and Applicable Mathematics, 2010 (to appear).

[2] Bede, B., Nobuhara, H., Fodor, J., Hirota, K., *Max-product Shepard approximation operators*, Journal of Advanced Computational Intelligence and Intelligent Informatics, **10**(2006), 494–497.

[3] Bede, B., Nobuhara, H., Daňková, M., Di Nola, A., *Approximation by pseudo-linear operators*, Fuzzy Sets and Systems, **159**(2008), 804–820.

[4] Finta, Z., *Direct local and global approximation theorems for some linear positive operators*, Anal. Theory Appl., **20** (2004), No. 4, 307-322.

[5] Gal, S. G., *Shape-Preserving Approximation by Real and Complex Polynomials*, Birkhäuser, Boston-Basel-Berlin, 2008.

[6] Popoviciu, T., *Deux remarques sur les fonctions convexes*, Bull. Soc. Sci. Acad. Roumaine, **220**(1938), 45-49.

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WEIGHTED CONVERGENCE OF SOME POSITIVE LINEAR OPERATORS ON THE REAL SEMIAXIS

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Abstract. Some Shepard and Grünwald type operators are introduced on the semiaxis and their convergence, in suitable weighted spaces equipped with the uniform norm, is investigated.

1. Introduction

The purpose of this paper is the approximation of functions defined on $(0,\infty)$ by means of some positive linear operators based on the zeros of Laguerre polynomials. We will examine functions continuous on $(0,\infty)$, having singularities at the origin and increasing exponentially for $x \to +\infty$. In definitive functions belonging to the weighted space $L^\infty_{w_\gamma}$, $w_\gamma(x) = x^\gamma e^{-x}$, $\gamma \geq 0$ equipped with the uniform norm (see (2.4)) will be considered.

First we will examine the Shepard operator defined as

$$\mathcal{S}_m(f,x) = \frac{\displaystyle\sum_{k=1}^{m}(x-x_k)^{-2}f(x_k)}{\displaystyle\sum_{k=1}^{m}(x-x_k)^{-2}}, \quad x_k = -1+\frac{2k}{m}$$

and introduced by D. Shepard in [20]. It has been widely used in approximation theory and is simple to implement in applications like interpolation of scattered data, curves and surfaces, fluid dynamics etc. (see, for instance, [2], [1] and references therein). For this reason, it has been several papers (see, for instance, [19], [4], [8], [6], [7], [15]) investigating on $\mathcal{S}_m$ on the equidistant knots and on the zeros of Jacobi polynomials.

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In this paper we consider the following

$$\mathcal{S}_m^{(\alpha)}(f,x) := \frac{\displaystyle\sum_{k=1}^{j}(x-x_k)^{-2}f(x_k)}{\displaystyle\sum_{k=1}^{j}(x-x_k)^{-2}} \tag{1.1}$$

where $f \in L^\infty_{w_\gamma}$, $x_1 < x_2 < ... < x_m$ are the zeros of the Laguerre polynomial $p_m(w_\alpha)$ with $w_\alpha(x) = x^\alpha e^{-x}$, $\alpha > -1$ and

$$x_j = \min_{1\le k\le m}\{x_k : x_k \ge 2m\}. \tag{1.2}$$

Then, in Theorem 3.1, the convergence of $\mathcal{S}_m^{(\alpha)}(f)$ to f is showed and an estimate of the error $|f(x) - S_m^{(\alpha)}(f,x)|$ in the norm of the space $L^\infty_{w_\gamma}$ is given. Moreover, Theorem 3.2 proves that this error does not improve for smoother functions.

We will also consider the following Hermann-Vertési type operator

$$\mathcal{V}_m^{(\alpha)}(f,x) = \phi_m(x)\sum_{k=1}^{j} l_k^2(x)f(x_k), \quad \phi_m(x) = \left[\sum_{k=1}^{j} l_k^2(x)\right]^{-1}, \tag{1.3}$$

where j is defined in (1.2), x_k are the Laguerre zeros and l_k are the fundamental Lagrange polynomials based on the zeros of $p_m(w_\alpha)$. This operator, introduced in a more general form by T. Hermann and P. Vértesi in [11], was investigated in the case when $x_k \in (-1,1)$ (see, for instance, [11], [4]). Here we examine $\mathcal{V}_m^{(\alpha)}$ at the Laguerre zeros and in Theorem 3.3 we will show that, $\mathcal{V}_m^{(\alpha)}$ has a behavior similar to the Shepard operator.

Finally we will examine the Grünwald operator (see (3.11))

$$G_{m+1}^{*(\alpha)}(f,x) = \sum_{k=1}^{j} \tilde{l}_k^2(x)f(x_k),$$

where j is defined in (1.2) and $\tilde{l}_k(x) = l_k(x)\frac{4m-x}{4m-x_k}$, $\forall k = 1,...,j$ are the fundamental Lagrange polynomials based on $m+1$ points. The Grünwald operator was firstly introduced by G. Grünwald in [10] and was investigated by several authors in the case when x_k are the zeros of polynomials which are orthonormal with respect to Jacobi weights or some Freud-type weights (see, for instance, [10], [22]). In this paper, the Grünwald operator $G_{m+1}^{*(\alpha)}(f)$ based on the Laguerre zeros is considered and in Theorem 3.4 the convergence of $G_{m+1}^{*(\alpha)}(f)$ to f in $L^\infty_{w_\gamma}$ is showed and an error estimate is given.

The paper is structured as follows. In Section 2 some basic facts are given. In Section 3 the main results are presented. Then in Subsection 3.1 the Shepard operator $\mathcal{S}_m^{(\alpha)}$ is considered, in Subsection 3.2 the Hermann-Vertési operator $\mathcal{V}_m^{(\alpha)}$ is examined and in Subsection 3.3 the Grünwald operator $G_{m+1}^{*(\alpha)}$ is investigated . In Section 4 the proofs of the main results conclude the paper.

2. **Basic facts**

First we give some notations and preliminary results. In the following we will denote by $\mathcal{C}$ a positive constant which may assume different values in different formulae and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, ...)$ to indicate that $\mathcal{C}$ is independent of the parameters $a, b, ...$ Moreover if A and B are two positive quantities depending on some parameters, we will write $A \sim B$ if and only if $(A/B)^{\pm 1} \leq \mathcal{C}$, where $\mathcal{C}$ is a positive constant independent of the above parameters.

Now let $w_\alpha(x) = x^\alpha e^{-x}$, $x > 0$, $\alpha > -1$ a Laguerre weight and let $\{p_m(w_\alpha)\}$ the sequence of orthonormal Laguerre polynomials defined as

$$p_m(w_\alpha, x) = \gamma_m x^m + ..., \quad \gamma_m > 0,$$
$$\int_0^\infty p_m(w_\alpha, x) p_n(w_\alpha, x) w_\alpha(x) dx = \delta_{m,n}.$$

If we denote by $x_k := x_{m,k}(w_\alpha)$ the zeros of $p_m(w_\alpha)$, $m \geq 1$ then we have (see, for instance, [23], [9], [12]),

$$\frac{\mathcal{C}}{m} < x_1 < ... < x_m = 4m + 2\alpha + 2 - \mathcal{C}(4m)^{\frac{1}{3}}, \quad x_k \sim \frac{k^2}{m} \tag{2.1}$$

and $\forall k = 1, 2, ..., m-1$

$$\Delta x_k := x_{k+1} - x_k \sim \sqrt{\frac{x_k}{4m - x_k}}, \qquad \Delta x_k \sim \Delta x_{k+1} \tag{2.2}$$

where $\mathcal{C}$ and the constants involved in $\sim$ are independent of m and k.

Moreover, denoted by x_d a node closest to x, and by

$$x_j = \min_{1 \leq k \leq m} \{x_k : x_k \geq 2m\},$$

it results that if $0 \leq x \leq 4m$, $k \neq \{d, d \pm 1\}$, $k = 1, ...j$, $d \leq m$ and $x \neq x_k$, (see, for instance, [16, Lemma 5]) then

$$|x - x_k| \geq (|d - k| + 1) \min(\Delta x_d, \Delta x_k) \geq \mathcal{C}(|d - k| + 1)\Delta x_k \tag{2.3}$$

where $\mathcal{C} \neq \mathcal{C}(m, k)$.

Setting $w_\gamma(x) = x^\gamma e^{-x}$, $\gamma \geq 0$ and denoting by $C^\circ(B)$, $B \subseteq [0, \infty)$ the set of all continuous functions on B, we introduce the space $L^\infty_{w_\gamma}$ as follows

$$L^\infty_{w_\gamma} = \left\{ f \in C^\circ((0, \infty)) \ : \ \lim_{\substack{x \to 0 \\ x \to +\infty}} (f w_\gamma)(x) = 0 \right\} \tag{2.4}$$

equipped with the norm

$$\|f\|_{L^\infty_{w_\gamma}} := \|fw_\gamma\|_\infty = \sup_{x\geq 0} |(fw_\gamma)(x)|.$$

If $\gamma > 0$, then $L^\infty_{w_\gamma}$ denotes the set of all continuous functions on $[0,\infty)$ such that $\lim_{x\to+\infty}(fw_\gamma)(x) = 0$. In other words, when $\gamma > 0$, the functions $f \in L^\infty_{w_\gamma}$ could take very large values, with algebraic growth, as x approaches zero from the right, and could have an exponential growth as $x \to \infty$.

In order to characterize these type of functions we introduce the following modulus of smoothness (see, for instance, [5], [13, p. 175])

$$\omega_\varphi(f,t)_{w_\gamma} = \Omega_\varphi(f,t)_{w_\gamma} + \inf_{\mathcal{C}} \|[f-\mathcal{C}]w_\gamma\|_{L^\infty((0,4t^2))} + \inf_{\mathcal{C}} \|[f-\mathcal{C}]w_\gamma\|_{L^\infty\left(\left(\frac{1}{t^2},\infty\right)\right)},$$

where

$$\Omega_\varphi(f,t)_{w_\gamma} = \sup_{0<h\leq t} \sup_{x\in\left[4h^2,\frac{1}{h^2}\right]} |f(x+h\varphi(x)) - f(x)| w_\gamma(x), \quad \varphi(x) = \sqrt{x}$$

and we recall the following properties (see, for instance, [13, p. 169], [17], [5])

$$\Omega_\varphi(f,\lambda\,\delta_1)_{w_\gamma} \leq ([\lambda]+1)\;\Omega_\varphi(f,\delta_1)_{w_\gamma}, \tag{2.5}$$

$$\Omega_\varphi(f,\delta)_{w_\gamma} \leq \omega_\varphi(f,\delta)_{w_\gamma}, \tag{2.6}$$

$$\frac{\Omega_\varphi(f,\delta_1)}{\delta_1} \leq \frac{\Omega_\varphi(f,\delta_2)}{\delta_2}, \quad \delta_1 > \delta_2 \tag{2.7}$$

where $[\lambda]$ denotes the integer part of λ.

Finally, we define the error of best approximation of $f \in L^\infty_{w_\gamma}$ by means of polynomials of degree at most m ($P_m \in \mathbb{P}_m$) as

$$E_m(f)_{w_\gamma} = \inf_{P\in\mathbb{P}_m} \|(f-P)w_\gamma\|_\infty,$$

and we recall the following Jackson's inequality (see, for instance, [17], [5])

$$E_m(f)_{w_\gamma} \leq \mathcal{C}\omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma} \tag{2.8}$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$.

3. **Main Results**

3.1. **Shepard-type operator.** Let us consider the Shepard-type operator defined in (1.1)

$$\mathcal{S}_m^{(\alpha)}(f,x) := \frac{\displaystyle\sum_{k=1}^{j}(x-x_k)^{-2}f(x_k)}{\displaystyle\sum_{k=1}^{j}(x-x_k)^{-2}}.$$

By the definition, it follows that $\mathcal{S}_m^{(\alpha)}$ is a rational linear positive average operator such that

$$\begin{aligned} \mathcal{S}_m^{(\alpha)}(e_0,x) &= e_0, \quad e_0 = 1 \\ \mathcal{S}_m^{(\alpha)}(f,x_i) &= f(x_i), \quad \forall i = 1,...,j. \end{aligned} \tag{3.1}$$

By considering $\mathcal{S}_m^{(\alpha)} : L^\infty_{w_\gamma} \to L^\infty_{w_\gamma}$ we can state the following.

Theorem 3.1. *For every $f \in L^\infty_{w_\gamma}$, we have*

$$\|[f - \mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_\infty \le \mathcal{C}\left[\sum_{i=1}^{j}\frac{1}{i^2}\omega_\varphi\left(f, \frac{i}{\sqrt{m}}\right)_{w_\gamma} + e^{-Am}E_0(f)_{w_\gamma}\right] \tag{3.2}$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$ and $A \neq A(m,f)$.

Note that both members of (3.2) vanish if f is a constant function.

Moreover by Theorem 3.1 it follows that, if $f \in Z_r(w_\gamma)$ with

$$Z_r(w_\gamma) = \left\{ f \in L^\infty_{w_\gamma} \ : \ \|f\|_{Z_r(w_\gamma)} = \|fw_\gamma\|_\infty + \sup_{t>0}\frac{\omega_\varphi(f,t)_{w_\gamma}}{t^r} < \infty \right\}, \tag{3.3}$$

it results

$$\|[f - \mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_\infty = \begin{cases} \mathcal{O}\left(\frac{1}{m^{r/2}}\right), & 0 < r < 1; \\ \\ \mathcal{O}\left(\frac{\log m}{\sqrt{m}}\right), & \text{r=1}. \end{cases} \tag{3.4}$$

In other words $\mathcal{S}_m^{(\alpha)}$ converges to f with the same order of the polynomial of best approximation of functions in Zygmund spaces in $(0,+\infty)$.

Furthermore we mention that, if in the definition (1.1) of the Shepard operator we consider the sums until m and not until j, estimate (3.2) is not true (see relation (4.11)).

The following theorem shows that the error $|f(x) - \mathcal{S}_m^{(\alpha)}(f,x)|$ does not improve for smoother functions.

Theorem 3.2. *The asymptotic relation*

$$\frac{\sqrt{m}}{\log m}[\mathcal{S}_m^{(\alpha)}(f,x)-f(x)]w_\gamma(x)=o(1), \quad m\to\infty$$

is not valid for every x and for every non-constant function with continue first derivative.

Finally we underline that, thanks to the interpolatory character of $\mathcal{S}_m^{(\alpha)}(f)$, there exists a subsequence $\{m_k\}$ of natural numbers and a nonconstant function f (see, for instance, [8]) such that

$$\|[f-\mathcal{S}_{m_k}^{(\alpha)}(f)]w_\gamma\|_\infty \le \epsilon_{m_k} \tag{3.5}$$

where $\{\epsilon_m\}_{m=1}^\infty$ is an arbitrary positive fixed sequence.

3.2. **Hermann-Vertési type operator.** Let us consider the following Hermann-Vertési type operator

$$\mathcal{V}_m^{(\alpha)}(f,x)=\frac{\displaystyle\sum_{k=1}^{j} l_k^2(x)f(x_k)}{\displaystyle\sum_{k=1}^{j} l_k^2(x)}$$

where as before the index j is defined in (1.2) and $l_k(x):=l_{m,k}(x)=\frac{p_m(w_\alpha,x)}{(x-x_k)p'_m(w_\alpha,x)}$ are the fundamental Lagrange polynomials based on the zeros of $p_m(w_\alpha)$.

By definition one can easily deduce some properties of $\mathcal{V}_m^{(\alpha)}(f)$. It is a positive linear operator having degree of exactness 1 i.e. $\mathcal{V}_m^{(\alpha)}(1,x)=1$ interpolating the function at the nodes x_i $\forall i=1,...,j$. Moreover, we underline that, on the contrary of the Shepard operator introduced in the previous subsection, it is a polynomial operator.

Next theorem shows that, under suitable conditions on the parameter γ of the weight of the space, $\mathcal{V}_m^{(\alpha)}(f)$ converges to f in $L^\infty_{w_\gamma}$.

Theorem 3.3. *Let $f\in L^\infty_{w_\gamma}$ with*

$$\max\left\{0,\alpha+\frac{1}{2}\right\}\le\gamma\le\alpha+1.$$

Then

$$\|[f-\mathcal{V}_m^{(\alpha)}(f)]w_\gamma\|_\infty \le \mathcal{C}\left[\sum_{i=1}^{j}\frac{1}{i^2}\omega_\varphi\left(f,\frac{i}{\sqrt{m}}\right)_{w_\gamma}+e^{-Am}E_0(f)_{w_\gamma}\right]$$

where $\mathcal{C}\neq\mathcal{C}(m,f)$ and $A\neq A(m,f)$.

Thus, by the previous result, we can deduce that $\mathcal{V}_m^{(\alpha)}$ has the same behavior of the Shepard operator $\mathcal{S}_m^{(\alpha)}$. Consequently, if f belongs to the Zygmund space defined in (3.3), estimate (3.4) still holds true with $\mathcal{V}_m^{(\alpha)}(f)$ in place of $\mathcal{S}_m^{(\alpha)}(f)$.

Moreover, we mention that, in virtue of the interpolating character of $\mathcal{V}_m^{(\alpha)}$, relation (3.5) still satisfied with $\mathcal{V}_{m_k}^{(\alpha)}$ in place of $\mathcal{S}_{m_k}^{(\alpha)}(f)$.

3.3. **Grünwald operator.** In 1942 in [10] G. Grünwald introduced the following operator

$$G_m(f,x) = \sum_{k=1}^{m} l_k^2(x) f(x_k)$$

where f is a continuous function in $(-1,1)$, x_k are the zeros of Jacobi polynomials and l_k are the fundamental Lagrange polynomials based on the nodes x_k.

In 1999 in [22] V. E. S. Szabó investigated on $G_m(f)$ in the case when x_k are the roots of orthogonal polynomials with respect to any weight w belonging to the class of Freud weights W defined in [3]. Thus he proved that if $f \in C_{w^2}$ with

$$C_{w^2} = \left\{ f \ : \ f \ \text{ is continuous on } \mathbb{R} \mid \lim_{|x|\to\infty} (fw^2)(x) = 0 \right\},$$

then

$$\lim_{m\to\infty} \|[f - G_m(f)]w^2\|_\infty = 0$$

and this is the only case in which we can have an homogeneous error estimate in $\mathbb{R}$ for the Grünwald operator. Indeed, if we consider more general weights on $\mathbb{R}$ (see, for instance [21]) then it possible to have the following

$$\forall f \in C_{w_2} \quad \lim_{m\to\infty} \|[f - G_m(f)]w_1\|_\infty = 0$$

with $w_1 \neq w_2$.

In this subsection we will introduce a Grünwald operator at the Laguerre zeros and we will prove that an homogeneous error estimate can be given.

To this end let us consider the zeros of the orthonormal Laguerre polynomial $p_m(w_\alpha)$, $x_1 < ... < x_m$, and let $x_{m+1} := 4m$. We denote by $\tilde{l}_k$ the fundamental Lagrange polynomials based on the zeros $x_1 < x_2 < ... < x_m < x_{m+1}$

$$\tilde{l}_k(x) := \tilde{l}_{m,k}(x) = \frac{4m - x}{4m - x_k}\, l_{m,k}(x), \quad k \le m, \tag{3.6}$$

$$\tilde{l}_{m+1}(x) := \tilde{l}_{m,m+1}(x) = \frac{p_m(w_\alpha, x)}{p_m(w_\alpha, 4m)} \tag{3.7}$$

and we introduce the Grünwald operator

$$G_{m+1}^{(\alpha)}(f,x) = \sum_{k=1}^{m+1} \tilde{l}_k^2(x) f(x_k). \tag{3.8}$$

By the definition it easily results

$$G_{m+1}^{(\alpha)}(1,x)=1,\quad G_{m+1}^{(\alpha)}(f,x_i)=f(x_i),\quad \forall i=1,...,m+1.$$

Now if

$$x_j=\min_{1\le k\le m}\{x_k : x_k\ge 2m\},\tag{3.9}$$

for any $f\in C^{\circ}((0,\infty))$ we introduce the following function

$$f_j(x)=\begin{cases} f(x), & x\le x_j,\\ 0, & x>x_j\end{cases}\tag{3.10}$$

and we define the operator

$$G_{m+1}^{*(\alpha)}(f,x):=G_{m+1}^{(\alpha)}(f_j,x)=\sum_{k=1}^{j}\tilde{l}_k^2(x)f(x_k).\tag{3.11}$$

Next theorem gives an error estimate for $G_{m+1}^{*(\alpha)}(f)$.

Theorem 3.4. *Let $f\in L^{\infty}_{w_\gamma}$ with*

$$\max\left\{0,\alpha+\frac{1}{2}\right\}\le\gamma\le\alpha+1.$$

Then

$$\|[f-G_{m+1}^{*(\alpha)}(f)]w_\gamma\|_\infty\le\mathcal{C}\left[\sum_{i=1}^{j}\frac{1}{i^2}\omega_\varphi\left(f,\frac{i}{\sqrt{m}}\right)_{w_\gamma}+\frac{\|fw_\gamma\|_\infty}{m}\right]\tag{3.12}$$

where $\mathcal{C}\ne\mathcal{C}(m,f)$.

4. **Proofs**

In order to prove Theorem 3.1 we need the following lemma.

Lemma 4.1. *For every $f\in L^{\infty}_{w_\gamma}$ we have*

$$\|[f-\mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_{L^\infty((0,x_j])}\le\mathcal{C}\sum_{i=1}^{j}\frac{1}{i^2}\omega_\varphi\left(f,\frac{i}{\sqrt{m}}\right)_{w_\gamma}\tag{4.1}$$

where $\mathcal{C}\ne\mathcal{C}(m,f)$.

Proof. Since $\mathcal{S}_m^{(\alpha)}(e_0, x) = e_0$ with $e_0 = 1$ we can write

$$|[f(x) - \mathcal{S}_m^{(\alpha)}(f,x)]w_\gamma(x)| = |[f(x)\mathcal{S}_m^{(\alpha)}(e_0,x) - \mathcal{S}_m^{(\alpha)}(f,x)]w_\gamma(x)| \le \frac{\sum_{k=1}^{j}(x-x_k)^{-2}|f(x)-f(x_k)|}{\sum_{k=1}^{j}(x-x_k)^{-2}}.$$

Now we denote by x_d a node closest to x and assume $x_{d-1} < x < x_d < x_{d+1}$, $d \ge 2$. Then since

$$\sum_{k=1}^{j} \frac{1}{(x-x_k)^2} > \frac{1}{(x-x_d)^2}$$

we have

$$\begin{aligned}
|[f(x) - \mathcal{S}_m^{(\alpha)}(f,x)]w_\gamma(x)| &\le \sum_{k=1}^{j}\left(\frac{x-x_d}{x-x_k}\right)^2 |f(x)-f(x_k)|w_\gamma(x) \\
&= |f(x)-f(x_d)|w_\gamma(x) + \sum_{k=1}^{d-1}\left(\frac{x-x_d}{x-x_k}\right)^2 |f(x)-f(x_k)|w_\gamma(x) \\
&\quad + \sum_{k=d+1}^{j}\left(\frac{x-x_d}{x-x_k}\right)^2 |f(x)-f(x_k)|w_\gamma(x) \\
&:= A_1(x) + A_2(x) + A_3(x). \qquad (4.2)
\end{aligned}$$

In order to estimate $A_1(x)$ it is sufficient to observe that, being $x_{d-1} < x < x_d$ and $x \in (0, x_j]$, by (2.2) we have

$$x_d - x < x_d - x_{d-1} = \Delta x_{d-1} \le \mathcal{C}\sqrt{\frac{x_d}{m}}. \qquad (4.3)$$

Therefore, setting $\varphi(y) = \sqrt{y}$ and taking into account (2.5) we get

$$\begin{aligned}
A_1(x) &= |f(x + (x_d - x)) - f(x)|w_\gamma(x) \\
&\le \sup_{0<h\le\frac{1}{\sqrt{2m}}} \sup_{y\in\left(4h^2, \frac{1}{h^2}\right)} |f(y+h\varphi(y)) - f(y)|w_\gamma(y) \\
&\le \mathcal{C}\ \Omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma}. \qquad (4.4)
\end{aligned}$$

Now we estimate $A_2(x)$. Since $x_k < x < x_d$, $\forall k$ then $w_\gamma(x) \le \mathcal{C}w_\gamma(x_k)$ and

$$x - x_k < x_d - x_k < x_{d+1} - x_k = \sum_{i=k}^{d} \Delta x_i \le (d-k+1)\Delta x_d.$$

Thus, by using (4.3),(2.7) and (2.5) we can write

$$
\begin{aligned}
A_2(x) &\le \mathcal{C}\sum_{k=1}^{d-1}\left(\frac{x-x_d}{x-x_k}\right)^2 |f(x_k+(x-x_k))-f(x_k)|w_\gamma(x_k)\\
&\le \mathcal{C}\sum_{k=1}^{d-1}\left(\frac{x-x_d}{x-x_k}\right)^2 \Omega_\varphi\left(f,\frac{d-k+1}{\sqrt{m}}\right)_{w_\gamma}\\
&\le \mathcal{C}\left[\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma}+\sum_{k=1}^{[\frac{d}{2}]}\left(\frac{x-x_d}{x-x_k}\right)^2\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\right.\\
&\qquad\left.+\sum_{k=[\frac{d}{2}]+1}^{d-2}\left(\frac{x-x_d}{x-x_k}\right)^2\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\right]. \qquad (4.5)
\end{aligned}
$$

If $1 \le k \le [\frac{d}{2}]$ since in general it results

$$
x_i - x_r = \sum_{\ell=r}^{i-1}\Delta x_\ell > \Delta x_r(i-r), \quad \forall i > r \qquad (4.6)
$$

we have

$$
\begin{aligned}
(x-x_k) > (x_{d-1}-x_k) &= (x_{d-1}-x_{[\frac{d}{2}]})+(x_{[\frac{d}{2}]}-x_k)\\
&> \frac{d}{2}\Delta x_{[\frac{d}{2}]}+\left(\left[\frac{d}{2}\right]-k\right)\Delta x_k\\
&> \frac{d}{2}\Delta x_{[\frac{d}{2}]} > \mathcal{C}\frac{d}{2}\Delta x_d. \qquad (4.7)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sum_{k=1}^{[\frac{d}{2}]}\left(\frac{x-x_d}{x-x_k}\right)^2\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma} &\le \mathcal{C}\sum_{k=1}^{[\frac{d}{2}]}\frac{1}{d^2}\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\\
&\le \mathcal{C}\frac{1}{d}\Omega_\varphi\left(f,\frac{d}{\sqrt{m}}\right)_{w_\gamma}\\
&\le \mathcal{C}\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma}, \qquad (4.8)
\end{aligned}
$$

in virtue of (2.5).

If $[\frac{d}{2}]+1 \le k \le d-2$, then by applying (4.6) and by using (2.2) and (2.1) we have

$$\begin{aligned}\sum_{k=[\frac{d}{2}]+1}^{d-2} \left(\frac{x-x_d}{x-x_k}\right)^2 \Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma} &\le \sum_{k=[\frac{d}{2}]+1}^{d-2} \left(\frac{\Delta x_d}{(d-k)\Delta x_k}\right)^2 \Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma} \\ &\le \mathcal{C} \sum_{k=[\frac{d}{2}]+1}^{d-2} \frac{d^2}{(d-k)^2k^2}\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma} \\ &\le \mathcal{C} \sum_{k=[\frac{d}{2}]+1}^{d-2} \frac{1}{(d-k)^2}\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}. \qquad (4.9)\end{aligned}$$

Thus by replacing (4.8) and (4.9) in (4.5) we obtain

$$A_2(x) \le \mathcal{C}\left[\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma} + \sum_{k=[\frac{d}{2}]+1}^{d-2} \frac{1}{(d-k)^2}\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\right]. \qquad (4.10)$$

It remains to estimate $A_3(x)$. To this end it is sufficient to note that

$$x_k - x < x_k - x_{d-1} < (k-d)\Delta x_k \le \mathcal{C}(k-d)\sqrt{\frac{x_k}{m}}, \quad k>d \qquad (4.11)$$

$$x_k - x_d > (k-d)\Delta x_d,$$

and to proceed as done for $A_2(x)$. Hence we have

$$A_3(x) \le \mathcal{C}\left[\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma} + \sum_{k=d+2}^{j} \frac{1}{(k-d)^2}\Omega_\varphi\left(f,\frac{k-d}{\sqrt{m}}\right)_{w_\gamma}\right]. \qquad (4.12)$$

Thus by replacing (4.4), (4.10) and (4.12) in (4.2) we get

$$\|[f-\mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_{L^\infty((0,x_j])} \le \mathcal{C}\sum_{i=1}^{j}\frac{1}{i^2}\Omega_\varphi\left(f,\frac{i}{\sqrt{m}}\right)_{w_\gamma}$$

from which the thesis follows by using (2.6). □

Proof. [Proof of Theorem 3.1] Let $\Psi \in C^\infty(\mathbb{R})$ non decreasing such that

$$\Psi(x) = \begin{cases} 1, & \text{if } x \ge 1, \\ 0, & \text{if } x \le 0, \end{cases} \qquad (4.13)$$

and we introduce the function

$$\Psi_j(x) = \Psi\left(\frac{x-x_j}{x_{j+1}-x_j}\right) \qquad (4.14)$$

where $x_j = \min\{x_k : x_k \geq 2m\}$. Moreover for any function $f \in C^\circ((0,\infty))$ we define $f_j = (1-\Psi_j)f$. Obviously $f_j = f$ in $(0, x_j]$ and $f_j = 0$ in $[x_{j+1}, \infty)$. Now since $\mathcal{S}_m^{(\alpha)}(f) = \mathcal{S}_m^{(\alpha)}(f_j)$ we can write

$$\begin{aligned}\|[f - \mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_\infty &\leq \|[f - f_j]w_\gamma\|_\infty + \|[f_j - \mathcal{S}_m^{(\alpha)}(f_j)]w_\gamma\|_\infty \\ &\leq \mathcal{C}\Big[\|fw_\gamma\|_{L^\infty((x_j,+\infty))} + \|[f - \mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_{L^\infty((0,x_j])} \\ &\qquad + \|\mathcal{S}_m^{(\alpha)}(f)w_\gamma\|_{L^\infty((x_j,+\infty))}\Big] \\ &:= \mathcal{C}[N_1 + N_2 + N_3]. \end{aligned} \tag{4.15}$$

In order to estimate N_1 we denote by Q_M the near best approximant polynomial of f i.e. $\|[f - Q_M]w_\gamma\|_\infty \leq \mathcal{C}E_M(f)_{w_\gamma}$ with $M = \frac{1}{3}m$ and we write

$$\begin{aligned} N_1 &\leq \|[f - Q_M]w_\gamma\|_{L^\infty((x_j,+\infty))} + \|Q_M w_\gamma\|_{L^\infty((x_j,+\infty))} \\ &\leq \mathcal{C}E_M(f)_{w_\gamma} + \|Q_M w_\gamma\|_{L^\infty((x_j,+\infty))} \\ &\leq \mathcal{C}E_M(f)_{w_\gamma} + \|Q_M w_\gamma\|_{L^\infty([2m,+\infty))} \end{aligned} \tag{4.16}$$

being $x_j \geq 2m$. Now we recall that (see, for instance, [17]) $\forall P_m \in \mathbb{P}_m$ and $\forall \delta > 0$ it results

$$\|P_m w_\gamma\|_{L^\infty([4m(1+\delta),+\infty))} \leq \mathcal{C}e^{-Am}\|P_m w_\gamma\|_\infty \tag{4.17}$$

where $\mathcal{C}$ and A are positive constants independent of m and P_m.
Therefore, since for the choice of M, it results $2m = 4M(1+\frac{1}{2})$, by applying (4.17) to the last term of (4.16) we have

$$\begin{aligned} N_1 &\leq \mathcal{C}[E_M(f)_{w_\gamma} + e^{-Am}\|Q_M w_\gamma\|_\infty] \\ &\leq \mathcal{C}[E_M(f)_{w_\gamma} + e^{-Am}\|f w_\gamma\|_\infty] \end{aligned} \tag{4.18}$$

Consequently, by using (2.8) we deduce

$$N_1 \leq \mathcal{C}\left[\omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma} + e^{-Am}\|fw_\gamma\|_\infty\right]. \tag{4.19}$$

The term N_2 is estimated in Lemma 4.1 and then it remains only to bound N_3. Then let $x \in (x_j, +\infty)$, by the definition we get

$$\begin{aligned}
&|\mathcal{S}_m^{(\alpha)}(f,x)w_\gamma(x)| \\
&\le \sum_{k=1}^{j} \left(\frac{x-x_j}{x-x_k}\right)^2 |(fw_\gamma)(x_k)| \frac{w_\gamma(x)}{w_\gamma(x_k)} \\
&= \left\{ \sum_{x_1 \le x_k \le x_{[\frac{j}{2}]}} + \sum_{x_{[\frac{j}{2}]+1} \le x_k \le x_j} \right\} \left(\frac{x-x_j}{x-x_k}\right)^2 \left(\frac{x}{x_k}\right)^\gamma e^{-(x-x_k)} |(fw_\gamma)(x_k)|.
\end{aligned}$$

The first sum tends to zero exponentially. In fact, since in virtue of (2.1) it results $x_k \le x_{[\frac{j}{2}]} \le \frac{1}{4}\frac{j^2}{m} \le \frac{1}{4}x_j$, we deduce $e^{-(x-x_k)} \le e^{-\frac{3}{4}x_j} \le e^{-\frac{3}{2}m}$ being $x > x_j > 2m$. It remains to estimate the second sum. We first assume $x_j < x \le x_{[\frac{3}{2}j]}$. In virtue of (2.1) we have $(\frac{x}{x_k})^\gamma \le \mathcal{C}$. Moreover, taking into account that $\sup_x (x-x_j)^2 e^{-(x-x_j)} \le \mathcal{C}$, we have

$$\begin{aligned}
&\sum_{x_{[\frac{j}{2}]+1} \le x_k \le x_j} \left(\frac{x-x_j}{x-x_k}\right)^2 \left(\frac{x}{x_k}\right)^\gamma e^{-(x-x_k)} |(fw_\gamma)(x_k)| \\
&\qquad \le \mathcal{C}\ \|fw_\gamma\|_{L^\infty([x_{[\frac{j}{2}]+1}, x_j])} \sum_{x_{[\frac{j}{2}]+1} \le x_k \le x_j} \frac{1}{(x-x_k)^2} \\
&\qquad \le \mathcal{C}\ \|fw_\gamma\|_{L^\infty((\frac{m}{2},\infty))} \sum_{x_{[\frac{j}{2}]+1} \le x_k \le x_j} \frac{1}{(j-k+1)^2}
\end{aligned}$$

being $x_{[\frac{j}{2}]+1} > \frac{x_j}{2} > \frac{m}{2}$ and by applying (4.6). Hence by estimating the norm $\|fw_\gamma\|_{L^\infty((\frac{m}{2},\infty))}$ as already done for the term N_1 with $M = \frac{m}{12}$ we get

$$\sum_{x_{[\frac{j}{2}]+1} \le x_k \le x_j} \left(\frac{x-x_j}{x-x_k}\right)^2 \left(\frac{x}{x_k}\right)^\gamma e^{-(x-x_k)} |(fw_\gamma)(x_k)| \le \mathcal{C} e^{-Am} \|fw_\gamma\|_\infty.$$

Finally if $x > x_{[\frac{3}{2}j]}$ the sum tends to zero as e^{-Am}, $A > 0$. Thus we deduce

$$N_3 \le \mathcal{C} e^{-Am} \|fw_\gamma\|_\infty. \tag{4.20}$$

Hence by replacing (4.19), (4.1) and (4.20) in (4.15) we obtain

$$\|[f - \mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_\infty \le \mathcal{C} \left[\sum_{i=1}^{j} \frac{1}{i^2} \omega_\varphi \left(f, \frac{i}{\sqrt{m}}\right)_{w_\gamma} + e^{-Am} \|fw_\gamma\|_\infty \right]. \tag{4.21}$$

Now we consider $G(x) = f(x) - \mathcal{C}$ where $\mathcal{C}$ is a positive constant. By using (3.1), we can write

$$\|[f - \mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_\infty = \|[G - \mathcal{S}_m^{(\alpha)}(G)]w_\gamma\|_\infty.$$

Then by applying (4.21) to G we get

$$\|[f-\mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_\infty \le \mathcal{C}\left[\sum_{i=1}^{j}\frac{1}{i^2}\omega_\varphi\left(f-\mathcal{C},\frac{i}{\sqrt{m}}\right)_{w_\gamma}+e^{-Am}\|[f-\mathcal{C}]w_\gamma\|\right].$$

Thus being $\omega_\varphi\left(f-\mathcal{C},\frac{i}{\sqrt{m}}\right)_{w_\gamma} \le \omega_\varphi\left(f,\frac{i}{\sqrt{m}}\right)_{w_\gamma}$ and taking the infimum on $\mathcal{C}$ at the right-hand side we have

$$\|[f-\mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_\infty \le \mathcal{C}\left[\sum_{i=1}^{j}\frac{1}{i^2}\omega_\varphi\left(f,\frac{i}{\sqrt{m}}\right)_{w_\gamma}+e^{-Am}E_0(f)_{w_\gamma}\right]$$

and the theorem follows. □

Proof. [Proof of Theorem 3.2] Let f be a function with continue first derivative such that $f'(x)>0$ and $\int_{1/m}^{1}\frac{\Omega_\varphi(f',\sqrt{m}t)_{w_\gamma\varphi}}{t^2}dt<\infty$. Since by the Taylor formula, we have

$$f(x_k)-f(x)=(x_k-x)f'(x)+G_x(x_k) \tag{4.22}$$

where $G_x(x_k)$ is the error, we can write

$$\begin{aligned}\mathcal{S}_m^{(\alpha)}(f,x)-f(x) &= \frac{\sum_{k=1}^{j}(x-x_k)^{-2}[f(x_k)-f(x)]}{\sum_{k=1}^{j}(x-x_k)^{-2}}\\ &= \frac{\sum_{k=1}^{j}(x-x_k)^{-2}[f'(x)(x_k-x)+G_x(x_k)]}{\sum_{k=1}^{j}(x-x_k)^{-2}},\end{aligned}$$

and then in virtue of the linearity of $\mathcal{S}_m^{(\alpha)}$

$$[\mathcal{S}_m^{(\alpha)}(f,x)-f(x)]w_\gamma(x)=w_\gamma(x)f'(x)\mathcal{S}_m^{(\alpha)}(g_x,x)+w_\gamma(x)\mathcal{S}_m^{(\alpha)}(G_x,x) \tag{4.23}$$

where $g_x(t)=t-x$.

We estimate the last term of (4.23). We denote by x_d a node closest to x and assume $x_{d-1}<x<x_d$. By the definition of G_x we have

$$|G_x(x_k)|\le|x_k-x||f'(x+(\xi-x))-f'(x)|,\quad \xi\in(x_k,x)$$

and since

$$|\xi-x|\le|x_k-x|\le\Delta x_k(|k-d|+1)\le\sqrt{\frac{y}{m}}(|k-d|+1),\quad y=\sup x_k$$

we get

$$\begin{aligned}&|\mathcal{S}_m^{(\alpha)}(G_x,x)w_\gamma(x)|\\&\leq \mathcal{C}\sum_{k=1}^{j}\frac{(x-x_d)^2}{|x_k-x|}\sup_{0<h\leq\frac{k-d+1}{\sqrt{m}}}\ \sup_{y\in\left(4h^2,\frac{1}{h^2}\right)}|f'(y+h\varphi(y))-f'(y)|w_\gamma(y).\end{aligned}$$

Now, being $x-x_d\leq\frac{\varphi(x)}{\sqrt{m}}$, we have

$$\begin{aligned}&|\mathcal{S}_m^{(\alpha)}(G_x,x)w_\gamma(x)|\\&\leq\frac{\mathcal{C}}{\sqrt{m}}\sum_{k=1}^{j}\left|\frac{x-x_d}{x_k-x}\right|\sup_{0<h\leq\frac{k-d+1}{\sqrt{m}}}\ \sup_{y\in\left(4h^2,\frac{1}{h^2}\right)}|f'(y+h\varphi(y))-f'(y)|(w_\gamma\varphi)(y)\\&\leq\frac{\mathcal{C}}{\sqrt{m}}\sum_{k=1}^{j}\left|\frac{x-x_d}{x_k-x}\right|\Omega_\varphi\left(f',\frac{k-d+1}{\sqrt{m}}\right)_{w_\gamma\varphi}.\end{aligned}$$

Consequently, by proceeding as in the proof of Theorem 3.1, we found

$$\begin{aligned}|\mathcal{S}_m^{(\alpha)}(G_x,x)w_\gamma(x)|&\leq\frac{\mathcal{C}}{\sqrt{m}}\sum_{i=1}^{j}\frac{1}{i}\Omega_\varphi\left(f',\frac{i}{\sqrt{m}}\right)_{w_\gamma\varphi}\\&\leq\frac{\mathcal{C}}{\sqrt{m}}\int_{\frac{1}{m}}^{1}\frac{\Omega_\varphi(f',\sqrt{m}t)_{w_\gamma\varphi}}{t^2}dt\end{aligned}$$

from which in virtue of the assumptions on f', we can deduce

$$\lim_{m\to\infty}\frac{\sqrt{m}}{\log m}\mathcal{S}_m^{(\alpha)}(G_x,x)w_\gamma(x)=0.$$

Then by (4.23) we have

$$\lim_{m\to\infty}\frac{\sqrt{m}}{\log m}[\mathcal{S}_m^{(\alpha)}(f,x)-f(x)]w_\gamma(x)=\lim_{m\to\infty}\frac{\sqrt{m}}{\log m}\mathcal{S}_m^{(\alpha)}(g_x,x)w_\gamma(x)f'(x).$$

Now choose $x\in I:=(2m-(2m)^{1/4},2m]$. It results

$$\mathcal{S}_m^{(\alpha)}(g_x,x)=\frac{\displaystyle\sum_{k=1}^{j}\frac{1}{|x-x_k|}}{\displaystyle\sum_{k=1}^{j}\frac{1}{(x-x_k)^2}}\geq\mathcal{C}\frac{\log m}{\sqrt{m}},\quad\forall x\in I$$

from which

$$\lim_{m\to\infty}\frac{\sqrt{m}}{\log m}\mathcal{S}_m^{(\alpha)}(g_x,x)w_\gamma(x)f'(x)>w_\gamma(x)f'(x)>0$$

and the theorem follows. □

In order to prove Theorem 3.3 we recall that denoting by $\lambda_m(w_\alpha, x) = [\sum_{k=0}^{m-1} p_k^2(w_\alpha, x)]^{-1}$ the m-th Christoffel functions, it results (see, for instance, [23] and [18])

$$l_k^2(x) := l_{m,k}^2(w_\alpha, x) = p_m^2(w_\alpha, x)\frac{x_k \lambda_m(w_\alpha, x_k)}{(x - x_k)^2}, \quad \forall k = 1, ..., m. \tag{4.24}$$

and (see, for instance, [9], [12], [18])

$$\lambda_m(w_\alpha, x_k) \sim w_\alpha(x_k)\Delta x_k \sim w_\alpha(x_k)\sqrt{\frac{x_k}{4m - x_k}}. \tag{4.25}$$

Proof. [Proof of Theorem 3.3] We proceed as in the proof of Theorem 3.1. By estimate (4.15) with $\mathcal{V}_m^{(\alpha)}$ in place of $\mathcal{S}_m^{(\alpha)}$ we get

$$\begin{aligned}
\|[f - \mathcal{V}_m^{(\alpha)}(f)]w_\gamma\|_\infty &\leq \|[f - f_j]w_\gamma\|_\infty + \|[f_j - \mathcal{V}_m^{(\alpha)}(f_j)]w_\gamma\|_\infty \\
&\leq \mathcal{C}\Big[\|fw_\gamma\|_{L^\infty((x_j,+\infty))} + \|[f - \mathcal{V}_m^{(\alpha)}(f)]w_\gamma\|_{L^\infty((0,x_j])} \\
&\qquad + \|\mathcal{V}_m^{(\alpha)}(f)w_\gamma\|_{L^\infty((x_j,+\infty))}\Big] \\
&:= \mathcal{C}[N_1 + N_2 + N_3].
\end{aligned} \tag{4.26}$$

The term N_1 can be estimated by proceeding as in the proof of Theorem 3.1 (see estimate (4.18)).
Now we consider N_2. Since $\mathcal{V}_m^{(\alpha)}(f, 1) = 1$ we have

$$|[f(x) - \mathcal{V}_m^{(\alpha)}(f, x)]w_\gamma(x)| = \frac{\displaystyle\sum_{k=1}^{j} l_k^2(x)w_\gamma(x)|f(x) - f(x_k)|}{\displaystyle\sum_{k=1}^{j} l_k^2(x)}.$$

We denote by x_d a node closest to x and we assume $x_{d-1} < x < x_d < x_{d+1}$, $d \geq 2$ $x \in (0, x_j]$. Then we write

$$\begin{aligned}
|[f(x) - \mathcal{V}_m^{(\alpha)}(f, x)]w_\gamma(x)| &\leq \sum_{k=1}^{j} \frac{l_k^2(x)}{l_d^2(x)}|f(x) - f(x_k)|w_\gamma(x) \\
&= |f(x) - f(x_d)|w_\gamma(x) + \sum_{\substack{k=1\\k\neq d}}^{j} \frac{l_k^2(x)}{l_d^2(x)}|f(x) - f(x_k)|w_\gamma(x) \\
&\leq |f(x) - f(x_d)|w_\gamma(x) \\
&\quad + \sum_{\substack{k=1\\k\neq d}}^{j} \left(\frac{x - x_d}{x - x_k}\right)^2 \frac{x_k}{x_d}\frac{\lambda_m(w_\alpha, x_k)}{\lambda_m(w_\alpha, x_d)}w_\gamma(x)|f(x) - f(x_k)|
\end{aligned}$$

by using (4.24). Moreover, by applying (4.25) we found

$$\begin{aligned}
&|[f(x) - \mathcal{V}_m^{(\alpha)}(f,x)]w_\gamma(x)| \\
&\leq \mathcal{C}\left[|[f(x) - f(x_d)]w_\gamma(x)| \right.\\
&\left. + \left\{\sum_{k=1}^{d-1} + \sum_{k=d+1}^{j}\right\}\left(\frac{x-x_d}{x-x_k}\right)^2\left(\frac{x_k}{x}\right)^{1+\alpha-\gamma}\frac{\Delta x_k}{\Delta x_d}w_\gamma(x_k)|f(x)-f(x_k)|\right] \\
&:= \mathcal{C}\left[|[f(x) - f(x_d)]w_\gamma(x)| + S_1(x) + S_2(x)\right]. \qquad (4.27)
\end{aligned}$$

The first term is equal to that considered in (4.4) and it results

$$|f(x) - f(x_d)|w_\gamma(x) \leq \mathcal{C}\Omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma}. \qquad (4.28)$$

About S_1, we note that since $x_k < x$ for all k, then $\Delta x_k \leq \Delta x_d$ and taking into account that $\gamma \leq \alpha + 1$ we have

$$S_1(x) \leq \mathcal{C}\sum_{k=1}^{d-1}\left(\frac{x-x_d}{x-x_k}\right)^2 w_\gamma(x_k)|f(x)-f(x_k)|.$$

This sum was already estimated in the proof of Theorem 3.1 (see, estimates (4.5)-(4.10)) and it results

$$S_1(x) \leq \mathcal{C}\left[\Omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma} + \sum_{k=[\frac{d}{2}]+1}^{d-2}\frac{1}{(d-k)^2}\Omega_\varphi\left(f, \frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\right]. \qquad (4.29)$$

Now we give a bound for S_2. To this end we apply (2.2) getting

$$S_2(x) \leq \sum_{k=d+1}^{j}\left(\frac{x-x_d}{x-x_k}\right)^2\left(\frac{x_k}{x}\right)^{\frac{1}{2}+\alpha-\gamma}\frac{\Delta^2 x_k}{\Delta^2 x_d}\left(\frac{4m-x_k}{4m-x_d}\right)^{1/2}w_\gamma(x_k)|f(x)-f(x_k)|.$$

Now by observing that $\left(\frac{4m-x_k}{4m-x_d}\right) \leq \mathcal{C}$, taking into account that $\frac{x_k}{x} \leq 1$ being $\gamma \geq \alpha + \frac{1}{2}$, and by using (4.6) we have

$$\begin{aligned}
S_2 &\leq \mathcal{C}\sum_{k=d+1}^{j}\left(\frac{\Delta x_k}{x-x_k}\right)^2 w_\gamma(x_k)|f(x)-f(x_k)| \\
&\leq \mathcal{C}\sum_{k=d+1}^{j}\frac{1}{(k-d)^2}w_\gamma(x_k)|f(x)-f(x_k)|.
\end{aligned}$$

Thus by proceeding as done in Theorem 3.1 for the term A_3 we get

$$S_2(x) \leq \mathcal{C}\left[\Omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma} + \sum_{k=d+2}^{j}\frac{1}{(k-d)^2}\Omega_\varphi\left(f, \frac{k-d}{\sqrt{m}}\right)_{w_\gamma}\right]. \qquad (4.30)$$

Then in definitive by applying (4.28), (4.29) and (4.30) in (4.27) we found

$$N_2 \le \mathcal{C} \sum_{i=1}^{j} \frac{1}{i^2} \Omega_\varphi \left(f, \frac{i}{\sqrt{m}} \right)_{w_\gamma}. \tag{4.31}$$

Finally we consider N_3. By using (4.24),(4.25) and taking into account that $\gamma \le \alpha+1$ we get

$$\begin{aligned} \|\mathcal{V}_m^{(\alpha)}(f) w_\gamma\|_{L^\infty((x_j,+\infty))} &\le \sup_{x>x_j} w_\gamma(x) \sum_{k=1}^{j} \frac{l_k^2(x)}{l_j^2(x)} |f(x_k)| \\ &\le \mathcal{C} \sup_{x>x_j} \sum_{k=1}^{j} \left(\frac{x-x_j}{x-x_k} \right)^2 e^{-(x-x_j)} |(f w_\gamma)(x_k)| \end{aligned}$$

and this sum can be estimated as already done for the term N_3 in the proof of Theorem 3.1 (see estimate (4.20)). Thus by replacing (4.18), (4.31) and (4.20) in (4.26) and by applying (2.8) we deduce

$$\|[f - \mathcal{V}_m^{(\alpha)}(f)] w_\gamma\|_\infty \le \mathcal{C} \left[\sum_{i=1}^{j} \frac{1}{i^2} \omega_\varphi \left(f, \frac{i}{\sqrt{m}} \right)_{w_\gamma} + e^{-Am} \|f w_\gamma\|_\infty \right].$$

Now the thesis follows by introducing the function $G = f - \mathcal{C}$ where $\mathcal{C}$ is a positive constant and by proceeding as done in Theorem 3.1. □

In order to prove Theorem 3.4 we need the following Lemma.

Lemma 4.2. *Let $f \in L^\infty_{w_\gamma}$ with*

$$\max\left\{0, \alpha + \frac{1}{2}\right\} \le \gamma \le \alpha + 1.$$

Then

$$\|[f - G_{m+1}^{*(\alpha)}(f)] w_\gamma\|_{L^\infty((0,x_j])} \le \mathcal{C} \left[\sum_{i=1}^{j} \frac{1}{i^2} \omega_\varphi \left(f, \frac{i}{\sqrt{m}} \right)_{w_\gamma} + \frac{\|f w_\gamma\|_\infty}{m} \right] \tag{4.32}$$

where $\mathcal{C} \ne \mathcal{C}(m, f)$.

Proof. Since $G^{(\alpha)}_{m+1}(1,x) = 1$ we have

$$\begin{aligned} &\|[f - G^{*(\alpha)}_{m+1}(f)]w_\gamma\|_{L^\infty((0,x_j])} \qquad (4.33)\\ &\le \sup_{x\in(0,x_j]} \sum_{k=1}^{m+1} \tilde{l}^2_k(x) w_\gamma(x)|f_j(x) - f_j(x_k)| \\ &= \sup_{x\in(0,x_j]} \left\{ \sum_{k=1}^{j} \tilde{l}^2_k(x) w_\gamma(x)|f(x) - f(x_k)| + \sum_{k=j+1}^{m+1} \tilde{l}^2_k(x) w_\gamma(x)|f(x)| \right\} \\ &:= \sup_{x\in(0,x_j]} \{A_1(x) + A_2(x)\}. \qquad (4.34) \end{aligned}$$

We denote by x_d a node closest to x by assuming $x_{d-1} < x < x_d$, $d \ge 2$. We have

$$\begin{aligned} A_1(x) &:= \sum_{k=1}^{j} \tilde{l}^2_k(x) w_\gamma(x)|f(x) - f(x_k)| \\ &= \tilde{l}^2_d(x) w_\gamma(x)|f(x) - f(x_d)| + \sum_{\substack{k=1\\k\ne d}}^{j} \tilde{l}^2_k(x) w_\gamma(x)|f(x) - f(x_k)| \\ &\le \tilde{l}^2_d(x) \frac{w_\gamma(x)}{w_\gamma(x_d)} |f(x) - f(x_d)| w_\gamma(x_d) + \sum_{\substack{k=1\\k\ne d}}^{j} \tilde{l}^2_k(x) w_\gamma(x)|f(x) - f(x_k)| \\ &\le \tilde{l}^2_d(x) \frac{w_\gamma(x)}{w_\gamma(x_d)} |f(x + (x_d - x)) - f(x)| w_\gamma(x) + \sum_{\substack{k=1\\k\ne d}}^{j} \tilde{l}^2_k(x) w_\gamma(x)|f(x) - f(x_k)| \\ &\le \mathcal{C}\Omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma} + \sum_{\substack{k=1\\k\ne d}}^{j} \tilde{l}^2_k(x) w_\gamma(x)|f(x) - f(x_k)| \end{aligned}$$

being $\tilde{l}^2_d(x)\frac{w_\gamma(x)}{w_\gamma(x_d)} \le \mathcal{C}$ (see, for instance, [14, Lemma 3.2]) and by applying (4.4). Now, in order to estimate the sum, we recall that (see, for instance, [17])

$$|p_m(w_\alpha, x)\sqrt{w_\alpha(x)}| \le \frac{\mathcal{C}}{\sqrt[4]{x}\sqrt[4]{4m - x + \frac{4m}{m^{2/3}}}}, \quad x \in \left(\frac{\mathcal{C}}{m}, 4m\right) \qquad (4.35)$$

and then by relation (4.24) and (4.25) we get that for $x \in (\frac{\mathcal{C}}{m}, 4m)$ the following inequality holds true

$$\tilde{l}^2_k(x) w_\gamma(x) \le \mathcal{C}\left(\frac{4m - x}{4m - x_k}\right)^2 x^{\gamma-\alpha-\frac{1}{2}} \frac{\Delta x_k}{(x - x_k)^2} \frac{x_k w_\alpha(x_k)}{\sqrt{4m - x}}. \qquad (4.36)$$

Therefore

$$\begin{aligned}
&A_1(x)\\
&\leq \mathcal{C}\left[\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma}\right.\\
&\qquad\left.+\left\{\sum_{k=1}^{d-1}+\sum_{k=d+1}^{j}\right\}\left(\frac{4m-x}{4m-x_k}\right)^2 x^{\gamma-\alpha-\frac{1}{2}}\frac{\Delta x_k}{(x-x_k)^2}\,\frac{x_k w_\alpha(x_k)}{\sqrt{4m-x}}|f(x)-f(x_k)|\right]\\
&=\mathcal{C}\left[\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma}+\Sigma_1(x)+\Sigma_2(x)\right].
\end{aligned}$$

About Σ_1 we write

$$\begin{aligned}
\Sigma_1(x)&=\sum_{k=1}^{d-1}\left(\frac{4m-x}{4m-x_k}\right)^2 x^{\gamma-\alpha-\frac{1}{2}}\frac{\Delta x_k}{(x-x_k)^2}\,\frac{x_k^{1+\alpha-\gamma}}{\sqrt{4m-x}}|f(x)-f(x_k)|w_\gamma(x_k)\\
&=\sum_{k=1}^{d-1}\left(\frac{4m-x}{4m-x_k}\right)^2\sqrt{\frac{x}{4m-x}}\frac{\Delta x_k}{(x-x_k)^2}\left(\frac{x_k}{x}\right)^{1+\alpha-\gamma}|f(x)-f(x_k)|w_\gamma(x_k).
\end{aligned}$$

Now $\left(\frac{4m-x}{4m-x_k}\right)^2<\mathcal{C}$, $\left(\frac{x_k}{x}\right)^{1+\alpha-\gamma}<1$ being $\gamma\leq\alpha+1$ and $\sqrt{\frac{x}{4m-x}}\Delta x_k\leq\sqrt{\frac{x_d}{4m-x_d}}\Delta x_d\leq\Delta^2 x_d<(x-x_d)^2$, being $\Delta x_k\leq\Delta x_d$. Therefore

$$\Sigma_1(x)\leq\sum_{k=1}^{d-1}\left(\frac{x-x_d}{x-x_k}\right)^2|f(x)-f(x_k)|w_\gamma(x_k)$$

that is the sum appeared in (4.5). Then by (4.10) we get

$$\Sigma_1(x)\leq\mathcal{C}\left[\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma}+\sum_{k=[\frac{d}{2}]+1}^{d-2}\frac{1}{(d-k)^2}\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\right].$$

Now we consider Σ_2 and we write

$$\begin{aligned}
&\Sigma_2(x)\\
&\leq\sum_{k=d+1}^{j}\left(\frac{4m-x}{4m-x_k}\right)^2 x^{\gamma-\alpha-\frac{1}{2}}\frac{\Delta x_k}{(x-x_k)^2}\,\frac{x_k^{1+\alpha-\gamma}}{\sqrt{4m-x}}|f(x)-f(x_k)|w_\gamma(x_k)\\
&=\sum_{k=d+1}^{j}\left(\frac{4m-x}{4m-x_k}\right)^{3/2}\sqrt{\frac{x_k}{4m-x_k}}\frac{\Delta x_k}{(x-x_k)^2}\left(\frac{x}{x_k}\right)^{-\frac{1}{2}-\alpha+\gamma}|f(x)-f(x_k)|w_\gamma(x_k).
\end{aligned}$$

Then since $\Delta x_k \sim \sqrt{\frac{x_k}{4m-x_k}}$, $\frac{4m-x}{4m-x_k} < \mathcal{C}$, $w_\gamma(x_k) \le \mathcal{C} w_\gamma(x)$, $\left(\frac{x}{x_k}\right)^{-\frac{1}{2}-\alpha+\gamma} \le 1$ being $\gamma \ge \alpha + \frac{1}{2}$, we have by applying (2.3)

$$\Sigma_2(x) \le \mathcal{C} \sum_{k=d+1}^{j} \frac{1}{(k-d)^2} |f(x) - f(x_k)| w_\gamma(x),$$

that is the sum estimated in (4.12). Thus

$$\Sigma_2(x) \le \mathcal{C} \left[\Omega_\varphi \left(f, \frac{1}{\sqrt{m}} \right)_{w_\gamma} + \sum_{k=d+2}^{j} \frac{1}{(k-d)^2} \Omega_\varphi \left(f, \frac{k-d}{\sqrt{m}} \right)_{w_\gamma} \right].$$

Hence, by the previous estimates, we deduce

$$A_1(x) \le \mathcal{C} \sum_{i=1}^{j} \frac{1}{i^2} \Omega_\varphi \left(f, \frac{i}{\sqrt{m}} \right)_{w_\gamma} \le \mathcal{C} \sum_{i=1}^{j} \frac{1}{i^2} \omega_\varphi \left(f, \frac{i}{\sqrt{m}} \right)_{w_\gamma} \tag{4.37}$$

by using (2.8).

Now we estimate A_2. We write

$$A_2(x) = \tilde{l}_{m+1}^2(x) |(w_\gamma f)(x)| + \sum_{k=j+1}^{m} \tilde{l}_k^2(x) w_\gamma(x) |f(x)|.$$

The first term tends to zero exponentially. In fact

$$\begin{aligned} |l_{m+1}^2(x)(f w_\gamma)(x)| &= \left| \frac{w_\gamma(x)}{w_\gamma(4m)} \frac{p_m^2(w_\alpha, x)}{p_m^2(w_\alpha, 4m)} \frac{w_\gamma(4m)}{w_\gamma(x)} (f w_\gamma)(x) \right| \\ &\le \mathcal{C} e^{-2m} \|f w_\gamma\|_\infty \end{aligned}$$

being $\frac{w_\gamma(x)}{w_\gamma(4m)} \frac{p_m^2(w_\alpha, x)}{p_m^2(w_\alpha, 4m)} \le C$ (see, for instance, [14]).

It remains to estimate the sum. To this end we use relation (4.36). Hence we have

$$\begin{aligned} &\sum_{k=j+1}^{m} \tilde{l}_k^2(x) w_\gamma(x) |f(x)| \\ &\le \mathcal{C} \|f w_\gamma\|_\infty \sum_{k=j+1}^{m} \left(\frac{4m-x}{4m-x_k} \right)^2 \left(\frac{x}{x_k} \right)^{\gamma-\alpha-\frac{1}{2}} \frac{\Delta x_k}{(x_k-x)^2} \sqrt{\frac{x_k}{4m-x}} \left(\frac{x_k}{x} \right)^\gamma e^{-(x_k-x)}. \end{aligned}$$

Thus being $x < x_k < 4m$, $\gamma > \alpha + \frac{1}{2}$ and $\sqrt{\frac{x_k}{4m-x}} \le \mathcal{C}$ we have

$$\sum_{k=j+1}^{m} \tilde{l}_k^2(x) w_\gamma(x) |f(x)| \le \sum_{k=j+1}^{m} \frac{\Delta x_k}{(4m-x_k)^2} \left(\frac{x_k}{x} \right)^\gamma e^{-(x_k-x)}.$$

If $x < \frac{x_k}{2}$ then the sum tends to zero exponentially since $e^{-(x_k-x)} < e^{-\frac{x_k}{2}} < e^{-m}$. Then now we assume $\frac{x_k}{2} < x \le x_j$ and we write

$$\begin{aligned}&\sum_{k=j+1}^{m} \frac{\Delta x_k}{(4m-x_k)^2}\left(\frac{x_k}{x}\right)^{\gamma} e^{-(x_k-x)} \\ &\le \mathcal{C}\|fw_\gamma\|_\infty \left[\sum_{j+1\le k\le[\frac{3}{2}j]} + \sum_{[\frac{3}{2}j]+1\le k\le m}\right] \frac{\Delta x_k}{(4m-x_k)^2} e^{-(x_k-x)}.\end{aligned}$$

The second sum tends again to zero exponentially. In fact $x < x_j$, $x_k > x_{[\frac{3}{2}j]} > \mathcal{C}\frac{9}{4}\frac{j^2}{m} > \mathcal{C}\frac{9}{4}x_j$ being $x_i \sim \frac{i^2}{m}$ and consequently $e^{-(x_k-x)} \le e^{-\mathcal{C}\frac{5}{2}m}$.
About the first sum we have that it is bounded by

$$\mathcal{C}\frac{e^{-(x_{j+1}-x_j)}}{m^2} \sum_{x_{j+1}\le x_k\le \frac{3}{2}x} \int_{x_k}^{x_{k+1}} dt \le \frac{\mathcal{C}}{m}.$$

In definitive

$$A_2(x) \le \mathcal{C}\frac{\|fw_\gamma\|_\infty}{m}.$$

Thus by replacing this estimate and (4.37) in (4.33) we have the assertion. ☐

Proof. [Proof of Theorem 3.4] Let x_j, f_j, Ψ and Ψ_j be as in (3.9), (3.10), (4.13) and (4.14), respectively. Then being
$G^{*(\alpha)}_{m+1}(f) = G^{(\alpha)}_{m+1}(f_j)$ we have

$$\begin{aligned}\|[f - G^{*(\alpha)}_{m+1}(f)]w_\gamma\|_\infty &\le \|[f-f_j]w_\gamma\|_\infty + \|[f_j - G^{(\alpha)}_{m+1}(f_j)]w_\gamma\|_\infty \\ &\le \mathcal{C}\Big[\|fw_\gamma\|_{L^\infty((x_j,\infty))} + \|[f - G^{*(\alpha)}_{m+1}(f)]w_\gamma\|_{L^\infty((0,x_j])} \\ &\qquad + \|G^{*(\alpha)}_{m+1}(f)w_\gamma\|_{L^\infty((x_j,+\infty))}\Big].\end{aligned} \tag{4.38}$$

The first term is estimated in Theorem 3.1 (see estimate (4.19)) while the second one is given by (4.32). Therefore we have

$$\begin{aligned}\|[f - G^{*(\alpha)}_{m+1}(f)]w_\gamma\|_\infty \le \mathcal{C}&\left[\sum_{i=1}^{j}\frac{1}{i^2}\omega_\varphi\left(f, \frac{i}{\sqrt{m}}\right)_{w_\gamma} + \frac{\|fw_\gamma\|_\infty}{m}\right] \\ &+ \|G^{*(\alpha)}_{m+1}(f)w_\gamma\|_{L^\infty((x_j,+\infty))}\end{aligned}$$

by using (2.8) and (2.6). Consequently we have only to bound the last term. To this end it is sufficient to observe that G^*_{m+1} is a bounded operator and $G^*_{m+1}(f)$ is a polynomial of degree $2m$. Hence by applying (4.17) we get

$$\|G^{*(\alpha)}_{m+1}(f)w_\gamma\|_{L^\infty((x_j,+\infty))} \le \mathcal{C}e^{-Am}\|fw_\gamma\|_\infty$$

and the proof is complete. ☐

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References

[1] Barnhill, R. E., *Representation and approximation of surfaces*, Mathematical software, III, (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1977), Academic Press, New York, 1977, pp. 69-120. Publ. Math. Res. Center Univ. Wisconsin, No. 39.

[2] Barnhill, R. E., Dube, R. P., Little, F. F., *Properties of Shepard's surfaces*, Rocky Mountain J. Math., **13** (1983), no. 2, 365-382.

[3] Criscuolo, G., Della Vecchia, B., Lubinsky, D. S., Mastroianni, G., *Functions of the second kind for Freud weights and series expansions of Hilbert transforms*, J. Math. Anal. and Appl., **189** (1995), 256-296.

[4] Criscuolo, G., Mastroianni, G., *Estimates of the Shepard interpolatory procedure*, Acta Math. Hungar., **61** (1993), no. 1-2, 79-91.

[5] De Bonis, M. C., Mastroianni, G., Viggiano, M., *K-functionals, moduli of smoothness and weighted best approximation of the semiaxis*, Functions, series, operators (Budapest, 1999), Janos Bolyai Math. Soc., Budapest, 2002, 181-211.

[6] Della Vecchia, B., Mastroianni, G., *A Nevai-Shepart type operator*, Atti Sem. Mat. Fis. Univ. Modena, **39** (1991), no. 2, 537-549.

[7] Della Vecchia, B., Mastroianni, G., Szabados, J., *Uniform approximation on the semiaxis by rational operators*, East J. Approx., **4** (1998), no. 4, 435-458.

[8] Della Vecchia, B., Mastroianni, G., Totik, V., *Saturation of the Shepard operators*, Approx. Theory Appl., **6** (1990), no. 4, 76-84.

[9] Freud, G., *A certain class of orthogonal polynomials* (Russian), Mat. Zametki, **9** (1971), 511-520.

[10] Grünwald, G., *On the theory of interpolation*, Acta Math., **75** (1942), 219-245.

[11] Hermann, T., Vértesi, P., *On an interpolatory operator and its saturation*, Acta Math. Acad. Sci. Hungar., **37** (1981), no. 1-3, 1-9.

[12] Joó, I., *On some problems of M. Horváth (saturation theorems for Walsh-Fourier expansions)*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., **31** (1988), 243-260.

[13] Mastroianni, G., Milovanovic, G. V., *Interpolation processes, basic theory and applications*, Springer, 2008.

[14] Mastroianni, G., Occorsio, D., *Lagrange interpolation at laguerre zeros in some weighted uniform spaces*, Acta Math. Hungar., **91** (2001), 27-52.

[15] Mastroianni, G., Szabados, J., *Balázs-Shepard operators on infnite intervals*, J. Approx. Theory, **90** (1997), 1-8.

[16] Mastroianni, G., Szabados, J., *Polynomial approximation on infnite intervals with weights having inner zeros*, Acta. Math. Hungar., **96** (2002), 221-258.

[17] Mastroianni, G., Szabados, J., *Polynomial approximation on the real semiaxis with generalized Laguerre weights*, Stud. Univ. Babes-Bolyai Math., **52** (2007), no. 4, 105-128.

[18] Nevai, P., *A Laguerre polinomok gyökein alapuló Lagrange-féle*, Math. Lapok, **22** (1971), 149-164.

[19] Newman, D. J., Rivlin, T. J., *Optimal universally stable interpolation*, Analysis, **3** (1983), no. 1-4, 355-367.

[20] Shepard, D., *A two dimentional interpolation function for irregularly spaced data*, Proc. 23rd Nat. Conf. ACM (1968), 517-523.

[21] Szabados, J., *Weighted Lagrange and Hermite-Fejér Interpolation on the Real Line*, J. of Inequal and Appl., **1** (1997), 99-123.

[22] Szabó, V. E. S., *Weighted interpolation: the l_∞ theory. I*, Acta Math. Hungar., **83** (1999), 131-159.

[23] Szegó, G., *Orthogonal polynomials*, fourth ed., American Mathematical Society, Providence, R.I., 1975, American Mathematical Society, Colloquium Publications, Vol. XXIII.

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FIXED POINT AND INTERPOLATION POINT SET OF A POSITIVE LINEAR OPERATOR ON $C(\overline{D})$

IOAN A. RUS

Abstract. Let $D \subset \mathbb{R}^p$ be a compact convex subset with nonempty interior. If $A : C(D) \to C(D)$ is a positive linear operator with $\Pi_0(D) \subset F_A$ or $\Pi_1(D) \subset F_A$ then we establish some relations between the mixed-extremal point set of D and the interpolation point set of A. Our results include some well known results (see I. Raşa, *Positive linear operators preserving linear functions*, Ann. T. Popoviciu Seminar of Funct. Eq. Approx. Conv., **7**(2009), 105-109) and the proofs are directly and elementarely.

1. Introduction

In the iteration theory of a positive linear operator on a linear space of functions, the interpolation set of the operator has a fundamental part (U. Abel and M. Ivan [1], O. Agratini [2], [3], O. Agratini and I.A. Rus [5], [6], S. Andras and I.A. Rus [8], I. Gavrea and M. Ivan [12], H. Gonska and P. Piţul [14], I. Raşa [17], I.A. Rus [19], [20]).

A well known result is the following ([12],[14],[17], ...)

Theorem 1.1. *Let $L : C[0,1] \to C[0,1]$ be a positive linear operator such that*

$$L(e_i) = e_i,\ \ i = 0, 1$$

where $e_i(x) = x^i$, $x \in [0,1]$.

Then:

$$L(f)(0) = f(0) \text{ and } L(f)(1) = f(1),\ \ \forall f \in C[0,1].$$

There exist different proofs of this result. One proof uses some estimations (Mamedov [16], Raşa [17], Gonska and Piţul [14], ...). Another proof uses a theorem by H. Bauer (H. Bauer [9], N. Boboc and Gh. Bucur [10], F. Altomare and M. Campiti [7], I. Raşa [17], ...). In [17], I. Raşa gives a directly and elementary proof.

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Let $D \subset \mathbb{R}^p$ be a bounded open convex subset and $A : C(\overline{D}) \to C(\overline{D})$ be a positive linear operator. The aim of this paper is to establish some relations between the mixed-extremal point set of D, the fixed point set and the interpolation point set of A. In this paper we shall use the notations in [7] and [20].

2. **Mixed-extremal point set: Examples**

Let $D \subset \mathbb{R}^p$ be a convex closed subset of $\mathbb{R}^p$ with nonempty interior.

Definition 2.1. A point $x^0 = (x_1^0, \ldots, x_p^0) \in \partial D$ is mixed-extremal point of D iff for each $i \in \{1, \ldots, p\}$, x_i^0 is an extremal (i.e., maximal or minimal) point of the ordered set

$$\big(\{x_i \mid (x_1, \ldots, x_p) \in D\}, \leq_{\mathbb{R}}\big).$$

We shall denote by $(ME)_D$ the mixed-extremal point set of D.

For a better understanding of this notion we shall give some examples.

Example 2.2. If $D_1 := [0, 1] \subset \mathbb{R}$, then $(ME)_{D_1} = \{0, 1\}$.

Example 2.3. If $D_2 := \mathbb{R}_+$, then $(ME)_{D_2} = \{0\}$.

Example 2.4. If D_3 is the simplex $\overline{P_1P_2P_3}$ in $\mathbb{R}^2$ with $P_1 = (0, 0)$, $P_2 = (1, 0)$ and $P_3 = (0, 1)$, then $(ME)_{D_3} = \{P_1, P_2, P_3\}$.

Example 2.5. If D_4 is the simplex $\overline{P_1P_2P_3}$ in $\mathbb{R}^2$ with $P_1 = (0, 0)$, $P_2 = (2, 0)$ and $P_3 = (1, 1)$, then $(ME)_{D_4} = \{P_1, P_2\}$.

Example 2.6. If D_5 is the polytope $\overline{P_1P_2P_3P_4}$ with $P_1 = (0, 0)$, $P_2 = (1, 0)$, $P_3 = (2, 1)$ and $P_4 = (1, 1)$, then $(ME)_{D_5} = \{P_1, P_3\}$.

Example 2.7. If $D_6 := \big\{x \in \mathbb{R}^p \mid x_1^2 + \ldots + x_p^2 \leq 1\big\}$, then $(ME)_{D_6} = \emptyset$.

3. **Interpolation points and fixed points of positive linear operators**

Let $D \subset \mathbb{R}^p$ be a bounded open convex subset of $\mathbb{R}^p$. Let $A : C(\overline{D}) \to C(\overline{D})$ be a positive linear (i.e., increasing linear) operator.

Definition 3.1. A point $x \in \overline{D}$ is an interpolation point of A iff $A(f)(x) = f(x)$, for all $f \in C(\overline{D})$. A subset $E \subset \overline{D}$ is an interpolation set of A iff $A(f)\big|_E = f\big|_E$. The subset

$$(IP)_D := \big\{x \in \overline{D} \mid A(f)(x) = f(x),\ \forall f \in C(\overline{D})\big\}$$

is by definition the interpolation point set of A.

Remark 3.2. Let us denote by $\xrightarrow{p}$, the pointwise convergence. Let $Y \subset C(\overline{D})$ be a dense subset of $(C(\overline{D}), \xrightarrow{p})$. If for a point $x \in \overline{D}$ we have

$$A(f)(x) = f(x),\ \forall f \in Y$$

then x is an interpolation point of A.

Remark 3.3. If $A : (C(\overline{D}), \xrightarrow{p}) \to (C(\overline{D}), \xrightarrow{p})$ is weakly Picard operator and $x \in \overline{D}$ is an interpolation point of A, then x is an interpolation point of A^∞.

The main results of this paper are the following

Theorem 3.4. *We suppose that:*

(i) *A is an increasing linear operator;*
(ii) $\Pi_1(\overline{D}) \subset F_A$.

Then $(ME)_D$ is an interpolation set of A.

Proof. Let us denote by $\Pi(\overline{D}) \subset C(\overline{D})$ the set of polynomial functions on $\overline{D}$.

Since $\Pi(\overline{D})$ is a dense subset of $(C(\overline{D}), \xrightarrow{unif})$, it is sufficient to prove that

$$A(f)\big|_{(ME)_D} = f\big|_{(ME)_D},\ \forall f \in \Pi(\overline{D}).$$

Let $x^0 \in (ME)_D$. From the mean-value theorem we have

$$f(x) - f(x^0) = \sum_{i=1}^{p} (x_i - x_i^0) \frac{\partial f(x_0 + \theta(x - x_0))}{\partial x_i},\ \forall x \in \overline{D}.$$

Since $\overline{D}$ is compact and x^0 is a mixed-extremal element of $\overline{D}$, there exist $\alpha_i, \beta_i \in \mathbb{R}$, $i \in \{1, \dots, p\}$ such that

$$\sum_{i=1}^{p} \alpha_i (x_i - x_i^0) \le f(x) - f(x^0) \le \sum_{i=1}^{p} \beta_i (x_i - x_i^0),\ \forall x \in \overline{D}.$$

From this we have

$$\sum_{i=1}^{p} \alpha_i (q_i - x_i^0 \tilde{1}) \le f - f(x^0)\tilde{1} \le \sum_{i=1}^{p} \beta_i (q_i - x_i^0 \tilde{1}). \tag{3.1}$$

Here

$$q_i : \overline{D} \to \mathbb{R},\ x \mapsto x_i,\ i \in \{1, \dots, p\},$$

and

$$\tilde{1} : \overline{D} \to \mathbb{R},\ x \mapsto 1.$$

Since A is an increasing linear operator and $\tilde{1}$, q_1, ... , $q_p \in F_A$, from (3.1) we have

$$\sum_{i=1}^{p} \alpha_i (q_i - x_i^0 \tilde{1}) \le A(f) - f(x^0)\tilde{1} \le \sum_{i=1}^{p} \beta_i (q_i - x_i^0 \tilde{1}).$$

For $x := x^0$, we have

$$A(f)(x^0) = f(x^0),\ \forall f \in \Pi(\overline{D})$$

and, from Remark 3.2, for all $f \in C(\overline{D})$. □

More general we have

Theorem 3.5. *We suppose that*

(i) *A is an increasing linear operator;*

(ii) $\Pi_0(\overline{D}) \subset F_A$.

Then

$$E := \left\{x \in (ME)_D \mid A(q_i)(x) = x_i\right\}$$

is an interpolation set of A.

Proof. Let $x^0 \in E$. From (3.1) we have

$$\sum_{i=1}^{p} \alpha_i(A(q_i) - x_i^0\tilde{1}) \leq A(f) - f(x^0)\tilde{1} \leq \sum_{i=1}^{p} \beta_i(A(q_i) - x_i^0\tilde{1})$$

For $x := x^0$, it follows

$$A(f)(x^0) = f(x^0),\ \forall f \in C(\overline{D}),$$

□

In a similar way we have

Theorem 3.6. *We suppose that:*

(i) *A is an increasing linear operator;*

(ii) $q_1, \dots, q_p \in F_A$.

Then

$$E := \left\{x \in (ME)_D \mid A(\tilde{1})(x) = 1\right\}$$

is an interpolation set of A.

Example 3.7. Let $\overline{\Omega} = [0,1] \times [0,1]$ and

$$A(f)(x_1, x_2) := f(0,0) + f(1,0)x_1 + f(0,1)x_2.$$

In this case A is an increasing linear operator with

$$\tilde{1} \notin F_A \text{ and } q_1, q_2 \in F_A$$

and

$$(IP)_A = \{(0,0)\}.$$

We remark that

$$A(\tilde{1})(0,0) = 1,\ A(\tilde{1})(0,1) = 2,\ A(\tilde{1})(1,0) = 2 \text{ and } A(\tilde{1})(1,1) = 3.$$

In the case $p = 1$ and $\overline{D} = [a, b]$, let us denote $e_i(x) := x^i$, $x \in [a, b]$, $i \in \mathbb{N}$. We have

Theorem 3.8. *We suppose that:*

(i) $A : C[a, b] \to C[a, b]$ *is an increasing linear operator;*
(ii) e_0 *and* $e_2 \in F_A$.

Then:

(1) *If* $A(e_1)(a) = a$*, then* a *is an interpolation point of* A.
(2) *If* $A(e_1)(b) = b$*, then* b *is an interpolation point of* A.

Example 3.9. Let us consider the following operator of J.P. King (see [14])

$$A : C[0, 1] \to C[0, 1],$$
$$A(f)(x) := (1 - x^2)f(0) + x^2 f(1),\ x \in [0, 1].$$

In this case:

(1) $e_0, e_2 \in F_A$;
(2) $(IP)_A = \{0, 1\}$;
(3) $A(e_1)(0) = 0$, $A(e_1)(1) = 1$.

4. Open problems

From the above considerations the following problems arise:

Problem 4.1. To extend the above results to the case when D is an open convex subset of $\mathbb{R}^p$, not necessarily bounded.

Problem 4.2. Let $D \subset \mathbb{R}^p$ be an open convex subset of $\mathbb{R}^p$. Let $A : C(\overline{D}) \to C(\overline{D})$ be an increasing linear operator. We suppose that $E \subset \overline{D}$ is a strong Volterra set of A ([20], [6]), i.e.,

$$f, g \in C(\overline{D}),\ f\big|_E = g\big|_E \Rightarrow A(f) = A(g).$$

We consider the operator

$$A_{\overline{co}E} : C(\overline{co}E) \to C(\overline{co}E),\ A_{\overline{co}E}\big(f\big|_{\overline{co}E}\big) := A(f)\big|_{\overline{co}E}.$$

It is clear that $A_{\overline{co}E}$ is an increasing linear operator.
If $\Pi_0(\overline{D}) \subset F_A$ or $\Pi_1(\overline{D}) \subset F_A$, in which conditions we have that $(IP)_{A_{\overline{co}E}} \neq \emptyset$?

Problem 4.3. Could our results be derived from the H. Bauer principle of the barycenter of a probability Radon measure (Theorem 2.1 in Raşa [17])?

References

[1] Abel, U., Ivan, M., *Over-iterates of Bernstein's operators: A short and elementary proof*, The Amer. Math. Monthly, **116** (2009), 535-538.

[2] Agratini, O., *Positive Approximation Processes*, Hiperboreea Press, 2001.

[3] Agratini, O., *Stancu modified operators revisited*, Revue D'Anal. Num. Theorie L'Approx., **31** (2002), No., 9-16.

[4] Agratini, O., Rus, I. A., *Nonlinear Analysis Forum*, **8** (2003), 159-168.

[5] Agratini, O., Rus, I. A., *Iterates of a class of discrete linear operators via contraction principle*, Comment. Math. Univ. Caroline, 44, **3** (2003), 555-563.

[6] Agratini, O., Rus, I. A., *Iterates of positive linear operators preserving constant functions, via fixed point principles*, to appear.

[7] Altomare, F., Campiti, M., *Korovkin-type Approximation Theory and its Applications*, de Gruyter, 1994.

[8] Andras, S., Rus, I. A., *Iterates of Cesàro operators via fixed point principle*, Fixed Point Theory, **11** (2010), No. 2, 171-178.

[9] Bauer, H., *Silovscher and Dirichletsches Problem*, Ann. de l'Institut Fourier, **11** (1961), 89-136.

[10] Boboc, N., Bucur, Gh., *Conuri convexe de funcţii continue pe spaţii compacte*, Ed. Academiei R.S.R., Bucureşti, 1976.

[11] Gavrea, I., Ivan, M., (Eds.), *Mathematical Analysis and Approximation Theory*, Mediamira, Cluj-Napoca, 2005.

[12] Gavrea, I., Ivan, M., *On the over-iterates of Meyer-König and Zeller operators*, J. Math. Anal. Appl.

[13] Giaquinta, M., Modica, G., *Mathematical Analysis*, Birkhäuser, 2009.

[14] Gonska, H., Piţul, P., *Remarks on an article of J.P. King*, Comment. Math. Univ. Caroline, 46, **4** (2005), 645-652.

[15] Leoni, G., *A First Course in Sobolev Spaces*, AMS, Providence, 2009.

[16] Mamedov, R. G., *On the order of approximation of functions by sequences of linear positive operators (Russian)*, Dokl. Akad. Nauk SSSR, **128** (1959), 674-676.

[17] Raşa, I., *Positive linear operators preserving linear functions*, Ann. T. Popoviciu Seminar of Funct. Eq. Approx. Conv., **7** (2009), 105-109.

[18] Rus, I. A., *Iterates of Bernstein operators via contraction principle*, J. Math. Anal. Appl., **292** (2004), 259-261.

[19] Rus, I. A., *Iterates of Stancu operators, via contraction principle*, Studia Univ. "Babeş-Bolyai", Math., **47** (2002), No. 4, 101-104.

[20] Rus, I. A., *Iterates of Stancu operators (via fixed point principles) revisited*, Fixed Point Theory, **11** (2010), No. 2, 369-374.

[21] Vladislav, T., Raşa, I., *Analiză Numerică*, Ed. Tehnică, Bucureşti, 1999.

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ON THE COMBINATORIAL IDENTITIES OF ABEL-HURWITZ TYPE AND THEIR USE IN CONSTRUCTIVE THEORY OF FUNCTIONS

ELENA IULIA STOICA

Abstract. This paper is concerned with the problem of approximation of multivariate functions by means of the Abel-Hurwitz-Stancu type linear positive operators. Inspired by the work of D. D. Stancu [13], we continue the discussions of the approximation of trivariate functions by a class of Abel-Hurwitz-Stancu operators in the case of trivariate variables, continues on the unit cub $K_3 = [0,1]^3$. In this paper there are three sections. In Section 1, which is the Introduction, is mentioned the generalization given by N. H. Abel [1] in 1826, for the Newton binomial formula and then the very important extension of this formula given by A. Hurwitz in 1902, in the paper [3]. Here is mentioned an interesting combinatorial significance in a cycle-free directed graphes given by D. E. Knuth [5]. Then is presented a main result given in 2002 by D. D. Stancu [13], where is used a variant of the Hurwitz identity in order to construct and investigate a new linear positive operator, which was used in the theory of approximation univariate functions. In Section 2 is discussed in detail the trivariate polynomial operator of Stancu-Hurwitz type $S_{m,n,r}^{(\beta),(\gamma),(\delta)}$ associated to a function $f \in C(K_3)$, where K_3 is the unit cub $[0,1]^3$. Section 3 is devoted to the evaluation of the remainder term of the approximation formula (3.1) of the function $f(x,y,z)$ by means of the Stancu-Hurwitz type operator $S_{m,n,r}^{(\beta),(\gamma),(\delta)}$. Firstly is presented an integral form of this remainder, based on the Peano-Milne-Stancu result [12]. Then we give a Cauchy type form for this remainder. By using a theorem of T. Popoviciu [8] we gave an expression, using the divided differences of the first three orders. When the coordinates of the vectors (β), (γ), (δ) have respectively the same values we are in the case of the second operator of Cheney-Sharma [2]. In this case we obtain an extension of the results from the papers [13] and [17].

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1. **Introduction**

By means of the D. D. Stancu [13] trivariate polynomial operator of Stancu-Hurwitz type $S^{(\beta),(\gamma),(\delta)}_{m,n,r}$ associated to a function $f \in (K_3)$, where K_3 is the unit cub $K_3 = [0,1]^3$, we construct some linear positive operators, useful in constructive theory of functions.

It is known that by using the celebrated generalization of Newton binomial formula, given in 1826 by the outstanding Norvegian Niels Henrik Abel [1], namely

$$(u+v)^m = \sum_{k=0}^{m} \binom{m}{k} u(u-k\beta)^{k-1}(v+k\beta)^{m-k}, \tag{1.1}$$

where β is a nonnegative parameter, the German mathematician Adolf Hurwitz has given in 1902, in the paper [3], a generalization of the Abel identity (1.1), represented by the equality

$$(x+y)(x+y+z_1+\ldots+z_m)^{m-1} \tag{1.2}$$

$$=\sum x(x+\varepsilon_1 z_1+\ldots+\varepsilon_m z_m)^{\varepsilon_1+\ldots+\varepsilon_m-1} y(y+(1-\varepsilon_1)z_1+\ldots+(1-\varepsilon_m)z_m)^{m-1-\varepsilon_1-\ldots-\varepsilon_m}$$

summed over all 2^m choices of $\varepsilon_1, \ldots, \varepsilon_m$ independently taking the values 0 and 1. This is an identity in $2m+2$ variables, and Abel's binomial formula is the special case $z_1 = z_2 = \ldots = z_m$.

The famous specialist in computer science D. E. Knuth has given in [5] an interesting combinatorial significance in cycle-free directed graphs.

By using the following variant of Hurwitz identity

$$(u+v)(u+v+\beta_1+\ldots+\beta_m)^{m-1}$$

$$=\sum u(u+\beta_{i_1}+\ldots+\beta_{i_k})v(b+\beta_{j_1}+\ldots+\beta_{j_{m-k}}),$$

professor D. D. Stancu has constructed in the paper [13] a general linear positive operator useful for uniform approximation of continuous functions, namely

$$(S^{(\beta_1,\ldots,\beta_m)}f(x) = \frac{1}{1+\beta_1+\ldots+\beta_m} \sum_{k=0}^{m} w^{(\beta_1,\ldots,\beta_m)}_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.3}$$

where

$$w^{(\beta_1,\ldots,\beta_m)}_{m,k}(x) = \sum x(x+\beta_1+\ldots+\beta_m)^{k-1}(1-x)(1-x+\beta_{j_1}+\ldots+\beta_{j_{m-k}})^{m-k-1}.$$

Because we can write

$$(1+\beta_1+\ldots+\beta_m)^{m-1}(S_m^{(\beta_1,\ldots,\beta_m)}f)(x)=(1-x)(1-x+\beta_1+\ldots+\beta_m)^m f(0)$$

$$+x(1-x)\sum_{k=1}^{m-1} w_{m,k}^{(\beta_1,\ldots,\beta_m)}(x)f\left(\frac{k}{m}\right)+x(x+\beta_1+\ldots+\beta_m)^{m-1}f(1),$$

it is easy to see that the polynomial defined at (1.3) is interpolatory at both sides of the interval $[0,1]$, for any nonnegative values of the parameters $\beta_,\ldots,\beta_m$.

Consequently we can write

$$(S_m^{(\beta_1,\ldots,\beta_m)}f)(0)=f(0),\quad (S_m^{(\beta_1,\ldots,\beta_m)}f)(1)=f(1)$$

and we conclude that the operator of Stancu-Hurwitz defined at (1.3) reproduces the linear functions.

2. **The trivariate polynomial of Stancu-Hurwitz type**

For simplicity we restrict ourselves here to the case of the space of real-valued functions $f(x,y,z)$, continuous on the unit cub $K_3=[0,1]^3$. We associate to the function $f\in(K_3)$ the Stancu-Hurwitz trivariate polynomial $S_m^{(\beta),(\gamma),(\delta)}$, defined by the formula

$$(S_{m,n,r}^{(\beta),(\gamma),(\delta)})(x,y,z)=\sum_{k=0}^{m}\sum_{j=0}^{n}\sum_{\nu=0}^{r} u_{m,k}^{(\beta)}(x)v_{m,j}^{(\gamma)}(y)w_{r,\nu}^{(\delta)}(z)f\left(\frac{k}{m},\frac{j}{n},\frac{\nu}{r}\right), \tag{2.1}$$

where we have the vectors with nonnegative coordinates $\beta=(\beta_1,\ldots,\beta_m)$, $\gamma=(\gamma_1,\ldots,\gamma_n)$, $\delta=(\delta_1,\ldots,\delta_r)$, while the basic polynomials are given by the formulas

$$(1+\beta_1+\ldots+\beta_m)^{m-1}u_{m,k}^{(\beta)}(x)$$

$$=\sum x(x+\beta_1+\ldots+\beta_{i_k})^{k-1}(1-x)(1-x+\beta_{j_1}+\ldots+\beta_{j_{m-k}})^{m-k-1},$$

$$(1+\gamma_1+\ldots+\gamma_n)^{n-1}v_{n,j}^{(\gamma)}(y)$$

$$=\sum y(y+\gamma_1+\ldots+\gamma_{s_\nu})^{j-1}(1-y)(1-y+\gamma_{t_1}+\ldots+\gamma_{t_{n-\nu}})^{n-j-1},$$

$$(1+\delta_1+\ldots+\delta_r)^{r-1}w_{r,\nu}^{(\delta)}(z)$$

$$=\sum z(z+\delta_1+\ldots+\delta_{\gamma_\tau})^{\nu-1}(1-z)(1-z+\delta_{\tau_1}+\ldots+\delta_{\tau_{r-1}})^{r-\nu-1}.$$

In the special cases $\beta_i = \beta$ $(i = \overline{1,m})$, $\gamma_j = \gamma$ $(j = \overline{1,n})$, $\delta_s = \delta$ $(s = \overline{1,r})$, we obtain the Cheney-Sharma-Stancu type trivariate linear positive operator defined by the following formula

$$(S^{(\beta,\gamma,\delta)}_{m,n,r} f)(x,y,z) = \sum_{k=0}^{m}\sum_{j=0}^{n}\sum_{\nu=0}^{r} u^{(\beta)}_{m,k}(x) v^{(\gamma)}_{m,j}(y) w^{(\delta)}_{r,\nu}(z) f\left(\frac{k}{m}, \frac{j}{n}, \frac{\nu}{r}\right),$$

where now we have

$$(1+m\beta)^{m-1} u^{(\beta)}_{m,k}(x) = \binom{m}{k} x(x+k\beta)^{k-1}(1-x)(1-x+(m-k)\beta)^{m-k-1},$$

$$(1+n\gamma)^{n-1} v^{(\gamma)}_{n,j}(y) = \binom{n}{j} y(y+j\gamma)^{j-1}(1-y+(n-j)\gamma)^{n-j-1},$$

$$(1+r\delta) w^{(\delta)}_{r,\nu}(z) = \binom{r}{\nu} z(z+\nu\delta)^{\nu-1}(1-z+(r-\nu)\delta)^{r-\nu-1}.$$

This operator $S^{(\beta,\gamma,\delta)}_{m,n,r}$ represents an extension to three variables of the second operator of Cheney-Sharma [2].

3. The remainder of the approximation formula of the function $f(x,y,z)$ by means of the operator of Stancu-Hurwitz type $S^{(\beta),(\gamma),(\delta)}_{m,n,r}$

If we use a theorem of Peano-Milne-Stancu type, given in the paper of D. D. Stancu [12], we can present an integral representation of the remainder term of the approximation formula

$$f(x,y,z) = (S^{(\beta),(\gamma),(\delta)}_{m,n,r} f)(x,y,z) + (R^{(\beta),(\gamma),(\delta)}_{m,n,r} f)(x,y,z), \tag{3.1}$$

having the degree of exactness $(1,1,1)$.

We can state the following

Theorem 3.1. *If the function f of three variables has continuous second-order partial derivatives on the unit cub K_3, then the remainder of the above approximation formula can be represented under the following integral form*

$$(R^{(\beta),(\gamma),(\delta)}_{m,n,r} f)(x,y,z) = \int_0^1 L^{(\beta)}_m(u,y,z) f^{(2,0,0)}(u,y,z)du$$

$$+ \int_0^1 M^{(\gamma)}_n(x,v,z) f^{(0,2,0)}(x,v,z)dv + \int_0^1 N^{(\delta)}_r(x,y,w) f^{(0,0,2)}(x,y,w)dw$$

$$- \int_0^1\int_0^1 L^{(\beta)}_m(u,v,z) N^{(\delta)}_r(u,y,w) f^{(2,0,2)}(\xi,y,\zeta)dudw$$

$$-\int_0^1\int_0^1 L_m^{(\beta)}(u,v,z)M_n^{(\gamma)}(u,v,z)f^{(2,2,0)}(\xi,\eta,z)dudv$$

$$-\int_0^1\int_0^1 L_m^{(\beta)}(u,y,v)N_r^{(\delta)}(u,y,w)f^{(0,2,2)}(x,v,w)dudw$$

$$+\int_0^1\int_0^1\int_0^1 L_m^{(\beta)}(u,v,w)M_n^{(\gamma)}(u,v,w)N_r^{(\delta)}(u,v,w)f^{(2,2,2)}(u,v,w)dudvdw$$

where the Peano kernels are

$$L_m^{(\beta)}(u,y,z)=(R_m^{(\beta)}\varphi_x)(u),$$

$$M_n^{(\gamma)}(x,v,z)=(R_n^{(\gamma)}\psi_y)(v),$$

$$N_r^{(\delta)}(x,y,w)=(R_r^{(\delta)}\tau_z)(w),$$

with

$$\varphi_x(u)=\frac{x-u+|x-u|}{2}=(x-u)_+,$$

$$\psi_y(v)=\frac{y-v+|y-v|}{2}=(y-v)_+,$$

$$\tau_z(w)=\frac{z-w+|z-w|}{2}=(z-w)_+.$$

For the partial derivatives we have used the notation

$$F^{(r,s,t)}(u,v,w)=\frac{\partial^{r+s+t}F(u,v,w)}{\partial u^r\partial v^s\partial w^t}.$$

It follows that we can write explicitly

$$L_m^{(\beta)}(u,y,z)=(x-u)_+-\sum_{k=0}^{m} w_{m,k}^{(\beta)}(x)\left(\frac{k}{m}-u\right)_+,$$

$$M_n^{(\gamma)}(x,v,z)=(y-v)_+-\sum_{j=0}^{n} v_{n,j}^{(\gamma)}(y)\left(\frac{j}{n}-v\right)_+,$$

$$N_r^{(\delta)}(x,y,w)=(z-w)_+-\sum_{\nu=0}^{r} w_{r,\nu}^{(\delta)}(z)\left(\frac{i}{r}-w\right)_+.$$

By using these explicit expressions for the partial Peano kernels, we can see that they represent polygonal lines situated beneath the u axis, respectively the v axis and the w axis, which joins the points $(0,0,0)$ and $(0,1,0)$, respectively $(0,0,1)$.

Now if we take into account that on the cub K_3 we have $L_m^{(\beta)} \leq 0$, $M_n^{(\gamma)} \leq 0$ and $N_r^{(\delta)} \leq 0$, we can apply the first law of the mean to the integrals and we can find that

$$(R_{m,n,r}^{(\beta),(\gamma),(\delta)}f)(x,y,z) = f^{(2,0,0)}(\xi,y,z)\int_0^1 L_m^{(\beta)}(u,y,z)du$$
$$+f^{(0,2,0)}(x,\eta,z)\int_0^1 M_n^{(\gamma)}(x,v,z)dv + f^{(0,0,2)}(x,y,\zeta)\int_0^1 N_r^{(\delta)}(x,y,w)dw$$
$$-f^{(2,2,0)}(\xi,\eta,z)\int_0^1 L_m^{(\beta)}(u,v,z)M_n^{(\gamma)}(u,v,z)dudx$$
$$-f^{(2,0,2)}(\xi,y,\zeta)\int_0^1\int_0^1 L_m^{(\beta)}(u,y,w)N_r^{(\delta)}(u,y,w)dudw$$
$$-f^{(0,2,2)}(x,\eta,\zeta)\int_0^1\int_0^1 L_m^{(\beta)}(u,v,z)N_r^{(\delta)}(u,y,w)dvdw$$
$$+f^{(2,2,2)}(\xi,\eta,\zeta)\int_0^1\int_0^1\int_0^1 L_m^{(\beta)}(u,v,w)M_n^{(\gamma)}(u,v,w)N_r^{(\delta)}(u,v,w)dudvdw,$$

where ξ, η and ζ are certain points from the cub K_3.

It is easy to see that we have

$$\int_0^1 L_m^{(\beta)}(u,y,z)du = \frac{1}{2}(R_m^{(\beta)}e_{2,0,0})(x),$$
$$\int_0^1 M_n^{(\gamma)}(x,v,z)dv = \frac{1}{2}(R_n^{(\gamma)}e_{0,2,0})(y),$$
$$\int_0^1 N_r^{(\delta)}(x,y,w)dw = \frac{1}{2}(R_r^{(\delta)}e_{2,0,2})(z),$$

where we have considered the univariate remainders

$$R_m^{(\beta)} = I - S_m^{(\beta)}, \quad R_n^{(\gamma)} = I - S_m^{(\gamma)}, \quad R_r^{(\delta)} = I - S_r^{(\delta)}.$$

Now we can state the following result concerning the remainder of the approximation formula (3.1).

Theorem 3.2. *If $f \in C^{(2,2,2)}(K_3)$, then the remainder of the approximation formula (3.1) can be represented under the following Cauchy type form*

$$(R_{m,n,r}^{(\beta),(\gamma),(\delta)}f)(x,y,z) \tag{3.2}$$
$$= \frac{1}{2}(R_m^{(\beta)}e_{2,0,0})(x)f^{(2,0,0)}(\xi,y,z)$$
$$+\frac{1}{2}(R_n^{(\gamma)}e_{0,2,0})(y)f^{(0,2,0)}(x,\eta,z)$$

$$+\frac{1}{2}(R_r^{(\delta)}e_{0,0,2})(z)f^{(0,0,2)}(x,y,\zeta)$$
$$-\frac{1}{4}(R_m^{(\beta)}e_{2,0,0})(x)(R_n^{(\gamma)}e_{0,2,0})(y)f^{(2,2,0)}(\xi,\eta,z)$$
$$-\frac{1}{4}(R_m^{(\beta)}e_{2,0,0})(x)(R_r^{(\delta)}e_{0,0,2})(z)f^{(2,0,2)}(\xi,\eta,\zeta)$$
$$-\frac{1}{4}(R_m^{(\beta)}e_{2,0,0})(x)(R_r^{(\delta)}e_{2,0,2})(z)f^{(0,2,2)}(\xi,y,\zeta)$$
$$+\frac{1}{8}(R_m^{(\beta)}e_{2,0,0})(x)(R_n^{(\gamma)}e_{0,2,0})(z)(R_r^{(\delta)}e_{0,0,2})(z)f^{(2,2,2)}(\xi,\eta,\zeta).$$

Since $(S_m^{(\beta)}f)(x)$, $(S_n^{(\gamma)}f)(y)$, $(S_r^{(\delta)}f)(z)$ are interpolatory at both sides of the interval $[0,1]$, we can conclude that $(R_m^{(\beta)}e_{2,0,0})(x)$ contains the factor $x(x-1)$, then $(R_n^{(\gamma)}e_{0,2,0})(y)$ contains the factor $\eta(\eta-1)$, while $(R_r^{(\delta)}e_{0,0,2})(z)$ contains the factor $z(z-1)$.

Because $(R_{m,n,r}^{(\beta),(\gamma),(\delta)}e_{0,0,0})(x,y,z)=0$ and the remainder is different from zero for any convex function f of the first order, we can apply a criterion of T. Popoviciu [8] and we can conclude that the remainder of the approximation formula (3.1) is of simple form and we can state

Theorem 3.3. *If the second-order divided differences of the function $f\in C(K_3)$ are bounded on the unit cub K_3, then we can give an expression in terms of divided differences of the remainder of the approximation formula (3.1), namely*

$$(R_{m,n,r}^{(\beta),(\gamma),(\delta)}f)(x,y,z)$$
$$=(R_m^{(\beta)}e_{2,0,0}(x)[x_{m,1},x_{m,2},x_{m,3};f(t_1,y,z)]$$
$$+(R_n^{(\gamma)}e_{0,2,0})(y)[y_{n,1},y_{n,2},y_{n,3};f(x,t_2,z)]$$
$$+(R_r^{(\delta)}e_{0,0,2})(z)[z_{r,1},z_{r,r},z_{r,3};f(x,y,t_3)]$$
$$-(R_m^{(\beta)}e_{2,0,0})(x)(R_n^{(\gamma)}e_{0,2,0})(y)\left[\begin{array}{c}x_{m,1},x_{m,2},x_{m,3}\\ y_{n,1},y_{n,2},y_{n,3}\end{array};f(t_1,t_2,z)\right]$$
$$-(R_m^{(\beta)}e_{2,0,0})(x)(R_r^{(\delta)}e_{0,0,2})(y)\left[\begin{array}{c}x_{m,1},x_{m,2},x_{m,3}\\ z_{r,1},z_{r,2},z_{r,3}\end{array};f(t_1,y,t_3)\right]$$
$$-(R_n^{(\gamma)}e_{0,2,0})(y)(R_r^{(\delta)}e_{0,0,2})(z)\left[\begin{array}{c}y_{n,1},y_{n,2},y_{n,3}\\ z_{r,1},z_{r,2},z_{r,3}\end{array};f(x,t_2,t_3)\right]$$

$$+(R_m^{(\beta)}e_{2,0,0})(x)(R_n^{(\gamma)}e_{0,2,0})(y)(R_r^{(\delta)}e_{0,0,2})(z)\begin{bmatrix} x_{m,1},x_{m,2},x_{m,3} & \\ y_{n,1},y_{n,2},y_{n,3} & ;f(t_1,t_2,t_3) \\ z_{r,1},z_{r,2},z_{r,3} & \end{bmatrix},$$

where $x_{m,1}, x_{m,2}, x_{m,3}$*, respectively* $y_{n,1}, y_{n,2}, y_{n,3}$ *and* $z_{n,1}, z_{n,2}, z_{n,3}$ *are certain points in the interval* $[0,1]$.

Now if we consider that $f \in C^{2,2,2}(K_3)$, then we can apply the mean value theorems to the divided differences which occur above and we arrive at the expression (3.2) for the remainder of the approximation formula (3.1).

Clearly, the results of this paper can be extended to functions of more than three variables without any difficulty.

References

[1] Abel, N. H., *Démonstration d'une expression de laquelle la formule binôme est un cas particulier*, Journ. für Reine und Angewandte Mathematick, Oeuvres Complètes, Christiania, Groendahls, 1839. **1** (1826), 159-160.

[2] Cheney, E. W., Sharma, A., *On a generalization of Bernstein polynomials*, Riv. Mat. Univ., Parma, **5** (1964), 77-84.

[3] Hurwitz, A., *Über Abel's Verallgemeinerung der binomischen Formel*, Acta Mathematica, **26** (2002), 199-203.

[4] Jensen, J. L. W., *Sur une identité d'Abel et sur d'autre formules analogues*, Acta Mathematica, **26** (1902), 307-318.

[5] Knuth, D. E., *The Art of Computer-Programming*, vol. 1, *Fundamental Algorithms*, Addison-Wesley Publ. Comp., Reading, Massachusetts, USA, 1968, p. 75.

[6] Milne, W. E., *The remainder in linear methods of approximation*, J. Res. Mat. Bur. Standards, **43** (1949), 501-511.

[7] Peano, G., *Resto nelle formule di quadratura expresso con un integrale definito*, Atti Acad. Naz. Lincei Rend., **22** (1913), 562-569.

[8] Popoviciu, T., *Sur le reste dans certaines formules linéaires d'approximation de l'analyse*, Mathematica, Cluj, **1**(**24**) (1959), 95-142.

[9] Riordan, J., *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958.

[10] Riordan, J., *Combinatorial Identities*, Wiley, New York, 1968.

[11] Stancu, D. D., *Evaluation of the remainder term in approximation formulas by Bernstein polynomials*, Math. Comp., **17** (1963), 137-163.

[12] Stancu, D. D., *The remainder of certain linear approximation formulas in two variables*, J. SIAM Numer. Anal. B, **1** (1964), 137-163.

[13] Stancu, D. D., *Use of an identity of A. Hurwitz for construction of a linear positive operator of approximation*, Rev. Analyse Numér. Théorie de l'Approximation, **31** (2002), 115-118.

[14] Stancu, D. D., *Methods for construction of linear positive operators of approximation*, In: Numerical Analysis and Approximation Theory, Proc. Internat. Symposium (R. T. Trîmbiţaş, ed.), Cluj University Press, 2002, 23-45.

[15] Stancu, D. D., Cismaşiu, C., *On an approximating linear positive operator of Cheney-Sharma*, Rev. Anal. Numér. Théor. Approx., **26** (1997), 221-227.

[16] Stoica-Laze, E. I., *On the use of Abel-Jensen type combinatorial formulas for construction and investigation of some algebraic polynomial operators of approximation*, Studia Univ. "Babeş-Bolyai", Mathematica, **54** (2009), no. 4, 167-182.

[17] Taşcu, I., *Approximation of bivariate functions by operators of Stancu-Hurwitz type*, Facta Math. Inform., **20** (2005), 33-39.

[18] Toader, S., *An approximation operator of Stancu type*, Rev. Anal. Numér. Théorie de l'Approx., **34** (2005), 115-121.

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BOOK REVIEWS

Alexandru Kristály, Vicenţiu Rădulescu and Csaba György Varga, ***Variational Principles in mathematics, Physics, Geometry, and Economics - Qualitative Analysis of Nonlinear Equations and Unilateral Problems,*** xv+368 pp, Encyclopedia of Mathematics and its Applications, vol. 136, Cambridge University Press, 2010, ISBN: 978-0-521-11782-1.

The use of variational principles has a long and fruitful history in mathematics and physics, both in solving problems and shaping theories, and it has been introduced recently in economics. The corresponding literature is enormous and several monographs are already classical. The present book, Variational Principles in Mathematical Physics, Geometry, and Economics, by Kristály, Rădulescu and Varga, is original in several ways.

In Part I, devoted to variational principles in mathematical physics, unavoidable classical topics such as the Ekeland variational principle, the mountain pass lemma, and the Ljusternik-Schnirelmann category, are supplemented with more recent methods and results of Ricceri, Brezis-Nirenberg, Szulkin, and Pohozaev. The chosen applications cover variational inequalities on unbounded strips and for area-type functionals, nonlinear eigenvalue problems for quasilinear elliptic equations, and a substantial study of systems of elliptic partial differential equations. These are challenging topics of growing importance, with many applications in natural and human sciences, such as demography.

Part II demonstrates the importance of variational problems in geometry. Classical questions concerning geodesics or minimal surfaces are not considered, but instead the authors concentrate on a less standard problem, namely the transformation of classical questions related to the Emden-Fowler equation into problems defined on some four-dimensional sphere. The combination of the calculus of variations with group theory provides interesting results. The case of equations with critical exponents, which is of special importance in geometrical problems since Yamabe's work, is also treated.

Part III deals with variational principles in economics. Some choice is also necessary in this area, and the authors first study the minimization of cost-functions on manifolds, giving special attention to the Finslerian-Poincar disc. They then consider best approximation problems on manifolds before approaching Nash equilibria through variational inequalities.

The high level of mathematical sophistication required in all three parts could be an obstacle for potential readers more interested in applications. However, several appendices recall in a precise way the basic concepts and results of convex analysis, functional analysis, topology, and set-valued analysis. Because the present in science depends upon its past and shapes its future, historical and bibliographical notes are complemented by perspectives. Some exercises are proposed as complements to the covered topics.

Among the wide recent literature on critical point theory and its applications, the authors have had to make a selection. Their choice has of course been influenced by their own tastes and contributions. It is a happy one, because of the interest and beauty of selected topics, because of their potential for applications, and because of the fact that most of them have not been covered in existing monographs. Hence I believe that the book by Kristály, Rădulescu, and Varga will be appreciated by all scientists interested in variational methods and in their applications.

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NOTE. This text was published as a Foreword to the above mentioned book. It is reproduced here with the kind permission of the author and with the agreement of Cambridge University Press.

Jaroslav Lukeš, Jan Malý, Ivan Netuka, Jiří Spurný, *Integral Representation Theory - Applications to Convexity, Banach Spaces and Potential Theory*, Walter de Gruyter, Berlin - New York, 2010,715 pages, ISBN: 978-3-11-020320-2.

The Krein-Milman theorem asserts that any compact convex subset X of a locally convex space agrees with the closed convex hull of the set ext X of its extreme points, and if X =cl conv(Z), for some $Z \subset X$, then ext $X \subset \overline{Z}$. G. Choquet gave a reformulation of this result, showing that any point of a compact convex set X is the barycenter of a Radon measure supported by the closure of the set of extreme points of the set X, initiating a fruitful area of investigation, known as the Choquet theory. The aim of the present book is to present a more general approach to integral representation theory based on the notion of function space, with applications to the study of convex sets, Banach spaces and potential theory.

The book starts with a chapter headed *Prologue* (there is not an *Epilogue* to the book), containing a proof of Korovkin approximation theorem in the case $C[0,1]$, a more general approach being proposed in Chapter 3. The second chapter, *Compact convex sets*, is devoted to the study of the extremal structure of compact convex sets and of representing measures with support contained in the closure of the set of extreme points. Exposed points, their connection with farthest points (Strasewicz theorem) as well as measure convex and measure extremal sets are also considered.

In Chapter 3, *The Choquet theory of function spaces,* the Choquet theory is developed within the framework of function spaces. A function space if a subspace $\mathcal{H}$ of the space $C(K)$ of all continuous functions on the compact topological space K, containing the constants and separating the points of K. Many examples of concrete function spaces fit in this general scheme - the space $C(K)$ itself, the space of quadratic polynomials in $C[0,1]$, the space of affine functions on K (K a compact convex set), spaces of harmonic functions ($K = \overline{U}$, $U \subset \mathbb{R}^n$ open and bounded), spaces of Markov operators on $C(K)$. The results of this chapter, including Choquet boundary, Choquet ordering, Bauer maximum principle, a.o., are essential for the whole book. The next chapter, 4, *Affine functions on compact convex sets,* presents the classical Choquet theory for spaces of affine functions on compact convex sets. Chapter 5, *Perfect classes of functions and representation of affine functions,* concerned with a hierarchy of Borel sets and functions, is crucial for subsequent applications to descriptive set theory.

Other results are contained in the chapters 6, *Simplicial function spaces* (Choquet and Bauer simplices, the Daugavet property for simplicial spaces are included), 7. *Choquet theory for function cones* (based on a key Lemma containing a Hahn-Banach type extension theorem for measures), 8, *Choquet-like sets,* 9, *Topologies and boundaries*, 10, *Deeper results on function spaces and compact convex sets* (James and Shilov boundaries, isometries of spaces of affine functions, embedding of ℓ^1 in Banach spaces, metrizability of compact convex sets, topological properties of the sets of extreme points), 11, *Continuous and measurable selectors* (Lazar selection theorem and applications, measurable selectors), 12, *Constructions of function spaces* (products and inverse limits of function spaces), 13, *Function spaces in potential theory and the Dirichlet problem* (Bauer and Cornea approaches to harmonic spaces and Dirichlet problem).

The reward for the effort done in reading the book comes in Chapter 14, *Applications*, where a lot of classical representation results are obtained using the tools of the generalized Choquet theory developed in the previous chapters - integral representations for convex and for concave functions (Bauer's results), doubly stochastic matrices (in the finite dimensional case Krein-Milman suffices), the Riesz-Herglotz theorem on the Poisson kernel, typically real holomorphic functions and holomorphic functions with positive real part, Bernstein's representation theorem for completely monotone functions, Lyapunov theorem on the convexity of the range of a nonatomic measure, the Stone-Weierstrass approximation theorem, the existence of invariant and ergodic measures. In all of these cases, more complicated objects are expressed as mixtures (or averages) of simpler ones, which turn to be the extreme points of some appropriate compact convex sets.

Each chapter contains a set of exercises completing the main text. All are accompanied by sufficiently detailed hints, so that the reader can handle them without excessive effort. Historical and bibliographical notes and comments, as well as references to further work, are supplied at the end of each chapter. A consistent Appendix (60 pages) collects the basic notion and results from functional analysis, measure theory, descriptive set theory, partial differential equations and axiomatic potential theory, used in the main body of the book.

The book is clearly written and very well organized - a list of symbols, a notion index and a well documented bibliography counting 479 items, help the reader to navigate through the book and to find further bibliographical information.

Incorporating many original results of the authors, the book is addressed both to students who can find a clear and thorough presentation of the basics of the integral representation theory, as well as for the advanced readers who find a substantial amount of recent results, appearing for the first time in book form.

S. Cobzaş

A. L. Dontchev and R. T. Rockafellar, ***Implicit Functions and Solution Mappings. A View from Variational Analysis,*** Springer Monographs in Mathematics, Springer, New York, 2009, xii + 375 pp, ISBN: 978-0-387-87820-1,

The implicit function theorem is in the gallery of basic theorems in analysis and its many variants are basic tools in various parts of mathematics, including partial differential equations, nonsmooth analysis, and numerical analysis. The book starts with the classical framework of implicit function theorem and then is largely focusing on properties of solution mapping of variational problems.

The goal of the authors was to provide a reference on the topic and to present a unified collection of results scattered throughout the literature. The authors fully realized their goal.

In the classical framework the main issue in the subject is whether the solution to an equation depending on parameters may be considered as a function of those parameters and if so, what properties that function might have. This is addressed by the classical theory of implicit functions, which began with the single real variable and developed through several variables to equations in infinite dimensions, such as equations associated with integral and differential operators.

An important feature of the book is to lay out lesser known variants, for instance where standard assumptions of differentiability are relaxed in different senses. On the same vain the book shows how the same circle of ideas, when articulated in a suitable framework, can deal successfully may other problems than just solving equations, as min or max problems with inequality constraints. In such a case a question is addressed if a solution can be expressed as a function of parameters of the problem. Mathematical models resting on equations are replaced by "variational inequality" models. At this level of generality the main concept is that of the set-valued solution mapping which assigns to each instance of the parameter element in the model all the corresponding solutions, if any. The main question is whether a solution mapping can be localized graphically in order to achieve single-valuedness and in that sense produce a function, the searched implicit function.

The book under review concentrates primarily on local properties of solution mappings that can be captured metrically, rather than on results derived from topological considerations or involving lesser used spaces.

The first chapter concerns with the implicit function paradigm in the classical case of the solution mapping associated with a parameterized equation. Two proofs of the classical inverse function theorem are given and then two equivalent forms of it are derived: the implicit function theorem and the correction function theorem. The differentiability assumption is gradually relaxed in various ways and even completely exit from it. Instead it is substituted by the Lipschitz condition.

The second chapter was inspired by optimization problems and models of competitive equilibrium. The questions are essentially the same as in the first chapter, namely, when a solution mapping can be localized to a function with some continuity properties. Here the authors are dealing with generalized equations which captures a more complicated dependence and covers, among others, variational inequality conditions formulated in terms of the set-valued normal cone mapping associated with a convex set.

In the third chapter there are introduced the notions of Painlevé-Kuratowski convergence and Pompeiu-Hausdorff convergence for sequences of set, and used them in proving properties of continuity and Lipschitz continuity for set-valued mappings. Important results are obtained regarding the solution mapping associated with constraint systems.

In the fourth chapter graphical differentiation of a set-valued mapping is defined through the variational geometry of the mapping's graph. A characterization of the Aubin property is derived and applied to the case of a solution mapping. Strong metric subregularity is characterized and applications are introduced to parameterized constraint systems and special features of solution mappings for variational inequalities.

The fifth chapter is devoted to the study of regularity in infinite dimensions. It presents extensions of the Banach open mapping theorem which are shown to fit infinite-dimensionally into the paradigm of the theory developed finite dimensionally in Chapter 3.

The sixth chapter contains applications in numerical variational analysis. It is illustrated how some of the implicit function/mapping theorems from the earlier of the book can be used in the study of problems in numerical analysis.

Each chapter ends with a section of commentary. By these sections the authors exhibits connections of the results just introduced with other results by other authors. The commentaries are deep, pertinent and very useful.

The book ends with a rich list of references, a glossary of notation, and a subject index. This references reflect from one side the authors's contribution to this topic and from other side the contributions of many other researchers all over the world.

Certainly this wonderful work will be included in many libraries all over the world.

Marian Mureşan

William J. Terrell, ***Stability and Stabilization. An Introduction***, Princeton University Press, 2009, XV+457 pp, ISBN-13: 978-0-691-13444-4.

William Terrell's book is a first intermediate textbook that covers stability and feedback stabilization of equilibria for both linear and nonlinear autonomous systems of ordinary differential equations. It covers a portion of the core of mathematical control theory, including the concepts of linear systems theory and Lyapunov stability theory for nonlinear systems, with applications to feedback stabilization of control systems. This book takes a unique modern approach that bridges the gap between linear and nonlinear systems.

The book is designed for advanced undergraduates and beginning graduate students in the sciences, engineering, and mathematics. The minimal prerequisite for reading it is a working knowledge of elementary ordinary differential equations and elementary linear algebra.

The author structured his book in seventeen chapters and six appendixes. There are five chapters on linear systems (2–7) and nine chapters on nonlinear systems (8–16); an introductory chapter (1); a mathematical background chapter (2); and a short final chapter on further reading (17). Appendixes cover notations, basic analysis, ordinary differential equations, manifolds and the Frobenius theorem, and comparison functions and their use in differential equations. The introduction to linear system theory presents the full framework of basic state-space theory, providing just enough detail to prepare students for the material on nonlinear systems.

Clear formulated definitions and theorems, correct proofs and many interesting examples and exercises make this textbook very attractive.

Ferenc Szenkovits

Pankaj Sharan, ***Spacetime, Geometry and Gravitation***, Progress in Mathematical Physics 56, Birkhäuser, Basel - Boston - Berlin, 2009, XIV+355 pp, ISBN: 978-3-7643-9970-24.

This readable introductory textbook on the general theory of relativity presents a solid foundation for those who want to learn about relativity. The author offers us a physically intuitive, but mathematically rigorous presentation of the subject.

The book is structured into three parts. The first part, containing four chapters, offers preliminary topics on general relativity. This part begins with a general background and introduction, followed by an introduction to curvature through Gauss's Theorema Egregium, continue with basics on general relativity, and presents also simpler topics in general relativity like the Newtonian limit, red shift, the Schwarzschild solution, precession of the perihelion and bending of line in a gravitational field.

The second part is a comprehensive and accurate incursion in the mathematical background of relativity theory. Separate chapters are dedicated to the next topics: vectors and tensors, inner product, elementary differential geometry, connection and curvature, Riemannian geometry and varied additional topics in geometry are also presented.

The last part, entitled Gravitation, contains six chapters dealing with: the Einstein equation, general features of spacetime, weak gravitational fields, Schwarzschild and Kerr solutions, cosmology and special topics. Interesting exercises with complete answers simplify the understanding of the presented material. This textbook is recommended to advanced graduates and researchers.

Ferenc Szenkovits

John J. Benedetto and Wojciech Czaja, ***Integration and Modern Analysis***, xix+575 pp,Birkhäuser Advanced Texts, Birkhäuser, Boston -Basel - Berlin, 2009, ISBN: 978-0-8176-4306-5, e-ISBN: 978-0-8176-4656-1

The aim of the present book is to emphasize how the modern integration theory evolved from some classical problems in function theory, related mainly to Fourier analysis. It is worth to mention that some problems that arose in the study of Fourier series in the nineteenth century lay at the basis of many modern branches of mathematics as, for instance, set theory. For this reason the first chapter of the book, Ch. 1, *Classical real variables,* contains some classical results related to differentiation (e.g., continuous nowhere differentiable functions) and its imperfect relations with the Riemann integral, culminating in the new theory of integration developed by Lebesgue, which put the things in their right places. In fact, one of the main ideas of the book is the study of the relations between integrals and the a.e. derivatives (the Fundamental Theorem of Calculus - FTC), realized by the key notion of absolute continuity, viewed by the authors as a unifying concept for the FTC, Lebesgue dominated convergence theorem (LDC) and Radon-Nikodym theorem.

Another major topic of the book is that of switching limits in various processes of analysis (differentiation, integration), considered as a fundamental problem in the whole analysis, being related to compactness and weak compactness in function spaces (Dunford-Pettis theorem) or in spaces of measures (Grothendieck's results).

The first five chapters of the book, the first one mentioned above, 2. *Lebesgue measure and general measure theory,* 3. *The Lebesgue integral,* 4. *The relationship between differentiation and integration,* and 5. *Spaces of measures and the Radon-Nikodym theorem*, can serve as a basic one-semester course in real analysis.

The second part, containing the chapters 6. *Weak convergence of measures,* 7. *Riesz representation theorem,* 8. *Lebesgue differentiation theorem on* $\mathbb{R}^d$*,* 9. *Self-similar sets and fractals,* contains finer points, providing material for the second semester of a full-year course. Each chapter in the first part ends with a set of exercises and problems, ranging from routine to challenging (dedicated by the authors to the *"mathochistic"* student). The second part of the book contains no problems.

The book has also two appendices, one on basic results in functional analysis, and a consistent survey (42 pages) on harmonic analysis, a field in which the authors are experts, called by them one of the goddesses of mathematics. In fact, the book is written in the idea to provide the reader with the basic concepts and results in measure theory and integration, as a basic acting tool in several branches of mathematics as potential theory, harmonic analysis, probability theory, nonlinear dynamics, etc.

Each chapter ends with a section entitled *Potpourri and titillation,* dedicated to historical remarks and references for further reading, meant to be informative and fun, providing some breadth without depths, but which are strongly recommended to be read by the students. Beside these sections, the book contains short biographical sketches of some of the great figures of the domain and some disputes (e.g. that between Borel and Lebesgue).

A special tribute is paid to Vitali for his outstanding contributions, somewhat neglected by the mathematical community, who gave 100 years ago modern proofs to some fundamental results in real-analysis - the notion of absolute continuity, the first example of a nonmeasurable set, the first proof of Luzin's measurability theorem (Luzin 1912, Vitali 1905).

The bibliography (550 items plus 23 on the history of integration, listed at the end of first chapter) contains references to original works, to recent papers and books, as well as various folklore topics collected from classical books or from the American Mathematical Monthly.

A name index, containing full names (a misprint - Chebyshev should be "Pafnutii" not "Patnutii") with references to specific pages where they are quoted, a subject index and a notation index make the book very well organized.

The result is a nice book, containing a lot of results in measure theory and integration theory, making a good connection between classical and modern ones. The live style of exposition make the reading both instructive and agreeable. It can be recommended to instructors for one-semester or two semester courses in real analysis, to students for self-study and to researchers in various domains of analysis as a reference text.

S. Cobzaş

Steven G. Krantz, *Explorations in Harmonic Analysis- With Applications to Complex Function Theory and the Heisenberg Group*, xiv+360 pp, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston -Basel - Berlin, 2009, ISBN: 978-0-8176-4668-4, e-ISBN: 978-0-8176-4669-1

This is a self-contained introduction to modern harmonic analysis, starting with the classical theory of Fourier series and culminating in the theory of pseudodifferential operators and analysis on Heisenberg group. The idea of studying harmonic analysis on Heisenberg group (fractional integrals and singular integrals) belongs to E. M. Stein and his school in the 1970s. This turned out to be a powerful device for developing sharp estimates for integral operators (the Bergman projections, the Szegö projections) that arise naturally in the several complex variable setting. (A chapter of the book is dedicated to a crash course in several complex variables).

The book starts with a short historical account and comments on the evolution of the idea of Fourier expansion and continues with an introduction to the Hilbert transform (considered by the author as the most important operator in analysis), the Fourier transform, fractional and singular integrals, canonical integral operators (Bergman and Szegö kernels).

Hardy spaces are discussed in Chapter 8, including a sketch of the real variable method that provides a bridge between classical holomorphic-functions Hardy spaces and the more modern real-variable approach to Hardy and BMO spaces.

The analysis on the Heisenberg group is presented in Chapters 9. *Introduction to Heisenberg group*, 10. *Analysis on the Heisenberg group,* and 11. *A coda on domains of finite type.*

Three appendices: A1. *Rudiments of Fourier series*, A2. *The Fourier transform,* and A3. *Pseudodifferential operators,* complete the main text, making the book as self-contained as possible (but not more).

The text is very well organized: each chapter begins with an introductory *Prologue*, each section with a *Capsule* giving a quick preview of the material, and each key theorem is preceded by a *Prelude* providing motivation and putting the result in an adequate context. Some details of the proofs are left as *Exercises for the reader.*

Written by an expert in analysis, understood in a broad and unitary sense, this new book gives the reader a panoramic and, at a same time, detailed, view of the subject, showing how basic analytic tools, ranging from real analysis, several complex variables, Lie theory, differential equations, differential geometry, dynamical systems and other parts of mathematics, interact into this modern area of investigation which is harmonic analysis.

The book will be useful for advanced courses on harmonic analysis, singular integrals, as well as reference text for researchers in various domains of analysis, both pure and applied.

Gabriela Kohr

Mariano Giaquinta and Giuseppe Modica, *Mathematical Analysis - An Introduction to Functions of Several Variables*, xii+348 pp, Birkhäuser, Boston -Basel - Berlin, 2010, ISBN: 978-0-8176-4509-0 (hardcover), 978-0-8176-4507-6 (softcover) e-ISBN: 978-0-8176-4612-7

This is a part of an ampler project of the authors:

GM1. *Mathematical Analysis - Functions of one variable,* Birkhäuser 2003;

GM2. *Mathematical Analysis - Approximation and discrete processes,* Birkhäuser 2004;

GM3. *Mathematical Analysis - Linear and metric structures and continuity,* Birkhäuser 2007.

Another volume, GM5. *Mathematical Analysis - Foundation and advanced techniques for functions of several variables,* is announced to be published with Birkhäuser, too (the present volume is numbered as GM4). All these volumes were first published in Italian by Pitagora Editors, Bologna, these ones being revised translations of the original Italian versions.

The first chapter of the book, *Differential calculus,* contains the standard results of the subject, including differential calculus in Banach spaces and two proofs for the inverse mapping theorem. The first section of the second chapter, *Integral calculus,* contains the basic definitions and results on Lebsegue measure in $\mathbb{R}^n$ and Lebesgue integral. This section contains no proofs, the reader being referred to GM5 for a complete treatment. Accepting these basic results, one proves then results on mollifiers and smooth approximation, Luzin's theorem on the characterization of measurability, the properties of the integrals depending on parameter, the Hausdorff measure, the area and co-area formulas and Green-Gauss formula.

The aim of Chapter 3, *Curves and differential forms*, is to prove Poincaré's Lemma and Stokes' Theorem. Chapter 4, *Holomorphic functions,* is a good introduction (with complete proofs) to holomorphic functions of one complex variable and their applications. Some more special topics, as, for instance, the Mellin transform, are also included.

Chapter 5, *Surfaces and level sets,* is concerned with r-dimensional surfaces in $\mathbb{R}^n$ - parameterizations of maximal rank, various versions of the implicit function theorem, Sard's theorem, Morse Lemma, gradient flows, curvature, the first and the second fundamental form of a surface.

The last chapter of the book, 6, *Systems of ordinary differential equations* (ODE) is concerned with first order linear systems of ODEs, higher-order linear differential equations, and a short discussion on stability - Lyapunov's method and Cantor-Bedixson's theorem.

The applications and the examples included in the book make it more attractive. There are also exercises at the end of each chapter. When referring to a classical result, a picture of the author as well as an excerpt (the first page or the cover) of the original publication are included.

A word must be said about the elegant layout of the book.

Undoubtedly that finished, this ambitious project will supply the reader with a fairly complete account of the fundamental results in mathematical analysis and applications, including Lebesgue integration in $\mathbb{R}^n$ and complex analysis of one variable.

The books can be used for courses in real or complex analysis and their applications.

Tiberiu Trif

Martin Schechter, *Minimax Systems and Critical Point Theory,* xiv+239 pp, Birkhäuser, Boston -Basel - Berlin, 2009, ISBN: 978-0-8176-4805-3, e-ISBN: 978-0-8176-4902-9

Many problems in differential equations come from variational considerations, involving a real-valued functional G defined on a Banach space E. This means to minimize or maximize the functional G, the original approach to this problem being the solving of the Euler-Lagrange equation (1): $G'(u) = 0$, i.e., finding the critical points of the functional G. This works well in the one-dimensional case, but in higher dimensions it is more difficult to solve the Euler-Lagrange equations than to find minima or maxima, and so this approach was abandoned for many years. It was reconsidered again, when nonlinear partial differential equations and systems arose in applications leading to the question whether these equations and systems are the Euler-Lagrange equations corresponding to some functionals.

The minima and maxima approach in finding critical points works only for functionals bounded from below or from above (semibounded). If not, then other conditions must be imposed which can substitute the semiboundedness of the functional G. One such notion is that of linking pair, meaning two subsets A, B of E such that

(2): $a_0 := \sup G(A) < \inf G(B) =: b_0$, a topic treated by the author in a previous book, *Linking Methods in Critical Point Theory,* Birkhäuser, Boston - Basel-Berlin, 1999.

A Palais-Smale sequence for a bounded from below C^1-functional G is a sequence (u_k) in E such that (PS): $G(u_k) \to a := \inf G$ and $G'(u_k) \to 0$. A sequence (u_k) such that $G(u_k) \to \inf G$ is called a minimizing sequence. One obtains the stronger notion of Cerami sequence by replacing the second condition in (PS) by the stronger one $(1 + \|u_k\|)G'(u_k) \to 0$. The interesting case is when a PS-sequence, or a Cerami sequence, contains a convergent subsequence, a fact easier to be shown for these kind of sequences than for arbitrary minimizing sequences.

The aim of the present book is to expose in a unified way some new methods and results which appeared since the publication of that book. One of the main notions used in the book is that of linking pair $A, B \subset E$: one says that A links B if (2) implies (3): $\exists u$, $G(u) \geq b_0$, $G'(u) = 0$. Also one says that G satisfies the PS-condition if (PS) always implies (3). In this case, every functional that satisfies the PS-condition and is separated by a pair of linking sets has a critical point satisfying (3).

Other important notion is that of minimax systems for a subset A of E, meaning a family $\mathcal{K}$ of subsets of E such that $\sup G(A) < \inf\{\sup G(K) : K \in \mathcal{K}\}$. Again, a minimax system has the same advantages as a semibounded functional. The role played by the linking pairs and the minimax systems in finding critical points is shown in Chapters 1 and 2. The proofs of these results, based on some existence theorems for differential equations in abstract spaces given in Chapter 4, are postponed to Chapter 5. Examples of linking pairs and minimax methods, as well as some abstract methods of finding them, are given in Chapters 3 and 6.

In Chapter 7 the author introduces the notion of sandwich pair, meaning a pair $A, B \subset E$ such that $-\infty < \inf G(B) \leq \sup G(A) < \infty$. An important particular case of sandwich pair is formed by two subspaces M, N of a Hilbert space, one of them being finite dimensional, such that $M = N^{\perp}$. The author shows that infinite dimensional subspaces qualify as well. Weak linking and weak sandwich pairs are considered in Chapters 10, respectively 11.

The rest of the book is dedicated to applications, many of which are more involved than those existing in the literature: semilinear problems (Ch. 9), semilinear wave equations (Ch. 13), the Fučik spectrum (Chapters 11 and 14), multiple solutions (Ch. 16), second-order periodic systems (Ch. 17).

The book is rather elementary, being accessible to students with a background in functional analysis.

Largely based on the results obtained by the author, alone or in cooperation with other mathematicians, this book, together with the 1999 volume, cover a broad spectrum of applications of critical point method in solving various nonlinear differential and partial differential equations. The author proposes some very general and unitary approaches to find critical points and exposes them in a clear and sequential way. The book can be recommended for researchers in applied functional analysis, partial differential equations and their applications.

It is worth to mention the spectacular evolution of author's family: the book by the author and Wenming Zou, *Critical Point Theory and its Applications,* Springer 2006, was dedicated to his 22 grandchildren and 1 great grandchild, the present volume records 24 grandchildren and 6 great grandchildren (so far). Let them all live in peace and happiness.

Cornel Pintea

Manuel González and Antonio Martínez-Abejón, ***Tauberian Operators,*** xii+245 pp, Birkhäuser, Boston -Basel - Berlin, 2010, ISBN: 978-3-7643-8997-0, e-ISBN: 978-3-7643-8998-7

Abel theorem asserts that

(1) $\sum_{n=0}^{\infty} a_n = \lambda$ implies (3) $\lim_{x\to 1-} \sum_{n=0}^{\infty} a_n x^n = \lambda$.

Tauber proved in 1897 a conditioned converse of this result: the condition (3) plus (2) $\lim_n a_n/n = 0$ imply (1). Tauberian operators were introduced by Kalton and Wilanski in 1976 as an abstract counterpart of some operators associated to conservative summability matrices. An infinite matrix $A = (a_{ij})$ is called conservative if $Ax := (\sum_j a_{ij}x_j)_i$ is well defined and convergent for every convergent sequence x. Based on the Toeplitz-Silverman characterization of conservative matrices, one can associate with every conservative matrix A two operators $S_A : c \to c$ and $T_A : \ell_\infty \to \ell_\infty$, both defined by the same value Ax. J. P. Crawford in his thesis (1966) proved that $T_A^{-1}(c) \subset c$ iff $S^{**-1}(c) \subset c$, paving the way to the abstract definition (given by Kalton and Wilanski) of tauberian operators as those continuous linear operators $T : X \to Y$, X, Y Banach spaces for which $T^{**-1}(Y) \subset X$. This turned out to be a very important and useful class of operators in Banach space theory: Davis-Figiel-Johnson-Pelczynski factorization theorem, the study of exact sequences of Banach spaces, some summability problems of tauberian type, the study of the equivalence between Krein-Milman and Radon-Nikodym theorems, some sequels of James' characterization of reflexivity, extensions of the principle of local reflexivity to operators, the study of Calkin algebras associated with the weakly compact operators. All these applications are presented in the third chapter of the book.

The historical roots of tauberian operators are explained in Chapter 1, *The origins of tauberian operators.*

The basic properties of tauberian operators, as well as some important characterizations and connections with weakly compact and semi-Fredholm operators, are given in Chapter 2. *Tauberian operators. Basic properties,* a study continued in chapters 3. *Duality and examples of tauberian operators*, and 4. *Tauberian operators on spaces of integrable functions.* Some generalizations of tauberian operators, as well as connections with operator ideals and ultrapower classes of operators are studied in Chapter 6. *Tauberian-like operators.*

The prerequisites are a basic course in functional analysis and some familiarity with Fredholm theory, ultraproducts, operator ideals is recommended. For the convenience of the reader, these more specialized topics are summarized in a consistent appendix (40 pages) at the end of the volume.

The book present in a clear and unified way the basic properties of tauberian operators and their applications in functional analysis scattered throughout the literature. Some open problems are included, too.

Written by to experts in the field, the book is addressed to graduate students and researchers in functional analysis and operator theory, but it can be used also as a basic text for advanced graduate courses.

V. Anisiu

Lluís Puig Editors, *Frobenius categories versus Brauer blocks. The Grothendieck group of the Frobenius category of a Brauer block*. Progress in Mathematics 274. Birkhäuser Basel-Boston-Berlin, 2009, 498 p., ISBN: 978-3-7643-9997-9/hbk; ISBN: 978-3-7643-9998-6/ebook.

The concept of Frobenius category (also called saturated fusion system) has been developed by Lluís Puig many years ago. Such a category has as objects the subgroups of a given p-group S (where p is a prime), the morphisms between two subgroups are injective, and contain all conjugations and inclusions of subgroups of S; all these satisfy certain axioms. This is a generalization the p-local structure of a finite group G, and it turns out this point of view is the correct framework for modular representation theory of finite groups. Indeed, the theory of Brauer pairs, due to J. Alperin and M. Broué associates to a subgroup of the defect group of a block a conjugacy relation similar to that between the subgroups of the Sylow p-subgroup. Moreover, group-theoretic notions, such as normalizers and centralizers, can be generalized to Frobenius categories. Later, Broto, Levi and Oliver realized that Puig's concepts are useful in p-local homotopy theory, particularly for the proof of the Martino-Priddy conjecture. Therefore, the study of fusion systems has been very active recently. Let me briefly present the content of the book. Chapter 1 introduces the general background, and Chapter 2 present the first definitions concerning Frobenius P-categories. Chapter 3 presents the definition of the Frobenius P-category of a p-block of a finite groups, by using Brauer pairs. Chapter 4 introduces self-centralizing objects, by mimicking the definition of selfcentralizing Brauer pairs, and also related concepts. Chapter 5 develops, in the context of Frobenius categories, a refinement of Alperin's fusion theorem , by introducing the so-called essential objects. Chapter 6 introduces the exterior quotient of the subcategory of selfcentralizing objects. Chapter 7 analyzes the selfcentralizing and nilcentralized objects of the Frobenius category of a block. Chapters 8, 9 and 10 discuss Dade P-algebras, introduce the concept of polarization and present a gluing theorem for Dade algebras. Chapter 11 provides a tool for defining later in Chapter 14 the Grothendieck group of a Frobenius category of a block, and proves a lifting theorem for the nil-centralizing subcategory. Chapter 12 discusses quotients and normal subcategories, mimicking quotients and normal subgroups of a group. Chapter 13 introduces the hyperfocal subcategory of a Frobenius category. Chapters 14, 15 and 16 define and study in detail the Grothendieck group of a Frobenius category, and develops a strategy towards reducing certain questions to quasi-simple groups. Chapters 17 and 18 are devoted to coherent perfect localities, which have important applications to group cohomology. Chapter 19 generalizes the notion of solvability to Frobenius categories. Chapter 20 investigates the extendability of partial perfect localities of the nil-centralized Frobenius system to perfect localities of the original Frobenius category. Chapter 21 gives a definition of Frobenius categories in terms of bisets, and the final three chapters investigate the basic locality of a Frobenius category in these terms. The book ends with Appendix developing the author's point of view on cohomology of small categories. This volume is a research monograph containing many original results which appear for the first time in print (although Puig's ideas have widely circulated in manuscript and preprint form). The book is, in principle, accessible to advanced graduate students, but the technicalities and the complicated notation make it quite difficult to read. Having said that, the book is a necessary tool for any researcher in the field.

Andrei Marcus