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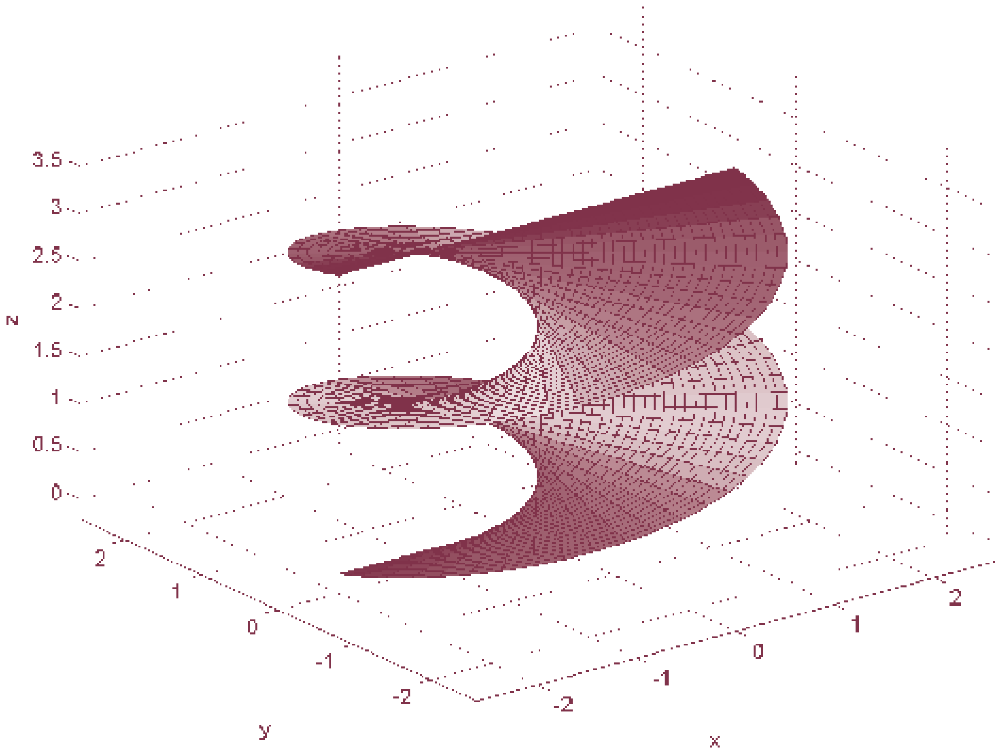

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VARIATIONAL-HEMIVARIATIONAL INEQUALITIES ON UNBOUNDED DOMAINS

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Abstract. This paper is a survey about hemivariational and variational-hemivariational inequalities defined on unbounded domains motivated by certain non-smooth phenomena appearing in Mathematical Physics. The paper contains various results obtained by the authors in the last few years. It is divided into six sections: the first section is a short introduction; in the second section we present some critical points results for locally Lipschitz functions; the third section is dedicated to Motreanu-Panagiotopoulos functionals; in the fourth section we provide some existence results for hemivariational inequalities; in the fifth section we give a multiplicity result for a special class of hemivariational inequalities; and in the last section we give some applications to hemivariational and variational-hemivariational inequalities.

1. Introduction

The study of *variational inequalities* began in the sixties with the pioneering work of Lions and Stampacchia [35]. The connection of this theory with the notion of the subdifferential of a convex function was achieved by Moreau [43], who introduced the notion of convex superpotentials which permitted the formulation and study in the weak form of a wide ranging class of complicated problems in Mechanics and Engineering (see Duvaut and Lions [12]). All the inequality problems studied in that period were related to convex energy functions and therefore were linked with the notion of monotonicity. Motivated by some problems from mechanics, Panagiotopoulos introduced in [50, 51] the notion of nonconvex superpotential by using the generalized gradient of Clarke. Due to the lack of convexity, new types of variational expressions

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were obtained; these are the so-called *Hemivariational Inequalities.* The hemivariational inequalities appears as a generalization of the variational inequalities, but actually they are much more general than these ones, because they are not equivalent to minimum problems. They are no longer connected with monotonicity, but since the main ingredient of their study is based on the notion of Clarke subdifferential of a locally Lipschitz funtion, the theory of hemivariational inequalities appears as a new field of Non-smooth Analysis. For a comprehensive treatment of the hemivariational inequality problems we refer to the monographs Naniewicz and Panagiotopoulos [48] (based on pseudomonotonicity), Motreanu and Panagiotopoulos [46], Motreanu and Rădulescu [47] (based on compactness arguments). In the above works (and in references therein) there are studied elliptic problems on *bounded domains.*

In this paper we treat hemivariational and variational-hemivariational inequalities problems on *unbounded domains* based on the authors' results in the last few years. Note that in the unbounded case the problem is more delicate, due to the lack of compactness in the Sobolev embeddings. First, some old and new results are recalled from critical points theory for locally Lipschitz functions and Motreanu-Panagiotopoulos functionals see [9], [44], [45], [33], [28], [38], [46], [47], [29] with applications to hemivariational and variational-hemivariational inequalities, see [66], [11], [28], [36], [30], [31], [27], [29]. Then, we present for locally Lipschitz functions the Mountain Pass Theorem (MPT) of "zero altitude", the version of MPT which satisfies the Cerami condition, and a version of the three critical points theorem of Ricceri [58]. In the third section we present some critical points results as well as the principle of symmetric criticality for Motreanu-Panagiotopoulos functionals. In the fourth section we give some existence results for a general class of hemivariational inequalities. In section five we prove a multiplicity result for a particular class of hemivariational inequalities while the last section is dedicated to various applications.

2. **Critical points results for locally Lipschitz functions**

In this section we present some critical points results for locally Lipschitz functions. These results appear in the papers of Motreanu, Varga [44], [45], Kristály, Motreanu and Varga [33] and Kristály, Marzantowicz and Varga [28].

2.1. **Elements of nonsmooth analysis.** Let $(X, \|\cdot\|)$ a real Banach space and $U \subset X$ an open subset. We denote by $\langle \cdot, \cdot \rangle$ the duality mapping between $X^\star$ and X.

Definition 2.1. A function $f : X \to \mathbb{R}$ is **locally Lipschitz** if, for every $x \in X$, there exist a neighborhood U of x and a constant $L > 0$ such that

$$|f(y) - f(z)| \leq L\|y - z\| \quad for\ all \quad y, z \in U.$$

Although it is not necessarily differentiable in the classical sense, a locally Lipschitz function admits a derivative, defined as follows:

Definition 2.2. The **generalized directional derivative** of f at the point $x \in X$ in the direction $y \in X$ is

$$f^\circ(x; y) = \limsup_{z \to x,\ \tau \to 0^+} \frac{f(z + \tau y) - f(z)}{\tau}.$$

The **generalized gradient** of f at $x \in X$ is the set

$$\partial f(x) = \{x^\star \in X^\star :\ \langle x^\star, y\rangle \leq f^\circ(x; y) \text{ for all } y \in X\}.$$

For all $x \in X$, the functional $f^\circ(x, \cdot)$ is subadditive and positively homogeneous: thus, due to the Hahn-Banach theorem, the set $\partial f(x)$ is nonempty. The next Lemma resumes the main properties of the generalized derivatives, which will be useful in the sequel:

Lemma 2.3. *Let $f, g : X \to \mathbb{R}$ be locally Lipschitz functions. Then,*

(f_1) $f^\circ(x; y) = \max\{\langle \xi, y\rangle :\ \xi \in \partial f(x)\}$;
(f_2) $(f + g)^\circ(x; y) \leq f^\circ(x; y) + g^\circ(x; y)$;
(f_3) $(-f)^\circ(x; y) = f^\circ(x; -y)$.
(f_4) *The function* $(x, y) \mapsto \Phi^\circ(x; y)$ *is upper semicontinuous.*

This notion extends both that of Gâteux derivative, and that of directional derivative for convex functionals. In particular:

Lemma 2.4. *Let $f : X \to \mathbb{R}$ be a convex, continuous, Gâteaux differentiable functional. Then, f is locally Lipschitz and*

$$\langle f'(x), y\rangle = f^\circ(x; y) \quad \textit{for all} \quad x, y \in X.$$

The next definition generalizes the notion of critical point to the non-smooth context:

Proposition 2.5. *The function $\lambda_f(u) = \inf_{w \in \partial f(u)} ||w||_{X^\star}$ is well defined and is lower semicontinuous, i.e. $\liminf_{u \to u_0} \lambda_f(u) \geq \lambda_f(u_0)$.*

Definition 2.6. Let $f : X \to \mathbb{R}$ be a locally Lipschitz function. We say that $u \in X$ is a critical point (in the sense of Chang) of f, if $\lambda_f(u) = 0$, which is equivalent with the fact that $0 \in \partial f(u)$.

Remark 2.7. A point $u \in X$ is critical point of f if $f^\circ(x; y) \geq 0$ for all $y \in X$.

Remark 2.8. Note that every local extremum of f is a critical point of f in the sense above.

Throughout in this paper we use the following notations for the locally Lipschitz function $f : X \to \mathbb{R}$ and a number $c \in \mathbb{R}$:

$$f^c = \{u \in X : f(u) \leq c\};$$

$$f_c = \{u \in X : f(u) \geq c\};$$

$$K_c = \{u \in X : \lambda_f(u) = 0, f(u) = c\};$$

$$(K_c)_\delta = \{u \in X : d(u, K_c) < \delta\};$$

$$(K_c)_\delta^c = X \setminus (K_c)_\delta.$$

In the sequel we introduce the notion of Palais-Smale condition.

Definition 2.9. We say that the locally Lipschitz function $f : X \to \mathbb{R}$ satisfies the Palais-Smale condition at the level c (shortly, $(PS)_c$), if every sequence $\{x_n\} \subset X$ with $f(x_n) \to c$, and $\lambda_f(x_n) \to 0$ when $n \to \infty$, contains a convergent subsequence in X. If we replace the condition $f(x_n) \to c$ with $\{f(x_n)\}$ is bounded we say that the function f satisfies the (PS) condition.

Remark 2.10. The (PS) condition has the following equivalent formulation: The function h satisfies the Palais-Smale condition, if every sequence $\{x_n\}$ in X such that

(PS_1)**:** $\{f(x_n)\}$ bounded;
(PS_2)**:** there exists a sequence $\{\varepsilon_n\}$ in $]0, +\infty[$ with $\varepsilon_n \to 0$ such that
$f^\circ(x_n; y - x_n) + \varepsilon_n \|y - x_n\| \geq 0$ for all $y \in X$, $n \in \mathbb{N}$

admits a convergent subsequence.

The following variant of Palais-Smale condition is an extension to the locally Lipschitz case of the one introduced by Ghoussoub and Preiss [20]. We consider a locally Lipschitz function $f : X \to \mathbb{R}$, a real number $c \in \mathbb{R}$ and a subset $B \subset X$.

Definition 2.11. We say that the locally Lipschitz function f satisfies the Palais-Smale condition around B at level c (shortly, $(PS)_{B,c}$), if every sequence $\{x_n\} \subset X$ with $f(x_n) \to c$, $\text{dist}(x_n, B) \to 0$ and $\lambda_f(x_n) \to 0$ when $n \to \infty$, contains a convergent subsequence in X.

In particular, we put $(PS)_c = (PS)_{X,c}$ and simply (PS) if $(PS)_c$ holds for every $c \in \mathbb{R}$.

For a fixed $B \subseteq X$ and a fixed number $\delta > 0$, we denote the closed δ-neighborhood of B by $N_\delta(B)$, that is,

$$N_\delta(B) = \{x \in X : \ \text{dist}(x, B) \leq \delta\}.$$

Definition 2.12. A generalized normalized pseudo-gradient vector field of the locally Lipschitz $f : X \to \mathbb{R}$ with respect to a subset $B \subset X$ and a number $c \in \mathbb{R}$ is a locally Lipschitz mapping $v : N_\delta(B) \cap f^{-1}[c - \delta, c + \delta] \to X$ with some $\delta > 0$, such that $||v(x)|| \leq 1$ and

$$\langle y^*, v(x) \rangle > \frac{1}{2} \inf_{x \in \text{dom}v} \lambda_f(x) > 0$$

for all $y^* \in \partial f(x)$ and $x \in \text{dom}v := N_\delta(B) \cap f^{-1}[c - \delta, c + \delta]$.

The existence of a generalized normalized pseudo-gradient vector field in the sense of Definition 2.12 is given by the result below. For the proof, see Motreanu-Varga [45].

Lemma 2.13. (Motreanu-Varga [45]) *Let $f : X \to \mathbb{R}$ be a locally Lipschitz function, $c \in \mathbb{R}$ and a closed subset B of X, such that $(PS)_{B,c}$ is satisfied together with $B \cap K_c(f) = \emptyset$ and $B \subset f^c$. Then there exists $\delta > 0$ and a generalized normalized pseudo-gradient vector field $v : N_\delta(B) \cap f^{-1}[c - \delta, c + \delta] \to X$ of f with respect to B and c.*

The following deformation result has been proved by Motreanu and Varga [45].

Theorem 2.14. (Motreanu-Varga [45]) *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional, $c \in \mathbb{R}$ and a closed subset B of X provided on has $(PS)_{B,c}, B \cap K_c(f) = \emptyset$* and *$B \subset f^c$. Let v be a generalized normalized pseudo-gradient vector field of f with respect to B and c. Then for every $\overline{\varepsilon} > 0$ there exist an $\varepsilon \in (0, \overline{\varepsilon})$ and a number $\delta < c$ such that for each closed subset A of X with $A \cap B = \emptyset$ and $A \subset f_{c-\varepsilon_A}$, where*

$$\varepsilon_A := \min(\varepsilon, \varepsilon d(A, B)), \tag{2.1}$$

and $d(A, B) := \inf\{\|x - y\| : \ x \in A, \ y \in B\}$, there is a continuous mapping $\eta_A : \mathbb{R} \times X \to X$ with the properties below

(i) *$\eta_A(\cdot, x)$ is the solution of the vector field $V_A = -\varphi_A v$ with the initial condition $x \in X$ for some locally Lipschitz function $\varphi_A : X \to [0, 1]$ whose support is contained in the set $(X \setminus A)$;*
(ii) *$\eta_A(t, x) = x$ for all $t \in \mathbb{R}$ and $x \in A \cup f^{c-\overline{\varepsilon}} \cup f_{c+\overline{\varepsilon}}$;*
(iii) *for every $\delta \leq d \leq c$ one has $\eta_A(1, B \cap f^d) \subset f^{d-\varepsilon}$.*

Proof. Let us note that the existence of a normalized generalized pseudo-gradient vector field $v : N_{3\delta_1}(B) \cap f^{-1}[c - 3\varepsilon_1, c + 3\varepsilon_1] \to X$ of f with respect to B and c is assured by Lemma 2.13, for some constants $\delta_1 > 0$ and $\varepsilon_1 > 0$. Consequently, a constant, $\sigma_1 > 0$ can be found such that

$$\langle y^*, v(x) \rangle > \frac{1}{2}\sigma_1, \ \forall y^* \in \partial f(x), \ x \in N_{3\delta_1}(B) \cap f_{c-3\varepsilon_1} \cap f^{c+3\varepsilon_1}. \tag{2.2}$$

We claim that the result of Theorem 2.14 holds for every $\varepsilon > 0$ with

$$\varepsilon < \min\{\overline{\varepsilon}, \varepsilon_1, \frac{1}{2}\sigma_1, \frac{1}{2}\sigma_1\delta_1\}. \tag{2.3}$$

In order to check the claim in (2.3) let us fix two locally Lipschitz functions $\varphi,\ \psi : X \to [0,1]$ satisfying

$$\varphi = 1 \text{ on } N_{\delta_1}(B) \cap f^{c+\varepsilon_1} \cap f_{c-\varepsilon_1};$$
$$\varphi = 0 \quad \text{on} \quad X \setminus (N_{2\delta_1}(B) \cap f^{c+2\varepsilon_1} \cap f_{c-2\varepsilon_1});$$
$$\psi = 0 \quad \text{on} \quad f^{c-\overline{\varepsilon}} \cup f_{c+\overline{\varepsilon}};$$
$$\psi = 1 \quad \text{on} \quad f^{c+\varepsilon_0} \cap f_{c-\varepsilon_0},$$

for some ε_0 with

$$\varepsilon < \varepsilon_0 < \min(\overline{\varepsilon}, \varepsilon_1). \tag{2.4}$$

Then we are able to construct the locally Lipschitz vector field $V : X \to X$ by setting

$$V(x) = \begin{cases} -\delta_1\varphi(x)\psi(x)v(x), & \forall x \in N_{3\delta_1}(B) \cap f_{c-3\varepsilon_1} \cap f^{c+3\varepsilon_1}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Using (2.5) we see that the vector field V is locally Lipschitz and bounded, namely

$$\|V(x)\| \leq \delta_1, \quad x \in X. \tag{2.6}$$

From (2.2), (2.5) and (2.6) we derive

$$-\langle y^*, V(x)\rangle = \delta_1\langle y^*, v(x)\rangle \geq \frac{1}{2}\delta_1\sigma_1, \ \ \forall x \in N_{\delta_1}(B) \cap f_{c-\varepsilon_0} \cap f^{c+\varepsilon_0},\ y^* \in \partial f(x). \tag{2.7}$$

In view of (2.6) we may consider the global flow $\gamma : \mathbb{R} \times X \to X$ of V defined by (2.5), i.e.

$$\frac{d\gamma}{dt}(t,x) = V(\gamma(t,x)),\ \forall (t,x) \in \mathbb{R} \times X,$$
$$\gamma(0,x) = x,\ \forall x \in X.$$

In the next we set

$$B_1 := \gamma([0,1]) \times B). \tag{2.8}$$

We notice that B_1 in (2.8) is a closed subset of X. To see this let $y_n = \gamma(t_n, x_n) \in B_1$ be a sequence with $t_n \in [0,1]$, $x_n \in B$ and $y_n \to y$ in X. Passing to a subsequence we can suppose that $t_n \to t \in [0,1]$ in $\mathbb{R}$. Putting $u_n = \gamma(t, x_n)$ we get

$$\|u_n - y_n\| = \|\gamma(t,x_n) - \gamma(t_n,x_n)\| = \|\int_{t_n}^{t} \frac{d}{dt}\gamma(\tau,x_n)d\tau\| \leq \delta_1|t_n - t|,$$

where (2.6) has been used. Since $u_n \to y$ in X, it turns out that $x_n \to \gamma(-t,y) \in B$. Finally, we obtain $y = \gamma(t, \gamma(-t,y)) \in B_1$ which establishes the closedness of B_1.

The next step is to justify that $f(\gamma(t,x))$ is a decreasing function of $t \in \mathbb{R}$, for each $x \in X$. Toward this, by applying Lebourg's mean value theorem and the chain rule for generalized gradients we infer for arbitrary real numbers $t > t_0$ the following inclusions

$$f(t,x) - f(t_0,x) \in \partial_t(f(\gamma(t,x)))\Big|_{t=\tau}$$
$$\subset \partial f(\gamma(\tau,x))\frac{d\gamma}{dt}(\tau,x)(t-t_0) = \partial f(\gamma(\tau,x))V(\gamma(\tau,x))(t-t_0)$$

with some $\tau \in (t_0,t)$, where the notation ∂_t stands for the generalized gradient with respect to t. By (2.2) and (2.5) we derive that $f(t,x) \le f(t_0,x)$. Now we prove the relation

$$A \cap B_1 = \emptyset. \tag{2.9}$$

To check (2.9), we admit by contradiction that there exist $x_0 \in B$ and $t_0 \in [0,1]$ provided $\gamma(t_0,x_0) \in A$. Since A and B are disjoint we have necesarilly that $t_0 > 0$.

From the relations $A \subset f_{c-\varepsilon_A}$ and $B \subset f^c$ we deduce

$$c - \varepsilon_A \le f(\gamma(t_0,x_0)) \le f(\gamma(t,x_0)) \le f(x_0) \le c, \ \forall t \in [0,t_0]. \tag{2.10}$$

It turns out that

$$\gamma(t,x_0) \in N_{\delta_1}(B) \cap f^c \cap f_{c-\varepsilon_A}, \quad \forall t \in [0,t_0].$$

On the other hand from (2.6) we infer the estimate

$$d(A,B) \le \|\gamma(t_0,x_0) - x_0\| = \|\int_0^{t_0} V(\gamma(s,x_0))ds\| \le \delta_1 t_0.$$

If we denote $h(t) = f(\gamma(t,x_0))$, then h is a locally Lipschitz function, and (2.5), (2.7) allow to write

$$h'(s) \le \max\{\langle y^*, \frac{d\gamma}{ds}(s,x)\rangle : \ y^* \in \partial f(\gamma(s,x))\}$$
$$= \max\{\langle y^*, V(\gamma(s,x))\rangle : \ y^* \in \partial f(\gamma(s,x))\} \le -\frac{1}{2}\delta_1\sigma_1$$

for a.e. $s \in [0,t_0]$. Therefore, by virtue of (2.3), we have the following estimate

$$f(\gamma(t_0,x_0)) - f(x_0) = h(t_0) - h(0) = \int_0^{t_0} h'(s)ds \le$$
$$-\frac{1}{2}\delta_1\sigma_1 t_0 < -\delta_1\varepsilon t_0 \le -\varepsilon d(A,B) \le -\varepsilon_A. \tag{2.11}$$

The contradiction between (2.10) and (2.11) shows that the property (2.9) is actually true. Taking into account (2.9) there is a locally Lipschitz function $\psi_A : X \to \mathbb{R}$ veryfying $\psi_A = 0$ on a neighborhood of A and $\psi_A = 1$ on B_1. Then we define the homotopy $\eta_A : \mathbb{R} \times X \to X$ as being the global flow of the vector field $V_A = \psi_A V$. The assertion (i) is clear from the construction of η_A because one can take $\varphi_A = -\delta_1\psi_A\varphi\psi$. Assertion (ii) follows easily because $V_A = 0$ on $A \cup f^{c-\overline{\varepsilon}} \cup f_{c+\overline{\varepsilon}}$. We show that (iii) is valid for $\delta = c+\varepsilon-\varepsilon_0$ with ε described in (2.3) and ε_0 in (2.4). To this end we argue by contradiction. Suppose that for some $d \in [\delta, c]$ there exists $x \in B \cap f^d$ such that

$$f(\eta_A(1,x)) > d - \varepsilon. \tag{2.12}$$

Using the fact that $\psi_A = 1$ on B_1 we deduce

$$\eta_A(t,x) = \gamma(t,x) \in N_{\delta_1}(B) \cap f^d \cap f_{d-\varepsilon}, \quad \forall t \in [0,1].$$

Then a reasoning similar to the one in (2.11) can be carried out to write

$$f(\eta_A(1,x)) - f(x) \leq -\frac{1}{2}\delta_1\sigma_1 < -\varepsilon.$$

This contradicts the relation (2.12) because $f(x) \leq d$. The proof of the assertion (iii) is complete. □

In this section we present a general minimax principle for locally Lipschitz functions. This result appears in the paper of Motreanu and Varga [45].

Theorem 2.15. (Motreanu-Varga [45]) *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional and $B \subseteq X$ a closed set such that $c := \inf_B f > -\infty$ and f satisfies $(PS)_{B,c}$. Let $\mathcal{M}$ be a nonempty family of subsets M of X such that*

$$c = \inf_{M\in\mathcal{M}} \sup_{x\in M} f(x). \tag{2.13}$$

Assume that for a generalized normalized pseudo-gradient vector field $\widehat{v}$ of f with respect to B and c the following hypothesis holds

(H) *for each set $M \in \mathcal{M}$ and each number $\varepsilon > 0$ with $f|_M < c+\varepsilon$ there exists a closed subset A of X with $f|_A \leq c+\varepsilon_A$ (see* (2.1)*), and $A \cap B = \emptyset$ such that for each locally Lipschitz function $\varphi_A : X \to [0,1]$ with supp $\varphi_A \subset (X \setminus A) \cap$ supp $\widehat{v}$ the global flow ξ_A of $\varphi_A\widehat{v}$ satisfies $\xi_A(1,M) \cap B \neq \emptyset$.*

Then the assertions below are true

(i) *$c = \inf_B f$ is attained;*
(ii) *$K_c(f) \setminus A \neq \emptyset$ for each set A entering* (H)*;*
(iii) *$K_c(f) \cap B \neq \emptyset$.*

Proof. The assertions (i) and (ii) are direct consequences of the property (iii). The proof of (iii) is achieved arguing by contradiction. Accordingly, we suppose $K_{-c}(-f) \cap B = \emptyset$. By hypothesis we know that $B \subset (-f)_{-c}$, so Theorem 2.14 can be applied for $-f$ and $-c$ (in place of f and c, respectively). Thus Theorem 2.14 yields an $\varepsilon > 0$ with the properties there stated. Then from the minimax description of c, by means of $\mathcal{M}$, we obtain the existence of a set $M \in \mathcal{M}$ satisfying $f|_M < c+\varepsilon$. Corresponding to M, assumption (H) allows to find a closed set $A \subset X \setminus B$ which satisfies $A \subset (-f)^{-c-\varepsilon_A}$ and the linking property formulated in (H). Theorem 2.14 gives rise to the deformation $\eta_A \in \mathcal{C}(\mathbb{R} \times X, X)$ which verifies $\eta_A(1, B \cap (-f)_{-c}) \subset (-f)_{-c-\varepsilon}$. This reads as

$$\eta_A(1, B) \subset f^{c+\varepsilon}. \tag{2.14}$$

By Theorem 2.14 and assumption (H) it is seen that

$$\xi_A(t, x) = \eta_A(-t, x), \tag{2.15}$$

for all $(t, x) \in \mathbb{R} \times X$. As shown in (H) one has the intersection property

$$\xi(1, M) \cap B \neq \emptyset.$$

Combining with (2.15) it turns out

$$\eta_A(1, B) \cap M \neq \emptyset.$$

Taking into account (2.13) we obtain the existence of some point $x_0 \in M$ with $f(x_0) \geq c + \varepsilon$. This contradicts the choice of the set M. □

Corollary 2.16. (Motreanu-Varga [45]) *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional satisfying (PS) and let a family $\mathcal{M}$ of subsets M of X be such that c defined by (2.13) is a real number. Assume that the hypothesis below holds*

(H') *for each $M \in \mathcal{M}$ there exists a closed set A in X with $f|_A < c$ such that for every homeomorphism h of X with $h|_A = id_A$ one has $h(M) \cap f^c \neq \emptyset$.*

Then c in (2.13) is a critical value of f and $K_c(f) \cap A = \emptyset$ for every A in (H').

Proof. We consider the global flow ξ_A (see (2.14)) and we apply Theorem 2.15 with $B = f^c$. It is clear that (H') implies (H) because $A \subset M \setminus B$ and $\xi_A(1, \cdot)$ is a homeomorphism of X with $\xi_A(1, \cdot) = id$ on A. Then Theorem 2.15 concludes the proof. □

Theorem 2.15 is suitable for applications to multiple linking problems.

Definition 2.17. Let Q, Q_0 be closed subsets of X, with $Q_0 \neq \emptyset$, $Q_0 \subset Q$, and let S be a subset of X such that $Q_0 \cap S = \emptyset$. We say that the pair (Q, Q_0) links with S if for each mapping $g \in \mathcal{C}(Q, X)$ with $g|_{Q_0} = id|_{Q_0}$ one has $g(Q) \cap S \neq \emptyset$.

Corollary 2.18. (Motreanu-Varga [45]) *Given the subsets Q, Q_0, S of the real Banach space X we assume that (Q, Q_0) links with S in X in the sense above. Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional such that $\sup_Q f < \infty$ and, for some number $\alpha \in \mathbb{R}_+$,*

$$Q_0 \subset f_\alpha, \quad S \subset f^\alpha.$$

Then assuming that for the minimax value

$$c = \inf_{g \in \Gamma} \sup_{x \in Q} f(g(x)),$$

where

$$\Gamma = \{g \in \mathcal{C}(Q, X) : g|_{Q_0} = id|_{Q_0}\},$$

$(PS)_{S,c}$ *is satisfied, the following properties hold*

(i) $c \geq \alpha$;
(ii) $K_c(f) \setminus Q_0 \neq \emptyset$;
(iii) $K_c(f) \cap S \neq \emptyset$ *if* $c = \alpha$.

Proof. Since the case $\alpha < c$ follows immediately we discuss only the situation where $\alpha = c$. The conclusion is readily obtained from Theorem 2.15 by choosing $\mathcal{M} = \{g(Q) : g \in \Gamma\}$ and $B = S$. □

A direct consequence of this corollary is the following.

Corollary 2.19. (Mountain pass theorem; zero altitude) *Let* $f : X \to \mathbb{R}$ *be a locally Lipschitz function on a Banach space satisfying* $(PS)_c$ *for every* $c \in \mathbb{R}$ *and the conditions:*

(i) $f(x) \geq \alpha \geq f(0)$ *for all* $||x|| = \rho$ *where* α *and* $\rho > 0$ *are constants;*
(ii) *there is* $e \in X$ *with* $||e|| > \rho$ *and* $f(e) \leq \alpha$.

Then the number

$$c = \inf_{g \in \Gamma} \max_{u \in [0,e]} f(g(u)),$$

where $[0, e]$ *is the closed line segment in* X *joining* 0 *and* e *and*

$$\Gamma = \{g \in \mathcal{C}([0, e], X) \; : \; g(0) = 0, \quad g(e) = e\},$$

is a critical value of f *with* $c \geq \alpha$.

Proof. It is sufficient to take in Corollary 2.18 the following choices $Q = [0, e], Q_0 = \{0, e\}$ and $S = \{x \in X \; : \; ||x|| = \rho \}$. □

A direct consequence of the above corollary is locally Lipschitz version of Pucci-Serrin Mountain Pass theorem, see [52].

Theorem 2.20. *Let* X *be a Banach space,* $h : X \to \mathbb{R}$ *a locally Lipschitz functional, satisfying the Palais-Smale condition,* x *and* y *two local minima of* h*. Then,* h *has a critical point in* X *different from* x *and* y*.*

In the next we prove a common generalization of some results of Chang [9] and Kourogenis-Papageorgiou [23]. For this see the paper of Kristály-Motreanu-Varga [33]. Let us consider $f : X \to \mathbb{R}$ to be a locally Lipschitz function.

Definition 2.21. We say that f satisfies the (C)-condition at level c (in short $(C)_c$) if every sequence $\{x_n\} \subset X$ such that $f(x_n) \to c$ and $(1 + \|x_n\|)\lambda_f(x_n) \to 0$ has a convergent subsequence.

It is clear that $(PS)_c$ implies $(C)_c$. Our approach is based on the following idea. We consider a globally Lipschitz functional $\varphi : X \to \mathbb{R}$ such that $\varphi(x) \geq 1, \forall x \in X$ (or, $\varphi(x) \geq \alpha$, for some $\alpha > 0$).

Definition 2.22. We say that the function f satisfies the $(\varphi - C)$-condition at level c (in short, $(\varphi - C)_c$) if every sequence $\{x_n\} \subset X$ such that $f(x_n) \to c$ and $\varphi(x_n)\lambda_f(x_n) \to 0$ has a convergent subsequence.

The $(\varphi - C)_c$-condition contains the $(PS)_c$ and $(C)_c$ compactness conditions, respectively. Indeed if $\varphi \equiv 1$ we get the $(PS)_c$-condition and if $\varphi(x) = 1 + \|x\|$ we have the $(C)_c$-condition.

We need the following result in order to obtain the existence of a suitable locally Lipschitz vector field.

Lemma 2.23. (Kristály-Motreanu-Varga [33]) *Let X be a Banach space and let $f : X \to \mathbb{R}$ be a locally Lipschitz function satisfying the $(\varphi - C)_c$-condition, where $\varphi : X \to \mathbb{R}$ is a globally Lipschitz function such that $\varphi(x) \geq 1, \quad \forall x \in X$. Then for each $\delta > 0$ there exist constants $\gamma, \varepsilon > 0$ and a locally Lipschitz vector field*

$$v : f^{-1}([c - \varepsilon, c + \varepsilon]) \cap (K_c)^c_\delta \to X$$

such that for each $x \in f^{-1}([c - \varepsilon, c + \varepsilon]) \cap (K_c)^c_\delta$ one has

$$||v(x)|| \leq \varphi(x) \tag{2.16}$$

$$\langle y^*, v(x)\rangle \geq \frac{\gamma}{2} \text{ for all } y^* \in \partial f(x). \tag{2.17}$$

In the sequel we shall prove a very general deformation result which unifies several results of this kind it appears in the paper of Kristály, Motreanu and Varga [33].

Theorem 2.24. (Kristály-Motreanu-Varga [33]) *Let $f : X \to \mathbb{R}$ be a locally Lipschitz function on the Banach space X satisfying the $(\varphi - C)_c$-condition, with $c \in \mathbb{R}$ and a globally Lipschitz function $\varphi : X \to \mathbb{R}$ with Lipschitz constant $L > 0$ and $\varphi(x) \geq 1$, $\forall\ x \in X$. Then for every $\varepsilon_0 > 0$ and every neighborhood U of K_c (if $K_c = \emptyset$, then we choose $U = \emptyset$) there exist a number $0 < \varepsilon < \varepsilon_0$ and a continuous function $\eta : X \times [0, 1] \to X$, such that for every $(x, t) \in X \times [0, 1]$ we have:*

(a) $\|\eta(x, t) - x\| \leq \varphi(x)te^{Lt}$;

(b) $\eta(x, t) = x$ *for every* $x \notin f^{-1}([c - \varepsilon_0, c + \varepsilon_0])$ *and* $t \in [0, 1]$;

(c) $f(\eta(x, t)) \leq f(x)$;

(d) $\eta(x,t) \neq x \Rightarrow f(\eta(x,t)) < f(x)$.
(e) $\eta(f^{c+\varepsilon},1) \subset f^{c-\varepsilon} \cup U$;
(f) $\eta(f^{c+\varepsilon} \setminus U,1) \subset f^{c-\varepsilon}$.

Proof. Fix $\varepsilon_0 > 0$ and a neighborhood U of K_c. From the compactness of K_c we can find $\delta > 0$ such that $(K_c)_{3\delta} \subseteq U$. Moreover, the proof of Lemma 2.23 guarantees the existence of $\gamma > 0$ and $0 < \overline{\varepsilon} < \varepsilon_0$ such that $\varphi(x)\lambda_f(x) \geq \gamma$ for all $x \in f^{-1}([c-\overline{\varepsilon}, c+\overline{\varepsilon}]) \cap (K_c)^c_\delta$. We consider the following two closed sets:

$$A = \{x \in X : |f(x) - c| \geq \overline{\varepsilon}\} \cup \overline{(K_c)_\delta} \tag{2.18}$$

$$B = \{x \in X : |f(x) - c| \leq \frac{\overline{\varepsilon}}{2}\} \cap (K_c)^c_{2\delta}. \tag{2.19}$$

Because $A \cap B = \emptyset$ there exists a locally Lipschitz function $\psi : X \to [0,1]$ such that $\psi = 0$ on a closed neighborhood of A, say $\tilde{A}$, disjoint of B, $\psi|_B = 1$ and $0 \leq \psi \leq 1$. For instance, we can take $\psi(x) = \dfrac{d(x,\tilde{A})}{d(x,\tilde{A}) + d(x,B)}$, $\forall\, x \in X$.

Let $V : X \to X$ be defined by

$$V(x) = \begin{cases} -\psi(x) \cdot v(x), & x \in f^{-1}([c-\overline{\varepsilon}, c+\overline{\varepsilon}]) \cap (K_c)^c_\delta; \\ 0, & \text{otherwise}, \end{cases} \tag{2.20}$$

where $v(x)$ is constructed in Lemma 2.23. The vector field V is locally Lipschitz and by the same lemma, for $x \in f^{-1}([c-\overline{\varepsilon}, c+\overline{\varepsilon}]) \cap (K_c)^c_\delta$ we have

$$\|V(x)\| = \psi(x) \cdot \|v(x)\| \leq \varphi(x) \tag{2.21}$$

$$\langle y^*, V(x)\rangle = -\psi(x) \cdot \langle y^*, v(x)\rangle \leq -\psi(x)\frac{\gamma}{2},\ \forall y^* \in \partial f(x). \tag{2.22}$$

Since V is locally Lipschitz and $\|V(x)\| \leq \varphi(0) + L\|x\|$, the following Cauchy problem:

$$\begin{cases} \dot{\eta}(x,t) = V(\eta(x,t)) & \text{a.e. on } [0,1] \\ \eta(x,0) = x \end{cases} \tag{2.23}$$

has a unique solution $\eta(x,\cdot)$ on $\mathbb{R}$, for each $x \in X$. By (2.21) we have that:

$$\|\eta(x,t) - x\| \leq \int_0^t \|V(\eta(x,s))\| ds \leq \int_0^t \varphi(\eta(x,s)) ds =$$

$$= \int_0^t [\varphi(\eta(x,s)) - \varphi(x)] ds + \int_0^t \varphi(x) ds \leq$$

$$\leq L \cdot \int_0^t \|\eta(x,s) - x\| ds + \varphi(x)t.$$

Using Gronwall's inequality we get $\|\eta(x,t) - x\| \leq \varphi(x)t \cdot e^{Lt}$, therefore the assertion (a) is proved. If $x \notin f^{-1}([c-\overline{\varepsilon}, c+\overline{\varepsilon}])$, then $x \in A$, so $\psi(x) = 0$. By (2.20) it follows that $V(x) = 0$ and from (2.23) we obtain that $\eta(x,t) = x$, for each $t \in [0,1]$. This yields (b).

Next, for a fixed $x \in X$, let us consider the function $h_x : [0,1] \to \mathbb{R}$ given by $h_x(t) = f(\eta(x,t))$. Using the chain rule we have

$$\frac{d}{dt}h_x(t) \leq \max\Big\{\langle y^*, \frac{d}{dt}\eta(x,t)\rangle : y^* \in \partial f(\eta(x,t))\Big\} =$$

$$= \max\Big\{\langle y^*, V(\eta(x,t))\rangle : y^* \in \partial f(\eta(x,t))\Big\} \text{ a.e. on } [0,1].$$

Therefore, taking into account (2.22), we infer

$$\frac{d}{dt}h_x(t) \leq -\psi(\eta(x,t))\frac{\gamma}{2} \leq 0 \text{ if } \eta(x,t) \in f^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}]) \cap (K_c)^c_\delta, \tag{2.24}$$

and clearly, by (2.20)

$$\frac{d}{dt}h_x(t) \leq 0, \quad \text{if } \eta(x,t) \notin f^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}]) \cap (K_c)^c_\delta.$$

Hence property (c) holds true.

In order to prove property (d), suppose that $\eta(x,t) \neq x$. First, we show that

$$\eta(x,s) \in f^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}]) \cap (K_c)^c_\delta, \quad \forall s \in [0,t]. \tag{2.25}$$

On the contrary, there would exist $s_0 \in [0,t]$ such that $\eta(x,s_0) \in A$. This implies that $V(\eta(x,s_0)) = 0$. Using the uniqueness of solution to the Cauchy problem formed by the equation in (2.23) and the initial condition with the initial value $\eta(x,s_0)$, we see that

$$\eta(x,\tau + s_0) = \eta(x,s_0), \quad \forall \tau \in \mathbb{R}.$$

Letting $\tau = t - s_0$ and $\tau = -s_0$ one obtains $\eta(x,t) = x$, which contradicts our assumption. Thus the claim in (2.25) is true.

Using (2.24) and (2.25) it follows that

$$f(x) - f(\eta(x,t)) = -\int_0^t \frac{d}{ds}h_x(s)\,ds \geq \frac{\gamma}{2}\int_0^t \psi(\eta(x,s))ds. \tag{2.26}$$

We show that there is $s \in [0,t]$ such that

$$\psi(\eta(x,s)) \neq 0. \tag{2.27}$$

For, otherwise, if $\psi(\eta(x,s)) = 0$, $\forall s \in [0,t]$, then $V(\eta(x,s)) = 0$, $\forall s \in [0,t]$. By (2.23), we get that $\eta(x,\cdot)$ is constant on $[0,t]$, which contradicts $\eta(x,t) \neq x$. It results that (2.27) is valid. Since $\psi \geq 0$, from (2.26) and (2.27) we infer that $f(\eta(x,t)) < f(x)$, which proves assertion (d).

We show now assertion (e). Let $\rho > 0$ such that $\overline{(K_c)_{3\delta}} \subset B(0,\rho)$. We choose

$$0 < \varepsilon \leq \min\left\{\frac{\bar{\varepsilon}}{2}, \frac{\gamma}{4}, \frac{\delta\gamma}{8}\mathrm{e}^{-\mathrm{L}}(\varphi(0) + \mathrm{L}\rho)^{-1}\right\}. \tag{2.28}$$

We argue by contradiction. Let $x \in f^{c+\varepsilon}$ such that $f(\eta(x,1)) > c-\varepsilon$ and $\eta(x,1) \notin U$. Since, by (c), $f(\eta(x,t)) \le f(x) \le c+\varepsilon$ and $f(\eta(x,t)) \ge f(\eta(x,1))$ for each $t \in [0,1]$, we get

$$c-\varepsilon < f(\eta(x,t)) \le c+\varepsilon, \quad \forall t \in [0,1]. \tag{2.29}$$

We claim that

$$\eta(\{x\} \times [0,1]) \cap (K_c)_{2\delta} \neq \emptyset. \tag{2.30}$$

Suppose that (2.30) does not hold. This means that

$$\eta(\{x\} \times [0,1]) \cap (K_c)_{2\delta} = \emptyset. \tag{2.31}$$

First, we show that

$$\eta(x,t) \in B, \quad \forall t \in [0,1]. \tag{2.32}$$

The fact that $\eta(x,t) \in f^{-1}([c-\frac{\bar{\varepsilon}}{2}, c+\frac{\bar{\varepsilon}}{2}])$ follows from (2.28) and (2.29). By (2.31) one has that $\eta(x,t) \in (K_c)^c_{2\delta}$. Consequently, from (2.19) we conclude that (2.32) is established. On the basis of (2.32) and (2.24) we may write

$$f(x) - f(\eta(x,1)) = h_x(0) - h_x(1) = -\int_0^1 \frac{d}{dt} h_x(t)dt \ge \int_0^1 \frac{\gamma}{2}\psi(\eta(x,t))dt.$$

Then, combining (2.32) and the definition of ψ it is clear that

$$f(x) - f(\eta(x,1)) \ge \frac{\gamma}{2}. \tag{2.33}$$

On the other hand, from (2.29) we obtain that

$$f(x) - f(\eta(x,1)) < 2\varepsilon. \tag{2.34}$$

From (2.33) and (2.34) we get $\frac{\gamma}{2} < 2\varepsilon$, which contradicts (2.28). This justifies (2.30).

The next step in the proof is to show that there exist $0 \le t_1 < t_2 \le 1$ such that

$$\mathrm{dist}(\eta(x,t_1), K_c) = 2\delta, \quad \mathrm{dist}(\eta(x,t_2), K_c) = 3\delta \tag{2.35}$$

and

$$2\delta < \mathrm{dist}(\eta(x,t), K_c) < 3\delta, \quad \forall t_1 < t < t_2. \tag{2.36}$$

Denote $g(t) = \mathrm{dist}(\eta(x,t), K_c)$, $\forall t \in [0,1]$. In view of (2.30) we have that $\{t \in [0,1] : g(t) \le 2\delta\} \neq \emptyset$. Thus it is permitted to consider

$$t_1 = \sup\{t \in [0,1] \,:\, g(t) \le 2\delta\}.$$

Since it is known that $(K_c)_{3\delta} \subset U$ and $\eta(x,1) \notin U$, we derive that $\eta(x,1) \notin (K_c)_{3\delta}$. This means that $g(1) \ge 3\delta$. Since $g(t_1) \le 2\delta$ it is necessary to have $t_1 < 1$. The definition of t_1 implies $g(t) > 2\delta$ for all $t \in (t_1, 1]$ (which is the first inequality in (2.36)). Letting $t \downarrow t_1$ we deduce that $g(t_1) \ge 2\delta$. We obtain that $g(t_1) = 2\delta$, so the first part in (2.35) is proved. Taking into account that $g(1) \geq 3\delta$, we see that $\{t \in [t_1, 1] : g(t) \geq 3\delta\}$ is nonempty. Then we can define

$$t_2 = \inf\{t \in [t_1, 1] : g(t) \geq 3\delta\}.$$

Since $g(t_2) \geq 3\delta$ and $g(t_1) = 2\delta$ it is clear that $t_1 < t_2$. By the definition of t_2 we have that $g(t) < 3\delta$ for all $t_1 \leq t < t_2$, so (2.36) holds. In addition, letting $t \uparrow t_2$, we get $g(t_2) = 3\delta$, so (2.35) holds, too.

Let us show that

$$t_2 - t_1 < \frac{4\varepsilon}{\gamma}. \tag{2.37}$$

From (2.36) it follows that $\eta(x,t) \notin (K_c)_{2\delta}$, $\forall t \in [t_1, t_2]$, while (2.29) and (2.28) imply $\eta(x,t) \in f^{-1}([c - \frac{\bar{\varepsilon}}{2}, c + \frac{\bar{\varepsilon}}{2}])$, $\forall t \in [t_1, t_2]$. The definition of the set B in (2.19) yields

$$\eta(x,t) \in B, \quad \forall t \in [t_1, t_2].$$

Using the definition of ψ, (2.24) and (2.29) we see that

$$\begin{aligned} \frac{\gamma}{2}(t_2 - t_1) &= \frac{\gamma}{2} \int_{t_1}^{t_2} \psi(\eta(x,t))dt \leq - \int_{t_1}^{t_2} \frac{d}{dt} h_x(t)dt \\ &= h_x(t_1) - h_x(t_2) = f(\eta(x,t_1)) - f(\eta(x,t_2)) < 2\varepsilon. \end{aligned}$$

Therefore (2.37) is proved.

We need the following inequality

$$\|\eta(x,t_2) - \eta(x,t_1)\| \geq \delta. \tag{2.38}$$

To check (2.38) consider a point $v \in K_c$ so that

$$\operatorname{dist}(\eta(x,t_1), K_c) = \|\eta(x,t_1) - v\| = 2\delta.$$

Here the compactness of K_c and the first part in (2.35) have been used. Then, on the basis of the second part in (2.35) we can write

$$\|\eta(x,t_2) - \eta(x,t_1)\| \geq \|\eta(x,t_2) - v\| - \|\eta(x,t_1) - v\| \geq 3\delta - 2\delta = \delta.$$

Therefore (2.38) holds.

Using (2.23), (2.21) and the Lipschtzianess of φ we can write

$$\begin{aligned} \|\eta(x,t_2) - \eta(x,t_1)\| &\leq \int_{t_1}^{t_2} \|V(\eta(x,s))\| ds \leq \int_{t_1}^{t_2} \varphi(\eta(x,s))ds \\ &= \int_{t_1}^{t_2} [\varphi(\eta(x,s)) - \varphi(\eta(x,t_1))]ds + \varphi(\eta(x,t_1))(t_2 - t_1) \\ &\leq \int_{t_1}^{t_2} L\|\eta(x,s) - \eta(x,t_1)\| ds + \varphi(\eta(x,t_1))(t_2 - t_1). \end{aligned} \tag{2.39}$$

By (2.39) and Gronwall's inequality we get

$$\|\eta(x,t_2)-\eta(x,t_1)\| \le \varphi(\eta(x,t_1))(t_2-t_1)e^{L(t_2-t_1)}. \tag{2.40}$$

From (2.38), (2.40), (2.37) and the Lipschtzianess of φ we deduce that

$$\begin{aligned}\delta \le \|\eta(x,t_2)-\eta(x,t_1)\| &< \frac{4\varepsilon}{\gamma}e^L\varphi(\eta(x,t_1)) \\ &\le \frac{4\varepsilon}{\gamma}e^L(\varphi(0)+L\|\eta(x,t_1)\|). \end{aligned}\tag{2.41}$$

In view of (2.35) and the choice of ρ to satisfy $\overline{(K_c)_{3\delta}} \subset B(0,\rho)$ we have $\eta(x,t_1) \in (K_c)_{3\delta} \subset B(0,\rho)$. This property and (2.28) yield from (2.41) that

$$\delta \le \frac{4\varepsilon}{\gamma}e^L(\varphi(0)+L\rho) \le \frac{\delta}{2},$$

which is a contradiction. This proves (e).

In order to show (f), since $(K_c)_{3\delta} \subset U$ it is enough to prove that

$$\eta(f^{c+\varepsilon} \setminus (K_c)_{3\delta}, 1) \subset f^{c-\varepsilon}. \tag{2.42}$$

Let us denote

$$C = (f^{c+\varepsilon} \setminus f^{c-\varepsilon}) \cap (K_c)^c_{3\delta}.$$

To check (2.42), we note that it is sufficient to verify that

$$\eta(x,1) \in f^{c-\varepsilon}, \quad \forall x \in C, \tag{2.43}$$

because for $x \in f^{c-\varepsilon}$ we have $f(\eta(x,1)) \le f(x) \le c-\varepsilon$, due to the nondecreasing monotonicity of $f(\eta(x,\cdot))$.

To show (2.43), denote

$$D = (f^{c+\varepsilon} \setminus f^{c-\varepsilon}) \cap (K_c)^c_{\frac{5}{2}\delta}.$$

First, we verify that

$$\forall x \in C,\ \exists\, t_x \in (0, \frac{4\varepsilon}{\gamma}] \ \text{ such that } \ \eta(x,t_x) \notin D. \tag{2.44}$$

To this end, we prove the inclusion below

$$\{t>0 \,:\, \eta(x,\tau) \in D,\ \forall \tau \in [0,t]\} \subset (0, \frac{4\varepsilon}{\gamma}),\ \forall x \in C. \tag{2.45}$$

Indeed, if $\eta(x,\tau)$ is in $D \subset B$, $\forall \tau \in [0,t]$, we have $\psi(\eta(x,\tau)) = 1$, $\forall \tau \in [0,t]$. Therefore, by (2.24), we have $\frac{d}{d\tau}h_x(\tau) \le -\frac{\gamma}{2}$, $\forall \tau \in [0,t]$. From this and (2.29) we obtain

$$2\varepsilon > h_x(0) - h_x(t) = -\int_0^t \frac{d}{d\tau}h_x(\tau)d\tau \ge \frac{\gamma}{2}t,$$

so $t < \frac{4\varepsilon}{\gamma}$. Thus (2.45) is satisfied.

We are now in the position to prove (2.44). We proceed arguing by contradiction. Assuming that there exist $x \in C$ such that $\eta(x,t) \in D$, $\forall t \in (0, \frac{4\varepsilon}{\gamma}]$, by (2.45), we arrive at the contradiction

$$\frac{4\varepsilon}{\gamma} \in \{t > 0 \,:\, \eta(x,\tau) \in D,\ \forall \tau \in [0,t]\} \subset (0, \frac{4\varepsilon}{\gamma}),$$

which proves (2.44).

Let us show that for every $x \in C$, it is true that

$$\eta(\{x\} \times [0,1]) \cap (K_c)_{\frac{5}{2}\delta} \neq \emptyset \Rightarrow \exists\, t_0 \in (0,t_3] \text{ such that } \eta(x,t_0) \in f^{c-\varepsilon}, \tag{2.46}$$

with

$$t_3 = \inf\{t \in [0,1] : \operatorname{dist}(\eta(x,t), K_c) \leq \frac{5}{2}\delta\},$$

where the set $\{t \in [0,1] :\ \operatorname{dist}(\eta(x,t),K_c) \leq \frac{5}{2}\delta\}$ is nonempty in view of (2.36). If (2.46) were not true it would exist $x \in C$ with $\eta(\{x\} \times [0,1]) \cap (K_c)_{\frac{5}{2}\delta} \neq \emptyset$ and $f(\eta(x,t)) > c - \varepsilon$, $\forall t \in [0,t_3]$. Hence $\eta(x,t) \in D$, $\forall t \in [0,t_3]$. This follows from the definition of t_3 and since $x \in C$. The inclusion in (2.45) implies that

$$t_3 < \frac{4\varepsilon}{\gamma}. \tag{2.47}$$

Introduce

$$t_4 = \sup\{t \in [0,t_3] :\ \operatorname{dist}(\eta(x,t),K_c) \geq 3\delta\}.$$

Since $x \in C$, then $x \in (K_c)^c_{3\delta}$, thus the set $\{t \in [0,t_3] :\ \operatorname{dist}(\eta(x,t),K_c) \geq 3\delta\}$ is nonempty. By the definitions of t_3 and t_4 it follows that

$$\eta(x,t) \in (f^{c+\varepsilon} \setminus f^{c-\varepsilon}) \cap ((K_c)_{3\delta} \setminus (K_c)_{\frac{5}{2}\delta}),\ \forall t \in [t_4,t_3].$$

We remark that

$$\|\eta(x,t_3) - \eta(x,t_4)\| \geq \frac{\delta}{2}\,. \tag{2.48}$$

Indeed, by the definition of t_4 we have

$$\|\eta(x,t_3) - \eta(x,t_4)\| \geq \|\eta(x,t_4) - v\| - \|\eta(x,t_3) - v\|$$

$$\geq 3\delta - \|\eta(x,t_3) - v\|, \quad \forall v \in K_c.$$

This leads to

$$\|\eta(x,t_3) - \eta(x,t_4)\| \geq 3\delta - \operatorname{dist}(\eta(x,t_3),K_c) = 3\delta - \frac{5}{2}\delta = \frac{\delta}{2}\,,$$

so (2.48) is verified.

Using (2.23), (2.21) and the Lipschtzianess of φ we can write

$$\|\eta(x,t_3) - \eta(x,t_4)\| \leq \int_{t_4}^{t_3} \|V(\eta(x,s))\| ds \leq \int_{t_4}^{t_3} \varphi(\eta(x,s)) ds$$

$$= \int_{t_4}^{t_3} [\varphi(\eta(x,s)) - \varphi(\eta(x,t_4))]ds + \varphi(\eta(x,t_4))(t_3 - t_4)$$

$$\leq \int_{t_4}^{t_3} L\|\eta(x,s) - \eta(x,t_4)\|ds + \varphi(\eta(x,t_4))(t_3 - t_4).$$

By Gronwall's inequality we get

$$\|\eta(x,t_3) - \eta(x,t_4)\| \leq \varphi(\eta(x,t_4))(t_3 - t_4)e^{L(t_3 - t_4)}. \tag{2.49}$$

Using (2.48), (2.49), the Lipschitzianess of φ, the inclusion $\overline{(K_c)_{3\delta}} \subset B(0,\rho)$ and (2.47), we have that

$$\frac{\delta}{2} \leq \|\eta(x,t_3) - \eta(x,t_4)\| \leq e^{L(t_3-t_4)}\varphi(\eta(x,t_4))(t_3 - t_4)$$

$$\leq e^L(\varphi(0) + L\|\eta(x,t_4)\|)t_3 < e^L(\varphi(0) + L\rho)\frac{4\varepsilon}{\gamma}\,.$$

This contradicts the choice of ε in (2.28), therefore (2.46) is true.

In order to complete the proof of (f), let $x \in C$. From (2.44), there exists $t_x \in (0, \frac{4\varepsilon}{\gamma}]$ such that $\eta(x,t_x) \notin D$. This means that

$$\eta(x,t_x) \in (X \setminus f^{c+\varepsilon}) \cup f^{c-\varepsilon} \cup (K_c)_{\frac{5}{2}\delta}\,.$$

On the other hand, $\eta(x,t_x) \in f^{c+\varepsilon}$ since, as $x \in C$, $f(\eta(x,t_x)) \leq f(x) \leq c+\varepsilon$. Consequently, we deduce that $\eta(x,t_x) \in f^{c-\varepsilon} \cup (K_c)_{\frac{5}{2}\delta}$. Two cases arise:

1) $\eta(x,t_x) \in f^{c-\varepsilon}$;

2) $\eta(x,t_x) \in (K_c)_{\frac{5}{2}\delta}$.

In case 1) we have directly that

$$f(\eta(x,1)) \leq f(\eta(x,t_x)) \leq c - \varepsilon,$$

which ensures the desired conclusion.

It remains to treat case 2). In this situation, we make use of property (2.46). Therefore, we find $t_0 \in (0,t_3]$ such that $\eta(x,t_0) \in f^{c-\varepsilon}$. Thus we may write $f(\eta(x,1)) \leq f(\eta(x,t_0)) \leq c - \varepsilon$. The proof is complete. □

Remark 2.25. If we choose $\varphi(x) = 1$ or $\varphi(x) = 1 + \|x\|$ then we obtain the deformation lemmas of Chang [9] and Kourogenis-Papageorgiou [24], respectively.

In the next we present a a general linking type result for locally Lipschitz functions which satisfy the generalized $(\varphi - C)_c$ condition. Let X be a Banach space and $A, C \subseteq X$ two sets.

Definition 2.26. We say that C links A, if $A \cap C = \emptyset$, and C is not contractible in $X \setminus A$.

Theorem 2.27. (Kristály-Motreanu-Varga [33]) *If $A, C \subseteq X$ are nonempty, A is closed, C links A, Γ_C is the set of all contractions of C, and $f : X \to \mathbb{R}$ is a locally Lipschitz which satisfies the $(\varphi - C)_c$-condition with*

$$c = \inf_{h\in\Gamma_C} \sup_{[0,1]\times C} f\circ h < \infty \quad \textit{and} \quad \sup_{x\in C} f(x) \le \inf_{x\in A} f(x),$$

then $c \ge \inf_{x\in A} f(x)$ and c is a critical value of f. Moreover, if $c = \inf_{x\in A} f(x)$, then there exists $x \in A$ such that $x \in K_c$.

Proof. Since by hypothesis C links A, for every $h \in \Gamma_C$ we have $h([0,1]\times C) \neq \emptyset$. So we infer that $c \ge \inf_{x\in A} f(x)$.

First we assume that $\inf_{x\in A} f(x) < c$. Suppose that $K_c = \emptyset$. Let $U = \emptyset$ and let $\varepsilon > 0$ and $\eta : [0,1]\times X \to X$ be as in Theorem 2.24. Also from the definition of c, we can find $h \in \Gamma_C$ such that

$$f(h(t,x)) \le c + \varepsilon \text{ for all } t \in [0,1] \text{ and } x \in C. \tag{2.50}$$

Let $H : [0,1]\times C \to X$ defined by

$$H(t,x) = \begin{cases} \eta(2t,x), & \text{if } \ 0 \le t \le \dfrac{1}{2} \\[2ex] \eta(1, h(2t-1,x)), & \text{if } \ \dfrac{1}{2} \le t \le 1. \end{cases}$$

It is easy to check that $H \in \Gamma_C$ and from d) and c) of Theorem 2.24 we obtain that for every $x \in C$ we have

$$f(H(t,x)) = f(\eta(2t,x)) \le f(x) \le \sup_{x\in C} f(x) < c, \text{ if } t \in \left[0, \frac{1}{2}\right]$$

$$f(H(t,x)) = f(\eta(1, h(2t-1,x))) \le c - \varepsilon < c, \text{ if } t \in \left[\frac{1}{2}, 1\right]$$

and from (2.50) we get

$$h(t,x) \in f^{c+\varepsilon} \text{ for every } t \in [0,1].$$

So we have contradicted the definition of c. This proves that $K_c \neq \emptyset$, when $c > \inf_{x\in A} f(x)$.

Next assume that $c = \inf_{x\in A} f(x)$. We need to show that $K_c \cap A \neq \emptyset$. Suppose the contrary and let U be a neighborhood of K_c with $U \cap A = \emptyset$. Let $\varepsilon > 0$ and $\eta : [0,1]\times X \to X$ be as in Theorem 2.24. As before let $h \in \Gamma_C$ such that $f(h(t,x)) \le$

$c+\varepsilon$ for all $(t,x)\in[0,1]\times C$. Then we define $H:[0,1]\times C\to X$ by

$$H(t,x)=\begin{cases}\eta(2t,x), & \text{if } \ 0\le t\le \dfrac{1}{2}\\ \\ \eta(1,h(2t-1,x)), & \text{if } \ \dfrac{1}{2}\le t\le 1.\end{cases}$$

Again, we have $H\in\Gamma_C$. From Theorem 2.24 follows that for all $0\le t\le \dfrac{1}{2}$ and all $x\in C$, we have

$$\eta(2t,x)=x \text{ or } f(\eta(2t,x))<f(x)\le \inf_{x\in A} f(x)=c$$

which implies

$$\eta(2t,x)\in C_X A \text{ for all } t\in\left[0,\frac{1}{2}\right] \text{ and all } x\in C.$$

For all $t\in\left[\dfrac{1}{2},1\right]$ and all $x\in C$, we have from d) Theorem 2.24

$$\eta(1,h(2t-1,x))\subseteq f^{c-\varepsilon}\cup U$$

while $(f^{c-\varepsilon}\cup U)\cap A=\emptyset$.

So H is a contraction of C in $X\setminus A$, which is a contradiction. This completely proves the theorem. □

In the next we prove a variant of Mountain Pass Theorem.

Theorem 2.28. (Kristály-Motreanu-Varga [33]) *Let X be a Banach space, $f:X\to\mathbb{R}$ be a locally Lipschitz function and $\varphi:X\to\mathbb{R}$ a globally Lipschitz function such that $\varphi(x)\ge 1$, $\forall\ x\in X$. Suppose that there exist $x_1\in X$ and $r>0$ such that $\|x_1\|>r$ and*

(i) $\max\{f(0),f(x_1)\}\le\inf\{f(x):\ \|x\|=r\}$

(ii) the function f satisfies the $(\varphi-C)_c$-condition ($c\in\mathbb{R}$), where

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} f(\gamma(t)),$$

with

$$\Gamma=\{\gamma\in C([0,1],X):\ \gamma(0)=0,\ \gamma(1)=x_1\}.$$

Then the minimax value c in (ii) is a critical value of f. Moreover, if $c=\inf\{f(x):\ \|x\|=r\}$, there exist a critical point x of f with $f(x)=c$ and $\|x\|=r$.

Proof. We will apply Theorem 2.27 with $A=\{x\in X:\|x\|=r\}$ and $C=\{0,x_1\}$. Clearly C links A and $c<\infty$. Let $\gamma\in\Gamma$ and define

$$h(t,x)=\begin{cases}\gamma(t), & \text{if } \ x=0\\ x_1, & \text{if } \ x=x_1\end{cases}$$

Then $h \in \Gamma_C$. Therefore

$$\inf_{\overline{h}\in\Gamma_C} \sup_{[0,1]\times C} f(\overline{h}(t,x)) \leq f(h(t,x)) \leq c. \tag{2.51}$$

On the other hand, if $h \in \Gamma_C$, then

$$\gamma(t) = \begin{cases} h(2t,0), & \text{if} \quad t \in \left[0, \dfrac{1}{2}\right] \\ \\ h(2-2t,x_1), & \text{if} \quad t \in \left[\dfrac{1}{2}, 1\right] \end{cases}$$

belongs to Γ and so

$$\inf_{h\in\Gamma_C} \sup_{[0,1]\times C} f(h(t,x)) \geq c. \tag{2.52}$$

By (2.51) and (2.52) we have

$$c = \inf_{h\in\Gamma_C} \sup_{[0,1]\times C} f(h(t,x))$$

and so we can apply Theorem 2.27 and finish the proof. □

2.2. **Multiple critical points results.** In this subsection we present a generalization of the three critical points theorem of Ricceri [58] to locally Lipschitz functions which appears in the paper of Kristály-Marzantowicz-Varga [28]. To do this, we first recall a topological result of Ricceri [59].

Theorem 2.29. (Ricceri [59, Theorem 4]) *Let X be a real, reflexive Banach space, let $\Lambda \subseteq \mathbb{R}$ be an interval, and let $\varphi : X \times \Lambda \to \mathbb{R}$ be a function satisfying the following conditions:*

1. *$\varphi(x,\cdot)$ is concave in Λ for all $x \in X$;*
2. *$\varphi(\cdot,\lambda)$ is continuous, coercive and sequentially weakly lower semicontinuous in X for all $\lambda \in \Lambda$;*
3. $\beta_1 := \sup_{\lambda\in\Lambda} \inf_{x\in X} \varphi(x,\lambda) < \inf_{x\in X} \sup_{\lambda\in\Lambda} \varphi(x,\lambda) =: \beta_2$.

Then, for each $\sigma > \beta_1$ there exists a non-empty open set $\Lambda_0 \subset \Lambda$ with the following property: for every $\lambda \in \Lambda_0$ and every sequentially weakly lower semicontinuous function $\Phi : X \to \mathbb{R}$, there exists $\mu_0 > 0$ such that, for each $\mu \in]0, \mu_0[$, the function $\varphi(\cdot,\lambda) + \mu\Phi(\cdot)$ has at least two local minima lying in the set $\{x \in X : \varphi(x,\lambda) < \sigma\}$.

The main result of this subsection is the following.

Theorem 2.30. (Kristály-Marzantowicz-Varga [28]) *Let $(X, \|\cdot\|)$ be a real reflexive Banach space and $\tilde{X}_i$ $(i = 1, 2)$ be two Banach spaces such that the embeddings $X \hookrightarrow \tilde{X}_i$ are compact. Let Λ be a real interval, $h : [0,\infty) \to [0,\infty)$ be a non-decreasing convex function, and let $\Phi_i : \tilde{X}_i \to \mathbb{R}$ $(i = 1, 2)$ be two locally Lipschitz functions such* *that* $E_{\lambda,\mu} = h(\|\cdot\|) + \lambda\Phi_1 + \mu g \circ \Phi_2$ *restricted to* X *satisfies the* $(PS)_c$*-condition for every* $c \in \mathbb{R}$, $\lambda \in \Lambda$, $\mu \in [0, |\lambda| + 1]$ *and* $g \in \mathcal{G}_\tau$, $\tau \geq 0$. *Assume that* $h(\|\cdot\|) + \lambda\Phi_1$ *is coercive on* X *for all* $\lambda \in \Lambda$ *and that there exists* $\rho \in \mathbb{R}$ *such that*

$$\sup_{\lambda\in\Lambda} \inf_{x\in X} [h(\|x\|) + \lambda(\Phi_1(x) + \rho)] < \inf_{x\in X} \sup_{\lambda\in\Lambda} [h(\|x\|) + \lambda(\Phi_1(x) + \rho)]. \tag{2.53}$$

Then, there exist a non-empty open set $A \subset \Lambda$ *and* $r > 0$ *with the property that for every* $\lambda \in A$ *there exists* $\mu_0 \in]0, |\lambda| + 1]$ *such that, for each* $\mu \in [0, \mu_0]$ *the functional* $\mathcal{E}_{\lambda,\mu} = h(\|\cdot\|) + \lambda\Phi_1 + \mu\Phi_2$ *has at least three critical points in* X *whose norms are less than* r.

Proof. Since h is a non-decreasing convex function, $X \ni x \mapsto h(\|x\|)$ is also convex; thus, $h(\|\cdot\|)$ is sequentially weakly lower semicontinuous on X, see H. Brézis [7, Corollaire III.8]. From the fact that the embeddings $X \hookrightarrow \tilde{X}_i$ $(i = 1, 2)$ are compact and $\Phi_i : \tilde{X}_i \to \mathbb{R}$ $(i = 1, 2)$ are locally Lipschitz functions, it follows that the function $E_{\lambda,\mu}$ as well as $\varphi : X \times \Lambda \to \mathbb{R}$ (in the first variable) given by

$$\varphi(x, \lambda) = h(\|x\|) + \lambda(\Phi_1(x) + \rho)$$

are sequentially weakly lower semicontinuous on X.

The function φ satisfies the hypotheses of Theorem 2.29. Fix $\sigma > \sup_\Lambda \inf_X \varphi$ and consider a nonempty open set Λ_0 with the property expressed in Theorem 2.29. Let $A = [a, b] \subset \Lambda_0$.

Fix $\lambda \in [a, b]$; then, for every $\tau \geq 0$ and $g_\tau \in \mathcal{G}_\tau$, there exists $\mu_\tau > 0$ such that, for any $\mu \in]0, \mu_\tau[$, the functional $E^\tau_{\lambda,\mu} = h(\|\cdot\|) + \lambda\Phi_1 + \mu g_\tau \circ \Phi_2$ restricted to X has two local minima, say x^τ_1, x^τ_2, lying in the set $\{x \in X : \varphi(x, \lambda) < \sigma\}$.

Note that

$$\bigcup_{\lambda\in[a,b]} \{x \in X : \varphi(x, \lambda) < \sigma\} \quad \subset \quad \{x \in X : h(\|x\|) + a\Phi_1(x) < \sigma - a\rho\}$$
$$\cup \{x \in X : h(\|x\|) + b\Phi_1(x) < \sigma - b\rho\}.$$

Because the function $h(\|\cdot\|) + \lambda\Phi_1$ is coercive on X, the set on the right-side is bounded. Consequently, there is some $\eta > 0$, such that

$$\bigcup_{\lambda\in[a,b]} \{x \in X : \varphi(x, \lambda) < \sigma\} \subset B_\eta, \tag{2.54}$$

where $B_\eta = \{x \in X : \|x\| < \eta\}$. Therefore,

$$x^\tau_1, x^\tau_2 \in B_\eta.$$

Now, set $c^\star = \sup_{t\in[0,\eta]} h(t) + \max\{|a|,|b|\}\sup_{B_\eta}|\Phi_1|$ and fix $r > \eta$ large enough such that for any $\lambda \in [a,b]$ to have

$$\{x \in X : h(\|x\|) + \lambda\Phi_1(x) \le c^\star + 2\} \subset B_r. \tag{2.55}$$

Let $r^\star = \sup_{B_r}|\Phi_2|$ and correspondingly, fix a function $g = g_{r^*} \in \mathcal{G}_{r^*}$. Let us define $\mu_0 = \min\{|\lambda|+1, \frac{1}{1+\sup|g|}\}$. Since the functional $E_{\lambda,\mu} = E^{r^*}_{\lambda,\mu} = h(\|\cdot\|)+\lambda\Phi_1+\mu g_{r^*}\circ\Phi_2$ restricted to X satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$, $\mu \in [0,\mu_0]$, and $x_1 = x_1^{r^*}, x_2 = x_2^{r^*}$ are local minima of $E_{\lambda,\mu}$, we may apply Corollary 2.19, obtaining that

$$c_{\lambda,\mu} = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} E_{\lambda,\mu}(\gamma(s)) \ge \max\{E_{\lambda,\mu}(x_1), E_{\lambda,\mu}(x_2)\} \tag{2.56}$$

is a critical value for $E_{\lambda,\mu}$, where Γ is the family of continuous paths $\gamma : [0,1] \to X$ joining x_1 and x_2. Therefore, there exists $x_3 \in X$ such that

$$c_{\lambda,\mu} = E_{\lambda,\mu}(x_3) \;\text{ and }\; 0 \in \partial E_{\lambda,\mu}(x_3).$$

If we consider the path $\gamma \in \Gamma$ given by $\gamma(s) = x_1 + s(x_2 - x_1) \subset B_\eta$we have

$$\begin{aligned} h(\|x_3\|) + \lambda\Phi_1(x_3) &= E_{\lambda,\mu}(x_3) - \mu g(\Phi_2(x_3)) \\ &= c_{\lambda,\mu} - \mu g(\Phi_2(x_3)) \\ &\le \sup_{s\in[0,1]} (h(\|\gamma(s)\|) + \lambda\Phi_1(\gamma(s)) + \mu g(\Phi_2(\gamma(s)))) - \mu g(\Phi_2(x_3)) \\ &\le \sup_{t\in[0,\eta]} h(t) + \max\{|a|,|b|\}\sup_{B_\eta}|\Phi_1| + 2\mu_0 \sup|g| \\ &\le c^\star + 2. \end{aligned}$$

From (2.55) it follows that $x_3 \in B_r$. Therefore, x_i, $i = 1,2,3$ are critical points for $E_{\lambda,\mu}$, all belonging to the ball B_r. It remains to prove that these elements are critical points not only for $E_{\lambda,\mu}$ but also for $\mathcal{E}_{\lambda,\mu} = h(\|\cdot\|) + \lambda\Phi_1 + \mu\Phi_2$. Let $x = x_i$, $i \in \{1,2,3\}$. Since $x \in B_r$, we have that $|\Phi_2(x)| \le r^*$. Note that $g(t) = t$ on $[-r^*, r^*]$; thus, $g(\Phi_2(x)) = \Phi_2(x)$. Consequently, on the open set B_r the functionals $E_{\lambda,\mu}$ and $\mathcal{E}_{\lambda,\mu}$ coincide, which completes the proof.

At the end of this section we recall the following non-smooth version of Ricceri [62, Theorem 2.5] which is proved by Marano and Motreanu [37].

Theorem 2.31. (Marano-Motreanu, [37, Theorem 1.1]) *Let $(X, \|\cdot\|)$ be a reflexive real Banach space, and $\tilde{X}$ another real Banach spaces such that X is compactly embedded into $\tilde{X}$. Let $\Phi : \tilde{X} \to \mathbb{R}$ and $\Psi : X \to \mathbb{R}$ be two locally Lipschitz functions, such that Ψ is weakly sequentially lower semicontinuous and coercive. For every $\rho > \inf_X \Psi$,* *put*

$$\varphi(\rho) = \inf_{u \in \Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(u) - \inf_{v \in \overline{(\Psi^{-1}(]-\infty,\rho[))}_w} \Phi(v)}{\rho - \Psi(u)}, \tag{2.57}$$

where $\overline{(\Psi^{-1}(]-\infty,\rho[))}_w$ *is the closure of* $\Psi^{-1}(]-\infty,\rho[)$ *in the weak topology. Furthermore, set*

$$\gamma := \liminf_{\rho\to+\infty} \varphi(\rho), \qquad \delta := \liminf_{\rho\to(\inf_X \Psi)^+} \varphi(\rho). \tag{2.58}$$

Then, the following conclusions hold.

(A) *If* $\gamma < +\infty$ *then, for every* $\lambda > \gamma$, *either*
  (A1) $\Phi + \lambda\Psi$ *possesses a global minimum, or*
  (A2) *there is a sequence* $\{u_n\}$ *of critical points of* $\Phi + \lambda\Psi$ *such that* $\lim_{n\to+\infty} \Psi(u_n) = +\infty$.

(B) *If* $\delta < +\infty$ *then, for every* $\lambda > \delta$, *either*
  (B1) $\Phi + \lambda\Psi$ *possesses a local minimum, which is also a global minimum of* Ψ, *or*
  (B2) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points of* $\Phi + \lambda\Psi$, *with* $\lim_{n\to+\infty} \Psi(u_n) = \inf_X \Psi$, *weakly converging to a global minimum of* Ψ.

3. Motreanu-Panagiotopoulos functionals

In this section we present some results from the critical point theory for Motreanu-Panagiotopoulos type functionals. For details we refer the reader to the monographs of Motreanu-Panagiotopoulos [46], Motreanu-Rădulescu [47], Gasinski-Papageorgiou [18] and the papers of Marano and Motreamu [38], [37]. At the end of this section we present the Principle of Symmetric Criticality for this class of functionals following the paper of Kristály-Varga-Varga [29].

3.1. Critical point results.

Let $\mathcal{I} = h + \psi$, with $h : X \to \mathbb{R}$ locally Lipschitz and $\psi : X \to (-\infty, +\infty]$ convex, proper (i.e., $\psi \not\equiv +\infty$), and lower semicontinuous. $\mathcal{I}$ is a *Motreanu-Panagiotopoulos type* functional, see [46, Chapter 3].

Definition 3.1. ([46, Definition 3.1]) An element $u \in X$ is said to be a critical point of $\mathcal{I} = h + \psi$, if

$$h^0(u; v - u) + \psi(v) - \psi(u) \geq 0, \forall v \in X.$$

In this case, $\mathcal{I}(u)$ is a critical value of $\mathcal{I}$.

We have the following result, see Gasinski-Papgeourgiu [18], Remark 2.3.1.

Proposition 3.2. *An element $u \in X$ is a critical point of $\mathcal{I} = h + \psi$, if and only if $0 \in \partial h(u) + \partial \psi(u)$, where $\partial \psi(u)$ denotes the subdifferential of the convex function ψ at u, i.e.*

$$\partial \psi(u) = \{x^* \in X^* : \psi(v) - \psi(u) \geq \langle x^*, v - u\rangle_X \text{ for every } v \in X\}.$$

Definition 3.3. ([46, Definition 3.2]) The functional $\mathcal{I} = h + \psi$ is said to satisfy the Palais-Smale condition at level $c \in \mathbb{R}$ (*shortly*, $(PS)_c$), if every sequence (u_n) from X satisfying $\mathcal{I}(u_n) \to c$ and

$$h^0(u_n; v - u_n) + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \forall v \in X,$$

for a sequence (ε_n) in $[0, \infty)$ with $\varepsilon_n \to 0$, contains a convergent subsequence. If $(PS)_c$ is verified for all $c \in \mathbb{R}$, $\mathcal{I}$ is said to satisfy the Palais-Smale condition (*shortly*,(PS)).

The next result is a non-smooth version of the Mountain Pass Theorem, see Corollary 3.2 from [46].

Theorem 3.4. (Motreanu-Panagiotopoulos [46]) *Assume that the functional $I : X \to (-\infty, +\infty]$ defined by $I = h + \psi$, satisfies (PS), $I(0) = 0$, and*

(i) *there exist constants $\alpha > 0$ and $\rho > 0$, such that $I(u) \geq \alpha$ for all $||u|| = \rho$;*
(ii) *there exists $e \in X$, with $||e|| > \rho$ and $I(e) \leq 0$.*

Then, the number

$$c = \inf_{f \in \Gamma} \sup_{t \in [0,1]} I(f(t)),$$

where

$$\Gamma = \{f \in C([0, 1], X) \ : \ f(0) = 0, \ f(1) = e \ \},$$

is a critical value of I with $c \geq \alpha$.

In the next we present the three critical points theorem of Ricceri [55] for Motreanu-Panagiotopoulos functionals. This result was proved by Marano and Motreanu [38, Theorem B].

Let $h_1, h_2 : X \to \mathbb{R}$ be locally Lipschitz functions, and let $\psi_1 : X \to]-\infty, +\infty]$ be a convex, proper, lower semicontinuous function. Then the function $h_1 + \psi_1 + \lambda h_2$ is a Motreanu-Panagiotopoulos type functional for every $\lambda \in \mathbb{R}$.

Theorem 3.5. (Marano-Motreanu [38]) *Suppose that $(X, || \cdot ||)$ is a separable and reflexive Banach space. Let $I_1 = h_1 + \psi_1$, $I_2 = h_2$, and let $\Lambda \subseteq \mathbb{R}$ be an interval. We assume that:*

(a_1) *h_1 is weakly sequentially lower semicontinuous and h_2 is weakly sequentially continuous;*

(a_2) *for every* $\lambda \in \Lambda$ *the function* $I_1 + \lambda I_2$ *fulfils* $(PS)_c, c \in \mathbb{R}$, *and*

$$\lim_{||u|| \to +\infty} (I_1(u) + \lambda I_2(u)) = +\infty;$$

(a_3) *there exists a continuous concave function* $h : \Lambda \to \mathbb{R}$ *satisfying*

$$\sup_{\lambda \in \Lambda} \inf_{u \in X} (I_1(u) + \lambda I_2(u) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \in \Lambda} (I_1(u) + \lambda I_2(u) + h(\lambda)).$$

Then, there exists an open interval $\Lambda_0 \subset \Lambda$, *such that for each* $\lambda \in \Lambda_0$ *the function* $I_1 + \lambda I_2$ *has at least three critical points in* X.

3.2. **Principle of Symmetric Criticality.** We now prove the Principle of Symmetric Criticality for Motreanu-Panagiotopoulos functionals. This result simultaneously generalizes the Principle of Symmetric Criticality in its standard form, see Palais [49] for smooth functionals; the result of Krawcewicz and Marzantowicz [25] for locally Lipschitz functions; and the result of Kobayashi and Ôtani [22] for Szulkin-type functionals. The results of this subsection is contained in the paper of Kristály, Varga and Varga [29].

Let G be a topological group which acts *linearly* on X, i.e., the action $G \times X \to X$: $[g, u] \mapsto gu$ is continuous and for every $g \in G$, the map $u \mapsto gu$ is linear. The group G induces an action of the same type on the dual space X^* defined by $\langle gx^*, u\rangle_X = \langle x^*, g^{-1}u\rangle_X$ for every $g \in G$, $u \in X$ and $x^* \in X^*$. A function $h : X \to \mathbb{R} \cup \{+\infty\}$ is $G-$*invariant* if $h(gu) = h(u)$ for every $g \in G$ and $u \in X$. A set $K \subseteq X$ (or $K \subseteq X^*$) is $G-$*invariant* if $gK = \{gu : u \in K\} \subseteq K$ for every $g \in G$. Let

$$\Sigma = \{u \in X : gu = u \text{ for every } g \in G\}$$

the *fixed point set of* X *under* G.

Now we recall some facts from [22]. Let

$$\Phi(X) = \{\psi : X \to \mathbb{R} \cup \{\infty\} : \ \psi \text{ is convex, proper, lower semicontinuous}\};$$

$$\Phi_G(X) = \{\psi \in \Phi(X) : \ \psi \text{ is } G-\text{invariant}\};$$

$$\Gamma_G(X^*) = \{K \subseteq X^* : \ K \text{ is } G-\text{invariant, weak}^*\text{−closed, convex}\}.$$

Proposition 3.6. ([22, Theorem 3.16]) *Assume that a compact group* G *acts linearly on a reflexiv Banach space* X. *Then for every* $K \in \Gamma_G(X^*)$ *and* $\psi \in \Phi_G(X)$ *one has*

$$K|_\Sigma \cap \partial(\psi|_\Sigma)(u) \neq \emptyset \Rightarrow K \cap \partial\psi(u) \neq \emptyset, \ u \in \Sigma, \tag{3.1}$$

where $K|_\Sigma = \{x^*|_\Sigma : x^* \in K\}$ *with* $\langle x^*|_\Sigma, u\rangle_\Sigma = \langle x^*, u\rangle_X$, $u \in \Sigma$.

Let $A : X \to X$ be the *averaging operator* over G, defined by

$$Au = \int_G gud\mu(g), \ u \in X, \tag{3.2}$$

where μ is the normalized Haar measure on G. The relation (3.2) can reads as follows

$$\langle x^*, Au\rangle_X = \int_G \langle x^*, gu\rangle_X d\mu(g), \ u \in X, \ x^* \in X^*. \tag{3.3}$$

It is easy to verify that A is a continuous linear projection from X to Σ and for every G-invariant closed convex set $K \subseteq X$ we have $A(K) \subseteq K$. The adjoint operator $A^* : \Sigma^* \to X^*$ of $A : X \to \Sigma$ is defined by

$$\langle A^* w^*, z\rangle_X = \langle w^*, Az\rangle_\Sigma, \ z \in X, \ w^* \in \Sigma^*. \tag{3.4}$$

Lemma 3.7. *Let $h : X \to \mathbb{R}$ be a G-invariant locally Lipschitz function and $u \in \Sigma$. Then*

(a) $\partial(h|_\Sigma)(u) \subseteq \partial h(u)|_\Sigma$.
(b) $\partial h(u) \in \Gamma_G(X^*)$.

Proof. (a) Let us fix $w^* \in \partial(h|_\Sigma)(u)$. Then by definition, one has

$$\langle w^*, v\rangle_\Sigma \le (h|_\Sigma)^0(u; v) \text{ for every } v \in \Sigma.$$

First, a simple estimation shows that $(h|_\Sigma)^0(u; v) \le h^0(u; v)$ for every $v \in \Sigma$. Thus, applying the above inequality for $v = Az \in \Sigma$ with $z \in X$ arbitrarily fixed, by (3.4) one has

$$\langle A^* w^*, z\rangle_X = \langle w^*, Az\rangle_\Sigma \le h^0(u; Az). \tag{3.5}$$

Using [10, Proposition 2.1.2 (b)] and (3.3), we get

$$\begin{aligned} h^0(u; Az) &= \max\{\langle x^*, Az\rangle_X : \ x^* \in \partial h(u)\} \\ &= \max\{\int_G \langle x^*, gz\rangle_X d\mu(g) : \ x^* \in \partial h(u)\} \\ &\le \int_G h^0(u; gz) d\mu(g) = \int_G h^0(g^{-1}u; z) d\mu(g) = \int_G h^0(u; z) d\mu(g) \\ &= h^0(u; z). \end{aligned}$$

Combining this relation with (3.5), we conclude that $A^* w^* \in \partial h(u)$. Since $w^* = A^* w^*|_\Sigma$, we obtain that $w^* \in \partial h(u)|_\Sigma$, completing the proof of (a).

(b) Since $\partial h(u)$ is a nonempty, convex and weak*-compact subset of X^* (see [10, Proposition 2.1.2 (a)]), it is enough to prove that $\partial h(u)$ is G-invariant, i.e., $g\partial h(u) \subseteq \partial h(u)$ for every $g \in G$. To this end, let us fix $g \in G$ and $x^* \in \partial h(u)$. Then, for every $z \in X$ we have

$$\langle gx^*, z\rangle_X = \langle x^*, g^{-1}z\rangle_X \le h^0(u; g^{-1}z) = h^0(gu; z) = h^0(u; z),$$

i.e., $gx^* \in \partial h(u)$. □

Theorem 3.8. (Kristály-Varga-Varga [29]) *Let X be a reflexiv Banach space and $\mathcal{I} = h + \psi : X \to \mathbb{R} \cup \{+\infty\}$ be a Motreanu-Panagiotopoulos type functional. If a compact group G acts linearly on X, and the functionals h and ψ are $G-$invariant, then every critical point of $\mathcal{I}|_\Sigma$ is also a critical point of $\mathcal{I}$.*

Proof. Let $u \in \Sigma$ be a critical point of $\mathcal{I}|_\Sigma$. Thanks to Proposition 3.2 one has $0 \in \partial(h|_\Sigma)(u) + \partial(\psi|_\Sigma)(u)$. Moreover, due to Lemma 3.7(a) we have

$$\emptyset \neq -\partial(h|_\Sigma)(u) \cap \partial(\psi|_\Sigma)(u) \subseteq -\partial h(u)|_\Sigma \cap \partial(\psi|_\Sigma)(u).$$

By choosing $K = \partial h(u)$ in Proposition 3.6 and taking into account Lemma 3.7(b), relation (3.1) implies that $\emptyset \neq -\partial h(u) \cap \partial \psi(u)$. Thus, in particular $0 \in \partial h(u) + \partial \psi(u)$, i.e., u is indeed a critical point of $\mathcal{I}$. □

A direct consequence of this theorem is the following proved by Krawcewicz and Marzantowicz [25].

Remark 3.9. (Krawcewicz-Marzantowicz [25]) Let $f : X \to \mathbb{R}$ be a G-invariant locally Lipschitz function and $u \in X^G$ a fixed point. Then $u \in X^G$ is a critical point of f if and only if u is a critical point of $f^G = f|_{X^G} : X^G \to \mathbb{R}$.

4. **Application to hemivaritional inequalities**

4.1. **Formulation of the problem.** In this section we prove some existence results for a general class of hemivariational inequalities. These results appear in the paper of Kristály [27] and Dályai-Varga [11].

Let $(X, \|\cdot\|)$ be a real, separable, reflexive Banach space, and let $(X^\star, \|\cdot\|_\star)$ be its dual. We consider $\Omega \subset \mathbb{R}^N$ an unbounded domain. Also assume that the inclusion $X \hookrightarrow L^l(\Omega)$ is continuous with the embedding constants $C(l)$, where $l \in [p, p^\star]$ ($p \geq 2, p^\star = \frac{Np}{N-p}$).

Let us denote by $\|\cdot\|_l$ the norm of $L^l(\Omega)$. In this section we suppose that the following condition holds:

(CE)**:** X is compactly embedded in $L^r(\Omega)$ for some $r \in [p, p^\star[$

Let $A : X \to X^\star$ be a potential operator with the potential $a : X \to \mathbb{R}$, i.e. a is Gâteaux differentiable and

$$\lim_{t\to 0} \frac{a(u + tv) - a(u)}{t} = \langle A(u), v\rangle,$$

for every $u, v \in X$. Here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $X^\star$ and X. For a potential we always assume that $a(0) = 0$. We suppose that $A : X \to X^\star$ satisfies the following properties:

- A is hemicontinuous, i.e. A is continuous on line segments in X and $X^\star$ equipped with the weak topology.
- A is homogeneous of degree $p-1$, i.e. for every $u \in X$ and $t > 0$ we have $A(tu) = t^{p-1}A(u)$. Consequently, for a homogeneous hemicontinuous operator of degree $p-1$, we have $a(u) = \frac{1}{p}\langle A(u), u\rangle$.
- $A : X \to X^\star$ is a strongly monotone operator, i.e. there exists a function $\kappa : [0,\infty) \to [0,\infty)$ which is positive on $(0,\infty)$ and $\lim_{t\to\infty} \kappa(t) = \infty$ and such that for all $u, v \in X$,

$$\langle A(u) - A(v), u - v\rangle \geq \kappa(\|u - v\|)\|u - v\| .$$

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable function which satisfies the following growth condition:

(F1) $|f(x,s)| \leq c(|s|^{p-1} + |s|^{r-1})$, for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$

Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$F(x,u) = \int_0^u f(x,s)ds, \quad \text{for a.e. } x \in \Omega,\ \forall s \in \mathbb{R}. \tag{4.1}$$

For a.e. $x \in \Omega$ and for every $u, v \in \mathbb{R}$, we have:

$$|F(x,u) - F(x,v)| \leq c_1|u - v|\left(|u|^{p-1} + |v|^{p-1} + |u|^{r-1} + |v|^{r-1}\right), \tag{4.2}$$

where c_1 is a constant which depends only of u and v. Therefore, the function $F(x,\cdot)$ is locally Lipschitz and we can define the partial Clarke derivative, i.e.

$$F_2^0(x,u;w) = \limsup_{y\to u,\ t\to 0^+} \frac{F(x,y+tw) - F(x,y)}{t}, \tag{4.3}$$

for every $u, w \in \mathbb{R}$ and for a.e. $x \in \mathbb{R}$.

Now, we formulate the hemivariational inequality problem that will be studied in the next:

Find $u \in X$ such that

$$\langle Au, v\rangle + \int_\Omega F_2^0(x, u(x); -v(x))dx \geq 0, \quad \forall\, v \in X. \tag{4.4}$$

To study the existence of solutions of the problem (4.4) we introduce the energy functional $\Psi : X \to \mathbb{R}$ defined by

$$\Psi(u) = a(u) - \Phi(u),$$

where $a(u) = \frac{1}{p}\langle A(u), u\rangle$ and $\Phi(u) = \int_\Omega F(x, u(x))dx$.

Remark 4.1. In Proposition 4.6 we will prove that the critical points of the functional Ψ are solution of the problem (4.4).

To study the existence of the critical point of the function Ψ is necessary to impose some conditions on the function f:

(F2) There exists $\alpha > p$, $\lambda \in [0, \frac{\kappa(1)(\alpha-p)}{C^p(p)}[$ and a continuous function $g : \mathbb{R} \to \mathbb{R}_+$, such that for a.e. $x \in \mathbb{R}^N$ and for all $u \in \mathbb{R}$ we have

$$\alpha F(x,u) + F_2^0(x,u;-u) \leq g(u), \tag{4.5}$$

where $\lim_{|u|\to\infty} g(u)/|u|^p = \lambda$.

(F2') There exists $\alpha \in (\max\{p, p^\star \frac{r-p}{p^\star-p}\}, p^\star)$ and a constant $C > 0$ such that for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}$ we have

$$-C|u|^\alpha \geq F(x,u) + \frac{1}{p} F_2^0(x,u;-u). \tag{4.6}$$

Next, we impose further assumptions on f. First we define two functions by

$$\underline{f}(x,s) = \lim_{\delta\to 0^+} \operatorname{essinf}\{f(x,t) : |t-s| < \delta\},$$
$$\overline{f}(x,s) = \lim_{\delta\to 0^+} \operatorname{esssup}\{f(x,t) : |t-s| < \delta\},$$

for every $s \in \mathbb{R}$ and for a.e. $x \in \Omega$. It is clear that the function $\underline{f}(x,\cdot)$ is lower semicontinuous and $\overline{f}(x,\cdot)$ is upper semicontinuous. The following hypothesis on f was introduced by Chang [9].

(F3) The functions $\underline{f}, \overline{f}$ are N-measurable, i.e. for every measurable function $u : \Omega \to \mathbb{R}$ the functions $x \mapsto \underline{f}(x,u(x)), x \mapsto \overline{f}(x,u(x))$ are measurable.

(F4) For every $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ such that for a.e. $x \in \Omega$ and for every $s \in \mathbb{R}$ we have

$$|f(x,s)| \leq \varepsilon|s|^{p-1} + c(\varepsilon)|s|^{r-1}.$$

(F5) For the $\alpha \in (p, p^\star)$ from condition (F2), there exists a $c^\star > 0$ such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$ we have

$$F(x,u) \geq c^\star(|u|^\alpha - |u|^p).$$

Remark 4.2. We observe that if we impose the following condition on f,

(F4') $\lim_{\varepsilon\to 0^+} \operatorname{esssup}\{\frac{|f(x,s)|}{|s|^p} : (x,s) \in \Omega \times (-\varepsilon, \varepsilon)\} = 0$,

then this condition with (F1) imply (F4).

4.2. **Some basic lemmas.** Before to study the hemivariational inequality (4.4) we prove some auxiliary lemmas. The results of this subsection appear in the paper of Dályai-Varga [11]. So, we consider the function $\Phi : X \to \mathbb{R}$ by

$$\Phi(u) = \int_\Omega F(x,u(x))dx, \quad \forall\, u \in X, \tag{4.7}$$

where $F(x,u) = \int_0^u f(x,s)ds, \quad$ for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$..

Remark 4.3. For simplicity we denote $h(u) = c|u|^{p-1}$ and in the next two results we use only that the function h is monotone increasing, convex and $h(0) = 0$.

The following results appears in the paper of Kristály [27] and Dályai-Varga [11].

Proposition 4.4. *The function $\Phi : X \to \mathbb{R}$, defined by $\Phi(u) = \int_\Omega F(x, u(x))dx$ is locally Lipschitz on bounded sets of X.*

Proof. For every $u, v \in X$, with $\|u\|, \|v\| < r$, we have

$$\begin{aligned}
&\|\Phi(u) - \Phi(v)\| \\
&\leq \int_\Omega |F(x,u(x)) - F(x,v(x))|dx \\
&\leq c_1 \int_\Omega |u(x) - v(x)|[h(|u(x)|) + h(|v(x)|)] \\
&\leq c_2 \Big(\int_\Omega |u(x) - v(x)|^p \Big)^{1/p} \Big[\Big(\int_\Omega (h(|u(x)|)^{p'} dx \Big)^{1/p'} + \Big(\int_\Omega (h(|v(x)|)^{p'} dx \Big)^{1/p'} \Big] \\
&\leq c_2 \|u - v\|_p [\|h(|u|)\|_{p'} + \|h(|v|)\|_{p'}) \\
&\leq C(u,v)\|u - v\|,
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and we used the Hölder inequality, the subadditivity of the norm $\| \cdot \|_{p'}$ and the fact that the inclusion $X \hookrightarrow L^p(\Omega)$ is continuous. We observe that $C(u, v)$ is a constant which depends only of u and v. □

Proposition 4.5. (Kristály [27] and Dályai-Varga [11]) *If condition* (F1) *holds, then for every $u, v \in X$, we have*

$$\Phi^0(u; v) \leq \int_\Omega F_2^0(x, u(x); v(x))dx. \tag{4.8}$$

Proof. It is sufficient to prove the proposition for the function f, which satisfies only the growth condition $|f(x,s)| \leq c|u|^{p-1}$ from Remark 4.3. Let us fix the elements $u, v \in X$. The function $F(x, \cdot)$ is locally Lipschitz and therefore continuous. Thus $F_2^0(x, u(x); v(x))$ can be expressed as the upper limit of $\big(F(x, y+tv(x)) - F(x,y)\big)/t$, where $t \to 0^+$ takes rational values and $y \to u(x)$ takes values in a countable subset of $\mathbb{R}$. Therefore, the map $x \to F_2^0(x, u(x); v(x))$ is measurable as the "countable limsup" of measurable functions in x. From condition (F1) we get that the function $x \to F_2^0(x, u(x); v(x))$ is from $L^1(\mathbb{R}^N)$.

Using the fact that the Banach space X is separable, there exists a sequence $w_n \in X$ with $\|w_n - u\| \to 0$ and a real number sequence $t_n \to 0^+$, such that

$$\Phi^0(u,v) = \lim_{n\to\infty} \frac{\Phi(w_n + t_n v) - \Phi(w_n)}{t_n}. \tag{4.9}$$

Since the inclusion $X \hookrightarrow L^p(\mathbb{R}^N)$ is continuous, we get $\|w_n - u\|_p \to 0$. Using [7, Theorem IV.9], there exists a subsequence of (w_n) denoted in the same way, such that $w_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$. Now, let $\varphi_n : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be the function defined by

$$\varphi_n(x) = -\frac{F(x, w_n(x) + t_n v(x)) - F(x, w_n(x))}{t_n} + c_1|v(x)|[h(|w_n(x) + t_n v(x)|) + h(|w_n(x)|)].$$

We see that the the functions φ_n are measurable and non-negative. If we apply Fatou's lemma, we get

$$\int_\Omega \liminf_{n\to\infty} \varphi_n(x)dx \le \liminf_{n\to\infty} \int_\Omega \varphi_n(x)dx.$$

This inequality is equivalent to

$$\int_\Omega \limsup_{n\to\infty} [-\varphi_n(x)]dx \ge \limsup_{n\to\infty} \int_\Omega [-\varphi_n(x)]dx. \tag{4.10}$$

For simplicity in the calculus we introduce the following notation:

(i) $\varphi_n^1(x) = \frac{F(x,w_n(x)+t_nv(x))-F(x,w_n(x))}{t_n}$;
(ii) $\varphi_n^2(x) = c_1|v(x)|[h(|w_n(x) + t_n v(x)|) + h(|w_n(x)|)]$.

With these notation, we have $\varphi_n(x) = -\varphi_n^1(x) + \varphi_n^2(x)$.

Now we prove the existence of limit $b = \lim_{n\to\infty} \int_\Omega \varphi_n^2(x)dx$. Using the facts that the inclusion $X \hookrightarrow L^p(\Omega)$ is continuous and $\|w_n - u\| \to 0$, we get $\|w_n - u\|_p \to 0$. Using [7, Theorem IV.9], there exist a positive function $g \in L^p(\Omega)$, such that $|w_n(x)| \le g(x)$ a.e. $x \in \Omega$. Considering that the function h is monotone increasing, we get

$$|\varphi_n^2(x)| \le c_1|v(x)|[h(g(x) + |v(x)|) + h(g(x))], \quad \text{a.e. } x \in \Omega.$$

Moreover, $\varphi_n^2(x) \to 2c_1|v(x)|h(|u(x)|)$ for a.e. $x \in \Omega$. Thus, using the Lebesque dominated convergence theorem, we have

$$b = \lim_{n\to\infty} \int_\Omega \varphi_n^2(x)dx = \int_\Omega 2c_1|v(x)|h(|u(x)|)dx. \tag{4.11}$$

If we denote by $I_1 = \limsup_{n\to\infty} \int_\Omega [-\varphi_n(x)]dx$, then using (4.9) and (4.11), we have

$$I_1 = \limsup_{n\to\infty} \int_\Omega [-\varphi_n(x)]dx = \Phi^0(u;v) - b. \tag{4.12}$$

Next we estimate the expression $I_2 = \int_\Omega \limsup_{n\to\infty}[-\varphi_n(x)]dx$. We have the inequality

$$\int_\Omega \limsup_{n\to\infty}[\varphi_n^1(x)]dx - \int_\Omega \lim_{n\to\infty} \varphi_n^2(x)dx \geq I_2. \tag{4.13}$$

Using the fact that $w_n(x) \to u(x)$ a.e. $x \in \Omega$ and $t_n \to 0^+$, we get

$$\int_\Omega \lim_{n\to\infty} \varphi_n^2(x)dx = 2c_1 \int_\Omega |v(x)|h(|u(x)|)dx.$$

On the other hand,

$$\begin{aligned}\int_\Omega \limsup_{n\to\infty} \varphi_n^1(x)dx &\leq \int_\Omega \limsup_{y\to u(x),\, t\to 0^+} \frac{F(x,y+tv(x)) - F(x,y)}{t}dx \\ &= \int_\Omega F_2^0(x,u(x);v(x))dx.\end{aligned}$$

Using relations (4.10), (4.12), (4.13) and the above estimates, we obtain the desired result. □

Now we prove that the critical points of the function $\Psi : X \to \mathbb{R}$ defined by $\Psi(u) = a(u) - \Phi(u)$ are solutions of problem (4.4).

Proposition 4.6. *If* $0 \in \partial\Psi(u)$, *then* u *solves the problem* (4.4).

Proof. Because $0 \in \partial\Psi(u)$, we have $\Psi^0(u;v) \geq 0$ for every $v \in X$. Using the Proposition 4.5 and a property of Clarke derivative we obtain

$$\begin{aligned}0 \leq \Psi^0(u;v) &\leq \langle u,v\rangle + (-\Phi)^0(u;v) \\ &= \langle A(u),v\rangle + \Phi^0(u;-v) \\ &\leq \langle A(u),v\rangle + \int_{\mathbb{R}^N} F_2^0(x,u(x),-v(x))dx,\end{aligned}$$

for every $v \in X$. □

4.3. **The Palais-Smale and Cerami compactness conditions.** In this subsection we study the situation when the function Ψ satisfies the $(PS)_c$ and $(CPS)_c$ conditions. We have the following result.

Proposition 4.7. *Let* $(u_n) \subset X$ *be a* $(PS)_c$ *sequence for the function* $\Psi : X \to \mathbb{R}$. *If the conditions (F1) and (F2) are fulfilled, then the sequence* (u_n) *is bounded in* X.

Proof. Because $(u_n) \subset X$ is a $(PS)_c$ sequence for the function Ψ, we have $\Psi(u_n) \to c$ and $\lambda_\Psi(u_n) \to 0$. From the condition $\Psi(u_n) \to c$ we get $c+1 \geq \Psi(u_n)$ for sufficiently large $n \in \mathbb{N}$.

Because $\lambda_\Psi(u_n) \to 0$, $\|u_n\| \geq \|u_n\|\lambda_\Psi(u_n)$ for every sufficiently large $n \in \mathbb{N}$. From the definition of $\lambda_\Psi(u_n)$ results the existence of an element $z_{u_n}^\star \in \partial\Psi(u_n)$, such that $\lambda_\Psi(u_n) = \|z^\star_{u_n}\|_\star$. For every $v \in X$, we have $|z^\star_{u_n}(v)| \leq \|z^\star_{u_n}\|_\star \|v\|$, therefore $\|z^\star_{u_n}\|_\star\|v\| \geq -z^\star_{u_n}(v)$. If we take $v = u_n$, then $\|z^\star_{u_n}\|_\star\|u_n\| \geq -z^\star_{u_n}(u_n)$.

Using the properties $\Psi^0(u,v) = \max\{z^\star(v) : z^\star \in \partial\Psi(u)\ \}$ for every $v \in X$, we have $-z^\star(v) \geq -\Psi^0(u,v)$ for all $z^\star \in \partial\Psi(u)$ and $v \in X$. If we take $u = v = u_n$ and $z^\star = z^\star_{u_n}$, we get $-z^\star_{u_n}(u_n) \geq -\Psi^0(u_n,u_n)$. Therefore, for every $\alpha > 0$, we have

$$\frac{1}{\alpha}\|u_n\| \geq \frac{1}{\alpha}\|z^\star_{u_n}\|_\star\|u_n\| \geq -\frac{1}{\alpha}\Psi^0(u_n,u_n).$$

When we add the above inequality with $c+1 \geq \Psi(u_n)$, we obtain

$$c+1+\frac{1}{\alpha}\|u_n\| \geq \Psi(u_n) - \frac{1}{\alpha}\Psi^0(u_n;u_n).$$

Using the above inequality, $\Psi^0(u,v) \leq \langle A(u),v\rangle + \Phi^0(u,-v)$, and Proposition 4.5 we get

$$\begin{aligned}
&c+1+\frac{1}{\alpha}\|u_n\| \\
&\geq \Psi(u_n) - \frac{1}{\alpha}\Psi^0(u_n;u_n) \\
&= \frac{1}{p}\langle A(u_n),u_n\rangle - \Phi(u_n) - \frac{1}{\alpha}\big(\langle A(u_n),u_n\rangle + \Phi^0(u_n;-u_n)\big) \\
&\geq (\frac{1}{p}-\frac{1}{\alpha})\langle A(u_n),u_n\rangle - \int_\Omega \big[F(x,u_n(x)) + \frac{1}{\alpha}F_2^0(x,u_n(x);-u_n(x))\big]dx \\
&\geq (\frac{1}{p}-\frac{1}{\alpha})\langle A(u_n),u_n\rangle - \frac{1}{\alpha}\int_\Omega g(u_n(x))dx.
\end{aligned}$$

The relation $\lim_{|u|\to\infty}\frac{g(u)}{|u|^p} = \lambda$ assures the existence of a constant M, such that $\int_\Omega g(u_n(x))dx \leq M + \lambda\int_\Omega |u_n(x)|^p dx$. We use again that the inclusion $X \hookrightarrow L^p(\Omega)$ is continuous, that $a(u) = \frac{1}{p}\langle A(u),u\rangle$ and that

$$a(u) = \|u\|^p\langle A(\frac{u}{\|u\|}),\frac{u}{\|u\|}\rangle \geq \kappa(1)\|u\|^p,$$

to obtain

$$\begin{aligned}
c+1+\|u_n\| &\geq (\frac{1}{p}-\frac{1}{\alpha})\langle A(u_n),u_n\rangle - \frac{\lambda C^p(p)}{\alpha}\|u_n\|^p - \frac{M}{\alpha} \\
&\geq \frac{\kappa(1)(\alpha-p)-\lambda C^p(p)}{\alpha}\|u_n\|^p - \frac{M}{\alpha}.
\end{aligned}$$

From the above inequality, it results that the sequence (u_n) is bounded. □

Proposition 4.8. *If conditions (F1), (F2') and (F4) hold, then every* $(CPS)_c(c>0)$ *sequence* $(u_n) \subset X$ *for the function* $\Psi : X \to \mathbb{R}$ *is bounded in* X.

Proof. Let $(u_n) \subset X$ be a $(CPS)_c$ $(c > 0)$ sequence for the function Ψ, i.e. $\Psi(u_n) \to c$ and $(1 + \|u_n\|)\lambda_\Psi(u_n) \to 0$. From $(1 + \|u_n\|)\lambda_\Psi(u_n) \to 0$, we get $\|u_n\|\lambda_\Psi(u_n) \to 0$ and $\lambda_\Psi(u_n) \to 0$. As in Proposition 4.7, there exists $z^\star_{u_n} \in \partial\Psi(u_n)$ such that

$$\frac{1}{p}\|z^\star_{u_n}\|_\star\|u_n\| \geq -\Psi^0(u_n; \frac{1}{p}u_n).$$

From this inequality, Proposition 4.5, condition (F2') and the property $\Psi^0(u; v) \leq \langle Au, v\rangle + \Phi^0(u; -v)$ we get

$$\begin{aligned} c + 1 &\geq \Psi(u_n) - \frac{1}{p}\Psi^0(u_n; u_n) \\ &\geq a(u_n) - \Phi(u_n) - \frac{1}{p}\left[\langle Au_n, u_n\rangle + \Phi^0(u_n; -u_n)\right] \\ &\geq -\int_\Omega \left[F(x, u_n(x)) + \frac{1}{p}F_2^0(x, u_n(x); -u_n(x))\right]dx \\ &\geq C\|u_n\|_\alpha^\alpha. \end{aligned}$$

Therefore, the sequence (u_n) is bounded in $L^\alpha(\Omega)$. From the condition (F4) follows that, for every $\varepsilon > 0$, there exists $c(\varepsilon) > 0$, such that for a.e. $x \in \mathbb{R}^N$,

$$F(x, u(x)) \leq \frac{\varepsilon}{p}|u(x)|^p + \frac{c(\varepsilon)}{r}|u(x)|^r.$$

After integration, we obtain

$$\Phi(u) \leq \frac{\varepsilon}{p}\|u\|_p^p + \frac{c(\varepsilon)}{r}\|u\|_r^r.$$

Using the above inequality, the expression of Ψ, and $\|u\|_p \leq C(p)\|u\|$, we obtain

$$\frac{\kappa(1) - \varepsilon C^p(p)}{p}\|u\|^p \leq \Psi(u) + \frac{c(\varepsilon)}{r}\|u\|_r^r \leq c + 1 + \|u\|_r^r.$$

Now, we study the behaviour of the sequence $(\|u_n\|_r)$. We have the following two cases:

(i) If $r = \alpha$, then it is easy to see that the sequence $(\|u_n\|_r)$ is bounded in $\mathbb{R}$.
(ii) If $r \in (\alpha, p^\star)$ and $\alpha > p^\star\frac{r-p}{p^\star-p}$, then we have

$$\|u\|_r^r \leq \|u\|_\alpha^{(1-s)\alpha} \cdot \|u\|_{p^\star}^{sp^\star},$$

where $r = (1 - s)\alpha + sp^\star$, $s \in (0, 1)$.

Using the inequality $\|u\|_{p^\star}^{sp^\star} \leq C^{sp^\star}(p)\|u\|^{sp^\star}$, we obtain

$$\frac{\kappa(1) - \varepsilon C^p(p)}{p}\|u\|^p \leq c + 1 + \frac{c(\varepsilon)}{r}\|u\|_\alpha^{(1-s)\alpha}\|u\|^{sp^\star}. \tag{4.14}$$

When in the inequality (4.14) we take $\varepsilon \in \left(0, \frac{\kappa(1)}{C^p(p)}\right)$ and use b), we obtain that the sequence (u_n) is bounded in X. □

The main result of this section is as follows.

Theorem 4.9. (Dályai-Varga [11])

1. *If the conditions (CE),(F1)-(F4) hold, then* Ψ *satisfies the* $(PS)_c$ *condition for every* $c \in \mathbb{R}$.
2. *If the conditions (CE),(F1), (F2'), (F3), and (F4) hold, then* Ψ *satisfies the* $(CPS)_c$ *condition for every* $c > 0$.

Proof. Let $(u_n) \subset X$ be a $(PS)_c (c \in \mathbb{R})$ or a $(CPS)_c (c > 0)$ sequence for the function $\Psi(u_n)$. Using Propositions 4.7, 4.8 it follows that the sequence (u_n) is a bounded in X. Because X is reflexive Banach space follows the existence of an element $u \in X$, such that $u_n \rightharpoonup u$ weakly in X. Because the inclusions $X \hookrightarrow L^r(\mathbb{R}^N)$ is compact, we have that $u_n \to u$ strongly in $L^r(\mathbb{R}^N)$.

Next we estimate the expressions $I_n^1 = \Psi^0(u_n; u_n - u)$ and $I_n^2 = \Psi^0(u; u - u_n)$. First we estimate the expression $I_n^2 = \Psi^0(u; u - u_n)$. We know that $\Psi^0(u; v) = \max\{z^\star(v) : z^\star \in \partial\Psi(u)\}, \ \forall v \in X$. Therefore, there exists $z_u^\star \in \partial\Psi(u)$, such that $\Psi^0(u; v) = z_u^\star(v)$ for all $v \in X$. From the above relation and from the fact that $u_n \rightharpoonup u$ weakly in X, we get $\Psi^0(u; u - u_n) = z_u^\star(u - u_n) \to 0$.

Now, we estimate the expression $I_n^1 = \Psi^0(u_n; u_n - u)$. From $\lambda_\Psi(u_n) \to 0$ follows the existence of a positive real numbers sequence $\mu_n \to 0$, such that $\Psi^0(u_n, u_n - u) + \mu_n \|u_n - u\| \geq 0$.

Now, we estimate the expression $I_n = \Phi^0(u_n; u - u_n) + \Phi^\circ(u; u - u_n)$. For the simplicity in calculus we introduce the notations $h_1(s) = |s|^{p-1}$ and $h_2(s) = |s|^r$. For this we observe that if we use the continuity of the functions h_1 and h_2, the condition (F4) implies that for every $\varepsilon > 0$, there exists a $c(\varepsilon) > 0$ such that

$$\max\left\{|\underline{f}(x,s)|, |\overline{f}(x,s)|\right\} \leq \varepsilon h_1(s) + c(\varepsilon) h_2(s), \tag{4.15}$$

for a.e. $x \in \mathbb{R}^N$ and for all $s \in \mathbb{R}$. Using this relation and Proposition 4.5, we have

$$\begin{aligned}
I_n &= \Phi^0(u_n; u - u_n) + \Phi(u; u - u_n) \\
&\leq \int_\Omega \left[F_2^0(x, u_n(x); u_n(x) - u(x)) + F_2^0(x, u(x); u(x) - u_n(x))\right] dx \\
&\leq \int_\Omega \left[\underline{f}(x, u_n(x))(u_n(x) - u(x)) + \overline{f}(x, u(x))(u(x) - u_n(x))\right] dx \\
&\leq 2\varepsilon \int_\Omega \left[h_1(u(x)) + h_1(u_n(x))\right] |u_n(x) - u(x)| dx \\
&\quad + 2c_\varepsilon \int_\Omega \left[(h_2(u(x)) + h_2(u_n(x))\right] |u_n(x) - u(x)| dx.
\end{aligned}$$

Using Hölder inequality and that the inclusion $X \hookrightarrow L^p(\Omega)$ is continuous, we get

$$I_n \le 2\varepsilon C(p)\|u_n - u\|(\|h_1(u)\|_{p'} + \|h_1(u_n)\|_{p'}) \\ + 2c(\varepsilon)\|u_n - u\|_r(\|h_2(u)\|_{r'} + \|h_2(u_n)\|_{r'}),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Using the fact that the inclusion $X \hookrightarrow L^r(\Omega)$ is compact, we get that $\|u_n - u\|_r \to 0$ as $n \to \infty$. For $\varepsilon \to 0^+$ and $n \to \infty$ we obtain that $I_n \to 0$.

Finally, we use the inequality $\Psi^0(u; v) \le \langle A(u), v\rangle + \Phi^0(u; -v)$. If we replace v with $-v$, we get $\Psi^0(u, -v) \le -\langle A(u), v\rangle + \Phi^0(u; v)$, therefore $\langle A(u), v\rangle \le \Phi^0(u; v) - \Psi^0(u, -v)$.

In the above inequality we replace u and v by $u = u_n, v = u - u_n$ then $u = u, v = u_n - u$ and we get

$$\langle A(u_n), u - u_n\rangle \le \Phi^0(u_n, u - u_n) - \Psi^0(u_n; u_n - u),$$
$$\langle A(u), u_n - u\rangle \le \Phi^0(u, u_n - u) - \Psi^0(u, u - u_n).$$

Adding these relations, we have the following key inequality:

$$\|u_n - u\|\kappa(u_n - u) \le \langle A(u_n - u), u_n - u\rangle$$

$$\le \left[\Phi^0(u_n; u - u_n) + \Phi(u; u - u_n)\right] - \Psi^0(u_n; u_n - u) - \Psi^0(u; u - u_n) = I_n - I_n^1 - I_n^2.$$

Using the above relation and the estimations of I_n, I_n^1 and I_n^2, we obtain

$$\|u_n - u\|\kappa(u_n - u) \le I_n + \mu_n\|u_n - u\| - z_u^\star(u_n - u).$$

If $n \to \infty$, from the above inequality we obtain the assertion of the theorem. □

4.4. **Existence result.** The main result of this subsection is the following.

Theorem 4.10. (Dályai-Varga [11])

1. *If conditions (CE),(F1)-(F5) hold, then problem* (4.4) *has a nontrivial solution.*
2. *If conditions (CE), (F1),(F2'), (F3), and (F4) hold, then problem* (4.4) *has a nontrivial solution.*

Proof. Using (1) in Theorem 4.9, and conditions (F1)-(F4), it follows that the functional $\Psi(u) = \frac{1}{p}\langle A(u), u\rangle - \Phi(u)$ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$. From Corollary 2.19 we verify the following geometric hypotheses:

$$\exists\, \alpha, \rho > 0, \quad \text{such that } \Psi(u) \ge \beta \text{ on } B_\rho(0) = \{u \in X : \|u\| = \rho\}, \tag{4.16}$$

$$\Psi(0) = 0 \quad \text{and there exists } v \in H \setminus B_\rho(0) \text{ such that } \Psi(v) \le 0. \tag{4.17}$$

For the proof of relation (4.16), we use the relation (F4), i.e. $|f(x,s)| \leq \varepsilon|s|^{p-1} + c(\varepsilon)|s|^{r-1}$. Integrating this inequality and using that the inclusions $X \hookrightarrow L^p(\mathbb{R}^N)$, $X \hookrightarrow L^r(\mathbb{R}^N)$ are continuous, we get that

$$\begin{aligned}\Psi(u) &\geq \frac{\kappa(1) - \varepsilon C(p)}{p}\langle A(u), u\rangle - \frac{1}{r}c(\varepsilon)C(r)\|u\|_r^r \\ &\geq \frac{\kappa(1) - \varepsilon C(p)}{p}\|u\|^p - \frac{1}{r}c(\varepsilon)C(r)\|u\|^r.\end{aligned}$$

The right member of the inequality is a function $\chi : \mathbb{R}_+ \to \mathbb{R}$ of the form $\chi(t) = At^p - Bt^r$, where $A = \frac{\kappa(1)-\varepsilon C(p)}{p}$, $B = \frac{1}{r}c(\varepsilon)C(r)$. The function χ attains its global maximum in the point $t_M = (\frac{pA}{rB})^{\frac{1}{r-p}}$. When we take $\rho = t_M$ and $\beta \in]0, \chi(t_M)]$, it is easy to see that the condition (4.16) is fulfilled.

From (F5) we have $\Psi(u) \leq \frac{1}{p}\langle A(u), u\rangle + c^\star\|u\|_p^p - c^\star\|u\|_\alpha^\alpha$. If we fix an element $v \in H \setminus \{0\}$ and in place of u we put tv, then we have

$$\Psi(tv) \leq (\frac{1}{p}\langle A(v), v\rangle + c^\star\|v\|_p^p)t^p - c^\star t^\alpha\|v\|_\alpha^\alpha.$$

From this we see that if t is large enough, $tv \notin B_\rho(0)$ and $\Psi(tv) < 0$. So, the condition (4.17) is satisfied and Corollary 2.19 assures the existence of a nontrivial critical point of Ψ.

Now when we use (2) in Theorem 4.9, from conditions (F1), (F2'), (F3), and (F4), we get that the function Ψ satisfies the condition $(CPS)_c$ for every $c > 0$. Now, we use Theorem 2.28, which assures the existence of a nontrivial critical point for the function Ψ. It is sufficient to prove only the relation (4.17), because (4.16) is proved in the same way.

To prove the relation (4.17) we fix an element $u \in X$ and we define the function $h : (0, +\infty) \to \mathbb{R}$ by $h(t) = \frac{1}{t}F(x, t^{1/p}u) - C\frac{p}{\alpha-p}t^{\frac{\alpha}{p}-1}|u|^\alpha$. The function h is locally Lipschitz. We fix a number $t > 1$, and from the Lebourg's main value theorem follows the existence of an element $\tau \in (1, t)$ such that

$$h(t) - h(1) \in \partial_t h(\tau)(t-1),$$

where ∂_t denotes the generalized gradient of Clarke with respect to $t \in \mathbb{R}$. From the Chain Rules we have

$$\partial_t F(x, t^{1/p}u) \subset \frac{1}{p}\partial F(x, t^{1/p}u)t^{\frac{1}{p}-1}u.$$

Also we have

$$\partial_t h(t) \subset -\frac{1}{t^2}F(x, t^{1/p}u) + \frac{1}{t}\partial F(x, t^{1/p}u)t^{\frac{1}{p}-1}u - Ct^{\frac{\alpha}{p}-2}|u|^\alpha.$$

Therefore,

$$\begin{aligned} h(t)-h(1) &\subset \partial_t h(\tau)(t-1) \\ &\subset -\frac{1}{t^2}\left[F(x,t^{1/p}u)-t^{1/p}u\partial F(x,t^{1/p}u)+C|t^{1/p}u|^{\alpha}\right](t-1). \end{aligned}$$

Using the relation (F2'), we obtain that $h(t)\geq h(1)$; therefore,

$$\frac{1}{t}F(x,t^{1/p}u)-C\frac{p}{\alpha-p}t^{\frac{\alpha}{p}-1}|u|^{\alpha}\geq F(x,u)-C\frac{p}{\alpha-p}|u|^{\alpha}.$$

From this inequality, we get

$$F(x,t^{1/p})\geq tF(x,u)+C\frac{p}{\alpha-p}[t^{\alpha/p}-t]|u|^{\alpha}, \tag{4.18}$$

for every $t>1$ and $u\in\mathbb{R}$. Let us fix an element $u_0\in X\setminus\{0\}$; then for every $t>1$, we have

$$\begin{aligned} \Psi(t^{1/p}u_0) &= \frac{1}{p}\langle A(t^{1/p}u_0),t^{1/p}u_0\rangle-\int_{\mathbb{R}^N}F(x,t^{1/p}u_0(x))dx \\ &\leq \frac{t}{p}\langle Au_0,u_0\rangle - t\int_{\mathbb{R}^N}F(x,u_0(x))dx - C\frac{p}{\alpha-p}[t^{\alpha/p}-t]\|u_0\|_{\alpha}^{\alpha}. \end{aligned}$$

If t is sufficiently large, then for $v_0=t^{1/p}u_0$ we have $\Psi(v_0)\leq 0$. This ends the proof. □

In general the inclusion $X\hookrightarrow L^r(\Omega)$ is not compact and we impose some invariant properties. So, let G be the compact topological group $O(N)$ or a subgroup of $O(N)$. We suppose that G acts continuously and linear isometrically on the Banach space X. We denote by

$$X^G=\{u\in H : gx=x \text{ for all } g\in G\}$$

the fixed point set of the action G on X. It is well known that X^G is a closed subspace of X. In several applications the condition (CE) is replaced by the condition

(CEG) The embeddings $X^G\hookrightarrow L^r(\mathbb{R}^N)$ are compact $(p<r<p^{\star})$.

We suppose that the potential $a:X\to\mathbb{R}$ of the operator $A:X\to X^{\star}$ is G-invariant and the next condition for the function $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ holds:

(F6) For a.e. $x\in\mathbb{R}^N$ and for every $g\in G, s\in\mathbb{R}$ we have $f(gx,s)=f(x,s)$.

If we use the Principle of Symmetric Criticality for locally Lipschitz functions, see Remark 3.9, from the above theorem we obtain the following corollary, which is useful in the applications.

Corollary 4.11. *We suppose that the potential $a:X\to\mathbb{R}$ is G-invariant and (F6) is satisfied. Then the following assertions hold.*

(a) *If the conditions (CEG),(F1)-(F5) are fulfilled, then problem* (4.4) *has a nontrivial solution.*

(b) *If the conditions (CEG), (F1), (F2'), F3), and (F4) are fulfilled, then problem* (4.4) *has a nontrivial solution.*

5. **A multiplicity result for hemivariational inequalities**

In this section we state a multiplicity result for a particular hemivariational inequality. These results appear in the paper of Faraci, Iannizzotto, Lisei and Varga [15]. Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be an unbounded domain with smooth boundary $\partial\Omega$, $p \in]1, N[$ be a real number. Throughout in this section X denotes a separable, uniformly convex Banach space with strictly convex topological dual; moreover, we assume that the condition (CE) holds. In the sequel, X will denote a (real) Banach space (with norm $\|\cdot\|$) and $X^\star$ its topological dual (with norm $\|\cdot\|_\star$); by $\langle\cdot,\cdot\rangle$ we will denote the duality pairing between $X^\star$ and X.

The next Lemma introduces the *duality mapping* on the space X, related to the weight function $t \to t^{p-1}$:

Lemma 5.1. ([8], Propositions 2.2.2, 2.2.4) *Let X be a Banach space with strictly convex dual, $p > 1$ a real number. Then, there exists a mapping $A : X \to X^\star$ such that for all $x \in X$*

(DM_1)**:** $\|A(x)\|_\star = \|x\|^{p-1}$;

(DM_2)**:** $\langle A(x), x\rangle = \|A(x)\|_\star \|x\|$.

Moreover, for all $x, y \in X$

$$\langle A(x) - A(y), x - y\rangle \geq \left(\|x\|^{p-1} - \|y\|^{p-1}\right)(\|x\| - \|y\|).$$

The functional $x \to \dfrac{\|x\|^p}{p}$ is Gâteaux differentiable with derivative A.

Let $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz, non-zero function such that $F(0) = 0$ and

(F)**:** there exist $k > 0$, $q \in]0, p-1[$ such that $|\xi| \leq k|s|^q$ for all $s \in \mathbb{R}$, $\xi \in \partial F(s)$.

Let $b : \Omega \to \mathbb{R}$ be a non-negative, not zero function such that

(b)**:** $b \in L^1(\Omega) \cap L^\infty(\Omega) \cap L^\nu(\Omega)$, where $\nu = \dfrac{r}{r-(q+1)}$.

The problem studied in this section is the following.

Find $u_0 \in X$, $\lambda > 0$ such that

$$(P_\lambda) \qquad \langle A(u - u_0), v\rangle + \lambda \int_\Omega b(x) F^\circ(u(x); -v(x))dx \geq 0 \ \text{ for all } v \in X$$

Our approach to problem (P_λ) is variational. Given $u_0 \in X$ and $\lambda > 0$, the energy functional $I : X \to \mathbb{R}$ associated to the problem (P_λ) is defined by

$$I(u) = \frac{\|u - u_0\|^p}{p} - \lambda J(u).$$

As in Proposition 4.6 follows that the critical points of I are solutions of the problem (P_λ).

Let us define the functional $J : X \to \mathbb{R}$ by

$$J(u) = \int_\Omega b(x)F(u(x))dx$$

for all $u \in X$.

Lemma 5.2. *The functional J is well-defined, locally Lipschitz, sequentially weakly continuous and satisfies*

$$J^\circ(u; v) \le \int_\Omega b(x)F^\circ(u(x); v(x))dx \quad \text{for all } u, v \in X.$$

Proof. In the same way as in Proposition 4.4 follows that J is locally Lipschitz and from Proposition 4.5 follows the inequality. We prove now that J is sequentially weakly continuous: let $\{u_n\}$ be a sequence in X, weakly convergent to some $\bar{u} \in X$. Due to condition (CE), there is a subsequence, still denoted by $\{u_n\}$, such that $\|u_n - \bar{u}\|_r \to 0$; then, by well-known results, we may assume that $u_n \to \bar{u}$ a.e. in Ω and there exists a positive function $g \in L^r(\Omega)$ such that $|u_n(x)| \le g(x)$ for all $n \in \mathbb{N}$ and almost all $x \in \Omega$. By the Lebesgue Theorem, $\{J(u_n)\}$ tends to $J(\bar{u})$. ☐

Before to prove the main result of this section we recall two results.

Theorem 5.3. ([60, Theorem 1 and Remark 1]) *Let X be a topological space, Λ a real interval, and $f : X \times \Lambda \to \mathbb{R}$ a function satisfying the following conditions:*

- (A_1) *for every $x \in X$, the function $f(x, \cdot)$ is quasi-concave and continuous;*
- (A_2) *for every $\lambda \in \Lambda$, the function $f(\cdot, \lambda)$ is lower semicontinuous and each of its local minima is a global minimum;*
- (A_3) *there exist $\rho_0 > \sup_\Lambda \inf_X f$ and $\lambda_0 \in \Lambda$ such that $\{x \in X : f(x, \lambda_0) \le \rho_0\}$ is compact.*

Then,

$$\sup_\Lambda \inf_X f = \inf_X \sup_\Lambda f.$$

Theorem 5.4. ([65, Theorem 2], [13, Lemma 1]) *Let X be a uniformly convex Banach space, with strictly convex topological dual, M a sequentially weakly closed, non-convex subset of X.*

Then, for any convex, dense subset S of X, there exists $x_0 \in S$ such that the set

$$\{y \in M \ : \ \|y - x_0\| = d(x_0, M)\}$$

has at least two points.

The main result of this section is the following and appear in the paper of Faraci, Iannizzotto, Lisei, and Varga [15].

Theorem 5.5. (Faraci-Iannizzotto-Lisei-Varga [15]) *Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain with smooth boundary $\partial\Omega$ ($N \geq 2$), $p \in]1, N[$ be a real number, X be a separable, uniformly convex Banach space with strictly convex topological dual, satisfying (E). Let $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz, non-zero function satisfying $F(0) = 0$ and (F), $b : \Omega \to \mathbb{R}$ be a non-negative, not zero function satisfying (b).*

Then, for every $\sigma \in]\inf_X J, \sup_X J[$ and every $u_0 \in J^{-1}(]-\infty, \sigma[)$ one of the following conditions is true:

- (B_1) *there exists $\lambda > 0$ such that the problem (P_λ) has at least three solutions in X;*
- (B_2) *there exists $v \in J^{-1}(\sigma)$ such that, for all $u \in J^{-1}([\sigma, +\infty[)$, $u \neq v$,*

$$\|u - u_0\| > \|v - u_0\|.$$

Proof. Fix σ and u_0 as in the thesis, and assume that (B_1) does not hold: we shall prove that (B_2) is true.

Putting $\Lambda = [0, +\infty[$ and endowing X with the weak topology, we define the function $f : X \times \Lambda \to \mathbb{R}$ by

$$f(u, \lambda) = \frac{\|u - u_0\|^p}{p} + \lambda(\sigma - J(u)),$$

which satisfies all the hypotheses of Theorem 5.3. Indeed, conditions (A_1), (A_3) are trivial.

In examining condition (A_2), let $\lambda \geq 0$ be fixed: we first observe that, by Lemma 5.2, the functional $f(\cdot, \lambda)$ is sequentially weakly lower semicontinuous (l.s.c.).

Moreover, $f(\cdot, \lambda)$ is coercive: indeed, for all $u \in X$ we have

$$f(u, \lambda) \geq \|u\|^p \left(\frac{\|u - u_0\|^p}{p\ \|u\|^p} - \lambda\ k\ c_r^{q+1} \|b\|_\nu \|u\|^{(q+1)-p} \right) + \lambda\sigma,$$

and the latter goes to $+\infty$ as $\|u\| \to +\infty$. As a consequence of the Eberlein-Smulyan theorem, the outcome is that $f(\cdot, \lambda)$ is weakly l.s.c..

We need to check that every local minimum of $f(\cdot, \lambda)$ is a global minimum. Arguing by contradiction, suppose that $f(\cdot, \lambda)$ admits a local, non global minimum; besides, being coercive, it has a global minimum too, that is, it has two strong local minima.

We now prove that $f(\cdot,\lambda)$ fulfills the Palais-Smale condition: let $\{u_n\}$ be a sequence satisfying (PS_1), (PS_2). From (PS_1), together with the coercivity of $f(\cdot,\lambda)$, it follows that $\{u_n\}$ is bounded, hence we can find a subsequence, which we still denote $\{u_n\}$, weakly convergent to a point $\bar{u}\in X$. By condition (CE) we can choose $\{u_n\}$ to be convergent to $\bar{u}$ with respect to the norm of $L^r(\Omega)$.

Fix $\varepsilon>0$. As the sequence $\{\varepsilon_n\}$ from (PS_2) tends to 0, for $n\in\mathbb{N}$ big enough we have

$$\varepsilon_n\|u_n-\bar{u}\|<\frac{\varepsilon}{2},$$

so, from (PS_2) and Lemma 5.2 it follows

$$\begin{aligned} 0 &\le f^\circ(u_n,\lambda;\bar{u}-u_n)+\frac{\varepsilon}{2} \\ &\le \langle A(u_n-u_0),\bar{u}-u_n\rangle+\lambda\int_\Omega b(x)F^\circ(u_n(x);u_n(x)-\bar{u}(x))dx+\frac{\varepsilon}{2} \end{aligned}$$

($f^\circ(\cdot,\lambda;\cdot)$ denotes the generalized directional derivative of the locally Lipschitz functional $f(\cdot,\lambda)$). Moreover, for n big enough

$$\begin{aligned} \left|\int_\Omega b(x)F^\circ(u_n(x);u_n(x)-\bar{u}(x))dx\right| &\le k\int_\Omega b(x)|u_n(x)|^q|u_n(x)-\bar{u}(x)|dx \\ &\le k\ c_r^q\|b\|_\nu\|u_n\|^q\|u_n-\bar{u}\|_r<\frac{\varepsilon}{2\lambda}. \end{aligned}$$

Hence

$$\langle A(u_n-u_0),u_n-\bar{u}\rangle<\varepsilon$$

for $n\in\mathbb{N}$ big enough. On the other hand, $\langle A(\bar{u}-u_0),u_n-\bar{u}\rangle$ tends to zero as n goes to infinity. From the previous computations, it follows that

$$\limsup_n\langle A(u_n-u_0)-A(\bar{u}-u_0),u_n-\bar{u}\rangle\le 0. \tag{5.1}$$

Applying Lemma 5.1, we obtain that

$$\begin{aligned} &\langle A(u_n-u_0)-A(\bar{u}-u_0),u_n-\bar{u}\rangle \\ &\ge\left(\|u_n-u_0\|^{p-1}-\|\bar{u}-u_0\|^{p-1}\right)\left(\|u_n-u_0\|-\|\bar{u}-u_0\|\right)\ge 0. \end{aligned}$$

From the previous inequality and (5.1), we deduce that $\|u_n-u_0\|\to\|\bar{u}-u_0\|$ and this, together with the weak convergence, implies that $\{u_n\}$ tends to $\bar{u}$ in X: that is, the Palais-Smale condition is fulfilled.

Then, we can apply Theorem 2.20, deducing that $f(\cdot,\lambda)$ (or equivalently the energy functional I) admits a third critical point: by Proposition 4.6, the inequality (P_λ) should have at least three solutions in X, against our assumption. Thus, condition (A_2) is fulfilled.

Now Theorem 5.3 assures that

$$\sup_{\lambda\in\Lambda}\inf_{u\in X} f(u,\lambda) = \inf_{u\in X}\sup_{\lambda\in\Lambda} f(u,\lambda) =: \alpha. \tag{5.2}$$

Notice that the function $\lambda \to \inf_{u\in X} f(u,\lambda)$ is upper semicontinuous in Λ, and tends to $-\infty$ as $\lambda \to +\infty$ (since $\sigma < \sup_X J$): hence, it attains its supremum in $\lambda^\star \in \Lambda$, that is

$$\alpha = \inf_{u\in X}\left(\frac{\|u-u_0\|^p}{p} + \lambda^\star(\sigma - J(u))\right). \tag{5.3}$$

The infimum in the right hand side of (5.2) is easily determined as

$$\alpha = \inf_{u\in J^{-1}([\sigma,+\infty[)} \frac{\|u-u_0\|^p}{p} = \frac{\|v-u_0\|^p}{p}$$

for some $v \in J^{-1}([\sigma,+\infty[)$.

It is easily seen that $v \in J^{-1}(\sigma)$. Hence

$$\alpha = \inf_{u\in J^{-1}(\sigma)} \frac{\|u-u_0\|^p}{p} \qquad \text{(in particular } \alpha > 0). \tag{5.4}$$

By (5.3) and (5.4) it follows that

$$\inf_{u\in X}\left(\frac{\|u-u_0\|^p}{p} - \lambda^\star J(u)\right) = \inf_{u\in J^{-1}(\sigma)}\left(\frac{\|u-u_0\|^p}{p} - \lambda^\star J(u)\right). \tag{5.5}$$

We deduce that $\lambda^\star > 0$: if $\lambda^\star = 0$, indeed, (5.5) would become $\alpha = 0$, against (5.4).

Now we can prove (B_2). Arguing by contradiction, let $w \in J^{-1}([\sigma,+\infty[)\setminus\{v\}$ be such that $\|w-u_0\| = \|v-u_0\|$. As above, we have that $w \in J^{-1}(\sigma)$, and so both w and v are global minima of the functional I (for $\lambda = \lambda^\star$) over $J^{-1}(\sigma)$, hence, by (5.5), over X. Thus, applying Theorem 2.20, we obtain that I has at least three critical points, against the assumption that (B_1) does not hold (recall that $\lambda^\star$ is positive). This concludes the proof. □

In the next Corollary, the alternative of Theorem 5.5 is resolved, under a very general assumption on the functional J, and so we are led to a multiplicity result for the hemivariational inequality (P_λ) (for suitable data u_0, λ).

Corollary 5.6. (Faraci-Iannizzotto-Lisei-Varga [15]) *Let Ω, p, X, F, b be as in Theorem 5.5 and let S be a convex, dense subset of X. Moreover, let $J^{-1}([\sigma,+\infty[)$ be not convex for some $\sigma \in]\inf_X J, \sup_X J[$.*
Then, there exist $u_0 \in J^{-1}(]-\infty,\sigma[) \cap S$ and $\lambda > 0$ such that problem (P_λ) admits at least three solutions in X.

Proof. Since J is sequentially weakly continuous (Lemma 5.2), the set $M = J^{-1}([\sigma,+\infty[)$ is sequentially weakly closed.

By Theorem 5.4 we get that, for some $u_0 \in S$, there exist two distinct points $v_1, v_2 \in M$ satisfying

$$\|v_1 - u_0\| = \|v_2 - u_0\| = \operatorname{dist}(u_0, M).$$

Clearly $u_0 \notin M$, that is, $J(u_0) < \sigma$. In the framework of Theorem 5.5, condition (B_2) is false, so (B_1) must be true: there exists $\lambda > 0$ such that (P_λ) has at least three solutions in X. □

6. **Applications**

6.1. **Existence results for a particular hemivariational inequality.**

In this subsection we give some concrete applications of Theorem 4.10. In the first two examples we suppose that X is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$.

Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a measurable function as in the section 4, i.e. satisfies the conditions (F1), (F2), (F'2) and (F3)-(F5).

Application 1. We consider the function $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ which satisfies the following conditions:

(a) $V(x) > 0$ for all $x \in \mathbb{R}^N$

(b) $V(x) \to +\infty$ as $|x| \to +\infty$.

Let X be the Hilbert space defined by

$$X = \{u \in H^1(\mathbb{R}^N) : \int (|\nabla u(x)|^2 + V(x)|u(x)|^2) dx < \infty\},$$

with the inner product

$$\langle u, v \rangle = \int (\nabla u \nabla v + V(x) uv) dx.$$

It is well known that if the conditions (a) and (b) are fulfilled then the inclusion $X \hookrightarrow L^2(\mathbb{R}^N)$ is compact, see [17], therefore the condition (CE) is satisfied.

Now we formulate the problem.

Find a positive $u \in X$ such that for every $v \in X$ we have

$$\int_{\mathbb{R}^N} (\nabla u \nabla v + V(x) uv) dx + \int_{\mathbb{R}^N} F_2^0(x, u(x); -v(x)) dx \geq 0. \tag{6.1}$$

We have the following result.

Corollary 6.1. 1. *If conditions (F1)-(F5) and (a)-(b) hold, then problem* (6.1) *has a nontrivial positive solution.*

2. *If conditions (F1),(F2'), (F3), (F4) and (a)-(b) hold, then problem* (6.1) *has a nontrivial positive solution.*

Proof. We replace the function f by $f_+ : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ defined by

$$f_+(x,u) = \begin{cases} f(x,u) & \text{if } u \geq 0; \\ 0, & \text{if } u < 0 \end{cases} \tag{6.2}$$

and use (2) in Theorem 4.10. □

Application 2. Now, we consider $Au := -\triangle u + |x|^2 u$ for $u \in D(A)$, where

$$D(A) := \{u \in L^2(\mathbb{R}^N) : Au \in L^2(\mathbb{R}^N)\}.$$

Here $|\cdot|$ denotes the Euclidian norm of $\mathbb{R}^N$. In this case the Hilbert space X is defined by

$$X = \{\ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + |x|^2 u^2) dx < \infty\},$$

with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + |x|^2 uv) dx.$$

The inclusion $X \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $s \in [2, \frac{2N}{N-2})$, see Kavian [21, Exercise 20, pp. 278]. Therefore, the condition (CE) is satisfied.

Now, we formulate the next problem.

Find a positive $u \in X$ such that for every $v \in X$ we have

$$\int_{\mathbb{R}^N} (\nabla u \nabla v + |x|^2 uv) dx + \int_{\mathbb{R}^N} F_2^0(x, u(x); -v(x)) dx \geq 0. \tag{6.3}$$

Corollary 6.2.
1. *If conditions (F1)-(F5) hold, then problem* (6.3) *has a positive solution.*
2. *If conditions (F1),(F2'), (F3), and (F4) hold, then problem* (6.3) *has a positive solution.*

Application 3. In this example we suppose that G is a subgroup of the group $O(N)$. Let Ω be an unbounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and the elements of G leave Ω invariant, i.e. $g(\Omega) = \Omega$ for every $g \in G$. We suppose that Ω is compatible with G, see the book of Willem [67, Definition 1.22]. The action of G on $X = W_0^{1,p}(\Omega)$ is defined by

$$gu(x) := u(g^{-1}x).$$

The subspace of invariant function X^G is defined by

$$X^G := \{u \in X : gu = u,\ \forall g \in G\ \}.$$

The norm on X is defined by

$$\|u\| = \Big(\int_\Omega (|\nabla u|^p + |u|^p) dx\Big)^{1/p}.$$

If Ω is compatible with G, then the embeddings $X \hookrightarrow L^s(\Omega)$, with $p < s < p^\star$ are compact, see the paper of Kobayashi and Otani [22]. Therefore the condition (CEG) is satisfied.

We consider the potential $a : X \to \mathbb{R}$ defined by $a(u) = \frac{1}{p}\|u\|^p$. This function is G-invariant because the action of G is isometric on X. The Gateaux differential $A : X \to X^\star$ of the function $a : X \to \mathbb{R}$ is given by

$$\langle Au, v\rangle = \int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv\right) dx.$$

The operator A is homogeneous of degree $p-1$ and strongly monotone, because $p \geq 2$.

Now, we formulate the following problem.

Find $u \in X \setminus \{0\}$ such that for every $v \in X$ we have

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv\right) dx + \int_\Omega F_2^0(x, u(x); -v(x))dx \geq 0. \tag{6.4}$$

We have the following result.

Corollary 6.3. (a) *If conditions (F1)-(F6) are fulfilled, then problem* (6.4) *has a nontrivial symmetric solution.*

(b) *If conditions (F1), (F2'), (F3), (F4) and (F6) are fulfilled, then problem* (6.4) *has a nontrivial symmetric solution.*

6.2. **Multiplicity results for some hemivariational inequalities.** In this subsection we state a multiplicity result for a particular hemivariational inequality as application of Corollary 5.6. Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be an unbounded domain with smooth boundary $\partial\Omega$, $p \in]1, N[$ be a real number. As in Section 5, let $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz, non-zero function such that $F(0) = 0$ and

(F)**:** there exist $k > 0$, $q \in]0, p-1[$ such that $|\xi| \leq k|s|^q$ for all $s \in \mathbb{R}$, $\xi \in \partial F(s)$.

Let $b : \Omega \to \mathbb{R}$ be a non-negative, not zero function such that

(b)**:** $b \in L^1(\Omega) \cap L^\infty(\Omega) \cap L^\nu(\Omega)$, where $\nu = \dfrac{r}{r-(q+1)}$.

We suppose that F is not a quasi-concave function, that is:

(C)**:** there exists $\rho \in]\inf_\mathbb{R} F, \sup_\mathbb{R} F[$ such that $F^{-1}([\rho, +\infty[)$ is not convex.

6.2.1. *First application.* Let $V : \Omega \to \mathbb{R}$ be a continuous potential satisfying the following conditions:

(V_1) $\inf_\Omega V > 0$;

(V_2) for every $M > 0$ the set $\{x \in \Omega \ : \ V(x) \leq M\}$ has finite Lebesgue measure (note that in particular, condition (V_2) is fulfilled whenever V is coercive). We introduce the space

$$X = \left\{ u \in W^{1,p}(\Omega) \ : \ \int_\Omega (|\nabla u(x)|^p + V(x)|u(x)|^p)dx < \infty \right\}$$

endowed with the norm

$$\|u\| = \left(\int_\Omega (|\nabla u(x)|^p + V(x)|u(x)|^p)dx \right)^{\frac{1}{p}}.$$

With the definitions above, for all $u_0 \in X$, $\lambda > 0$, our problem (P_λ) reads as follows:

$$\int_\Omega (|\nabla(u(x) - u_0(x))|^{p-2}\nabla(u(x) - u_0(x)) \cdot \nabla v(x)$$
$$+V(x)|u(x) - u_0(x)|^{p-2}(u(x) - u_0(x))v(x))dx$$
$$+\lambda \int_\Omega b(x)F^\circ(u(x); -v(x))dx \geq 0 \text{ for all } v \in X.$$

We can state the following multiplicity result:

Corollary 6.4. *Let Ω, p, V, X be as above; F, b be as in Theorem 5.5 (with $\nu = p/(p-(q+1))$ in condition (b)); S be a convex, dense subset of X. Moreover, assume that condition (C) is satisfied. Then, there exist $u_0 \in S$ and $\lambda > 0$ such that the problem (P_λ) admits at least three solutions in X.*

Proof. We observe that X is a separable, uniformly convex Banach space with strictly convex topological dual, and that $C_c^\infty(\Omega) \subset X$; moreover, the conditions (V_1), (V_2) guarantee that the space X is compactly embedded in $L^p(\Omega)$ (see [4] for the case $p = 2$), so condition (E) is satisfied with $r = p$. Since b is not zero, there exist a point $x_0 \in \Omega$ and $R > 0$ such that

$$b_1 = \int_B b(x)dx > 0,$$

where B is the open ball centered in x_0 with radius R, contained in Ω.

By condition (C), we can assume, without loss of generality, that there exist real numbers $s_1 < s_2 < s_3$ such that $F(s_1), F(s_3) > \rho$, $F(s_2) < \rho$. Now we prove that the functional J admits a non-convex superlevel set. Choose $\varepsilon > 0$, $R_1 > R$ with

$$\|b\|_\infty M \text{meas}(A) < \varepsilon < b_1|F(s_i) - \rho| \quad (i = 1, 2, 3),$$

where $A = \{x \in \Omega : R < |x - x_0| < R_1\}$ and $M = \max\{|F(t)| : |t| \leq |s_i|,\ i = 1, 2, 3\}$. There exists $u_1 \in C_c^\infty(\Omega)$ such that

$$u_1(x) = \begin{cases} s_1 & \text{if } x \in B \\ 0 & \text{if } x \in \Omega \setminus (A \cup B) \end{cases}$$

and $\|u_1\|_\infty = |s_1|$; define, also, $u_2, u_3 \in C_c^\infty(\Omega)$ by putting $u_2 = (s_2/s_1)u_1$, $u_3 = (s_3/s_1)u_1$ (we assume $s_1 \neq 0$). Thus,

$$\begin{aligned} J(u_1) &= \int_B b(x)F(s_1)dx + \int_A b(x)F(u_1(x))dx \\ &\geq b_1F(s_1) - M\|b\|_\infty \mathrm{meas}(A) \\ &\geq b_1F(s_1) - \varepsilon \\ &> b_1\rho. \end{aligned}$$

Analogously, we get

$$J(u_2) < b_1\rho, \ J(u_3) > b_1\rho.$$

Then, since u_2 lies on the segment joining u_1 and u_3, it is proved that $J^{-1}([b_1\rho, +\infty[)$ is not convex. An application of Corollary 5.6 yields the existence of a function $u_0 \in J^{-1}(]-\infty, b_1\rho[) \cap S$ and $\lambda > 0$ such that (P_λ) has at least three solutions in X. □

Example 6.5. In this example we prove the existence of a continuous function $g : \mathbb{R}^N \to \mathbb{R}$ and a positive λ such that the equation

$$(E_\lambda) \qquad -\Delta u + V(x)u = \lambda b(x)H(u-1)(\ln u - 1) + g(x) \quad \text{in } \mathbb{R}^N$$

(where V is a positive and coercive potential and H is the Heaviside function) admits at least three solutions in $H^2(\mathbb{R}^N)$ More precisely, let $V : \mathbb{R}^N \to \mathbb{R}$ be a continuous, positive and coercive function, X be as above with $p = 2 < N$, b be as in Theorem 5.5. Recall that the Heaviside function $H : \mathbb{R} \to \mathbb{R}$ is defined by

$$H(s) = \begin{cases} 0 & \text{if } s \leq 0 \\ 1 & \text{if } s > 0 \end{cases},$$

and put

$$f(s) = H(s-1)(\ln s - 1) \quad \text{for all } s \in \mathbb{R}$$

(with obvious meaning for $s \leq 0$). We denote, for all $s \in \mathbb{R}$,

$$f_-(s) = \lim_{\delta\to 0^+} \inf_{|t-s|<\delta} f(t), \quad f_+(s) = \lim_{\delta\to 0^+} \sup_{|t-s|<\delta} f(t).$$

Following Chang [9], for all continuous $g : \mathbb{R}^N \to \mathbb{R}$ and $\lambda > 0$, by a weak solution of (E_λ) we mean a function $u \in H^2(\mathbb{R}^N)$ such that, for almost every $x \in \mathbb{R}^N$,

$$-\Delta u(x) + V(x)u(x) \in g(x) + \lambda b(x)[f_-(u(x)), f_+(u(x))]. \tag{6.5}$$

It is easily seen that the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(s) = \int_0^s f(t)dt$$

is locally Lipschitz and satisfies the condition (F) with arbitrary $q \in]0,1[$ for k big enough; moreover, for all $\rho \in]2-e,0]$ the set $F^{-1}([\rho,+\infty[)$ is not convex, so condition (C) is fulfilled. Taking $S = C_c^\infty(\mathbb{R}^N)$, we can apply Corollary 6.4: thus, we find $u_0 \in S$ and $\lambda > 0$ such that the hemivariational inequality

$$\int_{\mathbb{R}^N} (\nabla(u(x) - u_0(x)) \cdot \nabla v(x) + V(x)(u(x) - u_0(x))v(x))\, dx +$$

$$+\lambda \int_{\mathbb{R}^N} b(x) F^\circ(u(x); -v(x)) dx \geq 0 \quad \text{for all } v \in X$$

admits at least three solutions in X. Let u be one of these: by standard regularity results, we get $u \in H_0^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$; arguing as in [9], we find that u satisfies (6.5) with

$$g(x) = -\Delta u_0(x) + V(x) u_0(x) \;\text{ for all } x \in \mathbb{R}^N.$$

Thus, (E_λ) has at least three weak solutions.

6.2.2. *Second application.* Here we give an application of Corollary 5.6 combined with the Principle of Symmetric Criticality for locally Lipschitz functions. Let Ω be an unbounded domain in $\mathbb{R}^N$ $(N > 2)$ with smooth boundary, such that $0 \in \Omega$, and G be a closed subgroup of $O(N)$ which leaves Ω invariant, i.e. $g(\Omega) = \Omega$ for all $g \in G$. We assume that Ω is *compatible* with G, that is, there exists $r > 0$ such that

$$m(x, r, G) \to \infty \text{ as } \operatorname{dist}(x, \Omega) \leq r, \quad |x| \to \infty,$$

where

$$m(x, r, G) = \sup \{ n \in \mathbb{N} : \; \exists\, g_1, g_2, \cdots g_n \in G \text{ s.t. } B(g_i x, r) \cap B(g_j x, r) = \emptyset \text{ if } i \neq j \}.$$

We consider the space $X = W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega (|\nabla u(x)|^p + |u(x)|^p) dx \right)^{\frac{1}{p}}.$$

Our problem is the following: *For $u_0 \in X$, $\lambda > 0$, find $u \in X$ such that*

$$\int_\Omega (|\nabla(u(x) - u_0(x))|^{p-2} \nabla(u(x) - u_0(x)) \cdot \nabla v(x)$$

$$+ |u(x) - u_0(x)|^{p-2}(u(x) - u_0(x)) v(x)) dx$$

$$+\lambda \int_\Omega b(x) F^\circ(u(x); -v(x)) dx \geq 0 \;\text{ for all } v \in X.$$

We define the action of the group G over the space X as follows:

$$gu(x) = u(g^{-1}x) \text{ for all } g \in G,\; u \in X, x \in \Omega.$$

We observe that G acts *linearly* and *isometrically* on X, i.e., the action $G \times X \to X$ which maps (g, u) into gu is continuous and, for every $g \in G$, the map $u \to gu$ is linear and $||gu|| = ||u||$ for every $u \in X$. The group G induces an action of the same type on the dual space $X^\star$ defined by $\langle gu^\star, u\rangle = \langle u^\star, g^{-1}u\rangle$ for every $g \in G$, $u \in X$ and $u^\star \in X^\star$.

We introduce the set

$$X^G = \{u \in X \ : \ gu = u \text{ for all } g \in G\}$$

of the fixed points of X under the action of G, and observe that X^G is a Banach space (which inherits all the properties of X), whose dual coincides with the fixed point set of $X^\star$ under the action of G, denoted $(X^G)^\star$. From [22, Proposition 4.2], follows that X^G is compactly embedded in $L^r(\Omega)$ for all $r \in]p, p^\star[$.

We have the following result.

Corollary 6.6. *Let Ω, p, X, G be as above, S be a convex, dense subset of X^G. Let F be as in Theorem 5.5 and satisfying condition (C). Also, let $b : \Omega \to \mathbb{R}$ be a non-negative, G-invariant function (that is, $b(gx) = b(x)$ for all $g \in G$, $x \in \Omega$) satisfying condition (b) and such that*

$$\int_B b(x)dx > 0 \qquad (B = B(0, R) \text{ for some } R > 0 \text{ small enough}).$$

Then, there exist $u_0 \in S$ and $\lambda > 0$ such that the problem (P_λ) admits at least three solutions lying in X^G.

Proof. We are going to apply Corollary 5.6 to the space X^G and to the functional $J|_{X^G}$: first, we note that X^G is separable and uniformly convex, and that $(X^G)^\star$ is strictly convex (as a subspace of $X^\star$); moreover, the space X^G satisfies condition (CEG) for any $r \in]p, p^\star[$.

In order to see that $J|_{X^G}$ admits a non-convex superlevel set, we argue as in the proof of Corollary 6.4, putting $x_0 = 0$ and choosing the functions $u_1, u_2, u_3 \in C_c^\infty(\Omega)$ radially symmetric (so, in particular, lying in X^G).

Thus, by Corollary 5.6 , there exist $u_0 \in S$ and $\lambda > 0$ such that the energy functional $I|_{X^G}$ has at least three critical points in X^G.

Now we prove that I is G-invariant on X. Let $g \in G$ and $u \in X$; recalling that $u_0 \in X^G$, G acts isometrically over X and b is G-invariant, we obtain the following equalities:

$$\begin{aligned} I(gu) &= \frac{1}{p}\|gu - u_0\|^p - \int_\Omega b(x)F(gu(x))dx \\ &= \frac{1}{p}\|g(u - u_0)\|^p - \int_\Omega b(x)F(u(g^{-1}x))dx \\ &= \frac{1}{p}\|u - u_0\|^p - \int_\Omega b(y)F(u(y))dy = I(u). \end{aligned}$$

Then, applying Theorem 6.13, we deduce that the critical points of $I|_{X^G}$ are actually critical points of I. We can conclude that problem (P_λ) has at least three symmetric solutions. □

Next we give an example, in order to highlight the generality of our hypotheses:

Example 6.7. Put $N = 3$ and define the unbounded domain

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ |x_3| < x_1^2 + x_2^2 + 1\}.$$

Then, consider the closed subgroup of $O(3)$ defined by $G = O(2) \times \{\text{id}\}$, whose action on $X = W_0^{1,p}(\Omega)$ $(1 < p < N)$ is expressed as follows: for all $g = (\tilde{g}, \text{id}) \in G$, and for all $u \in X$, $(x_1, x_2, x_3) \in \Omega$ we set

$$gu(x_1, x_2, x_3) = u(\tilde{g}^{-1}(x_1, x_2), x_3).$$

It is easily seen that Ω is G-invariant and compatible with G, and that the subspace X^G of the fixed points of X under the action of G is the set of all $u \in X$ with a *cylindric symmetry*, that is,

$$u(x_1, x_2, x_3) = u(y_1, y_2, x_3) \ \text{ if } \ x_1^2 + x_2^2 = y_1^2 + y_1^2.$$

Let $q \in]0, p-1[$ be a real number, $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(s) = 1 - \big||s|^{q+1} - 1\big| \qquad \text{for all } s \in \mathbb{R}.$$

It is easily seen that F is a locally Lipschitz function, satisfying $F(0) = 0$ and conditions (F) (with $k = q + 1$) and (C) (for all $\rho \in]0, 1]$).

Moreover, we consider a non-negative function $b : \Omega \to \mathbb{R}$, having a cylindric symmetry and satisfying condition (b) and we assume that b is positive in a neighborhood of 0.

In such a setting, Corollary 6.6 applies: thus, there exist $u_0 \in X^G$, $\lambda > 0$ such that the hemivariational inequality (P_λ) admits at least three solutions, and each of them has a cylindric symmetry.

6.3. **Some differential inclusion problems in $\mathbb{R}^N$.** In this subsection we give two applications for some differential inclusions problems. The first application is a differential inclusion problem with two parameters. This result appears in the paper of Kristály, Marzantowicz and Varga [28].

Let $p > 2$ and $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function such that

($\tilde{\mathbf{F}}$1) $\displaystyle\lim_{t \to 0} \frac{\max\{|\xi| : \xi \in \partial F(t)\}}{|t|^{p-1}} = 0;$

($\tilde{\mathbf{F}}$2) $\displaystyle\limsup_{|t| \to +\infty} \frac{F(t)}{|t|^p} \leq 0;$

($\tilde{\mathbf{F}}$3) There exists $\tilde{t} \in \mathbb{R}$ such that $F(\tilde{t}) > 0$, and $F(0) = 0$.

Here we study the differential inclusion problem

$$(\tilde{P}_{\lambda,\mu}) \qquad \begin{cases} -\triangle_p u + |u|^{p-2}u \in \lambda\alpha(x)\partial F(u(x)) + \mu\beta(x)\partial G(u(x)) & \text{on } \mathbb{R}^N, \\ u(x) \to 0 \text{ as } |x| \to \infty, \end{cases}$$

where $p > N \geq 2$, the numbers λ, μ are positive, and $G : \mathbb{R} \to \mathbb{R}$ is any locally Lipschitz function. Furthermore, we assume that $\beta \in L^1(\mathbb{R}^N)$ is any function, and
($\tilde{\alpha}$) $\alpha \in L^1(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$, $\alpha \geq 0$, and $\sup_{R>0} \operatorname{essinf}_{|x|\leq R}\alpha(x) > 0$.

The functional space where our solutions are going to be sought is the usual Sobolev space $W^{1,p}(\mathbb{R}^N)$, endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p + \int_{\mathbb{R}^N} |u(x)|^p\right)^{1/p}.$$

Definition 6.8. We say that $u \in W^{1,p}(\mathbb{R}^N)$ is a solution of problem $(\tilde{P}_{\lambda,\mu})$, if there exist $\xi_F(x) \in \partial F(u(x))$ and $\xi_G(x) \in \partial G(u(x))$ for almost every $x \in \mathbb{R}^N$ such that for all $v \in W^{1,p}(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)dx = \lambda \int_{\mathbb{R}^N} \alpha(x)\xi_F v dx + \mu \int_{\mathbb{R}^N} \beta(x)\xi_G v dx. \quad (6.6)$$

Remark 6.9. (a) The terms in the right hand side of (6.6) are well-defined. Indeed, due to Morrey's embedding theorem, i.e., $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is continuous ($p > N$), we have $u \in L^\infty(\mathbb{R}^N)$. Thus, there exists a compact interval $I \subset \mathbb{R}$ such that $u(x) \in I$ for a.e. $x \in \mathbb{R}^N$. Since the set-valued mapping ∂F is upper-semicontinuous, the set $\partial F(I) \subset \mathbb{R}$ is bounded; let $C_F = \sup |\partial F(I)|$. Therefore,

$$|\int_{\mathbb{R}^N} \alpha(x)\xi_F v dx| \leq C_F \|\alpha\|_{L^1} \|v\|_\infty < \infty.$$

Similar argument holds for the function G.

(b) Since $p > N$, any element $u \in W^{1,p}(\mathbb{R}^N)$ is homoclinic, i.e., $u(x) \to 0$ as $|x| \to \infty$, see Brézis [7, Théorème IX.12].

Remark 6.10. An upper bound for the embedding constant c_∞ of $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, is $2p(p-N)^{-1}$ (see [7]), i.e. $c_\infty \leq 2p(p-N)^{-1}$.

Remark 6.11. Every function $u \in W^{1,p}(\mathbb{R}^N)$ ($p > N$) admits a continuous representation, see [7, p. 166]; in the sequel, we will replace u by this element.

Note that no hypothesis on the growth of G is assumed; therefore, the last term in $(\tilde{P}_{\lambda,\mu})$ may have an arbitrary growth. However, assumption ($\tilde{\alpha}$) together with ($\tilde{\mathbf{F}}$3) guarantee the existence of non-trivial solutions for $(\tilde{P}_{\lambda,\mu})$. The embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is continuous (due to Morrey's theorem ($p > N$)), bit it is not compact. We overcome this gap by introducing the subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$. The action of the orthogonal group $O(N)$ on $W^{1,p}(\mathbb{R}^N)$ can be defined by $(gu)(x) = u(g^{-1}x)$, for every $g \in O(N)$, $u \in W^{1,p}(\mathbb{R}^N)$, $x \in \mathbb{R}^N$. It is clear that this group acts linearly and isometrically; in particular $\|gu\| = \|u\|$ for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$.

We denote by

$$W^{1,p}_{\text{rad}}(\mathbb{R}^N) = \{u \in W^{1,p}(\mathbb{R}^N) : gu = u \text{ for all } g \in O(N)\},$$

the subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$.

We have the following result, which is contained in the paper of Kristály [30].

Proposition 6.12. (Kristály [30]) *The embedding* $W^{1,p}_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ *is compact whenever* $2 \le N < p < \infty$.

Proof. Let u_n be a bounded sequence in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$. Up to a subsequence, $u_n \rightharpoonup u$ in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ for some $u \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$. Let $\rho > 0$ be an arbitrarily fixed number. Due to the radially symmetric properties of u and u_n, we have

$$\|u_n - u\|_{W^{1,p}(B_N(g_1 y,\rho))} = \|u_n - u\|_{W^{1,p}(B_N(g_2 y,\rho))} \tag{6.7}$$

for every $g_1, g_2 \in O(N)$ and $y \in \mathbb{R}^N$. For a fixed $y \in \mathbb{R}^N$, we can define

$$\begin{aligned} m(y,\rho) = \sup\{n \in \mathbb{N} \quad : \quad & \exists g_i \in O(N),\ i \in \{1,...,n\} \text{ such that} \\ & B_N(g_i y,\rho) \cap B_N(g_j y,\rho) = \emptyset,\ \forall\, i \neq j\}. \end{aligned}$$

By virtue of (6.7), for every $y \in \mathbb{R}^N$ and $n \in \mathbb{N}$, we have

$$\|u_n - u\|_{W^{1,p}(B_N(y,\rho))} \le \frac{\|u_n - u\|_{W^{1,p}}}{m(y,\rho)} \le \frac{\sup_{n\in\mathbb{N}} \|u_n\|_{W_{1,p}} + \|u\|_{W^{1,p}}}{m(y,\rho)}.$$

The right hand side does not depend on n, and $m(y,\rho) \to +\infty$ whenever $|y| \to +\infty$ (ρ is kept fixed, and $N \ge 2$). Thus, for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that for every $y \in \mathbb{R}^N$ with $|y| \ge R_\varepsilon$ one has

$$\|u_n - u\|_{W^{1,p}(B_N(y,\rho))} < \varepsilon(2S_\rho)^{-1} \quad \text{for every } n \in \mathbb{N}, \tag{6.8}$$

where $S_\rho > 0$ is the embedding constant of $W^{1,p}(B_N(0,\rho)) \hookrightarrow C^0(B_N[0,\rho])$. Moreover, we observe that the embedding constant for $W^{1,p}(B_N(y,\rho)) \hookrightarrow C^0(B_N[y,\rho])$ can be chosen S_ρ as well, *independent* of the position of the point $y \in \mathbb{R}^N$. This fact can be concluded either by a simple translation of the functions $u \in W^{1,p}(B_N(y,\rho))$ into $B_N(0,\rho)$, i.e. $\tilde{u}(\cdot) = u(\cdot - y) \in W^{1,p}(B_N(0,\rho))$ (thus $\|u\|_{W^{1,p}(B_N(y,\rho))} = \|\tilde{u}\|_{W^{1,p}(B_N(0,\rho))}$ and $\|u\|_{C^0(B_N[y,\rho])} = \|\tilde{u}\|_{C^0(B_N[0,\rho])}$); or, by the invariance with respect to rigid motions of the cone property of the balls $B_N(y,\rho)$ when ρ is kept fixed.

Thus, in view of (6.8), one has that

$$\sup_{|y|\geq R_\varepsilon} \|u_n - u\|_{C^0(B_N[y,\rho])} \leq \varepsilon/2 \quad \text{for every } n \in \mathbb{N}. \tag{6.9}$$

On the other hand, since $u_n \rightharpoonup u$ in $W^{1,p}_{\rm rad}(\mathbb{R}^N)$, then in particular, by Rellich theorem it follows that $u_n \to u$ in $C^0(B_N[0,R_\varepsilon])$, i.e., there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\|u_n - u\|_{C^0(B_N[0,R_\varepsilon])} < \varepsilon \quad \text{for every } n \geq n_\varepsilon. \tag{6.10}$$

Combining (6.9) with (6.10), one concludes that $\|u_n - u\|_{L^\infty} < \varepsilon$ for every $n \geq n_\varepsilon$, i.e., $u_n \to u$ in $L^\infty(\mathbb{R}^N)$. This ends the proof. □

An alternate proof of Proposition 6.12. Lions [34, Lemme II.1] provided us with a Strauss-type estimation (see [63]) for radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$; namely, for every $u \in W^{1,p}_{\rm rad}(\mathbb{R}^N)$ we have

$$|u(x)| \leq p^{1/p}(\text{Area}S^{N-1})^{-1/p}\|u\|_{W^{1,p}}|x|^{(1-N)/p}, \quad x \neq 0, \tag{6.11}$$

where S^{N-1} is the N-dimensional unit sphere.

Now, let $\{u_n\}$ be a sequence in $W^{1,p}_{\rm rad}(\mathbb{R}^N)$ which converges weakly to some $u \in W^{1,p}_{\rm rad}(\mathbb{R}^N)$. By applying inequality (6.11) for $u_n - u$, and taking into account that $\|u_n - u\|_{W^{1,p}}$ is bounded, and $N \geq 2$, then for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\|u_n - u\|_{L^\infty(|x|\geq R_\varepsilon)} \leq C|R_\varepsilon|^{(1-N)/p} < \varepsilon, \quad \forall n \in \mathbb{N},$$

where $C > 0$ does not depend on n. The rest is similar as above. □

Let $\Phi_1, \Phi_2 : L^\infty(\mathbb{R}^N) \to \mathbb{R}$ be defined by

$$\Phi_1(u) = -\int_{\mathbb{R}^N} \alpha(x)F(u(x))dx \quad \text{and} \quad \Phi_2(u) = -\int_{\mathbb{R}^N} \beta(x)G(u(x))dx.$$

Since $\alpha, \beta \in L^1(\mathbb{R}^N)$, the functionals Φ_1, Φ_2 are well-defined and locally Lipschitz, see Clarke [10, p. 79-81]. Moreover, we have

$$\partial\Phi_1(u) \subseteq -\int_{\mathbb{R}^N} \alpha(x)\partial F(u(x))dx, \quad \partial\Phi_2(u) \subseteq -\int_{\mathbb{R}^N} \beta(x)\partial G(u(x))dx.$$

The energy functional $\mathcal{E}_{\lambda,\mu} : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ associated to problem $(\tilde{P}_{\lambda,\mu})$, is given by

$$\mathcal{E}_{\lambda,\mu}(u) = \frac{1}{p}\|u\|^p + \lambda\Phi_1(u) + \mu\Phi_2(u), \quad u \in W^{1,p}(\mathbb{R}^N).$$

It is clear that the critical points of the functional $\mathcal{E}_{\lambda,\mu}$ are solutions of the problem $(\tilde{P}_{\lambda,\mu})$ in the sense of Definition 6.8.

Since α, β are radially symmetric, then $\mathcal{E}_{\lambda,\mu}$ is $O(N)$-invariant, i.e. $\mathcal{E}_{\lambda,\mu}(gu) = \mathcal{E}_{\lambda,\mu}(u)$ for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$. Therefore, we may apply a non-smooth version of the *principle of symmetric criticality,* proved by Krawcewicz-Marzantowicz [25], for locally Lipschitz functions, see Remark 3.9.

Proposition 6.13. *Any critical point of $\mathcal{E}_{\lambda,\mu}^{\rm rad} = \mathcal{E}_{\lambda,\mu}|_{W^{1,p}_{\rm rad}(\mathbb{R}^N)}$ will be also a critical point of $\mathcal{E}_{\lambda,\mu}$.*

In the proof of the main result we use, the following result.

Proposition 6.14. $\lim_{t\to 0^+} \frac{\inf\{\Phi_1(u):\ u\in W^{1,p}_{\rm rad}(\mathbb{R}^N),\ \|u\|^p<pt\}}{t} = 0.$

Proof. Due to ($\tilde{\mathbf{F}}$1), for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$|\xi| \le \varepsilon |t|^{p-1},\ \ \forall t \in [-\delta(\varepsilon), \delta(\varepsilon)],\ \forall \xi \in \partial F(t). \tag{6.12}$$

For any $0 < t \le \frac{1}{p}\left(\frac{\delta(\varepsilon)}{c_\infty}\right)^p$ define the set

$$S_t = \{\ u \in W^{1,p}_{\rm rad}(\mathbb{R}^N) : \|u\|^p < pt\},$$

where $c_\infty > 0$ denotes the best constant in the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$.

Note that $u \in S_t$ implies that $\|u\|_\infty \le \delta(\varepsilon)$; indeed, we have $\|u\|_\infty \le c_\infty \|u\| < c_\infty (pt)^{1/p} \le \delta(\varepsilon)$. Fix $u \in S_t$; for a.e. $x \in \mathbb{R}^N$, Lebourg's mean value theorem and (6.12) imply the existence of $\xi_x \in \partial F(\theta_x u(x))$ for some $0 < \theta_x < 1$ such that

$$F(u(x)) = F(u(x)) - F(0) = \xi_x u(x) \le |\xi_x| \cdot |u(x)| \le \varepsilon |u(x)|^p.$$

Consequently, for every $u \in S_t$ we have

$$\begin{aligned}\Phi_1(u) &= -\int_{\mathbb{R}^N} \alpha(x) F(u(x))dx \ge -\varepsilon \int_{\mathbb{R}^N} \alpha(x)|u(x)|^p dx\\ &\ge -\varepsilon\|\alpha\|_{L^1}\|u\|_\infty^p \ge -\varepsilon\|\alpha\|_{L^1} c_\infty^p \|u\|^p\\ &\ge -\varepsilon\|\alpha\|_{L^1} c_\infty^p pt.\end{aligned}$$

Therefore, for every $0 < t \le \frac{1}{p}\left(\frac{\delta(\varepsilon)}{c_\infty}\right)^p$ we have

$$0 \ge \frac{\inf_{u\in S_t}\Phi_1(u)}{t} \ge -\varepsilon\|\alpha\|_{L^1} c_\infty^p p.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the required limit. □

The main result of this subsection appear in the paper Kristály, Marzantowicz and Varga [28].

Theorem 6.15. (Kristály-Marzantowicz-Varga [28]) *Assume that $p > N \ge 2$. Let $\alpha, \beta \in L^1(\mathbb{R}^N)$ be two radial functions, α fulfilling $(\tilde{\alpha})$, and let $F, G : \mathbb{R} \to \mathbb{R}$ be two locally Lipschitz functions, F satisfying the conditions ($\tilde{\mathbf{F}}$1)-($\tilde{\mathbf{F}}$3). Then there exists a non-degenerate compact interval $[a, b] \subset]0, +\infty[$ and a number $\tilde{r} > 0$, such that for every $\lambda \in [a, b]$ there exists $\mu_0 \in]0, \lambda + 1]$ such that for each $\mu \in [0, \mu_0]$, the problem $(\tilde{P}_{\lambda,\mu})$ has at least three distinct, radially symmetric solutions with L^∞-norms less than $\tilde{r}$.*

Proof. We are going to apply Theorem 2.30 by choosing $X = W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, $\tilde{X}_1 = \tilde{X}_2 = L^\infty(\mathbb{R}^N)$, $\Lambda = [0, +\infty)$, $h(t) = t^p/p$, $t \geq 0$.

Fix $g \in \mathcal{G}_\tau$ ($\tau \geq 0$), $\lambda \in \Lambda$, $\mu \in [0, \lambda + 1]$, and $c \in \mathbb{R}$. We prove that the functional $E_{\lambda,\mu} : W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$E_{\lambda,\mu}(u) = \frac{1}{p}\|u\|^p + \lambda\Phi_1(u) + \mu(g \circ \Phi_2)(u), \quad u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N),$$

satisfies the $(PS)_c$ condition.

Note first that the function $\frac{1}{p}\|\cdot\|^p + \lambda\Phi_1$ is coercive on $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$. To prove this, let $0 < \varepsilon < (p\|\alpha\|_1 c_\infty^p \lambda)^{-1}$. Then, on account of ($\tilde{\mathbf{F}}2$), there exists $\delta(\varepsilon) > 0$ such that

$$F(t) \leq \varepsilon|t|^p, \quad \forall |t| > \delta(\varepsilon).$$

Consequently, for every $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ we have

$$\Phi_1(u) = -\int_{\mathbb{R}^N} \alpha(x)F(u(x))dx$$

$$= -\int_{\{x\in\mathbb{R}^N:|u(x)|>\delta(\varepsilon)\}} \alpha(x)F(u(x))dx - \int_{\{x\in\mathbb{R}^N:|u(x)|\leq\delta(\varepsilon)\}} \alpha(x)F(u(x))dx$$

$$\geq -\varepsilon\int_{\{x\in\mathbb{R}^N:|u(x)|>\delta(\varepsilon)\}} \alpha(x)|u(x)|^p dx - \max_{|t|\leq\delta(\varepsilon)} |F(t)| \int_{\{x\in\mathbb{R}^N:|u(x)|\leq\delta(\varepsilon)\}} \alpha(x)dx$$

$$\geq -\varepsilon\|\alpha\|_{L^1} c_\infty^p \|u\|^p - \|\alpha\|_{L^1} \max_{|t|\leq\delta(\varepsilon)} |F(t)|.$$

Now, we have

$$\frac{1}{p}\|u\|^p + \lambda\Phi_1(u) \geq \left(\frac{1}{p} - \varepsilon\lambda\|\alpha\|_{L^1} c_\infty^p\right)\|u\|^p - \lambda\|\alpha\|_{L^1} \max_{|t|\leq\delta(\varepsilon)} |F(t)|,$$

which clearly implies the coercivity of $\frac{1}{p}\|\cdot\|^p + \lambda\Phi_1$.

As an immediate consequence, the functional $E_{\lambda,\mu}$ is also coercive on $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$. Therefore, it is enough to consider a bounded sequence $\{u_n\} \subset W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that

$$E^\circ_{\lambda,\mu}(u_n; v - u_n) \geq -\varepsilon_n\|v - u_n\| \ \text{ for all } \ v \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N), \tag{6.13}$$

where $\{\varepsilon_n\}$ is a positive sequence such that $\varepsilon_n \to 0$. Since the sequence $\{u_n\}$ is bounded in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, one can find an element $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, and $u_n \to u$ strongly in $L^\infty(\mathbb{R}^N)$, due to Proposition 6.12.

Due to Proposition 2.3 for every $u, v \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ we have

$$E^\circ_{\lambda,\mu}(u; v) \leq \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u\nabla v + |u|^{p-2}uv) + \lambda\Phi_1^\circ(u; v) + \mu(g \circ \Phi_2)^\circ(u; v). \tag{6.14}$$

Put $v = u$ in (6.13) and apply relation (6.14) for the pairs $(u, v) = (u_n, u-u_n)$ and $(u, v) = (u, u_n - u)$, we have that

$$\begin{aligned} I_n &\leq \varepsilon_n\|u-u_n\| - E^\circ_{\lambda,\mu}(u; u_n-u) + \lambda[\Phi_1^\circ(u_n; u-u_n) + \Phi_1^\circ(u; u_n-u)] \\ &\quad +\mu[(g\circ\Phi_2)^\circ(u_n; u-u_n) + (g\circ\Phi_2)^\circ(u; u_n-u)], \end{aligned}$$

where

$$\begin{aligned} I_n &\stackrel{\text{not.}}{=} \int_{\mathbb{R}^N}(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u) \\ &\quad + \int_{\mathbb{R}^N}(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u). \end{aligned}$$

Since $\{u_n\}$ is bounded in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$, we have that $\lim_{n\to\infty}\varepsilon_n\|u-u_n\| = 0$. Fixing $z^* \in \partial E^\circ_{\lambda,\mu}(u)$ arbitrarily, we have $\langle z^*, u_n - u\rangle \leq E^\circ_{\lambda,\mu}(u; u_n-u)$. Since $u_n \rightharpoonup u$ weakly in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$, we have that $\liminf_{n\to\infty} E^\circ_{\lambda,\mu}(u; u_n-u) \geq 0$. The functions $\Phi_1^\circ(\cdot;\cdot)$ and $(g\circ\Phi_2)^\circ(\cdot;\cdot)$ are upper semicontinuous functions on $L^\infty(\mathbb{R}^N)$. Since $u_n \to u$ strongly in $L^\infty(\mathbb{R}^N)$, the upper limit of the last four terms is less or equal than 0 as $n\to\infty$, see Proposition 2.3 (f_4).
Consequently,

$$\limsup_{n\to\infty} I_n \leq 0. \tag{6.15}$$

Since $|t-s|^p \leq (|t|^{p-2}t - |s|^{p-2}s)(t-s)$ for every $t, s \in \mathbb{R}^m$ $(m\in\mathbb{N})$ we infer that $\|u_n - u\|^p \leq I_n$. The last inequality combined with (6.15) leads to the fact that $u_n \to u$ strongly in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$, as claimed.

It remains to prove relation (2.53) from Theorem 2.30. First, we construct the function $u_0 \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ such that $\Phi_1(u_0) < 0$.

On account of $(\tilde{\alpha})$, one can fix $R > 0$ such that $\alpha_R = \text{essinf}_{|x|\leq R}\alpha(x) > 0$. For $\sigma \in]0,1[$ define the function

$$w_\sigma(x) = \begin{cases} 0, & \text{if} \quad x \in \mathbb{R}^N \setminus B_N(0,R); \\ \tilde{t}, & \text{if} \quad x \in B_N(0,\sigma R); \\ \frac{\tilde{t}}{R(1-\sigma)}(R-|x|), & \text{if} \quad x \in B_N(0,R)\setminus B_N(0,\sigma R), \end{cases}$$

where $B_N(0,r)$ denotes the $N-$dimensional open ball with center 0 and radius $r > 0$, and $\tilde{t}$ comes from $(\tilde{\mathbf{F}}3)$. Since $\alpha \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, then $M(\alpha, R) = \sup_{x\in B_N(0,R)}\alpha(x) < \infty$. A simple estimate shows that

$$-\Phi_1(w_\sigma) \geq \omega_N R^N[\alpha_R F(\tilde{t})\sigma^N - M(\alpha,R)\max_{|t|\leq|\tilde{t}|}|F(t)|(1-\sigma^N)].$$

When $\sigma \to 1$, the right hand side is strictly positive; choosing σ_0 close enough to 1, for $u_0 = w_{\sigma_0}$ we have $\Phi_1(u_0) < 0$.

Let us define the function for every $t > 0$ by

$$\beta(t) = \inf\{\Phi_1(u) : u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N),\ \frac{\|u\|^p}{p} < t\}.$$

We have that $\beta(t) \leq 0$, for $t > 0$, and Proposition 6.14 yields that

$$\lim_{t\to 0^+} \frac{\beta(t)}{t} = 0. \tag{6.16}$$

We consider the $u_0 \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, for wchich $\Phi_1(u_0) < 0$. Therefore it is possible to choose a number $\eta > 0$ such that

$$0 < \eta < -\Phi_1(u_0)\left[\frac{\|u_0\|^p}{p}\right]^{-1}.$$

By (6.16) we get the existence of a number $t_0 \in (0, \frac{\|u_0\|^p}{p})$ such that $-\beta(t_0) < \eta t_0$. Thus

$$\beta(t_0) > \left[\frac{\|u_0\|^p}{p}\right]^{-1}\Phi_1(u_0)t_0. \tag{6.17}$$

Due to the choice of t_0 and using (6.17), we conclude that there exists $\rho_0 > 0$ such that

$$-\beta(t_0) < \rho_0 < -\Phi_1(u_0)\left[\frac{\|u_0\|^p}{p}\right]^{-1} t_0 < -\Phi_1(u_0). \tag{6.18}$$

Define now the function $\varphi : W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \times \mathbb{I} \to \mathbb{R}$ by

$$\varphi(u,\lambda) = \frac{\|u\|^p}{p} + \lambda\Phi_1(u) + \lambda\rho_0,$$

where $\mathbb{I} = [0, +\infty)$. We prove that the function φ satisfies the inequality

$$\sup_{\lambda\in\mathbb{I}}\ \inf_{u\in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N))} \varphi(u,\lambda) < \inf_{u\in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)}\ \sup_{\lambda\in\mathbb{I}} \varphi(u,\lambda). \tag{6.19}$$

The function

$$\mathbb{I} \ni \lambda \mapsto \inf_{u\in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)} \left[\frac{\|u\|^p}{p} + \lambda(\rho_0 + \Phi_1(u))\right]$$

is obviously upper semicontinuous on $\mathbb{I}$. It follows from (6.18) that

$$\lim_{\lambda\to+\infty}\ \inf_{u\in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)} \varphi(u,\lambda) \leq \lim_{\lambda\to+\infty}\left[\frac{\|u_0\|^p}{p} + \lambda(\rho_0 + \Phi_1(u_0))\right] = -\infty.$$

Thus we find an element $\overline{\lambda} \in \mathbb{I}$ such that

$$\sup_{\lambda\in\mathbb{I}}\ \inf_{u\in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)} \varphi(u,\lambda) = \inf_{u\in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)} \left[\frac{\|u\|^p}{p} + \overline{\lambda}(\rho_0 + \Phi_1(u))\right]. \tag{6.20}$$

Since $-\beta(t_0) < \rho_0$, it follows from the definition of β that for all $u \in W^{1,p}_{\rm rad}(\mathbb{R}^N)$ with $\frac{\|u\|^p}{p} < t_0$ we have $-\Phi_1(u) < \rho_0$. Hence

$$t_0 \leq \inf\{\frac{\|u\|^p}{p} : u \in W^{1,p}_{\rm rad}(\mathbb{R}^N),\ -\Phi_1(u) \geq \rho_0\ \}. \tag{6.21}$$

On the other hand,

$$\begin{aligned}\inf_{u\in W^{1,p}_{\rm rad}(\mathbb{R}^N)} \sup_{\lambda\in\mathbb{I}} \varphi(u,\lambda) &= \inf_{u\in W^{1,p}_{\rm rad}(\mathbb{R}^N)} \left[\frac{\|u\|^p}{p} + \sup_{\lambda\in\mathbb{I}} (\lambda(\rho_0 + \Phi_1(u)))\right] \\ &= \inf_{u\in W^{1,p}_{\rm rad}(\mathbb{R}^N)} \left\{\frac{\|u\|^p}{p} : -\Phi_1(u) \geq \rho_0\right\}.\end{aligned}$$

Thus inequality (6.21) is equivalent to

$$t_0 \leq \inf_{u\in W^{1,p}_{\rm rad}(\mathbb{R}^N)} \sup_{\lambda\in\mathbb{I}} \varphi(u,\lambda). \tag{6.22}$$

We consider two cases. First, when $0 \leq \overline{\lambda} < \frac{t_0}{\rho_0}$, then we have that

$$\inf_{u\in W^{1,p}_{\rm rad}(\mathbb{R}^N)} \left[\frac{\|u\|^p}{p} + \overline{\lambda}(\rho_0 + \Phi_1(u))\right] \leq \varphi(0,\overline{\lambda}) = \overline{\lambda}\rho_0 < t_0.$$

Combining this inequality with (6.20) and (6.22) we obtain (6.19).

Now, if $\frac{t_0}{\rho_0} \leq \overline{\lambda}$, then from (6.17) and (6.18), it follows that

$$\begin{aligned}\inf_{u\in W^{1,p}_{\rm rad}(\mathbb{R}^N)} \left[\frac{\|u\|^p}{p} + \overline{\lambda}(\rho_0 + \Phi_1(u))\right] &\leq \frac{\|u_0\|^p}{p} + \overline{\lambda}(\rho_0 + \Phi_1(u_0)) \\ &\leq \frac{\|u_0\|^p}{p} + \frac{t_0}{\rho_0}(\rho_0 + \Phi_1(u_0)) < t_0.\end{aligned}$$

It remains to apply again (6.20) and (6.22), which concludes the proof of (6.19).

Due to Theorem 2.30, there exist a non-empty open set $A \subset \Lambda$ and $r > 0$ with the property that for every $\lambda \in A$ there exists $\mu_0 \in]0, \lambda + 1]$ such that, for each $\mu \in [0, \mu_0]$ the functional $\mathcal{E}^{\rm rad}_{\lambda,\mu} = \frac{1}{p}\|\cdot\|^p + \lambda\Phi_1 + \mu\Phi_2$ defined on $W^{1,p}_{\rm rad}(\mathbb{R}^N)$ has at least three critical points in $W^{1,p}_{\rm rad}(\mathbb{R}^N)$ whose $\|\cdot\|$-norms are less than r. Applying Proposition 6.13, the critical points of $\mathcal{E}^{\rm rad}_{\lambda,\mu}$ are also critical points of $\mathcal{E}_{\lambda,\mu}$, thus, radially weak solutions of problem $(\tilde{P}_{\lambda,\mu})$. Due to the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, if $\tilde{r} = c_\infty r$, then the L^∞-norms of these elements are less than $\tilde{r}$ which concludes our proof. □

The second problem studied in this subsection is the following differential inclusion problem:

$$\text{(DI)} \qquad \begin{cases} -\triangle_p u + |u|^{p-2}u \in \alpha(x)\partial F(u(x)), & x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases}$$

where $2 \leq N < p < +\infty$, $\alpha \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is radially symmetric, and ∂F stands for the generalized gradient of a locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$. By a solution of (DI) it will be understood an element $u \in W^{1,p}(\mathbb{R}^N)$ for which there corresponds a mapping $\mathbb{R}^N \ni x \mapsto \zeta_x$ with $\zeta_x \in \partial F(u(x))$ for almost every $x \in \mathbb{R}^N$ having the property that for every $v \in W^{1,p}(\mathbb{R}^N)$, the function $x \mapsto \alpha(x)\zeta_x v(x)$ belongs to $L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)dx = \int_{\mathbb{R}^N} \alpha(x)\zeta_x v(x)dx. \tag{6.23}$$

Under suitable oscillatory assumptions on the potential F at zero or at infinity, we show the existence of infinitely many, radially symmetric solutions of (DI). These results appear in the paper of Kristály [30].

For $l = 0$ or $l = +\infty$, set

$$F_l := \limsup_{|\rho| \to l} \frac{F(\rho)}{|\rho|^p}. \tag{6.24}$$

Problem (DI) will be studied in the following four cases:

- $0 < F_l < +\infty$, whenever $l = 0$ or $l = +\infty$ and
- $F_l = +\infty$, whenever $l = 0$ or $l = +\infty$.

In the next in this subsection we assume that:

(H) • $F : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, $F(0) = 0$, and $F(s) \geq 0$, $\forall s \in \mathbb{R}$;
• $\alpha \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is radially symmetric, and $\alpha(x) \geq 0$, $\forall x \in \mathbb{R}^N$.

Let $\mathcal{F} : L^\infty(\mathbb{R}^N) \to \mathbb{R}$ be a function defined by

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} \alpha(x)F(u(x))dx.$$

Since F is continuous and $\alpha \in L^1(\mathbb{R}^N)$, we easily seen that $\mathcal{F}$ is well-defined. Moreover, if we fix a $u \in L^\infty(\mathbb{R}^N)$ arbitrarily, there exists $k_u \in L^1(\mathbb{R}^N)$ such that for every $x \in \mathbb{R}^N$ and $v_i \in \mathbb{R}$ with $|v_i - u(x)| < 1$, $(i \in \{1, 2\})$ one has

$$|\alpha(x)F(v_1) - \alpha(x)F(v_2)| \leq k_u(x)|v_1 - v_2|.$$

Indeed, if we fix some small open intervals I_j $(j \in J)$, such that $F|_{I_j}$ is Lipschitz function (with Lipschitz constant $L_j > 0$) and $[-\|u\|_{L^\infty} - 1, \|u\|_{L^\infty} + 1] \subset \cup_{j\in J} I_j$, then we choose $k_u = \alpha \max_{j\in J} L_j$. (Here, without losing the generality, we supposed that card$J < +\infty$.) Thus, we are in the position to apply Theorem 2.7.3 from [10, p. 80]; namely, $\mathcal{F}$ is a locally Lipschitz function on $L^\infty(\mathbb{R}^N)$ and for every closed subspace E of $L^\infty(\mathbb{R}^N)$ we have

$$\partial(\mathcal{F}|_E)(u) \subseteq \int_{\mathbb{R}^N} \alpha(x)\partial F(u(x))dx, \quad \text{for every } u \in E, \tag{6.25}$$

where $\mathcal{F}|_E$ stands for the restriction of $\mathcal{F}$ to E. The interpretation of (6.25) is as follows (see also [10]): For every $\zeta \in \partial(\mathcal{F}|_E)(u)$ there corresponds a mapping $\mathbb{R}^N \ni x \mapsto \zeta_x$ such that $\zeta_x \in \partial F(u(x))$ for almost every $x \in \mathbb{R}^N$ having the property that for every $v \in E$ the function $x \mapsto \alpha(x)\zeta_x v(x)$ belongs to $L^1(\mathbb{R}^N)$ and

$$\langle \zeta, v\rangle_E = \int_{\mathbb{R}^N} \alpha(x)\zeta_x v(x)dx.$$

Now, let $\mathcal{E} : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ be the energy functional associated to our problem (DI), i.e., for every $u \in W^{1,p}(\mathbb{R}^N)$ set

$$\mathcal{E}(u) = \frac{1}{p}\|u\|^p_{W^{1,p}} - \mathcal{F}(u).$$

It is clear that $\mathcal{E}$ is locally Lipschitz on $W^{1,p}(\mathbb{R}^N)$ and we have

Proposition 6.16. *Any critical point $u \in W^{1,p}(\mathbb{R}^N)$ of $\mathcal{E}$ is a solution of* (DI).

Proof. Combining $0 \in \partial\mathcal{E}(u) = -\triangle_p u + |u|^{p-2}u - \partial(\mathcal{F}|_{W^{1,p}(\mathbb{R}^N)})(u)$ with the interpretation of (6.25), the desired requirement yields, see (6.23). □

Since α is radially symmetric, then $\mathcal{E}$ is $O(N)$-invariant, i.e. $\mathcal{E}(gu) = \mathcal{E}(u)$ for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$, we are in the position to apply the Principle of Symmetric Criticality for locally Lipschitz functions, see Remark 3.9. Therefore we have

Proposition 6.17. *Any critical point of $\mathcal{E}_r = \mathcal{E}|_{W^{1,p}_{\rm rad}(\mathbb{R}^N)}$ will be also a critical point of $\mathcal{E}$.*

Remark 6.18. In view of Propositions 6.16 and 6.17, it is enough to find critical points of $\mathcal{E}_r$ in order to guarantee solutions for (DI). This fact will be carried out by means of Theorem 2.31, setting

$$X := W^{1,p}_{\rm rad}(\mathbb{R}^N),\ \tilde{X} := L^\infty(\mathbb{R}^N),\ \Phi := -\mathcal{F},\ \text{ and } \Psi := \|\cdot\|^p_r, \tag{6.26}$$

where the notation $\|\cdot\|_r$ stands for the restriction of $\|\cdot\|_{W^{1,p}}$ into $W^{1,p}_{\rm rad}(\mathbb{R}^N)$. A few assumptions are already verified. Indeed, the embedding $X \hookrightarrow \tilde{X}$ is compact (cf. Theorem 6.12), $\Phi = -\mathcal{F}$ is locally Lipschitz, while $\Psi = \|\cdot\|^p_r$ is of class C^1 (thus, locally Lipschitz as well), coercive and weakly sequentially lower semicontinuous (see [7, Proposition III.5]). Moreover, $\mathcal{E}_r \equiv \Phi|_{W^{1,p}_{\rm rad}(\mathbb{R}^N)} + \frac{1}{p}\Psi$. According to (6.26), the function φ (defined in (2.57)) becomes

$$\varphi(\rho) = \inf_{\|u\|^p_r<\rho} \frac{\sup_{\|v\|^p_r\le\rho}\mathcal{F}(v) - \mathcal{F}(u)}{\rho - \|u\|^p_r}, \quad \rho > 0. \tag{6.27}$$

The investigation of the numbers γ and δ (defined in (2.58)), as well as the cases (A) and (B) from Theorem 2.31 constitute the objective of the next.

Theorem 6.19. (A. Kristály [30]; The case $0 < F_l < +\infty$) *Let $l = 0$ or $l = +\infty$, and let $2 \leq N < p < +\infty$. Let $F : \mathbb{R} \to \mathbb{R}$ and $\alpha : \mathbb{R}^N \to \mathbb{R}$ be two functions which satisfy the hypotheses (H) and $0 < F_l < +\infty$. Assume that $\|\alpha\|_{L^\infty} F_l > 2^N p^{-1}$ and there exists a number $\beta_l \in]2^N (pF_l)^{-1}, \|\alpha\|_{L^\infty}[$ such that*

$$\frac{2}{(2^{-N} p \beta_l F_l - 1)^{1/p}} < \sup\{r : \mathrm{meas}(B_N(0,r) \setminus \alpha^{-1}(]\beta_l, +\infty[)) = 0\}. \tag{6.28}$$

Assume further that there are sequences $\{a_k\}$ and $\{b_k\}$ in $]0, +\infty[$ with $a_k < b_k$, $\lim_{k\to+\infty} b_k = l$, $\lim_{k\to+\infty} \frac{b_k}{a_k} = +\infty$ such that

$$\sup\{\mathrm{sign}(s)\xi : \xi \in \partial F(s), |s| \in]a_k, b_k[\} \leq 0. \tag{6.29}$$

Then, problem (DI) *possesses a sequence $\{u_n\}$ of solutions which are radially symmetric and*

$$\lim_{n\to+\infty} \|u_n\|_{W^{1,p}} = l.$$

In addition, if $F(s) = 0$ for every $s \in]-\infty, 0[$, then the elements u_n are non-negative.

Proof. Since $\lim_{k\to+\infty} b_k = +\infty$, instead of the sequence $\{b_k\}$, we may consider a non-decreasing subsequence of it, denoted again by $\{b_k\}$. Fix an $s \in \mathbb{R}$ such that $|s| \in]a_k, b_k]$. By using Lebourg's mean value theorem (see [10, Theorem 2.3.7]), there exists $\theta \in]0,1[$ and $\xi_\theta \in \partial F(\theta s + (1-\theta)\mathrm{sign}(s) a_k)$ such that

$$\begin{aligned} F(s) - F(\mathrm{sign}(s)a_k) &= \xi_\theta (s - \mathrm{sign}(s)a_k) = \mathrm{sign}(s)\xi_\theta(|s| - a_k) \\ &= \mathrm{sign}(\theta s + (1-\theta)\mathrm{sign}(s)a_k)\xi_\theta(|s| - a_k). \end{aligned}$$

According now to (6.29), we obtain that $F(s) \leq F(\mathrm{sign}(s)a_k)$ for every $s \in \mathbb{R}$ complying with $|s| \in]a_k, b_k]$. In particular, we are led to $\max_{[-a_k,a_k]} F = \max_{[-b_k,b_k]} F$ for every $k \in \mathbb{N}$. Therefore, one can fix a $\overline{\rho}_k \in [-a_k, a_k]$ such that

$$F(\overline{\rho}_k) = \max_{[-a_k,a_k]} F = \max_{[-b_k,b_k]} F. \tag{6.30}$$

Moreover, since $\{b_k\}$ is non-decreasing, the sequence $\{|\overline{\rho}_k|\}$ can be chosen non-decreasingly as well. In view of (6.28) we can choose a number μ such that

$$\frac{2}{(2^{-N} p \beta_\infty F_\infty - 1)^{1/p}} < \mu < \tag{6.31}$$

$$< \sup\{r : \mathrm{meas}(B_N(0,r) \setminus \alpha^{-1}(]\beta_\infty, +\infty[)) = 0\}.$$

In particular, one has

$$\alpha(x) > \beta_\infty, \quad \text{for a.e. } x \in B_N(0,\mu). \tag{6.32}$$

For every $k \in \mathbb{N}$ we define

$$u_k(x) = \begin{cases} 0, & \text{if} \quad x \in \mathbb{R}^N \setminus B_N(0,\mu); \\ \overline{\rho}_k, & \text{if} \quad x \in B_N(0,\frac{\mu}{2}); \\ \frac{2\overline{\rho}_k}{\mu}(\mu - |x|), & \text{if} \quad x \in B_N(0,\mu) \setminus B_N(0,\frac{\mu}{2}). \end{cases} \tag{6.33}$$

It is easy to see that u_k belongs to $W^{1,p}(\mathbb{R}^N)$ and it is radially symmetric. Thus, $u_k \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$. Let $\rho_k = (\frac{b_k}{c_\infty})^p$, where c_∞ is the embedding constant of $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$.

Claim 1. *There exists a $k_0 \in \mathbb{N}$ such that $\|u_k\|_r^p < \rho_k$, for every $k > k_0$.*

Since $\lim_{k\to+\infty} \frac{b_k}{a_k} = +\infty$, there exists a $k_0 \in \mathbb{N}$ such that

$$\frac{b_k}{a_k} > c_\infty(\mu^N \omega_N K(p,N,\mu))^{1/p}, \quad \text{for every } k > k_0, \tag{6.34}$$

where ω_N denotes the volume of the N-dimensional unit ball and

$$K(p,N,\mu) := \frac{2^p}{\mu^p}\left(1 - \frac{1}{2^N}\right) + 1. \tag{6.35}$$

Thus, for every $k > k_0$ one has

$$\begin{aligned} \|u_k\|_r^p &= \int_{\mathbb{R}^N} |\nabla u_k|^p dx + \int_{\mathbb{R}^N} |u_k|^p dx \\ &\leq \left(\frac{2|\overline{\rho}_k|}{\mu}\right)^p (\text{vol}B_N(0,\mu) - \text{vol}B_N(0,\frac{\mu}{2})) + |\overline{\rho}_k|^p \text{vol}B_N(0,\mu) \\ &= |\overline{\rho}_k|^p \mu^N \omega_N K(p,N,\mu) \leq a_k^p \mu^N \omega_N K(p,N,\mu) \\ &< (\frac{b_k}{c_\infty})^p = \rho_k, \end{aligned}$$

which proves Claim 1.

Now, let φ from (6.27) and $\gamma = \liminf_{\rho\to+\infty} \varphi(\rho)$ defined in (2.58).

Claim 2. $\gamma = 0$.

By definition, $\gamma \geq 0$. Suppose that $\gamma > 0$. Since $\lim_{k\to+\infty} \frac{\rho_k}{|\overline{\rho}_k|^p} = +\infty$, there is a number $k_1 \in \mathbb{N}$ such that for every $k > k_1$ we have

$$\frac{\rho_k}{|\overline{\rho}_k|^p} > \frac{2}{\gamma}(F_\infty + 1)(\|\alpha\|_{L^1} - \beta_\infty \overline{\mu}^N \omega_N) + \mu^N \omega_N K(p,N,\mu), \tag{6.36}$$

where $\overline{\mu}$ is an arbitrary fixed number complying with

$$0 < \overline{\mu} < \min\left\{ \left(\frac{\|\alpha\|_{L^1}}{\beta_\infty \omega_N}\right)^{1/N}, \frac{\mu}{2} \right\}. \tag{6.37}$$

Moreover, since $|\overline{\rho}_k| \to +\infty$ as $k \to +\infty$ (otherwise we would have $F_\infty = 0$), by the definition of F_∞, see (6.24), there exists a $k_2 \in \mathbb{N}$ such that

$$\frac{F(\overline{\rho}_k)}{|\overline{\rho}_k|^p} < F_\infty + 1, \quad \text{for every } k > k_2. \tag{6.38}$$

Now, let $v \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ arbitrarily fixed with $\|v\|_r^p \leq \rho_k$. Due to the continuous embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, we have $\|v\|_{L^\infty}^p \leq c_\infty^p \rho_k = b_k^p$. Therefore, one has

$$\sup_{x \in \mathbb{R}^N} |v(x)| \leq b_k.$$

In view of (6.30), we obtain

$$F(v(x)) \leq \max_{[-b_k, b_k]} F = F(\overline{\rho}_k), \quad \text{for every } x \in \mathbb{R}^N. \tag{6.39}$$

Hence, for every $k > \max\{k_0, k_1, k_2\}$, one has

$$\begin{aligned}
\sup_{\|v\|_r^p \leq \rho_k} \mathcal{F}(v) \quad - & \quad \mathcal{F}(u_k) \\
= & \quad \sup_{\|v\|_r^p \leq \rho_k} \int_{\mathbb{R}^N} \alpha(x) F(v(x)) dx - \int_{\mathbb{R}^N} \alpha(x) F(u_k(x)) dx \\
\leq & \quad F(\overline{\rho}_k) \|\alpha\|_{L^1} - \int_{B_N(0,\overline{\mu})} \alpha(x) F(u_k(x)) dx \\
\leq & \quad F(\overline{\rho}_k)(\|\alpha\|_{L^1} - \beta_\infty \overline{\mu}^N \omega_N) \\
\leq & \quad (F_\infty + 1)|\overline{\rho}_k|^p (\|\alpha\|_{L^1} - \beta_\infty \overline{\mu}^N \omega_N) \\
\leq & \quad \frac{\gamma}{2}(\rho_k - |\overline{\rho}_k|^p \mu^N \omega_N K(p, N, \mu)) \\
\leq & \quad \frac{\gamma}{2}(\rho_k - \|u_k\|_r^p).
\end{aligned}$$

Since $\|u_k\|_r^p < \rho_k$ (cf. Claim 1), and $\rho_k \to +\infty$ as $k \to +\infty$, we obtain

$$\gamma = \liminf_{\rho \to +\infty} \varphi(\rho) \leq \liminf_{k \to +\infty} \varphi(\rho_k) \leq \liminf_{k \to +\infty} \frac{\sup_{\|v\|_r^p \leq \rho_k} \mathcal{F}(v) - \mathcal{F}(u_k)}{\rho_k - \|u_k\|_r^p} \leq \frac{\gamma}{2},$$

contradiction. This proves Claim 2.

CLAIM 3. *$\mathcal{E}_r$ is not bounded below on $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$.*

By (6.31), we find a number ε_∞ such that

$$0 < \varepsilon_\infty < F_\infty - \frac{2^N}{p\beta_\infty}\left(\left(\frac{2}{\mu}\right)^p + 1\right). \tag{6.40}$$

In particular, for every $k \in \mathbb{N}$, $\sup_{|\rho| \geq k} \frac{F(\rho)}{|\rho|^p} > F_\infty - \varepsilon_\infty$. Therefore, we can fix $\tilde{\rho}_k$ with $|\tilde{\rho}_k| \geq k$ such that

$$\frac{F(\tilde{\rho}_k)}{|\tilde{\rho}_k|^p} > F_\infty - \varepsilon_\infty. \tag{6.41}$$

Now, define $w_k \in W^{1,p}_{\rm rad}(\mathbb{R}^N)$ in the same way as u_k, see (6.33), replacing $\overline{\rho}_k$ by $\tilde{\rho}_k$. We obtain

$$\begin{aligned}\mathcal{E}_r(w_k) &= \frac{1}{p}\|w_k\|_r^p - \mathcal{F}(w_k)\\ &\leq \frac{1}{p}|\tilde{\rho}_k|^p\mu^N\omega_N K(p,N,\mu) - \int_{B_N(0,\frac{\mu}{2})}\alpha(x)F(w_k(x))dx\\ &\leq \frac{1}{p}|\tilde{\rho}_k|^p\mu^N\omega_N K(p,N,\mu) - (F_\infty - \varepsilon_\infty)|\tilde{\rho}_k|^p\beta_\infty\omega_N\left(\frac{\mu}{2}\right)^N\\ &= |\tilde{\rho}_k|^p\mu^N\omega_N\left(\frac{1}{p}K(p,N,\mu) - \frac{1}{2^N}(F_\infty-\varepsilon_\infty)\beta_\infty\right) < -\frac{1}{p}|\tilde{\rho}_k|^p\omega_N\left(\frac{2}{\mu}\right)^{p-N}.\end{aligned}$$

Since $|\tilde{\rho}_k| \to +\infty$ as $k \to +\infty$, we obtain $\lim_{k\to+\infty}\mathcal{E}_r(w_k) = -\infty$, which ends the proof of Claim 3.

The case $0 < F_\infty < +\infty$. It is enough to apply Remark 6.18. Indeed, since $\gamma = 0$ (cf. Claim 2) and the function $\mathcal{E}_r \equiv -\mathcal{F}|_{W^{1,p}_{\rm rad}(\mathbb{R}^N)} + \frac{1}{p}\|\cdot\|_r^p$ is not bounded below (cf. Claim 3), the alternative (A1) from Theorem 2.31, applied to $\lambda = \frac{1}{p}$, is excluded. Thus, there exists a sequence $\{u_n\} \subset W^{1,p}_{\rm rad}(\mathbb{R}^N)$ of critical points of $\mathcal{E}_r$ with $\lim_{n\to+\infty}\|u_n\|_r = +\infty$.

Now, let us suppose that $F(s) = 0$ for every $s \in]-\infty, 0[$, and let u be a solution of (DI). Denote $S = \{x \in \mathbb{R}^N : \ u(x) < 0\}$, and assume that $S \neq \emptyset$. In virtue of Remark 6.11, the set S is open. Define $u_S : \mathbb{R}^N \to \mathbb{R}$ by $u_S = \min\{u, 0\}$. Applying (6.23) for $v := u_S \in W^{1,p}(\mathbb{R}^N)$ and taking into account that $\zeta_x \in \partial F(u(x)) = \{0\}$ for every $x \in S$, one has

$$0 = \int_{\mathbb{R}^N}(|\nabla u|^{p-2}\nabla u\nabla u_S + |u|^{p-2}uu_S)dx = \int_S(|\nabla u|^p + |u|^p)dx = \|u\|^p_{W^{1,p}(S)},$$

which contradicts the choice of the set S. This ends the proof in this case.

Remark 6.20. A closer inspection of the proof allows us to replace hypothesis (6.28) by a weaker, but a more technical condition. More specifically, it is enough to require that $p\|\alpha\|_{L^\infty}F_l > 1$, and instead of (6.28), put

$$\sup_M\left\{N_{\beta_l} - \frac{1}{(1-\sigma)(p\beta_l F_l\sigma^N - 1)^{1/p}}\right\} > 0, \tag{6.42}$$

where

$$M = \{(\sigma,\beta_l) : \sigma \in](p\|\alpha\|_{L^\infty}F_l)^{-1/N}, 1[,\ \beta_l \in](pF_l\sigma^N)^{-1}, \|\alpha\|_{L^\infty}[\}$$

and

$$N_{\beta_l} = \sup\{r : \text{meas}(B_N(0,r)\setminus\alpha^{-1}(]\beta_l, +\infty[)) = 0\}.$$

Now, in the construction of the functions w_k we replace the radius $\frac{\mu}{2}$ of the ball by $\sigma\mu$, where σ is chosen according to (6.42).

The case $0 < F_0 < +\infty$. The proof works similarly as in the case $0 < F_\infty < +\infty$ and we will show only the differences. The sequence $\{\rho_k\}$ defined as above, converges now to 0, while the same holds for $\{\overline{\rho}_k\}$. Instead of Claim 2, we can prove that $\delta = \liminf_{\rho\to 0^+} \varphi(\rho) = 0$. Since 0 is the unique global minimum of $\Psi = \|\cdot\|_r^p$, it would be enough to show that 0 is not a local minimum of $\mathcal{E}_r \equiv -\mathcal{F}|_{W^{1,p}_{\rm rad}(\mathbb{R}^N)} + \frac{1}{p}\|\cdot\|_r^p$, in order to exclude alternative (B1) from Theorem 2.31. To this end, we fix $\tilde{\rho}_k$ with $|\tilde{\rho}_k| \le \frac{1}{k}$ such that

$$\frac{F(\tilde{\rho}_k)}{|\tilde{\rho}_k|^p} > F_0 - \varepsilon_0,$$

where ε_0 is fixed in a similar manner as in (6.40), replacing β_∞, F_∞ by β_0, F_0, respectively. If we take w_k as in case $0 < F_\infty < +\infty$, then it is clear that $\{w_k\}$ strongly converges now to 0 in $W^{1,p}_{\rm rad}(\mathbb{R}^N)$, while $\mathcal{E}_r(w_k) < -\frac{1}{p}|\tilde{\rho}_k|^p\omega_N\,(2/\mu)^{p-N} < 0 = \mathcal{E}_r(0)$. Thus, 0 is not a local minimum of $\mathcal{E}_r$. So, there exists a sequence $\{u_n\} \subset W^{1,p}_{\rm rad}(\mathbb{R}^N)$ of critical points of $\mathcal{E}_r$ such that $\lim_{n\to+\infty}\|u_n\|_r = 0 = \inf_{W^{1,p}_{\rm rad}(\mathbb{R}^N)} \Psi$. This concludes completely the proof of Theorem 6.19.

In the next result we trait the case when the function F has oscillation at infinity. We have the following result.

Theorem 6.21. (A. Kristály [30]; The case $F_l = +\infty$) *Let $l = 0$ or $l = +\infty$, and let $2 \le N < p < +\infty$. Let $F : \mathbb{R} \to \mathbb{R}$ and $\alpha : \mathbb{R}^N \to \mathbb{R}$ be two functions which satisfy (H) and $F_l = +\infty$. Assume that $\|\alpha\|_{L^\infty} > 0$, and there exist $\mu > 0$ and $\beta_l \in]0, \|\alpha\|_{L^\infty}[$ such that*

$$\text{meas}(B_N(0,\mu) \setminus \alpha^{-1}(]\beta_l, +\infty[)) = 0, \tag{6.43}$$

and there are sequences $\{a_k\}$ and $\{b_k\}$ in $]0,+\infty[$ with $a_k < b_k$, $\lim_{k\to+\infty} b_k = l$, $\lim_{k\to+\infty}\frac{b_k}{a_k} = +\infty$ such that

$$\sup\{\text{sign}(s)\xi : \xi \in \partial F(s), |s| \in]a_k, b_k[\} \le 0,$$

and

$$\limsup_{k\to+\infty} \frac{\max_{[-a_k,a_k]} F}{b_k^p} < (pc_\infty^p\|\alpha\|_{L^1})^{-1}, \tag{6.44}$$

where c_∞ is the embedding constant of $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$. Then the conclusions of Theorem 6.19 hold.

Proof. **The case** $F_\infty = +\infty$. Due to (6.43),

$$\alpha(x) > \beta_\infty, \quad \text{for a.e. } x \in B_N(0,\mu). \tag{6.45}$$

Let $\overline{\rho}_k$ and ρ_k as in the proof of Theorem 6.19, as well as u_k, defined this time by means of $\mu > 0$ from (6.45).

Claim 1'. *There exists a $k_0 \in \mathbb{N}$ such that $\|u_k\|_r^p < \rho_k$, for every $k > k_0$.*

The proof is similarly as in the proof of Theorem 6.19.

Claim 2'. $\gamma < \frac{1}{p}$.

Note that $F(\overline{\rho}_k) = \max_{[-a_k,a_k]} F$, cf. (6.30). Since $|\overline{\rho}_k| \leq a_k$, then $\lim_{k\to+\infty} \frac{|\overline{\rho}_k|}{b_k} = 0$. Combining this fact with (6.44), and choosing $\varepsilon > 0$ sufficiently small, one has

$$\limsup_{k\to+\infty} \frac{F(\overline{\rho}_k) + |\overline{\rho}_k|^p \mu^N \omega_N p^{-1} \|\alpha\|_{L^1}^{-1} K(p,N,\mu)}{b_k^p} < ((p+\varepsilon)c_\infty^p \|\alpha\|_{L^1})^{-1},$$

where $K(p,N,\mu)$ is from (6.35). According to the above inequality, there exists $k_3 \in \mathbb{N}$ such that for every $k > k_3$ we readily have

$$\begin{aligned} F(\overline{\rho}_k)\|\alpha\|_{L^1} &\leq (p+\varepsilon)^{-1} c_\infty^{-p} b_k^p - p^{-1} |\overline{\rho}_k|^p \mu^N \omega_N K(p,N,\mu) \\ &\leq \frac{1}{p+\varepsilon}\left(\rho_k - \frac{p+\varepsilon}{p}\|u_k\|_r^p\right) < \frac{1}{p+\varepsilon}\left(\rho_k - \|u_k\|_r^p\right). \end{aligned}$$

Thus, for every $k > k_3$, one has

$$\sup_{\|v\|_r^p \leq \rho_k} \mathcal{F}(v) - \mathcal{F}(u_k) < F(\overline{\rho}_k)\|\alpha\|_{L^1} < \frac{1}{p+\varepsilon}\left(\rho_k - \|u_k\|_r^p\right).$$

Hence $\gamma \leq \frac{1}{p+\varepsilon} < \frac{1}{p}$, which concludes the proof of Claim 2'.

Claim 3'. *$\mathcal{E}_r$ is not bounded below on $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$.*

Since $F_\infty = +\infty$, for an arbitrarily large number $M > 0$, we can fix $\tilde{\rho}_k$ with $|\tilde{\rho}_k| \geq k$ such that

$$\frac{F(\tilde{\rho}_k)}{|\tilde{\rho}_k|^p} > M. \tag{6.46}$$

Define $w_k \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ as in (6.33), putting $\tilde{\rho}_k$ instead of $\overline{\rho}_k$. We obtain

$$\begin{aligned} \mathcal{E}_r(w_k) &= \frac{1}{p}\|w_k\|_r^p - \mathcal{F}(w_k) \\ &\leq \frac{1}{p}\mu^N \omega_N |\tilde{\rho}_k|^p K(p,N,\mu) - \int_{B_N(0,\frac{\mu}{2})} \alpha(x) F(w_k(x))dx \\ &\leq |\tilde{\rho}_k|^p \mu^N \omega_N \left(\frac{1}{p}K(p,N,\mu) - \frac{1}{2^N} M \beta_\infty\right). \end{aligned}$$

Since $|\tilde{\rho}_k| \to +\infty$ as $k \to +\infty$, and M is large enough we obtain that

$$\lim_{k\to+\infty} \mathcal{E}_r(w_k) = -\infty.$$

The proof of Claim 3' is concluded.

Proof concluded. Since $\gamma < \frac{1}{p}$ (cf. Claim 2'), we can apply Theorem 2.31 (A) for $\lambda = \frac{1}{p}$. The rest is the same as in Theorem 6.19.

The case $F_0 = +\infty$.

We follow the line of $F_\infty = +\infty$. The sequences $\{\rho_k\}$, $\{\overline{\rho}_k\}$ are defined as above; they converge to 0. Let $\mu > 0$ be as in (6.45), replacing β_∞ by β_0. Instead of

Claim 2', we may prove that $\delta = \liminf_{\rho\to 0^+} \varphi(\rho) < \frac{1}{p}$. Now, we are in the position to apply Theorem 2.31 (B) with $\lambda = \frac{1}{p}$. Since $F_0 = +\infty$, for an arbitrarily large number $M > 0$, we may choose $\tilde{\rho}_k$ with $|\tilde{\rho}_k| \le \frac{1}{k}$ such that

$$\frac{F(\tilde{\rho}_k)}{|\tilde{\rho}_k|^p} > M.$$

Define $w_k \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ by means of $\tilde{\rho}_k$ as above. It is clear that $\{w_k\}$ strongly converges to 0 in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ while

$$\mathcal{E}_r(w_k) \le |\tilde{\rho}_k|^p \mu^N \omega_N \left(\frac{1}{p} K(p,N,\mu) - \frac{1}{2^N} M\beta_0 \right) < 0 = \mathcal{E}_r(0).$$

Consequently, in spite of the fact that 0 is the unique global minimum of $\Psi = \|\cdot\|_r^p$, it is not a local minimum of $\mathcal{E}_r$; thus, (B1) can be excluded. The rest is the same as in the proof of Theorem 6.19. This completes the proof of Theorem 6.21.

In the next we give some example. We suppose that $2 \le N < p < +\infty$.

Example 6.22. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(s) = \frac{2^{N+p+3}}{p} |s|^p \max\{0, \sin \ln(\ln(|s|+1)+1)\},$$

and $\alpha : \mathbb{R}^N \to \mathbb{R}$ by

$$\alpha(x) = \frac{1}{(1+|x|^N)^2}. \tag{6.47}$$

Then (DI) has an unbounded sequence of radially symmetric solutions.

Proof. The functions F and α clearly fulfill (H). Moreover, $F_\infty = \frac{2^{N+p+3}}{p}$. Since $\|\alpha\|_{L^\infty} = 1$, we may fix $\beta_\infty = 1/4$ which verifies (6.28). For every $k \in \mathbb{N}$ let

$$a_k = e^{e^{(2k-1)\pi}-1} - 1 \quad \text{and} \quad b_k = e^{e^{2k\pi}-1} - 1.$$

If $a_k \le |s| \le b_k$, then $(2k-1)\pi \le \ln(\ln(|s|+1)+1) \le 2k\pi$, thus $F(s) = 0$ for every $s \in \mathbb{R}$ complying with $a_k \le |s| \le b_k$. So, $\partial F(s) = \{0\}$ for every $|s| \in]a_k, b_k[$ and (6.29) is verified. Thus, all the assumptions of Theorem 6.19 are satisfied. □

Example 6.23. Fix $\sigma \in \mathbb{R}$. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(s) = \begin{cases} \frac{8^{N+1}}{p} s^{p-\sigma} \max\{0, \sin \ln \ln \frac{1}{s}\}, & s \in]0, e^{-1}[; \\ 0, & s \notin]0, e^{-1}[, \end{cases}$$

and let $\alpha : \mathbb{R}^N \to \mathbb{R}$ be as in (6.47). Then, for every $\sigma \in [0, \min\{p-1, p(1-e^{-\pi})\}[$, (DI) admits a sequence of non-negative, radially symmetric solutions which strongly converges to 0 in $W^{1,p}(\mathbb{R}^N)$.

Proof. Since $\sigma < p-1$, (H) is verified. We distinguish two cases: $\sigma = 0$, and $\sigma \in]0, \min\{p-1, p(1-e^{-\pi})\}[$.

Case 1. $\sigma = 0$. We have $F_0 = \frac{8^{N+1}}{p}$. If we choose $\beta_0 = (1+2^N)^{-2}$, this clearly verifies (6.28). For every $k \in \mathbb{N}$ set

$$a_k = e^{-e^{2k\pi}} \quad \text{and} \quad b_k = e^{-e^{(2k-1)\pi}}. \tag{6.48}$$

For every $s \in [a_k, b_k]$, one has $(2k-1)\pi \leq \ln\ln\frac{1}{s} \leq 2k\pi$; thus $F(s) = 0$. So, $\partial F(s) = \{0\}$ for every $s \in]a_k, b_k[$ and (6.29) is verified. Now, we apply Theorem 6.19.

Case 2. $\sigma \in]0, \min\{p-1, p(1-e^{-\pi})\}[$. We have $F_0 = +\infty$. In order to verify (6.43), we fix for instance $\beta_0 = (1+2^N)^{-2}$ and $\mu = 2$. Take $\{a_k\}$ and $\{b_k\}$ in the same way as in (6.48). The inequality in (6.44) becomes obvious since

$$\limsup_{k\to+\infty} \frac{\max_{[-a_k,a_k]} F}{b_k^p} \leq \frac{8^{N+1}}{p} \limsup_{k\to+\infty} \frac{a_k^{p-\sigma}}{b_k^p} =$$
$$= \frac{8^{N+1}}{p} \lim_{k\to+\infty} e^{[p-e^{\pi}(p-\sigma)]e^{(2k-1)\pi}} = 0.$$

Therefore, we may apply Theorem 6.21. □

Example 6.24. Let $\{a_k\}$ and $\{b_k\}$ be two sequences such that $a_1 = 1$, $b_1 = 2$ and $a_k = k^k$, $b_k = k^{k+1}$ for every $k \geq 2$. Define, for every $s \in \mathbb{R}$ the function

$$f(s) = \begin{cases} \frac{b_{k+1}^p - b_k^p}{a_{k+1} - b_k}, & \text{if } s \in [b_k, a_{k+1}[; \\ 0, & \text{otherwise.} \end{cases}$$

Then the problem

$$\begin{cases} -\triangle_p u + |u|^{p-2}u \in \frac{\sigma}{(1+|x|^N)^2}[\underline{f}(u(x)), \overline{f}(u(x))], & x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases}$$

possesses an unbounded sequence of non-negative, radially symmetric solutions whenever $0 < \sigma < \frac{N}{p}\left(\frac{p-N}{2p}\right)^p (\text{Area}S^{N-1})^{-1}$.

Proof. Let $F(s) = \int_0^s f(t)dt$. Since the function f is locally (essentially) bounded, F is locally Lipschitz. A more explicit expression of F is

$$F(s) = \begin{cases} b_k^p - b_1^p + \frac{b_{k+1}^p - b_k^p}{a_{k+1} - b_k}(s - b_k), & \text{if } s \in [b_k, a_{k+1}[; \\ b_k^p - b_1^p, & \text{if } s \in [a_k, b_k[; \\ 0, & \text{otherwise.} \end{cases}$$

An easy calculation shows, as we expect, that $\partial F(s) = [\underline{f}(s), \overline{f}(s)]$ for every $s \in \mathbb{R}$. Taking $\alpha(x) = \frac{\sigma}{(1+|x|^N)^2}$, (H) is verified, and $\|\alpha\|_{L^1} = \frac{\sigma}{N}\text{Area}S^{N-1}$. Moreover,

$$F_\infty = \limsup_{|s|\to+\infty} \frac{F(s)}{|s|^p} \geq \lim_{k\to+\infty} \frac{F(a_k)}{a_k^p} = \lim_{k\to+\infty} \frac{b_k^p - b_1^p}{a_k^p} = +\infty.$$

Choosing $\mu = 1$ and $\beta_\infty = \sigma/4$, (6.43) is verified, while (6.29) becomes trivial. Since $\max_{[-a_k,a_k]} F = F(a_k) = b_k^p - b_1^p$, relation (6.44) reduces to $pc_\infty^p \|\alpha\|_{L^1} < 1$ which is fulfilled due to the choice of σ and to Remark 6.10. It remains to apply Theorem 6.21. □

6.4. **An application to variational-hemivariational inequalities.** In this subsection we give two applications of the Principle of Symmetric Criticality for Motreanu-Panagiotopulos functionals. These results appear in the paper of Lisei and Varga [36].

First we formulate the problem. For this let $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, which is locally Lipschitz in the second variable (the real variable) and satisfies the following conditions:

($\overline{F}$1) $F(z, 0) = 0$ for all $z \in \mathbb{R}^L \times \mathbb{R}^M$ and there exist $c_1 > 0$ and $r \in]p, p^\star[$ such that

$$|\xi| \le c_1(|s|^{p-1} + |s|^{r-1}),\ \forall \xi \in \partial F(z, s),\ (z, s) \in \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R}.$$

We denoted by $\partial F(z, s)$ the generalized gradient of $F(z, \cdot)$ at the point $s \in \mathbb{R}$ and $p^\star = \dfrac{(L + M)p}{L + M - p}$ is the critical Sobolev exponent.

Let $a : \mathbb{R}^L \times \mathbb{R}^M \to \mathbb{R}$ $(L \ge 2)$ be a nonnegative continuous function satisfying the following assumptions:

(A_1) $a(x, y) \ge a_0 > 0$ if $|(x, y)| \ge R$ for a large $R > 0$;
(A_2) $a(x, y) \to +\infty$, when $|y| \to +\infty$ uniformly for $x \in \mathbb{R}^L$;
(A_3) $a(x, y) = a(x', y)$ for all $x, x' \in \mathbb{R}^L$ with $|x| = |x'|$ and all $y \in \mathbb{R}^M$.

Consider the following subspaces of $W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M)$

$$\tilde{E} = \{u \in W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M) : u(x, y) = u(x', y)\ \forall\ x, x' \in \mathbb{R}^L, |x| = |x'|, \forall y \in \mathbb{R}^M\},$$

$$E = \{u \in W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M) : \int_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz < \infty\},$$

$$E_a = \tilde{E} \cap E = \{u \in \tilde{E} : \int_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz < \infty\},$$

endowed with the norm

$$\|u\|^p = \int_{\mathbb{R}^{L+M}} |\nabla u(z)|^p dz + \int_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz$$

and the closed convex cone $\mathcal{K} = \{v \in E\ :\ v \ge 0 \text{ a.e. in } \mathbb{R}^L \times \mathbb{R}^M\}$.

The aim of this subsection is to study the following eigenvalue problem (P_λ): *For* $\lambda > 0$ *find* $u \in \mathcal{K}$ *such that*

$$\int_{\mathbb{R}^{L+M}} |\nabla u(z)|^{p-2}\nabla u(z)(\nabla v(z) - \nabla u(z))dz + \int_{\mathbb{R}^{L+M}} a(z)u^{p-1}(z)(v(z) - u(z))dz$$
$$+ \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u(z); v(z) - u(z))dz \geq 0$$

for all $v \in \mathcal{K}$, where $F^0(z, s; t)$ is the generalized directional derivative of $F(z, \cdot)$ at the point s in the direction t.

Let $\mathcal{I}_\lambda : E \to] - \infty, +\infty]$ be defined by

$$\mathcal{I}_\lambda(u) = \frac{1}{p}||u||^p - \lambda\mathcal{F}(u) + \psi_{\mathcal{K}}(u),$$

where $\psi_{\mathcal{K}}(u)$ denotes the indicator function of the closed convex cone $\mathcal{K}$,i.e.

$$\psi_{\mathcal{K}}(u) = \begin{cases} 0, & \text{if } x \in \mathcal{K} \\ +\infty, & \text{otherwise.} \end{cases}$$

Clearly $\psi_{\mathcal{K}}$ is convex and lower-semicontinuous on E.

Now we rewrite problem (P_λ) by using the duality map. By Theorem 3.5 from [1] it follows that E is a separable, reflexive and uniform convex Banach space. We denote by $E^\star$ its dual. Let $A : E \to E^\star$ the duality mapping corresponding to the weight function $\varphi : [0, +\infty[\to [0, +\infty[$ defined by $\varphi(t) = t^{p-1}$, where $p \in]1, +\infty[$. It is well known that the duality mapping J satisfies the following conditions:

$$||Au||_\star = \varphi(||u||) \text{ and } \langle Au, u\rangle = ||Au||_\star ||u|| \text{ for all } u \in E.$$

Moreover, the functional $\chi : E \to \mathbb{R}$ defined by $\chi(u) = \frac{1}{p}||u||^p$ is convex and Gateaux differentiable on E, and $d\chi = A$. The problem (P_λ) can be reformulated in the following way: *For* $\lambda > 0$ *find* $u \in \mathcal{K}$ *such that*

$$\langle Au, v - u\rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u(z); v(z) - u(z))dx \geq 0$$

for every $v \in \mathcal{K}$.

Lemma 6.25. *Fix* $\lambda > 0$ *arbitrary. Every critical point* $u \in E$ *of the functional* $\mathcal{I}_\lambda$ *is a solution of the problem* (P_λ).

Proof. Since $u \in E$ is a critical point of the functional $\mathcal{I}_\lambda$, one has

$$\langle Au, v - u\rangle + \lambda(-\mathcal{F})^0(u; v - u) + \psi_{\mathcal{K}}(v) - \psi_{\mathcal{K}}(u) \geq 0$$

for every $v \in E$. From Proposition 4.5 we obtain

$$\langle Au, v - u\rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u(z); u(z) - v(z))dz + \psi_{\mathcal{K}}(v) - \psi_{\mathcal{K}}(u) \geq 0$$

for every $v \in E$.

Therefore $u \in \mathcal{K}$ and for every $v \in \mathcal{K}$ we have

$$\langle Au, v - u\rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u(z); u(z) - v(z))dz \geq 0. \qquad \square$$

Let $a : \mathbb{R}^L \times \mathbb{R}^M \to \mathbb{R}$ $(L \geq 2)$ be a function, which satisfy the assumptions (A_1), (A_2), (A_3). We consider the following subspaces of $W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M)$

$$\tilde{E} = \{u \in W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M) : u(x, y) = u(x', y)\ \forall\ x, x' \in \mathbb{R}^L, |x| = |x'|, \forall y \in \mathbb{R}^M\},$$

$$E = \{u \in W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M) : \int_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz < \infty\},$$

$$E_a = \tilde{E} \cap E = \{u \in \tilde{E} : \int_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz < \infty\}$$

endowed with the norm

$$\|u\|^p = \int_{\mathbb{R}^{L+M}} |\nabla u(z)|^p dz + \int_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz.$$

The next result is proved by de Morais Filho, Souto, Marcos Do [42] and is a very useful tool in our investigations.

Theorem 6.26. *If (A_1), (A_2) and (A_3) hold, then the Banach space E_a is continuously embedded in $L^s(\mathbb{R}^L \times \mathbb{R}^M)$, if $p \leq s \leq p^*$, and compactly embedded if $p < s < p^*$.*

We have,

$$\|u\|_s \leq C(s)\|u\| \quad \text{for each } u \in E_a,$$

where $\|\cdot\|_s$ is the norm in $L^s(\mathbb{R}^L \times \mathbb{R}^M)$ and $C(s) > 0$ is the embedding constant.

Let

$$G = \left\{ g : E \to E : g(v) = v \circ \begin{pmatrix} R & 0 \\ 0 & Id_{\mathbb{R}^M} \end{pmatrix}, R \in O(\mathbb{R}^L) \right\},$$

where $O(\mathbb{R}^L)$ is the set of all rotations on $\mathbb{R}^L$ and $Id_{\mathbb{R}^M}$ denotes the $M \times M$ identity matrix. The elements of G leave $\mathbb{R}^{L+M}$ invariant, i.e. $g(\mathbb{R}^{L+M}) = \mathbb{R}^{L+M}$ for all $g \in G$.

The action of G over E is defined by

$$(gu)(z) = u(g^{-1}z), \quad g \in G, \quad u \in E, \quad \text{a.e. } z \in \mathbb{R}^{L+M}.$$

As usual we shall write gu in place of $\pi(g)u$.

A function u defined on $\mathbb{R}^{L+M}$ is said to be G-invariant, if

$$u(gz) = u(z), \ \ \forall \ \ g \in G, \ \text{a.e. } z \in \mathbb{R}^{L+M}.$$

Then $u \in E$ is G-invariant if and only if $u \in \Sigma$, where

$$\Sigma := E_a = \tilde{E} \cap E.$$

We observe that the norm

$$\|u\| = \left\{ \int_{\mathbb{R}^{L+M}} (|\nabla u(z)|^p + a(z)|u(z)|^p) dz \right\}^{\frac{1}{p}}$$

is G-invariant.

In order to study our problem we give the assumptions on the nonlinear function F. We assume that $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which is locally Lipschitz in the second variable, satisfying condition **($\overline{F}$1)** and moreover:

($\overline{F}$2) $\lim\limits_{s\to 0} \dfrac{\max\{|\xi| : \xi \in \partial F(z,s)\}}{|s|^{p-1}} = 0$ uniformly for every $z \in \mathbb{R}^{L+M}$.

($\overline{F}$3) There exists $\nu > p$ such that

$$\nu F(z,s) + F^0(z,s;-s) \leq 0, \ \forall (z,s) \in \mathbb{R}^{L+M} \times \mathbb{R}.$$

($\overline{F}$4) There exists $r > 0$ such that

$$\inf\{F(z,s) : (z,|s|) \in \mathbb{R}^{L+M} \times [r,\infty)\} > 0.$$

Remark 6.27. a) If $F : \mathbb{R}^{L+M} \times \mathbb{R} \to \mathbb{R}$ satisfies **($\overline{F}$1)** and **($\overline{F}$2)**, then for every $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ such that

i) $|\xi| \leq \varepsilon |s|^{p-1} + c(\varepsilon)|s|^{r-1}, \ \forall\, \xi \in \partial F(z,s), \ (z,s) \in \mathbb{R}^{L+M} \times \mathbb{R};$

ii) $|F(z,s)| \leq \varepsilon |s|^p + c(\varepsilon)|s|^r, \ \forall\, (z,s) \in \mathbb{R}^{L+M} \times \mathbb{R}.$

b) If $F : \mathbb{R}^{L+M} \times \mathbb{R} \to \mathbb{R}$ satisfies **($\overline{F}$1)**, **($\overline{F}$3)** and **($\overline{F}$4)**, then there exist $c_2, c_3 > 0$ and $\nu \in]p, p^\star[$ such that

$$F(z,s) \geq c_2|s|^\nu - c_3|s|^p.$$

To study the existence of the solutions of problem (P_λ), it is sufficient to prove the existence of critical points of the functional $\mathcal{I}_\lambda$ (see Lemma 6.25).

We have the following result, which appear in the paper of Lisei-Varga [36].

Theorem 6.28. (Lisei-Varga [36]) *Let $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ be a function, which satisfies **($\overline{F}$1)-($\overline{F}$4)** and $F(\cdot,s)$ is G-invariant for every $s \in \mathbb{R}$. Then for every $\lambda > 0$ problem (P_λ) has a nontrivial positive solution.*

Before to prove this result we introduce some notations and we prove some auxiliary results. We have that the cone $\mathcal{K}$ is G-invariant, it follows that $\psi_\mathcal{K}$ is G-invariant. Taking into account that the action of G is linear and isometric on E, we deduce that the function $\chi(u) = \frac{1}{p}||u||^p$ is G-invariant. The function $\mathcal{F}$ is also G-invariant, because $F(\cdot,s)$ is G-invariant for every $s \in \mathbb{R}$. If we apply Theorem 3.8, it is sufficient to prove that the functional $\mathcal{I}_\Sigma := \mathcal{I}_\lambda\Big|_\Sigma$ has critical points, which implies that the functional $\mathcal{I}_\lambda$ has critical points, which are solutions for problem (P_λ). We introduce the following notations:

$$||\cdot||_\Sigma = ||\cdot||\Big|_\Sigma, \ \mathcal{F}_\Sigma = \mathcal{F}\Big|_\Sigma, \ \psi_\Sigma = \psi_\mathcal{K}\Big|_\Sigma$$

and the restricted duality map $A_\Sigma : \Sigma \to \Sigma^*$ with $A_\Sigma = A\Big|_\Sigma$. Therefore we have

$$\mathcal{I}_\Sigma(u) = \frac{1}{p}||u||_\Sigma^p - \lambda\mathcal{F}_\Sigma(u) + \psi_\Sigma(u).$$

In the next we verify that the conditions of Theorem 3.4 are satisfied by the functional $\mathcal{I}_\Sigma$.

Proposition 6.29. *If $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ verifies the conditions **($\overline{F}$1)-($\overline{F}$3)** and $F(\cdot, s), s \in \mathbb{R}$ is G- invariant, then $\mathcal{I}_\Sigma$ satisfies the (PS) condition, for every $\lambda > 0$.*

Proof. Let $\lambda > 0$ and $c \in \mathbb{R}$ be some fixed numbers and let $(u_n) \subset \Sigma$ be a sequence such that

$$\mathcal{I}_\Sigma(u_n) = \frac{1}{p}||u_n||_\Sigma^p - \lambda\mathcal{F}_\Sigma(u_n) + \psi_\Sigma(u_n) \to c \tag{6.49}$$

and for every $v \in \Sigma$ we have

$$\langle A_\Sigma u_n, v-u_n\rangle+\lambda\int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); u_n(z)-v(z))dz+\psi_\Sigma(v)-\psi_\Sigma(u_n) \geq -\varepsilon_n||v-u_n||_\Sigma, \tag{6.50}$$

for a sequence (ε_n) in $[0, +\infty[$ with $\varepsilon_n \to 0$.

By (6.49) one concludes that $(u_n) \subset \mathcal{K} \cap \Sigma$. Setting $v = 2u_n$ in (6.50), we obtain

$$\langle A_\Sigma u_n, u_n\rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); -u_n(z))dz \geq -\varepsilon_n||u_n||_\Sigma. \tag{6.51}$$

By (6.49) one has for large $n \in \mathbb{N}$ that

$$c + 1 \geq \frac{1}{p}||u_n||_\Sigma^p - \lambda\mathcal{F}_\Sigma(u_n). \tag{6.52}$$

We multiply inequality (6.51) with ν^{-1} and use Proposition 4.5 to obtain

$$\varepsilon_n\frac{||u_n||_\Sigma}{\nu} \geq -\frac{\langle A_\Sigma u_n, u_n\rangle}{\nu} - \frac{\lambda}{\nu}\int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); -u_n(z))dz. \tag{6.53}$$

Adding the inequalities (6.52) and (6.53), and using (F3) we get

$$\begin{aligned} c + 1 + \frac{\varepsilon_n}{\nu}||u_n||_\Sigma &\geq \left(\frac{1}{p} - \frac{1}{\nu}\right)||u_n||_\Sigma^p \\ &\quad - \lambda\int_{\mathbb{R}^{L+M}} [F(z, u_n(z)) + \frac{1}{\nu}F^0(z, u_n(z); -u_n(z))]dz \\ &\geq \left(\frac{1}{p} - \frac{1}{\nu}\right)||u_n||_\Sigma^p. \end{aligned}$$

From this, we get that the sequence $(u_n) \subset \mathcal{K} \cap \Sigma$ is bounded. Because E is reflexive, it follows that Σ is reflexive too and there exists an element $u \in \Sigma$ such that $u_n \rightharpoonup u$ weakly. Since $\mathcal{K} \cap \Sigma$ is closed and convex, we get $u \in \mathcal{K} \cap \Sigma$. Moreover, from (6.50) with $v = u$ we obtain

$$\langle A_\Sigma u_n, u - u_n \rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); u_n(z) - u(z))dz \geq -\varepsilon_n ||u_n - u||_\Sigma. \quad (6.54)$$

From this we get

$$\langle A_\Sigma u_n, u_n - u \rangle \leq \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); u_n(z) - u(z))dz + \varepsilon_n ||u_n - u||_\Sigma$$

$$\leq \lambda \int_{\mathbb{R}^{L+M}} \max\{\xi_n(z)(u_n(z) - u(z)) \ : \ \xi_n(z) \in \partial F(z, u_n(z))\}dz + \varepsilon_n ||u_n - u||_\Sigma$$

$$\leq \lambda \int_{\mathbb{R}^{L+M}} \left(\varepsilon |u_n(z)|^{p-1} + c(\varepsilon)|u_n(z)|^{r-1}\right)|u_n(z) - u(z)|dz + \varepsilon_n ||u_n - u||_\Sigma.$$

Hence, by Hölder's inequality and the fact that the inclusion $\Sigma \hookrightarrow L^p(\mathbb{R}^{L+M})$ is continuous (see Theorem 6.26), we obtain

$$\langle A_\Sigma u_n, u_n - u \rangle \leq \lambda \varepsilon C(p)||u_n - u||_\Sigma ||u_n||_p^{p-1} + \lambda c(\varepsilon)||u_n - u||_r ||u_n||_r^{r-1} + \varepsilon_n ||u_n - u||_\Sigma.$$

Moreover, the inclusion $\Sigma \hookrightarrow L^r(\mathbb{R}^{L+M})$ is compact for $r \in]p, p^*[$ (see Theorem 6.26), therefore $||u_n - u||_r \to 0$ as $n \to +\infty$. For $\varepsilon \to 0^+$ and $n \to +\infty$ we obtain that $\limsup_{n \to +\infty} \langle A_\Sigma u_n, u_n - u \rangle \leq 0$. Finally, since the duality operator J_Σ has the (S_+) property we obtain $u_n \to u$ in $\mathcal{K}$, because $\mathcal{K}$ is closed. □

Proposition 6.30. *If $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ verifies ($\overline{F}$1)-($\overline{F}$4) and $F(\cdot, s)$ is G-invariant for every $s \in \mathbb{R}$, then for every $\lambda > 0$ the following assertions are true:*

i) *there exist constants $\alpha_\lambda > 0$ and $\rho_\lambda > 0$ such that $\mathcal{I}_\Sigma(u) \geq \alpha_\lambda$ for all $||u||_\Sigma = \rho_\lambda$;*
ii) *there exists $e_\lambda \in \mathcal{K}$ with $||e_\lambda|| > \rho_\lambda$ and $\mathcal{I}_\Sigma(e_\lambda) \leq 0$.*

Proof. From Remark 6.27 and from the fact that the embedding $\Sigma \hookrightarrow L^l(\mathbb{R}^{L+M})$ is continuous for $l \in [p, p^\star]$, it follows that

$$\mathcal{F}_\Sigma(u) \leq \varepsilon C^p(p)||u||_\Sigma^p + c(\varepsilon)C^r(r)||u||_\Sigma^r,$$

for every $u \in \Sigma$. It is suffices to restrict our attention to elements u which belong to $\mathcal{K} \cap \Sigma$, otherwise $\mathcal{I}_\Sigma(u)$ will be $+\infty$, i.e. i) holds trivially.

Let $\lambda > 0$ be arbitrary. We choose $\varepsilon \in]0, \frac{1}{p\lambda C^p(p)}[$ and for $u \in \mathcal{K} \cap \Sigma$ we have

$$\mathcal{I}_\Sigma(u) = \frac{1}{p}||u||_\Sigma^p - \lambda \mathcal{F}_\Sigma(u) \geq \left(\frac{1}{p} - \lambda \varepsilon C^p(p)\right) ||u||_\Sigma^p - \lambda c(\varepsilon)C^r(r)||u||_\Sigma^r.$$

We denote by $M = \frac{1}{p} - \lambda\varepsilon C^p(p)$ and $N = \lambda c(\varepsilon)C^r(r)$ and we consider the function $g : \mathbb{R}_+ \to \mathbb{R}$ given by $g(t) = Mt^p - Nt^r$. The function g attains its global maximum in the point $t_\lambda = \left(\frac{pM}{rN}\right)^{\frac{1}{r-p}}$. If we take $\rho_\lambda = t_\lambda$ and $\alpha_\lambda \in]0, g(t_\lambda)[$, the condition i) is fulfilled.

To prove ii) from b) Remark 6.27 we observe that for every $u \in \mathcal{K} \cap \Sigma$ we have

$$\mathcal{I}_\Sigma(u) \leq \frac{1}{p}||u||_\Sigma^p + \lambda c_3 C^p(p)||u||_\Sigma^p - \lambda c_2||u||_\nu^\nu.$$

If we fix an element $v \in (\mathcal{K} \cap \Sigma) \setminus \{0\}$ and in place of u we put tv, then we have

$$\mathcal{I}_\Sigma(tv) \leq \left(\frac{1}{p} + \lambda c_3 C^p(p)\right) ||v||_\Sigma^p t^p - \lambda c_2||v||_\nu^\nu t^\nu.$$

From this we see that if t is large enough, then $||tv||_\Sigma > \rho_\lambda$ and $\mathcal{I}_\Sigma(tv) < 0$. If we take $e_\lambda = tv$ we obtain the desired results. □

Proof of Theorem 6.28. Now we prove that the conditions of Theorem 3.4 are satisfied by the functional $\mathcal{I}_\Sigma$. Because $F(z, 0) = 0$, it follows that

$$\mathcal{I}_\Sigma(0) = \int_{\mathbb{R}^{L+M}} F(z, 0)dz = 0.$$

From Proposition 6.29 we get that $\mathcal{I}_\Sigma$ satisfies the (PS) condition. Proposition 6.30 implies that $\mathcal{I}_\Sigma$ satisfies the conditions (i) and (ii) from Theorem 3.4, hence the number

$$c_\lambda = \inf_{f\in\Gamma} \sup_{t\in[0,1]} I_\Sigma(f(t)),$$

where

$$\Gamma_\lambda = \{f \in C([0, 1], \Sigma) \ : \ f(0) = 0, \ f(1) = e_\lambda \},$$

is a critical value of $\mathcal{I}_\Sigma$ with $c_\lambda \geq \alpha_\lambda$. □

In the next we replace (F3) and (F4) with the following two conditions

$(\overline{F}'3)$ There exist $q \in]0, p[, \nu \in [p, p^\star], \alpha \in L^{\frac{\nu}{\nu-q}}(\mathbb{R}^{L+M}), \beta \in L^1(\mathbb{R}^{L+M})$ such that

$$F(z, s) \leq \alpha(z)|s|^q + \beta(z)$$

for all $s \in \mathbb{R}$ and a.e. $z \in \mathbb{R}^{L+M}$;

$(\overline{F}'4)$ There exists $u_0 \in \mathcal{K}$ such that $\displaystyle\int_{\mathbb{R}^{L+M}} F(z, u_0(z))dz > 0$.

We have the following result.

Theorem 6.31. (Lisei-Varga [36]) *Let $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ be a function which satisfies $(\overline{F}1)$, $(\overline{F}2)$, $(\overline{F}'3)$, $(\overline{F}'4)$ and $F(\cdot, s)$ is G-invariant for all $s \in \mathbb{R}$. Then there exists an open interval $\Lambda_0 \subset \Lambda$ such that for each $\lambda \in \Lambda_0$ problem (P_λ) has at least three distinct solutions which are axially symmetric.*

To prove Theorem 6.31 we combine Theorem 3.5 with Theorem 3.8. First we consider the functional $f : E \times \Lambda \to]-\infty, +\infty]$ given by $f(u, \lambda) = I_1(u) + \lambda I_2(u)$, where

$$I_1(u) = \frac{1}{p}||u||^p + \psi_{\mathcal{K}}(u), \quad I_2(u) = -\mathcal{F}(u) = -\int_{\mathbb{R}^{L+M}} F(z, u(z))dz.$$

As in Lemma 6.25 we have that every critical point of the function $f = I_1 + \lambda I_2$ is a solution of problem (P_λ). Using Theorem 3.8 it is sufficient to prove that the functional $f_\Sigma = \Big(I_1 + \lambda I_2\Big)\Big|_\Sigma$ satisfies conditions from Theorem 3.5, where we choose $h_1, \Psi_1, h_2 : \Sigma \to \mathbb{R}$

$$h_1(u) = \frac{1}{p}||u||_\Sigma^p, \ \Psi_1(u) = \psi_\Sigma(u), \ h_2(u) = -\mathcal{F}_\Sigma(u) = -\int_{\mathbb{R}^{L+M}} F(z, u(z))dz, \ u \in \Sigma,$$

and take

$$I_1 = h_1 + \Psi_1, \ I_2 = h_2.$$

First we prove that (a_1) holds.

Proposition 6.32. *If $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ verifies the conditions **($\overline{F}$1)** and **($\overline{F}$2)**, then h_1 is weakly sequentially lower semicontinuous and h_2 is weakly sequentially continuous.*

Proof. The weakly sequentially lower semicontinuity of $h_1 = \frac{1}{p}||\cdot||_\Sigma^p$ is standard (every convex lower semicontinuous function is sequentially lower semicontinuous, see e.g. [7]).

In order to prove the weakly sequentially continuity of h_2 we assume that (u_n) is a sequence in Σ such that $u_n \rightharpoonup u$ (in Σ). We will prove that $\mathcal{F}_\Sigma(u_n) \to \mathcal{F}_\Sigma(u)$.

By Lebourg's Mean Value Theorem (see [10]) it follows that there exist $\theta_n \in [0, 1]$ and $v_n \in \partial\mathcal{F}_\Sigma(u + \theta_n(u_n - u))$ such that

$$\mathcal{F}_\Sigma(u_n) - \mathcal{F}_\Sigma(u) = \langle v_n, u_n - u\rangle.$$

We denote $w_n = u + \theta_n(u_n - u)$. Using the definition of $\mathcal{F}_\Sigma^0$, Proposition 4.5 it follows that

$$\begin{aligned}\mathcal{F}_\Sigma(u_n) - \mathcal{F}_\Sigma(u) &\leq (\mathcal{F}_\Sigma)^0(w_n; u_n - u) \leq \int_{\mathbb{R}^{L+M}} F^\circ(z, w_n(z); u_n(z) - u(z))dz \\ &= \int_{\mathbb{R}^{L+M}} \max\Big\{\langle v(z), u_n(z) - u(z)\rangle : v \in \partial F(z, w_n(z))\Big\}.\end{aligned}$$

Now we use Remark 6.27 to get

$$\mathcal{F}_\Sigma(u_n) - \mathcal{F}_\Sigma(u) \leq \int_{\mathbb{R}^{L+M}} \Big(\varepsilon|w_n(z)|^{p-1} + c(\varepsilon)|w_n(z)|^{r-1}\Big)|u_n(z) - u(z)|dz.$$

We use Hölder's inequality and the fact that the inclusion $\Sigma \hookrightarrow L^p(\mathbb{R}^{L+M})$ is continuous (see Theorem 6.26) to obtain

$$\mathcal{F}_\Sigma(u_n) - \mathcal{F}_\Sigma(u) \leq \varepsilon C(p)\|u_n - u\|_\Sigma \|w_n\|_p^{p-1} + c(\varepsilon)C(r)\|u_n - u\|_r \|w_n\|_r^{r-1}. \quad (6.55)$$

Now we use the same ideas as before for $-\mathcal{F}_\Sigma$ and find the existence of $\tau_n \in [0,1]$ and $\hat{v}_n \in \partial(-\mathcal{F}_\Sigma)(u + \tau_n(u_n - u))$ such that

$$\mathcal{F}_\Sigma(u) - \mathcal{F}_\Sigma(u_n) = \langle \hat{v}_n, u_n - u\rangle.$$

We denote $\hat{w}_n = u + \tau_n(u_n - u)$. Using the definition of $-\mathcal{F}_\Sigma^0$, and properties of the generalized gradient (see [10]), it follows that

$$\mathcal{F}_\Sigma(u) - \mathcal{F}_\Sigma(u_n) \quad \leq \quad (-\mathcal{F}_\Sigma)^0(\hat{w}_n; u_n - u) = (\mathcal{F}_\Sigma)^0(\hat{w}_n; u - u_n).$$

Analogously to (6.55) we get

$$\mathcal{F}_\Sigma(u) - \mathcal{F}_\Sigma(u_n) \leq \varepsilon C(p)\|u_n - u\|_\Sigma \|\hat{w}_n\|_p^{p-1} + c(\varepsilon)C(r)\|u_n - u\|_r \|\hat{w}_n\|_r^{r-1}. \quad (6.56)$$

Using (6.55) and (6.56) we have

$$\begin{aligned} |\mathcal{F}_\Sigma(u_n) - \mathcal{F}_\Sigma(u)| \quad \leq \quad & \varepsilon C(p)\|u_n - u\|_\Sigma(\|w_n\|_p^{p-1} \\ & +\|\hat{w}_n\|_p^{p-1}) + c(\varepsilon)C(r)\|u_n - u\|_r(\|w_n\|_r^{r-1} + \|\hat{w}_n\|_r^{r-1}). \end{aligned} \quad (6.57)$$

The inclusion $\Sigma \hookrightarrow L^r(\mathbb{R}^{L+M})$ is compact for $r \in]p, p^*[$ (see Theorem 6.26), then we get that $||u_n - u||_r \to 0$ as $n \to +\infty$, while the sequences (w_n) and $(\hat{w}_n)$ are bounded in the $\|\cdot\|_p$ and $\|\cdot\|_r$ norms. Then in (6.57) we get $\mathcal{F}_\Sigma(u_n) \to \mathcal{F}_\Sigma(u)$. Hence h_2 is weakly sequentially continuous. □

Proof of Theorem 6.31. For this let $u \in \mathcal{K} \cap \Sigma$, from condition $(\overline{F}'3)$ and from the fact that the embedding $\Sigma \hookrightarrow L^\nu(\mathbb{R}^{L+M})$ is continuous and $q < p$ it follows that

$$\begin{aligned} f_\Sigma(u,\lambda) \quad \geq \quad & \frac{1}{p}||u||_\Sigma^p - \lambda \int_{\mathbb{R}^{L+M}} \alpha(z)|u(z)|^q dz - \lambda \int_{\mathbb{R}^{L+M}} \beta(z) dz \\ \geq \quad & \frac{1}{p}||u||_\Sigma^p - \lambda||\alpha||_{\frac{\nu}{\nu-q}}||u||_\nu^q - \lambda||\beta||_1 \\ \geq \quad & \frac{1}{p}||u||_\Sigma^p - \lambda||\alpha||_{\frac{\nu}{\nu-q}} C^q(q)||u||_\Sigma^q - \lambda||\beta||_1. \end{aligned}$$

Therefore, if $||u||_\Sigma \to +\infty$, we have $f_\Sigma(u,\lambda) \to +\infty$. Let $(u_n) \subset \mathcal{K} \cap \Sigma$ be a sequence such that

$$f_\Sigma(u_n, \lambda) \to c \quad (6.58)$$

and for every $v \in \Sigma$ we have

$$\langle A_\Sigma u_n, v - u_n\rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); u_n(z) - v(z)) dz + \psi_\Sigma(v) - \psi_\Sigma(u_n) \geq -\varepsilon_n ||v - u_n||_\Sigma, \quad (6.59)$$

for a sequence (ε_n) in $[0,+\infty[$ with $\varepsilon_n \to 0$. From (6.58) follows that the sequence (u_n) is bounded in $\mathcal{K}\cap\Sigma$ and as in Proposition 6.29 we get that there exists an element $u \in \mathcal{K}\cap\Sigma$ such that $u_n \to u$. Let us define the function

$$g(t) = \sup\left\{\mathcal{F}_\Sigma(u) : \ \frac{1}{p}||u||_\Sigma^p \le t\right\}.$$

Using ii) from Remark 6.27 and the fact that the inclusion $\Sigma \hookrightarrow L^l(\mathbb{R}^{L+M}), l \in [p, p^\star]$ is continuous, it follows that

$$g(t) \le \varepsilon C^p(p)t + c(\varepsilon)C^r(r)t^{\frac{r}{p}}. \tag{6.60}$$

On the other hand $g(t) \ge 0$ for each $t > 0$, then from the above relation we get

$$\lim_{t\to 0^+} \frac{g(t)}{t} = 0. \tag{6.61}$$

By $(\overline{F}'4)$ it is clear that $u_0 \ne 0$ (since $\mathcal{F}(0) = 0$). Therefore it is possible to choose a number η such that

$$0 < \eta < \mathcal{F}_\Sigma(u_0)\left[\frac{1}{p}||u_0||_\Sigma^p\right]^{-1}.$$

From $\lim\limits_{t\to 0^+} \dfrac{g(t)}{t} = 0$ it follows the existence of a number $t_0 \in \left]0, \frac{1}{p}||u_0||_\Sigma^p\right[$ such that $g(t_0) < \eta t_0$. Thus

$$g(t_0) < \left[\frac{1}{p}||u_0||_\Sigma^p\right]^{-1}\mathcal{F}_\Sigma(u_0)t_0.$$

Let $\rho_0 > 0$ such that

$$g(t_0) < \rho_0 < \left[\frac{1}{p}||u_0||_\Sigma^p\right]^{-1}\mathcal{F}_\Sigma(u_0)t_0. \tag{6.62}$$

Due to the choice of t_0 and (6.62) we have

$$\rho_0 < \mathcal{F}_\Sigma(u_0). \tag{6.63}$$

Define $h : \Lambda = [0,+\infty[\to \mathbb{R}$ by $h(\lambda) = \rho_0\lambda$. We prove that the function h satisfies the inequality

$$\sup_{\lambda\in\Lambda}\inf_{u\in\mathcal{K}\cap\Sigma}(f_\Sigma(u,\lambda) + h(\lambda)) < \inf_{u\in\mathcal{K}\cap\Sigma}\sup_{\lambda\in\Lambda}(f_\Sigma(u,\lambda) + h(\lambda)).$$

The function

$$\Lambda \ni \lambda \mapsto \inf_{u\in\mathcal{K}\cap\Sigma}\left[\frac{1}{p}||u||_\Sigma^p + \lambda(\rho_0 - \mathcal{F}_\Sigma(u))\right]$$

is obviously upper semicontinuous on Λ.

From (6.63) it follows that

$$\lim_{\lambda\to+\infty}\inf_{u\in\mathcal{K}\cap\Sigma}[f_\Sigma(u,\lambda)+\rho_0\lambda]\leq\lim_{\lambda\to+\infty}\left[\frac{1}{p}||u_0||_\Sigma^p+\lambda(\rho_0-\mathcal{F}_\Sigma(u_0))\right]=-\infty. \quad (6.64)$$

Thus we find an element $\overline{\lambda}\in\Lambda$ such that

$$\sup_{\lambda\in\Lambda}\inf_{u\in\mathcal{K}\cap\Sigma}(f_\Sigma(u,\lambda)+\rho_0\lambda)=\inf_{u\in\mathcal{K}\cap\Sigma}\left[\frac{1}{p}||u||_\Sigma^p+\overline{\lambda}(\rho_0-\mathcal{F}_\Sigma(u))\right]. \quad (6.65)$$

From $g(t_0)<\rho_0$ it follows that for all $u\in\Sigma$ with $\frac{1}{p}||u||_\Sigma^p\leq t_0$, we have $\mathcal{F}_\Sigma(u)<\rho_0$. Hence

$$t_0\leq\inf\left\{\frac{1}{p}||u||_\Sigma^p\ :\ \mathcal{F}_\Sigma(u)\geq\rho_0\right\}. \quad (6.66)$$

On the other hand,

$$\begin{aligned}\inf_{u\in\mathcal{K}\cap\Sigma}\sup_{\lambda\in\Lambda}(f_\Sigma(u,\lambda)+\rho_0\lambda) &= \inf_{u\in\mathcal{K}\cap\Sigma}\left[\frac{1}{p}||u||_\Sigma^p+\sup_{\lambda\in\Lambda}(\lambda(\rho_0-\mathcal{F}_\Sigma(u)))\right]\\ &= \inf\left\{\frac{1}{p}||u||_\Sigma^p\ :\ \mathcal{F}_\Sigma(u)\geq\rho_0\right\}.\end{aligned}$$

Thus (6.66) is equivalent with

$$t_0\leq\inf_{u\in\mathcal{K}\cap\Sigma}\sup_{\lambda\in\Lambda}[f_\Sigma(u,\lambda)+\rho_0\lambda]. \quad (6.67)$$

There are two distinct cases:

(I) If $0\leq\overline{\lambda}<\frac{t_0}{\rho_0}$, we have

$$\inf_{u\in\mathcal{K}\cap\Sigma}\left[\frac{1}{p}||u||_\Sigma^p+\overline{\lambda}(\rho_0-\mathcal{F}_\Sigma(u))\right]\leq f_\Sigma(0,\overline{\lambda})=\overline{\lambda}\rho_0<t_0.$$

Combining the above inequality with (6.65) and (6.67) we obtain the inequality from (a_2) Theorem 3.5.

(II) If $\frac{t_0}{\rho_0}\leq\overline{\lambda}$, then from $\rho_0<\mathcal{F}_\Sigma(u_0)$ and (6.62) it follows

$$\begin{aligned}\inf_{u\in\mathcal{K}\cap\Sigma}\left[\frac{1}{p}||u||_\Sigma^p+\overline{\lambda}(\rho_0-\mathcal{F}_\Sigma(u))\right] &\leq \frac{1}{p}||u_0||_\Sigma^p+\overline{\lambda}(\rho_0-\mathcal{F}_\Sigma(u_0))\\ &\leq \frac{1}{p}||u_0||_\Sigma^p+\frac{t_0}{\rho_0}(\rho_0-\mathcal{F}_\Sigma(u_0))<t_0.\end{aligned}$$

Theorem 3.5 implies that there exists an open interval $\Lambda_0\subset\Lambda$, such that for each $\lambda\in\Lambda_0$, the function $f_\Sigma(\cdot,\lambda)$ has at least three critical points in $\mathcal{K}\cap\Sigma$. Therefore, problem (P_λ) has at least three distinct solutions for every $\lambda\in\Lambda_0$. This ends the proof. □

We conclude this subsection with two examples for which Theorem 6.28 and 6.31 can be applied.

Example 6.33. Let $k \in \mathbb{R}, k > 1$. We define the sequence of real numbers (A_n) by $A_0 = 0$, and

$$A_n = \frac{1}{1^k} + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k}, \quad n \geq 1.$$

Let $r > p > 2$. We consider the functions $f, F : \mathbb{R} \to \mathbb{R}$ given respectively by

$$f(s) = s|s|^{p-2}(|s|^{r-p} + A_n) \text{ for } s \in]-n-1, -n] \cup [n, n+1[, \; n \in \mathbb{N},$$

$$F(u) = \int_0^u f(s)ds \text{ for } u \in]-n-1, -n] \cup [n, n+1[, \; n \in \mathbb{N}.$$

Clearly F satisfies $(\overline{F}\mathbf{1})$, $(\overline{F}\mathbf{2})$, $(\overline{F}\mathbf{3})$ and $(\overline{F}\mathbf{4})$, hence owing to Theorem 6.28 problem (P_λ) has a nontrivial positive solution.

Example 6.34. Let $A : \mathbb{R}^L \to \mathbb{R}$ be a continuous, nonnegative, not identically zero, axially symmetric function with compact support in $\mathbb{R}^L$. We consider $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ defined by

$$F((x, y), s) = A(x) \min\{s^r, |s|^q\} \quad \text{for } (x, y) \in \mathbb{R}^L \times \mathbb{R}^M, \; s \in \mathbb{R},$$

where $r \in \left]p, \frac{(L+M)p}{L+M-p}\right[$ is an odd number and $q \in]0, p[$. The function F satisfies the assumptions $(\overline{F}\mathbf{1})$, $(\overline{F}\mathbf{2})$, $(\overline{F}'\mathbf{3})$ and $(\overline{F}'\mathbf{4})$ and $F(\cdot, s)$ is G-invariant for all $s \in \mathbb{R}$. Theorem 6.31 implies that there exists an open interval $\Lambda_0 \subset \Lambda$ such that for each $\lambda \in \Lambda_0$ problem (P_λ) has at least three distinct solutions which are axially symmetric.

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ON THE REGULARITY OF SOLUTIONS OF A BOUNDARY VALUE PROBLEM USING DECOMPOSITION AND LOCALIZATION TECHNIQUES IN A CORNER

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Abstract. The subject of this work is the study of the singular behaviour of solutions for the Lamé system with Dirichlet, mixed and Neumann conditions in a bounded domain. A technique of localization of the problem in a corner is presented. The method is an adaptation of that of Kondratiev [8] extended to the weighted Sobolev spaces. This method have been considered by many authors.

1. **Introduction**

Questions of existence and uniqueness have been considered in Grisvard [6] for the Lamé system in the classical framework of weighted Sobolev spaces with weight in a polygon. The Sobolev spaces with double weight have been introduced in Dauge [3] for the Stokes system in a polygon.

In [1], Benseridi and Dilmi have used the complex Fourier transform with respect to the first variable in an infinite sector for a class of double weighted Sobolev spaces, to study problems of existence, unicity, regularity, and singularity of solutions of the Lamé system.

In their paper, Benseridi and Merouani [2], have studied some transmission problems related to the Lamé system in a polyhedron for a class of double weighted Sobolev spaces. They have given an explicit description of singularities of the variational solutions for the homogeneous case, by the same they have shown that the singular behaviour of the solutions is governed by a sequence of transcendental equations.

Here, we give an extension for some results previously obtained by the above mentioned authors. This paper is organized as follows: In section 1 we give some

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basic tools and properties related to the weighted Sobolev spaces which will be useful for the next. Section 2 is concerned with the notations and the formulation of our problem (P_1), while section 3, we study the regularity of the weak solution of the mixed problem (P_1) by using the technique of localization in a corner and this is done by means of the weighted Sobolev spaces. The solution is expressed as a some of a regular and a singular part. Finally, we state our main result by giving an explicit calculus of the singular functions that appear in the singular part of the solutions of the three problems (Dirichlet, Neumann and mixed). To do this, we compute the eigenvalues and the corresponding eigenvectors.

2. **Overview on the weighted Sobolev spaces**

In this section we give some basic tools and properties related to the weighted Sobolev spaces which will be useful in the next.

In what follows Ω is an infinite plane-sector of an opening ω

$$\Omega = \{(x, y) : x + iy = re^{i\theta}\ ,\ r > 0,\ 0 < \theta < \omega\}.$$

B is the strip defined by: $B = \mathbb{R}\times]0\,,\,\omega[$, θ_0, θ_∞ are two reals: $\theta_0 \leq \theta_\infty$.

Definition 2.1. *Let Ω be an open bounded set of $\mathbb{R}^n$ with closure $\overline{\Omega}$, and boundary Γ. Let $\rho \in C^\infty(\mathbb{R}^n)$, $\rho > 0$ on Ω, $\rho = 0$ on Γ and* gradiant(ρ) *is nonnull on Γ. For a positive integer l, α and p two real numbers such that $p > 1$, $W^{l,p}_\alpha(\Omega)$ is the Banach space of the distributions u on Ω such that $\rho^\alpha D^\beta u \in L^p(\Omega)$, for $|\beta| \leq l$, equipped with the norm*

$$\|u\|_{W^{l,p}_\alpha(\Omega)} = \left[\sum_{|\beta|\leq l} \left\|\rho^\alpha D^\beta u\right\|^p_{L^p(\Omega)}\right]^{\frac{1}{p}}.$$

Definition 2.2. *Let $s \in \mathbb{N}$, we define the space $V^s(B)$ by*

$$V^s(B) = \{u \in L^2(B)\ /\ (1+\xi^2)^{\frac{k}{2}} u \in L^2(\mathbb{R}\ H^{s-k}(]0,\omega[)\ ),\ k = 0, ..., s\}.$$

$V^s(B)$ is a Hilbert space for which the scalar product is given by

$$\langle u, v\rangle = \sum_{k=0}^{s} \int\int_B (1+\xi^2)^k \left|D^{s-k}_\theta u\right| \left|D^{s-k}_\theta v\right| d\theta d\xi.$$

Lemma 2.3. ([5, 8]). *Let $\eta_1, \eta_2 \in \mathbb{R}$ such that, $\eta_1 \leq \eta_2$. If $f \in L^2_{\eta_1,\eta_2}(B)$, then*

1) $\forall\eta \in [\eta_1, \eta_2]$, $e^{\eta\, t} f \in L^2(B)$, *and* $\left\|e^{\eta\, t} f\right\|_{L^2(B)} \leq \|f\|_{L^2_{\eta_1,\eta_2}(B)}$;

2) $\forall\eta \in]\eta_1, \eta_2[$, $e^{\eta\, t} f \in L^1(B)$, *and* $\left\|e^{\eta\, t} f\right\|_{L^1(B)} \leq c\,\|f\|_{L^2_{\eta_1,\eta_2}(B)}$.

Definition 2.4. *We denote by T the partial Fourier transform with respect to the first variable on B, then*

$$T\,(f)\,(\varphi,\theta) = T\left(e^{\eta\,t} f\right)(\xi,\theta), \text{ with } \varphi = \xi + i\eta,$$

where $T\,(e^{\eta\,t} f)$ denotes the real Fourier transform of $e^{\eta\,t} f$ with respect to the first variable.

Clearly f admits a complex Fourier transform, if and only if, $e^{\eta\,t} f$ admits a real Fourier transform.

For simplicity we write: $T\,(f)\,(\varphi,\theta) = \widehat{f}\,(\varphi,\theta)$.

Property 2.5. *Let $f \in H^{s}_{\eta_1,\eta_2}(B)$, then, for every $k, j \in \mathbb{N}$ such that $k + j \leq s$, we have*

$$T\left(\frac{\partial^{k+j}\, f}{\partial\, t^k\, \partial\, \theta^j}\right) = (i\varphi)^k\, \frac{\partial^j}{\partial\theta^j} T\,(f)\,(\varphi,\theta),\ (i^2 = -1),$$

for every φ in $\mathbb{C}$ and $\mathrm{Im}\varphi \in [\eta_1,\eta_2]$.

3. Notations and formulation of the problem

Ω denotes an homogeneous body, elastic and isotrope, occupying a bounded domain of $\mathbb{R}^2$ with a polygonal rectilignal boundary $\Gamma = \underset{j\in J}{\cup} \Gamma_j$, where Γ_j are open piecewise lines. $\{J_1, J_2\}$ is a partition of J, s_j will be the origin of Γ_{j+1}, and s_{j+1} its extremity according to the usual orientation.

The opening of the angle formed by Γ_j and Γ_{j+1} towards the interior of Ω will be denoted ω_j, with $0 < \omega_j < 2\pi$ for all $j \in J$. Ω then defined is consequently an open bounded domain with Lipschitz boundary. All results on this kind of domain are valid here.

It is more convenient to work at the origin with polar coordinates. Therefore by a translation first and then by a rotation, we can bring back s_j, Γ_j, Γ_{j+1} to O, OX, O_ω (ω is the angle formed by OX and O_ω towards the interior of Ω).

Our interest is to study the properties of regularity for a weak solution of the following mixed problem (Dirichlet-Neumann)

$$\begin{cases} \mu\Delta u + (\lambda+\mu)\,\nabla\,(\mathrm{div} u) = f \ \text{ in } \Omega \\ u = 0 \qquad\qquad \text{on } \underset{j\in J_1}{\cup} \Gamma_j \\ \sigma\,(u)\,.\tau = 0 \qquad \text{on } \underset{j\in J_2}{\cup} \Gamma_j \end{cases} \qquad (P_1)$$

where λ and μ are the elasticity coefficients with $\lambda > 0$ and $\lambda + \mu \geq 0$, (u), (f) designate respectively the displacement vector and the density of external powers. σ denote the stress tensor with $\sigma = (\sigma_{hk})$, h, $k = 1, 2$. The σ_{hk} elements are given by the Hooke's law

$$\sigma_{hk}(u) = 2\mu\varepsilon_{hk}(u) + \lambda \operatorname{div}(u)\delta_{hk},$$

where $\varepsilon_{hk}(u) = \frac{1}{2}(\partial_k u_h + \partial_h u_k)$ the symmetric deformation velocity tensor. τ is the normal vector.

Definition 3.1. *We denote by V the closure of the set*

$$\left\{v \in C^{\infty}\left(\overline{\Omega}\right)^2, \ v/_{\Gamma_j} = 0 \ \textit{ for every } j \in J_1\right\} \ \textit{in } H^1(\Omega)^2.$$

In order to define a weak solution, we introduce a symmetric bilinear form on V^2 by considering the scalar product of system (P_1). More explicitly

$$\begin{array}{rcl} l & : & V^2 \longrightarrow \mathbb{R} \\ (u,v) & \longmapsto & l(u,v) = -\displaystyle\int_{\Omega} \sum_{h=1}^{2}\sum_{k=1}^{2} \sigma_{hk}(u)\,\varepsilon_{hk}(v)\,dx. \end{array}$$

Definition 3.2. *The function $u \in V$ is a weak solution for problem (P_1) if*

$$l(u,v) = \int_{\Omega} \sum_{h=1}^{2} f_h v_h dx, \ \forall v \in V.$$

There is no particular problem to apply the variational method for the resolution of (P_1) because the Korn inequality is still valid in a polygon, moreover it is known that there exists a unique weak solution $u \in V$, if the bilinear form is bounded in V^2 and coercive. These conditions are verified if $mes\,(\underset{j\in J_1}{\cup}\Gamma_j) > 0$. When it is the Neumann problem ($J_1 = \emptyset$), we suppose that the necessary condition of existence is verified, the orthogonality of the rigid displacements data, i.e. $\int_{\Omega}\sum_{h=1}^{2} f_h v_h dx = 0$, for every v of the form $v(x,y) = (a + cy, b - cx)$, with a, b, c arbitrary reals.

4. **Localization of the problem in a corner**

The analysis of the existence, the unicity and the regularity for the boundary value problem (P_1) is more developed when the domain Ω is sufficiently smooth. Many results has been obtained by many authors. The principal regularity is in the interior of the domain Ω and on $\Gamma / \underset{j\in J}{\cup} V_j$, where V_j is a closed neighbourhood of a vertex s_j. (s_j), $j \in J$, are called singular points.

In the sequel, we only envisage the singular behaviour of the solution of (P_1) in a neighbourhood of a singular point, then we transpose the results to the weak solution; for this aim we consider the function $\rho(r)$ such that

$$\begin{aligned} 0 &\leq \rho(r) \leq 1,\ \rho(r) \in C^{\infty}(]0,\omega[) \\ \rho(r) &= \begin{cases} 1, & if \quad 0 \leq r \leq \delta \\ 0, & if \quad r \geq 2\delta \end{cases}, \end{aligned}$$

where δ is the smallest positive real for which no singular point of Γ is in the circle $\{x : |x| \leq 3\delta\}$. We denote by $\mathbb{K}$ an infinite plane sector.

We set $w = \rho u$, the problem (P_1) will be

$$\begin{cases} \mu \triangle w + (\lambda+\mu) \nabla (\mathrm{div} w) = F \ in \ \mathbb{K} & (4.1), \\ w = 0, on \ \Gamma_0 \cup \Gamma_\omega & (4.2), \\ \qquad or & \\ \begin{cases} w = 0 \ on \ \Gamma_0 \ \ and \, \sigma(w).\tau = 0, \ on \ \Gamma_\omega & (4.3), \\ \sigma(w).\tau = 0 \ \ on \ \ \Gamma_0 \cup \Gamma_\omega & (4.4), \end{cases} \end{cases} \qquad (P_2)$$

where F is depending on f, ρ and u.

Since we are interested with the solution in a neighbourhood of the vertex $(\rho = 1)$, we can suppose for simplicity, that F is an arbitrary given data (which does not depend on the solution u). Under this hypothesis, we have $F = \rho f$.

5. **The regularity in the weighted Sobolev spaces**

This section is concerned with the decomposition of the solution in a regular part and a singular part. We denote by $A(D_x)$ the differential operator for system (4.1)

$$A(D_x) = \begin{pmatrix} (\lambda+2\mu)\frac{\partial^2}{\partial x^2} + \mu\frac{\partial^2}{\partial y^2} & (\lambda+\mu)\frac{\partial^2}{\partial x \partial y} \\ (\lambda+\mu)\frac{\partial^2}{\partial x \partial y} & (\lambda+2\mu)\frac{\partial^2}{\partial y^2} + \mu\frac{\partial^2}{\partial x^2} \end{pmatrix},$$

and $B(D_x)$ the boundary operator (4.2), (4.3), (4.4).

For (4.4) we have

$$B(D_x) = \begin{pmatrix} (2\mu+\lambda)\tau_1\frac{\partial}{\partial x} + \mu\tau_2\frac{\partial}{\partial y} & \lambda\tau_1\frac{\partial}{\partial y} + \mu\tau_2\frac{\partial}{\partial x} \\ \mu\tau_1\frac{\partial}{\partial y} + \lambda\tau_2\frac{\partial}{\partial x} & \mu\tau_1\frac{\partial}{\partial x} + (2\mu+\lambda)\tau_2\frac{\partial}{\partial y} \end{pmatrix}.$$

Let $a(D_x) = [A(D_x), B(D_x)]$ be the operator defined by

$$a(D_x) : \ H^2_{\beta,\beta}(\mathbb{K})^2 \longrightarrow L^2_{\beta,\beta}(\mathbb{K})^2 \times H^{2-m-\frac{1}{2}}_{\beta,\beta}(\Gamma_0)^2 \times H^{2-m-\frac{1}{2}}_{\beta,\beta}(\Gamma_\omega)^2,$$

where m represents the order of the trace operator, $m = 0$ for the Dirichlet condition and $m = 1$ for the Neumann condition.

By passing to the polar coordinates, and we apply the complex Fourier transform with respect to the first variable, the boundary value problem $a\left(D_X\right) w = F$ will be

$$a(z, D_\theta)\widehat{\widehat{w}} = \widehat{\widehat{F}},$$

where

$$a(z, D_\theta) = [A(z, D_\theta), B(z, D_\theta)] : H^2\left(]0, \omega[\right)^2 \longrightarrow L^2\left(]0, \omega[\right)^2 \times \mathbb{C}^2 \times \mathbb{C}^2,$$

with $A(z, D_\theta) = (A_{hk})$, $i^2 = -1$ and

$$\begin{aligned}
A_{11} &= -\mu z^2 + (\lambda+\mu)\left(\left(\frac{-z^2}{2} - iz\right)\cos 2\theta - \frac{-z^2}{2}\right) + \\
&\quad (\lambda+\mu)(1-iz)\sin 2\theta \frac{d}{d\theta} + \left(\mu + (\lambda+\mu)\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)\right)\frac{d^2}{d\theta^2}. \\
A_{12} &= A_{21} = (\lambda+\mu)\left(\left(\frac{-z^2}{2} - iz\right)\sin 2\theta + (iz-1)\cos 2\theta \frac{d}{d\theta} - \frac{1}{2}\frac{d^2}{d\theta^2}\right). \\
A_{22} &= -\mu z^2 + (\lambda+\mu)\left(\left(\frac{z^2}{2} + iz\right)\cos 2\theta - \frac{z^2}{2}\right) + \\
&\quad (\lambda+\mu)(iz-1)\sin 2\theta \frac{d}{d\theta} + (\lambda+\mu)\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\frac{d^2}{d\theta^2}.
\end{aligned}$$

$B(z, D_\theta)$ is the boundary operator.

For condition (4.2) and $\theta = 0$, we get

$$B(z, D_\theta) = \begin{pmatrix} \dfrac{d}{d\theta} & iz \\ 2\nu iz & 2(1-\nu)\dfrac{d}{d\theta} \end{pmatrix},$$

where

$$\nu = \frac{\lambda}{2(\lambda+\mu)}.$$

Definition 5.1. *The complex number $z = z_0$ is called an eigenvalue of $a(z, D_\theta)$ if there exists a nontrivial solution $e^0(z_0, \theta) \in H^2\left(]0, \omega[\right)^2$ for the equation*

$$a(z, D_\theta) e(z, \theta)\,|_{z=z_0} = 0.$$

$e^0(z_0, \theta)$ is called the eigenvector of $a(z, D_\theta)$ corresponding to z_0. The function $e^1(z_0, \theta)$ is an associated vector to z_0 if

$$-i\frac{da(z, D_\theta)}{dz}\,|_{z=z_0}\, e^0(z_0, \theta) + a(z_0, D_\theta) e^1(z_0, \theta) = 0.$$

Theorem 5.2. *The operator $A(D_x)$ is an isomorphism if and only if $a(z, D_\theta)$ has no eigenvalue with imaginary part $\beta - 1$.*

Theorem 5.3. *Let θ_0, θ_∞ two reals such that $\theta_0 \leq \theta_\infty$. We suppose that the operator $a(z, D_\theta)$ has no eigenvalue on the lines $\mathbb{R} + i(\theta_0 - 1)$, $\mathbb{R} + i(\theta_\infty - 1)$, then for every $F \in L^2_{\theta_0,\theta_\infty}(\mathbb{K})^2$ the solution $w \in H^2_{\theta_\infty,\theta_\infty}(\mathbb{K})^2$ of problem (P_2) is written in the following form*

$$\overline{w}(r,\theta) = \sum_{l=1}^{N}\sum_{\sigma=1}^{I_l}\sum_{k=0}^{\delta_{\sigma\, l}} C_{\sigma k l}\overline{u}_{k,l}^{(\sigma)}(r,\theta) + \overline{V}(r,\theta),$$

where $V \in H^2_{\theta_0,\theta_0}(\mathbb{K})^2$, $z_1, z_2, .., z_N$ are the eigenvalues of $a(z, D_\theta)$ such that $\theta_0 - 1 \leq$ $\mathrm{Im} z_l \leq \theta_\infty - 1$,

$$I_l = \dim\left(span\left\{e_1^0(z_0,\theta),\ e_2^0(z_0,\theta), ...\right\}\right),$$

$$\delta_{\sigma\, l} = \begin{cases} 1, \text{ if an associated vector exists for } z_l \text{ and } e^0(z_0,\theta), \\ 0, \text{ otherwise.} \end{cases}$$

$C_{\sigma k l}$ are constants,

$$\overline{u}_{k,l}^{(\sigma)}(r,\theta) = r^{iz_l}\sum_{s=0}^{k}(\log r)^s e_\sigma^{k-s}(z_l,\theta)$$

are called singular functions.

Proof. See A. M. Sandig, U. Richter, R. Sandig [10]. □

We consider a weak solution $u \in V$ of problem (P_1).

Lemma 5.4. *Let $f \in L^2_{1+\varepsilon\,,1+\varepsilon}(\Omega)^2$, where ε is a small positive real, then*

$$\rho u \in H^2_{1+\varepsilon\,,1+\varepsilon}(\mathbb{K})^2.$$

Proof. We consider a sequence of domains Ω_h, $h = 1, 2, ...$ where $\Omega_h = \Omega \cap R_h$, with

$$R_h = \left\{x \,:\, \frac{\delta}{2^{h+1}} \leq |x| \leq \frac{\delta}{2^h}\right\}.$$

For $\widehat{\delta} = 2\delta$, we consider the function $\widehat{\rho}(r)$ such that

$$\begin{aligned} \widehat{\rho}(r) &\in C^\infty(]0,\infty[), 0 \leq \widehat{\rho}(r) \leq 1 \\ \widehat{\rho}(r) &= \begin{cases} 1 & \text{if } \ 0 \leq r \leq \widehat{\delta} \\ 0 & \text{if } \ r \geq 2\widehat{\delta} \end{cases} \end{aligned}$$

we have

$$\underset{h}{\cup}\Omega_h = \mathbb{K}_0 \subset \mathbb{K}.$$

The standard theorems of regularity give, for $|\gamma| = 2$

$$\iint\limits_{\Omega_h} |D^\gamma u|^2\, dx \leq c\left[\iint\limits_{\Omega_{h-1}\cup\ \Omega_h \cup\ \Omega_{h+1}} |f|^2\, dx + \iint\limits_{\Omega_{h-1}\cup\ \Omega_h\cup\ \Omega_{h+1}} r^{-4}|u|^2\, dx\right]. \tag{5.1}$$

Multiplying by $(\frac{\widehat{\delta}}{2^h})^{2(1+\varepsilon)}$, we obtain

$$\iint\limits_{\Omega_h} r^{2(1+\varepsilon)} \left|D^\gamma u\right|^2 dx \quad \leq \quad c(\iint\limits_{\Omega_{h-1}\cup\ \Omega_h\cup\ \Omega_{h+1}} r^{2(1+\varepsilon)} \left|f\right|^2 dx + $$

$$+ \iint\limits_{\Omega_{h-1}\cup\ \Omega_h\cup\ \Omega_{h+1}} r^{2(-1+\varepsilon)} \left|u\right|^2 dx)$$

and by summing with respect to h, from 0 to ∞, the inequality (5.1) becomes

$$\iint\limits_{\mathbb{K}_0} r^{2(1+\varepsilon)} \left|D^\gamma u\right|^2 dx \leq c \left[\iint\limits_{\mathbb{K}_0} r^{2(1+\varepsilon)} \left|f\right|^2 dx + \iint\limits_{\mathbb{K}_0} r^{2(-1+\varepsilon)} \left|u\right|^2 dx \right].$$

Clearly

$$\iint\limits_{\mathbb{K}_0} r^{2(-1+\varepsilon)} \left|u\right|^2 dx = \iint\limits_{\mathbb{K}_0} r^{2(-1+\varepsilon)} \left|\rho u\right|^2 dx.$$

Now, passing to polar coordinates and using Hardy inequality, we get

$$\int\limits_0^\infty \left|f(t)\right|^2 t^{(\varepsilon'-2)} dt \leq \left(\frac{2}{|\varepsilon'-1|}\right)^2 \int\limits_0^\infty \left|f'(t)\right|^2 t^{\varepsilon'} dt,$$

for $\varepsilon' > 1$ and $\lim\limits_{t\to\infty} f(t) = 0$. We get for $\overline{u}(r,\theta) = u(x,y)$

$$\iint\limits_{\mathbb{K}_0} r^{-2+2\varepsilon+1} \left|\widehat{\rho u}\right|^2 drd\theta \ \leq \int\limits_0^\omega \int\limits_0^\infty r^{-2+2\varepsilon+1} \left|\widehat{\rho u}\right|^2 drd\theta$$

$$\leq \int\limits_0^\omega (\frac{2}{2\varepsilon})^2 \int\limits_0^\infty r^{2\varepsilon} \left|\frac{\partial}{\partial r}\widehat{\rho u}\right|^2 rdrd\theta \leq \ c \iint\limits_{\Omega\ \cap\ \sup p\ \widehat{\eta}} r^{2\varepsilon} \left|u\right|^2 dx$$

$$+c \iint\limits_{\Omega\ \cap\ \sup p\ \widehat{\eta}} r^{2\varepsilon} (\left|\text{gradiant} u_1\right|^2 + \left|\text{gradiant} u_2\right|^2) dx \leq c \left\|u\right\|_{H^1(\Omega)^2}.$$

For $|\gamma| = 2$ we have

$$\iint\limits_{\mathbb{K}_0} r^{2(1+\varepsilon)} \left|D^\gamma \rho u\right|^2 dx \leq c \sum_{|\gamma'|\leq\ 2} \iint\limits_{\mathbb{K}_0} r^{2(1+\varepsilon)} \left|D^{\gamma'} \rho u\right|^2 dx,$$

therefore, $\rho u \in H^2_{1+\varepsilon\ ,1+\varepsilon}(\mathbb{K})^2$. □

We can now give the following theorem.

Theorem 5.5. ([10]) *Let $u \in V$ a weak solution of problem (P_1). Let ε a real positive small number such that the operator $a(z, D_\theta)$ has no eigenvalue with imaginary part ε or (-1). We suppose that $\rho f \in L^2(\mathbb{K})^2$, then*

$$\rho^2 \overline{u}(r,\theta) = \rho \sum_{l=1}^{N} \sum_{\sigma=1}^{I_l} \sum_{k=0}^{\delta_{\sigma\, l}} C_{\sigma k l} \overline{u}_{k,l}^{(\sigma)}(r,\theta) + \rho \overline{V}(r,\theta),$$

where $\rho V \in H^2(\mathbb{K})^2$.

6. Computation of the singular functions

Our goal is to compute the functions $\overline{u}_{k,l}^{(\sigma)}(r,\theta)$ for the three problems (Dirichlet, Neumann, and mixed). To do this, we have to know the eigenvalues z_l of $a(z, D_\theta)$, the corresponding eigenvectors, and the associated vectors.

6.1. Dirichlet problem.

Lemma 6.1. *If z_l is an eigenvalue of $a(z, D_\theta)$, for the angle ω, $\omega \notin \{\pi, 2\pi\}$, we get $I_l = 1$ and*

$$e_1^0(z_l,\theta) = C_3(z_l) Y_3(z_l,\theta) + C_4(z_l) Y_4(z_l,\theta),$$

where

$$Y_3(z_l,\theta) = \begin{pmatrix} -\cosh(z_l\theta) + \cosh(z_l + 2i)\theta \\ (1 - \frac{2(3-4\nu)}{iz_l}) i \sinh(z_l\theta) - i\sinh(z_l + 2i)\theta \end{pmatrix}.$$

$$Y_4(z_l,\theta) = \begin{pmatrix} -(1 + \frac{2(3-4\nu)}{iz_l}) i \sinh(z_l\theta) + i\sinh(z_l + 2i)\theta \\ -\cosh(z_l\theta) + \cosh(z_l + 2i)\theta \end{pmatrix}.$$

$$C_3(z_l) = -\cosh(z_l\omega) + \cosh(z_l + 2i)\omega.$$

$$C_4(z_l) = -\left(1 - \frac{2(3-4\nu)}{iz_l}) i \sinh(z_l\omega\right) + i\sinh(z_l + 2i)\omega.$$

For $\omega = \pi$ or $\omega = 2\pi$ we have

$$z_l = -il \text{ or } z_l = -\tfrac{il}{2},\ l = 1, 2, ...$$

For the two cases $I_l = 2$ and $e_1^0(z_l,\theta) = Y_3(z_l,\theta)$, $e_2^0(z_l,\theta) = Y_4(z_l,\theta)$ are two eigenvectors linearly independent.

Proof. Note that the eigenvectors of $a(z, D_\theta)$ are the zeros of the transcendental function $D_1(z)$ defined by

$$D_1(z) = 4\sin^2\omega + \left(\tfrac{2(3-4\nu)}{iz}\right)^2 \sinh^2(z\omega).$$

The general solution for the equation $A(z,D_\theta)e\,(z,\theta)=0$ is given by

$$
\begin{aligned}
e\,(z,\theta) \;=\;& C_1(z)\begin{pmatrix}\cosh(z\theta)\\ -i\sinh(z\theta)\end{pmatrix}+C_2(z)\begin{pmatrix} i\sinh(z\theta)\\ \cosh(z\theta)\end{pmatrix}+\\
& C_3(z)\begin{pmatrix}\cosh(z+2i)\theta\\ -i\sinh(z+2i)\theta-\frac{2(3-4\nu)}{iz}i\sinh(z\theta)\end{pmatrix}+\\
& C_4(z)\begin{pmatrix} i\sinh(z+2i)\theta\\ \cosh(z+2i)\theta+\frac{2(3-4\nu)}{iz}\cosh(z\theta)\end{pmatrix}.
\end{aligned}
$$

The condition $B(z_l,D_\theta)e\,(z_l,\theta)=0$ for $\theta=0$ shows that $C_1(z_l)=-C_3(z_l)$ and $C_2(z_l)=-C_4(z_l)(1+\frac{2(3-4\nu)}{iz_l})$. From the condition $B(z_l,D_\theta)e\,(z_l,\theta)=0$ for $\theta=\omega$, it comes that $M(z_l,\omega)C(z_l)=0$, where

$$
\begin{aligned}
M(z_l,\omega) \;=\;& \begin{pmatrix} -\cosh(z_l\omega) & -(1+\frac{2(3-4\nu)}{iz_l})i\sinh(z_l\omega)\\ -\frac{2(3-4\nu)}{iz_l}\,i\sinh(z_l\omega) & -\cosh(z_l\omega)\end{pmatrix}+\\
& \begin{pmatrix}\cosh(z_l+2i)\omega & i\sinh(z_l+2i)\omega\\ 0 & i\cosh(z_l+2i)\omega\end{pmatrix},\\
C(z) \;=\;& \begin{pmatrix} C_3(z_l)\\ C_4(z_l)\end{pmatrix}.
\end{aligned}
$$

The determinant of the matrix $M(z_l,\omega)$ is equal to $D_1(z_l)$ which is null, we can then choose $C_3(z_l),C_4(z_l)$ such that

$$C_4(z_l)=-(1-\frac{2(3-4\nu)}{iz_l})i\sinh(z_l\omega)+i\sinh(z_l+2i)\omega,$$
$$C_3(z_l)=-\cosh(z_l\omega)+\cosh(z_l+2i)\omega.$$

Replacing $C_3(z_l),C_4(z_l)$ by their values in the expression of solution $e\,(z,\theta)$, we obtain

$$e_1^0(z_l,\theta)=C_3(z_l)Y_3(z_l,\theta)+C_4(z_l)Y_4(z_l,\theta).$$

Now if $\omega=\pi$ or $\omega=2\pi$, the rank of the matrix of the system which results from the boundary condition $B(z_l,D_\theta)e\,(z_l,\theta)=0$ for $\theta=0$, $\theta=\omega$, is equal to 2, consequently $I_l=2$, we can choose $C_3(z_l)=0,C_4(z_l)=1$ or $C_3(z_l)=1,C_4(z_l)=0$, which proves that $e_1^0(z_l,\theta),e_2^0(z_l,\theta)$ are the linearly independant eigenvectors. □

Remark 6.2. *For $z=0$ we have $D_1(0)=2-2\cos 2\omega-4\,(3-4\nu)^2\,\omega^2$, then $D_1(0)$ is null if and only if $\omega=0$, consequently $z=0$ is not an eigenvalue of $a(z,D_\theta)$.*

In the sequel, we are going to study the correlation between the order of multiplicity of an eigenvalue z_l of the operator $a(z,D_\theta)$ and the existence of an associated vector. For this, we denote by $m(z_l)$ the order of multiplicity of z_l.

The two following propositions are similar to [10].

Proposition 6.3. *Denote by $m(z_l)$ the order of multiplicity of (z_l), then*

$$m(z_l) = \sum_{\sigma=1}^{I_l} (\delta_{\sigma l} + 1) \geq I_l.$$

Proposition 6.4. *If $m(z_l) = 2$ and $I_l = 1$, then there exists an associated unique vector and if $\omega = \pi$ or $\omega = 2\pi$ if there is any associated vector.*

Lemma 6.5. *Suppose that*

$$(H) \begin{cases} \tanh(z_l\omega) = \omega z_l, \\ \left(\dfrac{\sin\omega}{\omega}\right)^2 = [(3-4\nu)\cosh(z_l\omega)]^2, \\ \sinh(z_l\omega)\cosh(z_l\omega) \quad \text{is nonnull.} \end{cases}$$

Then $m(z_l) = 2$ and moreover the associated vector to the eigenvalue z_l is

$$e_1^1(z_l, \theta) = -i\frac{de_1^0(z,\theta)}{dz}\Big|_{z=z_l},$$

$e_1^0(z_l, \theta)$ is the eigenvector as defined in Lemma 6.1 by substituting z_l by z.

Proof. The hypothesis (H) is verified if and only if

$$D_1(z_l) = 0 \text{ and } D_1'(z_l) = 0.$$

Then $m(z_l) = 2$; which insures the existence of an associated vector.
We know that the associated vectors are the solutions of the equation

$$-i\frac{da(z, D_\theta)}{dz}\Big|_{z=z_l} e_1^0(z_l, \theta) + a(z_l, D_\theta)e_1^1(z_l, \theta) = 0 \tag{6.1}$$

i.e., for $z = z_l$

$$-i\frac{dA(z, D_\theta)}{dz} e_1^0(z, \theta) + A(z, D_\theta)e_1^1(z, \theta) = 0 \tag{6.2}$$

and

$$-i\frac{dB(z,\theta)}{dz} + B(z, D_\theta)e_1^1(z, \theta) = 0 \text{ for } \theta = 0,\ \theta = \omega.$$

$A(z, D_\theta)e_1^0(z, \theta) = 0$, for all z in a neighbourhood of z_l, then

$$\frac{d}{dz}\left[A(z, D_\theta)e_1^0(z, \theta)\right] = 0.$$

But

$$\frac{d}{dz}\left[A(z, D_\theta)e_1^0(z, \theta)\right] = A(z, D_\theta)\frac{de_1^0(z,\theta)}{dz} + \frac{dA(z, D_\theta)}{dz}e_1^0(z, \theta). \tag{6.3}$$

We multiply (6.2) by i, and then we compare it with (6.3), we find

$$e_1^1(z_l, \theta) = -i\frac{de_1^0(z,\theta)}{dz}\Big|_{z=z_l}.$$

In a similar way we prove that $\theta = 0$

$$-i\frac{dB(z, D_\theta)}{dz}\Big|_{z=z_l} + B(z_l, D_\theta)e_1^1(z_l, \theta) = 0.$$

For $\theta = \omega$, we have

$$B(z, D_\theta)e_1^0(z,\theta) = \begin{pmatrix} D_1(z_l) \\ 0 \end{pmatrix},$$

then

$$\left.\frac{dB(z, D_\theta)e_1^0(z,\theta)}{dz}\right|_{z=z_l} = \begin{pmatrix} D_1'(z_l) \\ 0 \end{pmatrix}.$$

On the other hand

$$\begin{aligned} \left.\frac{dB(z, D_\theta)e_1^0(z,\theta)}{dz}\right|_{z=z_l} &= \left.\frac{dB(z, D_\theta)}{dz}\right|_{z=z_l} e_1^0(z_l,\theta) + B(z_l, D_\theta)\left.\frac{de_1^0(z_l,\theta)}{dz}\right|_{z=z_l} \\ &= i\left[-i\left.\frac{dB(z, D_\theta)}{dz}\right|_{z=z_l} + B(z_l, D_\theta)e_1^1(z_l,\theta)\right] = 0. \end{aligned}$$

The proof is complete. □

The following theorem gives a summary for the results concerning the singular functions.

Theorem 6.6. *The singular functions of the weak solution $u \in V$ of the Dirichlet problem are given as follows:*
(1) *If $\omega \notin \{\pi, 2\pi\}$ and z_l is a simple nonnull zero of $D_1(z)$, then there exists a unique singular function*

$$\overline{u}_{0,l}^{(1)}(r,\theta) = r^{iz_l} e_1^0(z_l,\theta).$$

(2) *If $\omega \notin \{\pi, 2\pi\}$ and z_l is a nonnull double zero of $D_1(z)$, then there exist two singular functions*

$$\begin{aligned} \overline{u}_{0,l}^{(1)}(r,\theta) &= r^{iz_l} e_1^0(z_l,\theta), \\ \overline{u}_{1,l}^{(1)}(r,\theta) &= r^{iz_l}\left[e_1^1(z_l,\theta) + (\log r)e_1^0(z_l,\theta)\right]. \end{aligned}$$

(3) *If $\omega = \pi$, then $z_l = -il$, $l = 1, 2, ...$*

$$\begin{aligned} \overline{u}_{0,l}^{(1)}(r,\theta) &= r^l e_1^0(z_l,\theta), \\ \overline{u}_{0,l}^{(2)}(r,\theta) &= r^l e_2^0(z_l,\theta). \end{aligned}$$

(4) *If $\omega = 2\pi$, then $z_l = -\frac{il}{2}$, $l = 1, 2, ...$*

$$\begin{aligned} \overline{u}_{0,l}^{(1)}(r,\theta) &= r^{\frac{l}{2}} e_1^0(z_l,\theta), \\ \overline{u}_{0,l}^{(2)}(r,\theta) &= r^{\frac{l}{2}} e_2^0(z_l,\theta). \end{aligned}$$

Remark 6.7. *Notice that in* (1) *and* (2)

$$e_1^0(z_l,\theta) = C_3(z_l)Y_3(z_l,\theta) + C_4(z_l)Y_4(z_l,\theta),$$

and in (3) *and* (4)

$$e_1^0(z_l,\theta) = Y_3(z_l,\theta),\ e_2^0(z_l,\theta) = Y_4(z_l,\theta).$$

6.2. **The mixed problem.** We have previously found the expression for the general solution of equation $A(z, D_\theta)e\,(z,\theta) = 0$.

The Dirichlet-Neumann condition $B(z, D_\theta)e\,(z,\theta) = 0$ for $\theta = 0$ and $\theta = \omega$ gives a system of Cramer of order 4 with determinant

$$D_2(z) = -16\mu\left[z^2\sin^2\omega + 4\,(1-\nu)^2 + (3-4\nu)\sinh^2(z\omega)\right],\ \forall\, z,\ z \neq 0.$$

Therefore the eigenvalues of the operator $a(z, D_\theta)$ are the zeros of the transcendental equation

$$z^2\sin^2\omega + 4\,(1-\nu)^2 + (3-4\nu)\sinh^2(z\omega) = 0.$$

Remark 6.8. *If $z = 0$, then the determinant $D_2\,(0)$ is given by*

$$D_2(0) = 4(1-(3-4\nu)^2)(\lambda + 2\mu).$$

In this case there is no eigenvalue.

Lemma 6.9. *If z_l is a zero of $D_2\,(z)$ then $I_l = 1$ and*

$$e_1^0(z_l,\theta) = C_3(z_l)Y_3(z_l,\theta) + C_4(z_l)Y_4(z_l,\theta)$$

is an eigenvector, where

$$\begin{aligned}
Y_3(z_l,\theta) &= \begin{pmatrix} \left(\frac{-4(1-\nu)}{iz_l} - 1\right)\cosh(z_l\omega) + \cosh(z_l+2i)\omega \\ \left(\frac{-2(1-2\nu)}{iz_l} + 1\right) i\sinh(z_l\omega) - i\sinh(z_l+2i)\omega \end{pmatrix}, \\
Y_4(z_l,\theta) &= \begin{pmatrix} \left(\frac{-2(1-2\nu)}{iz_l} - 1\right) i\sinh(z_l\omega) + i\sinh(z_l+2i)\omega \\ \left(\frac{4(1-\nu)}{iz_l} - 1\right)\cosh(z_l\omega) + \cosh(z_l+2i)\omega \end{pmatrix}, \\
C_3(z_l) &= \left(\frac{4(1-\nu)}{iz_l} - 1\right)\cosh(z_l\omega) + \cosh(z_l+2i)\omega, \\
C_4(z_l) &= \left(\frac{2(1-2\nu)}{iz_l} - 1\right) i\sinh(z_l\omega) + i\sinh(z_l+2i)\omega.
\end{aligned}$$

Proof. The rank of the matrix of $D_2\,(z)$ is equal to 3, consequently I_l=1. We use the same idea for the proof as in Lemma 6.1. We consider the general solution $e(z_l,\theta)$ of equation $A(z_l, D_\theta)e\,(z_l,\theta) = 0$, and we determine the constants $C_1(z_l)$, $C_2(z_l)$, $C_3(z_l)$ and $C_4(z_l)$ such that they verify the boundary condition $B(z_l, D_\theta)e\,(z_l,\theta) = 0$ for $\theta = 0$ and $\theta = \omega$.

Finally, we obtain the result $e_1^0(z_l,\theta) = C_3(z_l)Y_3(z_l,\theta) + C_4(z_l)Y_4(z_l,\theta)$.

Now we seek for the associated vectors.

Lemma 6.10. *Let z_l be a zero of $D_2\,(z)$.*

(1) If $m\,(z_l) = 2$, then the associated vectors exist.

(2) The equalities

$$\begin{aligned}(4\nu-3)\frac{\sinh(z_l\omega)\cosh(z_l\omega)}{z_l\omega} &= \frac{\sin^2\omega}{\omega^2},\\ (z_l\omega)\ \sinh(z_l\omega)\cosh(z_l\omega) &= \sinh^2(z_l\omega)+\frac{4(1-\nu)^2}{(3-4\nu)},\end{aligned}$$

are necessary and sufficient so that $m\,(z_l)=2$.
(3) The associated vectors are given by

$$e_1^1(z_l,\theta)=-i\frac{de_1^0(z,\theta)}{dz}\Big|_{z=z_l},$$

where $e_1^0(z,\theta)$ *is the eigenvector defined in the previous lemma by replacing* z_l *by* z.

Proof. (1) From proposition (1).

(2) The two equations are verified if and only if $D_2\,(z_l)=D_2'\,(z_l)=0$.

(3) The proof is similar to that of Lemma 6.5.

The following theorem is similar to Theorem 6.6 for the mixed problem.

Theorem 6.11. *The singular functions of the weak solution* $u\in V$ *for the mixed problem have the following forms*
(1) If z_l *is a simple zero of* $D_2\,(z)$, *then there exists only one singular function*

$$\overline{u}_{0,l}^{(1)}(r,\theta)=r^{iz_l}e_1^0\,(z_l,\theta)\,.$$

(2) If z_l *is a double zero of* $D_2\,(z)$, *then there exist two singular functions*

$$\begin{aligned}\overline{u}_{0,l}^{(1)}(r,\theta) &= r^{iz_l}e_1^0\,(z_l,\theta)\,,\\ \overline{u}_{1,l}^{(1)}(r,\theta) &= r^{iz_l}\left[\ e_1^1(z_l,\theta)+(\log r)e_1^0\,(z_l,\theta)\right],\end{aligned}$$

where $e_1^0\,(z_l,\theta)$ *is the eigenvector defined in Lemma 6.9 and* $e_1^1(z_l,\theta)$ *is the associated vector defined in Lemma 6.10.*

6.3. **Neumann problem.** In this case we consider the boundary conditions of Neumann $B(z,D_\theta)e\,(z,\theta)=0$. These conditions give a system of four equations with determinant is

$$D_3(z)=-32\mu^2z^2\left[-z^2\sin^2\omega+\sinh^2(z\omega)\right],\ \forall\ z,\ z\neq 0.$$

Therefore the boundary problem have a nontrivial solution, if and only if $D_3(z)=0$, consequently the eigenvalues of the operator $a(z,D_\theta)$ are the zeros of $D_3(z)$.

Remark 6.12. *For* $z=0$, *we have* $D_3(0)=0$ *for all* ω.

Lemma 6.13. *(1) We suppose that* z_l *is a zero of* $D_3(z)$, $z_l\notin\{0,-i\}$, $\omega\notin\{\pi,2\pi\}$, *then* $I_l=1$ *and*

$$e_1^0(z_l,\theta)=C_3(z_l)Y_3(z_l,\theta)+C_4(z_l)Y_4(z_l,\theta)$$

is an eigenvector, where

$$Y_3(z_l,\theta)=\begin{pmatrix} \left(\frac{-4(1-\nu)}{iz_l}-1\right)\cosh(z_l\omega)+\cosh(z_l+2i)\omega \\ \left(\frac{-2(1-2\nu)}{iz_l}+1\right)i\sinh(z_l\omega)-i\sinh(z_l+2i)\omega \end{pmatrix},$$

$$\begin{aligned} Y_4(z_l,\theta) &= \begin{pmatrix} \left(\frac{-2(1-2\nu)}{iz_l}-1\right)i\sinh(z_l\omega)+i\sinh(z_l+2i)\omega \\ \left(\frac{4(1-\nu)}{iz_l}-1\right)\cosh(z_l\omega)+\cosh(z_l+2i)\omega \end{pmatrix}, \\ C_3(z_l) &= \mu(z_l+2i)\sinh(z_l\omega)-z_l\sinh(z_l+2i)\omega, \\ C_4(z_l) &= iz_l\left[\cosh(z_l\omega)-\cosh(z_l+2i)\omega\right]. \end{aligned}$$

(2) The number $z_l=(-i)$ is an eigenvalue of $a(z,D_\theta)$ for all $\omega\in\,]0,2\pi]$ and its eigenvector is

$$e_1^0(-i,\theta)=\begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}.$$

(3) If $\omega=\pi$ (resp. $\omega=2\pi$), then $z_l=-il$ (resp. $z_l=\frac{-il}{2}$), $l=1,2,...,$ and $I_l=2$. The eigenvectors are

(a) If $z_l=-i$

$$e_1^0(-i,\theta)=\begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix},\ e_2^0(-i,\theta)=Y_3(-i,\theta).$$

(b) If z_l is different from $-i$

$$e_1^0(z_l,\theta)=Y_4(z_l,\theta)\ ,\ e_2^0(z_l,\theta)=Y_3(z_l,\theta).$$

Proof. (1) We consider the general solution $e(z,\theta)$ for the equation

$$A(z,D_\theta)e(z,\theta)=0,$$

then using conditions $B(z_l,D_\theta)e(z_l,\theta)=0$, for $\theta=0$ and $\theta=\omega$, we obtain (1).
(2) It is easy to check that $z_l=(-i)$ is a zero of $D_3(z)$ for all $\omega\in\,]0,2\pi[$.

The boundary conditions $B(z_l,D_\theta)e(z_l,\theta)=0$ for $\theta=0,\theta=\omega$ give the following system

$$\begin{pmatrix} -4\sin\omega & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C_3 \\ C_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.4}$$

We choose $C_3=0, C_4=\frac{-1}{4(1-\nu)}$, we obtain

$$e_1^0(-i,\theta)=\begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}.$$

(3) When $\omega = \pi$ or $\omega = 2\pi$, the matrix in (6.4) will be null, we can then choose $C_3 = 1, C_4 = 0$ or $C_3 = 0, C_4 = 1$ thus (a). The proof of part (b) is similar to the last part in Lemma 6.1.

The following lemma illustrates the correlation between the order of multiplicity of an eigenvalue and the existence of an associated vector.

Lemma 6.14. *(1) If z_l is different from $-i$, $I_l = 1$ and $m(z_l) = 2$, then there exists an associated vector.*
(2) The conditions

$$\begin{aligned} \tanh(z_l\omega) &= \tanh(z_l\omega), \\ \cosh^2(z_l\omega) &= \frac{\sin^2\omega}{\omega^2},\ z_l \neq -i, \end{aligned}$$

are necessary and sufficient to $m(z_l) = 2$, and in this case the associated vectors are given by

$$e_1^1(z_l, \theta) = -i\frac{de_1^0(z,\theta)}{dz}\Big|_{z=z_l},$$

where $e_1^0(z, \theta)$ is the eigenvector defined in part (1) of the previous lemma by replacing z_l by z.

To prove this lemma, it suffices to compare with Lemma 6.5 and Lemma 6.10.

To close this section, we give a similar theorem as in 6.11 which corresponds to the Neumann case.

Theorem 6.15. *The singular functions of the weak solution $u \in V/\mathrm{Im}$ of the Neumann problem are*
(1) If $\omega \notin \{\pi, 2\pi\}$ and z_l is a simple zero of $D_3(z)$, then there exists a unique singular function

$$\overline{u}_{0,l}^{(1)}(r,\theta) = r^{iz_l} e_1^0(z_l, \theta).$$

(2) If $\omega \notin \{\pi, 2\pi\}$ and z_l is a double zero of $D_3(z)$, then there exist two singular functions

$$\begin{aligned} \overline{u}_{0,l}^{(1)}(r,\theta) &= r^{iz_l} e_1^0(z_l, \theta), \\ \overline{u}_{1,l}^{(1)}(r,\theta) &= r^{iz_l}\left[\, e_1^1(z_l,\theta) + (\log r) e_1^0(z_l,\theta)\right]. \end{aligned}$$

(3) If $\omega = \pi$, then $z_l = -il$, $l = 1, 2, \ldots$, and

$$\begin{aligned} \overline{u}_{0,l}^{(1)}(r,\theta) &= r^l e_1^0(-il, \theta), \\ \overline{u}_{0,l}^{(2)}(r,\theta) &= r^l e_2^0(-il, \theta). \end{aligned}$$

(4) If $\omega = 2\pi$*, then* $z_l = \frac{-il}{2}$*,* $l = 1, 2, \ldots$*, and*

$$\overline{u}_{0,l}^{(1)}(r,\theta) = r^{\frac{l}{2}} e_1^0 \left(\frac{-il}{2}, \theta \right),$$

$$\overline{u}_{0,l}^{(2)}(r,\theta) = r^{\frac{l}{2}} e_2^0 \left(\frac{-il}{2}, \theta \right).$$

Remark 6.16. *Note that*

- *In (1) and (2),* $e_1^0(z_l,\theta) = C_3(z_l)Y_3(z_l,\theta) + C_4(z_l)Y_4(z_l,\theta)$.
- *In (3) and (4),* $e_1^0(z_l,\theta) = Y_4(z_l,\theta)$, $e_2^0(z_l,\theta) = Y_3(z_l,\theta)$.
- $\mathrm{Im} = \mathit{span}\{(0,\ 1),\ (0,\ 1),\ (-x_2,\ x_1)\}$.

Aknowledgment. This work is dedicated to the memory of our colleague N. Mosbah.

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SIMPLE CRITERIA FOR STARLIKENESS OF ORDER β

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Abstract. In this paper we obtain a new criterion for starlikeness of order β for an analytic function $f \in \mathcal{A}_n$. This criterion involves only the second derivative of the given function and generalizes a well-known result due to P. T. Mocanu.

1. **Introduction**

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in the unit disc

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

For n a positive integer and $a \in \mathbb{C}$ let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + ...\}.$$

Let $\mathcal{A}_n$ denote the class of functions

$$f(z) = z + a_{n+1}z^{n+1} + ..., n \geq 1$$

that are analytic on the unit disc and let $\mathcal{A}_1 = \mathcal{A}$.

Let $\mathcal{D}$ be a domain in $\mathbb{C}$. A function $f : \mathcal{D} \to \mathbb{C}$ is called univalent on $\mathcal{D}$ if $f \in \mathcal{H}(\mathcal{D})$ and f is injective on $\mathcal{D}$.

The analytic function f, with $f(0) = 0$ and $f^{'}(0) \neq 0$ is starlike on U (i.e. f is univalent on U and $f(U)$ is starlike with respect to origin) if and only if $\Re\left[\frac{zf^{'}(z)}{f(z)}\right] > 0$, for $z \in U$.

An analytic function f with $f(0) = 0$ and $f^{'}(0) \neq 0$ is starlike of order β, $\beta \geq 0$ if and only if $\Re\left[\frac{zf^{'}(z)}{f(z)}\right] > \beta$, for $z \in U$, $\beta \geq 0$.

Let denote S^* and $S^*(\beta)$ the subclasses of $\mathcal{A}$ consisting of functions f which are starlike and starlike of order β.

Let $\mathcal{D}$ be a domain in $\mathbb{C}$. A function $f : \mathcal{D} \to$ is convex on $\mathcal{D}$ if f is univalent on $\mathcal{D}$ and $f(\mathcal{D})$ is a convex domain in $\mathbb{C}$.

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If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$,or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$, for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

We shall use the following results to prove our main results.

Lemma 1.1. [8] *Let h be a starlike function with $h(0) = 0$. If the function $p \in \mathcal{H}[a, n]$ satisfies the differential subordination*

$$zp'(z) \prec h(z) \tag{1.1}$$

then

$$p(z) \prec q(z) = a + \frac{1}{n}\int_0^z \frac{h(t)}{t}dt.$$

Function q is the best (a,n)-dominant of subordination.

Lemma 1.2. [3] *Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}^*$ with $\Re\gamma \geq 0$. If the function $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma}zp'(z) \prec h(z) \tag{1.2}$$

then

$$p(z) \prec q(z) \prec h(z)$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^1 h(t)t^{\frac{\gamma}{n}-1}dt. \tag{1.3}$$

Lemma 1.3. [4] *Let be a function $p \in \mathcal{H}[a, n]$.*

1. *If $\Psi \in \Psi_n\{\Omega, a\}$ then*

$$\Psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \Rightarrow \Re\, p(z) > 0, z \in U$$

2. *If $\Psi \in \Psi_n\{\Omega, a\}$ then*

$$\Re\, \Psi(p(z), zp'(z), z^2p''(z); z) > 0, z \in U \Rightarrow \Re\, p(z) > 0, z \in U$$

Lemma 1.4. [7] *Let n be a positive integer and*

$$\alpha_n = \frac{n+2}{C_n} \tag{1.4}$$

where

$$C_n = 2\Big[1 + \frac{n+2}{n}\ln 2 - \int_0^1 \frac{t^{\frac{1}{n}}}{1+t}dt\Big]. \tag{1.5}$$

If $f \in \mathcal{A}_n$ and

$$\Re\big[zf''(z)\big] > -\alpha_n, z \in U \tag{1.6}$$

then $f \in S^*$.

In this paper we obtain a new criterion for starlikeness of order β for an analytic function $f \in \mathcal{A}_n$. This criterion involves only the second derivative of the given function and generalizes a well-known result due to P.T. Mocanu.

2. Main results

Theorem 2.1. *Let n be a positive integer, $\beta \in [0, \frac{1}{2}]$ and*

$$\alpha_n(\beta) := \frac{(n+2)-(n+4)\beta}{2\left[\frac{n+2}{n}\ln 2 - \frac{n+4}{n}\beta ln2 - (1-\beta)\int_0^1 \frac{t^{\frac{1}{n}}}{1+t}dt + 1\right]}.$$

If $f \in \mathcal{A}_n$ and

$$\Re\left[zf^{"}(z)\right] > -\alpha_n(\beta), z \in U, n \in \mathbb{N}$$

then $f \in S^*(\beta)$.

Proof. We will show first that f is univalent on U. From the definition of $\alpha_n = \alpha_n(\beta)$ we have that $\alpha_n(\beta) > 0$. If $\alpha \in [0, \alpha_n]$ the inequality $\Re[zf''(z)] > -\alpha$, $z \in U$ is equivalent with the following subordination

$$zf''(z) \prec -\frac{2\alpha z}{1+z} = h(z).$$

Since the function f is starlike and $f' \in \mathcal{H}[1, n]$ by applying Lemma 1.1 we obtain that

$$f''(z) \prec 1 + \frac{1}{n}\int_0^z \frac{h(t)}{t}dt = 1 - \frac{2\alpha}{n}log(1+z) = q(z)$$

where the function q is convex.

Due to the fact that the function q is convex and has real coefficients we get that:

$$\Re f'(z) > \gamma = \gamma(\alpha) = q(1) = 1 - \frac{2\alpha}{n}\ln 2, z \in U. \tag{2.1}$$

We prove the following inequality:

$$\alpha_n \leq \frac{n}{\ln 4}. \tag{2.2}$$

We have:

$$\begin{aligned} \alpha_n &= \frac{(n+2)-(n+4)\beta}{2\left[\frac{n+2}{n}\ln 2 - \frac{n+4}{n}\beta\ln 2 - (1-\beta)\int_0^1 \frac{t^{\frac{1}{n}}}{1+t}dt + 1\right]} \\ &\leq \frac{n+2}{2\frac{n+2}{n}\ln 2} = \frac{n}{2\ln 2} = \frac{n}{\ln 4}, \end{aligned}$$

as desired.

Since $\alpha_n \leq \frac{n}{\ln 4}$ then

$$\Re f'(z) > \gamma(\alpha) \geq 0, z \in U. \tag{2.3}$$

So f is univalent on U.

Next, we prove that $f \in S^*(\beta)$. If

$$P(z) := \frac{f(z)}{z} \tag{2.4}$$

then P satisfies the differential subordination

$$zP'(z) + P(z) = f'(z) \prec q(z).$$

From the previous relation, by using Lemma 1.2 for $\gamma = 1$ we obtain the exact subordination $P(z) \prec Q(z)$, where the function Q is convex and is defined by

$$Q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z q(t) t^{\frac{1}{n}-1} dt = 1 - \frac{2\alpha}{n^2 t^{\frac{1}{n}}} \int_0^1 t^{\frac{1}{n}-1} log(1+t) dt. \tag{2.5}$$

Because the function Q is convex, from differential subordination $P \prec Q$ we have that

$$\Re\, P(z) > \delta = \delta(\alpha) = Q(1) = 1 - \frac{2\alpha}{n}\Big[\ln 2 - \int_0^1 \frac{t^{\frac{1}{n}}}{1+t} dt\Big]. \tag{2.6}$$

If we denote by

$$p(z) := \frac{\frac{zf'(z)}{f(z)} - \beta}{1-\beta} \tag{2.7}$$

then

$$zf'(z) = (1-\beta)p(z)f(z) + \beta f(z).$$

By differentiating the previous equality we get that:

$$zf''(z) + (1-\beta)f'(z) = (1-\beta)p'(z)f(z) + (1-\beta)p(z)f'(z)$$

and hence

$$zf''(z) + (1-\beta)f'(z) = \frac{f(z)}{z}\Big[(1-\beta)zp'(z) + (1-\beta)p(z)\frac{zf'(z)}{f(z)}\Big].$$

The previous equality can also be written as

$$zf''(z) + (1-\beta)f'(z) = P(z)[(1-\beta)zp'(z) + (1-\beta)^2 p^2(z) + \beta(1-\beta)p(z)] \tag{2.8}$$

where by P we denoted the function $P(z) = \frac{f(z)}{z}$.

Since

$$\beta(1-\beta)p(z)P(z) = \beta f'(z) - \beta^2 P(z) \tag{2.9}$$

the equality (2.8) becomes

$$zf''(z) + (1-2\beta)f'(z) = P(z)[(1-\beta)zp'(z) + (1-\beta)^2 p^2(z) - \beta^2]. \tag{2.10}$$

It is obvious that

$$\begin{aligned} \Re[zf''(z)+(1-2\beta)f'(z)] &= \Re[zf''(z)]+(1-2\beta)\Re f'(z) \\ &> -\alpha+(1-2\beta)\gamma(\alpha). \end{aligned} \tag{2.11}$$

By using the first part of Lemma 1.3 and the inequality (2.11) we will show that $\Re\, p(z) > 0$, $z \in U$.

In order to do that it is sufficient to show that the function

$$\Psi(r,s,z) = P(z)[(1-\beta)s+(1-\beta)^2r^2-\beta^2]$$

is an admissible function.

We have that

$$\begin{aligned} \Re\ \Psi(\delta i,\sigma,z) &= \Re\{P(z)[(1-\beta)\sigma-(1-\beta)^2\delta^2-\beta^2]\} = \\ &= [(1-\beta)\sigma-(1-\beta)^2\delta^2-\beta^2]\Re\ P(z) \\ &\leq -\alpha+(1-2\beta)\gamma(z) \end{aligned} \tag{2.12}$$

By using Lemma 1.3 we want to show that $\Re\ P(z) > 0, z \in U$.

Since,

$$\sigma \leq -\frac{n(1+\delta^2)}{2}, \delta,\sigma \in \mathbb{R}. \tag{2.13}$$

Next, we will verify that $0 \leq \alpha \leq \alpha_n, \Re\ P(z) > 0, z \in U$. By using the relation (2.13) we obtain that

$$[(1-\beta)\sigma-(1-\beta)^2\delta^2-\beta^2]\Re\ P(z) \leq \Big[-\frac{(1-\beta)n}{2}(1+\delta^2)-(1-\beta)^2\delta^2-\beta^2\Big]$$

$$\Re\ P(z) = -\frac{(1-\beta)n}{2}\Re\ P(z) - \Big[\frac{n(1-\beta)}{2}\delta^2+(1-\beta)^2\delta^2+\beta^2\Big]$$

$$\Re\ P(z) \leq -\frac{(1-\beta)n}{2}\Re\ P(z) = \frac{n(1-\beta)}{2}[-\Re\ P(z)] \leq -\frac{n(1-\beta)}{2}\delta(\alpha).$$

In order that the relation (2.12) to be satisfied it is sufficient that the following inequality to be true:

$$-\frac{n(1-\beta)}{2}\delta(z) \leq -\alpha+(1-2\beta)\gamma(\alpha).$$

If $\alpha \leq \alpha_0$ then the previous inequality is satisfied.

By using Lemma 1.3 and the relation (2.11) we obtain that $\Re\, p(z) > 0, z \in U$.

Hence, applying the analytical characterization for starlike functions of order β we proved that $f \in S^*(\beta)$.

If we take $\beta = 0$ in Lemma 2.1 we obtain a well-known result, due to P.T. Mocanu [7].

Next, for $n = 1$, we obtain the following particular result.

Corolarry 2.2. *Let* $\beta \in [0, \frac{1}{2}]$ *and* $\alpha_1(\beta) := \frac{3-5\beta}{2[4\ln 2 - 6\beta \ln 2 + \beta]}$. *If* $f \in \mathcal{A}_1$ *and* $\Re\big[zf''(z)\big] > -\alpha_1(\beta), z \in U$ *then* $f \in S^*(\beta)$.

Proof. We put $n = 1$ in the previous result and obtain:

$$\begin{cases} \delta(1) = 1 - 2\alpha[\ln 2 - \int_0^1 \frac{t}{1+t}dt] = 1 - 2\alpha[2\ln 2 - 1] \\ \gamma(1) = 1 - 2\alpha \ln 2 \end{cases}$$

We have that:

$$\begin{cases} \delta(1) = 1 - 2\alpha[2\ln 2 - 1] \\ \gamma(1) = 1 - 2\alpha \ln 2 \end{cases}$$

By using the following equality: $-\frac{1-\beta}{2}\delta(1) = -\alpha + (1-2\beta)\gamma(1)$ we obtain

$$-\frac{1-\beta}{2} + 2\alpha(1-\beta)\ln 2 - \alpha(1-\beta) = -\alpha + 1 - 2\alpha\ln 2 - 2\beta + 4\alpha\beta\ln 2$$

and hence

$$\alpha = \frac{3-5\beta}{2[4\ln 2 - 6\beta\ln 2 + \beta]}, \alpha = \alpha_1(\beta)$$

where

$$\alpha_1(\beta) = \frac{3-5\beta}{2[4\ln 2 - 6\beta\ln 2 + \beta]} \le \frac{1}{2\ln 2}.$$

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THE ORDER OF CONVEXITY OF TWO INTEGRAL OPERATORS

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Abstract. In this paper, we obtain the order of convexity of the integral operators $\int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\beta_i}} dt$ and $\int_0^z \left(te^{f(t)}\right)^{\gamma} dt$, where f_i and f satisfy the condition $\left| f'(z) \left(\frac{z}{f(z)}\right)^{\mu} - 1 \right| < 1 - \alpha$.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1.1)$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. A function $f(z)$ belonging to $\mathcal{S}$ is said to be starlike of order α if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \qquad (1.2)$$

for some $\alpha (0 \leq \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order α in $\mathcal{U}$. Also, a function $f(z)$ belonging to $\mathcal{S}$ is said to be convex of order α if it satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \qquad (1.3)$$

for some $\alpha (0 \leq \alpha < 1)$. We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are convex of order α in $\mathcal{U}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}(\alpha)$ iff

$$\operatorname{Re}\left(f'(z)\right) > \alpha, \qquad (z \in \mathcal{U}). \qquad (1.4)$$

It is well known that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}$.

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Very recently, Frasin and Jahangiri [4] define the family $\mathcal{B}(\mu,\alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$ so that it consists of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1 \right| < 1-\alpha \qquad (z \in \mathcal{U}). \tag{1.5}$$

The family $\mathcal{B}(\mu,\alpha)$ is a comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, $\mathcal{B}(1,\alpha)\equiv\mathcal{S}^*(\alpha)$, and $\mathcal{B}(0,\alpha)\equiv\mathcal{R}(\alpha)$. Another interesting subclass is the special case $\mathcal{B}(2,\alpha) \equiv \mathcal{B}(\alpha)$ which has been introduced by Frasin and Darus [3](see also [1, 2]).

In this paper, we will obtain the order of convexity of the following integral operators:

$$\int_0^z \left(\frac{f_1(t)}{t}\right)^{\frac{1}{\beta_1}} \cdots \left(\frac{f_n(t)}{t}\right)^{\frac{1}{\beta_n}} dt \tag{1.6}$$

and

$$\int_0^z \left(te^{f(t)}\right)^{\gamma} dt \tag{1.7}$$

where the functions $f_1(t), f_2(t), ..., f_n(t)$ and $f(t)$ are in $\mathcal{B}(\mu,\alpha)$.

In order to prove our main results, we recall the following lemma:

Lemma 1.1. *(Schwarz Lemma). Let the analytic function $f(z)$ be regular in the unit disc $\mathcal{U}$, with $f(0) = 0$. If $|f(z)| \leq 1$, for all $z \in \mathcal{U}$, then*

$$|f(z)| \leq |z|, \qquad \text{for all } z \in \mathcal{U}$$

and equality holds only if $f(z) = \varepsilon z$, where $|\varepsilon| = 1$.

2. Main results

Theorem 2.1. *Let $f_i(z) \in \mathcal{A}$ be in the class $\mathcal{B}(\mu,\alpha)$, $\mu \geq 1$, $0 \leq \alpha < 1$ for all $i = 1, 2, \cdots, n$. If $|f_i(z)| \leq M \quad (M \geq 1;\ z \in \mathcal{U})$ then the integral operator*

$$\int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\beta_i}} dt \tag{2.1}$$

is in $\mathcal{K}(\delta)$, where

$$\delta = 1 - \sum_{i=1}^{n} \frac{1}{|\beta_i|}\left((2-\alpha)\, M^{\mu-1} + 1\right) \tag{2.2}$$

and $\sum_{i=1}^{n} \frac{1}{|\beta_i|}\left((2-\alpha)\, M^{\mu-1} + 1\right) < 1$, $\beta_i \in \mathbb{C} - \{0\}$ for all $i = 1, 2, \cdots, n$.

Proof. Define the function $F(z)$ by

$$F(z)=\int_0^z \prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\frac{1}{\beta_i}}dt$$

for $f_i(z)\in\mathcal{B}(\mu,\alpha)$. Since

$$F'(z)=\prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\frac{1}{\beta_i}}$$

we see that

$$\frac{zF''(z)}{F'(z)}=\sum_{i=1}^{n}\frac{1}{\beta_i}\left(\frac{zf_i'(z)}{f_i(z)}-1\right). \tag{2.3}$$

It follows from (2.3) that

$$\begin{aligned}\left|\frac{zF''(z)}{F'(z)}\right| &\le \sum_{i=1}^{n}\frac{1}{|\beta_i|}\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)\\ &= \sum_{i=1}^{n}\frac{1}{|\beta_i|}\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu}\right|\left|\left(\frac{f_i(z)}{z}\right)^{\mu-1}\right|+1\right).\end{aligned} \tag{2.4}$$

Since $|f_i(z)|\le M \quad (z\in\mathcal{U})$, applying the Schwarz lemma, we have

$$\left|\frac{f_i(z)}{z}\right|\le M \qquad (z\in\mathcal{U}).$$

Therefore, from (2.4), we obtain

$$\left|\frac{zF''(z)}{F'(z)}\right|\le\sum_{i=1}^{n}\frac{1}{|\beta_i|}\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu}\right|M^{\mu-1}+1\right). \tag{2.5}$$

From (2.5) and (1.5), we see that

$$\begin{aligned}\left|\frac{zF''(z)}{F'(z)}\right| &\le \sum_{i=1}^{n}\frac{1}{|\beta_i|}\left(\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu}-1\right|+1\right)M^{\mu-1}+1\right)\\ &\le \sum_{i=1}^{n}\frac{1}{|\beta_i|}\left((2-\alpha)M^{\mu-1}+1\right)\\ &= 1-\delta.\end{aligned}$$

This completes the proof. □

Corollary 2.2. *Let $f_i(z)\in\mathcal{A}$ be in the class $\mathcal{B}(\mu,\alpha)$, $\mu\ge 1$, $0\le\alpha<1$ for all $i=1,2,\cdots,n$. If $|f_i(z)|\le M$ $(M\ge 1; z\in\mathcal{U})$ then the integral operator $\int_0^z\prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\beta_i}dt$ is convex function in $\mathcal{U}$, where*

$$\sum_{i=1}^{n}\frac{1}{|\beta_i|}=1/\left((2-\alpha)M^{\mu-1}+1\right),\quad \beta_i\in C-\{0\}$$

for all $i = 1, 2, \cdots, n$.

Letting $\mu = 1$ in Theorem 2.1, we have

Corollary 2.3. *Let* $f_i(z) \in \mathcal{A}$ *be in the class* $\mathcal{S}^*(\alpha)$, $0 \leq \alpha < 1$ *for all* $i = 1, 2, \cdots, n$. *If* $|f_i(z)| \leq M \quad (M \geq 1;\ z \in \mathcal{U})$ *then the integral operator* $\int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\beta_i} dt \in \mathcal{K}(\delta)$, *where*

$$\delta = 1 - \sum_{i=1}^{n} \frac{1}{|\beta_i|} (3 - \alpha) \tag{2.6}$$

where $\sum_{i=1}^{n} \frac{1}{|\beta_i|} (3 - \alpha) < 1$, $\beta_i \in \mathbb{C} - \{0\}$ *for all* $i = 1, 2, \cdots, n$.

Letting $n = 1$ and $\alpha = \delta = 0$ in Corollary 2.3, we have

Corollary 2.4. *Let* $f(z) \in \mathcal{A}$ *be starlike function in* $\mathcal{U}$. *If* $|f(z)| \leq M \quad (M \geq 1;\ z \in \mathcal{U})$ *then the integral operator* $\int_0^z \left(\frac{f(t)}{t}\right)^{\frac{1}{\beta}} dt$ *is convex in* $\mathcal{U}$ *where* $|\beta| = 3$, $\beta \in \mathbb{C}$.

Theorem 2.5. *Let* $f \in \mathcal{A}$ *be in the class* $\mathcal{B}(\mu, \alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$. *If* $|f(z)| \leq M \ (M \geq 1;\ z \in \mathcal{U})$ *then the integral operator*

$$G(z) = \int_0^z \left(te^{f(t)}\right)^{\gamma} dt \tag{2.7}$$

is in $\mathcal{K}(\delta)$, *where*

$$\delta = 1 - |\gamma| \left((2 - \alpha) M^{\mu} + 1\right) \tag{2.8}$$

and $|\gamma| < \frac{1}{(2-\alpha)M^{\mu}+1}$, $\gamma \in \mathbb{C}$.

Proof. Let $f \in \mathcal{A}$ be in the class $\mathcal{B}(\mu, \alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$. It follows from (2.7) that

$$\frac{G''(z)}{G'(z)} = \gamma \left(\frac{1}{z} + f'(z)\right)$$

and hence

$$\begin{aligned} \left|\frac{zG''(z)}{G'(z)}\right| &= |\gamma| \left(|1 + zf'(z)|\right) \\ &\leq |\gamma| \left(1 + \left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}\right| \left|\left(\frac{f(z)}{z}\right)^{\mu}\right| |z|\right). \end{aligned} \tag{2.9}$$

Applying the Schwarz lemma once again, we have

$$\left|\frac{f(z)}{z}\right| \leq M \qquad (z \in \mathcal{U}).$$

Therefore, from (2.9), we obtain

$$\left|\frac{zG''(z)}{G'(z)}\right| \leq |\gamma| \left(1 + \left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}\right| M^{\mu}\right) \qquad (z \in \mathcal{U}). \tag{2.10}$$

From (2.5) and (2.10), we see that

$$\begin{aligned} \left| \frac{zG''(z)}{G'(z)} \right| &\leq |\gamma| \left((2-\alpha) M^{\mu} + 1 \right) \\ &= 1 - \delta. \end{aligned}$$

□

Letting $\mu = 0$, in Theorem 2.5, we have

Corollary 2.6. *Let $f \in \mathcal{A}$ be in the class $\mathcal{R}(\alpha)$, $0 \leq \alpha < 1$. Then the integral operator $\int_0^z \left(te^{f(t)}\right)^{\gamma} dt \in \mathcal{K}(\delta)$, where*

$$\delta = 1 - |\gamma| (3 - \alpha) \tag{2.11}$$

and $|\gamma| < \frac{1}{3-\alpha}$, $\gamma \in \mathbb{C}$.

Letting $\mu = 1$, in Theorem 2.5, we have

Corollary 2.7. *Let $f \in \mathcal{A}$ be in the class $\mathcal{S}^*(\alpha)$, $0 \leq \alpha < 1$. If $|f(z)| \leq M$ $(M \geq 1;\ z \in \mathcal{U})$ then the integral operator $\int\limits_0^z \left(te^{f(t)}\right)^{\gamma} dt \in \mathcal{K}(\delta)$, where*

$$\delta = 1 - |\gamma| \left((2-\alpha) M + 1 \right) \tag{2.12}$$

and $|\gamma| < \frac{1}{(2-\alpha)M+1}$, $\gamma \in \mathbb{C}$.

Letting $\alpha = \delta = 0$ in Corollary2.7, we have

Corollary 2.8. *Let $f(z) \in \mathcal{A}$ be starlike function in $\mathcal{U}$. If $|f(z)| \leq M$ $(M \geq 1;\ z \in \mathcal{U})$ then the integral operator $\int\limits_0^z \left(te^{f(t)}\right)^{\gamma} dt$ is convex in $\mathcal{U}$ where $|\gamma| = \frac{1}{2M+1}$, $\gamma \in \mathbb{C}$.*

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AN INVERSION OF ONE CLASS OF INTEGRAL OPERATOR BY L. A. SAKHNOVICH'S OPERATOR IDENTITY METHOD

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Abstract. An inversion problem of integral operator in the form

$$Sf = \frac{d^3}{dx^3} \int_0^{\omega} S(x,t) f(t) dt$$

under the condition that the kernel $S(x,t)$ satisfies the equation

$$(\partial_x^3 + \partial_t^3) S(x,t) = 0$$

is investigated. It was proved that the operator $A_0 S - S A_0^*$ is finite if $A_0 = J^3$, where $Jf = i \int_0^x f(t) dt$. Presentation for the inverse operator $T = S^{-1}$ is obtained and it's structure is studied.

1. **Introduction**

An inversion of some classes of the integral operators S is based on use of operator identities in the form $A_0 S - S A_0^*$ or $S - T_0 S T_0^*$. The main idea of the operator identity method lies in the fact, that, if the operator $B = A_0 S - S A_0^*$ is a projector on a finite-dimensional subspace, then the inversion of the integral operator reduced to the inversion on a finite number of specific functions, the number of function is equal to the dimension of the finite-dimensional subspace, mentioned above. Thus, in general case the inversion of the integral operator is reduced to the selection of the operator A_0 and is determined by the finite number of partial solutions of corresponding integral equation.

The concept, first, was realized by V. A. Ambartzumyan. However, as the operator A_0, for the integral equation with kernel, depending on the difference, he used the operator of differentiation that leads to some difficulties in verification of

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the operator identity method. For the kernel, depending on the difference, L. A. Sakhnovich [3] proposed to use

$$A_0 = i \int_0^x f(t)dt,$$

the integral operator acting in $L^2[0, \omega]$ space.
The significant point here is the fact that the integral equation kernel, depending on the difference, satisfies the equation

$$\frac{\partial}{\partial x} S(x,t) = -\frac{\partial}{\partial t} S(x,t),$$

that allows to use the operator identity method effectively and to find the structure of the inverse integral operator.
Later Sakhnovich's idea was generalized in different directions [4]-[2].

The problem, concerning the inversion of the integral operator in the form

$$Sf = \frac{d^3}{dx^3} \int_0^\omega S(x,t)f(t)dt,$$

is investigated in this article under the condition that the kernel $S(x,t)$ satisfies the equation

$$\frac{\partial^3}{\partial x^3} S(x,t) + \frac{\partial^3}{\partial t^3} S(x,t) = 0.$$

It was proved that if the operator A_0 is in the form

$$A_0 f = -i \int_0^x \int_0^y \int_0^z f(t)dtdzdy = -\frac{i}{2} \int_0^x (x-t)^2 f(t)dt,$$

then the operator

$$A_0 S - S A_0^*,$$

is finite-dimensional.

The representation of the inverse operator is obtained and it is structure is investigated.

2. The operator identity

The general idea of this method can be summarized as follows. Consider an operator kernel S such that, $S(x,t) \in L^2([0,\omega] \times [0,\omega])$ and satisfying the equation

$(D_x \pm D_t)S(x,t) = 0$, where D_x is differential or integro-differential operator. Then, if

$$Sf = D_x \int_0^\omega S(x,t)f(t)dt$$

and the corresponding form of the operator A is chosen (often the operator D_x^{-1} may be used as operator A) so that $A_0S - SA_0$ is finite-dimensional. Then the evaluation of the inverse operator S^{-1} is reduced to the inversion of the operator S on the finite numbers of functions.

Currently, we suppose that,

$$\begin{aligned} f(x) &\in L^2[0,\omega], \\ S(x,t) &\in L^2([0,\omega] \times [0,\omega]), \end{aligned}$$

and that

$$g(x) = \int_0^\omega f(t)S(x,t)\,dt,$$

is absolutely continuous on the segment $[0,\omega]$.

Let

$$Jf = i\int_0^x f(t)dt,$$

then

$$J^*f = -i\int_x^\omega f(t)\,dt, \qquad J^2f = \int_0^x (t-x)\,f(t)\,dt,$$

$$J^{*^2}f = \int_x^\omega (x-t)\,f(t)\,dt, \qquad J^3f = -\tfrac{i}{2}\int_0^x (x-t)^2\,f(t)\,dt, \text{ and}$$

$$J^{*^3}f = \tfrac{i}{2}\int_x^\omega (x-t)^2\,f(t)\,dt.$$

Lemma 2.1. (On representation of a linear bounded operator in $L^2[0,\omega]$) *Any bounded operator $S \in [L^2[0,\omega] \times L^2[0,\omega]]$ is representable in the form*

$$Sf = \frac{d^3}{dx^3}\int_0^\omega S(x,t)f(t)dt,$$

where $S(x,t) \in L^2[0,\omega]$ at any fixed x.

Proof. Consider the function

$$\ell_x(t) = \begin{cases} \frac{(x-t)^2}{2}, & \text{when } t \le x \\ 0, & \text{when } t > x. \end{cases}$$

Then for the scalar product we have

$$\langle Sf, \ell_x \rangle = \int_0^x (Sf)dt = \langle f, S^*\ell_x \rangle .$$

Let us denote $S(x,t)$ by $S^*\ell_x$ at any fixed x.
Then

$$\langle f, S^*\ell_x \rangle = \int_0^\omega S(x,t)f(t)dt.$$

On the other hand, denoting $g(x)$ by Sf, we get

$$\langle Sf, \ell_x \rangle = \langle g, \ell_x \rangle = \frac{1}{2}\int_0^x (x-t)^2 g(t)dt.$$

So that,

$$\frac{1}{2}\int_0^x (x-t)^2 g(t)dt = \int_0^\omega S(x,t)g(t)dt.$$

And differentiating by x three times we get the representation

$$g(x) = Sf = \frac{d^3}{dx^3}\int_0^\omega S(x,t)f(t)dt.$$ □

Let $Df = \frac{d}{dx}f(x)$, $A_0 f = J^3 f$. Consider the operator

$$Sf = \frac{d^3}{dx^3}\int_0^\omega S(x,t)f(t)dt. \tag{2.1}$$

Then the next theorem holds.

Theorem 2.2. *For a bounded operator of the form (2.1) with the kernel $S(x,t)$, satisfying the equation*

$$\left(D_x^3 + D_t^3\right) S(x,t) = 0, \tag{2.2}$$

there holds an equality (operator identity)

$$\begin{aligned} \left(A_0 S - S A_0^*\right) f \quad &= \quad i\int_0^\omega f(t)\left(\frac{x^2}{2}N''(t) - \frac{t^2}{2}M''(t) + xN'(t)\right. \\ &\qquad \left. - tM'(x) + N(t) - M(x)\right)dt, \end{aligned} \tag{2.3}$$

where,

$$\begin{aligned} M(x) &= S(x,0), & N(t) &= S(0,t),\\ M'_x(x) &= S'_t(x,0), & N'(t) &= S'_t(0,t),\\ M''_x(x) &= S''_t(x,0), & N''(t) &= S''_t(0,t). \end{aligned}$$

Proof. Integrating by parts and using the equation for the kernel we obtain

$$\begin{aligned} A_0Sf &= i\int_0^x \left(xt - \frac{x^2}{2} - \frac{t^2}{2}\right)\frac{d^3}{dt^3}\int_0^\omega S(t,y)f(y)dydt \\ &= i\int_0^\omega f(t)\left(\frac{x^2}{2}S''_{xx}(0,t) + xS'_x(0,t) + S(0,t) - S(x,t)\right)dt. \end{aligned}$$

Similarly,

$$\begin{aligned} SA_0^*f &= i\frac{d^3}{dx^3}\int_0^\omega\left(\int_t^\omega\left(\frac{y^2}{2} + \frac{t^2}{2} - t_y\right)f(y)dy\right)S(x,t)dt \\ &= i\int_0^\omega f(t)\left(\frac{t^2}{2}S''_{tt}(x,0) + tS'_t(x,0) - S(x,t)\right)dt. \end{aligned}$$

And subtracting the equalities obtained above and using (2.3) we get the assertion of the Theorem 2.2. □

From the above Theorem it follows that the operator $A_0S - SA_0^*$ maps $L^2[0,\omega]$ onto six-dimensional space, stretched on the functions

$$1, x, \frac{x^2}{2}, M(x), M'(x), M''(x).$$

Really,

$$\begin{aligned} (AS - SA_0^*)f &= i\Big\{\left(f,\overline{N''}\right)\frac{x^2}{2} - \left(f,\frac{t^2}{2}\right)M'' + \left(f,\overline{N'}\right)x \\ &\quad -(f,t)M'(x) + \left(f,\overline{N}\right)1 - (f,1)\,M(x)\Big\}. \end{aligned}$$

Corollary 2.3. *If there exists a bounded operator T, which is the inverse to the operator S, then the following equality holds*

$$(TA_0 - A_0^*T)\,f = i\int_0^\omega f(t)\sum_{i=1}^{6}\overline{M_i(t)}N_i(t)dt, \tag{2.4}$$

where,

$$\begin{array}{cc} S^*M_1(t) = \overline{N''}(x), & SN_1(t) = \frac{x^2}{2} \\ S^*M_2(t) = -\frac{x^2}{2}, & SN_2(t) = M''(x) \\ S^*M_3(t) = \overline{N'}(x), & SN_3(t) = x \\ S^*M_4(t) = -x, & SN_4(t) = M'(x) \\ S^*M_5(t) = \overline{N}(x), & SN_5(t) = 1 \\ S^*M_6(t) = -1, & SN_6(t) = M(x). \end{array} \tag{2.5}$$

Proof.

$$(TA_0 - A_0^*T)\, f = T\,(A_0S - SA_0^*)\, Tf$$

$$= iT\int_0^\omega Tf\Big[\frac{x^2}{2}N''(t) - \frac{t^2}{2}M''(x) + xN'(t) - tM'(x) + N(t) - M(x)\Big]dt$$

$$= iT\Big\{\left(Tf, \overline{N''}\right)\frac{x^2}{2} - (Tf, \frac{t^2}{2})M'' + (Tf, \overline{N'})x - (Tf, t)M'(x)$$

$$+(Tf, \overline{N}) - (Tf, 1)M(x)\Big\}$$

$$= iT\Big\{\left(f, T^*\overline{N''}\right)\frac{x^2}{2} - (f, T^*\frac{x^2}{2})M''(x) + (f, T^*\overline{N'})x$$

$$-(f, T^*x)M'(x) + (f, T^*\overline{N}) - (f, T^*1)M(x)\Big\}$$

$$= iT\int_0^\omega f(t)\Big[\overline{T^*\overline{N''}}\frac{x^2}{2} - \overline{T^*\frac{x^2}{2}}M''(x) + \overline{T^*\overline{N'}}x - \overline{T^*x}M'(x)$$

$$+\overline{T^*\overline{N}} - \overline{T^*1}M(x)\Big]dt$$

$$= i\int_0^\omega f(t)\Big[T\frac{x^2}{2}\overline{T^*\overline{N''}} - TM''(x)\overline{T^*\frac{x^2}{2}} + Tx\overline{T^*\overline{N'}}$$

$$-TM'(x)\overline{T^*x} + T1\overline{T^*\overline{N}} - TM(x)\overline{T^*1}\Big]dt$$

$$= i\int_0^\omega f(t)\sum_{i=1}^{6}\overline{M_i(t)}\; N_i(t)\; dt.$$

□

3. Representation for the inverse operator

Let $N_k(x)$, $M_k(x)$ $(k = \overline{1,6})$ be functions in $L^2[0,\omega]$.
Let us introduce the function

$$Q(x,t) = \sum_{i=1}^{6} \overline{M_i(t)} N_i(x), \tag{3.1}$$

then

$$Qf = \int_0^{\omega} f(t)Q(x,t)dt.$$

Theorem 3.1. *If a bounded operator T, acting in $L^2[0,\omega]$, satisfies the operator equation $TA_0 - A_0^*T = iQ$, then*

$$Tf = \frac{d^3}{dx^3} \int_0^{\omega} f(t) \frac{\partial^3}{\partial t^3} \ \Phi(x,t) \ dt, \tag{3.2}$$

holds, where $\frac{\partial^3}{\partial t^3}\Phi(x,t)$ is the solution of the equation

$$\frac{\partial^3 F(x,t)}{\partial x^3} - \frac{\partial^3 F(x,t)}{\partial t^3} = \frac{\partial^6 q(x,t)}{\partial t^3 \partial x^3}.$$

Proof. The operator T may be represented in the form

$$Tf = \frac{d^3}{dx^3} \int_0^{\omega} f(t)F(x,t)dt.$$

The operator equation $TA_0 - A_0^*T = iQ$ means, that

$$i \int_0^{\omega}\int_0^{t}\int_0^{y}\int_0^{z} f(s)F(x,t)dsdzdydt + i \int_x^{\omega}\int_y^{\omega}\int_x^{\omega}\int_0^{\omega} f(t)F(x,t)dtdsdzdy$$
$$= \quad i \int_0^{\omega} f(t)q(x,t)dt.$$

Consequently,

$$\frac{\partial^3 F(x,t)}{\partial x^3} - \frac{\partial^3 F(x,t)}{\partial t^3} = \frac{\partial^6 q(x,t)}{\partial t^3 \partial x^3}.$$

Then the solution is

$$F(x,t) = H(t,x,q(x,t)) = \frac{\partial^3}{\partial t^3}\Phi(x,t). \qquad \square$$

4. **Relation between $N_k(x)$ and $M_k(x)$**

Let us define the involution operator Uf by

$$Uf = \overline{f(\omega - x)}.$$

Lemma 4.1. *$USU = S^*$.*

Proof. Let

$$\begin{aligned} g(x) &\in C^3(0,\omega), \\ g(0) &= g(\omega) = 0, \\ g'(0) &= g'(\omega) = 0, \\ g''(0) &= g''(\omega) = 0, \end{aligned}$$

since

$$(Sf,g) = \int_0^\omega \frac{d^3}{dt^3} \int_0^\omega f(y)S(t,y)dy\overline{g(t)}dt = \begin{bmatrix} \overline{g(t)} = U, & \int_0^\omega f(y)S'''_{ttt}(t,y)dy = V'_t \\ \overline{g'(t)} = U', & \int_0^\omega f(y)S''_{tt}(t,y)dy = V \end{bmatrix}$$

$$= -\int_0^\omega \int_0^\omega f(y)S''_{tt}(t,y)dy\overline{g'(t)}dt = \begin{bmatrix} \overline{g'(t)} = U, & \int_0^\omega f(y)S''_{tt}(t,y)dy = V'_t \\ \overline{g''(t)} = U', & \int_0^\omega f(y)S'_{t}(t,y)dy = V \end{bmatrix}$$

$$= \int_0^\omega \int_0^\omega f(y)S'_{t}(t,y)dy\overline{g''(t)}dt = \begin{bmatrix} \overline{g''(t)} = U, & \int_0^\omega f(y)S'_{t}(t,y)dy = V'_t \\ \overline{g'''(t)} = U', & \int_0^\omega f(y)S(t,y)dy = V \end{bmatrix}$$

$$= -\int_0^\omega \int_0^\omega f(y)S(t,y)dy\overline{g'''(t)}dt = -\int_0^\omega \int_0^\omega f(y)S(t,y)\overline{g'''(t)}dtdy$$

$$= -\int_0^\omega f(y) \int_0^\omega S(t,y)\overline{g'''(t)}dtdy,$$

it follows that

$$\begin{aligned}
S^*g &= -\int_0^\omega \overline{S(t,x)}g'''(t)dt = \left[\begin{array}{ll} \overline{S(t,x)} = U, & g'''(t) = V' \\ \overline{S_t'(t,x)} = U_t', & g''(t) = V \end{array}\right] \\
&= \int_0^\omega \overline{S_t'(t,x)}g''(t)dt = \left[\begin{array}{ll} \overline{S_t'(t,x)} = U, & g''(t) = V' \\ \overline{S_{tt}''(t,x)} = U_t', & g'(t) = V \end{array}\right] \\
&= -\int_0^\omega \overline{S_{tt}''(t,x)}g'(t)dt = \left[\begin{array}{ll} \overline{S_{tt}''(t,x)} = U, & g'(t) = V' \\ \overline{S_{ttt}'''(t,x)} = U_t', & g(t) = V \end{array}\right] \\
&= \int_0^\omega \overline{S_{ttt}'''(t,x)}g(t)dt = -\frac{d^3}{dx^3}\int_0^\omega g(t)\overline{S(t,x)}dt.
\end{aligned}$$

Then it is easy to see, that

$$USUg = -\frac{d^3}{dx^3}\int_0^\omega g(t)\overline{S(t,x)}dt. \qquad \square$$

In what follows, for simplicity, we restrict our study to those solution of equation for the kernel $S(x,t)$ which depends only on the difference $x-t$. More general case, require cumbersome computations while the reasoning is the same as for the case when the kernel depends only on the difference.

Theorem 4.2. *Suppose that there exists such N_i $(i=\overline{1,6})$ from $L^2[0,\omega]$ such that*

$$\begin{aligned}
SN_1(t) &= \tfrac{x^2}{2}, \\
SN_2(t) &= M''(x), \\
SN_3(t) &= x, \\
SN_4(t) &= M'(x), \\
SN_5(t) &= 1, \\
SN_6(t) &= M(x),
\end{aligned}$$

holds, then

$$\begin{aligned}
S^*M_1(t) &= \overline{N''(t)}, \\
S^*M_2(t) &= -\tfrac{x^2}{2}, \\
S^*M_3(t) &= \overline{N'(t)}, \\
S^*M_4(t) &= -x, \\
S^*M_5(t) &= \overline{N(x)}, \\
S^*M_6(t) &= -1,
\end{aligned}$$

are valid, where

$$\begin{aligned} M(x) &= S(x), & N(t) &= S(-t),\\ M'(x) &= S'_t(x), & N'(t) &= S'_x(-t),\\ M''(x) &= S''_{tt}(x), & N''(t) &= S''_{xx}(-t), \end{aligned}$$

and

$$\begin{aligned} M_1(x) =&\ \overline{N_2(\omega-x)}-1,\\ M_2(x) =&\ \overline{N_1(\omega-x)}+\omega\overline{N_3(\omega-x)}+\tfrac{\omega^2}{2}\overline{N_6(\omega-x)},\\ M_3(x) =&\ \omega\overline{N_2(\omega-x)}+\overline{N_4(\omega-x)}+x,\\ M_4(x) =&\ \omega\overline{N_5(\omega-x)}-\overline{N_3(\omega-x)},\\ M_5(x) =&\ \overline{N_6(\omega-x)}-\tfrac{(\omega-x)^2}{2}+(\tfrac{\omega^2}{2}+\omega)(\omega\overline{N_2(\omega-x)}+\overline{N_4(\omega-x)}+x),\\ M_6(x) =&\ -\overline{N_5(\omega-x)}. \end{aligned}$$

Proof. By direct integration by parts we verify, that

$$\begin{aligned} S1 &= \frac{d^3}{dx^3}\int\limits_0^\omega S(x-t)dt = -\omega S''_{tt}(x-\omega)+S'_t(x-\omega)-S'_t(x)\\ &= -\omega U\overline{N''(x)}-U\overline{N'(x)}-M'(x). \end{aligned}$$

Similarly,

$$\begin{aligned} S\frac{t^2}{2} &= \frac{d^3}{dx^3}\int\limits_0^\omega \frac{t^2}{2}S(x-t)dt = -\int\limits_0^\omega \frac{t^2}{2}S'''_{ttt}(x-t)dt\\ &= -\frac{\omega^2}{2}U\overline{N''(x)}-\omega U\overline{N'(x)}-U\overline{N(x)}+M(x). \end{aligned}$$

That is

$$\begin{aligned} S1 &= SN_2-U\overline{N''(x)},\\ St &= -SN_4-\omega U\overline{N''(x)}-U\overline{N'(x)},\\ S\frac{t^2}{2} &= -\left(\frac{\omega^2}{2}+\omega\right)U\overline{N'(x)}-U\overline{N(x)}+SN_6. \end{aligned}$$

Consequently,

$$\begin{aligned} U\overline{N''(x)} &= S[N_2-1],\\ U\overline{N'(x)} &= -S[t+N_4]-\omega\left(U\overline{N(x)}\right)'' = S[\omega-\omega N_2-N_4-t],\\ U\overline{N(x)} &= S[N_6-\frac{t^2}{2}]-\left(\frac{\omega^2}{2}+\omega\right)\left(U\overline{N(x)}\right)'\\ &= S\left[N_6-\frac{t^2}{2}-\left(\frac{\omega^2}{2}+\omega\right)(\omega-\omega N_2-N_4-t)\right]. \end{aligned}$$

Then,

1)

$$
\begin{aligned}
M_1(x) &= \overline{N_2(\omega - x)} - 1, \\
M_1(x) &= U\left[N_2(x) - 1\right], \\
US^*M_1 &= US^*U\left[N_2 - 1\right], \\
US^*M_1 &= S\left[N_2 - 1\right], \\
US^*M_1 &= U\overline{N''(x)}, \\
S^*M_1 &= \overline{N''(x)}.
\end{aligned}
$$

2)

$$
\begin{aligned}
M_2(x) &= \left[\overline{N_1(\omega - x)} + \omega\overline{N_3(\omega - x)} + \tfrac{\omega^2}{2}\overline{N_5(\omega - x)}\right] \\
M_2(x) &= U\left[N_1(x) + \omega N_3(x) + \tfrac{\omega^2}{2}N_5(x)\right] \\
US^*M_2 &= US^*U\left[N_1 + \omega N_3 + \tfrac{\omega^2}{2}N_5\right] \\
US^*M_2 &= S\left[N_1 + \omega N_3 + \tfrac{\omega^2}{2}N_5\right] \\
US^*M_2 &= \tfrac{-x^2 + 2x\omega - \omega^2}{2} \\
US^*M_2 &= -\tfrac{(\omega - x)^2}{2} \\
S^*M_2 &= -\tfrac{x^2}{2}.
\end{aligned}
$$

3)

$$
\begin{aligned}
M_3(x) &= \omega\overline{N_2(\omega - x)} + \overline{N_4(\omega - x)} + x \\
M_3(x) &= U\left[\omega - \omega N_2(x) - N_4(x) - x\right] \\
US^*M_3 &= US^*U\left[\omega - \omega N_2 - N_4 - t\right] \\
US^*M_3 &= S\left[\omega - \omega N_2 - N_4 - t\right] \\
US^*M_3 &= U\overline{N'(x)} \\
S^*M_1 &= \overline{N'(x)}.
\end{aligned}
$$

4)

$$
\begin{aligned}
M_4(x) &= \omega\overline{N_5(\omega - x)} - \overline{N_3(\omega - x)} \\
M_4(x) &= U\left[\omega N_5(x) - N_3(x)\right] \\
US^*M_4 &= US^*U\left[\omega N_5 - N_3\right] \\
US^*M_4 &= S\left[\omega N_5 - N_3\right] \\
US^*M_4 &= \omega - x \\
S^*M_4 &= x.
\end{aligned}
$$

5)

$$\begin{aligned}
M_5(x) &= \overline{N_6(\omega,t)} - \tfrac{(\omega-x)^2}{2} + \left(\tfrac{\omega^2}{2}+\omega\right)\left(\omega\overline{N_2(\omega-x)} + \overline{N_4(\omega-x)} + x\right) \\
M_5(x) &= U\left[N_6(x) - \tfrac{x^2}{2} - \left(\tfrac{\omega^2}{2}+\omega\right)(\omega - \omega N_2 - N_4 - t)\right] \\
US^*M_5 &= US^*U\left[N_6 - \tfrac{t^2}{2} - \left(\tfrac{\omega^2}{2}+\omega\right)(\omega - \omega N_2 - N_4 - t)\right] \\
US^*M_5 &= S\left[N_6 - \tfrac{t^2}{2} - \left(\tfrac{\omega^2}{2}+\omega\right)(\omega - \omega N_2 - N_4 - t)\right] \\
US^*M_5 &= U\overline{N(x)} \\
S^*M_5 &= \overline{N(x)}.
\end{aligned}$$

6)

$$\begin{aligned}
M_6(x) &= -\overline{N_5(\omega-x)} \\
M_6(x) &= -UN_5(x) \\
US^*M_6 &= -US^*UN_5 \\
US^*M_6 &= -SN_5 \\
S^*M_6 &= -1.
\end{aligned}$$

□

If the operator S is invertible then from formula (3.1) it follows that

$$\begin{aligned}
Q(x,t) &= \sum_{i=1}^{6} \overline{M_i(t)} N_i(t) \\
&= \left[N_2(\omega-t) - 1\right] N_1(x) + \left[N_1(\omega-t) + \omega N_3(\omega-t)\right. \\
&\quad + \frac{\omega^2}{2} N_6(\omega-t)]N_2(x) + \left[\omega N_2(\omega-t) + N_4(\omega-t) + t\right] N_3(x) \\
&\quad + \left[\omega N_5(\omega-t) - N_3(\omega-t)\right] N_4(x) + \left[N_6(\omega-t) - \frac{(\omega-t)^2}{2}\right. \\
&\quad + \left(\frac{\omega^2}{2}+\omega\right)(\omega N_2(\omega-t) + N_4(\omega-t) + t)]N_5(x) \\
&\quad - \left[N_5(\omega-t)\right] N_6(x).
\end{aligned}$$

Using $Q(x,t)$ one may construct the operator T.
Thus to construct operator $T = S^{-1}$ it is sufficiently to know it's action upon

$$1, x, \frac{x^2}{2}, M(x), M'(x), M''(x).$$

Thus, a method, proposed by L. A. Sakhnovich, and it's generalizations are analogs of construction of the general solution for the linear differential equation by it's particular solutions.However, in the theory of differential equations there exist general methods for solutions representation by partial solutions for any linear differential equation with variable coefficients of any finite order, while it was not possible to extend Sakhnovich's method for linear integral equations with any arbitrary kernel, i.e, it was not possible to prove that the operator $A_0S - SA_0^*$ is finite dimensional, where $Sf = D_x \int_0^\omega S(x,t)f(t)dt$, and $A_0 = (D_x)^{-1}$, such that $f(x) \in L^2[0,\omega]$, D_x is a linear integro-differential operator, and the kernel $S(x,t)$ satisfies the equation

$$(D_x + D_t)\, S(x,t) = 0.$$

As it is obvious from the results obtained, Sakhnovich's method can be extended to include a case where D_x is a general linear differential operator of the order 3 as in the form

$$D_x = \sum_{k=0}^{3} a_k \frac{d^k}{dx^k}.$$

Sakhnovich's method may be also applied when $D_x = \frac{d^4}{dx^4}$.

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THE RATE OF APPROXIMATION OF FUNCTIONS IN AN INFINITE INTERVAL BY POSITIVE LINEAR OPERATORS

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Abstract. We obtain an estimation, in the uniform norm, of the rate of the approximation by positive linear operators of functions defined on the positive half line that have a finite limit at the infinity.

1. Introduction

Let us denote by $C^*[0,\infty)$, the Banach space of all real-valued continuous functions on $[0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists and is finite, endowed with the uniform norm. In [2], it is proved the following theorem:

Theorem 1.1. *If the sequence $A_n : C^*[0,\infty) \to C^*[0,\infty)$ of positive linear operators satisfies the conditions*

$$\lim_{n\to\infty} A_n(e^{-kt}, x) = e^{-kx}, \quad k = 0, 1, 2,$$

uniformly in $[0,\infty)$, then

$$\lim_{n\to\infty} A_n f(x) = f(x),$$

uniformly in $[0,\infty)$, for every $f \in C^*[0,\infty)$.

In [1], it is proved the above theorem in a more general setting. In the same book, the authors give the results for the particular operators of Szász-Mirakjan, of Baskakov and of Bernstein-Chlodovsky.

In the following, we obtain an estimation of the rate of convergence of operators satisfying the conditions from the above theorem, first, in the general form and then, for the particular cases presented above. For this estimation, we use the following modulus of continuity:

$$\omega^*(f,\delta) = \sup_{\substack{x,t\geq 0 \\ |e^{-x}-e^{-t}|\leq\delta}} |f(x) - f(t)|,$$

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defined for every $\delta \geq 0$ and every function $f \in C^*[0,\infty)$. This modulus can be expressed in terms of the usual modulus of continuity, by the relation:

$$\omega^*(f,\delta) = \omega(f^*,\delta),$$

where f^* is the continuous function defined on $[0,1]$ by

$$f^*(x) = \begin{cases} f(-\ln x), & x \in (0,1] \\ \lim_{t\to\infty} f(t), & x = 0. \end{cases}$$

Remark 1.2. Because $|e^{-t} - e^{-x}| \leq |t-x|$, for every $t, x \geq 0$, we have for $\delta \geq 0$

$$\omega(f,\delta) \leq \omega^*(f,\delta),$$

and because $|e^{-t} - e^{-x}| = e^{-\theta}|t-x| \geq e^{-M}|t-x|$, for every $t,x \in [0,M]$, we have

$$\omega^*(f,\delta) \leq \omega(f, e^M\delta) \leq (1+e^M)\cdot\omega(f,\delta).$$

2. Main result

Theorem 2.1. *If $A_n : C^*[0,\infty) \to C^*[0,\infty)$ is a sequence of positive linear operators with*

$$\begin{aligned} \|A_n 1 - 1\|_\infty &= a_n, \\ \|A_n(e^{-t},x) - e^{-x}\|_\infty &= b_n, \\ \left\|A_n(e^{-2t},x) - e^{-2x}\right\|_\infty &= c_n, \end{aligned}$$

where a_n, b_n and c_n tend to zero as n goes to the infinity, then

$$||A_n f - f||_\infty \leq \|f\|_\infty a_n + (2+a_n)\cdot\omega^*\left(f, \sqrt{a_n + 2b_n + c_n}\right),$$

for every function $f \in C^*[0,\infty)$.

Proof. Using the property of the usual modulus of continuity

$$|F(u) - F(v)| \leq \left(1 + \frac{(u-v)^2}{\delta^2}\right)\omega(F,\delta),$$

for the function $F = f^*$ and for $u = e^{-t}$ and $v = e^{-x}$ and using the relation $f^*(e^{-t}) = f(t)$, we obtain

$$|f(t) - f(x)| \leq \left(1 + \frac{(e^{-t}-e^{-x})^2}{\delta^2}\right)\omega^*(f,\delta).$$

Because

$$A_n((e^{-t}-e^{-x})^2,x) = [A_n(e^{-2t},x) - e^{-2x}] - 2e^{-x}[A_n(e^{-t},x) - e^{-x}] + e^{-2x}[A_n(1,x) - 1]$$

we obtain

$$\begin{aligned} A_n(|f(t)-f(x)|,x) &\leq \left(A_n(1,x)+\frac{A_n((e^{-t}-e^{-x})^2,x)}{\delta^2}\right)\omega^*(f,\delta) \\ &\leq \left(1+a_n+\frac{a_n+2b_n+c_n}{\delta^2}\right)\omega^*(f,\delta). \end{aligned}$$

Choosing $\delta=\sqrt{a_n+2b_n+c_n}$ and using the inequality

$$|A_nf(x)-f(x)|\leq |f(x)|\cdot|A_n(1,x)-1|+A_n(|f(t)-f(x)|,x),$$

we obtain, in the uniform norm, the estimation stated in the theorem. □

Remark 2.2. Because all positive linear operators L can be modified to preserve constant functions, $\tilde{L}f=\frac{1}{L1}Lf$, we can take $a_n=0$ in the theorem above and obtain:

$$||A_nf-f||_\infty\leq 2\cdot\omega^*(f,\sqrt{2b_n+c_n}).$$

Remark 2.3. If we restrict ourselves on a compact interval $[0,M]$ and if we use the Remark 1.2, we obtain an estimation using the usual modulus of continuity:

$$||A_nf-f||_\infty\leq C\cdot\omega\left(f,\sqrt{2b_n+c_n}\right).$$

We have used the Korovkin subset $\left\{1,e^{-x},e^{-2x}\right\}$ for $C^*[0,\infty)$, but as suggested in the article [3], we can use any other Korovkin subset for this space, such as for example $\left\{1,\frac{x}{1+x},\frac{x^2}{(1+x)^2}\right\}$. In this case we can introduce

$$\omega^{\#}(f,\delta)=\sup_{\substack{x,t\geq 0\\ \left|\frac{x}{1+x}-\frac{t}{1+t}\right|\leq\delta}}|f(x)-f(t)|,$$

defined for every $\delta\geq 0$ and every function $f\in C^*[0,\infty)$. This modulus can be expressed in terms of the usual modulus of continuity, by the relation:

$$\omega^{\#}(f,\delta)=\omega(f^{\#},\delta),$$

where $f^{\#}$ is the continuous function defined on $[0,1]$ by

$$f^{\#}(x)=\begin{cases} f\left(\frac{x}{1-x}\right), & x\in[0,1) \\ \lim_{t\to\infty}f(t), & x=1. \end{cases}$$

Because of $\left|\frac{x}{1+x}-\frac{t}{1+t}\right|\leq|x-t|$, where $x,t\geq 0$, we have

$$\omega(f,\delta)\leq\omega^{\#}(f,\delta),$$

and because $\left|\frac{x}{1+x}-\frac{t}{1+t}\right|\geq\frac{|x-t|}{(1+M)^2}$, for $x,t\in[0,M]$, we obtain

$$\omega^{\#}(f,\delta)\leq\omega(f,(1+M)^2\delta)\leq(1+M)^2\cdot\omega(f,\delta),$$

where $M>0$, is an integer. We have the following

Theorem 2.4. *If $A_n : C[0,\infty) \to C[0,\infty)$ is a sequence of positive linear operators which preserves linear functions and*

$$\sup_{x\geq 0} \frac{|A_n(t^2,x) - x^2|}{(1+x)^2} = d_n,$$

is a sequence which tends to zero as n goes to the infinity, then

$$||A_n f - f||_\infty \leq 2 \cdot \omega^\# \left(f, \sqrt{d_n}\right),$$

for every function $f \in C^*[0,\infty)$.

Proof. Using the property of the usual modulus of continuity

$$|F(u) - F(v)| \leq \left(1 + \frac{(u-v)^2}{\delta^2}\right) \omega(F,\delta),$$

for the function $F = f^\#$ and for $u = t/(1+t)$ and $v = x/(1+x)$ and using the relation $f^\#(t/(1+t)) = f(t)$, we obtain

$$|f(t) - f(x)| \leq \left[1 + \frac{1}{\delta^2}\left(\frac{t}{1+t} - \frac{x}{1+x}\right)^2\right] \omega^\#(f,\delta) \leq \left(1 + \frac{(t-x)^2}{\delta^2(1+x)^2}\right) \omega^\#(f,\delta).$$

Because

$$A_n(t-x)^2, x) = A_n(t^2,x) - x^2$$

we obtain

$$|A_n f(x) - f(x)| \leq A_n(|f(t) - f(x)|, x) \leq \left(1 + \frac{d_n}{\delta^2}\right) \omega^\#(f,\delta).$$

Choosing $\delta = \sqrt{d_n}$ we obtain, in the uniform norm, the estimation stated in the theorem. □

3. **Applications**

In order to obtain particular results, we use the following

Lemma 3.1. *For every $x > 0$ we have*

$$e^{-x\alpha_n} - e^{-x} < \frac{x_n}{2e}, \quad \textit{for every } n \geq 1,$$

where $\alpha_n = \frac{1-e^{-x_n}}{x_n}$ and $x_n > 0$, for every $n \geq 1$.

Proof. First, let us notice that

$$\max_{x>0} xe^{-cx} = \frac{1}{ec}, \text{ for every } c > 0. \tag{3.1}$$

Indeed, the point $t = 1/c$ is a maximum point for $f(t) = te^{-ct}$, $t > 0$.

Secondly, let us notice that $0 < a_n < 1$, for every $n \geq 1$. This is true, because of the inequality $1 - e^{-x} < x$, for $x \neq 0$.

Next, using the inequalities between geometric, logarithmic and arithmetic means

$$\sqrt{uv} < \frac{u-v}{\ln u - \ln v} < \frac{u+v}{2}, \quad \text{for } 0 < v < u,$$

for the values $u = e^{-x\alpha_n} > v = e^{-x} > 0$, we obtain

$$e^{-x\alpha_n} - e^{-x} < \frac{e^{-x\alpha_n} + e^{-x}}{2} \cdot x(1-\alpha_n) = \frac{1-\alpha_n}{2}\left(xe^{-x\alpha_n} + xe^{-x}\right).$$

Using (3.1), we obtain

$$e^{-x\alpha_n} - e^{-x} \leq \frac{1-\alpha_n}{2}\left(\frac{1}{e\alpha_n} + \frac{1}{e}\right) = \frac{1-\alpha_n^2}{2e\alpha_n}.$$

It remain to prove that $\frac{1-\alpha_n^2}{\alpha_n} < x_n$, which is a particular case of

$$\frac{1-\left(\frac{1-e^{-x}}{x}\right)^2}{\frac{1-e^{-x}}{x}} < x, \quad \text{for } x > 0.$$

This is equivalent with $x^2e^{-x} + 2e^{-x} - 1 - e^{-2x} < 0$, for $x > 0$, which is true by an elementary calculus argument. □

Corollary 3.2. *For the Szász-Mirakjan operators $M_n : C^*[0,\infty) \to C^*[0,\infty)$ defined by*

$$M_nf(x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),$$

we have for $f \in C^*[0,\infty)$, the estimations

$$||M_nf - f||_\infty \leq 2\cdot\omega^*\left(f, \frac{1}{\sqrt{n}}\right), \quad n \geq 1,$$

and

$$||M_nf - f||_\infty \leq 2\cdot\omega^\#\left(f, \frac{1}{2\sqrt{n}}\right), \quad n \geq 1.$$

Proof. We have $M_n(1,x) = 1$, so $a_n = 0$. We, also, have

$$M_n(e^{-\lambda t}, x) = e^{-\lambda x\frac{1-e^{-\lambda/n}}{\lambda/n}},$$

which gives, by Lemma 3.1

$$|M_n(e^{-\lambda t}, x) - e^{-\lambda x}| \leq \frac{\lambda}{2en}.$$

It follows that

$$b_n \leq \frac{1}{2en} \text{ and } c_n \leq \frac{1}{en}, \text{ for } n \geq 1,$$

and because

$$a_n + 2b_n + c_n \leq \frac{2}{2en} + \frac{1}{en} \leq \frac{1}{n}, \text{ for } n \geq 1,$$

we obtain the estimation stated in the theorem.

Because $M_n(t,x) = x$ and $M_n(t^2,x) = x^2 + \frac{x}{n}$, we obtain

$$d_n = \sup_{x\geq 0} \frac{|M_n(t^2,x) - x^2|}{(1+x)^2} = \sup_{x\geq 0} \frac{x}{n(1+x)^2} = \frac{1}{4n}.$$

□

Corollary 3.3. *For the Baskakov operators $V_n : C^*[0,\infty) \to C^*[0,\infty)$ defined by*

$$V_n f(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right),$$

we have for $f \in C^*[0,\infty)$, the estimations

$$||V_n f - f||_\infty \leq 2 \cdot \omega^* \left(f, \frac{5}{2\sqrt{n}}\right), \quad n \geq 2,$$

and

$$||V_n f - f||_\infty \leq 2 \cdot \omega^{\#} \left(f, \frac{1}{\sqrt{n}}\right), \quad n \geq 1.$$

Proof. From the identity $V_n(1,x) = 1$, we deduce $a_n = 0$. Computing

$$V_n(e^{-\lambda t},x) = \sum_{k=0}^{\infty} \binom{-n}{k} (-xe^{-\lambda/n})^k (1+x)^{-n-k} = \left(-xe^{-\lambda/n} + 1 + x\right)^{-n},$$

we obtain

$$\begin{aligned} |V_n(e^{-\lambda t},x) - e^{-\lambda x}| &= |[1 + x(1-e^{-\lambda/n})]^{-n} - e^{-\lambda x}| \\ &= e^{-\lambda x} \left| e^{-n\ln\left(1+x(1-e^{-\lambda/n})\right)+\lambda x} - 1 \right| \\ &\leq \left[-n\ln\left(1+x(1-e^{-\lambda/n})\right) + \lambda x\right] \cdot e^{-n\ln\left(1+x(1-e^{-\lambda/n})\right)}, \end{aligned}$$

where, we have used the inequality $e^t - 1 \leq te^t$ for

$$t = -n\ln\left(1+x(1-e^{-\lambda/n})\right) + \lambda x \geq -nx(1-e^{-\lambda/n}) + \lambda x \geq -nx \cdot \frac{\lambda}{n} + \lambda x = 0.$$

Because $\ln(1+t) \geq t/(1+t)$, for every $t \geq 0$, we obtain

$$\begin{aligned} |V_n(e^{-\lambda t},x) - e^{-\lambda x}| &\leq \frac{-nx(1-e^{-\lambda/n}) + \lambda x + \lambda x^2(1-e^{-\lambda/n})}{\left(1+x(1-e^{-\lambda/n})\right)^{n+1}} \\ &\leq \frac{-nx(1-e^{-\lambda/n}) + \lambda x + \lambda x^2(1-e^{-\lambda/n})}{1 + (n+1)x(1-e^{-\lambda/n}) + \frac{n(n+1)}{2}x^2(1-e^{-\lambda/n})^2}. \end{aligned}$$

Because $1 - e^{-\lambda/n} \geq \lambda/n - \lambda^2/(2n^2)$, we get from the above inequality

$$\sup_{x\geq 0} |V_n(e^{-\lambda t},x) - e^{-\lambda x}| \leq \frac{2\lambda}{n(n+1)(1-e^{-\lambda/n})}.$$

Using the same inequality, we obtain

$$b_n = \sup_{x \geq 0} |V_n(e^{-t}, x) - e^{-x}| \leq \frac{2}{n(n+1)\left(\frac{1}{n} - \frac{1}{2n^2}\right)} \leq \frac{2}{n}, \quad \text{for } n \geq 1$$

and using $1 - e^{-2/n} \geq 2/n - 2/n^2 + 4/(3n^3) - 2/(3n^4)$, we have

$$c_n = \sup_{x \geq 0} |V_n(e^{-2t}, x) - e^{-2x}| \leq \frac{4}{n(n+1)\left(\frac{2}{n} - \frac{2}{n^2} + \frac{4}{3n^3} - \frac{2}{3n^4}\right)} = \frac{h(n)}{n},$$

where $h(t) = 6t^4/((t+1)(3t^3 - 3t^2 + 2t - 1))$. Because

$$h'(t) = \frac{6t^3}{(t+1)^2(3t^3 - 3t^2 + 2t - 1)}(-2t^2 + 3t - 4) < 0, \quad t \geq 1,$$

we obtain $h(n) \leq h(2) = 32/15$, for $n \geq 2$. Finally, we obtain

$$\sqrt{a_n + 2b_n + c_n} \leq \frac{1}{\sqrt{n}}\sqrt{4 + \frac{32}{15}} \leq \frac{5}{2\sqrt{n}}.$$

Because $V_n(t, x) = x$ and $V_n(t^2, x) = x^2 + x(1 + x)/n$, we obtain

$$d_n = \sup_{x \geq 0} \frac{|V_n(t^2, x) - x^2|}{(1 + x)^2} = \sup_{x \geq 0} \frac{x}{n(1 + x)} = \frac{1}{n}.$$

□

Corollary 3.4. *For the Bernstein-Chlodovsky operators* $C_n : C^*[0, \infty) \to C^*[0, \infty)$ *defined by*

$$C_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\beta_n\right)\binom{n}{k}\left(\frac{x}{\beta_n}\right)^k\left(1 - \frac{x}{\beta_n}\right)^{n-k},$$

for $0 \leq x \leq \beta_n$ *and* $C_n f(x) = f(x)$, *for* $x > \beta_n$, *where* β_n *is a sequence of positive numbers such that*

$$\lim_{n\to\infty} \beta_n = \infty \text{ and } \lim_{n\to\infty} \frac{\beta_n}{n} = 0,$$

we have for $f \in C^*[0, \infty)$, *the estimations*

$$||C_n f - f||_\infty \leq 2 \cdot \omega^*\left(f, \sqrt{\frac{\beta_n}{n}}\right), \quad n \geq 1,$$

and

$$||C_n f - f||_\infty \leq 2 \cdot \omega^\#\left(f, \sqrt{\frac{\beta_n}{4n}}\right), \quad n \geq 1.$$

Proof. From the identity $C_n(1, x) = 1$, we deduce $a_n = 0$. Computing

$$C_n(e^{-\lambda t}, x) = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{\beta_n}e^{-\lambda\beta_n/n}\right)^k\left(1 - \frac{x}{\beta_n}\right)^{n-k} = \left(e^{-\lambda\beta_n/n}\frac{x}{\beta_n} + 1 - \frac{x}{\beta_n}\right)^n,$$

we obtain

$$\begin{aligned}|C_n(e^{-\lambda t},x)-e^{-\lambda x}| &= \left|\left(1-\lambda x\frac{1-e^{-\lambda\beta_n/n}}{\lambda\beta_n}\right)^n - e^{-\lambda x}\right| \\ &= \left|e^{n\ln\left(1-\frac{x}{\beta_n}(1-e^{-\lambda\beta_n/n})\right)} - e^{-\lambda x}\right| \\ &\leq e^{-\lambda x\frac{1-e^{-\lambda\beta_n/n}}{\lambda\beta_n/n}} - e^{-\lambda x},\end{aligned}$$

because $\ln(1-t)\leq -t$, for every $t\in(0,1)$. Using Lemma 3.1, we obtain

$$|C_n(e^{-\lambda t},x)-e^{-\lambda x}|\leq \frac{\lambda\beta_n}{2en}.$$

This gives the estimations

$$b_n\leq\frac{\beta_n}{2en}\ \text{and}\ c_n\leq\frac{\beta_n}{en},\ \text{so}\ a_n+2b_n+c_n\leq\frac{\beta_n}{n}.$$

Because $C_n(t,x)=x$ and $C_n(t^2,x)=x^2+\frac{x(\beta_n-x)}{n}$, we obtain

$$d_n=\sup_{x\geq 0}\frac{|C_n(t^2,x)-x^2|}{(1+x)^2}=\sup_{x\in[0,\beta_n]}\frac{x(\beta_n-x)}{n(1+x)^2}=\frac{\beta_n^2}{4n(1+\beta_n)}\leq\frac{\beta_n}{4n}.$$

□

Corollary 3.5. *For the Bleimann-Butzer-Hahn operators* $L_n\colon C^*[0,\infty)\to C^*[0,\infty)$ *defined by*

$$L_n(f,x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1+x)^{-n}f\left(\frac{k}{n-k+1}\right)$$

we have

$$\|L_nf-f\|_\infty\leq 2\cdot\omega^{\#}\left(f,\frac{2}{\sqrt{n+1}}\right),\quad n\geq 1,\quad f\in C^*[0,\infty).$$

Proof. For the proof, we use the argument from Theorem 2.1 for the test functions $x^k/(x+1)^k$ instead of e^{-kx} and the modulus $\omega^{\#}(f,\delta)$ instead of $\omega^*(f,\delta)$.

Because $L_n(1,x)=1$ we have $a_n=\|L_n1-1\|_\infty=0$. From the equalities (see [5])

$$\begin{aligned}L_n\left(\frac{t}{1+t},x\right) &= \frac{nx}{(1+n)(1+x)}\\ L_n\left(\left(\frac{t}{1+t}\right)^2,x\right) &= \frac{n^2x^2}{(1+n)^2(1+x)^2}+\frac{nx}{(1+n)^2(1+x)^2}\end{aligned}$$

we obtain

$$b_n=\sup_{x\geq 0}\left|L_n\left(\frac{t}{1+t},x\right)-\frac{x}{1+x}\right|=\frac{1}{n+1}$$

and

$$c_n = \sup_{x\geq 0}\left| L_n\left(\left(\frac{t}{1+t}\right)^2, x\right) - \left(\frac{x}{1+x}\right)^2\right| = \sup_{x\geq 0}\frac{\left|nx - x^2(2n+1)\right|}{(1+n)^2(1+x)^2}.$$

After some computations $c_n = \frac{2n+1}{(n+1)^2}$, which gives

$$a_n + 2b_n + c_n \leq \frac{4}{n+1},$$

and so the corollary is proved. □

Remark 3.6. In the papers [5] and [4], it is defined the space H_w: for a function w of the type of modulus of continuity, having the properties:

(i) w is non-negative increasing function on $[0,\infty)$,

(ii) $\lim_{\delta\to 0} w(\delta) = 0$,

the space H_w consists of all real-valued functions f defined on the semiaxis $[0,\infty)$, satisfying the following condition:

$$|f(x) - f(y)| \leq w\left(\left|\frac{x}{1+x} - \frac{y}{1+y}\right|\right), \quad \text{for all } x, y \geq 0.$$

It is proved that $H_w \subset C[0,\infty)\cap B[0,\infty)$ and $\|L_n f - f\|_\infty \to 0$, for $f \in H_w$. But, let us notice that $H_w \subset C^*[0,\infty)$. Indeed, considering $\varphi(x) = x/(1-x)$, $x \in [0,1)$, the inverse of the function $t \mapsto t/(1+t)$ and considering $f \in H_w$, we have

$$\left|f\left(\frac{u}{1-u}\right) - f\left(\frac{v}{1-v}\right)\right| \leq w(|u-v|), \quad \text{for all } u, v \in [0,1).$$

Using the property (ii) of w, we deduce that $f\circ\varphi$ is uniformly continuous on $[0,1)$. From this, it follows that $f\circ\varphi$ has finite limit at $x=1$, which proves that f has finite limit at infinity.

So, the result obtained in Corollary 3.5 for the space $C^*[0,\infty)$ is more general than the results obtained in the papers mentioned above.

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ON A NEW SEQUENCE SPACE DEFINED BY MUSIELAK-ORLICZ FUNCTIONS

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Abstract. In this paper we define a new sequence space $m(\mathcal{M}, \phi, p)$, which is a generalization of $m(\phi, p)$ (B. C. Tripathy and M. Sen [12]) by Musielak-Orlicz functions. We study some of the properties of this space.

1. Introduction

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by

$$M(x + y) \leq M(x) + M(y)$$

then this function is called a modular function, defined and discussed by Nakano [10] and Musielak [7] and others. It is well known that if M is a convex functions and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$ (see [1], [2], [9]).

Lindendstrauss and Tzafriri [5] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{ x = (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$||x|| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(x) = x^p$, $1 \leq p < \infty$, the space ℓ_M coincides with the classical sequence space l_p.

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A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function [See [3], [4], [6], [7]). In addition, a **Musielak-Orlicz function** $N = (N_k)$ is called a complementary function of a Musielak-Orlicz function $\mathcal{M}$ if

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, \cdots$$

For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $l_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$l_{\mathcal{M}} := \{x \in s : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\},$$

$$h_{\mathcal{M}} := \{x \in s : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k),\ x = (x_k) \in l_{\mathcal{M}}.$$

We consider $l_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\{k > 0 : I_{\mathcal{M}}(\frac{x}{k}) \leq 1\},$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\{\frac{1}{k}(1 + I_{\mathcal{M}}(kx)) : k > 0\}.$$

If $x = (x_n)$ is a sequence, then $S(x)$ denotes the set of all permutation of the elements of (x_n). A sequence space E is said to be symmetric if $S(x) \subset E$ for all $x \in E$. A sequence space E is said to be solid if $(y_n) \in E$ whenever $(x_n) \in E$ and $|y_n| \leq |x_n|$ for all $n \in \mathbb{N}$.

A BK-space is a Banach sequence space E in which the coordinate maps are continuous, i.e. if $(x_k^{(n)})_k \in E$, then

$$||(x_k^{(n)}) - (x_k)|| \to 0 \ as\ n \to \infty$$

$$\Rightarrow |(x_k^{(n)}) - (x_k)| \to \ 0 \ as\ n \to \infty, \quad \text{for each fixed k.}$$

Let $\mathcal{C}$ denote the space whose elements are finite sets of distinct positive integers. Given any element σ of $\mathcal{C}$, we denote by $c(\sigma)$ the sequence $\{c_n(\sigma)\}$ which is such that $c_n(\sigma) = 1$ if $n \in \sigma, c_n(\sigma) = 0$ otherwise. Further, let

$$\mathcal{C}_s = \left\{\sigma \in \mathcal{C} : \sum_{n=1}^{\infty} c_n(\sigma) \leq s\right\} \text{(cf.[8])},$$

be the set of those σ whose support has cardinality at most s. Throughout the paper ϕ_n denotes a non-decreasing sequence of positive numbers such that $n\phi_{n+1} \leq (n+1)\phi_n$ for all $n \in \mathbb{N}$.

The space $m(\phi)$ is defined as follows (Sargent [11]):

$$m(\phi) := \left\{ x = (x_k) \in \omega : \sup_{s\geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \sum_{k\in\sigma} |x_k| < \infty \right\}.$$

The space $m(\phi, p)$ is defined as follows (B.C. Tripathy and M. Sen [12]):
For $1 \leq p < \infty$,

$$m(\phi, p) := \left\{ x = (x_k) \in \omega : \sup_{s\geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \left\{ \sum_{k\in\sigma} |x_k|^p \right\}^{1/p} < \infty \right\}.$$

In this paper we introduce the space $m(\mathcal{M}, \phi, p)$ as follows:

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. We define the following sequence space

$$m(\mathcal{M}, \phi, p) := \left\{ x = (x_k) \in \omega : \sup_{s\geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \left\{ \sum_{k\in\sigma} \left[M_k \left(\frac{|x_k|}{\rho} \right) \right]^p \right\}^{1/p} < \infty, \text{ for some } \rho > 0 \right\}.$$

It is clear that if $M_k(x) = x$ then $m(\mathcal{M}, \phi, p) = m(\phi, p)$.

Throughout ω, l^p, l^1, l^∞ denote the spaces of all p-absolutely summable, absolutely summable and bounded sequences respectively. $\mathbb{N}$ and $\mathbb{C}$ denotes the set of all natural numbers and complex numbers, respectively.

2. **Main results**

Theorem 2.1. *The space $m(\mathcal{M}, \phi, p)$ is complete.*

Proof. Let $\{x^{(n)}\}$ be a Cauchy sequence in $m(\mathcal{M}, \phi, p)$. Then

$$\sup_{s\geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \left\{ \sum_{i\in\sigma} \left[M_i \left(\frac{|x_i|}{\rho} \right) \right]^p \right\}^{1/p} < \infty,$$

for some $\rho > 0$ and for all n $n = 1, 2, 3, \cdots$).

For each $\epsilon > 0$, there exists a positive integer n_0 such that

$$||x^{(m)} - x^{(n)}||_{m(\mathcal{M},\phi,p)} < \epsilon, \text{ for all } m, n \geq n_0.$$

This implies that

$$\sup_{s\geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \left\{ \sum_{i\in\sigma} \left[M_i \left(\frac{|x_i^{(m)} - x_i^{(n)}|}{\rho} \right) \right]^p \right\}^{1/p} < \epsilon, \tag{2.1}$$

for some $\rho > 0$ and for all $m, n \geq n_0$.

Hence

$$|x_i^{(m)} - x_i^{(n)}| < \epsilon\phi_1 \text{ for all } m, n \geq n_0 \textit{ and} \text{ for all } i \in \mathbb{N},$$

showing that for each fixed i $(1 \leq i < \infty)$, the sequence $\{x_i^{(n)}\}$ is a Cauchy sequence in $\mathbb{C}$.

Let $x_i^{(n)} \to x_i$ as $n \to \infty$. We define $x = (x_1, x_2, x_3, \cdots)$. We need to show that $x \in m(\mathcal{M}, \phi, p)$ and $x^{(n)} \to x$.

From (2.1) we get, for each fixed s

$$\sum_{i \in \sigma} \left[M_i \left(\frac{|x_i^{(m)} - x_i^{(n)}|}{\rho} \right) \right]^p < \epsilon^p {\phi_s}^p \text{ , for some } \rho > 0, \text{ for all } m, n \geq n_0 \text{ and } \sigma \in \mathcal{C}_s.$$

Taking $n \to \infty$ we get

$$\sum_{i \in \sigma} \left[M_i \left(\frac{|x_i^{(m)} - x_i^{(n)}|}{\rho} \right) \right]^p < \epsilon^p {\phi_s}^p \text{ , for some } \rho > 0, \text{ for all } m, n \geq n_0 \text{ and } \sigma \in \mathcal{C}_s.$$

This implies that

$$\sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \left\{ \sum_{i \in \sigma} \left[M_i \left(\frac{|x_i^{(m)} - x_i|}{\rho} \right) \right]^p \right\}^{1/p} < \epsilon, \tag{2.2}$$

for some $\rho > 0$ and for all $m, n \geq n_0$.

$$\Rightarrow x^{(n)} - x \in m(\mathcal{M}, \phi, p), \text{ for all } n \geq n_0.$$

Hence $x = x^{(n_0)} + x - x^{(n_0)} \in m(\mathcal{M}, \phi, p)$ as $m(\mathcal{M}, \phi, p)$ is a linear space.

From (2.2)

$$||x^{(n)} - x||_{m(\mathcal{M}, \phi, p)} < \epsilon, \text{ for all } n \geq n_0,$$

which implies that

$$||x^{(n)} - x||_{m(\mathcal{M}, \phi, p)} \to 0, \text{ as } n \to \infty.$$

Hence $m(\mathcal{M}, \phi, p)$ $(1 \leq p < \infty)$ is a Banach space.

Theorem 2.2. *The space $m(\mathcal{M}, \phi, p)$ is a BK-space.*

Proof. Suppose that

$$||x^{(n)} - x||_{m(\mathcal{M}, \phi, p)} \to 0 \text{ } as \text{ } n \to \infty.$$

For each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$||x^{(n)} - x|| < \epsilon \text{ for all } n \geq n_0.$$

This implies that

$$\sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \left\{ \sum_{k \in \sigma} \left[M_k \left(\frac{|x_k^{(n)} - x_k|}{\rho} \right) \right]^p \right\}^{1/p} < \epsilon, \text{ for some } \rho > 0 \text{ and for all } n \geq n_0.$$

Consequently

$$|x_k^{(n)} - x_k| < \epsilon\phi_1, \text{ for all } \ n \geq n_0 \ \text{ and for all } \ k.$$

So $|x_k^{(n)} - x_k| \to 0$ *as* $n \to \infty$ and the proof is complete. □

Proposition 2.3. *1. The space $m(\mathcal{M}, \phi, p)$ is a symmetric space. If $x \in m(\mathcal{M}, \phi, p)$ and $v \in S(x)$, then $||v||_{m(\mathcal{M},\phi,p)} = ||x||_{m(\mathcal{M},\phi,p)}$.*

2. The space $m(\mathcal{M}, \phi, p)$ is a normal space.

Proposition 2.4. $m(\phi) \subseteq m(\mathcal{M}, \phi, p)$.

Proof. Suppose that $x \in m(\phi)$. Then

$$||x||_{m(\phi)} = \sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \frac{1}{\phi_s} \left\{ \sum_{n\in\sigma} |x_n| \right\} = K < \infty.$$

Hence for each fixed s,

$$\sum_{n\in\sigma} |x_n| \leq K\phi_s, \ \sigma \in \mathcal{C}_s.$$

This implies that

$$\left\{ \sum_{n\in\sigma} \left[M_n \left(\frac{|x_n|}{\rho} \right) \right]^p \right\}^{1/p} \leq K\phi_s, \ \sigma \in \mathcal{C}_s, \text{ for some } \rho > 0,$$

so that

$$\sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \left[\frac{1}{\phi_s} \left\{ \sum_{n\in\sigma} \left[M_n \left(\frac{|x_n|}{\rho} \right) \right]^p \right\}^{1/p} \right] \leq K, \text{ for some } \rho > 0.$$

Thus $x \in m(\mathcal{M}, \phi, p)$ and this completes the proof. □

Proposition 2.5. *$m(\mathcal{M}, \phi, p) \subseteq m(\mathcal{M}, \psi, p)$ if and only if $\sup_{s\geq 1}(\frac{\phi_s}{\psi_s}) < \infty$.*

*Proof.*Let $\sup_{s\geq 1}(\frac{\phi_s}{\psi_s}) = K < \infty$. Then $\phi_s \leq K\psi_s$. Now if $(x_k) \in m(\mathcal{M}, \phi, p)$, then

$$\sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \left[\frac{1}{\phi_s} \left\{ \sum_{n\in\sigma} \left[M_n \left(\frac{|x_n|}{\rho} \right) \right]^p \right\}^{1/p} \right] < \infty, \text{ for some } \rho > 0 \ .$$

This implies that

$$\sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \left[\frac{1}{K\psi_s} \left\{ \sum_{n\in\sigma} \left[M_n \left(\frac{|x_n|}{\rho} \right) \right]^p \right\}^{1/p} \right] < \infty, \text{ for some } \rho > 0,$$

so that

$$||x||_{m(\mathcal{M},\psi,p)} < \infty.$$

Hence $m(\mathcal{M}, \phi, p) \subseteq m(\mathcal{M}, \psi, p)$.

Conversely, suppose that $m(\mathcal{M},\phi,p) \subseteq m(\mathcal{M},\psi,p)$. We need to show that

$$\sup_{s\geq 1}(\frac{\phi_s}{\psi_s}) = \sup_{s\geq 1}(\eta_s) < \infty.$$

Let $\sup_{s\geq 1}(\eta_s) = \infty$. Then there exists a subsequence (η_{s_i}) of (η_s) such that

$$\lim_{i\to\infty}(\eta_{s_i}) = \infty.$$

Then for $(x_k) \in m(\mathcal{M},\phi,p)$ we have

$$\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\left[\frac{1}{\psi_s}\left\{\sum_{n\in\sigma}\left[M_n\left(\frac{|x_n|}{\rho}\right)\right]^p\right\}^{1/p}\right]$$

$$\geq \sup_{s_i\geq 1}\sup_{\sigma\in\mathcal{C}_{s_i}}\left[\psi_{s_i}\frac{1}{\phi_{s_i}}\left\{\sum_{n\in\sigma}\left[M_n\left(\frac{|x_n|}{\rho}\right)\right]^p\right\}^{1/p}\right] = \infty,$$

for some $\rho > 0$.

This implies that $(x_k) \notin m(\mathcal{M},\psi,p)$, a contradiction which completes the proof. □

Theorem 2.6. $l^p \subseteq m(\mathcal{M},\phi,p) \subset l^\infty$.

Proof. Since $m(\mathcal{M},\phi,p) = l^p$ for $M_k(x) = x$ and $\phi_n = 1$, for all $n \in \mathbb{N}$, it follows that $l^p \subseteq m(\mathcal{M},\phi,p)$.

Next, let $x \in m(\mathcal{M},\phi,p)$. Then

$$\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\left[\frac{1}{\phi_s}\left\{\sum_{n\in\sigma}\left[M_n\left(\frac{|x_n|}{\rho}\right)\right]^p\right\}^{1/p}\right] = K < \infty, \text{ for some } \rho > 0.$$

This implies that

$$|x_n| \leq K\phi_1, \text{ for all } n \in \mathbb{N},$$

so that $x \in l^\infty$. Thus $m(\mathcal{M},\phi,p) \subset l^\infty$.

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ALEXANDER TRANSFORM OF CLOSE-TO-CONVEX FUNCTIONS

PÁL AUREL KUPÁN AND RÓBERT SZÁSZ

Abstract. In this paper a result concerning the starlikeness of the image of the Alexander Operator is deduced. The technique of differential subordinations is used.

1. Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1$ be the open unit disc of the complex plane.

We denote by $\mathcal{A}$ the class of analytic functions defined on the unit disc U and having the form $f(z) = z + a_2z^2 + a_3z^3 + \dots$.

The subclass of $\mathcal{A}$ consisting of functions for which the domain $f(U)$ is starlike with respect to 0, is called the class of starlike functions, and is denoted by S^*. An analytic description of S^* is

$$S^* = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ (\forall)\ z \in U \right\}.$$

Let $\alpha \in [0, 1)$. The class of starlike functions of order α denoted by $S^*(\alpha)$, is defined by the equality:

$$S^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha,\ (\forall)\ z \in U \right\}.$$

Another subclass of $\mathcal{A}$ which we deal with, is defined by

$$C = \left\{ f \in \mathcal{A} \mid (\exists)\ g \in S^* : \operatorname{Re} \frac{zf'(z)}{g(z)} > 0,\ z \in U \right\}.$$

This is the class of close-to-convex functions.

We mention that $C,\ S^*$ and $S^*(\alpha)$ contain univalent functions.

The Operator of Alexander is defined by

$$F(z) = A(f)(z) = \int_0^z \frac{f(t)}{t} dt. \tag{1.1}$$

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In [3] it has been proved that $A(C) \not\subset S^*$.

This result put the problem to determine suitable conditions which ensure that subclasses of C are mapped by the Alexander operator to S^*.

In [2] (pg. 310-311), the authors proved the following theorem concerning this question:

Theorem 1.1. *Let A be the operator of Alexander defined by (1.1) and let $g \in \mathcal{A}$ satisfy*

$$\mathrm{Re}\,\frac{zg'(z)}{g(z)} \geq \left|\mathrm{Im}\,\frac{z(zg'(z))'}{g(z)}\right|,\ z \in U. \tag{1.2}$$

If $f \in \mathcal{A}$ satisfies

$$\mathrm{Re}\,\frac{zf'(z)}{g(z)} > 0,\ z \in U,$$

then $F = A(f) \in S^*$.

We will prove another result regarding this problem. We will need the following definitions and lemmas in our work.

2. Preliminaries

The class $\mathcal{P}$ is defined by the equality:

$$\mathcal{P} = \{f | f \text{ analytic in U},\ f(0) = 1,\ \text{and } \mathrm{Re} f(z) > 0,\ z \in \mathrm{U}\}.$$

Lemma 2.1. [1]*(The Herglotz formula) For every $f \in \mathcal{P}$ there exists a measure μ on the interval $[0, 2\pi]$ so that $\mu([0, 2\pi]) = 1$ (a probability measure) and*

$$f(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t),$$

or in developed form

$$f(z) = 1 + 2\sum_{n=1}^{\infty} \int_0^{2\pi} z^n e^{-in} d\mu(t).$$

The converse of the theorem is also valid.

Lemma 2.2. [2] *p.26 Let $p(z) = a + \sum_{k=n}^{\infty} a_k z^k$, $p(z) \not\equiv a$ and $n \geq 1$. If $z_0 \in U$ and*

$$\mathrm{Re}\, p(z_0) = \min\{\mathrm{Re}\, p(z) : |z| \leq |z_0|\},$$

then

(i) $z_0 p'(z_0) \leq -\dfrac{n}{2} \dfrac{|p(z_0) - a|^2}{\mathrm{Re}\,(a - p(z_0))}$

and

(ii) $\mathrm{Re}\,[z_0^2 p''(z_0)] + z_0 p'(z_0) \leq 0.$

Lemma 2.3. *If $f, g \in \mathcal{A}$ and*

$$\mathrm{Re}\Big[\frac{1}{g'(z)}\int_0^1\int_0^1 g'(uvz)\frac{1+uvze^{-it}}{1-uvze^{-it}}dudv\Big] \geq 0,\ \ z \in U,\ t \in \mathbb{R}, \tag{2.1}$$

then the inequality $\mathrm{Re}\frac{f'(z)}{g'(z)} > 0,\ \ z \in U$ *implies that*

$$\mathrm{Re}\frac{F(z)}{zg'(z)} > 0,\ \ z \in U, \tag{2.2}$$

where F is defined by (1.1).

Proof. The developments

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

hold for $z \in U$.

The conditions of the lemma imply $\frac{f'}{g'} \in \mathcal{P}$ and from the Herglotz formula it follows that:

$$\frac{f'(z)}{g'(z)} = 1 + 2\int_0^{2\pi}\left(\sum_{n=1}^{\infty} z^n e^{-in}\right)d\mu(t),\ \ z \in U$$

for a suitable probability measure μ.

Denoting $c_n = 2\int_0^{2\pi} e^{-in}d\mu(t)$, we get:

$$f'(z) = g'(z)(1 + \sum_{n=1}^{\infty} c_n z^n)$$

$$= (1 + \sum_{n=2}^{\infty} nb_n z^{n-1})(1 + \sum_{n=1}^{\infty} c_n z^n) = 1 + \sum_{n=1}^{\infty} d_n z^n, \tag{2.3}$$

$$f(z) = z + \sum_{n=1}^{\infty} \frac{d_n}{n+1} z^{n+1}$$

and

$$\frac{F(z)}{z} = 1 + \sum_{n=1}^{\infty} \frac{d_n}{(n+1)^2} z^n.$$

Thus we have

$$\frac{F(z)}{zg'(z)} = \frac{1}{g'(z)}\int_0^1\int_0^1 \big(1 + \sum_{n=1}^{\infty} d_n u^n v^n z^n\big)dudv,$$

and according to (2.3), this is equivalent to

$$\frac{F(z)}{zg'(z)} = \frac{1}{g'(z)}\int_0^{2\pi}\int_0^1\int_0^1 g'(uvz)\frac{1+uvze^{-it}}{1-uvze^{-it}}dudvd\mu(t),$$

and the proof is finished. □

Lemma 2.4. *The following inequality holds:*

$$1+r^4u^2\rho^2-r^2\rho^2\sqrt{(1+u^2)^2+\frac{4u^2\rho^2(1-u^2)^2\sin^2\alpha}{(1-u^2\rho^2)^2(1+u^2)^2}}\cos(2\theta+\gamma)\geq$$
$$1+r^4u^2-r^2\sqrt{1+6u^2+u^4},\quad \rho,u\in[0,1];\theta,\alpha\in\mathbb{R}.$$

Proof. It is easily seen that:

$$1+r^4u^2\rho^2-r^2\rho^2\sqrt{(1+u^2)^2+\frac{4u^2\rho^2(1-u^2)^2\sin^2\alpha}{(1-u^2\rho^2)^2(1+u^2)^2}}\cos(2\theta+\gamma)\geq$$
$$1+r^4u^2\rho^2-r^2\rho^2\sqrt{(1+u^2)^2+\frac{4u^2\rho^2(1-u^2)^2}{(1-u^2\rho^2)^2(1+u^2)^2}} \tag{2.4}$$

Since

$$1\geq\rho^2$$

and

$$-r^4u^2+r^2\sqrt{1+6u^2+u^4}\geq -r^4u^2+r^2\sqrt{(1+u^2)^2+\frac{4u^2\rho^2(1-u^2)^2}{(1-u^2\rho^2)^2(1+u^2)^2}}\geq 0$$
$$r,u,\rho\in[0,1]$$

it follows that

$$-r^4u^2+r^2\sqrt{1+6u^2+u^4}\geq -r^4u^2\rho^2+r^2\rho^2\sqrt{(1+u^2)^2+\frac{4u^2\rho^2(1-u^2)^2}{(1-u^2\rho^2)^2(1+u^2)^2}}$$
$$r,u,\rho\in[0,1].$$

Thus

$$1+r^4u^2\rho^2-r^2\rho^2\sqrt{(1+u^2)^2+\frac{4u^2\rho^2(1-u^2)^2}{(1-u^2\rho^2)^2(1+u^2)^2}}\geq$$
$$1+r^4u^2-r^2\sqrt{1+6u^2+u^4}\quad r,u,\rho\in[0,1]. \tag{2.5}$$

The desiderated inequality follows by (2.4) and (2.5). □

3. **Main result**

Theorem 3.1. *Let $g\in\mathcal{A}$ be a function having the property:*

$$Re\frac{g'(uz)}{g'(z)}\frac{1+uw}{1-uw}>0,\ \textit{for all}\ \ u\in(0,1)\ \textit{and}\ z,w\in U,|z|=|w|. \tag{3.1}$$

Provided that $f \in \mathcal{A}$, and the function h defined by $h(z) = zg'(z)$ satisfies the inequality

$$Re\frac{zf'(z)}{h(z)} > 0 \quad z \in U, \tag{3.2}$$

then $F = A(f) \in S^*$.

Proof. We differentiate twice the equality $F(z) = \int_0^z \frac{f(t)}{t}$ and we get: $zF''(z) + F'(z) = f'(z)$. If we set $p(z) = \frac{zF'(z)}{F(z)}$, then this equality can be rewritten as follows:

$$\frac{F(z)}{zg'(z)}(zp'(z) + p^2(z)) = \frac{zf'(z)}{h(z)}.$$

The conditions of the theorem imply:

$$\mathrm{Re}\big[\frac{F(z)}{zg'(z)}(zp'(z) + p^2(z))\big] > 0, \text{ for all } z \in U. \tag{3.3}$$

If the inequality $\mathrm{Re}p(z) > 0$ does not hold for all $z \in U$, then according to Lemma 2 (in case of $a = 1$) there is a point $z_0 \in U$ and there are two real numbers $x, y \in \mathbb{R}$ having the property:

$$p(z_0) = ix$$

$$z_0 p'(z_0) = y \leq -\frac{x^2+1}{2}.$$

Thus it follows that:

$$\mathrm{Re}\big[\frac{F(z_0)}{z_0 g'(z_0)}(z_0 p'(z_0) + p^2(z_0))\big] = \mathrm{Re}\frac{F(z_0)}{z_0 g'(z_0)}(y - x^2). \tag{3.4}$$

Since $\mathrm{Re}\frac{f'(z)}{g'(z)} = \mathrm{Re}\frac{zf'(z)}{h(z)} > 0, z \in U$, Lemma 3 and condition (3.1) lead to the inequality $\mathrm{Re}\frac{F(z)}{zg'(z)} > 0,\ z \in U$. This inequality and (3.4) imply

$$\mathrm{Re}\frac{z_0 f'(z_0)}{h(z_0)} = \mathrm{Re}\big[\frac{F(z_0)}{z_0 g'(z_0)}(z_0 p'(z_0) + p^2(z_0))\big] \leq 0$$

which contradicts (3.3). The contradiction shows that $\mathrm{Re}p(z) > 0$ for all $z \in U$, and this is equivalent to $F \in S^*$. □

Corollary 3.2. *If $Re\frac{f'(z)}{e^z} > 0$ for all $z \in U$, then $A(f) \in S^*$.*

Proof. We apply Theorem 2 to prove this assertion. In case of $g(z) = e^z - 1$, $z = re^{i\theta}$ and $w = re^{i\alpha}$, $r \in (0,1)$ the following equality holds:

$$\mathrm{Re}\frac{g'(uz)}{g'(z)}\frac{1+uw}{1-uw} = \frac{e^{r(u-1)\cos\theta}(1-u^2r^2)}{1+u^2r^2-2ur\cos\alpha}\big\{\cos[r(1-u)\sin\theta] + \frac{2ur\sin\alpha}{1-u^2r^2}\sin[r(1-u)\sin\theta]\big\} \tag{3.5}$$

There is a real number $v \in (-\frac{\pi}{2}, \frac{\pi}{2})$ having the property $\tan v = \frac{2ur\sin\alpha}{1-u^2r^2}$. Therefore the equality (3.5) can be rewritten in the following way:

$$\mathrm{Re}\frac{g'(uz)}{g'(z)}\frac{1+uw}{1-uw} = \frac{e^{r(u-1)\cos\theta}(1-u^2r^2)}{(1+u^2r^2-2ur\cos\alpha)\cos v}\cos[r(1-u)\sin\theta - v].$$

This means that in order to prove condition (3.1) of Theorem 2, we have to prove the inequality: $\cos[r(1-u)\sin\theta - v] > 0,\ r, u \in (0,1),\ \alpha, \theta \in \mathbb{R}$.

Since $|r(1-u)\sin\theta - v| = |r(1-u)\sin\theta - \arctan\frac{2ur\sin\alpha}{1-u^2r^2}| \leq r(1-u) + \arctan\frac{2ur}{1-u^2r^2} < 1-u+\arctan\frac{2u}{1-u^2}$, and $\varphi'(u) = \frac{1-u^2}{1+u^2} > 0$ where $\varphi : (0,1) \to \mathbb{R}$, $\varphi(u) = 1-u+\arctan\frac{2u}{1-u^2}$, the inequality follows $|r(1-u)\sin\theta - v| < \lim_{u\to 1}\varphi(u) = \frac{\pi}{2}$.

Thus condition (3.1) also holds, and applying Theorem 2 the proof is done. □

Remark 3.3. In case of $g(z) = e^z - 1$, it is easily seen that $g \in \mathcal{A}$ and $h(z) = zg'(z) = ze^z$ and $\mathrm{Re}\big(\frac{zh'(z)}{h(z)}\big) = \mathrm{Re}(1+z) > 0,\ z \in U$, consequently $h \in S^*$ holds. Thus the differential inequality $Re\frac{zf'(z)}{h(z)} = Re\frac{f'(z)}{e^z} > 0$, $z \in U$, defines a subclass of C and this subclass is mapped by the Operator of Alexander in S^*.

Corollary 3.4. *If* $0 < r \leq \big(3 - 8^{\frac{1}{2}}\big)^{\frac{1}{4}} = 0,643...$ *and*

$$Re(1-r^2z^2)f'(z) > 0, \quad z \in U, \tag{3.6}$$

then $A(f) \in S^*$.

Proof. We apply again Theorem 2 to prove this assertion. Let $g : U \to \mathbb{C}$ be the mapping defined by the equality: $g(z) = \frac{1}{2r}\log\frac{1+rz}{1-rz}$, $r \in (0,1]$, and $h(z) = zg'(z) = \frac{z}{1-r^2z^2}$. We have to prove condition (3.1) in case of $z = \rho e^{i\theta}$ and $w = \rho e^{i\alpha}$. The following equalities hold:

$$\mathrm{Re}\frac{g'(uz)}{g'(z)}\frac{1+uw}{1-uw} = \mathrm{Re}\frac{1-r^2\rho^2e^{2i\theta}}{1-r^2u^2\rho^2e^{2i\theta}}\frac{1+u\rho e^{i\alpha}}{1-u\rho e^{i\alpha}} =$$

$$\frac{(1-u^2\rho^2)[1+r^4u^2\rho^2 - r^2\rho^2(1+u^2)\cos 2\theta + 2\frac{1-u^2}{1-u^2\rho^2}ur^2\rho^3\sin 2\theta\sin\alpha]}{|1-r^2u^2e^{2i\theta}|^2|1-ue^{i\alpha}|^2}. \tag{3.7}$$

According to (3.7) condition (3.1) holds if and only if:

$$1+r^4u^2\rho^2 - r^2\rho^2(1+u^2)\cos 2\theta + 2\frac{1-u^2}{1-u^2\rho^2}ur^2\rho^3\sin 2\theta\sin\alpha \geq 0,$$
$$\rho, u \in [0,1]; \theta, \alpha \in \mathbb{R},$$

and this is equivalent to

$$1+r^4u^2\rho^2 - r^2\rho^2(1+u^2)\big[\cos 2\theta - 2\frac{1-u^2}{(1-u^2\rho^2)(1+u^2)}u\rho\sin 2\theta\sin\alpha\big] \geq 0,$$
$$\rho, u \in [0,1]; \theta, \alpha \in \mathbb{R}.$$

Using the notation $\tan\gamma = \frac{2u\rho(1-u^2)\sin\alpha}{(1-u^2\rho^2)(1+u^2)}$, $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ it can be rewritten as follows:

$$1 + r^4u^2\rho^2 - r^2\rho^2\sqrt{(1+u^2)^2 + \frac{4u^2\rho^2(1-u^2)^2\sin^2\alpha}{(1-u^2\rho^2)^2(1+u^2)^2}}\cos(2\theta+\gamma) \geq 0, \quad u, \rho \in [0,1]; \theta, \alpha \in \mathbb{R}. \tag{3.8}$$

According to Lemma 4 we have:

$$1 + r^4u^2\rho^2 - r^2\rho^2\sqrt{(1+u^2)^2 + \frac{4u^2\rho^2(1-u^2)^2\sin^2\alpha}{(1-u^2\rho^2)^2(1+u^2)^2}}\cos(2\theta+\gamma) \geq 1 + r^4u^2 - r^2\sqrt{1+6u^2+u^4}, \rho, u \in [0,1]; \theta, \alpha \in \mathbb{R}.$$

Inequality (3.8) holds provided that:

$$1 + r^4u^2 - r^2\sqrt{1+6u^2+u^4} \geq 0, \ u \in [0,1].$$

The last inequality is equivalent to

$$1 - r^4 - 4r^4u^2 - r^4(1-r^4)u^4 \geq 0, \ u \in [0,1],$$

which holds for all $u \in [0,1]$ if and only if:

$$1 - 6r^4 + r^8 \geq 0, \quad r \in (0,1]$$

and this leads to $0 < r \leq \left(3 - 8^{\frac{1}{2}}\right)^{\frac{1}{4}}$. □

Remark 3.5. 1. Since $g, h \in \mathcal{A}$ and

$$\mathrm{Re}\frac{zh'(z)}{h(z)} = \mathrm{Re}\frac{1+r^2z^2}{1-r^2z^2} > 0, z \in U, \ r \in [0,1],$$

follows that $h \in S^*$. Thus condition (3.6) defines a subclass of C.

2. It remains an interesting open question to determine the biggest $r \in [0,1]$ for which the class of analytic functions defined by the conditions

$$f \in \mathcal{A}, \ \mathrm{Re}(1 - r^2z^2)f'(z) > 0, \ z \in U$$

is mapped in S^*, by the Alexander Operator.

3. Since Corollary 1 and Corollary 2 can not be proved using Theorem 1, we may assert that Theorem 2 is independent from Theorem 1, in spite of the fact, that the ideas of their proofs are analogous.

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DATA DEPENDENCE FOR SOME INTEGRAL EQUATIONS

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Abstract. In the paper *Integral equations, periodicity and fixed points*, published in Fixed Point Theory, 9(2008), No 1, 47-65 the author T.A. Burton considered the equation

$$x(t) = g(t) + \int_{-\infty}^{t} K(t, s, x(s))ds.$$

In this paper we shall study the data dependence for this integral equations.

1. **Introduction**

Let $(P_T, \|\cdot\|)$ denote the Banach space of continuous scalar $T-$periodic functions with the supremum norm.

We consider the equation

$$x(t) = g(t) + \int_{-\infty}^{t} K(t, s, x(s))ds, \ t \in \mathbb{R} \tag{1.1}$$

under the conditions:

(C_1) there exists $T > 0$ such that

$$g(t + T) = g(t), \ K(t + T, s + T, u) = K(t, s, u)$$

for all $t, s, u \in \mathbb{R}$;

(C_2) for all $x \in P_T$ we have that $\int_{-\infty}^{(\cdot)} K((\cdot), s, x(s))ds \in P_T$

Now we define the operator

$$A : P_T \to P_T,$$

$$A(x)(t) = g(t) + \int_{-\infty}^{t} K(t, s, x(s))ds.$$

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In [1], T.A. Burton was considered the following conditions

(C_3) there exists a function $B(t,s)$ with $\int_{-\infty}^{t} B(t,s)$ defined such that

$$|K(t,s,u) - K(t,s,v)| \leq B(t,s)|u-v|,$$

for all $-\infty < s \leq t < \infty$, $u, v \in \mathbb{R}$;

(C_4) there exists $0 < \alpha < 1$ such that $\int_{-\infty}^{t} B(t,s) \leq \alpha$.

Under conditions $(C_1) - (C_4)$ we have that the operator A has a unique fixed point $x_A^\star$, and $A^n(x) \to x_A^\star$ for $n \to \infty$ and for all $x \in P_T$, so the operator A is Picard(I.A. Rus [3])

The purpose of this article is to establish a Gronwall type lemma corresponding to the equation (1.1) and also data dependence theorems, comparison theorems for the solutions of the equation (1.1). More results about nonlinear integral equations we find in [2].

2. A Gronwall type inequalities

We consider the following integral inequalities:

$$x(t) \leq g(t) + \int_{-\infty}^{t} K(t,s,x(s))ds,\ t \in \mathbb{R} \tag{2.1}$$

$$x(t) \geq g(t) + \int_{-\infty}^{t} K(t,s,x(s))ds\ t \in \mathbb{R}. \tag{2.2}$$

Throughout this section we use the following

Lemma 2.1. *I.A. Rus* [5] *Let (X,d) be an ordered metric space and $A : X \to X$ be such that:*

(i) *the operator A is Picard, with the set of fixed points $F_A = \{x_A^\star\}$;*

(ii) *the operator A is monotone increasing.*

Then

(a) *$x \leq A(x)$ implies $x \leq x_A^\star$;*

(b) *$x \geq A(x)$ implies $x \geq x_A^\star$;*

We have

Theorem 2.2. *We suppose that:*

(i) *the conditions $(C_1) - (C_4)$ hold;*

(ii) *the operator $K(t,s,\cdot)$ is monotone increasing, for all $-\infty < s \leq t < \infty$.*

Then

(a) *the equation (1.1) has a unique solution* $x^\star$;

(b) *for all solutions* $x \in P_T$ *of the inequality (2.1) we have that* $x \leq x^\star$*;*

(c) *for all solutions* $x \in P_T$ *of the inequality (2.2) we have that* $x \geq x^\star$,

Proof. (a) We consider the operator

$$A : P_T \to P_T,$$

$$A(x)(t) = g(t) + \int_{-\infty}^{t} K(t,s,x(s))ds.$$

T.A Burton [1] proves that the operator A is Picard operator, $F_A = \{x^\star\}$.

(b)+(c) From the condition (ii) we obtain that A is an increasing operator. Then, by Lemma 2.1 we have the conclusions.

3. **A comparison result**

Now we shall give a comparison result for the solution of the equation (1.1). For this study we need the following abstract result ([5]).

Lemma 3.1. *Let* $(X, d, \leq)$ *be an ordered metric space and* $A, B, C : X \to X$ *be such that:*

(i) $A \leq B \leq C$

(ii) A, B, C *are Picard operators,* $F_A = \{x_A^\star\}$, $F_B = \{x_B^\star\}$, $F_C = \{x_C^\star\}$*;*

(iii) *the operator* B *is increasing.*

Then

$$x_A^\star \leq x_B^\star \leq x_C^\star.$$

We consider the equations

$$(4)_i \quad x(t) = g_i(t) + \int_{-\infty}^{t} K(t,s,x(s))ds,\ t \in \mathbb{R},\ i = \overline{1,3},$$

We have

Theorem 3.2. *We consider the equation* $(4)_i$*. We suppose that:*

(i) g_i *and* K_i, $i = \overline{1,3}$, *satisfy the condition* (*i*) *in Theorem 2.2;*

(ii) $g_1(t) \leq g_2(t) \leq g_3(t)$ *and* $K_1(t,s,\cdot) \leq K_2(t,s,\cdot) \leq K_3(t,s,\cdot)$ *for all* $-\infty < s \leq t < \infty$;

(iii) $K_2(t,s,\cdot)$ *is monotone increasing for all* $-\infty < s \leq t < \infty$.

Then

(a) *the equations* $(4)_i$ *have a unique solution* $x_i^\star \in P_T$, $i = \overline{1,3}$

(b) $x_1^\star \leq x_2^\star \leq x_3^\star$.

Proof. (a) We consider the operator

$$A_i : P_T \to P_T,$$

$$A_i(x)(t) = g_i(t) + \int_{-\infty}^{t} K_i(t,s,x(s))ds, \; i = \overline{1,3}.$$

The condition (i) from Theorem 2.2 implies that the operators A_i are Picard with $F_{A_i} = \{x_i^\star\}, i = \overline{1,3}$.

(b) From the condition (ii) we have that $A_1 \leq A_2 \leq A_3$ and from (iii) we obtain that A_2 is an increasing operator. Then, from Lemma 2.1 we have the conclusion.

4. **Data dependence: Continuity**

Now we consider the equations

$$x(t) = g_1(t) + \int_{-\infty}^{t} K_1(t,s,x(s))ds, \; t \in \mathbb{R} \tag{4.1}$$

$$x(t) = g_2(t) + \int_{-\infty}^{t} K_2(t,s,x(s))ds, \; t \in \mathbb{R}. \tag{4.2}$$

We have

Theorem 4.1. *We suppose that*

(1) g_1, g_2, K_1, K_2 *satisfy the conditions* (i) *in Theorem 2.2;*

(2) *there exists* $\eta_1 >$ *such that*

$$|g_1(t) - g_2(t)| \leq \eta_1,$$

for all $t \in \mathbb{R}$*;*

(ii) *there exists a function* $\eta_2(t,s)$ *and* $\eta_3 > 0$ *such that*

$$\int_{-\infty}^{t} \eta_2(t,s)ds \leq \eta_3,$$

$$|K_1(t,s,u) - K_2(t,s,u)| \leq \eta_2(t,s),$$

for all $-\infty < s \leq t < \infty$, $u \in \mathbb{R}$.

Then

(a) *the equations (4.1), (4.2) have a unique solution* $x_1^\star$ *respectively* $x_2^\star$*;*

(b) $\|x_1^\star - x_2^\star\| \leq \frac{\eta_1+\eta_3}{1-\alpha}$.

Proof. (a) We define the operators

$$A_i : P_T \to P_T,$$

$$A_i(x)(t) = g_i(t) + \int_{-\infty}^{t} K_i(t,s,x(s))ds, i = \overline{1,2}.$$

The condition (i) from Theorem 2.2 implies that the operators A_i are Picard with $F_{A_i} = \{x_i^\star\}, i = \overline{1,2}$.

(b) Because

$$|A_1(x)(t) - A_2(x)(t)| \leq |g_1(t) - g_2(t)| + \int_{-\infty}^{t} |K_1(t,s,x(s)) - K_2(t,s,x(s))|ds \leq$$

$$\leq \eta_1 + \int_{-\infty}^{t} \eta_2(t,s)ds \leq \eta_1 + \eta_3$$

for all $x \in P_T$ and $t \in \mathbb{R}$, we obtain that

$$\|A_1(x) - A_2(x)\| \leq \eta_1 + \eta_3.$$

Now the proof follows from a well known abstract result([3], [4]).

5. **Smooth dependence on parameter**

Next we consider the following integral equation

$$x(t) = g(t,\lambda) + \int_{-\infty}^{t} K(t,s,x(s),\lambda)ds, t \in \mathbb{R}, \lambda \in J = [c,d] \subset \mathbb{R}. \quad (5.1)$$

Let $(P_T, \|\cdot\|)$ be the Banach space of continuous scalar $T-$periodic functions, defined on $\mathbb{R} \times J$, with the supremum norm.

We assume that

(H_1) $g, K \in C^1(\mathbb{R} \times J)$ and it verify the conditions (C_1), (C_2);

(H_2) there exists a function $B(t,s)$ such that

$$|\frac{\partial K}{\partial u}(t,s,u,\lambda)| \leq B(t,s),$$

for all $-\infty < s \leq t < \infty$, $u, v \in \mathbb{R}, \lambda \in J$;

(H_3) $\int_{-\infty}^{t} B(t,s)ds$ is defined and $\int_{-\infty}^{t} B(t,s)ds \leq \alpha < 1$.

We define the operator

$$B : P_T \to P_T,$$

$$B(x)(t,\lambda) = g(t,\lambda) + \int_{-\infty}^{t} K(t,s,x(s,\lambda),\lambda)ds.$$

It is clear that, in the conditions $(H_1) - (H_3)$ the operator B is Picard operator. Let $x^\star(\cdot,\lambda)$ be the unique fixed point of the operator B. Then

$$x^\star(t,\lambda) = g(t,\lambda) + \int_{-\infty}^{t} K(t,s,x^\star(s,\lambda),\lambda)ds \tag{5.2}$$

We suppose that there exists $\frac{\partial x^\star}{\partial \lambda}$. Then from (5.2) we have that

$$\frac{\partial x^\star}{\partial \lambda}(t,\lambda) = \frac{\partial g}{\partial \lambda}(t,\lambda) + \int_{-\infty}^{t} [\frac{\partial K}{\partial u}(t,s,x^\star(s,\lambda);\lambda)\frac{\partial x^\star(s,\lambda)}{\partial \lambda} + \frac{\partial K}{\partial \lambda}(t,s,x^\star(s,\lambda);\lambda)]ds$$

This relation suggest us to consider the following operator

$$C : P_T \times P_T \to P_T,$$

$$C(x,y)(t,\lambda) = \frac{\partial g}{\partial \lambda}(t,\lambda) + \int_{-\infty}^{t} [\frac{\partial K}{\partial u}(t,s,x(s,\lambda);\lambda)y(s,\lambda) + \frac{\partial K}{\partial \lambda}(t,s,x(s,\lambda);\lambda)]ds$$

In this way we have the triangular operator

$$A : P_T \times P_T \to P_T \times P_T,$$

$$A(x,y) = (B(x), C(x,y))$$

where B is a Picard operator and $C(x,\cdot) : P_T \to P_T$ is an $\alpha-$contraction.

From the theorem of fiber contraction (see I.A. Rus [5],[6]) we have that the operator A is Picard operator. So, the sequences

$$x_{n+1} = B(x_n), n \in \mathbb{N}$$

$$y_{n+1} = C(x_n,y_n), n \in \mathbb{N}$$

converges uniformly to $(x^\star,y^\star) \in F_A$, for all $x_0,y_0 \in P_T$.

If we take $x_0 = 0$, $y_0 = \frac{\partial x_0}{\partial \lambda} = 0$ then $y_1 = \frac{\partial x_1}{\partial \lambda}$ and by induction we prove that $y_n = \frac{\partial x_n}{\partial \lambda}$, for all $n \in \mathbb{N}^\star$.

Thus

$$x_n \to x^\star, \quad uniform \;\; as \;\; by \; n \to \infty$$

$$\frac{\partial x_n}{\partial \lambda} \to y^\star, \quad uniform \;\; as\; by\; n \to \infty$$

These imply that there exists $\frac{\partial x^\star}{\partial \lambda}$ and $\frac{\partial x^\star}{\partial \lambda} = y^\star$

From the above considerations, we have the following result

Theorem 5.1. *We consider the integral equation (5.1) in the hypothesis* $(H_1) - (H_3)$. *Then*

(i) *the equation (5.1) has a unique solution* $x^\star(t, \cdot) \in P_T$;

(ii) $x^\star(t, \cdot) \in C^1(J)$, *for all* $t \in \mathbb{R}$.

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MIXED CONVECTION IN A VERTICAL CHANNEL SUBJECT TO ROBIN BOUNDARY CONDITION

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Abstract. The steady mixed convection flow in a vertical channel is investigated for laminar and fully developed flow regime. In the modelling of the heat transfer the viscous dissipation term was also considered. Temperature on the right wall is assumed constant while a mixed boundary condition (Robin boundary condition) is considered on the left wall. The governing equations are expressed in non-dimensional form and then solved both analytically and numerically. It was found that there is a decrease in reversal flow with an increase in the mixed convection parameter.

1. **Introduction**

Heat transfer in channels occurs in many industrial processes and natural phenomena. It has been, therefore, the subject of many detailed, mostly numerical studies for different flow configurations. Most of the interest in this subject is due to its practical applications, for example, in the design of cooling systems for electronic devices and in the field of solar energy collection. Some of the published papers, such as by Aung [1], Aung et al. [2], Aung and Worku [3, 4], Barletta [5, 6], and Boulama and Galanis [7], are concerned with the evaluation of the temperature and velocity profiles for the vertical parallel-flow fully developed regime. As is well known, heat exchangers technology involves convective flows in vertical channels. In most cases, these flows imply conditions of uniform heating of a channel, which can be modelled either by uniform wall temperature (UWT) or uniform wall heat flux (UHF) thermal boundary conditions. In the present paper, new types of boundary conditions are considered. The right wall is kept at constant temperature while a convective heat flux is considered on the left wall (see, Bejan[8]):

$$\left(k\frac{\partial T}{\partial y}\right)_{y=0} + h_a\left(T_a - T\right)_{y=0} = 0 \tag{1.1}$$

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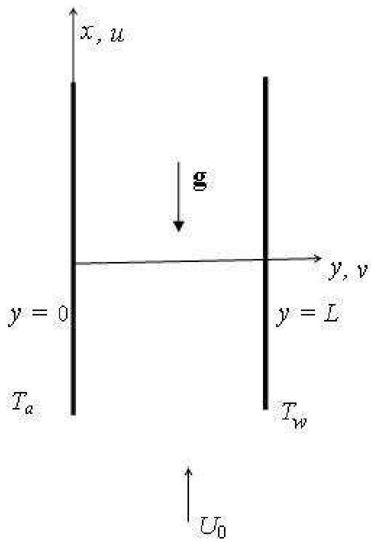

FIGURE 1. Geometry of the problem and the co-ordinate system

where k is the thermal conductivity, h_a is the external heat transfer coefficient and T_a is the external temperature (see Figure 1). This kind of boundary condition is appropriate to express mathematically heat loosing in insullation problems. In addition we have taken in account in this paper the effect of viscous dissipation, see Barletta [9].

2. Basic equations

Consider a viscous and incompressible fluid, which steadily flows between two infinite vertical and parallel plane walls. At the entrance of the channel the fluid has an entrance velocity U_0 parallel to the vertical axis of the channel. The geometry of the problem, the boundary conditions, and the coordinate system are shown in Fig. 1. The variation of density with temperature is given by the Boussinesq approximation and the fluid rises in the duct driven by buoyancy forces and initial velocity. Hence, the flow is due to difference in temperature and in the pressure gradient. The flow being fully developed the following relations apply here $v = 0$, $\partial v/\partial y = 0$, $\partial p/\partial y = 0$, where v is the velocity in the transversal direction and p is the pressure. Thus, from the continuity equation, we get $\partial u/\partial x = 0$ so that the velocity component along x-axis depends only by y, $u = u(y)$. Based on the fact that the flow is fully developed we can assume that the temperature $T = T(y)$. Under these assumptions the momentum and energy equations for the flow and heat transfer have the following form:

$$\nu \frac{d^2u}{dy^2} - \frac{1}{\rho}\frac{dp}{dx} + g\beta\,(T - T_0) = 0 \tag{2.1}$$

$$\alpha \frac{d^2T}{dy^2} + \frac{\nu}{c_p}\left(\frac{du}{dy}\right)^2 = 0 \tag{2.2}$$

subject to the boundary condition given by Eq. (1.1), noslip condition for for velocity at the walls and constant temperature at the left wall:

$$u(0) = 0, u(L) = 0, T(L) = T_w \tag{2.3}$$

where α is the thermal diffusivity of the viscous fluid, ρ is the fluid density and c_p is the specific heat at constant pressure. In the system (2.1) and (2.2) there is an additional unknown, the gradient of pressure, dp/dx. In order to close the above system subject to the boundary conditions (1.1) and (2.3) it is necessary to consider the equation of the mass flux conservation:

$$U_0 = \frac{1}{L}\int_0^L u(y)dy \tag{2.4}$$

where L is the channel width. Further, we introduce the following dimensionless variables (see Pop and Ingham[10] or Kohr and Pop[11]):

$$U = \frac{u}{U_0}, X = \frac{xRe}{L}, Y = \frac{y}{L}, \theta = \frac{T - T_0}{T_w - T_0}, P = \frac{L^2}{\rho\nu^2}p \tag{2.5}$$

where $Re = U_0L/\nu$ is the Reynolds number and $T_0 = (T_a + T_w)/2$ is a characteristic temperature. Using (2.5) in the equations (2.1)-(2.2), in the boundary conditions (1.1) and (2.3) and in the mass flux conservation (2.4) we obtain:

$$\frac{d^2U}{dY^2} + \lambda\theta - \gamma = 0 \tag{2.6}$$

$$\frac{d^2\theta}{dY^2} + Br(\frac{dU}{dY})^2 = 0 \tag{2.7}$$

$$U(0) = 0, U(1) = 0, \left(\frac{d\theta}{dY}\right)_{Y=0} = \kappa(1 + \theta)_{Y=0}, \theta(1) = 1 \tag{2.8}$$

$$\int_0^1 U(Y)dY = 1; \tag{2.9}$$

In Eqs. (2.6)-(2.9) γ is the pressure gradient in X direction, Br is the Brinkman number, λ is the mixed convection parameter and κ is the convection heat transfer parameter given by

$$\gamma = \frac{dP}{dX}, Br = PrEc = \frac{\mu U_0^2}{k(T_w - T_0)}, \lambda = \frac{Gr}{Re} = \frac{g\beta(T_w - T_0)L^2}{U_0\nu}, \kappa = \frac{h_aL}{k} \tag{2.10}$$

and Pr, Ec, Gr and Re are the Prandtl number, Eckert number, Grashoff number and Reynolds number,respectively, defined as:

$$Pr = \frac{\nu}{\alpha}, Ec = \frac{U_0^2}{c_p(T_w - T_0)}, Gr = \frac{g\beta(T_w - T_0)L^3}{\nu^2}, Re = \frac{U_0 L}{\nu} \tag{2.11}$$

The physical quantity of interest in this problem are the skin friction coefficient C_f and the Nusselt number Nu, which are defined as:

$$C_f = \frac{\mu}{\rho U_0^2}\left(\frac{du}{dy}\right)_{y=0,L}, Nu = \left(\frac{h_f L}{k}\right)_{y=0,L} \tag{2.12}$$

In Eq. (2.12) h_f is the internal heat transfer coefficient which can be calculated from the heat transfer balance at the wall:

$$\left(k\frac{\partial T}{\partial \mathbf{n}}\right)_{wall} = h_f\left(T_{wall} - T_{fluid}\right)$$

where $\mathbf{n}$ is the normal to the wall. Using dimensionless variables (2.5) we obtain:

$$C_f Re = \left(\frac{dU}{dY}\right)_{Y=0,1}, Nu|_{Y=0} = \kappa\left(\frac{\theta(0)+1}{\theta(0)-1}\right), Nu|_{Y=1} = -\frac{\left(\frac{d\theta}{dY}\right)_{Y=1}}{\theta(0)-1} \tag{2.13}$$

3. **Results and discussions**

Equations (2.6) to (2.9) admit an analytical solution in two particular cases:

i) Case $Br = 0$

In this case the system (2.6) and (2.7) becomes:

$$\frac{d^2U}{dY^2} - \frac{dP}{dX} + \lambda\theta = 0 \tag{3.1}$$

$$\frac{d^2\theta}{dY^2} = 0 \tag{3.2}$$

subject to the boundary conditions (2.8). Further, from Eq. (3.1), (3.2) and condition (2.9) we obtain

$$\theta(Y) = \frac{2\kappa}{1+\kappa}Y + \frac{1-\kappa}{1+\kappa}$$

$$U(Y) = -\frac{\kappa\lambda}{1+\kappa}\frac{Y^3}{3} + (\gamma + \frac{1-\kappa}{1+\kappa}\lambda)\frac{Y^2}{2} + (\frac{\kappa\lambda}{3(1+\kappa)} - \frac{1}{2}(\gamma + \frac{1-\kappa}{1+\kappa}))Y \tag{3.3}$$

$$\gamma = -12 + \frac{\lambda}{1+\kappa}$$

ii) Case $\lambda = 0$

For $\lambda = 0$ the forced convection only is considered. The system (2.6) and (2.7) takes the following form:

$$\frac{d^2U}{dY^2} - \gamma = 0 \tag{3.4}$$

$$\frac{d^2\theta}{dY^2} + Br(\frac{dU}{dY})^2 = 0 \tag{3.5}$$

Taking in account that γ is constant, using the boundary conditions (2.8) and mass flux conservation (2.9) we have:

$$U(Y) = -6Y^2 + 6Y$$

$$\theta(Y) = -12BrY^4 + 24BrY^3 - 18BrY^2 + \frac{2\kappa}{1+\kappa}(1+3Br)Y + \frac{1+6Br-\kappa}{1+\kappa} \tag{3.6}$$

$$\gamma = -12$$

Equations (2.6) and (2.7) subject to (2.8) and (2.9) were solved numerically for different values of the parameters, λ, κ and Br (λ = 0, 100, 250, 500; κ = 0.01, 0.1, 1, 10; Br = 0, 0.001, 0.01, 0.025) using an implicit finite-difference method for velocity and a Gauss-Seidel iteration for temperature. Dimensionless velocity profiles, U(Y), and temperature profiles, $\theta(Y)$, are presented in Figs. 2 to 7 for different values of the above parameters. Analytical solutions (λ= 0, Br = 0) are also presented on figures with a circle marker.

The variation of the velocity $U(Y)$ and temperature $\theta(Y)$ with the mixed convection parameter λ is presented in Figs. 2 and 5.

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λ	κ	$Br = 0$	$Br = 0.001$	$Br = 0.01$
0	0.1	5.940594	5.940594	5.940594
	1	5.940594	5.940594	5.940594
	10	5.940594	5.940594	5.940594
100	0.1	4.470594	4.501963	4.799567
	1	-2.144405	-2.134825	-2.047795
	10	-8.759405	-8.784130	-9.001301
500	0.1	-1.409405	-1.27279	-0.125966
	1	-34.484405	-32.905781	-25.265381
	10	-67.559405	-68.912398	-68.791864

TABLE 1. Friction coefficient $C_f Re|_{Y=0}$

λ	κ	$Br = 0$	$Br = 0.001$	$Br = 0.01$
0	0.1	-5.940594	-5.940594	-5.940594
	1	-5.940594	-5.940594	-5.940594
	10	-5.940594	-5.940594	-5.940594
100	0.1	-7.410594	-7.370828	-6.999151
	1	-14.025594	-13.992628	-13.700667
	10	-20.640594	-20.606004	-20.288425
500	0.1	-13.290594	-13.053063	-11.199059
	1	-46.365594	-43.025955	-27.986534
	10	-79.440594	-74.606586	-42.694129

TABLE 2. Friction coefficient $C_f Re|_{Y=1}$

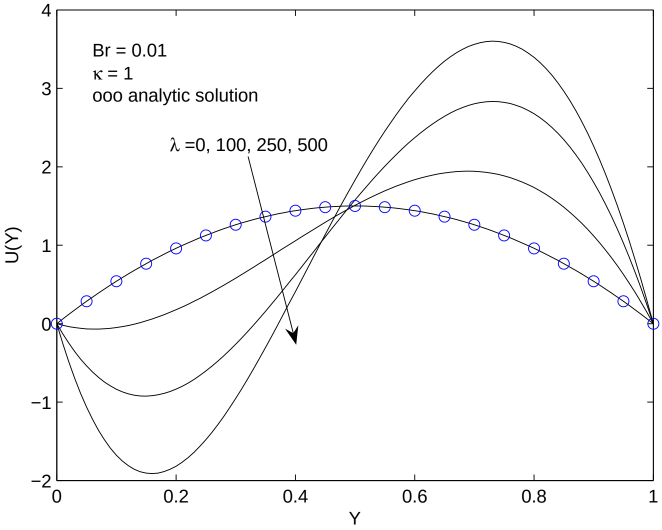

FIGURE 2. Velocity profiles for different values of λ

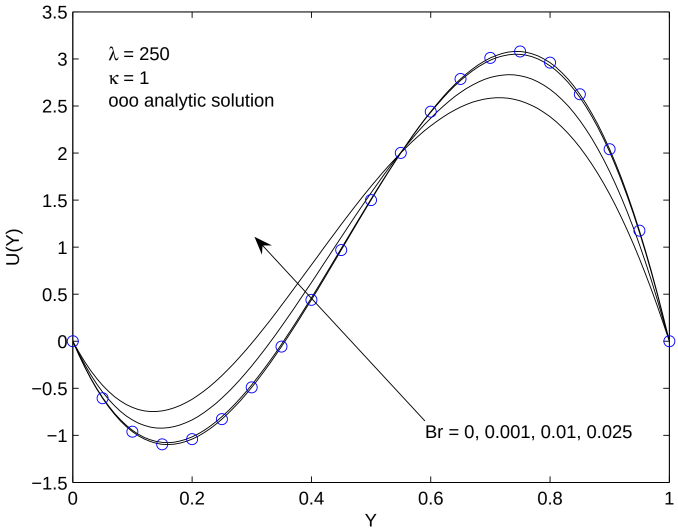

FIGURE 3. Velocity profiles for different values of Br

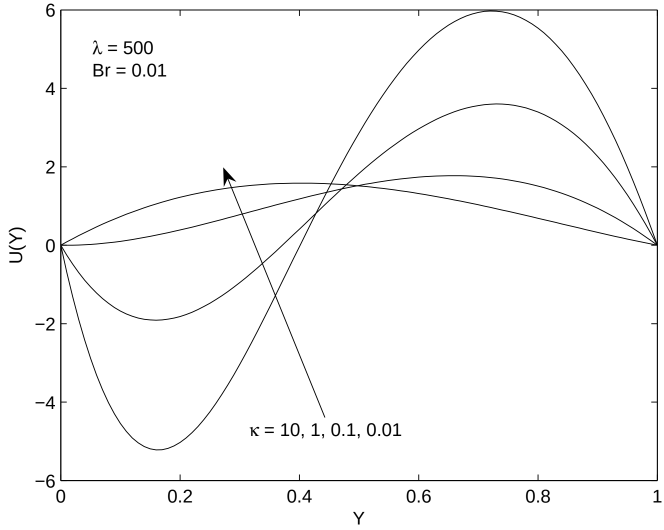

FIGURE 4. Velocity profiles for different values of κ

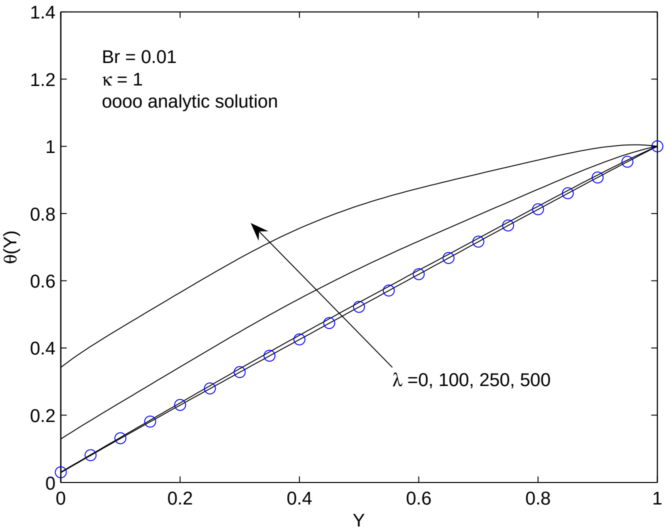

FIGURE 5. Temperature profiles for different values of λ

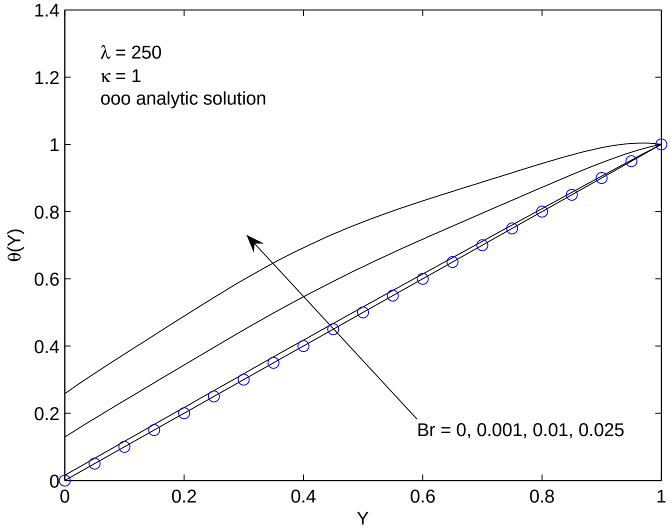

FIGURE 6. Temperature profiles for different values of Br

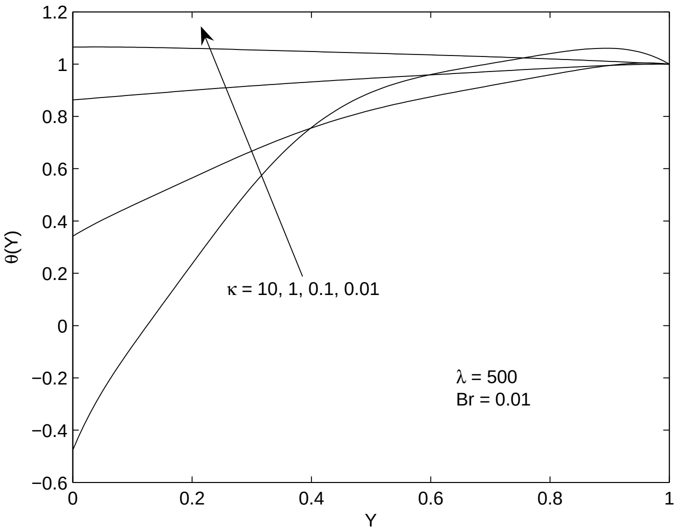

FIGURE 7. Temperature profiles for different values of κ

λ	κ	$Br = 0$	$Br = 0.001$	$Br = 0.01$
0	0.1	-0.999999	-1.032983	-1.451739
	1	-1.000000	-1.005839	-1.059969
	10	-1.000000	-1.003203	-1.032116
100	0.1	-0.999999	-1.027827	-1.377468
	1	-1.000000	-1.006256	-1.063425
	10	-1.000000	-1.009053	-1.092552
500	0.1	-0.999999	-1.031473	-1.360956
	1	-1.000000	-1.133834	-2.040945
	10	-1.000000	-1.276971	-3.552673

TABLE 3. Nusselt number on the left wall $Nu|_{Y=0}$

λ	κ	$Br = 0$	$Br = 0.001$	$Br = 0.01$
0	0.1	1.000000	0.967016	0.548260
	1	1.000000	0.994160	0.940030
	10	1.000000	0.996796	0.967883
100	0.1	1.000000	0.959813	0.498805
	1	1.000000	0.981394	0.813917
	10	0.999999	0.978624	0.787469
500	0.1	1.000000	0.906917	0.033431
	1	1.000000	0.810584	-0.328300
	10	0.999999	0.683490	-0.854004

TABLE 4. Nusselt number on the right wall $Nu|_{Y=1}$

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A COLLOCATION METHOD USING CUBIC B-SPLINES FUNCTIONS FOR SOLVING SECOND ORDER LINEAR VALUE PROBLEMS WITH CONDITIONS INSIDE THE INTERVAL $[0,1]$

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Abstract. Consider the problem:

$$y''(x) - Q(x)y(x) = R(x), \qquad x \in [0,1]$$
$$y(a) = \alpha$$
$$y(b) = \beta, \qquad a, b \in (0,1).$$

where $Q(x), R(x) \in C[0,1]; y \in C^2[0,1]$. The aim of this paper is to present an approximate solution of this problem based on cubic B-splines. The approximate solution uses a mesh based on Legendre points.A numerical solution is also given.

1. Introduction

Consider the problem(PVP):

$$y''(x) - Q(x)y(x) = R(x), \qquad x \in [0,1] \tag{1.1}$$
$$y(a) = \alpha$$
$$y(b) = \beta, \qquad a, b \in (0,1).$$

where $Q(x), R(x) \in C[0,1]; y \in C^2[0,1], a, b, \alpha, \beta \in \mathbb{R}$.This is not a two point boundary value problem (BVP), since $a, b \in (0,1)$.

If the solution of the two-point boundary value problem (BVP):

$$y''(x) - Q(x)y(x) = r(x), \qquad x \in [a,b]$$
$$y(a) = \alpha \tag{1.2}$$
$$y(b) = \beta,$$

exists and it is unique, then the requirement $y \in C^2[0,1]$ assures the existence and the uniqueness of (1.1).

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I have two initial value problems on $[0, a]$ and $[b, 1]$, respectively, and the existence and the uniqueness for (1.2) assure existence and uniqueness of these problems. It is possible to solve this problem by dividing it into the three above-mentioned problems and to solve each of these problem separately, but I am interested to a unitary approach that solve it as a whole.

Remark 1.1. • If $a = 0$ and $b = 1$ the problem (PVP) becomes a classical (BVP).

• If $a = 0$ or $b = 1$ the problem (PVP) may be decomposed into an (BVP) and one initial value problem(IVP).

Historical Note

In 1966, two researchers from *Tiberiu Popoviciu Institute of Romanian Academy Cluj Napoca*, D. Rîpianu and O. Aramă published a paper on polylocal problem (see [10]).

2. Preliminaries

Consider a partition of $[0, 1]$ like:

$$\pi : 0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1, \tag{2.1}$$

and the step sizes:

$$H_i := x_{i+1} - x_i, \qquad i = 0, \ldots, N. \tag{2.2}$$

In each subinterval $[x_i, x_{i+1}]$ we construct the collocation points as follows

$$\xi_{ij} := x_i + H_i\rho_j;\ i = 0, 1, ..., N,\ j = 0, 1, 2, ..., k, \tag{2.3}$$

where

$$0 \leq \rho_0 < \rho_1 < \rho_2 < ... < \rho_k \leq 1 \tag{2.4}$$

are the roots of k-th Legendre polynomial on each subintervals:$[x_i, x_{i+1}], i = 0, 1, ..., N$ with the stepsize given by (2.2) (see [1] for more details). I insert the points a, b so I obtained $N(k+1)+2$ points. One renumbers the collocation points such that the first is $\xi_0 := x_0 + H_0\rho_0 = 0$, and the last is $\xi_{n+2} := x_N + H_N\rho_k = 1$, where $n = N(K+1)$. Therefore the partition of $[0, 1]$ becomes:

$$\Delta := 0 \leq \xi_0 < \xi_1 < ... < \xi_{n+2} = 1$$

We augment the above partition Δ to form:

$$\overline{\Delta} : \xi_{-2} < \xi_{-1} < \xi_0 = 0 < \xi_1 < ... < \xi_{n+2} = 1 < \xi_{n+3} < \xi_{n+4} \tag{2.5}$$

where: $\xi_l := a$; $\xi_{l+p} := b$; $0 < l < n+1$; $1 < l+p < n+2$, $\xi_{-1} - \xi_{-2} = \xi_0 - \xi_{-1} = \xi_1 - \xi_0$, $\xi_{n+4} - \xi_{n+3} = \xi_{n+3} - \xi_{n+2} = \xi_{n+2} - \xi_{n+1}$.

Remark 2.1. If $a = \xi_i$ or $b = \xi_{i+p}$, $1 \leq i \leq n-2$, $1 < p < n+1-i$ we increment k.

Notation 2.2.

$$Q_i := Q(\xi_i)\ ;\ h_i := \xi_{i+1} - \xi_i;\ H := \max_{0\leq\ i\leq\ n+1} (\xi_{i+1} - \xi_i);\ h := \min_{0\leq\ i\leq n+1} (\xi_{i+1} - \xi_i).$$

Definition 2.3. Given the meshpoint (2.5) I define the vector space:

$$S\left(\overline{\Delta}\right) = \{p(x) \in C^2[0,1] : p(x) \text{ is a cubic polynomial of each}$$

subinterval $[\xi_{i-2}\ ,\xi_{i+2}],\ 0 \leq i \leq n+2\}$.

$\dim S\left(\overline{\Delta}\right) = n + 2$ (numbers of subintervals, see [12, pp. 73])

Definition 2.4. For $x \in \mathbb{R}\ ; 0 \leq i \leq n$, the cubic B-splines with the five knots: ξ_{i-2}, ξ_{i-1}, ξ, ξ, ξ are given by:

$$B_{i,3}(x) = \frac{x - \xi_{\ i-2}}{h_{i-2} + h_{i-1} + h_i} B_{i,2}(x) + \frac{\xi_{\ i+2} - x}{h_{i+1} + h_i + h_{i-1}} B_{i+1,2}(x) \tag{2.6}$$

where

$$B_{i,0} = \begin{cases} 1 \text{ if } \xi_{i-2} \leq x < \xi_{i-1} \\ 0 \text{ otherwise} \end{cases}$$

$$B_{i,2}(x) = \begin{cases} \frac{(x-\xi_{i-2})^2}{h_{i-2}\ (h_{i-2}+h_{i-1})},\ \text{if } \xi_{i-2} \leq\ x\ \leq \xi_{i-1} \\ \frac{(x-\xi_{i-2})(\xi_i - x)}{h_{i-1}(h_{i-1}+h_{i-2})} + \frac{(\xi_{i\ +1}-x)(x-\xi_{i-1})}{h_{i-1}(h_{i-1}+h_i)}, \text{if } \xi_{i-1} \leq x \leq \xi_i \\ \frac{(\xi_{i+1}-x)^2}{(h_{i-1}+h_i)h_i},\ \text{if } \xi_i \leq x \leq \xi_{i+1} \\ 0 \qquad ,\quad \text{otherwise .} \end{cases}$$

We need a bases from $S(\overline{\Delta})$ having $(n+2)$ cubic B-splines.Our choice is based on some special properties of cubic B-splines (see [11, pp.19-21] for details):

- The set
$$\{B_i\}\ i = 0, ..., n+1 \tag{2.7}$$
form a basis for $S(\overline{\Delta})$.
- $$\{B_i\} \text{ is positive on } (\xi_{i-2}, \xi_{i+2}) \text{ and zero elsewhere.} \tag{2.8}$$
- $\{B_i\}$ has local support (ξ_{i-2}, ξ_{i+2}) so computations using B-splines lead to linear system of equations with banded matrices.
- $$\sum_{i\ =\ 0}^{n+1} B_{i,3}(x) = 1 \text{ for every } x \in [0,1] \tag{2.9}$$

I recall some results from matrix theory ([7, pp. 359-361], [8, pp. 50-55]):

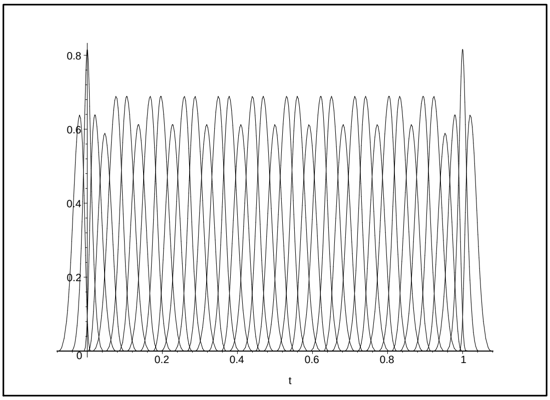

FIGURE 1. **B-spline bases**

Definition 2.5. A matrix $A = [a_{i\ j}], i = 1, 2, ..., m, j = 1, 2, ..., n$ is called *reducible* if there is a permutation that puts it into the form

$$\widetilde{A} = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where B and D are square matrices. Otherwise A is called *irreducible.*

Definition 2.6. A matrix $A = [a_{i\ j}], i = 1, 2, ..., m, j = 1, 2, ..., n$ is called *monotone* if $Az \geq 0$ implies $z \geq 0$.

Theorem 2.7. *A square tridiagonal matrix* $A = [a_{ij}]$ $i, j = 1, 2, ..., n$ *is* irreducible *iff:*

$$a_{i,i-1} \neq 0 \ (i = 2, 3, ..., n) \ \textit{and} \ a_{i,i+1} \neq 0 \ (i = 1, 2, ..., n-1)$$

and is reducible *iff:*

$$a_{i,i-1} = 0 \ \textit{or} \ a_{i,i+1} = 0 \ \textit{for some} \ i = 2, 3, ..., n$$

Theorem 2.8. *A monotone matrix is nonsingular.*

3. Main Results

3.1. **Consistency of the method.** I wish to find a approximate solution of the problem (1.1) in the following form:

$$u_{\overline{\Delta}}(x) = \sum_{i=\ 0}^{n+1} c_i B_{i,3}(x). \tag{3.1}$$

where $B_{i,3}(x)$ is a cubic B-splines with knots $\{\xi_{i+k}\}_{k\ =\ -2}^{2}$.

Remark 3.1. My approximation method is inspired from ([3], chap. 2,5)

I impose the conditions:

(c1) The approximate solution (3.1) verifies the differential equation (1.1) at ξ_j, $j = 1, ..., n+2$, $j \neq l$, $j \neq l+p$.

(c2) The solution verifies $u_{\overline{\Delta}}(\xi_l) = \alpha, u_{\overline{\Delta}}(\xi_{l+p}) = \beta$ (we recall that $a = \xi_l, b = \xi_{l+p}$).

Conditions **(c1)** and **(c2)** yield to a linear system:

$$A \cdot c = \gamma \tag{3.2}$$

with $(n+2)$ equations and $(n+2)$ unknowns c_i, $i = 0, ..., n+1$.The system matrix A is tridiagonal with 3 nonzero elements on each row.

We denote by:

$$f_i(x) := B"_{i,3}(x) - Q(x)B_{i,3}(x),\ i = 0, 1, ..., n+1;$$

then

$$A = \begin{bmatrix} f_i(\xi_j); i \in \{0, 1, 2..., n+1\}, \quad j \in \{1, 2, ..., n+2\} \backslash \{l, l+p\} \\ B_{i,3}(\xi_l); i = l-1, l, l+1 \\ B_{i,3}(\xi_{l+p}); i = l+p-1, l+p, l+p+1 \end{bmatrix}$$

The right hand side of (3.2) is:

$$\gamma = [R(\xi_1), ..., R(\xi_{l-1}), \alpha, R(\xi_{l+1}), ..., R(\xi_{l+p-1}), \beta, R(\xi_{l+p+1}), ..., R(\xi_{n+2})]$$

Lemma 3.2. (see [11, p. 23]) *For each $l > 0$, and $x \in [0,1]$, we have $B_{i,l}(x) \in C^1[0,1]$ and*

$$B'_{i,l}(x) = l \left[\frac{B_{i,l-1}(x)}{\xi_{i+l-2} - \xi_{i-2}} - \frac{B_{i+1,l-1}(x)}{\xi_{i+l-1} - \xi_{i-1}} \right]. \tag{3.3}$$

First I prove the next lemmas:

Lemma 3.3. *For each $x \in [0,1]$, $B_{i,3}(x) \in C^2[0,1]$ and*

$$B^{"}_{i,3}(x) = 3! \left[\frac{B_{i,1}(x)}{(h_i + h_{i-1} + h_{i-2})(h_{i-1} + h_{i-2})} - \right. \tag{3.4a}$$

$$- \frac{B_{i+1,1}(x)(h_{i-2} + 2h_{i-1} + 2h_i + h_{i+1})}{(h_i + h_{i-1})(h_i + h_{i-1} + h_{i-2})(h_{i+1} + h_i + h_{i-1})} + \tag{3.4b}$$

$$\left. + \frac{B_{i+2,1}(x)}{(h_{i+1} + h_i + h_{i-1})(h_i + h_{i+1})} \right], \tag{3.4c}$$

where

$$B_{i,1}(x) = \begin{cases} \frac{x - \xi_{i-2}}{h_{i-2}}, \textit{if } \xi_{i-2} \le x < \xi_{i-1} \\ \frac{\xi_i - x}{h_{i-1}}, \textit{if } \xi_{i-1} \le x < \xi_i \\ 0, \quad \textit{otherwise.} \end{cases}$$

Proof. For $l = 3$ we obtain from (3.3)

$$B'_{i,3}(x) = 3\left[\frac{B_{i,2}(x)}{h_i + h_{i-1} + h_{i-2}} - \frac{B_{i+1,2}(x)}{(h_{i+1} + h_i + h_{i-1}}\right].$$

Then

$$B''_{i,3}(x) = 3\left[\frac{B'_{i,2}(x)}{h_i + h_{i-1} + h_{i-2}} - \frac{B'_{i+1,2}(x)}{h_{i+1} + h_i + h_{i-1}}\right]. \tag{3.5}$$

Using again (3.3) for $l = 2$, it results:

$$B'_{i,2}(x) = 2\left[\frac{B_{i,1}(x)}{h_{i-1} + h_{i-2}} - \frac{B_{i+1,1}(x)}{h_i + h_{i-1}}\right], \tag{3.6}$$

$$B'_{i+1,2}(x) = 2\left[\frac{B_{i+1,1}(x)}{h_i + h_{i-1}} - \frac{B_{i+2,1}(x)}{h_{i+1} + h_i}\right]. \tag{3.7}$$

By substituting (3.6) and (3.7) into (3.5), I obtain (3.4a), □

Lemma 3.4. *For every $i = 0, 1, ..., n + 1$, it holds*

$$\frac{h^2}{3H^2} < B_{i,3}(\xi_i) < \frac{H^2}{3h^2} \tag{3.8}$$

$$-\frac{2}{h^2} < B''_{i,3}(\xi_i) < -\frac{2}{H^2} \tag{3.9}$$

Proof. By substituting ξ_i into (2.6) I obtain:

$$B_{i,3}(\xi_i) = \frac{1}{(h_{i-1} + h_i)}\left[\frac{h_i(h_{i-1} + h_{i-2})}{(h_i + h_{i-1} + h_{i-2})} + \frac{h_{i-1}(h_{i+1} + h_i)}{(h_i + h_{i-1} + h_{i+1})}\right]$$

But since

$$h \leq h_i \leq H, \text{ for every } i = 0, 1, ..., n \tag{3.10}$$

we obtain (3.8). Also substituting ξ_i into (3.4a) we have:

$$B''_{i,3}(\xi_i) = -\frac{1}{(h_{i-1} + h_i)}\left[\frac{1}{(h_i + h_{i-1} + h_{i-2})} + \frac{1}{(h_i + h_{i-1} + h_{i+1})}\right]$$

Using again (3.10), it results (3.9). □

Lemma 3.5. *If $Q(x) < -1$ for all $x \in [0, 1]$, then the elements of the matrix A are strictly positive.*

Proof. From (2.8)

$$B_{i,3}(\xi_l) > 0; i = l - 1, l, l + 1$$
$$B_{i,3}(\xi_{l+p}) > 0; i = l + p - 1, l + p, l + p + 1.$$

Using (3.4a)

$$B''_{i,3}(\xi_{i-1}) = \frac{3!}{(h_i + h_{i-1} + h_{i-2})(h_{i-1} + h_{i-2})} > 0,$$
$$B''_{i,3}(\xi_{i+1}) = \frac{3!}{(h_{i+1} + h_i + h_{i-1})(h_i + h_{i+1})} > 0$$

and:

$$Q(x) < 0, B_{i,3}(\xi_{i-1}) > 0, B_{i,3}(\xi_{i+1}) > 0 \text{ then } f_i(\xi_{i-1}) > 0, f_i(\xi_{i+1}) > 0.$$

Also since

$$f_i(\xi_i) = B''_{i,3}(\xi_i) - Q_i \cdot B_{i,3}(\xi_i)$$

it follows:

If $Q_i < \frac{B''_{i,3}(\xi_i)}{B_{i,3}(\xi_i)} < -\frac{2}{H^2}\frac{3H^2}{h^2} < -\frac{1}{h^2} < -1$; then for all $i = 0, 1, 2, ..., n : f_i(\xi_i) > 0$

□

Lemma 3.6. *If* $A = [a_{i,j}]$ *is a square tridiagonal matrix with all elements strict positive then* A *is* monotone.

Proof. By hypothesis $a_{i,i-1} > 0; a_{i,i} > 0; a_{i,i+1} > 0$ then, cf. Theorem 2.7, the matrix A is irreducible, and moreover

$$a_{i,i-1} + a_{i,i} + a_{i,i+1} > 0 \tag{3.11}$$

*Reductio ad absurdum.*I assume that there exists a vector z with a negative component $z_q < 0$ but such $Az \geq 0$. This assumption is equivalent to assuming that A is not monotone.I shall show that this contradicts the assumption that A is irreducible. Denote by $W := \{1, 2, ..., n\}$ and e the vector whose components are all 1. Then from (3.11) we have

$$A \cdot e > 0, A \cdot e \neq 0. \tag{3.12}$$

Since the sum of two nonnegative vectors is nonnegative, it follows that for $0 \leq \lambda \leq 1$

$$\lambda Az + (1 - \lambda)Ae = A[\lambda z + (1 - \lambda)e] > 0 \tag{3.13}$$

Consider the vector $w_\lambda = \lambda z + (1 - \lambda)e$ as a function of λ.For $\lambda = 0$ all components w_λ are positive, namely +1. For $\lambda = 1$ there is a least one negative component, namely $z_q, q \in W$. The components of w_λ are continuous functions of λ. Since $0 \leq \lambda \leq 1$, at least one component of w_λ must pass thought the value 0. Let δ the smallest value of λ such that w_λ has a zero component $(0 < \delta < 1)$. Now let S be a set of indices of zero components of w_λ and let $T = W - S$. (By construction,

$S \neq \Phi, T \neq \Phi$). For if all components of w_λ were zero, then the vectors z and e would be proportional:

$$e = -\frac{\delta}{1-\delta} z, \tag{3.14}$$

and from $Az \geq 0$ it would followed that:

$$Ae = -\frac{\delta}{1-\delta} Az \leq 0$$

contradicting (3.12). By (3.13), $Aw_\delta \geq 0$, so in particular, if $i \in S$:

$$(Aw_\delta)_i = \sum_{j \in T} a_{i\ j} w_{\delta\ j} \geq 0 \tag{3.15}$$

by construction $w_{\delta\ j} > 0$, if $j \in T$. In view of $a_{i,j} > 0$ if $j \in T$, (3.15) is thus possible if $a_{i,i-1} = a_{i,i} = a_{i,i+1} = 0$.Then A is reducible, contradicting our assumption, that implies A is monotone. □

Theorem 3.7. *If $Q(x) < -1$ the system(3.2) has a unique solution.*

Proof. Using above lemmas the system matrix A is monotone. By Theorem 2.8 A is nonsingular and moreover $\det A \neq 0$. □

To solve the system (3.2), I use *Crout Reduction for Tridiagonal Linear Systems Algorithm* (see [5, pp. 336-340]). This algorithm requires only $(5n-4)$ multiplications/divisions and $(3n-3)$ addition/subtractions, and consequently it has considerable computational advantages over the methods that do not consider the tridiagonality of the matrix, especially for large values of n.

3.2. **Error analysis.** I recall ([2, pp. 58-62]):

Theorem 3.8. *If the exact solution of (PVP) $y(x) \in C^2[0,1]$, then there exists a B-spline $B(x) \in S(\overline{\Delta})$ determined locally as follows*

$$\max_{\xi_{i-2} \leq x \leq \xi_{i+2}} |y(x) - B_i(x)| := \|y - B_i\|_{[\xi_{i-2}, \xi_{i+2}]} \leq K \cdot H_1^2 \cdot \left\| y^{(2)} \right\|_{[\xi_{i-2}, \xi_{i+2}]}, \tag{3.16}$$

where $H_1 := \max\{h_{i-2}, h_{i-1}, h_i, h_{i+1}\}$ and K is a real constant independent of $\overline{\Delta}$ and $y(x)$.

Since the points of $\overline{\Delta}$, except $\xi_l = a$ and $\xi_{l+p} = b$ are the roots of the kth Legendre polynomial, the orthogonality relation

$$\int_0^1 \rho(t) \prod_{j=1}^{k} (t - \rho_j) dt = 0$$

holds for all polynomials $\rho(t)$ of degree $q (2 \leq q \leq k)$, and then the superconvergence occurs at the meshpoints:

$$\left| y^{(j)}(\xi_i) - u_{\overline{\Delta}}^{(j)}(\xi_i) \right| = \mathbb{O}(H^{k+q}); 0 \leq i \leq n+2, 0 \leq j \leq 1 \tag{3.17}$$

(see [1], [4]). I use as collocation points the Gaussian points taking $q = k$.Then the *superconvergence* of my method at the meshpoints $\xi_i, i \in \{0, 1, 2, ..., n+2\} \setminus \{l, l+p\}$ is assured.

$$\left| y^{(j)}(\xi_i) - u^{(j)}_{\bar{\Delta}}(\xi_i) \right| = \mathbb{O}(H^{2k}); 0 \le i \le n+2, 0 \le j \le 1$$

Since $Q(x) \in C^1[0,1]$, then there exists $N = \max_{0 \le x \le 1} |Q(x)|$ such that

$$\left| y''(\xi_i) - u^{"}_{\bar{\Delta}}(\xi_i) \right| \le N \left| y(\xi_i) - u_{\bar{\Delta}}(\xi_i) \right| = N \cdot \mathbb{O}(H^{2k}).$$

In $\xi_l = a, \xi_{l+p} = b$ cf(3.16)

$$|y(\xi_l) - B_i(\xi_l)|_{[\xi_{l-2}, \xi_{l+2}]} \le K_1 \cdot H^2 \cdot \left\| y^{(2)} \right\|_{[\xi_{l-2}, \xi_{l+2}]}$$

$$|y(\xi_{l+p}) - B_i(\xi_{l+p})|_{[\xi_{l+p-2}, \xi_{l+p+2}]} \le K_1 \cdot H^2 \cdot \left\| y^{(2)} \right\|_{[\xi_{l+p-2}, \xi_{l+p+2}]}$$

where K_1, K_2 are constants, independent of $\overline{\Delta}$ and $y(x)$. It follows that my method is *superconvergent* of order $\mathbb{O}(H^2)$.

3.3. **Numerical examples.** I shall give one example. For this example, I plot the approximate solution, error in semilogarithmic scale and I generate the execution profile with the pair `profile` $-$ `showprofile`, see ([6]).

I want to approximate the oscillating solution of the following problem:

$$Z''(t) - 50 \cdot Z(t) = \sin(t); 0 \le t \le 1 \tag{3.18}$$

with conditions:

$$Z\left(\frac{1}{6}\right) = \frac{1}{49} \frac{-\sin(\frac{5\sqrt{2}}{6}) \sin 1 + \sin \frac{1}{6} \sin(5\sqrt{2})}{\sin(5\sqrt{2})} \tag{3.19}$$

$$Z\left(\frac{3}{4}\right) = \frac{1}{49} \frac{-\sin(\frac{15\sqrt{2}}{4}) \sin 1 + \sin \frac{3}{4} \sin(5\sqrt{2})}{\sin(5\sqrt{2})}$$

The exact solution provided by `dsolve` is:

$$Z(t) = \frac{1}{49} \frac{-\sin(5\sqrt{2}t) \sin 1 + \sin t \sin(5\sqrt{2})}{\sin(5\sqrt{2})}$$

Since

$$\int_0^1 |Q(x)| \, dx > 4,$$

due to disconjugate criteria given by *Lyapunov* (1893), the problem (3.18) has an oscillatory solution. I used Maple 8 to solve the problem exactly and to approximate the solution, for $n = 10$ and $k = 3$. I obtained a very good approximation, but I must increase the number of decimals with Maple command:

```
> Digits := 18;
```

If I use a method based on orthogonal polynomials, for example first kind Chebyshev polynomials, I observe that the B-spline method is faster and requires less memory. The reason is that for the B-spline method the matrix of the system that provides the coefficients is a band matrix with at most 3 nonzero elements per line, while for Chebyshev method the matrix is dense. This example with oscillating solution supports this conclusion (see for more details [9]).

Here are the profiles for the procedures `genspline` and `genceb` in the case of oscillating solution to problem (3.18):

function	depth	calls	time	time	bytes	bytes
`genspline`	1	1	7.691	100.0	156424156	100.00
`genceb`	1	1	17115	100.0	156424156	100.00

The the graphs of approximate solution and the error in semilogarithmic scale are given in Figure 2 and Figure 3, respectively.

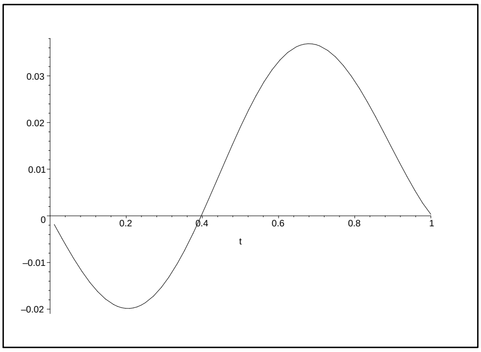

FIGURE 2. **Approximate solution** $n = 10$**,** $k = 3$

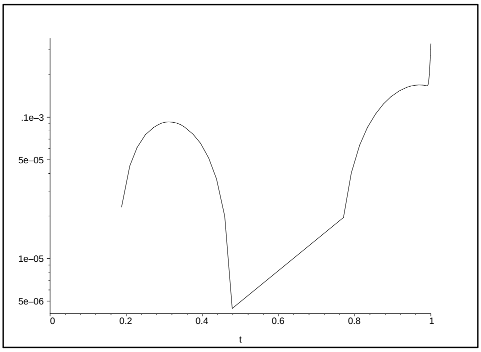

FIGURE 3. **Error plot,** $n = 10$**,** $k = 3$

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THE SOLVABILITY AND PROPERTIES OF SOLUTIONS OF ONE WIENER-HOPF TYPE EQUATION IN THE SINGULAR CASE

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Abstract. The work defines the conditions of solvability of one integral convolutional equation with degreely difference kernels in a singular case. This type of integral equations was not studied earlier, and it turned out that all methods used for the investigation of such equations with the help of Riemann boundary problem at the real axis are not applied there. The investigation of such type equations is based on the investigation of the equivalent singular integral equation with the Cauchy type kernel at the real axis in a singular case. It is determined that the equation is not a Noetherian one. Besides, there are shown the number of the linear independent solutions of the homogeneous equation and the number of conditions of solvability for the heterogeneous equation in the singular case. The general form of these conditions is also shown and there are determined the spaces of solutions of the equation. Thus the convolutional equation that wasn't studied earlier is presented in this work and the theory of its solvability in the singular case is built here. So some new and interesting theoretical results are got in this paper.

The present work is devoted to studying the next Wiener-Hopf type integral equation such as

$$P_m(x)\varphi(x) + \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} k(t, x-t)\varphi(t)\, dt = h(x), x \in \mathbf{R}, \tag{1.1}$$

where $\mathbf{R}$ is the real axis;

$$k(t, x-t) = \sum_{j=0}^{n} k_j(x-t)t^j,$$

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and

$$P_m(x)=\sum_{k=0}^{m}A_kx^k$$

is the known polynomial of degree m and $k_j(x)\in\mathbf{L}$, $j=\overline{1,n}$, $h(x)\in\mathbf{L}_2$ are known functions.

The theory of solvability of Wiener-Hopf equations with difference kernels was constructed in [2] and there were made quite wide assumptions concerning their kernels and right parts. This theory was based on the investigation of the Riemann boundary-value problem on the real axis, that was obtained with the help of the Fourier transformation. But we can't use methods from [2] to study the equation (1.1), because the investigation of it with the help of the Fourier transformation goes to the investigation of the Riemann differential boundary-value problem on the real axis. The ordinary Wiener-Hopf equation was studied in details in [6], where the conditions of solvability and some properties of solutions in the normal and singular cases were determined. The number of the linear independent solutions and the number of conditions of solvability for the both cases were also established there. Now we are studying the Wiener-Hopf type equation with more complicated kernel.

Let $D^+=\{z\in\mathbf{C}:\mathbf{Im\,z}>\mathbf{0}\}$ be an upper half plane and $D^-=\{z\in\mathbf{C}:\mathbf{Im\,z}<\mathbf{0}\}$ be a lower half plane of the complex plane $\mathbf{C}$. According to the properties of the Fourier transformation [3], [2] the investigation of the equation (1.1) reduces to the investigation of the following Riemann differential boundary-value problem

$$\left[\sum_{k=0}^{m}A_k(-1)^k\Phi^{+(k)}(x)+\sum_{j=0}^{n}(-1)^jK_j(x)\Phi^{+(j)}(x)\right]-\Phi^-(x)=H(x),x\in\mathbf{R},\quad(1.2)$$

where $K_j(x),H(x)$ are accordingly the Fourier transforms of functions $k_j(x),h(x)$, $j=\overline{1,n}$. $\Phi^{+(p)}(x)$ and $\Phi^-(x)$ are the boundary values at $\mathbf{R}$ of functions $\Phi^{+(p)}(z)$ and $\Phi^-(z)$ accordingly, where $\Phi^+(z)$, $\Phi^-(z)$ are unknown functions, which are analytical in the domains D^+ and D^- accordingly. As all the transformations of the Riemann differential boundary-value problem (1.2) and the equation (1.1) are identical, then the problem (1.2) and the equation (1.1) are equivalent in such a sense that they are simultaneously solvable or are not, and there is one and only one solution $\Phi^{\pm}(x)$ of the Riemann differential boundary-value problem (1.2) that corresponds to one and only one solution $\varphi(x)$ of the equation (1.1) and vice versa. The solutions of the equation (1.1) are expressed over the solutions of the problem (1.2) by the formula

$$\varphi(x)=\frac{1}{\sqrt{2\pi}}\int\limits_{\mathbf{R}}\Phi^+(t)e^{-ixt}\,dt,x>0.\quad(1.3)$$

We consider the functions $K_j(x) \in \mathbf{H}_\alpha^{(r)}$, $r \geq 0, 0 < \alpha \leq 1$, $\mathbf{H}_\alpha^{(0)} = \mathbf{H}_\alpha$, $j = \overline{1,n}$ and the function $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq 0$, $\mathbf{L}_2^{(0)} = \mathbf{L}_2$. As functions $k_j(x) \in \mathbf{L}$, then according to Riemann-Lebesgue theorem [3] $\lim\limits_{x\to\infty} K_j(x) = 0$, $j = \overline{1,n}$. The investigation of the equation (1.1) we will do basing on the investigation of the Riemann differential boundary-value problem (1.2). The investigation of the Riemann differential boundary-value problem (1.2) reduces to the investigation of the singular integral equation with Cauchy kernel at the real axis with the help of integral representations for functions and derivatives of them built in [8]. Let construct functions $\Phi^+(z)$ and $\Phi^-(z)$ such that they are analytic in the domains D^+, D^- respectively and decay at infinity. Besides, the boundary values on $\mathbf{R}$ of functions $\Phi^{+(p)}(z)$ and $\Phi^-(z)$ satisfy the following condition $\Phi^{+(p)}(x), \Phi^-(x) \in \mathbf{L}_2^{(r)}$, $r \geq 0, p \geq 0$.These conditions satisfy such functions as:

$$\Phi^{\pm}(z) = (2\pi\imath)^{-1} \int\limits_{\mathbf{R}} P^{\pm}(x,z)\rho(x)\,dx, z \in D^{\pm}, \tag{1.4}$$

where

$$P^+(x,z) = \frac{(-1)^p(x+\imath)^{-p}}{(p-1)!}\left[(x-z)^{p-1}\ln\left(1-\frac{x+\imath}{z+\imath}\right) - \sum_{k=0}^{p-2} d_{p-k-2}(x+\imath)^{k+1}(z+\imath)^{p-k-2}\right],$$

$$x \in \mathbf{R}, \mathbf{z} \in \mathbf{D}^+;$$

$$P^-(x,z) = \frac{1}{x-z}$$

$$x \in \mathbf{R}, \mathbf{z} \in \mathbf{D}^-;$$

$$d_{p-k-2} = (-1)^{k+1}\sum_{j=0}^{k} C_{p-1}^{p-1-j}(k-j+1)^{-1},$$

where C_n^m are binomial coefficients and the function $\ln\left[1-\frac{x+\imath}{z+\imath}\right]$ is the main branch ($\ln 1 = 0$) of the logarithmic function in the complex plane with the cut that connects such points as $z = -\imath$ and $z = \infty$, following the negative direction of the axis of ordinate. It's easy to verify, that defined by (1.4) functions $\Phi^+(z)$ and $\Phi^-(z)$ are the unique analytic functions in domains D^+, D^- respectively. It is easy to verify that the function $\rho(x) \in \mathbf{L}_2$ is defined uniquely by the functions $\Phi^+(z)$ and $\Phi^-(z)$ and vice versa, so with the help of the given function $\rho(x) \in \mathbf{L}_2$ both functions $\Phi^+(z)$ and $\Phi^-(z)$ are constructed uniquely. The following representations take place:

$$\Phi^{+(p)}(z) = (2\pi\imath)^{-1}\int\limits_{\mathbf{R}} (z+\imath)^{-p}(x-z)^{-1}\rho(x)\,dx, z \in D^+,$$

$$\Phi^{-}(z) = (2\pi\imath)^{-1} \int\limits_{\mathbf{R}} (x-z)^{-1} \rho(x)\, dx, z \in D^{-}. \tag{1.5}$$

We consider the case, when $m = n$. Using the properties [8] of partial derivatives of functions $P^{+}(x,z)$ with respect to z and Sohotski formulas for derivatives from [1], with the help of the representations (1.4), (1.5), we transform the Riemann differential boundary-value problem (1.2) into the following singular integral equation

$$A(x)\rho(x) + B(x)(\pi\imath)^{-1} \int\limits_{\mathbf{R}} (t-x)^{-1} \rho(t)\, dt + (T\rho)(x) = H(x), x \in \mathbf{R}, \tag{1.6}$$

where

$$A(x) = 0,5(-1)^{m} \left\{[A_m + K_m(x)](x+\imath)^{-m} + 1\right\},$$

$$B(x) = 0,5(-1)^{m} \left\{[A_m + K_m(x)](x+\imath)^{-m} - 1\right\}, \tag{1.7}$$

$$(T\rho)(x) = \int\limits_{\mathbf{R}} K(x,t)\rho(t)\, dt, x \in \mathbf{R}, \tag{1.8}$$

$$K(x,t) = \frac{1}{2\pi\imath} \sum_{j=0}^{m-1} (-1)^{j} \left[A_j + K_j(x)\right] \frac{\partial^{j} P^{+}(t,x)}{\partial x^{j}}, \tag{1.9}$$

and $\frac{\partial^{j} P^{+}(t,x)}{\partial x^{j}}$ is a limiting value at $\mathbf{R}$ of the function $\frac{\partial^{j} P^{+}(t,z)}{\partial z^{j}}$, $j = \overline{0, m-1}$.

Lemma 1.1. *If functions $K_j(x) \in \mathbf{H}_{\alpha}^{(r)}$, $j = \overline{1,n}$, then the operator $T : \mathbf{L_2^{(r)}} \to \mathbf{L_2^{(r)}}$, $r \geq 0$, defined by the formula (1.8) is a compact operator.*

The proof of lemma follows from Frechet-Kolmogorov-Riesz criterion of compactness of integral operators on the real axis in the space $\mathbf{L_p}$, $p > 1$, the properties of functions $P^{\pm}(x,z)$ follow from the results of the papers [8] and [9].

According to the work [4], the problem (1.2) and the singular integral equation (1.6) are equivalent in such a sense that they are simultaneously solvable or are not, and for the every solution $\rho(x)$ of the equation (1.6) there exists may be an ununique solution $\Phi^{\pm}(x)$ of the problem (1.2) and vice versa. In order to make this solution to be the unique one, it is necessary to set initial conditions for the problem (1.2). As its solutions $\Phi^{\pm}(x)$ are found in spaces of decaying at infinity functions, then according to the properties of Cauchy type integral, solutions of the problem (1.2) are such that $\Phi^{\pm(j)}(\infty) = 0$, $j = \overline{0, m-1}$, thus we obtain trivial initial conditions of (1.2) and they are set automatically. So it follows that the Riemann differential boundary-value problem (1.2) and the singular integral equation (1.6) are equivalent in such a sense that they are simultaneously solvable or are not, and there is one and only one solution $\rho(x)$ of the equation (1.6) for the every solution $\Phi^{\pm}(x)$ of the problem (1.2) and vice versa. By the force of formula (1.4), the solutions of the problem (1.2) are expressed over solutions of the equation (1.6) according to the formula

$$\Phi^+(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} P^+(t,x)\rho(t)\,dt, x \in \mathbf{R}, \tag{1.10}$$

where $p = m$; $P^+(t,x)$ is the boundary value at $x \in \mathbf{R}$ of function $P^+(t,z)$, and $\rho(x)$ is the solution of the equation (1.6). As the equation (1.1) and the problem (1.2) are equivalent, the problem (1.2) and the singular integral equation (1.6) are equivalent, too, it follows that the equation (1.1) and the equation (1.6) are equivalent in such a sense that they are simultaneously solvable or are not, and there is one and only one solution $\varphi(x)$ of the equation (1.1) for the every solution $\rho(x)$ of the equation (1.6) and vice versa. Thus the solutions of the equation (1.1) are expressed over solutions of the equation (1.6) according to the formulas (1.10), (1.3). That is why the equation (1.1) we will call Noetherian if the equation (1.6) is Noetherian.

Theorem 1.2. *The equation (1.1) is not Noetherian.*

Proof. According to the work [4] the equation (1.6) is Noetherian if and only if when $A(x) + B(x) \neq 0$, $A(x) - B(x) \neq 0$ on $x \in \mathbf{R}$. From the formula (1.7) it follows that

$$A(x) + B(x) = (-1)^m [A_m + K_m(x)](x + \imath)^{-m},$$

$$A(x) - B(x) = 1.$$

So we have got that the function $A(x)+B(x)$ possesses a zero at infinity of at least the order m. It means that the equation (1.6) is not Noetherian. Then as the equations (1.1) and (1.6) are equivalent, the equation (1.1) is not Noetherian, too.

The theorem is proved.

Let determine $\chi = -ind\,[A_m + K_m(x)]$.

Here we don't study the case when $A(x) + B(x) \neq 0$ on $\mathbf{R}$ as this is the normal case and the results of [6] for it remain correct if in Theorems 1.3 and 1.4 in [6] we will study the function $A_m + K_m(x)$ instead of $A_m + B_m K(x)$.

Let's study the singular case when the condition $A_m + K_m(x) \neq 0$ at $\mathbf{R}$ is not executed. Then we suppose that the function $A_m + K_m(x)$ goes to zero on the real axis in such points as $a_1, a_2, \ldots, a_s$ with accordingly integer orders $\gamma_1, \gamma_2, \ldots, \gamma_s$. Then in virtue of [5] the following representation takes place

$$A(x) + B(x) = (x + \imath)^{-m} M(x) \prod_{k=1}^{s} \left(\frac{x - a_k}{x + \imath} \right)^{\gamma_k}, \tag{1.11}$$

where the function $M(x) \neq 0$ on $\mathbf{R}$, $M(x) \in \mathbf{H}_\alpha^{(r)}$.

Let

$$r_0 = \max\{\gamma_1, \gamma_2, \ldots, \gamma_s, m\}, \gamma = \sum_{k=1}^{s} \gamma_k, \chi = -indM(x). \tag{1.12}$$

Theorem 1.3. *Let the functions $k_j(x) \in \mathbf{L}$, $h(x) \in \mathbf{L}_2$; the functions $K_j(x) \in \mathbf{H}_\alpha^{(r)}$, $j = \overline{1,m}$, $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq r_0$, where the number r_0 is defined by the formula (1.12) and the representations (1.11) take place, where $M(x) \neq 0$ on $\mathbf{R}$.*
If $\chi + m + \gamma \leq 0$, where the numbers χ, γ are defined by formulas (1.12), then the homogeneous equation (1.1) has not less than $|\chi+m+\gamma|$ linear independent solutions; the heterogeneous equation (1.1) is unconditionally solvable and its general solution depends upon not less than $|\chi + m + \gamma|$ arbitrary constants.
If $\chi+m+\gamma > 0$, then generally speaking the heterogeneous equation (1.1) is unsolvable. It will be solvable when not less than $m + \gamma + \chi$ conditions of solvability

$$\int_R H(x)\psi_j(x)dx = 0, \tag{1.13}$$

will be executed. Here $H(x)$ is a right part of the equation (1.6), and $\psi_j(x)$ are the linear independent solutions of the homogeneous equation

$$A(x)\psi(x) - (\pi i)^{-1} \int_R \frac{B(t)\psi(t)}{t-x} dt + \int_R K(t,x)\psi(t)dt = 0,$$

allied to the equation (1.6), where $A(x), B(x), K(x,t)$ are the coefficients and the regular kernel of the singular integral equation (1.6).
Proof. According to [5], the index of the equation (1.6) is equal to $-(\chi + m + \gamma)$. Then due to [4], if $\chi + m + \gamma \leq 0$, then the homogeneous equation (1.6) has not less than $|\chi+m+\gamma|$ linear independent solutions; the heterogeneous equation (1.6) is unconditionally solvable and its general solution depends upon not less than $|\chi+m+\gamma|$ arbitrary constants. If $\chi + m + \gamma > 0$, then due to [4], the heterogeneous equation (1.6) is unsolvable. It will be solvable, when not less than $m + \gamma + \chi$ conditions of solvability (1.13) will be executed. As the equations (1.1) and (1.6) are equivalent, then the theorem is proved.
Theorem 1.4. *Let the functions $k_j(x) \in \mathbf{L}$, $h(x) \in \mathbf{L}_2$; the functions $K_j(x) \in \mathbf{H}_\alpha^{(r)}$, $j = \overline{1,m}$, $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq r_0$, where the number r_0 is defined by the formula (1.12), the representations (1.11) take place, where $M(x) \neq 0$ on $\mathbf{R}$, and the equation (1.1) is solvable. Then its solutions belong to the space $\mathbf{L}_2[-r - m + r_0; 0]$, $r \geq r_0$.*
Proof. According to [5], the solutions of the equation (1.6) $\rho(x) \in \mathbf{L}_2^{(r-r_0)}$, $r \geq r_0$. Then in virtue of the representations (1.5) and the properties of the Cauchy type integral the limit values $\Phi^+(x)$ on $\mathbf{R}$ of the function $\Phi^+(z)$ belong to the space $\mathbf{L}_2^{(r-r_0)}$.

From the properties of Fourier transformation [2] we obtain that the solutions of the equation (1.1) belong to the space $\mathbf{L}_2[-r-m+r_0;0]$, $r \geq r_0$, and the theorem is proved.

If the function $A_m + K_m(x)$ goes to zero on the real axis $\mathbf{R}$ in the points $a_1, a_2, \ldots, a_s$ with accordingly fractional orders $\gamma_1, \gamma_2, \ldots, \gamma_s$, then the representation (1.11) where $M(x) \neq 0$ at $\mathbf{R}$ and $M(x) \in H_\alpha^{(r)}$ is also fulfilled. But for the numbers γ_k, $k = \overline{1,s}$ the following representation takes place:

$$\gamma_k = [\gamma_k] + \{\gamma_k\}, \quad k = \overline{0,s}.$$

There $[a]$ means the integer part, and $\{a\}$ - the fractional part of the number a.
So here we will note

$$r_0 = max\{\alpha_1{}', ..., \alpha_s{}', m\}, \quad \alpha = \sum_{k=1}^{s} \alpha_k{}', \quad \varkappa = -indM(x), \tag{1.14}$$

$$\alpha_k{}' = \begin{cases} [\gamma_k], 0 < \{\gamma_k\} < \frac{1}{2} \\ [\gamma_k] + 1, \frac{1}{2} < \{\gamma_k\} < 1, \quad k = \overline{0,s}. \end{cases} \tag{1.15}$$

With such assumption the following theorem takes place.

Theorem 1.5. *Let the functions $k_j(x) \in L$, $j = \overline{1,m}$, $h(x) \in L_2$ and the functions $K_j(x) \in H_\alpha^{(r)}$, $j = \overline{1,m}$, $0 < \alpha \leqslant 1$, $H(x) \in L_2^{(r)}$, $r \geq r_0$, where the number r_0 is defined by (1.14), (1.15) and the representation (1.11) takes place, where $M(x) \neq 0$ on $\mathbf{R}$.*

If $\chi + m + \alpha \leq 0$, where the numbers χ, α are defined by formulas (1.14), then the homogeneous equation (1.1) has not less than $|\chi+m+\alpha|$ linear independent solutions; the heterogeneous equation (1.1) is unconditionally solvable and its general solution depends upon not less than $|\chi + m + \alpha|$ arbitrary constants.

If $\chi+m+\alpha > 0$, then generally speaking the heterogeneous equation (1.1) is unsolvable. It will be a solvable one when not less than $m + \alpha + \chi$ conditions of solvability (1.13) will be executed.

Proof. According to [5], [7] the index of the equation (1.6) is equal to $-(\chi + m + \alpha)$. Then due to [4], if $\chi + m + \alpha \leq 0$, then the homogeneous equation (1.6) has not less than $|\chi + m + \alpha|$ linear independent solutions; the heterogeneous equation (1.6) is unconditionally solvable and its general solution depends upon not less than $|\chi+m+\alpha|$ arbitrary constants. If $\chi + m + \alpha > 0$, then due to [4], the heterogeneous equation (1.6) is unsolvable. It will be solvable, when not less than $m + \alpha + \chi$ conditions of solvability (1.13) will be executed. As the equations (1.1) and (1.6) are equivalent, then the theorem is proved.

Theorem 1.6. *Let the functions* $k_j(x) \in \mathbf{L}$, $h(x) \in \mathbf{L}_2$; *the functions* $K_j(x) \in \mathbf{H}_\alpha^{(r)}$, $j = \overline{1, m}$, $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq r_0$, *where the number* r_0 *is defined by the formula (1.14), the representations (1.11) take place, where* $M(x) \neq 0$ *on* $\mathbf{R}$, *and the equation (1.1) is solvable. Then its solutions belong to the space* $\mathbf{L}_2[-r - m + r_0; 0]$, $r \geq r_0$.

The proof of the Theorem 1.6 coincides with the proof of the theorem 3.

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ANALYSIS OF A BILATERAL CONTACT PROBLEM WITH ADHESION AND FRICTION FOR ELASTIC MATERIALS

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Abstract. We consider a mathematical model which describes a contact problem between a deformable body and a foundation. The contact is bilateral and is modelled with Tresca's friction law in which adhesion is taken into account. The evolution of the bonding field is discribed by a first order differential equation and the material's behavior is modelled with a nonlinear elastic constitutive law. We derive a variational formulation of the mechanical problem and prove the existence and uniqueness result of the weak solution. Moreover, we prove that the solution of the contact problem can be obtained as the limit of the solution of a regularized problem as the regularizaton parameter converges to 0. The proof is based on arguments of time-dependent variational inequalities, differential equations and Banach fixed point theorem.

1. **Introduction**

Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. Because of the importance of this process a considerable effort has been made in its modelling and numerical simulations. A first study of frictional contact problems within the framework of variational inequalities was made in [5]. Recently a new book [18] was appeared such that the aim is to introduce the reader of the theory of variational inequalities with analysis to the study of contact mechanics, and, specifically, with study of antiplane contact problems with linearly elastic and viscoelastic materials. The mathematical, mechanical and numerical state of the art can be found in [15]. The frictional contact problem with normal compliance and adhesion for elastic materials was studied in [11]. In this paper we study a model of an elastic contact problem with Tresca's friction law in which adhesion into contact surfaces was taken into account. We recall that models for dynamic or quasistatic

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process of frictionless adhesive contact between a deformable body and a foundation have been studied in [2, 3, 15, 16]. In [1] the unilateral quasistatic contact problem with friction and adhesion was studied and an existence result for a friction coefficient small enough was established. As in [7, 8] we use the bonding field as an additional state variable β, defined on the contact surface of the boundary. The variable is restricted to values $0 \leq \beta \leq 1$, when $\beta = 0$ all the bonds are severed and there are no active bonds; when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$ it measures the fraction of active bonds and partial adhesion takes place. We refer the reader to the extensive bibliography on the subject in $[9, 12, 13, 14, 15, 16, 17]$. In this work we derive a variational formulation of the mechanical problem for which we prove the existence of a unique weak solution, and obtain a partial regularity result for the solution. Moreover, we study the behavior of the solution of a regularized problem as the regularization parameter converges to 0.
The paper is structured as follows. In Section 2 we present some notations and give the variational formulation. In Section 3 we state and prove our main existence and uniqueness result, Theorem 2.1. Finally, in Section 4, we show that the regularized problem admits a unique solution, Theorem 4.1, and prove a convergence result of this problem, Theorem 4.2.

2. Problem statement and variational formulation

Let $\Omega \subset \mathbf{R}^d; (d = 2, 3)$, be the domain occupied by a nonlinear elastic elastic body. Ω is supposed to be open, bounded, with a sufficiently regular boundary Γ. Γ is partitioned into three measurable parts $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ where $\Gamma_1, \Gamma_2, \Gamma_3$ are disjoint open sets and $meas\ \Gamma_1 > 0$. The body is acted upon by a volume force of density φ_1 on Ω and a surface traction of density φ_2 on Γ_2. On Γ_3 the body is in bilateral and adhesive contact with Tresca's friction law with a foundation.
Thus, the classical formulation of the mechanical problem is written as follows.
Problem P_1. Find a displacement field $u : \Omega \times [0, T] \to \mathbf{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0, T] \to [0, 1]$ such that

$$div\sigma + \varphi_1 = 0 \text{ in } \Omega \times (0, T), \tag{2.1}$$

$$\sigma = F\varepsilon(u) \text{ in } \Omega \times (0, T), \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{2.3}$$

$$\sigma\nu = \varphi_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.4}$$

$$\left.\begin{array}{c} u_\nu = 0 \\ \\ \left|\sigma_\tau + c_\tau \beta^2 R_\tau\left(u_\tau\right)\right| \leq g \\ \\ \left|\sigma_\tau + c_\tau \beta^2 R_\tau\left(u_\tau\right)\right| < g \Longrightarrow u_\tau = 0 \\ \\ \left|\sigma_\tau + c_\tau \beta^2 R_\tau\left(u_\tau\right)\right| = g \Longrightarrow \\ \\ \exists \lambda \geq 0 \text{ s.t. } u_\tau = -\lambda\left(\sigma_\tau + c_\tau \beta^2 R_\tau\left(u_\tau\right)\right) \end{array}\right\} \text{on } \Gamma_3 \times (0, T), \tag{2.5}$$

$$\dot{\beta} = -(c_\tau \beta \left|R_\tau\left(u_\tau\right)\right|^2 - \varepsilon_a)_+ \text{ on } \Gamma_3 \times (0, T), \tag{2.6}$$

$$\beta(0) = \beta_0 \text{ on } \Gamma_3. \tag{2.7}$$

We denote by u the displacement field, σ the stress field and $\varepsilon(u)$ the strain tensor. Equation (2.1) represents the equilibrium equation. Equation (2.2) represents the elastic constitutive law of the material in which F is a given nonlinear function. Here and below a dot above a variable represents a time derivative. We recall that in linear elasticity the stress tensor $\sigma = (\sigma_{ij})$ is given by

$$\sigma_{ij} = a_{ijkh} \varepsilon_{kh}(u),$$

where $F = (a_{ijkh})$ is the linear elasticity tensor, for $i, j, k, h = 1, ..., d$; (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress vector. Condition (2.5) represents the bilateral contact with Tresca's friction law in which adhesion is taken into account. Here g is a friction bound and the parameters c_τ and ε_a are adhesion coefficients which may depend on $x \in \Gamma_3$. As in [17], R_τ is a truncation operator defined by

$$R_\tau(v) = \begin{cases} v & \text{if } |v| \leq L \\ \\ L\dfrac{v}{|v|} & \text{if } |v| > L \end{cases},$$

where $L > 0$ is a characteristic length of the bonds. Equation (2.6) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [17] where $[s]_+ = \max(s, 0)$ $\forall s \in \mathbf{R}$. Since $\dot{\beta} \leq 0$ on $\Gamma_3 \times (0, T)$, once debonding occurs, bonding cannot be reestablished. Also we wish to make it clear that from [11] it follows that the model does not allow for complete debonding field in finite time. Finally, (2.7) represents the initial bonding field. We recall that the inner products and the corresponding norms on $\mathbf{R}^d$ and S_d are given by

$$u.v = u_i v_i, \quad |v| = (v.v)^{\frac{1}{2}} \quad \forall u, v \in \mathbf{R}^d,$$
$$\sigma.\tau = \sigma_{ij}\tau_{ij}, \ |\tau| = (\tau.\tau)^{\frac{1}{2}} \quad \forall \sigma, \tau \in S_d,$$

where S_d is the space of second order symmetric tensors on $\mathbf{R}^d$ $(d = 2, 3)$. Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H = \left(L^2(\Omega)\right)^d, \ H_1 = \left(H^1(\Omega)\right)^d, \ Q = \left\{\tau = (\tau_{ij}); \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\},$$

$$Q_1 = \{\sigma \in Q; div\sigma \in H\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$\langle u, v\rangle_H = \int_\Omega u_i v_i dx, \quad \langle \sigma, \tau\rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i,j} + u_{j,i});$$

$div\sigma = (\sigma_{ij,j})$ is the divergence of σ. For every element $v \in H_1$ we denote by v_ν and v_τ the normal and the tangential components of v on the boundary Γ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

Similarly, for a regular function $\sigma \in Q_1$, we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu$$

and we recall that the following Green's formula holds:

$$\langle \sigma, \varepsilon(v)\rangle_Q + \langle div\sigma, v\rangle_H = \int_\Gamma \sigma\nu.v da \ \ \forall v \in H_1,$$

where da is the surface measure element. Let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1, \, v_\nu = 0 \text{ on } \Gamma_3 \}.$$

Since $meas\ \Gamma_1 > 0$, the following Korn's inequality holds [5],

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.8}$$

where the constant $c_\Omega > 0$ depends only on Ω and Γ_1. We equip V with the inner product

$$(u, v)_V = \langle \varepsilon(u), \varepsilon(v)\rangle_Q$$

and $\|.\|_V$ is the associated norm. It follows from Korn's inequality (2.8) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent on V. Then $(V, \|.\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which depends only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.9}$$

For $p \in [1, \infty]$, we use the standard norm of $L^p(0, T; V)$. We also use the Sobolev space $W^{1,\infty}(0, T; V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

For every real Banach space $(X, \|.\|_X)$ and $T > 0$ we use the notation $C([0, T]; X)$ for the space of continuous functions from $[0, T]$ to X; recall that $C([0, T]; X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X.$$

We assume that the body forces and surface tractions have the regularity

$$\varphi_1 \in W^{1,\infty}(0, T; H), \qquad \varphi_2 \in W^{1,\infty}\left(0, T; \left(L^2(\Gamma_2)\right)^d\right) \tag{2.10}$$

and we denote by $f(t)$ the element of V defined by

$$(f(t), v)_V = \int_\Omega \varphi_1(t).v dx + \int_{\Gamma_2} \varphi_2(t).v da \quad \forall v \in V, \text{ for } t \in [0, T]. \tag{2.11}$$

Using (2.10) and (2.11) yield

$$f \in W^{1,\infty}(0, T; V).$$

Also we define the functional $j : V \to \mathbf{R}_+$ by

$$j(v) = \int_{\Gamma_3} g |v_\tau| da,$$

where g is assumed to satisfy

$$g \in L^\infty(\Gamma_3), \; g \geq 0 \; a.e. \text{ on } \Gamma_3. \tag{2.12}$$

In the study of Problem P_1 we assume that the elasticity operator F satisfies

$$\left.\begin{array}{l}(a)\ \ F:\Omega\times S_d\rightarrow S_d;\\ \\ (b)\ \text{there exists } M>0 \text{ such that}\\ \qquad |F(x,\varepsilon_1)-F(x,\varepsilon_2)|\leq M|\varepsilon_1-\varepsilon_2|,\\ \qquad \text{for all } \varepsilon_1,\varepsilon_2 \text{ in } S_d,\ a.e.\ x \text{ in } \Omega;\\ \\ (c)\ \text{there exists } m>0 \text{ such that}\\ \qquad (F(x,\varepsilon_1)-F(x,\varepsilon_2)).(\varepsilon_1-\varepsilon_2)\geq m|\varepsilon_1-\varepsilon_2|^2,\\ \qquad \text{for all } \ \varepsilon_1,\varepsilon_2 \text{ in } S_d,\ a.e.\ x \text{ in } \Omega;\\ \\ (d)\ \text{the mapping } x\rightarrow F(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega,\\ \qquad \text{for any } \varepsilon \text{ in } S_d;\\ \\ (e)\ x\rightarrow F(x,0)\in Q.\end{array}\right\}\tag{2.13}$$

As in [17] we assume that the adhesion coefficients c_τ and ε_a satisfy the conditions

$$c_\tau,\ \varepsilon_a\in L^\infty(\Gamma_3),\ c_\tau,\ \varepsilon_a\geq 0,\ a.e.\text{on } \Gamma_3.\tag{2.14}$$

Next, we define the functional $r:L^2(\Gamma_3)\times V\times V\rightarrow\mathbb{R}$ by

$$r(\beta,u,v)=\int_{\Gamma_3}c_\tau\beta^2R_\tau(u_\tau).v_\tau da.$$

Finally, we assume that the initial data satisfy

$$\beta_0\in L^2(\Gamma_3);\ 0\leq\beta_0\leq 1,\ a.e.\text{ on } \Gamma_3,\tag{2.15}$$

and we need the following set for the bonding field,

$$\mathcal{O}=\left\{\theta:[0,T]\rightarrow L^2(\Gamma_3);\ 0\leq\theta(t)\leq 1\ \forall t\in[0,T],\ a.e.\text{ on } \Gamma_3\right\}.$$

Now by assuming the solution to be sufficiently regular, we obtain by using Green's formula that the problem P_1 has the following variational formulation.

Problem P_2. Find a displacement field $u\in W^{1,\infty}(0,T;\Omega)$ and a bonding field $\beta\in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right)\cap\mathcal{O}$ such that

$$\begin{array}{l}\langle F\varepsilon(u(t)),\varepsilon(v)-\varepsilon(u(t))\rangle_Q+j(v)-j(u(t))\\ \\ +r(\beta(t),u(t),v-u(t))\geq(f(t),v-u(t))_V\quad\forall\ v\in V,\ t\in[0,T],\end{array}\tag{2.16}$$

$$\dot{\beta}(t)=-(c_\tau\beta(t)|R_\tau(u_\tau(t))|^2-\varepsilon_a)_+\ \ a.e.t\in(0,T),\tag{2.17}$$

$$\beta(0)=\beta_0 \text{ on } \Gamma_3.\tag{2.18}$$

Our main result of this section, which will be established in the next is the following theorem.

Theorem 2.1. *Let* $T > 0$ *and assume* (2.10), (2.12), (2.13), (2.14) *and* (2.15). *Then there exists a unique solution to Problem* P_2.

3. **Existence and uniqueness result**

The proof of Theorem 2.1 will be carried out in several steps. In the first step, let $k > 0$ and consider the space

$$X = \left\{ \beta \in C\left([0,T]\,; L^2\left(\Gamma_3\right)\right); \sup_{t\in[0,T]} \left[\exp\left(-kt\right) \left\|\beta\left(t\right)\right\|_{L^2(\Gamma_3)}\right] < +\infty \right\}.$$

X is a Banach space for the norm

$$\|\beta\|_X = \sup_{t\in[0,T]} \left[\exp\left(-kt\right) \left\|\beta\left(t\right)\right\|_{L^2(\Gamma_3)}\right],$$

which is equivalent for the standard norm $\|.\|_{C([0,T];L^2(\Gamma_3))}$, and for a given $\beta \in X$ we consider the following variational problem.

Problem $P_{1\beta}$. Find $u_\beta : [0,T] \to V$ such that

$$\begin{aligned} &\left\langle F\varepsilon\left(u_\beta\left(t\right)\right)), \varepsilon\left(v\right) - \varepsilon(u_\beta\left(t\right))\right\rangle_Q + j\left(v\right) - j\left(u_\beta\left(t\right)\right) \\ &+ r\left(\beta\left(t\right), u_\beta\left(t\right), v - u_\beta\left(t\right)\right) \geq \left(f\left(t\right), v - u_\beta\left(t\right)\right)_V \quad \forall\, v \in V,\, t \in [0,T]\,. \end{aligned} \tag{3.1}$$

Lemma 3.1. *There exists a unique solution to Problem* $P_{1\beta}$ *and it satisfies* $u_\beta \in C\left([0,T]\,; V\right)$.

Proof. Let $t \in [0,T]$ and let $A_t : V \to V$ be the operator given by

$$\left(A_t v, w\right)_V = \left\langle F\varepsilon\left(v\right), \varepsilon\left(w\right)\right\rangle_Q + r\left(\beta\left(t\right), v, w\right) \quad \forall v, w \in V.$$

Using (2.13) and the properties of R_τ (3.2) (see [15]) such that

$$\begin{aligned} &\left|R_\tau\left(u_\tau\right)\right| \leq L,\, \forall u \in V;\; \left|R_\tau\left(a\right) - R_\tau\left(b\right)\right| \leq \left|a - b\right|, \forall a, b \in \mathbf{R}^d, \\ &\left(R_\tau\left(u_\tau\right) - R_\tau\left(v_\tau\right)\right).\left(u_\tau - v_\tau\right) \geq 0,\, a.e. \text{ on } \Gamma_3,\, \forall u, v \in V, \end{aligned} \tag{3.2}$$

it follows that A_t is a strongly monotone and Lipschitz continuous operator. The functional j is a continuous semi-norm on V, then by a classical argument of elliptic variational inequalities [18], we deduce that there exists a unique element $u_\beta\left(t\right) \in V$ which satisfies (3.1). Let now, t_1, $t_2 \in [0,T]$. In inequality (3.1) written for $t = t_1$, take $w=u_{\beta}(t_2)$ and also in the same inequality written for $t=t_2$, take $w=u_{\beta}(t_1)$. We find after adding the resulting inequalities that

$$\begin{aligned}&\langle F\varepsilon(u_{\beta}(t_1)))-F\varepsilon(u_{\beta}(t_2)),\varepsilon(u_{\beta}(t_1))-\varepsilon(u_{\beta}(t_2))\rangle_Q\leq\\&r(\beta(t_1),u_{\beta}(t_1),u_{\beta}(t_2)-u_{\beta}(t_1))+r(\beta(t_2),u_{\beta}(t_2),u_{\beta}(t_1)-u_{\beta}(t_2)).\end{aligned}\tag{3.3}$$

We have

$$r(\beta(t_1),u_{\beta}(t_1),u_{\beta}(t_2)-u_{\beta}(t_1))+r(\beta(t_2),u_{\beta}(t_2),u_{\beta}(t_1)-u_{\beta}(t_2))=$$

$$\textstyle\int_{\Gamma_3}(c_\tau\beta^2(t_1)-\beta^2(t_2))R_\tau(u_\tau(t_1)).(u_{\beta\tau}(t_2)-u_{\beta\tau}(t_1))\,da$$

$$\textstyle+\int_{\Gamma_3}c_\tau\beta^2(t_2)(R_\tau(u_\tau(t_2))-R_\tau(u_\tau(t_1))).(u_{\beta\tau}(t_1)-u_{\beta\tau}(t_2))\,da.$$

Using (3.2), (2.13) (c) and, (2.9), it follows that exists a constant $c_1>0$ such that

$$\begin{aligned}&\|u_{\beta}(t_1)-u_{\beta}(t_2)\|_V\leq\\&c_1\left(\|f(t_1)-f(t_2)\|_V+\|\beta(t_1)-\beta(t_2)\|_{L^2(\Gamma_3)}\right).\end{aligned}\tag{3.4}$$

As $f\in C([0,T];V)$ and $\beta\in C\left([0,T];L^2(\Gamma_3)\right)$, then (3.4) implies that $u_\beta\in C([0,T];V)$. □

Next, we consider the following problem.

Problem $P_{2\beta}$. Find a bonding field $\beta^*:[0,T]\to L^2(\Gamma_3)$ such that

$$\dot\beta^*(t)=-\left(c_\tau\beta^*(t)\left|R_\tau(u_{\beta^*\tau}(t))\right|^2-\varepsilon_a\right)_+\quad a.e.\ t\in(0,T),\tag{3.5}$$

$$\beta^*(0)=\beta_0\text{ on }\Gamma_3.\tag{3.6}$$

We have the following result.

Lemma 3.2. *There exists a unique solution to Problem $P_{2\beta}$ and it satisfies*

$$\beta^*\in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right)\cap\mathcal{O}.$$

Proof. We consider the mapping $\mathcal{T}:X\to X$ given by

$$\mathcal{T}\beta(t)=\beta_0-\int_0^t(c_\tau\beta(s)\left|R_\tau(u_{\beta\tau}(s))\right|^2-\varepsilon_a)_+ds,$$

where u_β is a solution of Problem $P_{1\beta}$. Using that $|R_\tau(u_{\beta\tau})|\leq L$, it follows that there exists a constant $c_2>0$ such that

$$\|\mathcal{T}\beta_1(t)-\mathcal{T}\beta_2(t)\|_{L^2(\Gamma_3)}\leq c_2\textstyle\int_0^t\|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}\,ds.$$

Since

$$\int_0^t \|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}\,ds = \int_0^t e^{ks}(e^{-ks}\|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)})ds$$

$$\leq \|\beta_1-\beta_2\|_X \frac{e^{kt}}{k},$$

this inequality implies

$$e^{-kt}\|\mathcal{T}\beta_1(t)-\mathcal{T}\beta_2(t)\|_{L^2(\Gamma_3)} \leq \frac{c_2}{k}\|\beta_1-\beta_2\|_X \quad \forall t\in[0,T],$$

and then,

$$\|\mathcal{T}\beta_1-\mathcal{T}\beta_2\|_X \leq \frac{c_2}{k}\|\beta_1-\beta_2\|_X. \tag{3.7}$$

The inequality (3.7) shows that for $k > c_2$, $\mathcal{T}$ is a contraction. Then we deduce, by the Banach fixed point theorem that $\mathcal{T}$ has a unique fixed point β^* which satisfies (3.5) and (3.6). The regularity $\beta^* \in \mathcal{O}$ is a consequence of (3.6) and (2.15), see [15, 16] for details. □

Now, we provide the existence of the solution of Theorem 2.1. Indeed, let β^* be the fixed point of $\mathcal{T}$ and let u^* be the solution of Problem $P_{1\beta}$ for $\beta=\beta^*$, i-e., $u^* = u_{\beta^*}$. Take $v = u^*(t_2)$ in inequality (3.1) written for $t = t_1$, and also take $v = u^*(t_1)$ in the same inequality written for $t = t_2$ and adding the two inequalities, we obtain using similar arguments to those in the proof of (3.4), that there exists a constant $c_3 > 0$ such that

$$\|u^*(t_1)-u^*(t_2)\|_V \leq$$

$$c_3\left(\|\beta^*(t_1)-\beta^*(t_2)\|_{L^2(\Gamma_3)}+\|f(t_1)-f(t_2)\|_V\right) \quad \forall t_1,t_2\in[0,T]. \tag{3.8}$$

Now, as $T\beta^*=\beta^*$ we deduce from Lemma 3.2 that $\beta^* \in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right)$ and moreover as $f \in W^{1,\infty}(0,T;V)$, then (3.8) implies that $u^* \in W^{1,\infty}(0,T;V)$. Thus, we conclude by (3.1), (3.5) and (3.6) that (u^*,β^*) is a solution to Problem P_2. To prove the uniqueness of the solution, suppose that (u,β) is a solution of Problem P_2 which satisfies

$$(u,\beta)\in W^{1,\infty}(0,T;V)\times W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right)\cap\mathcal{O},$$

it follows that $\beta\in\mathcal{O}$. Moreover, we deduce from (3.1) that u is a solution to Problem $P_{1\beta}$, and as by Lemma 3.1, this problem has a unique solution denoted by u_β, we get $u = u_\beta$. Take $u = u_\beta$ in (2.16) and use the initial condition (2.18), we deduce that β is a solution of Problem $P_{2\beta}$. Therefore, we obtain from Lemma 3.2 that $\beta=\beta^*$ and we deduce that (u^*,β^*) is a unique solution to Problem P_2. □

4. The regularized problem

In this section we consider the frictional contact problem with adhesion in the case when the contact condition (2.5) is replaced by the contact conditions

$$u_{\delta\nu} = 0,\ \ \sigma_{\delta\tau} = -c_\tau \beta_\delta^2 R_\tau\left(u_{\delta\tau}\right) - g\frac{u_{\delta\tau}}{\sqrt{u_{\delta\tau}^2 + \delta^2}},\ \text{on}\ \Gamma_3 \times (0,T)\,, \tag{4.1}$$

where $\delta > 0$ is a regularization parameter. The friction law (4.1) describes situation when slip appears for a small shear, this is the case when the contact surfaces are lubricated by a thin layer or non-Newtonian fluid. Thus the regularized problem is defined as follows.

Problem $P_{1\delta}$. Find a displacement field $u_\delta : \Omega \times [0,T] \to \mathbf{R}^d$ and a bonding field $\beta_\delta : \Gamma_3 \times [0,T] \to [0,1]$ such that

$$div\sigma\left(u_\delta\right) + \varphi_1 = 0 \text{ in } \Omega \times (0,T)\,, \tag{4.2}$$

$$\sigma = F\varepsilon\left(u_\delta\right)\ \text{ in } \Omega \times (0,T)\,, \tag{4.3}$$

$$u_\delta = 0 \quad \text{ on }\ \Gamma_1 \times (0,T)\,, \tag{4.4}$$

$$\sigma\nu = \varphi_2 \quad \text{ on } \Gamma_2 \times (0,T)\,, \tag{4.5}$$

$$\left.u_{\delta\nu} = 0,\ \sigma_{\delta\tau} = -c_\tau \beta_\delta^2 R_\tau\left(u_{\delta\tau}\right) - g\frac{u_{\delta\tau}}{\sqrt{u_{\delta\tau}^2 + \delta^2}}\ \right\}\text{on } \Gamma_3 \times (0,T)\,, \tag{4.6}$$

$$\dot{\beta}_\delta = -(c_\tau \beta_\delta \left|R_\tau\left(u_{\delta\tau}\right)\right|^2 - \varepsilon_a)_+ \text{ on } \Gamma_3 \times (0,T)\,, \tag{4.7}$$

$$\beta_\delta\left(0\right) = \beta_0 \text{ on } \Gamma_3. \tag{4.8}$$

As in [18] let us define the functional $j_\delta : V \to \mathbf{R}$ by

$$j_\delta\left(v\right) = \int_{\Gamma_3} g\left(\sqrt{v_\tau^2 + \delta^2} - \delta\right) da,$$

then the problem (4.2) – (4.8) admits the following variational formulation.

Problem $P_{2\delta}$. Find $(u_\delta, \beta_\delta) \in W^{1,\infty}(0,T;V) \times W^{1,\infty}\left(0,T;L^2\left(\Gamma_3\right)\right)\right) \cap \mathcal{O}$ such that

$$\begin{aligned}&\left\langle F\varepsilon\left(u_\delta\left(t\right)\right), \varepsilon\left(v\right) - \varepsilon\left(u_\delta\left(t\right)\right)\right\rangle_Q + \ j_\delta\left(v\right) - \ j_\delta\left(u_\delta\left(t\right)\right)\\ &+r\left(\beta_\delta\left(t\right), u_\delta\left(t\right), v - u_\delta\left(t\right)\right) \geq \left(f\left(t\right), v - u_\delta\left(t\right)\right)_V \quad \forall v \in V,\ t \in [0,T]\,,\end{aligned} \tag{4.9}$$

$$\dot{\beta}_\delta\left(t\right) = -(c_\tau \beta_\delta\left(t\right)\left|R_\tau\left(u_{\delta\tau}\left(t\right)\right)\right|^2 - \varepsilon_a)_+\ a.e. \text{ on } \ (0,T)\,, \tag{4.10}$$

$$\beta_\delta\left(0\right) = \beta_0 \text{ on } \Gamma_3. \tag{4.11}$$

Now, our main is to study the behavior of the solution as $\delta \to 0$ and to prove that in the limit we obtain the solution of Problem P_2.

Theorem 4.1. *Assume that* (2.10), (2.12), (2.13), (2.14) *and* (2.15) *hold. Then for each* $\delta > 0$*, there exists a unique solution to Problem* $P_{2\delta}$.

Proof. The proof of Theorem 4.1 is similar to the proof of Theorem 2.1 and it is carried out in several steps. For this reason, we omit the details of the proof. The steps are:
(i) For any $\beta \in X$ we prove that there exists a unique $u_\delta \in C([0,T];V)$ such that

$$\begin{aligned}&\langle F\varepsilon(u_\delta(t)), \varepsilon(v) - \varepsilon(u_\delta(t))\rangle_Q + j_\delta(v) - j_\delta(u_\delta(t))\\ &+r(\beta(t), u_\delta(t), v - u_\delta(t)) \geq (f(t), v - u_\delta(t))_V \quad \forall v \in V,\ t \in [0,T].\end{aligned} \tag{4.12}$$

Indeed as the functional j_δ is proper convex and lower semicontinous on V, (see [18]), then using similar arguments to those in the proof of Lemma 3.1, we deduce (i).
(ii) There exists a unique β_δ such that

$$\beta_\delta \in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right) \cap \mathcal{O}, \tag{4.13}$$

$$\dot{\beta}_\delta(t) = -(c_\tau \beta_\delta(t) |R_\tau(u_{\delta\tau}(t))|^2 - \varepsilon_a)_+ \ a.e.\ t \in (0,T), \tag{4.14}$$

$$\beta_\delta(0) = \beta_0. \tag{4.15}$$

The proof of this step is based on Lemma 3.2.
(iii) Let β_δ defined in (ii) and denote again by u_δ the function obtained in step (i) for $\beta = \beta_\delta$. Then, using (4.12) – (4.15) we see that (u_δ, β_δ) is the unique solution to Problem $P_{2\delta}$ and it satisfies

$$(u_\delta, \beta_\delta) \in W^{1,\infty}(0,T;V) \times W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right) \cap \mathcal{O}.$$

We now study the convergence of the solution (u_δ, β_δ) as $\delta \to 0$.
Theorem 4.2. *Assume that* (2.10), (2.12), (2.13), (2.14) *and* (2.15) *hold. Then we have the following convergences:*

$$\lim_{\delta\to 0} \|u_\delta(t) - u(t)\|_V = 0, \ \textit{for all}\ t \in [0,T], \tag{4.16}$$

$$\lim_{\delta\to 0} \|\beta_\delta(t) - \beta(t)\|_{L^2(\Gamma_3)} = 0, \ \textit{for all}\ t \in [0,T]. \tag{4.17}$$

The proof is carried out in several steps. In the first step, we show the following lemma.
Lemma 4.3. *For each $t \in [0,T]$, there exists $\bar{u}(t) \in V$ such that after passing to a subsequence still denoted $(u_\delta(t))$ we have*

$$u_\delta(t) \to \bar{u}(t) \ \textit{weakly in}\ V\ \textit{as}\ \delta \to 0. \tag{4.18}$$

Proof. It is well known (see [18]) that the inequality (4.9) is equivalent to the equality

$$\begin{aligned}&\langle F\varepsilon(u_\delta(t)), \varepsilon(v)\rangle_Q + (\nabla j_\delta(u_\delta(t)), v)_{L^2(\Gamma_3)}\\ &+r(\beta_\delta(t), u_\delta(t), v) = (f(t), v)_V \quad \forall v \in V,\ t \in [0,T],\end{aligned} \tag{4.19}$$

where

$$(\nabla j_\delta (u_\delta (t)), v)_{L^2(\Gamma_3)} = \int_{\Gamma_3} g \frac{u_{\delta\tau}(t) v_\tau}{\sqrt{u_{\delta\tau}(t)^2 + \delta^2}} da.$$

Take $v = u_\delta (t)$ in (4.19), as

$$(\nabla j_\delta (u_\delta (t)), (u_\delta (t)))_{L^2(\Gamma_3)} \geq 0,\ r(\beta_\delta (t), u_\delta (t), u_\delta (t)) \geq 0,$$

then we get from (4.19) that

$$\langle F\varepsilon (u_\delta (t)), \varepsilon (u_\delta (t))\rangle_Q \leq (f(t), u_\delta (t))_V$$

and keeping, in mind (2.13) (c), it follows that there exists a constant $C > 0$ such that

$$\|u_\delta (t)\|_V \leq C \left(\|f(t)\|_V + \|F(0)\|_Q \right).$$

The sequence $(u_\delta (t))$ is bounded in V, then there exists $\bar{u}(t) \in V$ and a subsequence again denoted $(u_\delta (t))$ such that (4.18) holds. Let us now consider the auxiliary problem.

Problem P_a. Find $\beta : [0, T] \to L^2(\Gamma_3)$, such that

$$\dot{\beta}(t) = -(c_\tau \beta (t)) |R_\tau (\bar{u}_\tau (t))|^2 - \varepsilon_a)_+,\ a.e.\ t \in (0, T),$$

$$\beta(0) = \beta_0.$$

Using similar arguments to those in the proof of Lemma 3.2, we have the following result.

Lemma 4.4. *Problem P_a has a unique solution which satisfies*

$$\beta \in W^{1,\infty}\left(0, T; L^2(\Gamma_3)\right) \cap \mathcal{O}.$$

Next, we have the convergence result.

Lemma 4.5. *Let β be the solution to Problem P_a, then we have*

$$\lim_{\delta \to 0} \|\beta_\delta (t) - \beta (t)\|_{L^2(\Gamma_3)} = 0, \ \textit{for all}\ t \in [0, T]. \tag{4.20}$$

Proof. Using the properties of the operator R_τ, (see [15]), it follows that there exists a constant $C_1 > 0$ such that

$$\|\beta_\delta (t) - \beta (t)\|_{L^2(\Gamma_3)} \leq C_1 \int_0^t \|u_{\delta\tau}(s) - \bar{u}_\tau (s)\|_{(L^2(\Gamma_3))^d}\, ds. \tag{4.21}$$

From (4.18) we deduce that $u_{\delta\tau}(t) \to \bar{u}_\tau (t)$ strongly in $\left(L^2(\Gamma_3)\right)^d$, as $\delta \to 0$. On the other hand using (2.9), we have

$$\|u_{\delta\tau}(t) - \bar{u}_\tau (t)\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|u_\delta (t) - \bar{u}(t)\|_V$$

$$\leq d_\Omega (\|f(t)\|_V + \|\bar{u}(t)\|_V),$$

which implies that there exists a constant $C_2 > 0$ such that

$$\left\|u_{\delta\tau}(t) - \bar{u}_\tau(t)\right\|_{(L^2(\Gamma_3))^d} \leq C_2.$$

Then it follows from Lebesgue convergence theorem that

$$\lim_{\delta\to 0}\int_0^t \left\|u_{\delta\tau}(s) - \bar{u}_\tau(s)\right\|_{(L^2(\Gamma_3))^d} ds = 0.$$

So we deduce from (4.21) that

$$\left\|\beta_\delta(t) - \beta(t)\right\|_{L^2(\Gamma_3)} \to 0 \text{ as } \delta \to 0, \text{ for all } t \in [0,T],$$

and so (4.20) is proved.
Now it is necessary to show the following result.
Lemma 4.6. *We have* $\bar{u}(t) = u(t)$ *for all* $t \in [0,T]$.
Proof. Let $t \in [0,T]$. We have (see [18]),

$$\begin{aligned} &|j_\delta(v) - j(v)| \leq \delta \left\|g\right\|_{L^\infty(\Gamma_3)} meas\,\Gamma_3, \\ &|j_\delta(u_\delta(t)) - j(\bar{u}(t))| \leq \delta \left\|g\right\|_{L^\infty(\Gamma_3)} meas\,\Gamma_3 \\ &+ \left\|g\right\|_{L^\infty(\Gamma_3)} \left\|u_{\delta\tau}(t) - \bar{u}_\tau(t)\right\|_{(L^2(\Gamma_3))^d}. \end{aligned} \tag{4.22}$$

It follows from (4.22), as $\delta \to 0$, that

$$j_\delta(v) \to j(v), \text{ for all } v \in V, \tag{4.23}$$

and

$$j_\delta(u_\delta(t)) \to j(\bar{u}(t)). \tag{4.24}$$

On the other hand we have

$$\begin{aligned} &r(\beta_\delta(t), u_\delta(t), v - u_\delta(t)) \\ &= r(\beta_\delta(t), u_\delta(t), v - u_\delta(t)) - r(\beta(t), u_\delta(t), v - u_\delta(t)) \\ &+ r(\beta(t), u_\delta(t), v - u_\delta(t)) - r(\beta(t), \bar{u}(t), v - u_\delta(t)) \\ &+ r(\beta(t), \bar{u}(t), \bar{u}(t) - u_\delta(t)) + r(\beta(t), \bar{u}(t), v - \bar{u}(t)). \end{aligned} \tag{4.25}$$

Using that $0 \leq \beta(t) \leq 1$, $0 \leq \beta_\delta(t) \leq 1$, for all $t \in [0,T]$, and the properties of the operator R_τ, we have as $\delta \to 0$,

$$\begin{aligned} & r(\beta_\delta(t), u_\delta(t), v - u_\delta(t)) - r(\beta(t), u_\delta(t), v - u_\delta(t)) \to 0, \\ & r(\beta(t), u_\delta(t), v - u_\delta(t)) - r(\beta(t), \bar{u}(t), v - u_\delta(t)) \to 0, \\ & r(\beta(t), \bar{u}(t), \bar{u}(t) - u_\delta(t)) \to 0. \end{aligned} \tag{4.26}$$

So, we deduce from (4.25) and (4.26) that

$$r(\beta_\delta(t), u_\delta(t), v - u_\delta(t)) \to r(\beta(t), \bar{u}(t), v - \bar{u}(t)), \text{ as } \delta \to 0. \tag{4.27}$$

Therefore, using (4.23), (4.24), (4.27), and passing to the limit in (4.9) as $\delta \to 0$, we obtain

$$\begin{aligned} & \langle F\varepsilon(\bar{u}(t)), \varepsilon(v) - \varepsilon(\bar{u}(t))\rangle_Q + j(v) - j(\bar{u}(t)) \\ & + r(\beta(t), \bar{u}(t), v - \bar{u}(t)) \geq (f(t), v - \bar{u}(t))_V \quad \forall v \in V. \end{aligned} \tag{4.28}$$

Take now $v = u(t)$ in (4.28) and $v = \bar{u}(t)$ in (2.16) and add them up, we obtain using (2.13) (c) that

$$m\|\bar{u}(t) - u(t)\|_V^2 \leq r(\beta(t), \bar{u}(t), u(t) - \bar{u}(t)) + r(\beta(t), u(t), \bar{u}(t) - u(t)). \tag{4.29}$$

So as

$$r(\beta(t), \bar{u}(t), u(t) - \bar{u}(t)) + r(\beta(t), u(t), \bar{u}(t) - u(t)) \leq 0,$$

it follows from (4.29) that

$$\bar{u}(t) = u(t). \tag{4.30}$$

We have now all the ingredients to prove Theorem 4.2. Indeed, from (4.20) and (4.30), we deduce immediatly (4.17). To prove (4.16), take $v = u(t)$ in (4.28), and using (2.13) (c), it follows

$$\begin{aligned} & m\|u_\delta(t) - u(t)\|_V^2 \leq \\ & j_\delta(u(t)) - j_\delta(u_\delta(t)) + r(\beta_\delta(t), u_\delta(t), u(t) - u_\delta(t)) \\ & + \langle F\varepsilon(u(t)), \varepsilon(u(t) - u_\delta(t))\rangle_Q + (f(t), u_\delta(t) - u(t))_V. \end{aligned} \tag{4.31}$$

Passing to the limit as $\delta \to 0$ in the previous inequality and using the convergences

$$j_\delta(u(t)) - j_\delta(u_\delta(t)) \to 0,$$

$$r(\beta_\delta(t), u_\delta(t), u(t) - u_\delta(t)) \to 0,$$

$$\langle F\varepsilon(u(t)), \varepsilon(u(t) - u_\delta(t))\rangle_Q + (f(t), u_\delta(t) - u(t))_V \to 0,$$

we see immediately that (4.16) follows from (4.31).

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SUBCLASSES OF HARMONIC FUNCTIONS BASED ON GENERALIZED DERIVATIVE OPERATOR

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Abstract. Making use of Salagean and Ruscheweyh derivative operator we introduced a new class of complex-valued harmonic functions which are orientation preserving, univalent and starlike functions. We investigate the coefficient bounds, distortion inequalities, extreme points and inclusion results for the generalized class of functions.

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain Ω if both u and v are real and harmonic in Ω. In any simply connected domain $\mathcal{D} \subset \Omega$ we can write $f = h + \overline{g}$ where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$ (see [1]).

Denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g} \tag{1.1}$$

which are harmonic univalent and orientation preserving in the open unit disc $U = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$, we may express the analytic functions h and g in the forms $h(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=1}^{\infty} b_n z^n, (0 \leq b_1 < 1)$. Then

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \ \ |b_1| < 1. \tag{1.2}$$

We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class S of normalized univalent functions if the co-analytic part of $f = h + \overline{g}$ is identically zero that is $g \equiv 0$. Due to Silverman

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[6] we denote $\overline{\mathcal{H}}$ the subclass of $\mathcal{H}$ consisting of functions of the form $f = h + \overline{g}$ given by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1, a_n, b_n \geq 0. \tag{1.3}$$

In 1999 Jahangiri [2] introduced a subclass of $\mathcal{H}$ called the class of harmonic starlike functions of order α denoted by $S_H(\alpha)$ which consist of functions of the form (1.1) and satisfying the inequality:

$$\frac{\partial}{\partial \theta}\left(arg(f(z)\right) > \alpha \tag{1.4}$$

Equivalently

$$\text{Re}\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} \geq \alpha \tag{1.5}$$

where $z \in \mathcal{U}$.

Given two functions $\phi(z) = z + \sum\limits_{n=2}^{\infty} \phi_n z^n$ and $\psi(z) = z + \sum\limits_{n=2}^{\infty} \psi_n z^n$ in $\mathcal{S}$ their Hadamard product or convolution is defined by $(\phi * \psi)(z) = \phi(z) * \psi(z) = z + \sum\limits_{n=2}^{\infty} \phi_n \psi_n z^n$. Using the convolution, Ruscheweyh [5] introduced the derivative operator

$$D^m \phi(z) := \frac{z}{(1-z)^{m-1}} = z + \sum_{n=2}^{\infty} \left(\begin{array}{c} m+n-1 \\ n-1 \end{array}\right) \phi_n z^n, \quad (z \in U,\, m > -1). \tag{1.6}$$

Recently in [4] Jahangiri and etal. defined the Ruscheweyh derivative for harmonic functions, as given below

$$D^m f(z) := z + \sum_{n=2}^{\infty} \left(\begin{array}{c} m+n-1 \\ n-1 \end{array}\right) a_n z^n + \sum_{n=1}^{\infty} \left(\begin{array}{c} m+n-1 \\ n-1 \end{array}\right) \overline{b_n z^n}, \tag{1.7}$$

which was initially studied for the class of harmonic starlike functions $S_H(\alpha)$ in [4]. Further motivated by the works of Jahangiri et. al. [3] we define a new generalized derivative operator on harmonic function $f = h + \overline{g}$ in $\mathcal{H}$ as

$$D_k^m f(z) = D_k^m h(z) + (-1)^k \overline{D_k^m g(z)}, \quad m > -1, \text{ and } k \geq 0 \tag{1.8}$$

where

$$D_k^m h(z) = z + \sum_{n=2}^{\infty} n^k C(n,m) a_n z^n, D_k^m g(z) = \sum_{n=1}^{\infty} n^k C(n,m) b_n z^n,$$

and

$$C(n,m) = \binom{n+m-1}{n-1}.$$

For $0 \le \alpha < 1$, we let $\mathcal{HR}_k^m(\lambda, \alpha)$ a subclass of $\mathcal{H}$ of the form $f = h + \overline{g}$ given by (1.2) and satisfying the analytic criteria

$$\text{Re}\left\{\frac{z(D_k^m f(z))'}{(1-\lambda)D_k^m f(z) + \lambda z(D_k^m f(z))'}\right\} \ge \alpha \tag{1.9}$$

where $0 \le \lambda < 1$, $D_k^m f$ is given by (1.8) and $z \in U$. We also let $\overline{\mathcal{HR}}_k^m(\lambda, \alpha) = \mathcal{HR}_k^m(\lambda, \alpha) \cap \overline{\mathcal{H}}$.

We investigate the coefficient bounds,distortion inequalities, extreme points and inclusion results for the generalized class $\overline{\mathcal{HR}}_k^m(\lambda, \alpha)$

The Class $\mathcal{HR}_k^m(\lambda, \alpha)$

In our first theorem, we obtain a sufficient coefficient condition for harmonic functions in $\mathcal{HR}_k^m(\lambda, \alpha)$.

Theorem 1.1. *Let $f = h + \overline{g}$ be given by (1.2). If*

$$\sum_{n=1}^{\infty} n^k C(n,m)\left[(n - \alpha - \alpha\lambda(n-1))|a_n| + (n + \alpha - \alpha\lambda(n+1))|b_n|\right] \le 2(1-\alpha), \tag{1.10}$$

where $a_1 = 1$ and $0 \le \alpha < 1$, then $f \in \mathcal{HR}_k^m(\lambda, \alpha)$.

Proof. We first show that if (1.10) holds for the coefficients of $f = h + \overline{g}$, the required condition (1.9) is satisfied. From (1.9) we can write

$$\begin{aligned} &\text{Re}\left\{\frac{z(D_k^m h(z))' - \overline{z(D_k^m g(z))'}}{(1-\lambda)(D_k^m h(z) + \overline{D_k^m g(z)}) + \lambda(z(D_k^m h(z))' - \overline{z(D_k^m g(z))'})}\right\} \ge \alpha \\ = \;&\text{Re}\,\frac{A(z)}{B(z)} \ge \alpha, \end{aligned}$$

where

$$\begin{aligned} A(z) &= z(D_k^m h(z))' - \overline{z(D_k^m g(z))'} \\ &= z + \sum_{n=2}^{\infty} n^k C(n,m) a_n z^n - \sum_{n=1}^{\infty} n^k C(n,m)\overline{b_n}\overline{z}^n \\ \text{and } B(z) &= (1-\lambda)(D_k^m h(z) + \overline{D_k^m g(z)}) + \lambda(z(D_k^m h(z))' - \overline{z(D_k^m g(z))'}\ ) \\ &= z + \sum_{n=2}^{\infty} n^k C(n,m)(1-\lambda+n\lambda) a_n z^n + \sum_{n=1}^{\infty} n^k C(n,m)(1-\lambda-n\lambda)\overline{b_n}\overline{z}^n. \end{aligned}$$

Using the fact that Re $\{w\} \ge \alpha$ if and only if $|1 - \alpha + w| \ge |1 + \alpha - w|$, it suffices to show that

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \ge 0. \tag{1.11}$$

Substituting for $A(z)$ and $B(z)$ in (1.11), we get

$$|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|$$

$$=| \ (2-\alpha)z+\sum_{n=2}^{\infty}n^kC(n,m)[(n+1-\alpha)(1-\lambda+n\lambda)]a_nz^n$$

$$-\sum_{n=1}^{\infty}n^kC(n,m)[n-(1-\alpha)(1-\lambda+n\lambda)]\overline{b}_n\ \overline{z}^n \ \ |$$

$$-\ |\ -\alpha z+\sum_{n=2}^{\infty}n^kC(n,m)[n-(1+\alpha)(1-\lambda+n\lambda)a_nz^n$$

$$-\sum_{n=1}^{\infty}n^kC(n,m)[n+(1+\alpha)(1-\lambda+n\lambda)]\overline{b}_n\overline{z}^n \ |$$

$$\geq(2-\alpha)|z|-\sum_{n=2}^{\infty}n^kC(n,m)[n+(1-\alpha)(1-\lambda+n\lambda)|a_n||z|^n$$

$$-\sum_{n=1}^{\infty}n^kC(n,m)[n-(1-\alpha)(1-\lambda-n\lambda)]|b_n|\ |z|^n$$

$$-\alpha|z|-\sum_{n=2}^{\infty}n^kC(n,m)[n-(1+\alpha)(1-\lambda+n\lambda)]|a_n|\ |z|^n$$

$$-\sum_{n=1}^{\infty}n^kC(n,m)[n+(1+\alpha)(1-\lambda-n\lambda)]|b_n|\ |z|^n$$

$$\geq 2(1-\alpha)|z|\left\{2-\sum_{n=1}^{\infty}n^kC(n,m)\left[\frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}|a_n|\right.\right.$$

$$\left.\left.+\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}|b_n|\right]|z|^{n-1}\right\}$$

$$\geq 2(1-\alpha)\left\{2-\sum_{n=1}^{\infty}n^kC(n,m)\left[\frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}|a_n|+\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}|b_n|\right]\right\}.$$

The above expression is non negative by (1.10), and so $f(z)\in\mathcal{HR}_k^m(\lambda,\alpha)$. □

Corollary 1.2. *Let $f=h+\overline{g}$ be of the form (1.2) and satisfy the condition (1.10). Then each $D^i(z)$, $-1<i\leq m$, is orientation preserving, harmonic univalent and starlike of order α in U.*

Proof. Observe that $n^k C(n,m)$ is an increasing function of n. Therefore, by (1.10) for each i, $-1 < i \le m$, we can write

$$\begin{aligned}
&\sum_{n=1}^{\infty}[(n-\alpha-\alpha\lambda(n-1)]|a_n| + \sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(n+1)]|b_n|ht] \\
\le\ &\sum_{n=1}^{\infty} C(n,i)[n-\alpha-\alpha\lambda(n-1)]|a_n| + \sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(n+1)]|b_n| \\
\le\ &\sum_{n=1}^{\infty} n^k C(n,m)[n-\alpha-\alpha\lambda(n-1)]|a_n| + [n+\alpha-\alpha\lambda(n+1)]|b_n| \\
\le\ &2(1-\alpha).
\end{aligned}$$

Thus, by (1.10) each $D^i(z)$, $-1 < i \le m$, is orientation preserving, harmonic univalent and starlike of order α in U.

The harmonic function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1-\alpha}{n^k C(n,m)[n-\alpha-\alpha\lambda(n-1)]} x_n z^n + \sum_{n=1}^{\infty} \frac{1-\alpha}{n^k C(n,m)[n+\alpha-\alpha\lambda(n+1)]} \overline{y_n}(\overline{z})^n \tag{1.7}$$

where $\sum\limits_{n=2}^{\infty} |x_n| + \sum\limits_{n=1}^{\infty} |y_n| = 1$ shows that the coefficient bound given by (1.10) is sharp.

The functions of the form (1.7) are in $\mathcal{HR}_k^m(\lambda,\alpha)$ because

$$\sum_{n=1}^{\infty}\left(\frac{n^k C(n,m)[n-\alpha-\alpha\lambda(n-1)]}{1-\alpha}|a_n| + \frac{n^k C(n,m)[n+\alpha-\alpha\lambda(n+1)]}{1-\alpha}|b_n|\right)$$

$$=1 + \sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 2.$$

□

Next theorem establishes that such coefficient bounds cannot be improved further.

Theorem 1.3. *For $a_1 = 1$ and $0 \le \alpha < 1$, $f = h + \overline{g} \in \overline{\mathcal{H}\mathcal{R}}_k^m(\lambda,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} n^k C(n,m)\left\{[n-\alpha-\alpha\lambda(n-1)]|a_n| + [n+\alpha-\alpha\lambda(n+1)]|b_n|\right\} \le 2(1-\alpha). \tag{1.8}$$

Proof. Since $\overline{\mathcal{H}\mathcal{R}}_k^m(\lambda,\alpha) \subset \mathcal{HR}_k^m(\lambda,\alpha)$, we only need to prove the "only if" part of the theorem. To this end, for functions f of the form (1.3), we notice that the condition

$$\operatorname{Re}\left\{\frac{z(D_k^m f(z))'}{(1-\lambda)D_k^m f(z) + \lambda z(D_k^m f(z))'}\right\} \ge \alpha.$$

Equivalently,

$$\operatorname{Re}\left\{\frac{(1-\alpha)z-\sum\limits_{n=2}^{\infty}[n-\alpha-\alpha\lambda(n-1)]n^kC(n,m)a_nz^n-\sum\limits_{n=1}^{\infty}[n+\alpha-\alpha\lambda(n+1)]n^kC(n,m)\bar{b}_n\bar{z}^n}{z-\sum\limits_{n=2}^{\infty}n^kC(n,m)(1-\lambda+n\lambda)a_nz^n+\sum\limits_{n=1}^{\infty}n^kC(n,m)(1-\lambda-n\lambda)\bar{b}_n\bar{z}^n}\right\}\geq 0.$$

The above required condition must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$\frac{(1-\alpha)-\sum\limits_{n=2}^{\infty}[n-\alpha-\alpha\lambda(n-1)]n^kC(n,m)a_nr^{n-1}-\sum\limits_{n=1}^{\infty}[n+\alpha-\alpha\lambda(n+1)]n^kC(n,m)b_nr^{n-1}}{1-\sum\limits_{n=2}^{\infty}n^kC(n,m)(1-\lambda+n\lambda)a_nr^{n-1}+\sum\limits_{n=1}^{\infty}n^kC(n,m)(1-\lambda-n\lambda)b_nr^{n-1}}\geq 0. \quad (1.9)$$

If the condition (1.8) does not hold, then the numerator in (1.9) is negative for r sufficiently close to 1. Hence, there exists $z_0 = r_0$ in (0,1) for which the quotient of (1.9) is negative. This contradicts the required condition for $f(z) \in \overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$. This completes the proof of the theorem. □

Corollary 1.4. *Let $f = h + \overline{g}$ be given by (1.3). Then $D^i f(z)$, $-1 < i \leq m$ is orientation preserving, harmonic and starlike of order α, $0 \leq \alpha < 1$, if and only if the coefficient condition (1.8) holds.*

Next we determine the extreme points of closed convex hulls of $\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$ denoted by $clco\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$.

Theorem 1.5. *A function $f(z) \in \overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$ if and only if*

$$f(z) = \sum_{n=1}^{\infty}\left(X_nh_n(z) + Y_ng_n(z)\right),$$

where

$$h_1(z) = z, h_n(z) = z - \frac{1-\alpha}{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]}z^n; \quad (n \geq 2),$$

$$g_n(z) = z + \frac{1-\alpha}{n^kC(n,m)[+\alpha-\alpha\lambda(n+1)]}\overline{z}^n; \quad (n \geq 2),$$

$$\sum_{n=1}^{\infty}(X_n+Y_n) = 1, \quad X_n \geq 0 \text{ and } \ Y_n \geq 0.$$

In particular, the extreme points of $\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. First, we note that for f as in the theorem above, we may write

$$f(z) = \sum_{n=1}^{\infty}\left(X_nh_n(z) + Y_ng_n(z)\right)$$

$$= \sum_{n=1}^{\infty}(X_n+Y_n)z - \sum_{n=2}^{\infty}\frac{1-\alpha}{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]}X_nz^n$$

$$+\sum_{n=1}^{\infty}\frac{1-\alpha}{n^kC(n,m)[n+\alpha-\alpha\lambda(n+1)]}Y_n\overline{z}^n = z-\sum_{n=2}^{\infty}A_nz^n+\sum_{n=1}^{\infty}B_n\overline{z}^n,$$

where

$$A_n = \frac{1-\alpha}{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)])}X_n,$$

and

$$B_n = \frac{1-\alpha}{n^kC(n,m)[n+\alpha-\alpha\lambda(n+1)]}Y_n.$$

Therefore

$$\sum_{n=2}^{\infty}\frac{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]}{1-\alpha}A_n+\sum_{n=1}^{\infty}\frac{n^kC(n,m)[n+\alpha-\alpha\lambda(n+1)]}{1-\alpha}B_n$$

$$=\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n = 1-X_1 \le 1,$$

and hence $f(z)\in clco\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$.

Conversely, suppose that $f(z)\in clco\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$. Setting

$$X_n = \frac{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]}{1-\alpha}A_n, \quad (n\ge 2)$$

and

$$Y_n = \frac{n^kC(n,m)[n+\alpha-\alpha\lambda(n-1)]}{1-\alpha}B_n, \quad (n\ge 1)$$

where $\sum_{n=1}^{\infty}(X_n+Y_n)=1$. Then

$$\begin{aligned}
f(z) &= z-\sum_{n=2}^{\infty}a_nz^n+\sum_{n=1}^{\infty}\overline{b_n}\overline{z}^n, \quad a_n,\ b_n\ge 0.\\
&= z-\sum_{n=2}^{\infty}\frac{1-\alpha}{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]}X_nz^n\\
&+\sum_{n=1}^{\infty}\frac{1-\alpha}{n^kC(n,m)[n+\alpha-\alpha\lambda(n-1)]}Y_n\overline{z}^n\\
&= z-\sum_{n=2}^{\infty}(h_n(z)-z)X_n+\sum_{n=1}^{\infty}(g_n(z)-z)Y_n\\
&= \sum_{n=1}^{\infty}(X_nh_n(z)+Y_ng_n(z))
\end{aligned}$$

as required. □

The following theorem gives the distortion bounds for functions in $R_{\overline{H}}(m,\alpha)$ which yields a covering result for the class $\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$.

Theorem 1.6. *Let* $f \in \overline{\mathcal{HR}}_k^m(\lambda, \alpha)$. *Then for* $|z| = r < 1$, *we have*

$$(1 - b_1)r - \frac{1}{2^k C(2,m)}\left(\frac{1-\alpha}{2-\alpha-\alpha\lambda} - \frac{1+\alpha}{2-\alpha-\alpha\lambda}b_1\right) r^2 \le |f(z)|$$
$$\le (1 + b_1)r + \frac{1}{2^k C(2,m)}\left(\frac{1-\alpha}{2-\alpha-\alpha\lambda} - \frac{1+\alpha}{2-\alpha-\alpha\lambda}b_1\right) r^2.$$

Proof. We only prove the right hand inequality. Taking the absolute value of $f(z)$, we obtain

$$\begin{aligned}
|f(z)| &= \left| z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n \right| \\
&= \left| z + b_1\ \overline{z} + \sum_{n=2}^{\infty} (a_n z^n + \overline{b}_n \overline{z}^n) \right| \\
&\le (1 + b_1)|z| + \sum_{n=2}^{\infty} (a_n + b_n)|z|^n \\
&\le (1 + |b_1|)r + \sum_{n=2}^{\infty} (a_n + b_n)r^n \\
&\le (1 + b_1)r + \sum_{n=2}^{\infty} (a_n + b_n)r^2 \\
&\le (1 + b_1)r + \frac{1-\alpha}{2^k C(2,m)(2-\alpha-\alpha\lambda)} \\
&\quad \sum_{n=2}^{\infty}\left(\frac{2^k C(2,m)(2-\alpha-\alpha\lambda)}{1-\alpha}a_n + \frac{2^k C(2,m)(2-\alpha-\alpha\lambda)}{1-\alpha}b_n\right) r^2 \\
&\le (1 + b_1)r + \frac{1-\alpha}{2^k C(2,m)(2-\alpha-\alpha\lambda)}\left(1 - \frac{1+\alpha}{1-\alpha}b_1\right) r^2 \\
&\le (1 + b_1)r + \frac{1}{2^k C(2,m)}\left(\frac{1-\alpha}{2-\alpha-\alpha\lambda} - \frac{1+\alpha}{2-\alpha-\alpha\lambda}b_1\right) r^2.
\end{aligned}$$

□

The proof of the left hand inequality follows on lines similar to that of the right hand side inequality.
The covering result follows from the left hand inequality given in Theorem 1.6.

Corollary 1.7. *If* $f(z) \in \overline{\mathcal{HR}}_k^m(\lambda, \alpha)$. *Then*

$$\left\{ w : |w| < \frac{2^{k+1}C(2,m) - 1 - ((1+\lambda)2^k C(2,m) - 1)\alpha}{2^k C(2,m)(2-\alpha-\alpha\lambda)} \right.$$
$$\left. - \frac{2^{k+1}C(2,m) - 1 - ((1+\lambda)2^k C(2,m) - 1)\alpha}{2^k C(2,m)(2-\alpha-\alpha\lambda)} b_1 \right\} \subset f(U).$$

Proof. Using the left hand inequality of Theorem 1.6 and letting $r \to 1$, we prove that

$$\begin{aligned}
&(1-b_1) - \frac{1}{2^k C(2,m)}\left(\frac{1-\alpha}{2-\alpha-\alpha\lambda} - \frac{1+\alpha}{2-\alpha-\alpha\lambda}b_1\right)\\
&= (1-b_1) - \frac{1}{2^k C(2,m)(2-\alpha-\alpha\lambda)}[1-\alpha-(1+\alpha)b_1]\\
&= \frac{(1-b_1)2^k C(2,m)(2-\alpha-\alpha\lambda) - (1-\alpha) + (1+\alpha)b_1}{2^k C(2,m)(2-\alpha-\alpha\lambda)}\\
&= \frac{2^k C(2,m)(2-\alpha-\alpha\lambda) - 2^k C(2,m)(2-\alpha-\alpha\lambda)b_1 - (1-\alpha) + (1+\alpha)b_1}{2^k C(2,m)(2-\alpha-\alpha\lambda)}\\
&= \frac{2^k C(2,m)(2-\alpha-\alpha\lambda) - 1 + \alpha - [2^k C(2,m)(2-\alpha-\alpha\lambda) - (1+\alpha)]b_1}{2^k C(2,m)(2-\alpha-\alpha\lambda)}\\
&= \frac{2^{k+1} C(2,m) - 1 - \alpha[(1+\lambda)2^k C(2,m) - 1]}{2^k C(2,m)(2-\alpha-\alpha\lambda)}\\
&- \frac{[2C(2,m) - 1 - \alpha((1+\lambda)C(2,m) - 1)]}{2^k C(2,m)(2-\alpha-\alpha\lambda)}b_1 \subset f(U).
\end{aligned}$$

□

Now we show that $\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$ is closed under convex combinations of its member and also closed under the convolution product.

Theorem 1.8. *The family $\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$ is closed under convex combinations.*

Proof. For $i = 1, 2, \dots,$ suppose that $f_i \in \overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$ where

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{i,n} z^n + \sum_{n=2}^{\infty} \bar{b}_{i,n} \bar{z}^n.$$

Then, by Theorem 1.3

$$\sum_{n=2}^{\infty} \frac{n^k C(n,m)[n-\alpha-\alpha\lambda(n-1)]}{(1-\alpha)} a_{i,n} + \sum_{n=1}^{\infty} \frac{n^k C(n,m)[n+\alpha-\alpha\lambda(n+1)]}{(1-\alpha)} b_{i,n} \le 1. \tag{1.10}$$

For $\sum_{i=1}^{\infty} t_i$, $0 \le t_i \le 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{i,n}\right) z^n + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i \bar{b}_{i,n}\right)\bar{z}^n.$$

Using the inequality (1.8), we obtain

$$\sum_{n=2}^{\infty} \frac{n^k C(n,m)[n-\alpha-\alpha\lambda(n-1)]}{(1-\alpha)}\left(\sum_{i=1}^{\infty} t_i a_{i,n}\right) +$$

$$+\sum_{n=1}^{\infty}\frac{n^kC(n,m)[n+\alpha-\alpha\lambda(n+1)]}{(1-\alpha)}\left(\sum_{i=1}^{\infty}t_ib_{i,n}\right)$$

$$=\sum_{i=1}^{\infty}t_i\left(\sum_{n=2}^{\infty}\frac{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]}{(1-\alpha)}a_{i,n}+\sum_{n=1}^{\infty}\frac{n^kC(n,m)[n+\alpha-\alpha\lambda(n+1)]}{(1-\alpha)}b_{i,n}\right)$$

$$\le\sum_{i=1}^{\infty}t_i=1,$$

and therefore $\sum_{i=1}^{\infty}t_if_i\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$. □

Theorem 1.9. *For $0\le\beta\le\alpha<1$, let $f(z)\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$ and $F(z)\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\beta)$. Then $f(z)*F(z)\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha))\subset\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\beta)$.*

Proof. Let

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n+\sum_{n=1}^{\infty}\overline{b}_n\overline{z}^n\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$$

and

$$F(z)=z-\sum_{n=2}^{\infty}A_nz^n+\sum_{n=1}^{\infty}\overline{B}_n\overline{z}^n\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\beta).$$

Then $f(z)*F(z)$ is

$$f(z)*F(z)=z-\sum_{n=2}^{\infty}a_nA_nz^n+\sum_{n=1}^{\infty}\overline{b}_n\overline{B}_n\overline{z}^n.$$

For $f(z)*F(z)\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\beta)$ we note that $|A_n|\le 1$ and $|B_n|\le 1$.
Now by Theorem 1.3 we have

$$\sum_{n=2}^{\infty}\frac{n^kC(n,m)[n-\beta-\beta\lambda(n-1)]}{1-\beta}|a_n|\ |A_n|$$

$$+\sum_{n=1}^{\infty}\frac{n^kC(n,m)[n+\beta-\beta\lambda(n+1)]}{1-\beta}|b_n|\ |B_n|$$

$$\le\sum_{n=2}^{\infty}\frac{n^kC(n,m)[n-\beta-\beta\lambda(n-1)]}{1-\beta}|a_n|+\sum_{n=1}^{\infty}\frac{n^kC(n,m)[n+\beta-\beta\lambda(n+1)]}{1-\beta}|b_n|$$

and since $0\le\beta\le\alpha<1$

$$\sum_{n=2}^{\infty}\frac{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]}{1-\alpha}|a_n|+\sum_{n=1}^{\infty}\frac{n^kC(n,m)[n+\alpha-\alpha\lambda(n+1)]}{1-\alpha}|b_n|\le 1,$$

by Theorem 1.3 $f(z)\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$. Therefore

$$f(z)*F(z)\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)\subset\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\beta).$$ □

Concluding remarks. We observe that, if we specialize the parameter $\lambda = 0$, for suitable choice of $k = 0$ and $m = 0$; $m = 0$ and $k = 0$ we obtain the analogous results for the classes studied in [2, 3] and [4] respectively.

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A CLASS OF MULTIVALENT ANALYTIC FUNCTIONS INVOLVING THE DZIOK-SRIVASTAVA OPERATOR

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Abstract. By making use of subordination between analytic functions and the Dziok-Srivastava operator, we introduce a new subclass of multivalent analytic functions. Such results as inclusion relationship, integral presentations and convolution properties for this function class are proved.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Let $f, \ g \in \mathcal{A}_p$, where f is given by (1.1) and g is defined by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}.$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p} =: (g * f)(z).$$

For parameters

$$\alpha_j \in \mathbb{C} \ \ (j = 1, \ldots, l) \quad \text{and} \quad \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \ \ (\mathbb{Z}_0^- := \{0, -1, -2, \ldots\}; \ j = 1, \ldots, m),$$

the generalized hypergeometric function

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$$

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is defined by the following infinite series:

$$ {}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{z^n}{n!} $$

$$ (l \leqq m+1;\ l,m \in \mathbb{N}_0 := \mathbb{N}\cup\{0\};\ z\in\mathbb{U}), $$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$ (\lambda)_n := \begin{cases} 1, & (n=0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & (n\in\mathbb{N}). \end{cases} $$

Recently, Dziok and Srivastava [1] introduced a linear operator

$$ H_p(\alpha_1,\ldots,\alpha_l;\beta,\ldots,\beta_m):\ \mathcal{A}_p \longrightarrow \mathcal{A}_p $$

defined by the following Hadamard product:

$$ H_p(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)f(z) := [z^p\ {}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)] * f(z) \tag{1.2} $$

$$ (l \leqq m+1;\ l,m\in\mathbb{N}_0;\ z\in\mathbb{U}). $$

If $f\in\mathcal{A}_p$ is given by (1.1), then we have

$$ H_p(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)f(z) = z^p + \sum_{n=1}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\ a_{n+p}\frac{z^{n+p}}{n!} \quad (n\in\mathbb{N};\ z\in\mathbb{U}). $$

In order to make the notation simple, we write

$$ H_p^{l,m}(\alpha_j) := H_p(\alpha_1,\ldots,\alpha_j,\ldots,\alpha_l;\ \beta_1,\ldots,\beta_m) $$

$$ (l\leqq m+1;\ l,m\in\mathbb{N}_0;\ j\in\{1,2,\ldots,l\}). $$

It is easily verified from the definition (1.2) that

$$ z\left(H_p^{l,m}(\alpha_j)f\right)'(z) = \alpha_j H_p^{l,m}(\alpha_j+1)f(z) - (\alpha_j-p)H_p^{l,m}(\alpha_j)f(z) \qquad (f\in\mathcal{A}_p). \tag{1.3} $$

Let $\mathcal{P}$ denote the class of functions of the form:

$$ p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, $$

which are analytic and convex in $\mathbb{U}$ and satisfy the condition:

$$ \Re(p(z)) > 0 \qquad (z\in\mathbb{U}). $$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$ f(z) \prec g(z) \quad \text{or} \quad f\prec g, $$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

Throughout this paper, we assume that

$$p, k \in \mathbb{N}, \quad l, m \in \mathbb{N}_0, \quad \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right),$$

and

$$f_{p,k}^{l,m}(\alpha_j; z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon_k^{-\nu p}\left(H_p^{l,m}(\alpha_j)f\right)(\varepsilon_k^{\nu} z) = z^p + \cdots \qquad (f \in \mathcal{A}_p). \tag{1.4}$$

Clearly, for $k = 1$, we have

$$f_{p,1}^{l,m}(\alpha_j; z) = H_p^{l,m}(\alpha_j)f(z).$$

In a recent paper, Patel *et al.* [9] discussed the following subclass of multivalent analytic functions defined by Dziok-Srivastava operator $H_p^{l,m}(\alpha_1)$.

Definition 1.1. (See [9]) A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_p^{l,m}(\alpha_1; \beta; A, B)$ if it satisfies the following subordination condition:

$$\frac{1}{p-\beta}\left(\frac{z\left(H_p^{l,m}(\alpha_1)f\right)'(z)}{H_p^{l,m}(\alpha_1)f(z)} - \beta\right) \prec \frac{1+Az}{1+Bz} \qquad (0 \leqq \beta < p;\ -1 \leqq B < A \leqq 1).$$

In 2007, Polatoğlu *et al.* [8] introduced and investigated the following subclass of the class $\mathcal{A}_p$ of p-valent analytic functions.

Definition 1.2. (See [8]) A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{M}_p(\alpha)$ if it satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha \qquad (\alpha > p).$$

Motivated by the function classes $\mathcal{S}_p^{l,m}(\alpha_1; \beta; A, B)$ and $\mathcal{M}_p(\alpha)$, by making use of the operator $H_p^{l,m}(\alpha_j)$ and the above-mentioned principle of subordination between analytic functions, we introduce and investigate the following subclass of the class $\mathcal{A}_p$ of p-valent analytic functions.

Definition 1.3. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{M}_{p,k}^{l,m}(\alpha_j; \alpha; \phi)$ if it satisfies the following subordination condition:

$$\frac{1}{\alpha-p}\left(\alpha - \frac{z\left(H_p^{l,m}(\alpha_j)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_j; z)}\right) \prec \phi(z) \quad \left(\alpha \geqq 0,\ \alpha \neq p;\ \phi \in \mathcal{P};\ f_{p,k}^{l,m}(\alpha_j; z) \neq 0\right). \tag{1.5}$$

Remark 1.4. It is easy to see that, if we set

$$k = j = 1,\ 0 \leqq \alpha < p \quad \text{and} \quad \phi(z) = \frac{1+Az}{1+Bz} \quad (-1 \leqq B < A \leqq 1)$$

in the class $\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$, then it reduces to the class $\mathcal{S}_p^{l,m}(\alpha_1;\alpha;A,B)$. Furthermore, if we choose

$$l = 2,\ m = \alpha_1 = \alpha_2 = \beta_1 = 1,\ \alpha > p \quad \text{and} \quad \phi(z) = \frac{1+z}{1-z}$$

in the class $\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$, then it reduces to the class $\mathcal{M}_p^{(k)}(\alpha)$. We observe that the class $\mathcal{M}_p^{(1)}(\alpha) =: \mathcal{M}_p(\alpha)$ was discussed by Polatoğlu *et al.* [8]. Moreover, the class $\mathcal{M}_1^{(k)}(\alpha)$ was considered recently by Wang *et al.* [12], the class $\mathcal{M}_1^{(1)}(\alpha)$ was studied earlier by Nishiwaki and Owa [4], Owa and Nishiwaki [5], Owa and Srivastava [6], Srivastava and Attiya [10], Uralegaddi and Desai [11].

In this paper, we aim at proving such results as inclusion relationship, integral presentations and convolution properties for the function class $\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$.

2. **Preliminary results**

In order to prove our main results, we need the following lemmas.

Lemma 2.1. (See [2, 3]) *Let $\beta,\gamma \in \mathbb{C}$. Suppose that φ is convex and univalent in $\mathbb{U}$ with*

$$\varphi(0) = 1 \quad \textit{and} \quad \Re(\beta\varphi(z)+\gamma) > 0.$$

If $\mathfrak{p}$ is analytic in $\mathbb{U}$ with $\mathfrak{p}(0) = 1$, then the following subordination:

$$\mathfrak{p}(z) + \frac{z\mathfrak{p}'(z)}{\beta\mathfrak{p}(z)+\gamma} \prec \varphi(z)$$

implies that

$$\mathfrak{p}(z) \prec \varphi(z).$$

Lemma 2.2. (See [7]) *Let $\beta,\gamma \in \mathbb{C}$. Suppose that φ is convex and univalent in $\mathbb{U}$ with*

$$\varphi(0) = 1 \quad \textit{and} \quad \Re(\beta\varphi(z)+\gamma) > 0.$$

Also let

$$\mathfrak{q}(z) \prec \varphi(z).$$

If $\mathfrak{p} \in \mathcal{P}$ and satisfies the following subordination:

$$\mathfrak{p}(z) + \frac{z\mathfrak{p}'(z)}{\beta\mathfrak{q}(z)+\gamma} \prec \varphi(z),$$

then

$$\mathfrak{p}(z) \prec \varphi(z).$$

Lemma 2.3. *Let* $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. *Then*

$$\frac{1}{\alpha-p}\left(\alpha-\frac{z\left(f_{p,k}^{l,m}(\alpha_j;z)\right)'}{f_{p,k}^{l,m}(\alpha_j;z)}\right)\prec\phi(z). \tag{2.1}$$

Proof. By virtue of (1.4), we replace z by $\varepsilon_k^{\nu}z$ ($\nu=0,1,2,\ldots,k-1$) in $f_{p,k}^{l,m}(\alpha_j;z)$. Then

$$\begin{aligned} f_{p,k}^{l,m}(\alpha_j;\varepsilon_k^{\nu}z) &= \frac{1}{k}\sum_{n=0}^{k-1}\varepsilon_k^{-np}\left(H_p^{l,m}(\alpha_j)f\right)(\varepsilon_k^{n+\nu}z)\\ &= \varepsilon_k^{\nu p}\cdot\frac{1}{k}\sum_{n=0}^{k-1}\varepsilon_k^{-(n+\nu)p}\left(H_p^{l,m}(\alpha_j)f\right)(\varepsilon_k^{n+\nu}z)\\ &= \varepsilon_k^{\nu p}f_{p,k}^{l,m}(\alpha_j;z). \end{aligned} \tag{2.2}$$

Differentiating both sides of (1.4) with respect to z, we have

$$\left(f_{p,k}^{l,m}(\alpha_j;z)\right)'=\frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon_k^{-\nu(p-1)}\left(H_p^{l,m}(\alpha_j)f\right)'(\varepsilon_k^{\nu}z). \tag{2.3}$$

Thus, combining (2.2) and (2.3), we easily find that

$$\begin{aligned} \frac{1}{\alpha-p}\left(\alpha-\frac{z\left(f_{p,k}^{l,m}(\alpha_j;z)\right)'}{f_{p,k}^{l,m}(\alpha_j;z)}\right) &= \frac{1}{\alpha-p}\left(\alpha-\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{\varepsilon_k^{-\nu(p-1)}z\left(H_p^{l,m}(\alpha_j)f\right)'(\varepsilon_k^{\nu}z)}{f_{p,k}^{l,m}(\alpha_j;z)}\right)\\ &= \frac{1}{\alpha-p}\left(\alpha-\frac{1}{k}\sum_{\nu=0}^{k-1}\frac{\varepsilon_k^{\nu}z\left(H_p^{l,m}(\alpha_j)f\right)'(\varepsilon_k^{\nu}z)}{f_{p,k}^{l,m}(\alpha_j;\varepsilon_k^{\nu}z)}\right). \end{aligned} \tag{2.4}$$

Moreover, since $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$, it follows that

$$\frac{1}{\alpha-p}\left(\alpha-\frac{\varepsilon_k^{\nu}z\left(H_p^{l,m}(\alpha_j)f\right)'(\varepsilon_k^{\nu}z)}{f_{p,k}^{l,m}(\alpha_j;\varepsilon_k^{\nu}z)}\right)\prec\phi(z) \qquad (\nu\in\{0,1,2,\ldots,k-1\}). \tag{2.5}$$

Finally, by noting that ϕ is convex and univalent in $\mathbb{U}$, from (2.4) and (2.5), we conclude that the assertion (2.1) of Lemma 2.3 holds. □

3. Properties of the function class $\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$

We begin by stating the following inclusion relationship for the function class $\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$.

Theorem 3.1. *Let* $\phi \in \mathcal{P}$ *with*

$$\Re(\alpha+\alpha_j-p+(p-\alpha)\phi(z))>0 \qquad (\alpha \geqq 0,\ \alpha \neq p).$$

Then

$$\mathcal{M}_{p,k}^{l,m}(\alpha_j+1;\alpha;\phi) \subset \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi).$$

Proof. Making use of the relationships (1.3) and (1.4), we know that

$$\begin{aligned} z\left(f_{p,k}^{l,m}(\alpha_j;z)\right)' + (\alpha_j-p)f_{p,k}^{l,m}(\alpha_j;z) &= \frac{\alpha_j}{k}\sum_{\nu=0}^{k-1}\varepsilon_k^{-\nu p}\left(H_p^{l,m}(\alpha_j+1)f\right)(\varepsilon_k^{\nu}z) \\ &= \alpha_j f_{p,k}^{l,m}(\alpha_j+1;z). \end{aligned} \tag{3.1}$$

Let $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j+1;\alpha;\phi)$. Suppose also that

$$h(z) = z\left(\frac{f_{p,k}^{l,m}(\alpha_j;z)}{z^p}\right)^{1/(\alpha-p)} \qquad (\alpha \geqq 0,\ \alpha \neq p). \tag{3.2}$$

Then h is analytic in $\mathbb{U}$. By taking logarithmic differentiation in (3.2), it follows that

$$q(z) = \frac{zh'(z)}{h(z)} = \frac{1}{\alpha-p}\left(\alpha - \frac{z\left(f_{p,k}^{l,m}(\alpha_j;z)\right)'}{f_{p,k}^{l,m}(\alpha_j;z)}\right) \tag{3.3}$$

is analytic in $\mathbb{U}$ with $q(0)=1$. We now find from (3.1) and (3.3) that

$$\alpha+\alpha_j-p+(p-\alpha)q(z) = \alpha_j\frac{f_{p,k}^{l,m}(\alpha_j+1;z)}{f_{p,k}^{l,m}(\alpha_j;z)}. \tag{3.4}$$

Differentiating both sides of (3.4) with respect to z logarithmically and using (3.3), we have

$$q(z)+\frac{zq'(z)}{\alpha+\alpha_j-p+(p-\alpha)q(z)} = \frac{1}{\alpha-p}\left(\alpha - \frac{z\left(f_{p,k}^{l,m}(\alpha_j+1;z)\right)'}{f_{p,k}^{l,m}(\alpha_j+1;z)}\right). \tag{3.5}$$

From (3.5) and Lemma 2.3 (with α_j replaced by α_j+1), we conclude that

$$q(z)+\frac{zq'(z)}{\alpha+\alpha_j-p+(p-\alpha)q(z)} \prec \phi(z). \tag{3.6}$$

Since

$$\Re(\alpha+\alpha_j-p+(p-\alpha)\phi(z))>0,$$

an application of Lemma 2.1 to (3.6) yields

$$q(z) = \frac{1}{\alpha - p}\left(\alpha - \frac{z\left(f_{p,k}^{l,m}(\alpha_j; z)\right)'}{f_{p,k}^{l,m}(\alpha_j; z)}\right) \prec \phi(z). \tag{3.7}$$

We now suppose that

$$q_1(z) = \frac{1}{\alpha - p}\left(\alpha - \frac{z\left(H_p^{l,m}(\alpha_j)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_j; z)}\right). \tag{3.8}$$

Then $q_1(z)$ is analytic in $\mathbb{U}$ with $q_1(0) = 1$. It follows from (1.3) and (3.8) that

$$[(p-\alpha)q_1(z) + \alpha]f_{p,k}^{l,m}(\alpha_j; z) = \alpha_j H_p^{l,m}(\alpha_j + 1)f(z) - (\alpha_j - p)H_p^{l,m}(\alpha_j)f(z). \tag{3.9}$$

Differentiating both sides of (3.9) with respect to z and using (3.8), we have

$$zq_1'(z) + \left(\alpha_j - p + \frac{z\left(f_{p,k}^{l,m}(\alpha_j; z)\right)'}{f_{p,k}^{l,m}(\alpha_j; z)}\right)\left(q_1(z) + \frac{\alpha}{p-\alpha}\right) = \frac{\alpha_j}{p-\alpha}\frac{z\left(H_p^{l,m}(\alpha_j+1)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_j; z)}. \tag{3.10}$$

We now easily find from (3.3), (3.4) and (3.10) that

$$q_1(z) + \frac{zq_1'(z)}{\alpha + \alpha_j - p + (p-\alpha)q(z)} = \frac{1}{\alpha - p}\left(\alpha - \frac{z\left(H_p^{l,m}(\alpha_j+1)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_j+1; z)}\right) \prec \phi(z). \tag{3.11}$$

Since

$$q(z) \prec \phi(z)$$

and

$$\Re(\alpha + \alpha_j - p + (p-\alpha)\phi(z)) > 0,$$

it follows from (3.11) and Lemma 2.2 that

$$q_1(z) = \frac{1}{\alpha - p}\left(\alpha - \frac{z\left(H_p^{l,m}(\alpha_j)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_j; z)}\right) \prec \phi(z),$$

that is, that $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j; \alpha; \phi)$. This implies that

$$\mathcal{M}_{p,k}^{l,m}(\alpha_j + 1; \alpha; \phi) \subset \mathcal{M}_{p,k}^{l,m}(\alpha_j; \alpha; \phi).$$

The proof of Theorem 3.1 is thus completed. □

Next, we derive several integral representations for the function class $\mathcal{M}_{p,k}^{l,m}(\alpha_j; \alpha; \phi)$.

Theorem 3.2. *Let* $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. *Then*

$$f_{p,k}^{l,m}(\alpha_j;z) = z^p \cdot \exp\left(\frac{(p-\alpha)}{k}\sum_{\nu=0}^{k-1}\int_0^z \frac{\phi\left(\omega(\varepsilon_k^\nu \xi)\right)-1}{\xi}d\xi\right), \tag{3.12}$$

where $f_{p,k}^{l,m}(\alpha_j;z)$ *is defined by* (1.4), ω *is analytic in* $\mathbb{U}$ *with*

$$\omega(0)=0 \quad \text{and} \quad |\omega(z)|<1 \quad (z\in\mathbb{U}).$$

Proof. Suppose that $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. We know that the subordination condition (1.5) can be written as follows:

$$\frac{z\left(H_p^{l,m}(\alpha_j)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_j;z)} = (p-\alpha)\phi\left(\omega(z)\right)+\alpha, \tag{3.13}$$

where ω is analytic in $\mathbb{U}$ with

$$\omega(0)=0 \quad \text{and} \quad |\omega(z)|<1 \quad (z\in\mathbb{U}).$$

Replacing z by $\varepsilon_k^\nu z$ $(\nu = 0,1,2,\ldots,k-1)$ in the equation (3.13), we observe that (3.13) also holds, that is,

$$\frac{\varepsilon_k^\nu z\left(H_p^{l,m}(\alpha_j)f\right)'(\varepsilon_k^\nu z)}{f_{p,k}^{l,m}(\alpha_j;\varepsilon_k^\nu z)} = (p-\alpha)\phi\left(\omega(\varepsilon_k^\nu z)\right)+\alpha. \tag{3.14}$$

We note that

$$f_{p,k}^{l,m}(\alpha_j;\varepsilon_k^\nu z) = \varepsilon_k^{\nu p} f_{p,k}^{l,m}(\alpha_j;z).$$

Thus, by letting $\nu = 0,1,2,\ldots,k-1$ in (3.14), successively, and summing the resulting equations, we get

$$\frac{z\left(f_{p,k}^{l,m}(\alpha_j;z)\right)'}{f_{p,k}^{l,m}(\alpha_j;z)} = \frac{(p-\alpha)}{k}\sum_{\nu=0}^{k-1}\phi\left(\omega(\varepsilon_k^\nu z)\right)+\alpha. \tag{3.15}$$

We next find from (3.15) that

$$\frac{\left(f_{p,k}^{l,m}(\alpha_j;z)\right)'}{f_{p,k}^{l,m}(\alpha_j;z)} - \frac{p}{z} = \frac{(p-\alpha)}{k}\sum_{\nu=0}^{k-1}\frac{\phi\left(\omega(\varepsilon_k^\nu z)\right)-1}{z}, \tag{3.16}$$

which, upon integration, yields

$$\log\left(\frac{f_{p,k}^{l,m}(\alpha_j;z)}{z^p}\right) = \frac{(p-\alpha)}{k}\sum_{\nu=0}^{k-1}\int_0^z \frac{\phi\left(\omega(\varepsilon_k^\nu \xi)\right)-1}{\xi}d\xi. \tag{3.17}$$

The assertion (3.12) of Theorem 3.2 can now easily be derived from (3.17). □

Theorem 3.3. *Let* $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. *Then*

$$H_p^{l,m}(\alpha_j)f(z) = \int_0^z \zeta^{p-1}[(p-\alpha)\phi(\omega(\zeta))+\alpha]\cdot\exp\left(\frac{(p-\alpha)}{k}\sum_{\nu=0}^{k-1}\int_0^\zeta \frac{\phi(\omega(\varepsilon_k^\nu \xi))-1}{\xi}d\xi\right)d\zeta, \tag{3.18}$$

where ω *is analytic in* $\mathbb{U}$ *with*

$$\omega(0)=0 \quad \textit{and} \quad |\omega(z)|<1 \quad (z\in\mathbb{U}).$$

Proof. Suppose that $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. Then, by virtue of (3.12) and (3.13), we get

$$\left(H_p^{l,m}(\alpha_j)f\right)'(z) = \frac{f_{p,k}^{l,m}(\alpha_j;z)}{z}\cdot[(p-\alpha)\phi(\omega(z))+\alpha]$$

$$= z^{p-1}[(p-\alpha)\phi(\omega(z))+\alpha]\cdot\exp\left(\frac{(p-\alpha)}{k}\sum_{\nu=0}^{k-1}\int_0^z \frac{\phi(\omega(\varepsilon_k^\nu \xi))-1}{\xi}d\xi\right), \tag{3.19}$$

which, upon integration of (3.19), leads us easily to the assertion (3.18) of Theorem 3.3. □

In view of Lemma 2.3 and Theorem 3.1, we get another integral representation for the function class $\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$.

Theorem 3.4. *Let* $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. *Then*

$$H_p^{l,m}(\alpha_j)f(z) = \int_0^z \zeta^{p-1}[(p-\alpha)\phi(\omega_2(\zeta))+\alpha]\cdot\exp\left((p-\alpha)\int_0^\zeta \frac{\phi(\omega_1(\xi))-1}{\xi}d\xi\right)d\zeta, \tag{3.20}$$

where ω_t $(t=1,2)$ *are analytic in* $\mathbb{U}$ *with*

$$\omega_t(0)=0 \quad \textit{and} \quad |\omega_t(z)|<1 \quad (z\in\mathbb{U};\ t=1,2).$$

Proof. Suppose that $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. We then find from (2.1) that

$$\frac{z\left(f_{p,k}^{l,m}(\alpha_j;z)\right)'}{f_{p,k}^{l,m}(\alpha_j;z)} = (p-\alpha)\phi(\omega_1(z))+\alpha, \tag{3.21}$$

where ω_1 is analytic in $\mathbb{U}$ and $\omega_1(0)=0$. Thus, by similarly applying the method of proof of Theorem 3.2, we find that

$$f_{p,k}^{l,m}(\alpha_j;z) = z^p\cdot\exp\left((p-\alpha)\int_0^z \frac{\phi(\omega_1(\xi))-1}{\xi}d\xi\right). \tag{3.22}$$

It now follows from (3.13) and (3.22) that

$$\begin{aligned}\left(H_p^{l,m}(\alpha_j)f\right)'(z) &= \frac{f_{p,k}^{l,m}(\alpha_j;z)}{z}\cdot[(p-\alpha)\phi\big(\omega_2(z)\big)+\alpha]\\ &= z^{p-1}[(p-\alpha)\phi\big(\omega_2(z)\big)+\alpha]\cdot\exp\left((p-\alpha)\int_0^z\frac{\phi\big(\omega_1(\xi)\big)-1}{\xi}d\xi\right),\end{aligned} \tag{3.23}$$

where ω_t $(t=1,2)$ are analytic in $\mathbb{U}$ with

$$\omega_t(0)=0 \quad \text{and} \quad |\omega_t(z)|<1 \quad (z\in\mathbb{U};\ t=1,2).$$

Upon integrating both sides of (3.23), we readily arrive at the assertion (3.20) of Theorem 3.4. □

In the following we give some convolution properties for the function class $\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$.

Theorem 3.5. *Let $f\in\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. Then*

$$\begin{aligned}f(z) = &\left[\int_0^z \zeta^{p-1}[(p-\alpha)\phi\left(\omega(\zeta)\right)+\alpha]\cdot\exp\left(\frac{(p-\alpha)}{k}\sum_{\nu=0}^{k-1}\int_0^\zeta\frac{\phi\left(\omega(\varepsilon_k^\nu\xi)\right)-1}{\xi}d\xi\right)d\zeta\right]\\ &* \left(\sum_{n=0}^{\infty}\frac{n!(\beta_1)_n\cdots(\beta_m)_n}{(\alpha_1)_n\cdots(\alpha_j)_n\cdots(\alpha_l)_n}z^{n+p}\right),\end{aligned} \tag{3.24}$$

where ω is analytic in $\mathbb{U}$ with

$$\omega(0)=0 \quad \textit{and} \quad |\omega(z)|<1 \quad (z\in\mathbb{U}).$$

Proof. In view of (1.2) and (3.18), we find that

$$\begin{aligned}&\int_0^z \zeta^{p-1}[(p-\alpha)\phi\left(\omega(\zeta)\right)+\alpha]\cdot\exp\left(\frac{(p-\alpha)}{k}\sum_{\nu=0}^{k-1}\int_0^\zeta\frac{\phi\left(\omega(\varepsilon_k^\nu\xi)\right)-1}{\xi}d\xi\right)d\zeta\\ &\qquad = [z^p\ {}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)]*f(z).\end{aligned} \tag{3.25}$$

Thus, from (3.25), we easily get the assertion (3.24) of Theorem 3.5. □

Theorem 3.6. *Let $f\in\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. Then*

$$\begin{aligned}f(z) = &\left[\int_0^z \zeta^{p-1}[(p-\alpha)\phi\big(\omega_2(\zeta)\big)+\alpha]\cdot\exp\left((p-\alpha)\int_0^\zeta\frac{\phi\big(\omega_1(\xi)\big)-1}{\xi}d\xi\right)d\zeta\right]\\ &* \left(\sum_{n=0}^{\infty}\frac{n!(\beta_1)_n\cdots(\beta_m)_n}{(\alpha_1)_n\cdots(\alpha_j)_n\cdots(\alpha_l)_n}z^{n+p}\right),\end{aligned} \tag{3.26}$$

where ω_t $(t = 1, 2)$ *are analytic in* $\mathbb{U}$ *with*

$$\omega_t(0) = 0 \quad \textit{and} \quad |\omega_t(z)| < 1 \quad (z \in \mathbb{U};\ t = 1, 2).$$

Proof. By virtue of (1.2) and (3.20), we know that

$$\begin{aligned}\int_0^z \zeta^{p-1}[(p-\alpha)\phi\big(\omega_2(\zeta)\big) + \alpha] \cdot \exp\left((p-\alpha)\int_0^\zeta \frac{\phi\big(\omega_1(\xi)\big) - 1}{\xi} d\xi\right) d\zeta \\ = [z^p\ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)] * f(z).\end{aligned} \tag{3.27}$$

Thus, from (3.27), we easily arrive at the convolution property (3.26) asserted by Theorem 3.6. □

Theorem 3.7. *Let*

$$f \in \mathcal{A}_p \quad \textit{and} \quad \phi \in \mathcal{P}.$$

Then $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j; \alpha; \phi)$ *if and only if*

$$\begin{aligned}\frac{1}{z}\Bigg\{ f * \Bigg[\left(pz^p + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_j)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{n+p}{n!} z^{n+p} \right) \\ -[(p-\alpha)\phi(e^{i\theta}) + \alpha] \left(z^p + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_j)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{1}{n!} z^{n+p} \right) * \left(\frac{1}{k} \sum_{\upsilon=0}^{k-1} \frac{z^p}{1 - \varepsilon^\upsilon z} \right) \Bigg] \Bigg\} \neq 0\end{aligned} \tag{3.28}$$

$(z \in \mathbb{U};\ 0 \leqq \theta < 2\pi)$.

Proof. Suppose that $f \in \mathcal{M}_{p,k}^{l,m}(\alpha_j; \alpha; \phi)$. Since

$$\frac{z\left(H_p^{l,m}(\alpha_j)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_j; z)} \prec (p-\alpha)\phi(z) + \alpha$$

is equivalent to

$$\frac{z\left(H_p^{l,m}(\alpha_j)f\right)'(z)}{f_{p,k}^{l,m}(\alpha_j; z)} \neq (p-\alpha)\phi(e^{i\theta}) + \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \theta < 2\pi), \tag{3.29}$$

it is easy to see that the condition (3.29) can be written as follows:

$$\frac{1}{z}\left\{ z\left(H_p^{l,m}(\alpha_j)f\right)'(z) - f_{p,k}^{l,m}(\alpha_j; z)[(p-\alpha)\phi(e^{i\theta}) + \alpha] \right\} \neq 0 \quad (z \in \mathbb{U};\ 0 \leqq \theta < 2\pi). \tag{3.30}$$

On the other hand, we know from (1.2) that

$$z\left(H_p^{l,m}(\alpha_j)f\right)'(z) = \left(pz^p + \sum_{n=1}^{\infty} \frac{(\alpha_j)_n \cdots (\alpha_j)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{n+p}{n!} z^{n+p} \right) * f(z). \tag{3.31}$$

Moreover, from the definition of $f_{p,k}^{l,m}(\alpha_j;z)$, we have

$$\begin{aligned} f_{p,k}^{l,m}(\alpha_j;z) &= H_p^{l,m}(\alpha_j)f(z) * \left(\frac{1}{k}\sum_{\upsilon=0}^{k-1}\frac{z^p}{1-\varepsilon^\upsilon z}\right) \\ &= \left(z^p + \sum_{n=1}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_j)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{1}{n!}z^{n+p}\right) * \left(\frac{1}{k}\sum_{\upsilon=0}^{k-1}\frac{z^p}{1-\varepsilon^\upsilon z}\right) * f(z). \end{aligned} \tag{3.32}$$

Upon substituting (3.31) and (3.32) into (3.30), we easily deduce the convolution property (3.28) asserted by Theorem 3.7. □

Remark 3.8. By specializing the parameters in Theorems 3.1-3.7, we can get several interesting properties for some special function classes associated with the class $\mathcal{M}_{p,k}^{l,m}(\alpha_j;\alpha;\phi)$. Here, we choose to omit the details involved.

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