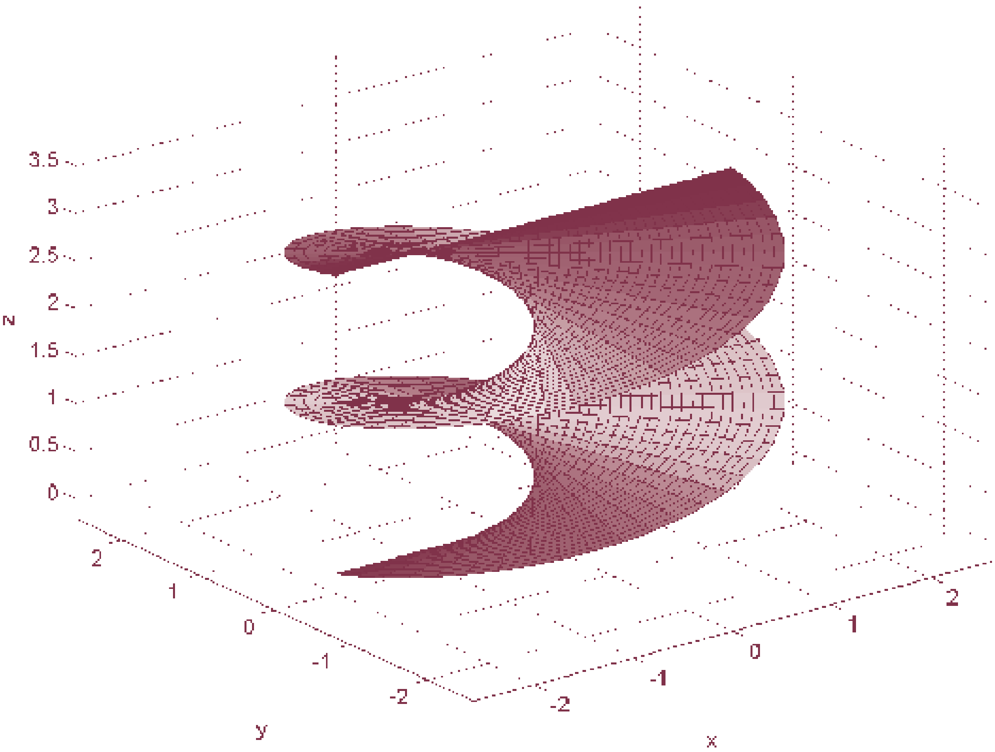
STUDIA UNIVERSITATIS
BABEŞ-BOLYAI



MATHEMATICA

3/2009

Anul LIV **2009**

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
3

Redacţia: 400084 Cluj-Napoca, str. M. Kogălniceanu nr. 1
Telefon: 405300

SUMAR - CONTENTS - SOMMAIRE

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 3, September 2009

FIXED POINT THEORY FOR MULTIVALUED GENERALIZED CONTRACTION ON A SET WITH TWO b-METRICS

MONICA BORICEANU

Abstract. The purpose of this paper is to present some fixed point results for multivalued generalized contraction on a set with two b-metrics. The data dependence and the well-posedness of the fixed point problem are also discussed.

1. **Introduction**

The concept of b-metric space was introduced by Czerwik in [2]. Since then several papers deal with fixed point theory for singelvalued and multivalued operators in b-metric spaces (see [1], [2], [7]). In the first part of the paper we will present a fixed point theorem for Ćirić-type multivalued operator on b-metric space endowed with two b-metrics. Then, a strict fixed point result for multivalued generalized contraction in b-metric spaces is proved. The last part contains several conditions under which the fixed point problem for a multivalued operator in a b-metric space is well-posed and a data dependence result is given.

2. **Preliminaries and auxiliary results**

The aim of this section is to present some notions and symbols used in the paper.

We will first give the definition of a b-metric space.

Received by the editors: 01.10.2008.

2000 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. Multivalued generalized contraction, fixed point, strict fixed point, b-metric space, data dependence, well-posedness.

This paper was presented at the 7-th Joint Conference on Mathematics and Computer Science, July 3-6, 2008, Cluj-Napoca, Romania.

Definition 2.1 (Czerwik [2]) Let X be a set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A pair (X, d) is called a b-metric space.

We give next some examples of b-metric spaces.

Example 2.2 (Berinde see [1])

The space $l_p(0 < p < 1)$,

$$l_p = \{(x_n) \subset \mathbb{R} | \sum_{n=1}^{\infty} |x_n|^p < \infty\},$$

together with the function $d : l_p \times l_p \to \mathbb{R}$,

$$d(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{1/p},$$

where $x = (x_n), y = (y_n) \in l_p$ is a b-metric space.

By an elementary calculation we obtain: $d(x, z) \leq 2^{1/p}[d(x, y) + d(y, z)]$.

Hence $a = 2^{1/p} > 1$.

Example 2.3 (Berinde see[1])

The space $L_p(0 < p < 1)$ of all real functions $x(t)$, $t \in [0, 1]$ such that:

$$\int_0^1 |x(t)|^p dt, \infty,$$

is a b-metric space if we take

$$d(x, y) = (\int_0^1 |x(t) - y(t)|^p dt)^{1/p}, \text{ for each } x, y \in L_p,$$

The constant a is as in the previous example $2^{1/p}$.

We continue by presenting the notions of convergence, compactness, closedness and completeness in a b-metric space.

Definition 2.4 Let (X, d) be a b-metric space. Then a sequence $(x_n)_{n \in \mathbb{N}}$ in X is called:

(a) Cauchy if and only if for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for each $n, m \geq n(\varepsilon)$ we have $d(x_n, x_m) < \varepsilon$.

(b) convergent if and only if there exists $x \in X$ such that for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$ we have $d(x_n, x) < \varepsilon$. In this case we write $\lim\limits_{n\to\infty} x_n = x$.

Remark 2.5

1. The sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy if and only if $\lim\limits_{n\to\infty} d(x_n, x_{n+p}) = 0$, for all $p \in \mathbb{N}^*$.
2. The sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to $x \in X$ if and only if $\lim\limits_{n\to\infty} d(x_n, x) = 0$.

Definition 2.6

1. Let (X, d) be a b-metric space. Then a subset $Y \subset X$ is called

   (i) compact if and only if for every sequence of elements of Y there exists a subsequence that converges to an element of Y.

   (ii) closed if and only if for each sequence $(x_n)_{n\in\mathbb{N}}$ in Y which converges to an element x, we have $x \in Y$.
2. The b-metric space is complete if every Cauchy sequence converges.

We consider next the following families of subsets of a b-metric space (X, d):

$$P(X) := \{Y \in \mathcal{P}(X) |\ Y \neq \emptyset\};$$

$$P_b(X) := \{Y \in P(X) |\ diam(Y) < \infty\},$$

where

$$diam : P(X) \to \mathbb{R}_+ \cup \{\infty\}, diam(Y) = sup\{d(a, b), a, b \in Y\}$$

is the generalized diameter functional;

$$P_{cp}(X) := \{Y \in P(X) |\ Y \text{ is compact}\};$$

$$P_{cl}(X) := \{Y \in P(X) |\ Y \text{ is closed}\};$$

$$P_{b,cl}(X) := P_b(X) \cap P_{cl}(X)$$

We will introduce the following generalized functionals on a b-metric space (X,d). Some of them were defined in [2].

1. $D : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$D(A,B) = \inf\{d(a,b) | a \in A, b \in B\},$$

for any $A, B \subset X$.

D is called the gap functional between A and B. In particular, if $x_0 \in X$ then $D(x_0, B) := D(\{x_0\}, B)$.

2. $\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\delta(A,B) = \sup\{d(a,b) |\ a \in A,\ b \in B\}.$$

3. $\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\rho(A,B) = \sup\{D(a,B) | a \in A\},$$

for any $A, B \subset X$.

ρ is called the (generalized) excess functional.

4. $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$H(A,B) = \max\left\{\sup_{x\in A} D(x,B), \sup_{y\in B} D(A,y)\right\},$$

for any $A, B \subset X$.

H is the (generalized) Pompeiu-Hausdorff functional.

Let (X,d) be a b-metric space. If $F : X \to P(X)$ is a multivalued operator, we denote by $FixF$ the fixed point set of F, i.e. $Fix(F) := \{x \in X | x \in F(x)\}$ and by $SFixF$ the strict fixed point set of F, i.e. $SFixF := \{x \in X | \{x\} = F(x)\}$.

Lemma 2.7 [4] *Let (X,d) be a b-metric space and let $A, B \in P(X)$. We suppose that there exists $\eta \in \mathbb{R}, \eta > 0$ such that:*

(i) for each $a \in A$ there is $b \in B$ such that $d(a,b) \le \eta$;

(ii) for each $b \in B$ there is $a \in A$ such that $d(a,b) \le \eta$.

Then

$$H(A,B) \le \eta.$$

Lemma 2.8 [4] *Let (X,d) be a b-metric space and let $A \in P(X)$ and $x \in X$. Then $D(x,A) = 0$ if and only if $x \in \bar{A}$.*

The following results are useful for some of the proofs in the paper.

Lemma 2.9 (Czerwik [2]) *Let (X,d) be a b-metric space. Then*

$$D(x,A) \leq s[d(x,y) + D(y,A)], \text{ for all } x,y \in X, A \subset X.$$

Lemma 2.10 (Czerwik [2]) *Let (X,d) be a b-metric space and let $\{x_k\}_{k=0}^n \subset X$. Then:*

$$d(x_n,x_0) \leq sd(x_0,x_1) + ... + s^{n-1}d(x_{n-2},x_{n-1}) + s^{n-1}d(x_{n-1},x_n).$$

Lemma 2.11 (Czerwik [2]) *Let (X,d) be a b-metric space and for all $A,B,C \in X$ we have:*

$$H(A,C) \leq s[H(A,B) + H(B,C)].$$

Lemma 2.12 (Czerwik [2])

(1) Let (X,d) be a b-metric space and $A,B \in P_{cl}(X)$. Then for each $\alpha > 0$ and for all $b \in B$ there exists $a \in A$ such that:

$$d(a,b) \leq H(A,B) + \alpha;$$

(2) Let (X,d) be a b-metric space and $A,B \in P_{cp}(X)$. Then for all $b \in B$ there exists $a \in A$ such that:

$$d(a,b) \leq sH(A,B).$$

3. **Main results**

The fist main result of this paper is a fixed point theorem.

Theorem 3.1 *Let X be a nonempty set, d and ρ two b-metrics on X with constants $t > 1$ and respectively $s > 1$ and let $F : X \to P(X)$ a multivalued operator. We suppose that:*

(i) *(X,d) is a complete b-metric space;*

(ii) *There exists $c > 0$ such that $d(x,y) \leq c \cdot \rho(x,y)$, for all $x,y \in X$;*

(iii) *$F : (X,d) \to (P(X), H_d)$ is closed;*

(iv) *There exists* $0 \le \alpha < \frac{1}{s}$ *such that*

$$H_\rho(F(x), F(y)) \le \alpha M_\rho^F(x, y),$$

for all $x, y \in X$, *where*

$$M_\rho^F(x,y) = max\left\{\rho(x,y), D_\rho(x, F(x)), D_\rho(y, F(y)), \frac{1}{2}\left[D_\rho(x, F(y)) + D_\rho(y, F(x))\right]\right\}.$$

Then we have:

1. $FixF \neq \emptyset$;
2. *For all* $x \in X$ *and each* $y \in F(x)$ *there exists* $(x_n)_{n\in\mathbb{N}}$ *such that:*
   (a) $x_0 = x, x_1 = y$;
   (b) $x_{n+1} \in F(x_n)$;
   (c) $d(x_n, x^*) \to 0$, *as* $n \to \infty$ *where* $x^* \in F(x^*)$;

Proof. Let $1 < q < \frac{1}{s\alpha}$ be arbitrary. For arbitrary $x_0 \in X$ and for $x_1 \in F(x_0)$ there exists $x_2 \in F(x_1)$ such that:

$$\rho(x_1, x_2) \le qH_\rho(F(x_0), F(x_1)) \le q\alpha M_\rho^F(x_0, x_1).$$

So we have

$$\rho(x_1,x_2)\le q\alpha\cdot max\left\{\rho(x_0,x_1), D_\rho(x_0,F(x_0)), D(x_1,F(x_1)), \frac{1}{2}\left[D_\rho(x_0,F(x_1)) + D_\rho(x_1,F(x_0))\right]\right\}.$$

Suppose that the $max = \rho(x_0, x_1)$. Then we have

$$\rho(x_1, x_2) \le q\alpha\rho(x_0, x_1).$$

Suppose that the $max = D_\rho(x_0, F(x_0))$. Then we have

$$\rho(x_1, x_2) \le q\alpha D_\rho(x_0, F(x_0)) \le q\alpha\rho(x_0, x_1).$$

Suppose that the $max = D_\rho(x_1, F(x_1))$. Then we have

$$\rho(x_1, x_2) \le q\alpha D_\rho(x_1, F(x_1)) \le q\alpha\rho(x_1, x_2).$$

So $\rho(x_1, x_2) = 0$ and thus $x_1 \in FixF$.

Suppose that the $max = \frac{1}{2}[D_\rho(x_0, F(x_1)) + D_\rho(x_1, F(x_0))]$. Then we have

$$\rho(x_1, x_2) \leq q\alpha\frac{1}{2}D_\rho(x_0, F(x_1)) \leq q\alpha\frac{1}{2}\rho(x_0, x_2)) \leq \frac{q\alpha}{2}s[\rho(x_0, x_1) + \rho(x_1, x_2)].$$

So we have $\rho(x_1, x_2) \leq \frac{q\alpha s}{2-q\alpha s}\rho(x_0, x_1)$.

For $x_2 \in F(x_1)$ there exists $x_3 \in F(x_2)$ such that:

$$\rho(x_2, x_3) \leq qH_\rho(F(x_1), F(x_2)) \leq q\alpha M_\rho^F \rho(x_1, x_2)$$

Suppose that the $max = \rho(x_1, x_2)$. Then we have

$$\rho(x_2, x_3) \leq q\alpha\rho(x_1, x_2) \leq (q\alpha)^2\rho(x_0, x_1).$$

Suppose that the $max = D_\rho(x_1, F(x_1))$. Then we have

$$\rho(x_2, x_3) \leq q\alpha D_\rho(x_1, F(x_1)) \leq q\alpha\rho(x_1, x_2) \leq (q\alpha)^2\rho(x_0, x_1).$$

Suppose that the $max = D_\rho(x_2, F(x_2))$. Then we have

$$\rho(x_2, x_3) \leq q\alpha D_\rho(x_2, F(x_2)) \leq q\alpha\rho(x_2, x_3).$$

So $\rho(x_2, x_3) = 0$ and thus $x_2 \in FixF$.

Suppose that the $max = \frac{1}{2}[D_\rho(x_1, F(x_2)) + D_\rho(x_2, F(x_1))]$. Then we have

$$\rho(x_1, x_2) \leq q\alpha\frac{1}{2}D_\rho(x_1, F(x_2)) \leq q\alpha\frac{1}{2}\rho(x_1, x_3)) \leq \frac{q\alpha}{2}s[\rho(x_1, x_2) + \rho(x_2, x_3)].$$

So we have

$$\rho(x_2, x_3) \leq \frac{q\alpha s}{2 - q\alpha s}\rho(x_1, x_2) \leq [\frac{q\alpha s}{2 - q\alpha s}]^2\rho(x_0, x_1).$$

We can construct by induction a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$\rho(x_n, x_{n+1}) \leq max\{(q\alpha)^n, [\frac{q\alpha s}{2 - q\alpha s}]^n\}\rho(x_0, x_1), \text{ for all } n \in \mathbb{N}.$$

We will prove next that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy, by estimating $\rho(x_n, x_{n+p})$.

We consider first that the maximum is $(q\alpha)^n$. So we have:

$$\begin{aligned}
\rho(x_n, x_{n+p}) &\leq s\rho(x_n, x_{n+1}) + s^2\rho(x_{n+1}, x_{n+2}) + ...+ \\
&+ s^{p-1}\rho(x_{n+p-2}, x_{n+p-1}) + s^{p-1}\rho(x_{n+p-1}, x_{n+p}) \\
&\leq s(q\alpha)^n\rho(x_0, x_1) + s^2(q\alpha)^{n+1}\rho(x_0, x_1) + ...+ \\
&+ s^{p-1}(q\alpha)^{n+p-2}\rho(x_0, x_1) + s^{p-1}(q\alpha)^{n+p-1}\rho(x_0, x_1) \\
&= s(q\alpha)^n\rho(x_0, x_1)[1 + sq\alpha + ... + (sq\alpha)^{p-2} + s^{p-2}(q\alpha)^{p-1}] \\
&\leq s(q\alpha)^n\rho(x_0, x_1)[1 + sq\alpha + ... + (sq\alpha)^{p-2} + s^{p-1}(q\alpha)^{p-1}] \\
&= s(q\alpha)^n\rho(x_0, x_1)\frac{1 - (sq\alpha)^p}{1 - sq\alpha}.
\end{aligned}$$

But $1 < q < \frac{1}{s\alpha}$. Hence we obtain that:

$$\rho(x_n, x_{n+p}) \leq s(q\alpha)^n\rho(x_0, x_1)\frac{1 - (sq\alpha)^p}{1 - sq\alpha} \to 0,$$

as $n \to \infty$. So $(x_n)_{n\in\mathbb{N}}$ is Cauchy and $x_n \to x \in X$.

We consider now the maximum $A := [\frac{q\alpha s}{2-q\alpha s}]^n$. So we have:

$$\begin{aligned}
\rho(x_n, x_{n+p}) &\leq s\rho(x_n, x_{n+1}) + s^2\rho(x_{n+1}, x_{n+2}) + ...+ \\
&+ s^{p-1}\rho(x_{n+p-2}, x_{n+p-1}) + s^{p-1}\rho(x_{n+p-1}, x_{n+p}) \\
&\leq sA^n\rho(x_0, x_1) + s^2A^{n+1}\rho(x_0, x_1) + ...+ \\
&+ s^{p-1}A^{n+p-2}\rho(x_0, x_1) + s^{p-1}A^{n+p-1}\rho(x_0, x_1) \\
&= sA^n\rho(x_0, x_1)[1 + sA + ... + (sA)^{p-2} + s^{p-2}A^{p-1}] \\
&\leq sA^n\rho(x_0, x_1)[1 + sA + ... + (sA)^{p-2} + s^{p-1}A^{p-1}] \\
&= sA^n\rho(x_0, x_1)\frac{1 - (sA)^p}{1 - sA}.
\end{aligned}$$

But $1 < q < \frac{1}{s\alpha}$ and we obtain that:

$$\rho(x_n, x_{n+p}) \leq sA^n\rho(x_0, x_1)\frac{1 - (sA)^p}{1 - sA} \to 0,$$

as $n \to \infty$. So $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, ρ).

From (ii) it follows that the sequence is Cauchy in (X, d). Denote by $x^* \in X$ the limit of the sequence. From (i) and (iii) we get that $d(x_n, x^*) \to 0$, as $n \to \infty$ where $x^* \in F(x^*)$. The proof is complete. $\square$

For the next results let us denote

$$N_\rho^F(x,y) = max\left\{\rho(x,y), D_\rho(y,F(y)), \frac{1}{2}\left[D_\rho(x,F(y)) + D_\rho(y,F(x))\right]\right\}.$$

The second main result of this paper is:

Theorem 3.2 *Let X be a nonempty set, d and ρ two b-metrics on X with constants $t > 1$ and respectively $s > 1$ and let $F : X \to P(X)$ a multivalued operator. We suppose that:*

(i) *(X, d) is a complete b-metric space;*

(ii) *There exists $c > 0$ such that $d(x, y) \le c \cdot \rho(x, y)$, for all $x, y \in X$;*

(iii) *$F : (X, d) \to (P(X), H_d)$ is closed;*

(iv) *There exists $0 \le \alpha < \frac{1}{s}$ such that*

$$H_\rho(F(x), F(y)) \le \alpha N_\rho^F(x, y),$$

for all $x, y \in X$;

(v) *$SFixF \neq \emptyset$.*

Then we have:

1. *$FixF = SFixF = \{x^*\}$;*
2. *$H_\rho(F^n(x), x^*) \le \alpha^n \rho(x, x^*)$, for all $n \in \mathbb{N}$ and for each $x \in X$;*
3. *$\rho(x, x^*) \le \frac{s}{1-s\alpha} H_\rho(x, F(x))$, for all $x \in X$;*
4. *The fixed point problem is well-posed for F with respect to D_ρ and with respect to H_ρ, too.*

Proof. 1. We suppose that $x^* \in SFixF$. Let $y \in SFixF$. Then we have

$$\begin{aligned}
\rho(x^*, y) &= H_\rho(F(x^*), F(y)) \\
&\le \alpha \cdot max\{\rho(x^*, y), D_\rho(y, F(y)), \frac{1}{2}[D_\rho(x^*, F(y)) + D_\rho(y, F(x^*))]\} \\
&\le \alpha \cdot max\{\rho(x^*, y), \frac{1}{2}[\rho(x^*, y) + \rho(y, x^*)]\} = \alpha\rho(x^*, y),
\end{aligned}$$

for all $x \in X$. So we have that $\rho(x^*, y) = 0$ and in conclusion $x^* = y$.

2. We take in the condition (iv) $y = x^*$. Then we have:

$$H_\rho(F(x), F(x^*)) \leq \alpha \cdot max\{\rho(x, x^*), D_\rho(x^*, F(x^*)), \frac{1}{2}[D_\rho(x, F(x^*)) + D_\rho(x^*, F(x))]\}$$
$$= \alpha \cdot max\{\rho(x, x^*), \frac{1}{2}[D_\rho(x, F(x^*)) + D_\rho(x^*, F(x))]\}.$$

If the maximum is $\rho(x, x^*)$ we have that $H_\rho(F(x), x^*) \leq \alpha\rho(x, x^*)$.

If the maximum is $\frac{1}{2}[D_\rho(x, F(x^*)) + D_\rho(x^*, F(x))]$ we have that

$$\begin{aligned} H_\rho(F(x), x^*) &\leq \frac{\alpha}{2}[D_\rho(x, F(x^*)) + D_\rho(x^*, F(x))] \\ &= \frac{\alpha}{2}[D_\rho(x, F(x^*)) + H_\rho(F(x^*), F(x))] \\ &\leq \frac{\alpha}{2}[\rho(x, F(x^*)) + H_\rho(F(x^*), F(x))]. \end{aligned}$$

So we obtain $H_\rho(F(x^*), F(x)) \leq \frac{\alpha}{2-\alpha}\rho(x, x^*)$.

We take now $max\{\alpha, \frac{\alpha}{2-\alpha}\} = \alpha$ and obtain $H_\rho(F(x^*), F(x)) \leq \alpha\rho(x, x^*)$, for all $x \in X$.

By induction we obtain

$$H_\rho(F^n(x), x^*) \leq \alpha^n \rho(x, x^*), \text{ for all } x \in X.$$

Consider now $y^* \in FixF$. Then $\rho(y^*, x^*) \leq H_\rho(F(y^*), x^*) \leq \alpha^n\rho(y^*, x^*) \to 0$, as $n \to \infty$. Hence $y^* = x^*$.

3. $\rho(x, x^*) \leq s[H_\rho(x, F(x)) + H_\rho(F(x), x^*)] \leq sH_\rho(x, F(x)) + s\alpha\rho(x, x^*)$.

So we obtain

$$\rho(x, x^*) \leq \frac{s}{1 - s\alpha} H_\rho(x, F(x)).$$

4. Let (x_n) be such that $D_\rho(x_n, F(x_n)) \to 0$, as $n \to \infty$. We will prove that $\rho(x_n, x^*) \to 0$, as $n \to \infty$.

Estimating $\rho(x_n, x^*)$ we have

$$\rho(x_n, x^*) \leq s[\rho(x_n, y_n) + D_\rho(y_n, F(x^*))] \leq s[\rho(x_n, y_n) + H_\rho(F(x_n), F(x^*))],$$

for all $y_n \in F(x_n)$ and for each $n \in \mathbb{N}$.

Taking $\inf\limits_{y_n \in F(x_n)}$ we obtain

$$\rho(x_n, x^*) \leq s[D(x_n, F(x_n)) + H(F(x_n), F(x^*))] \leq sD(x_n, F(x_n)) + s\alpha\rho(x_n, x^*).$$

Hence we have $\rho(x_n, x^*) \leq \frac{s}{1-s\alpha} D(x_n, F(x_n)) \to$ as $n \to \infty$. So $x_n \to x^*$. $\square$

We will next give a data dependence result.

Theorem 3.3 *Let X be a nonempty set, d and ρ two b-metrics on X with constants $t > 1$ and respectively $s > 1$ and let $F, T : X \to P(X)$ two multivalued operators. We suppose that:*

(i) *(X, d) is a complete b-metric space;*

(ii) *There exists $c > 0$ such that $d(x, y) \leq c \cdot \rho(x, y)$, for all $x, y \in X$;*

(iii) *$F : (X, d) \to (P(X), H_d)$ is closed;*

(iv) *There exists $0 \leq \alpha < \frac{1}{s}$ such that*

$$H_\rho(F(x), F(y)) \leq \alpha N_\rho^T(x, y),$$

for all $x, y \in X$;

(v) *$SFixF \neq \emptyset$;*

(vi) *$FixT \neq \emptyset$;*

(vii) *There exists $\eta > 0$ such that $H_\rho(F(x), T(x)) \leq \eta$, for all $x \in X$.*

Then

$$H_\rho(FixF, FixT) \leq \frac{s\eta}{1 - s\alpha}.$$

Proof. Let $x^* \in SFixF$ and $y^* \in FixT$. We have that

$$\rho(y^*, x^*) \leq H_\rho(T(y^*), x^*) \leq s[H_\rho(T(y^*), F(y^*)) + H_\rho(F(y^*), x^*)]$$
$$\leq s[\eta + H_\rho(F(y^*), F(x^*))] \leq s[\eta + \alpha\rho(y^*, x^*)].$$

Hence we have $\rho(y^*, x^*) \leq \frac{s\eta}{1-s\alpha}$. $\square$

References

[1] Berinde, V., *Generalized contractions in quasimetric spaces*, Seminar on fixed point theory, Preprint no. **3**(1993), 3-9.

[2] Czerwik, S., *Nonlinear set-valued contraction mappings in B-metric spaces*, Atti Sem. Mat. Univ. Modena, **46**(1998), 263-276.

[3] Petruşel, A., Rus, I.A., *Fixed point theoy for multivalued operators on a set with two metrics*, Fixed Point Theory, **8**(2007), 97-104.

[4] Petruşel, A., Mot, G., Petruşel, G., *Topics in Nonlinear Analysis and Applications to Mathematical Economics*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2007.

[5] Rus, I.A., Petruşel, A., Sîntămărian, A., *Data dependence of the fixed points set of multivalued weakly Picard operators*, Studia. Univ. "Babeş-Bolyai", Mathematica, **46**(2001), 111-121.

[6] Rus, I.A., *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[7] Singh, S.L., Bhatnagar, C., Mishra, S.N., *Stability of iterative procedures for multivalued maps in metric spaces*, Demonstratio Math., **37**(2005), 905-916.

Department of Applied Mathematics,
Babeş-Bolyai University,
Cluj-Napoca, Romania
E-mail address: bmonica@math.ubbcluj.ro

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 3, September 2009

FIRST ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT MODIFIED ARGUMENT

EDITH EGRI AND IOAN A. RUS

Abstract. The aim of our paper is to investigate the Cauchy problem constituting from the first order functional differential equation with state-dependent modified argument of the following form

$$x'(t) = f(t, x(t), x(g(t, x(t)))), \quad t \in [a, b],$$

where $x \in C([a-h, b], [a-h, b]) \cap C^1([a, b], [a-h, b])$, $h > 0$, and the associated generalized initial value $x|_{[a-h,a]} = \varphi$. We look for the solutions of the mentioned problem and deal with its properties, searching conditions for its existence and uniqueness, studying the data dependence: continuity, Lipschitz-continuity and differentiability regarding a parameter.

1. Introduction

Functional differential equations with state dependent modified argument was considered by numerous researchers, as they play an important role in applications. From the numerous works, which are related to functional differential equations, it is worth to mention V. R. Petuhov [12], R. D. Driver [3], R. J. Oberg [11], G. M. Dunkel [4], L. E. Elsgoltz and S. B. Norkin [7], B. Rzepecki [13], J. K. Hale [8], F. Hartung and J. Turi [9], V. Kalmanovskii and A. Myshkis [10], A. Buică [1]. For the application of the Picard operator's technique see I. A. Rus [14], [15], M. A. Şerban [16], E. Egri and I. A. Rus [6], E. Egri [5]. Some other results on iterative functional

Received by the editors: 24.10.2008.

2000 *Mathematics Subject Classification.* 47H10, 34K10, 34K20.

Key words and phrases. Iterative functional differential equations, state dependent modified argument, Cauchy problem, contraction principle, Schauder fixed point theorem, data dependence.

This paper was presented at the 7-th Joint Conference on Mathematics and Computer Science, July 3-6, 2008, Cluj-Napoca, Romania.

differential equations can be found in K. Wang [18], J. G. Si and S. S. Cheng [17], S. S. Cheng, J. G. Si and X. P. Wang [2].

The purpose of this paper is to study the following problem

$$x'(t) = f(t, x(t), x(g(t, x(t)))), \quad t \in [a, b], \tag{1}$$

$$x|_{[a-h,a]} = \varphi, \tag{2}$$

with $x \in C([a-h, b], [a-h, b]) \cap C^1([a, b], [a-h, b])$.

We suppose that

(C_1) $h > 0$;

(C_2) $f \in C([a, b] \times [a-h, b]^2, \mathbb{R})$;

(C_3) $g \in C([a, b] \times [a-h, b], [a-h, b])$;

(C_4) $\varphi \in C([a-h, a], [a-h, b])$;

(C_5) there exists $L_f > 0$ such that

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \le L_f(|u_1 - v_1| + |u_2 - v_2|),$$

$\forall t \in [a, b]$, $u_i, v_i \in [a-h, b]$, $i = 1, 2$;

(C_6) there exists $L_g > 0$ such that

$$|g(t, u) - g(t, v)| \le L_g |u - v|,$$

$\forall t \in [a, b]$, $u, v \in [a-h, b]$.

Realize that the problem $(1) + (2)$ is equivalent with the following fixed point equation

$$x(t) = \begin{cases} \varphi(t), & t \in [a-h, a], \\ \varphi(a) + \displaystyle\int_a^t f(s, x(s), x(g(s, x(s))))\,\mathrm{d}s, & t \in [a, b], \end{cases} \tag{3}$$

where $x \in C([a-h, b], [a-h, b])$.

2. **Existence**

Observe that the set $C([a-h,b],\mathbb{R})$ can be endowed with the Chebyshev norm

$$\|x\|_C = \max_{t\in[a-h,b]} |x(t)|.$$

Henceforth we consider on the set $C([a-h,b],[a-h,b])$ the metric induced by this norm.

Regarding our problem we define the following operator

$$A : C([a-h,b],[a-h,b]) \to C([a-h,b],\mathbb{R}),$$

where

$$A(x)(t) := \text{the right hand side of (3).} \tag{4}$$

In this manner we obtained the fixed point equation $x = A(x)$, which hereafter will be the subject of our research. Denote by F_A the fixed point set of the operator A.

Remark that the set $C([a-h,b],\mathbb{R})$ along with the Chebyshev norm, $\|\cdot\|_C$ constitutes a Banach space.

We have our first result.

Theorem 2.1. *We suppose that*

(i) *the conditions* $(C_1)-(C_4)$ *are satisfied;*

(ii) $m_f, M_f \in \mathbb{R}$ *are such that*

   (1) $m_f \le f(t,u_1,u_2) \le M_f,\ \forall\, t\in[a,b],\ u_i\in[a-h,b],\ i=1,2;$

   (2) $a \le h + \varphi(a) + \min\{0, m_f(b-a)\};$

   (3) $b \ge \varphi(a) + \max\{0, M_f(b-a)\}.$

Then the problem $(1)+(2)$ *has at least a solution.*

Proof. To justify the existence of the solution we will apply Schauder's theorem. For this purpose, to have a self-mapping operator, it is necessary to have satisfied the invariance property of the set $C([a-h,b],[a-h,b])$ for the operator $A : C([a-h,b],[a-h,b]) \to C([a-h,b],\mathbb{R})$. Therefore, it must hold the conclusion

$$x(t)\in[a-h,b] \Longrightarrow A(x)(t)\in[a-h,b],\ \forall t\in[a-h,b].$$

Taking into consideration the assumption (C_4), for $t \in [a-h, a]$ the condition above is realized. Moreover, from the definition of the operator A we have

$$\min_{t\in[a,b]} A(x)(t) = \varphi(a) + \min\{0, m_f(b-a)\},$$

$$\max_{t\in[a,b]} A(x)(t) = \varphi(a) + \max\{0, M_f(b-a)\}.$$

In this case we obtain

$$a - h \leq A(x)(t) \leq b, \, \forall t \in [a, b],$$

if the relations

$$a - h \leq \min_{t\in[a,b]} A(x)(t), \quad \max_{t\in[a,b]} A(x)(t) \leq b$$

are true. But these are fulfilled by the condition (ii). Therefore, it is right to consider the self-mapping operator

$$A : C([a-h, b], [a-h, b]) \to C([a-h, b], [a-h, b]).$$

Observe that the operator A is completely continuous, since the subset

$$C([a-h, b], [a-h, b]) \subset C([a-h, b], \mathbb{R})$$

is bounded, convex and closed, and what is more, the family of functions $A(C([a-h, b], [a-h, b]))$ is relatively compact. Consequently, it can be applied Schauder's fixed point theorem. Therefore, we have $F_A \neq \emptyset$, or equivalently, the problem $(1) + (2)$ has at least a solution. $\square$

3. **Existence and uniqueness**

To study the existence and uniqueness of the solution of the Cauchy problem $(1) + (2)$, take an arbitrary positive number L and construct the set

$$C_L([a-h, b], [a-h, b]) :=$$
$$\{x \in C([a-h, b], [a-h, b]) \mid |x(t_1) - x(t_2)| \leq L|t_1 - t_2|, \forall \, t_1, t_2 \in [a-h, b]\}.$$

Notice that the subset $C_L([a-h,b],[a-h,b]) \subset C([a-h,b],\mathbb{R})$ can be endowed with the Chebyshev metric defined by

$$\|x-y\|_C := \max_{t\in[a-h,b]} (|x(t)-y(t)|)\,, \tag{5}$$

and in this manner we obtain a complete metric space.

We have:

Theorem 3.1. *Consider the Cauchy problem* (1) + (2) *and suppose that*

(i) *the conditions* $(\mathrm{C}_1)-(\mathrm{C}_6)$ *are satisfied;*

(ii) $\varphi \in C_L([a-h,a],[a-h,b])$;

(iii) $m_f, M_f \in \mathbb{R}$ *are such that*

  (1) $m_f \le f(t,u_1,u_2) \le M_f,\ \forall\, t\in[a,b],\ u_i\in[a-h,b],\ i=1,2$;

  (2) $a \le h+\varphi(a)+\min\{0, m_f(b-a)\}$;

  (3) $b \ge \varphi(a)+\max\{0, M_f(b-a)\}$;

(iv) $\max\{|M_f|,|m_f|\} \le L$;

(v) $L_f(b-a)(2+LL_g) < 1$.

Then the problem (1) + (2) *has in* $C_L([a-h,b],[a-h,b])$ *a unique solution.*

Proof. Consider the operator

$$A : C_L([a-h,b],[a-h,b]) \to C([a-h,b],\mathbb{R})$$

given by (4). We want to apply the contraction principle for this operator. Therefore, first admit that A is self-mapping. Since all the conditions of the existence theorem hold, we have

$$a-h \le A(x)(t) \le b, \ \text{ when } a-h \le x(t) \le b,$$

for all $t\in[a-h,b]$. Moreover, from the condition (ii), if $t_1,t_2\in[a-h,a]$, we obtain

$$|A(x)(t_1)-A(x)(t_2)| = |\varphi(t_1)-\varphi(t_2)| \le L|t_1-t_2|.$$

On the other hand, if $t_1,t_2\in[a,b]$, due to (iv), we have

$$|A(x)(t_1)-A(x)(t_2)| = \left|\int_{t_1}^{t_2} f(s,x(s),x(g(s,x(s))))\,\mathrm{d}s\right| \le$$

$$\le \max\{|m_f|,|M_f|\}|t_1-t_2| \le L|t_1-t_2|,$$

which involves that the operator A is L-Lipschitz. Accordingly, we have $A(x) \in C_L([a-h,b],[a-h,b])$ for all $x \in C_L([a-h,b],[a-h,b])$.

Henceforward concede that from the condition (v) the operator A is an L_A-contraction, with

$$L_A := L_f(b-a)(2+LL_g).$$

Indeed, for all $t \in [a-h,a]$ we have $|A(x_1)(t) - A(x_2)(t)| = 0$. Furthermore, for $t \in [a,b]$ we successively get

$$\begin{aligned}
&|A(x_1)(t) - A(x_2)(t)| = \\
&= \left| \int_a^t [f(s,x_1(s),x_1(g(s,x_1(s)))) - f(s,x_2(s),x_2(g(s,x_2(s))))]\,\mathrm{d}s \right| \leq \\
&\leq \int_a^t |f(s,x_1(s),x_1(g(s,x_1(s)))) - f(s,x_2(s),x_2(g(s,x_2(s))))|\,\mathrm{d}s \leq \\
&\leq L_f \int_a^t \big[|x_1(s)-x_2(s)| + |x_1(g(s,x_1(s))) - x_2(g(s,x_2(s)))|\big]\mathrm{d}s \leq \\
&\leq L_f(b-a)\|x_1-x_2\|_C + \\
&+ L_f\int_a^t \big[|x_1(g(s,x_1(s))) - x_1(g(s,x_2(s)))| + |x_1(g(s,x_2(s))) - x_2(g(s,x_2(s)))|\big]\mathrm{d}s \leq \\
&\leq L_f(b-a)\|x_1-x_2\|_C + L_f\int_a^t \big[L\cdot|g(s,x_1(s)) - g(s,x_2(s))| + \|x_1-x_2\|_C\big]\mathrm{d}s \leq \\
&\leq 2L_f(b-a)\|x_1-x_2\|_C + LL_f\int_a^t |g(s,x_1(s)) - g(s,x_2(s))|\mathrm{d}s \leq \\
&\leq 2L_f(b-a)\|x_1-x_2\|_C + LL_f\int_a^t L_g|x_1(s)-x_2(s)|\mathrm{d}s \leq \\
&\leq \big[2L_f(b-a) + LL_fL_g(b-a)\big]\|x_1-x_2\|_C,
\end{aligned}$$

and it follows that

$$||A(x_1) - A(x_2)||_C \leq L_A||x_1-x_2||_C, \quad L_A = L_f(b-a)(2+LL_g).$$

From the condition (vi) we have $L_A < 1$, consequently the operator A is an L_A-contraction. By applying the contraction principle the operator A has a unique fixed point, i.e. the problem $(1)+(2)$ has in $C_L([a-h,b],[a-h,b])$ a unique solution.

□

4. **Data dependence: continuity**

In order to study the continuous dependence of the fixed points we will use the following result:

Lemma 4.1. (I. A. Rus [15]) *Let (X, d) be a complete metric space and*

$$A, B : X \to X$$

two operators. We suppose that

(i) *the operator A is a γ-contraction;*

(ii) $F_B \neq \emptyset$;

(iii) *there exists $\eta > 0$ such that*

$$d(A(x), B(x)) \leq \eta, \ \forall\, x \in X.$$

Then, if $F_A = \{x_A^*\}$ and $x_B^* \in F_B$, we have

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\gamma}.$$

Now, let f_i and φ_i as in Theorem 3.1. For $i = 1, 2$ we consider the following two Cauchy problems

$$x'(t) = f_i(t, x(t), x(g(t, x(t)))), \quad t \in [a, b], \tag{6}$$

$$x|_{[a-h,a]} = \varphi_i. \tag{7}$$

We assign to the problems (6) + (7) the operators

$$A_i : C_L([a-h, b], [a-h, b]) \to C_L([a-h, b], [a-h, b]),$$

given by

$$A_i(x)(t) := \begin{cases} \varphi_i(t), & t \in [a-h, a], \\ \varphi_i(a) + \displaystyle\int_a^t f_i(s, x(s), x(g(s, x(s))))\, \mathrm{d}s, & t \in [a, b], \end{cases} \tag{8}$$

$i = 1, 2$. From Theorem 3.1 the operators A_1 and A_2 are contractions. We will denote by x_1^*, x_2^* their unique fixed points.

Then, accordingly to Lemma 4.1 we have the result as follows.

Theorem 4.1. *We suppose the conditions of Theorem 3.1 concerning to the problems* $(6) + (7)$ *are satisfied and, moreover,*

(i) *there exists* η_1 *such that*

$$|\varphi_1(t) - \varphi_2(t)| \leq \eta_1, \quad \forall\, t \in [a-h, a]$$

(ii) *there exists* $\eta_2 > 0$ *such that*

$$|f_1(t, u_1, u_2) - f_2(t, u_1, u_2)| \leq \eta_2, \quad \forall\, t \in [a, b],\, \forall\, u_i \in [a-h, b],\, i = 1, 2.$$

Then the following estimation holds:

$$\|x_1^* - x_2^*\|_C \leq \frac{\eta_1 + \eta_2(b-a)}{1 - L_f(b-a)(2 + LL_g)},$$

where $L_f = \max\{L_{f_1}, L_{f_2}\}$ *and* $L_g = \max\{L_{g_1}, L_{g_2}\}$.

Proof. Observe that, since the assumptions of Theorem 3.1 are realized, the operators A_i $(i = 1, 2)$ given by (8) are L_{A_i}-contractions with

$$L_{A_i} := L_{f_i}(b-a)(2 + LL_{g_i}).$$

Consider $t \in [a-h, a]$. From the condition (ii) it follows that

$$\|A_1(x) - A_2(x)\|_C \leq \eta_1 \leq \eta_1 + \eta_2(b-a).$$

On the other hand, for $t \in [a, b]$, we obtain

$$|A_1(x)(t) - A_2(x)(t)| \leq$$

$$\leq |\varphi_1(a) - \varphi_2(a)| + \int_a^t |f_1(s, x(s), x(g(s, x(s)))) - f_2(s, x(s), x(g(s, x(s))))\, \mathrm{d}s| \leq$$

$$\leq \eta_1 + \eta_2(b-a).$$

Consequently,

$$\|A_1(x) - A_2(x)\|_C \leq \eta_1 + \eta_2(b-a), \quad \forall x \in C_L([a-h, b], [a-h, b]).$$

Now, the proof follows from Lemma 4.1. $\square$

5. Data dependence on parameter: Lipschitz-continuity

In this section we will use the following abstract result:

Lemma 5.1. *Let (X, d) be a complete metric space, $J \subset \mathbb{R}$ and $A : X \times J \to X$ an operator. We suppose that:*

(i) $\exists\, \alpha \in]0, 1[$ *such that*

$$d\big(A(x_1, \lambda), A(x_2, \lambda)\big) \leq \alpha d(x_1, x_2), \quad \forall\, x_1, x_2 \in X,\ \lambda \in J;$$

(ii) $\exists\, l > 0$ *such that*

$$d\big(A(x, \lambda_1), A(x, \lambda_2)\big) \leq l|\lambda_1 - \lambda_2|, \quad \forall\, x \in X,\ \lambda_1, \lambda_2 \in J.$$

Then

(a) $\forall\, \lambda \in J$, *the operator $A(\cdot, \lambda) : X \to X$ has a unique fixed point, $x^*(\lambda) \in X$;*

(b) $d(x^*(\lambda_1), x^*(\lambda_2)) \leq \dfrac{l}{1-\alpha}|\lambda_1 - \lambda_2|, \quad \forall \lambda_1, \lambda_2 \in J.$

Proof. Evidently, from the condition (i) the operator $A(\cdot, \lambda)$ is a contraction. Therefore, the fixed point equation $A(x, \lambda) = x$ has a unique solution $x^*(\lambda) \in X$, corresponding to an arbitrary value $\lambda \in J$. Moreover, for $\lambda_1, \lambda_2 \in J$ we have

$$\begin{aligned}
&d(x^*(\lambda_1), x^*(\lambda_2)) = d(A(x^*(\lambda_1), \lambda_1), A(x^*(\lambda_2), \lambda_2)) \leq \\
&\leq d(A(x^*(\lambda_1), \lambda_1), A(x^*(\lambda_1), \lambda_2)) + d(A(x^*(\lambda_1), \lambda_2), A(x^*(\lambda_2), \lambda_2)) \leq \\
&\leq l|\lambda_1 - \lambda_2| + \alpha \cdot d(x^*(\lambda_1), x^*(\lambda_2)),
\end{aligned}$$

and consequently

$$d(x^*(\lambda_1), x^*(\lambda_2)) \leq \frac{l}{1-\alpha}|\lambda_1 - \lambda_2|.$$

Accordingly, we have the proof. □

Now we consider the problem

$$\begin{cases} x'(t) = f(t, x(t), x(g(x, t)), \lambda), \quad t \in [a, b],\ \lambda \in J, \\ x(t) = \varphi(t, \lambda), \quad t \in [a-h, a],\ \lambda \in J,\ h > 0, \end{cases} \tag{9}$$

and, for $L > 0$, the corresponding operator A, given as follows:

$$A : C_L([a-h,b],[a-h,b]) \times J \to C_L([a-h,b],[a-h,b]) \times J,$$

$$A(x,\lambda) := \begin{cases} \varphi(t,\lambda), & t \in [a-h,a],\ \lambda \in J; \\ \varphi(a,\lambda) + \displaystyle\int_a^t f(s,x(s),x(g(s,x(s))),\lambda)\mathrm{d}s, & t \in [a,b], \quad \lambda \in J. \end{cases} \tag{10}$$

Based upon Lemma 5.1 we have the next result:

Theorem 5.1. *We suppose that*

(i) $f \in C([a,b] \times [a-h,b]^2 \times J, \mathbb{R})$;

(ii) $g \in C([a,b] \times [a-h,b],[a-h,b])$;

(iii) $\varphi \in C_L([a-h,a],[a-h,b]) \times J$, *and* $\exists l_\varphi > 0$ *such that*

$$|\varphi(t,\lambda_1) - \varphi(t,\lambda_2)| \le l_\varphi;$$

(iv) *there exists* $L_f > 0$ *such that*

$$|f(t,u_1,u_2,\lambda) - f(t,v_1,v_2,\lambda)| \le L_f(|u_1 - v_1| + |u_2 - v_2|),$$

$\forall t \in [a,b],\ u_i, v_i \in [a-h,b],\ \lambda \in J,\ i = 1,2;$

(v) *there exists* $L_f > 0$ *such that*

$$|f(t,u,v,\lambda_1) - f(t,u,v,\lambda_2)| \le l_f|\lambda_1 - \lambda_2|,$$

$\forall t \in [a,b],\ u,v \in [a-h,b],\ \lambda_i \in J,\ i = 1,2;$

(vi) *there exists* $L_g > 0$ *such that*

$$|g(t,u) - g(t,v)| \le L_g|u - v|,$$

$\forall t \in [a,b],\ u,v \in [a-h,b];$

(vii) $L_f(b-a)[2 + LL_g] < 1$.

Then

(a) $\forall\, \lambda \in J$, *the operator* $A(\cdot,\lambda) : X \to X$ *defined by* (10) *has a unique fixed point,* $x^*(\lambda) \in X$;

(b) $\|x^*(\lambda_1), x^*(\lambda_2)\|_C \le \dfrac{l_\varphi + l_f(b-a)}{1 - L_f(b-a)[2 + LL_g]} |\lambda_1 - \lambda_2|, \quad \forall \lambda_1, \lambda_2 \in J.$

Proof. From the proof of Theorem 3.1, for all $t \in [a, b]$ we have

$$|A(x_1, \lambda)(t) - A(x_2, \lambda)(t)| =$$
$$= \left| \int_a^t f(s, x_1(s), x_1(g(s, x_1(s))), \lambda)\mathrm{d}s - \int_a^t f(s, x_2(s), x_2(g(s, x_2(s))), \lambda)\mathrm{d}s \right| \le$$
$$\le L_f(b-a)[2 + LL_g]\|x_1 - x_2\|_C,$$

and taking $\alpha := L_f(b-a)[2 + LL_g]$, due to the condition (vi), the first assumption of Lemma 5.1 is satisfied.

Furthermore, for $t \in [a-h, a]$, we have:

$$|A(x, \lambda_1) - A(x, \lambda_2)| = |\varphi(t, \lambda_1) - \varphi(t, \lambda_2)| \le l_\varphi |\lambda_1 - \lambda_2|.$$

On the other hand, if $t \in [a, b]$, we obtain:

$$|A(x, \lambda_1) - A(x, \lambda_2)| \le |\varphi(a, \lambda_1) - \varphi(a, \lambda_2)| +$$
$$+ \int_a^t |f(s, x(s), x(g(s, x(s))), \lambda_1) - f(s, x(s), x(g(s, x(s))), \lambda_2)|\mathrm{d}s \le$$
$$\le l_\varphi|\lambda_1 - \lambda_2| + l_f(b-a)|\lambda_1 - \lambda_2| = [l_\varphi + l_f(b-a)]|\lambda_1 - \lambda_2|.$$

One can be observe that $l := l_\varphi + l_f(b-a)$ has the same property as the one from Lemma 5.1.

Consequently, the proof is complete. □

6. Data dependence: differentiability

Henceforward we will need the following result, which is very useful for proving solutions of operatorial equations to be differentiable with respect to parameters.

Theorem 6.1 (Fibre contraction principle (I. A. Rus [14])). *Let (X, d) and (Y, ρ) be two metric spaces and*

$$A : X \times Y \to X \times Y, \ (B : X \to X, \ C : X \times Y \to Y),$$
$$A(x, y) = (B(x), C(x, y))$$

a triangular operator.

We suppose that

(i) (Y, ρ) *is a complete metric space;*

(ii) *the operator* B *is a Picard operator;*

(iii) *there exists* $L_C \in [0, 1[$ *such that* $C(x, \cdot) : Y \to Y$ *is an* L_C*-contraction, for all* $x \in X$;

(iv) *if* $(x^*, y^*) \in F_A$, *then* $C(\cdot, y^*)$ *is continuous in* x^*.

Then the operator A *is a Picard operator.*

For some applications of the fibre contraction principle see I. A. Rus [15], E. Egri and I. A. Rus [6], E. Egri [5].

Consider the following problem with parameter:

$$x'(t; \lambda) = f(t, x(t; \lambda), x(g(t, x(t; \lambda)); \lambda); \lambda), \quad t \in [a, b], \tag{11}$$

$$x(t; \lambda) = \varphi(t; \lambda), \; t \in [a - h, a], \tag{12}$$

with $\lambda \in J \subset \mathbb{R}$ a compact subset.

We have:

Theorem 6.2. *Suppose that we have satisfied the conditions below:*

(P_1) $h > 0$, $J \subset \mathbb{R}$, *a compact interval;*

(P_2) $\varphi(t, \cdot) \in C^1(J, \mathbb{R})$, *for all* $t \in [a - h, a]$;
$\varphi(\cdot, \lambda) \in C^1_L([a - h, a], [a - h, a])$, *and*
$\varphi'(a, \lambda) = f(a, \varphi(a; \lambda), \varphi(g(a, \varphi(a; \lambda)); \lambda); \lambda)$;

(P_3) $f \in C^1([a, b] \times [a - h, b]^2 \times J, \mathbb{R})$;
$g \in C([a, b] \times [a - h, b], [a - h, b])$;

(P_4) *there exists* $L_f > 0$ *such that*

$$\left| \frac{\partial f(t, u_1, u_2; \lambda)}{\partial u_i} \right| \leq L_f,$$

for all $t \in [a, b]$, $u_i \in [a - h, b]$, $i = 1, 2$, $\lambda \in J$;

(P_5) $m_f, M_f \in \mathbb{R}$ *are such that*

(1) $m_f \leq f(t, u_1, u_2; \lambda) \leq M_f$, $\forall\, t \in [a, b]$, $u_1, u_2 \in [a - h, b]$, $\lambda \in J$;

(2) $a \leq h + \varphi(a;\lambda) + \min\{0, m_f(b-a)\}$;

(3) $b \geq \varphi(a;\lambda) + \max\{0, M_f(b-a)\}$;

(P_6) $\max\{|m_f|, |M_f|\} \leq L$;

(P_7) $L_f(b-a)(2 + LL_g) < 1$.

Then

(1) *the problem* (11) + (12) *has in* $C_L([a-h,b],[a-h,b])$ *a unique solution,* $x^*(\cdot,\lambda)$;

(2) $x^*(t,\cdot) \in C^1(J,\mathbb{R})$, $\forall t \in [a-h,b]$.

Proof. Since we are in the conditions of Theorem 3.1, we certainly have that the problem (11)+(12) has in $C_L([a-h,b],[a-h,b])$ a unique solution, $x^*(\cdot,\lambda)$. Therefore, statement (1) from the theorem is satisfied.

To justify the affirmation (2), first observe that the problem (11) + (12) is equivalent with the following fixed point equation

$$x(t;\lambda) = \begin{cases} \varphi(t;\lambda), & t \in [a-h,a],\ \lambda \in J, \\ \varphi(a;\lambda) + \displaystyle\int_a^t f(t, x(t;\lambda), x(g(t,x(t;\lambda));\lambda);\lambda)\,\mathrm{d}s, & t \in [a,b],\ \lambda \in J. \end{cases} \tag{13}$$

We try to fit in the fibre contraction principle. For this purpose we consider the operator

$$B : C_L([a-h,b] \times J, [a-h,b]) \to C_L([a-h,b] \times J, [a-h,b]),$$

where

$$B(x)(t;\lambda) := \text{ the right hand side of (13)}.$$

From the proof of the existence and uniqueness theorem 3.1 this operator is well-defined. We denote by X its domain (codomain). Observe that the set X endowed with the Chebyshev metric

$$d_C(x,y) = \max_{t,\lambda} |x(t;\lambda) - y(t;\lambda)|, \text{for all } x,y \in X,$$

is a Banach space.

From the contraction principle, in the conditions of the theorem, the operator B is a Picard operator with $F_B = \{x^*\}$.

We consider the subset $X_1 \subset X$,

$$X_1 := \left\{ x \in X \,\middle|\, \frac{\partial x}{\partial t} \in C([a-h,b] \times J, \mathbb{R}) \right\}.$$

Remark that $x^* \in X_1$, $B(X_1) \subset X_1$ and $B : X_1 \to X_1$ is Picard operator. Thus, B fulfills (ii) from the fibre contraction theorem.

Let $Y := C([a-h,b] \times J, \mathbb{R})$. We want to prove that there exists $\dfrac{\partial x^*}{\partial \lambda}$ and, moreover, $\dfrac{\partial x^*}{\partial \lambda} \in Y$. If we suppose that there exists $\dfrac{\partial x^*}{\partial \lambda}$, then from (13) we have

$$\frac{\partial x^*(t;\lambda)}{\partial \lambda} = \frac{\partial \varphi(a;\lambda)}{\partial \lambda} + \int_a^t \frac{\partial f(s, x^*(s;\lambda), x^*(g(s,x^*(s;\lambda));\lambda);\lambda)}{\partial u_1} \cdot \frac{\partial x^*(s;\lambda)}{\partial \lambda} \,\mathrm{d}s +$$
$$+ \int_a^t \frac{\partial f(s, x^*(s;\lambda), x^*(g(s,x^*(s;\lambda));\lambda);\lambda)}{\partial u_2} \cdot \frac{\partial x^*(g(s,x^*(s;\lambda));\lambda)}{\partial t} \cdot \frac{\partial g(s,x^*(s;\lambda))}{\partial v} \cdot$$
$$\cdot \frac{\partial x^*(s;\lambda)}{\partial \lambda} \,\mathrm{d}s + \int_a^t \frac{\partial f(s, x^*(s;\lambda), x^*(g(s,x^*(s;\lambda));\lambda);\lambda)}{\partial u_2} \cdot \frac{\partial x^*(g(s,x^*(s;\lambda));\lambda)}{\partial \lambda} \,\mathrm{d}s +$$
$$+ \int_a^t \frac{\partial f(s, x^*(s;\lambda), x^*(x^*(s;\lambda);\lambda);\lambda)}{\partial \lambda} \,\mathrm{d}s, \qquad t \in [a,b],\ \lambda \in J.$$

The obtained relation suggests us to consider the operator

$$C : X_1 \times Y \to Y$$

$$(x,y) \mapsto C(x,y)$$

defined by

$$C(x,y)(t;\lambda) := \frac{\partial \varphi(a;\lambda)}{\partial \lambda} + \int_a^t \frac{\partial f(s, x(s;\lambda), x(g(s,x(s;\lambda));\lambda);\lambda)}{\partial u_1} \cdot y(s;\lambda) \,\mathrm{d}s +$$
$$+ \int_a^t \frac{\partial f(s, x(s;\lambda), x(g(s,x(s;\lambda));\lambda);\lambda)}{\partial u_2} \cdot \frac{\partial x(g(s,x(s;\lambda));\lambda)}{\partial t} \cdot \frac{\partial g(s,x(s;\lambda))}{\partial v} \cdot$$
$$\cdot y(s;\lambda) \mathrm{d}s + \int_a^t \frac{\partial f(s, x(s;\lambda), x(g(s,x(s;\lambda));\lambda);\lambda)}{\partial u_2} \cdot \frac{\partial x(g(s,x(s;\lambda));\lambda)}{\partial \lambda} \mathrm{d}s +$$
$$+ \int_a^t \frac{\partial f(s, x(s;\lambda), x(g(x(s;\lambda));\lambda);\lambda)}{\partial \lambda} \,\mathrm{d}s, \qquad t \in [a,b],\ \lambda \in J.$$

and

$$C(x,y)(t,\lambda) := \frac{\partial \varphi(t;\lambda)}{\partial \lambda}, \text{ for } t \in [a_1, a],\ \lambda \in J.$$

In this way we can consider the triangular operator

$$A : X_1 \times Y \to X_1 \times Y$$

$$(x,y) \mapsto (B(x), C(x,y)).$$

As we have seen earlier, $B : X_1 \to X_1$ is a Picard operator. Realize that $C(x,\cdot) : Y \to Y$ is a contraction on $[a,b]$. Indeed, we have

$$\begin{aligned}
&|C(x,y_1)(t;\lambda) - C(x,y_2)(t;\lambda)| = \\
&\left|\int_a^t \frac{\partial f(s,x(s;\lambda),x(g(s,x(s;\lambda));\lambda);\lambda)}{\partial u_1} \cdot \big[y_1(s;\lambda) - y_2(s;\lambda)\big]\mathrm{d}s + \right.\\
&+ \int_a^t \frac{\partial f(s,x(s;\lambda),x(g(s,x(s;\lambda));\lambda);\lambda)}{\partial u_2} \cdot \frac{\partial x(g(s,x(s;\lambda));\lambda)}{\partial t} \cdot \frac{\partial g(s,x(s;\lambda))}{\partial v} \cdot \\
&\left.\cdot \big[y_1(s;\lambda) - y_2(s;\lambda)\big]\mathrm{d}s\right| \le \\
&\le L_f \int_a^t \left|1 + \frac{\partial x(g(s,x(s;\lambda));\lambda)}{\partial t} \cdot \frac{\partial g(s,x(s;\lambda))}{\partial v}\right| \cdot |y_1(s;\lambda) - y_2(s;\lambda)|\,\mathrm{d}s \le \\
&\le L_f(1+LL_g)\int_a^t |y_1(s;\lambda) - y_2(s;\lambda)|\,\mathrm{d}s \le \\
&\le L_f(1+LL_g)(b-a)||y_1 - y_2||_C \le \\
&\le L_f(b-a)(2+LL_g)||y_1 - y_2||_C.
\end{aligned}$$

In this way we got

$$||C(x,y_1) - C(x,y_2)||_C \le L_C \cdot ||y_1 - y_2||_C, \quad \text{with} \quad L_C := L_f(b-a)(2+LL_g).$$

So, we are in the condition of the fibre contraction principle, and consequently, A is a Picard operator, i.e. the sequences defined by

$$x_{n+1} := B(x_n),$$

$$y_{n+1} := C(x_n, y_n), \quad n \in \mathbb{N}$$

converges uniformly (with respect to $t \in [a-h,b],\ \lambda \in J$) to the unique fixed point of the operator A, $(x^*, y^*) \in F_A$, for all $x_0, y_0 \in C([a-h,b] \times J, [a-h,b])$.

Taking $x_0 = 0$, $y_0 = \dfrac{\partial x_0}{\partial \lambda} = 0$, we get $y_1 = \dfrac{\partial x_1}{\partial \lambda}$. By induction it can be proved that $y_n = \dfrac{\partial x_n}{\partial \lambda}$, $\forall\, n \in \mathbb{N}$. So,

$$x_n \overset{unif.}{\longrightarrow} x^* \text{ as } n \to \infty,$$

$$\frac{\partial x_n}{\partial \lambda} \overset{unif.}{\longrightarrow} y^* \text{ as } n \to \infty.$$

From these, using a theorem of Weierstrass we have that x^* is differentiable and $\dfrac{\partial x^*}{\partial \lambda} = y^* \in Y$. □

7. **Example**

To check our results consider the following Cauchy problem

$$x'(t) = \frac{1}{5}\left[x\left(\frac{1}{6}[t + x(t)]\right) - \frac{1}{6}x(t) - \frac{1}{24}t\right] + \frac{1}{2}, \qquad t \in [0,2], \tag{14}$$

$$x(t) = t, \qquad t \in [-1,0]. \tag{15}$$

We look for the solution $x \in C([-1,2],[-1,2]) \cap C^1([0,2],[-1,2])$ of the problem (14)+(15). For this purpose we apply Theorem 2.1. First observe that we have $a = 0$, $b = 2$, $h = 1$, $\varphi(t) = t$, and

$$g(t,u) = \frac{1}{6}(t+u), \text{ for all } t \in [0,2],\ u \in [-1,2],$$

$$f(t,u_1,u_2) = \frac{1}{5}\left[u_2 - \frac{1}{6}u_1 - \frac{1}{24}t\right] + \frac{1}{2}, \text{ for all } t \in [0,2],\ u_1,u_2 \in [-1,2],$$

with

$$m_f = \frac{13}{60},\ M_f = \frac{14}{15},\ L_f = \frac{1}{5},\ L_g = \frac{1}{6}.$$

Since all the conditions of Theorem 2.1 are fulfilled, the problem (14)+(15) has in $C([-1,2],[-1,2])$ at least a solution. Moreover, considering $\frac{1}{2} \le L < 3$, due to Theorem3.1, this solution is unique on the set $C_L([-1,2],[-1,2])$, and it is the limit of the sequence $(x_n)_{n\ge 0}$ of successive approximation, given by the recursive relation

$$x_{n+1} = \begin{cases} t, & t \in [-1,0],\\ \displaystyle\int_0^t \left\{\frac{1}{5}\left[x_n\left(\frac{1}{6}[s + x_n(s)]\right) - \frac{1}{6}x_n(s) - \frac{1}{24}s\right] + \frac{1}{2}\right\} \mathrm{d}s, & t \in [0,2], \end{cases}$$

Doing some calculations with Maple, it seems that the sequence of successive approximation is convergent to the function $x(t) = \frac{t}{2}$. Indeed, this is the unique solution of the problem, since it satisfies the functional differential equation (14).

References

[1] Buică, A., *Existence and continuous dependence of solutions of some functional-differential equations,* Seminar on Fixed Point Theory, Cluj-Napoca, 1995, 1-14.

[2] Cheng, S.S., Si, J.G., Wang, X.P., *An existence theorem for iterative functional differential equations,* Acta Math. Hungar., **94**(2002), 1-17.

[3] Driver, R.D., *Existence theory for a delay-differential system,* Contributions to Differential Equations, **1**(1963), 317-336.

[4] Dunkel, G.M., *Functional-differential equations: Examples and problems, Lecture Notes in Mathematics*, **144**(1970), Springer, Berlin, 49-63.

[5] Egri, E., *Cauchy problem for an iterative functional-differential equation, Nonlinear Analysis Forum.*

[6] Egri, E., Rus, I.A., *First order iterative functional-differential equation with parameter*, Studia Univ. "Babeş Bolyai", Math., **52**(2007), No.4, 67-80.

[7] Elsgoltz, L.E., Norkin, S.B., *Introduction to the Theory and Application of Differential Equations with Deviating Arguments,* Acad. Press, New York, 1973.

[8] Hale, J.K., Ladeira, L.A.C., *Differentiability with respect to delays,* J. Diff. Equations, **92**(1991), 14-26.

[9] Hartung, F., Turi, J., *On differentiability of solution with respect to parameters in state-dependent delay equations,* J. Diff. Equations, **135**(1997), 192-237.

[10] Kalmanovskii, V., Myshkis, A., *Applied Theory of Functional-Differential Equations*, Kluwer, Dordrecht, 1992.

[11] Oberg, R.J., *On the local existence of solutions of certain functional-differential equations,* Proc. AMS, **20**(1969), 295-302.

[12] Petuhov, V.R., *Functional perturbations of asymptotically and tangentially-asymptotically stable positions of equilibrium* (Russian), Izv. Zaved. Matematica, **2**(1966), 93-97.

[13] Rzepecki, B., *On some functional-differential equations,* Glasnik Mat., **19**(1984), 73-82.

[14] Rus, I.A., *Functional-differential equations of mixed type, via weakly Picard operators,* Seminar on fixed point theory, Cluj-Napoca, **3**(2002), 335-345.

[15] Rus, I.A., *Picard operators and applications,* Scientiae Math. Japonicae, **58**(2003), No. 1, 191-219.

[16] Şerban, M.A., *Iterative methods for some functional-differential equations.*

[17] Si, J.G., Cheng, S.S., *Smooth solutions of a nonhomogeneous iterative functional differential equation,* Proc. Royal Soc. Edinburgh, **128A**(1998), 821-831.

[18] Wang, K., *On the equation $x'(t) = f(x(x(t)))$,* Funkcialaj Ekvacioj, **33**(1990) 405-425.

Babeş-Bolyai University,
Department of Computer Science,
530241 Miercurea-Ciuc, Str. Topliţa, nr.20,
jud. Harghita, Romania
E-mail address: egriedit@yahoo.com

Babeş-Bolyai University,
Department of Applied Mathematics,
Str. M. Kogălniceanu Nr. 1,
400084 Cluj-Napoca, Romania
E-mail address: iarus@math.ubbcluj.ro

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 3, September 2009

FIXED POINT THEOREMS FOR MULTIVALUED WEAK CONTRACTIONS

ALEXANDRU-DARIUS FILIP AND PETRA TÜNDE PETRU

Abstract. The purpose of this work is to present some fixed point results for the so-called multivalued weak contractions. Our results are extensions of the theorems given by M. Berinde and V. Berinde in [1] and by C. Chifu and G. Petruşel in [2].

1. **Preliminaries**

Let us recall first some standard notations and terminologies which are used throughout the paper. For the following notions we consider the context of a metric space (X, d).

We denote by $\widetilde{B}(x_0, r)$ the closed ball centered in $x_0 \in X$ with radius $r > 0$, i.e., $\widetilde{B}(x_0, r) = \{x \in X \mid d(x, x_0) \leq r\}$.

Let $\mathcal{P}(X)$ be the set of all nonempty subsets of X. We also denote:

$$P(X) := \{Y \in \mathcal{P}(X) \mid Y \neq \emptyset\}; P_b(X) := \{Y \in P(X) \mid Y \text{ is bounded }\};$$

$$P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed }\}.$$

Let us define the gap functional between A and B by

$$D_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ D_d(A, B) = \inf\{d(a, b) \mid a \in A,\ b \in B\}$$

Received by the editors: 10.11.2008.

2000 *Mathematics Subject Classification.* 47H04, 47H10, 54C60.

Key words and phrases. Set with two metrics, multivalued operator, fixed point, weak contraction, metric space, Pompeiu-Hausdorff generalized distance.

This paper was presented at the 7-th Joint Conference on Mathematics and Computer Science, July 3-6, 2008, Cluj-Napoca, Romania.

(in particular, if $x_0 \in X$ then $D_d(x_0, B) := D_d(\{x_0\}, B)$) and the (generalized) Pompeiu-Hausdorff functional

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \quad H_d(A, B) = \max\{\sup_{a\in A} D_d(a, B), \sup_{b\in B} D_d(A, b)\}.$$

If $T : X \to P(X)$ is a multivalued operator, then $x \in X$ is called fixed point for T if and only if $x \in T(x)$ and strict fixed point if and only if $T(x) = \{x\}$. The set $Fix(T) := \{x \in X | x \in T(x)\}$ is called the fixed point set of T and $SFix(T) := \{x \in X | \{x\} = T(x)\}$ is the strict fixed point set of T.

If X is a metric space, then the multivalued operator $T : X \to P(X)$ is said to be closed if and only if its graph $Graph(F) := \{(x, y) \in X \times X : y \in F(x)\}$ is a closed subset of $X \times X$.

Let (X, d) be a metric space and $T : X \to P(X)$ be a multivalued operator. T is said to be a multivalued weak contraction or multivalued (θ, L)-weak contraction (see [1]) if and only if there exists $\theta \in]0, 1[$ and $L \geq 0$ such that

$$H(T(x), T(y)) \leq \theta \cdot d(x, y) + L \cdot D(y, T(x)), \text{ for all } x, y \in X.$$

The aim of this article is to extend some fixed point results for multivalued weak-contractions given by M. Berinde and V. Berinde in [1] and by C. Chifu and G. Petruşel in [2]. Our results are also in connection to some other theorems in this field, see [3], [5].

2. Main results

Our first result is a local one and it extends the theorem given by M. Berinde and V. Berinde in [1], to the case of a metric space endowed with two metrics.

Theorem 1. *Let X be a nonempty set, ρ and d two metrics on X, $x_0 \in X$, $r > 0$ and $T : \widetilde{B}_\rho(x_0, r) \to P(X)$ be a multivalued operator. We suppose that:*

(i) *(X, d) is a complete metric space;*

(ii) *there exists $c > 0$ such that $d(x, y) \leq c \cdot \rho(x, y)$, for each $x, y \in \widetilde{B}_\rho(x_0, r)$;*

(iii) *$T : (\widetilde{B}_\rho(x_0, r), d) \to (P(X), H_d)$ is closed;*

(iv) *T is a multivalued (θ, L)-weak contraction with respect to ρ;*

(v) $D_\rho(x_0, T(x_0)) < (1-\theta)r$.

Then we have:

(a) $Fix(T) \neq \emptyset$;

(b) *there exists a sequence* $(x_n)_{n\in\mathbb{N}} \subset \widetilde{B}_\rho(x_0, r)$ *such that:*

(b1) $x_{n+1} \in T(x_n),\ n \in \mathbb{N}$;

(b2) $x_n \xrightarrow{d} x^* \in Fix(T)$, *as* $n \to \infty$;

(b3) $d(x_n, x^*) \leq c \cdot \theta^n \cdot r$, *for each* $n \in \mathbb{N}$.

Proof. By (v), we have that there exists $x_1 \in T(x_0)$such that

$$\rho(x_0, x_1) < (1-\theta)r. \tag{1}$$

Since T is a (θ, L)-weak contraction we have that

$$H_\rho(T(x_0), T(x_1)) \leq \theta \cdot \rho(x_0, x_1) + L \cdot D_\rho(x_1, T(x_0)) = \theta \cdot \rho(x_0, x_1) < \theta \cdot (1-\theta) \cdot r.$$

Thus, for $x_1 \in T(x_0)$ there exists $x_2 \in T(x_1)$ such that

$$\rho(x_1, x_2) < \theta \cdot (1-\theta) \cdot r. \tag{2}$$

By (1) and (2) we obtain that

$$\rho(x_0, x_2) \leq \rho(x_0, x_1) + \rho(x_1, x_2) < (1-\theta) \cdot r + \theta \cdot (1-\theta) \cdot r = (1-\theta^2)r.$$

Hence $x_2 \in \widetilde{B}_\rho(x_0, r)$.

Proceeding inductively, we can construct a sequence $(x_n)_{n\in\mathbb{N}} \subset \widetilde{B}_\rho(x_0, r)$ having the following properties

$$x_{n+1} \in T(x_n),\ n \in \mathbb{N}, \tag{3}$$

$$\rho(x_n, x_{n+1}) < \theta^n \cdot (1-\theta) \cdot r. \tag{4}$$

We want to prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $p \in \mathbb{N}$. Then we have

$$\begin{aligned} \rho(x_n, x_{n+p}) &\leq \rho(x_n, x_{n+1}) + ... + \rho(x_{n+p-1}, x_{n+p}) \\ &< \theta^n \cdot (1-\theta) \cdot r \cdot (1 + \theta + ... + \theta^{p-1}) \\ &= \theta^n \cdot r \cdot (1-\theta^p). \end{aligned}$$

Letting $n \to \infty$, since $\theta \in]0,1[$, we have that $\rho(x_n, x_{n+p}) \to 0$. Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the metric ρ. By (ii) we have that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the metric d, too. Since (X,d) is a complete metric space, there exists $x^* \in X$ such that $x_n \overset{d}{\to} x^*$ as $n \to \infty$. It remains to show that $x^* \in Fix(T)$. Since $Graph(T)$ is closed with respect to (X,d) we get that $x^* \in Fix(T)$.

We already proved that $\rho(x_n, x_{n+p}) < \theta^n \cdot r \cdot (1-\theta^p)$, By (ii), we have that there exists $c > 0$ such that $d(x_n, x_{n+p}) \leq c \cdot \rho(x_n, x_{n+p}) < c \cdot \theta^n \cdot r \cdot (1-\theta^p)$. Letting $p \to \infty$ we obtain that $d(x_n, x^*) \leq c \cdot \theta^n \cdot r$, for each $n \in \mathbb{N}$. $\square$

We can state the above result on a set endowed with one metric.

Theorem 2. *Let (X,d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T : \widetilde{B}(x_0, r) \to P(X)$ a multivalued (θ, L)-weak contraction. We assume that*

$$D(x_0, T(x_0)) < (1-\theta)r.$$

Then we have:

(a) *$Fix(T) \neq \emptyset$;*

(b) *there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \widetilde{B}_\rho(x_0, r)$ such that:*

  (b1) *$x_{n+1} \in T(x_n)$, $n \in \mathbb{N}$;*

  (b2) *$x_n \overset{d}{\to} x^* \in Fix(T)$, as $n \to \infty$;*

  (b3) *$d(x_n, x^*) \leq \theta^n \cdot r$, for each $n \in \mathbb{N}$.*

In what follows we continue with a global version of Theorem 1 for multivalued (θ, L)-weak contractions on a set with two metrics.

Theorem 3. *Let X be a nonempty set, ρ and d twp metrics on X and $T : X \to P(X)$ a multivalued operator. We suppose that*

(i) *(X,d) is a complete metric space;*

(ii) *there exists $c > 0$ such that $d(x,y) \leq c \cdot \rho(x,y)$, for each $x, y \in X$;*

(iii) *$T : (X,d) \to (P(X), H_d)$ is closed;*

(iv) *T is a multivalued (θ, L)-weak contraction.*

Then we have:

(a) $Fix(T) \neq \emptyset$;

(b) *there exists a sequence* $(x_n)_{n\in\mathbb{N}} \subset X$ *such that:*

(b1) $x_{n+1} \in T(x_n),\ n \in \mathbb{N}$;

(b2) $x_n \xrightarrow{d} x^* \in Fix(T)$, *as* $n \to \infty$.

Proof. Fix $x_0 \in X$, choose $r > 0$ such that $D_\rho(x_0, T(x_0)) < (1-\theta)r$. The conclusion follows from Theorem 1. □

The following homotopy result extends some results given by M. Berinde, V. Berinde in [1] and C. Chifu, G. Petruşel in [2].

Theorem 4. *Let* (X, d) *be a complete metric space and* U *be an open subset of* X*. Let* $G : \overline{U} \times [0,1] \to P(X)$ *be a multivalued operator such that the following assumptions are satisfied:*

(i) $x \notin G(x,t)$, *for each* $x \in \partial U$ *and each* $t \in [0,1]$;

(ii) $G(\cdot, t) : \overline{U} \to P(X)$ *is a* (θ, L)*-weak contraction, for each* $t \in [0,1]$;

(iii) *there exists a continuous, increasing function* $\psi : [0,1] \to \mathbb{R}$ *such that*

$$H(G(x,t), G(x,s)) \leq |\psi(t) - \psi(s)|, \text{ for all } x \in \overline{U};$$

(iv) $G : \overline{U} \times [0,1] \to P(X)$ *is closed.*

Then $G(\cdot, 0)$ *has a fixed point if and only if* $G(\cdot, 1)$ *has a fixed point.*

Proof. Suppose that $z \in Fix(G(\cdot, 0))$. From (i) we have that $z \in U$. We define the following set:

$$E := \{(x,t) \in U \times [0,1] | x \in G(x,t)\}.$$

Since $(z, 0) \in E$, we have that $E \neq \emptyset$. We introduce a partial order on E defined by:

$$(x,t) \leq (y,s) \text{ if and only if } t \leq s \text{ and } d(x,y) \leq \frac{2}{1-\theta}[\psi(s) - \psi(t)].$$

Let M be a totally ordered subset of E, $t^* := \sup\{t \,|(x,t) \in M\}$ and $(x_n, t_n)_{n\in\mathbb{N}^*} \subset M$ be a sequence such that $(x_n, t_n) \leq (x_{n+1}, t_{n+1})$ and $t_n \to t^*$ as $n \to \infty$. Then

$$d(x_m, x_n) \leq \frac{2}{1-\theta}[\psi(t_m) - \psi(t_n)], \text{ for each } m, n \in \mathbb{N}^*,\ m > n.$$

Letting $m, n \to +\infty$ we obtain that $d(x_m, x_n) \to 0$, thus $(x_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence. Denote by $x^* \in X$ its limit. Since $x_n \in G(x_n, t_n)$, $n \in \mathbb{N}^*$ and G is closed, we have that $x^* \in G(x^*, t^*)$. From (i) we obtain that $x^* \in U$, so $(x^*, t^*) \in E$.

From the fact that M is totally ordered we have that $(x, t) \leq (x^*, t^*)$, for each $(x, t) \in M$. Thus (x^*, t^*) is an upper bound of M. We can apply Zorn's Lemma, so E admits a maximal element $(x_0, t_0) \in E$. We want to prove that $t_0 = 1$.

Suppose that $t_0 < 1$. Let $r > 0$ and $t \in]t_0, 1]$ such that $B(x_0, r) \subset U$ and $r := \frac{2}{1-\theta}[\psi(t) - \psi(t_0)]$. Then we have

$$\begin{aligned} D(x_0, G(x_0, t)) &\leq D(x_0, G(x_0, t_0)) + H(G(x_0, t_0), G(x_0, t)) \\ &\leq \psi(t) - \psi(t_0) = \frac{(1-\theta)\cdot r}{2} < (1-\theta)\cdot r. \end{aligned}$$

Since $\widetilde{B}(x_0, r) \subset \overline{U}$, the multivalued operator $G(\cdot, t) : \widetilde{B}(x_0, r) \to P_{cl}(X)$ satisfies the assumptions of Theorem 1 for all $t \in [0, 1]$. Hence there exists $x \in \widetilde{B}(x_0, r)$ such that $x \in G(x, t)$. Thus, by (i), we get that $(x, t) \in E$. Since $d(x_0, x) \leq r = \frac{2}{1-\theta}[\psi(t) - \psi(t_0)]$, we have that $(x_0, t_0,) < (x, t)$, which is a contradiction with the maximality of (x_0, t_0). Thus $t_0 = 1$.

Conversely, if $G(\cdot, 1)$ has a fixed point, by a similar approach we can obtain that $G(\cdot, 0)$ has a fixed point too. □

In 2006 A. Petruşel and I. A. Rus (see [4]) extended the notion of well-posed fixed point problem from singlevalued to multivalued operators, as follows.

Definition 1. *(A. Petruşel, I. A. Rus,* [4]*) Let (X, d) be a metric space, $Y \subset P(X)$ and $T : Y \to P_{cl}(X)$ be a multivalued operator. The fixed point problem is well-posed for T with respect to D iff:*

(a) *$Fix(T) = \{x^*\}$;*

(b) *If $x_n \in Y$, $n \in \mathbb{N}$ and $D(x_n, T(x_n)) \to 0$ as $n \to \infty$, then $x_n \to x^*$, as $n \to \infty$.*

The following result is a well-posed fixed point theorem for multivalued (θ, L)-weak contractions on a set endowed with one metric.

Theorem 5. *Let (X,d) be a complete metric space $T : X \to P_{cl}(X)$ is a multivalued (θ, L)-weak contraction with $\theta + L < 1$. Suppose that $SFix(T) \neq \emptyset$. Then the fixed point problem is well-posed for T with respect to D.*

Proof. First we want to prove that $Fix(T) = SFix(T) = \{x^*\}$. Let $x^* \in SFix(T)$. Clearly $SFix(T) \subset Fix(T)$. Thus, we only have to prove that $Fix(T) = \{x^*\}$. Let $x \in Fix(T)$ with $x^* \neq x$. Then

$$\begin{aligned} d(x^*, x) &= D(T(x^*), x) \leq H(T(x^*), T(x)) \\ &\leq \theta \cdot d(x^*, x) + L \cdot D(x, T(x^*)) \\ &= \theta \cdot d(x^*, x) + L \cdot d(x, x^*) = (\theta + L) \cdot d(x, x^*). \end{aligned}$$

Since $\theta + L < 1$ this is a contradiction, which proves that $Fix(T) = \{x^*\}$ and hence $Fix(T) = SFix(T) = \{x^*\}$.

Let $x^* \in SFix(T)$. Suppose $D(x_n, T(x_n)) \to 0$, as $n \to \infty$. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences such that $y_n \in T(x_n)$. Then we have

$$\begin{aligned} d(x_n, x^*) &\leq d(x_n, y_n) + d(y_n, x^*) = d(x_n, y_n) + D(y_n, T(x^*)) \\ &\leq d(x_n, y_n) + H(T(x_n), T(x^*)). \end{aligned}$$

Taking the infimum over $y_n \in T(x_n)$ we have

$$\begin{aligned} d(x_n, x^*) &\leq D(x_n, T(x_n)) + H(T(x_n), T(x^*)) \\ &\leq D(x_n, T(x_n)) + \theta d(x_n, x^*) + L D(x_n, T(x^*)) \\ &= D(x_n, T(x_n)) + \theta d(x_n, x^*) + L d(x_n, x^*). \end{aligned}$$

Thus $(1 - \theta - L) d(x_n, x) \leq D(x_n, T(x_n))$. Since $\theta + L < 1$, we have that

$$d(x_n, x^*) \leq \frac{1}{1 - \theta - L} D(x_n, T(x_n)) \to 0 \text{ as } n \to \infty.$$

□

Remark 1. *The above result give rise to the following open question: in which conditions the fixed point problem for (θ, L)-weak contractions is well-posed with respect to D, where $\theta \in]0, 1[$ and $L \geq 0$ (i.e., for $\theta + L \geq 1$, too).*

Remark 2. *It is also an open problem in the case of (θ, L)-weak contraction, in which conditions takes place the following implication*

$$SFix(T) \neq \emptyset \Rightarrow Fix(T) = SFix(T) = \{x^*\}.$$

References

[1] Berinde, M., Berinde, V., *On a general class of multi-valued weakly Picard mappings*, J. Math. Anal., **326**(2007), 772-782.

[2] Chifu, C., Petruşel, G., *Fixed point theorems for non-self operators on a set with two metrics*, Carpathian J. Math, to appear.

[3] Moţ, G., Petruşel, A., *Fixed point theory for a new type of contractive multivalued operators*, Nonlinear Anal., 2008, doi:10.1016/j.na.2008.05.005.

[4] Petruşel, A., Rus, I.A., *Well-posedness of the fixed point problem for multivalued operators*, in Applied Analysis and Differential Equations (O. Carja and I. I. Vrabie eds.), World Scientific, 2007, 295-306.

[5] Petruşel, A., Rus, I.A., *Fixed point theory for multivalued operators on a set with two metrics*, Fixed Point Theory, **8**(2007), 97-104.

[6] Petruşel, A., Rus, I.A., Yao, J.C., *Well-posedness in the generalized sense of the fixed point problems for multivalued operators*, Taiwanese J. Math., **11**(2007), 903-914.

[7] Rus, I.A., *Strict fixed point theory*, Fixed Point Theory, **4**(2003), 177-183.

Babeş-Bolyai University
Department of Applied Mathematics
Kogălniceanu 1, 400084, Cluj-Napoca, Romania
E-mail address: `filip_darius@yahoo.com, petra.petru@econ.ubbcluj.ro`

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 3, September 2009

STRONG AND CONVERSE FENCHEL DUALITY FOR VECTOR OPTIMIZATION PROBLEMS IN LOCALLY CONVEX SPACES

ANCA GRAD

Abstract. In relation to the vector optimization problem v-$\min_{x\in X}(f + g \circ A)(x)$, with f, g proper and cone-convex functions and $A : X \to Y$ a linear continuous operator between separated locally convex spaces, we define a general vector Fenchel-type dual problem. For the primal-dual pair we prove weak, and under appropriate regularity conditions, strong and converse duality. In the particular case when the image space is $\mathbb{R}^m$ we compare the new dual with two other duals, whose definitions were inspired from [9] and [10], respectively. The sets of Pareto efficient elements of the image sets of their feasible sets through the corresponding objective functions prove to be equal, despite the fact that among the image sets of the problems, strict inclusion usually holds. This equality allows us to derive weak, strong and converse duality results for the later two dual problems, from the corresponding results of the first mentioned one. Our results could be implemented in various practical areas, since they provide sufficient conditions for the existence of optimal solutions for vector optimization problems defined on very general spaces. They can be used in medical areas, for example in the study of chronical diseases and in oncology.

Received by the editors: 04.12.2008.
2000 *Mathematics Subject Classification.* 49N15, 32C37, 90C29.
Key words and phrases. conjugate functions, Fenchel dulity, vector optimization, weak, strong and converse duality.
This paper was presented at the 7-th Joint Conference on Mathematics and Computer Science, July 3-6, 2008, Cluj-Napoca, Romania.
This paper is supported in the framework of CRONIS. Project number is 11-003/2007, financed by National Programs Management Center through the 4th Programme "Partnerships in Proprietary Domains".

1. **Introduction**

Vector optimization problems have generated a great deal of interest during the last years, not only from a theoretical point of view, but also from a practical one, due to their applicability in different fields, such as economics, engineering and lately in medical areas. In general, when dealing with scalar optimization problems, the duality theory proves to be an important tool for giving dual characterizations of the optimal solutions of a primal problem. Similar characterizations can also be given for vector optimization problems.

An overview on the literature dedicated to this field shows that the general interest has been centered on vector problems having inequality constraints and on an extension of the classical Lagrange duality approach. We recall in this direction the concepts developed by Mond and Weir in [23], [24] (whose formulation is based on optimality conditions which follow from the scalar Lagrange duality). Tanino, Nakayama and Sawaragi examined in [21] the duality for vector optimization in finite dimensional spaces, using perturbations, which led them also to Lagrange-type duals. They extended Rockafellar's fully developed theory from [19] for scalar optimization to the vector case. In Jahn's paper [16] the Lagrange dual appears explicitly in the formulation of the feasible set of the multiobjective dual.

Another approach is due to Boţ and Wanka, who, in [8] constructed a vector dual using the Fenchel-Lagrange dual for scalar optimization problems, introduced by the authors in [3], [6], [7].

With respect to the vector duality based on Fenchel's duality concept, the bibliography is not very rich. We mention in this direction the works of Breckner and Kolumbán [10] and [11], continued by Breckner in [12], [13], Gerstewitz and Göpfert [15], Malivert [18] as well as the recent paper of Boţ, Dumitru (Grad) and Wanka [9].

In relation to the vector optimization problem v-$\min_{x \in X}(f + g \circ A)(x)$, with f, g proper and cone-convex functions and $A : X \to Y$ a linear continuous operator between separated locally convex spaces, we define a general vector Fenchel-type dual problem. For this dual pairs of problems we prove weak, and under appropriate, quite general regularity conditions, strong and converse duality. In the particular case when the image space is $\mathbb{R}^m$, we compare the new dual with two other duals, whose definitions were inspired from [9] and [10], respectively. Their sets of optimal solutions prove to be equal, despite the fact among the image sets of the problems, strict inclusion usually holds. This equality allows us to derive weak, strong and converse duality results for the later two dual problems, from the corresponding results of the first mentioned one.

The paper is organized as it follows. In Section 2 we recall some elements of convex analysis which are used later on. Using the formulation of the scalarized dual, we define in Section 3, the new vector dual problem. For it, we prove weak, strong and converse duality. In order to be able to understand the position of our dual, among other duals given in the literature, we present another Fenchel-type dual problems inspired by Breckner and Kolumbán's paper [11]. Weak, strong and converse duality for the later problem can be proved, using the corresponding theorems for the initial treated problems. Section 4 contains a further comparison, to a third dual problem, this time inspired from Boţ, Dumitru (Grad) and Wanka, (cf. [9]). The image sets of the three duals are tightly connected, as it is proved. Moreover, we illustrate by some examples that in general these inclusions are strict. Finally, we show that even though this happens, the sets of the maximal elements of the image sets of the feasible sets through the corresponding objective functions coincide.

The practical applicability of our results is vast, since they provide, among others, sufficient conditions for the existence of optimal solutions for a large area of optimization problems, in both finite and infinite dimensional spaces. Such results could be successfully applied in the study of chronical diseases, oncology, economy and the list could continue.

2. **Preliminaries**

Let X be a real separated locally convex space, and let X^* be its topological dual. By $\langle x^*, x\rangle$ we understand the value of the linear continuous functional $x^* \in X^*$ at $x \in X$.

Given a function $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, its *domain* is the set

$$\operatorname{dom} f := \{x \in X : f(x) < +\infty\}.$$

We call f *proper* if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. The *conjugate function* associated with f is $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) := \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\} \text{ for all } x^* \in X^*.$$

The function f is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in X$ and all $\lambda \in [0, 1]$.

Given a nonempty convex cone $C \subseteq X$, we denote by

$$C^+ := \{x^* \in X^* : \langle x^*, x\rangle \geq 0 \text{ for all } x \in C\}$$

its *dual cone* and by

$$C^{+0} := \{x^* \in X^* : \langle x^*, x\rangle > 0 \text{ for all } x \in C \setminus \{0\}\}$$

the *quasi-interior* of the dual cone. The convex cone C induces on X a partial ordering defined by $x \leqq_C y$ (denoted also by $y \geqq_C x$) if $y - x \in C$ for all $x, y \in X$. If $y - x \in C \setminus \{0\}$ we use the notation $x \leq_C y$ (denoted also by $y \geq_C x$).

There are notions referring to extended real-valued functions that can be generalized to functions taking values in infinite dimensional spaces. Thus, let Y be another real separated locally convex space partially ordered by the nonempty convex cone K. To Y we attach a greatest element ∞_Y with respect to $\leqq_K$, which does not belong to Y. Moreover, we set $Y^\bullet := Y \cup \{\infty_K\}$ and consider on $Y^\bullet$ the following operations: $y + \infty_K = \infty_K$, $t \cdot \infty_K = \infty_K$ and $\langle \lambda, \infty_K\rangle = +\infty$ for all $y \in Y$, $t \geq 0$ and $\lambda \in K^+$.

For a function $F : X \to Y^\bullet$ its *domain* is defined by

$$\operatorname{dom} F := \{x \in X : F(x) \in Y\}.$$

If $\operatorname{dom}(F) \neq \emptyset$, then F is said to be *proper*. The most common extension of the classical convexity of an extended real-valued function to a vector-valued function is the notion of cone-convexity. Thus, F is said to be $K-$ *convex* if

$$F(tx + (1-t)y) \leq_K tF(x) + (1-t)F(y)$$

for all $x, y \in X$ and all $t \in [0, 1]$.

For each $\lambda \in K^+$ we consider the function $(\lambda F) : X \to \overline{\mathbb{R}}$ defined by $(\lambda F)(x) = \langle \lambda, F(x) \rangle$ for all $x \in X$. In literature there are known several generalizations of the lower semicontinuity of extended real-valued functions to vector-valued functions. Here we mention one of them. The function F is said to be *star-K lower semicontinuous* if, for each $\lambda \in K^+$, the function (λF) is lower semicontinuous.

If U is a nonempty subset of X we denote by $\operatorname{lin} U$ its *linear hull* and by $\operatorname{cone} U := \cup_{\lambda \geq 0} \lambda U$ its *conic hull*. The *algebraic interior* associated with U is the set

$$\operatorname{core} U := \{u \in U : \forall x \in X, \exists \delta > 0 \text{ s.t. } \forall \lambda \in [0, \delta] : u + \lambda x \in U\}.$$

When U is a convex set, then $u \in \operatorname{core} U$ if and only if $\operatorname{cone}(U - x) = X$. In general, we have $\operatorname{int} U \subseteq \operatorname{core} U$, where $\operatorname{int} U$ denotes the *interior* of U. When U is convex then $\operatorname{int} U = \operatorname{core} U$ if one of the following conditions is satisfied: $\operatorname{int} U \neq \emptyset$; X is a Banach space and U is closed; X is finite dimensional (cf. [20]). Further, by maintaining the convexity assumption for U, one can define the *strong quasi-relative interior* of U, denoted by $\operatorname{sqri} U$, as

$$\operatorname{sqri} U := \{u \in U : \operatorname{cone}(U - u) \text{ is a closed linear subspace of } X\} \text{ (cf. [1]).}$$

We notice that $\operatorname{core} U \subseteq \operatorname{sqri} U$. If X is finite dimensional, then $\operatorname{sqri} U = \operatorname{ri} U$, where $\operatorname{ri} U$ denotes the *relative interior* of the set U, i.e. the set of the interior points of U relative to the affine hull of U.

3. **Fenchel-Type Vector Duality**

Let X, Y and V be real separated locally convex spaces, and let V be partially ordered by a nonempty pointed convex cone $K \subseteq V$. We shall study the general vector optimization problem

$$(P) \qquad \operatorname*{v-min}_{x \in X} (f + g \circ A)(x),$$

where $f : X \to V^\bullet = V \cup \{\infty_K\}$ and $g : Y \to V^\bullet$ are proper, K-convex functions and $A : X \to Y$ is a linear continuous operator such that $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g) \neq \emptyset$.

Due to the fact that the partial order induced on a vector space by a convex cone is not total, several notions of optimal solutions for vector optimization problems have been introduced during the years in the literature. For such definitions and their properties we refer the reader to [17]. In this paper we work with Pareto-efficient and properly efficient solutions. Particularly, for the problem (P) we study the existence of properly efficient solutions.

Definition 1. *An element $\overline{x} \in X$ is a properly efficient solution to (P) if there exists $v^* \in K^{+0}$ such that*

$$\langle v^*, (f + g \circ A)(\overline{x}) \rangle \leq \langle v^*, (f + g \circ A)(x) \rangle \text{ for all } x \in X.$$

Duality is an extremely used procedure in optimization. It consists in associating with a certain optimization problem, called primal problem, a new one, called dual problem, whose solutions may characterize the optimal solutions of the primal problem. In order to ensure strong and converse duality, respectively, certain regularity conditions have to be imposed on the functions and sets involved in the definition of the problems.

In this paper we treat three different types of dual problems associated with the vector optimization problem (P), for which which we prove weak, strong and converse duality. Furthermore, we shall compare the image sets of the feasible sets through the corresponding objective functions for the three problems.

The first dual associated with the primal vector optimization problem is

$$(D^{\leq}) \operatorname*{v-max}_{(v^*, y^*, v) \in \mathcal{B}^{\leq}} h^{\leq}(v^*, y^*, v),$$

where the feasible set is

$$\mathcal{B}^{\leq} = \{(v^*, y^*, v) \in K^{+0} \times Y^* \times \mathbb{R} : \langle v^*, v \rangle \leq -(v^* f)(-A^* y^*) - (v^* g)(y^*)\},$$

and the objective function is

$$h^{\leqq}(v^*, y^*, v) = v.$$

For this new optimization problem, we are interested in investigating the Pareto efficient solutions, defined below.

Definition 2. *An element $(\overline{v}^*, \overline{y}^*, \overline{v}) \in \mathcal{B}^{\leqq}$ is said to be an efficient (Pareto efficient) solution to $(D^{\leqq})$ if there exists no $(v^*, y^*, v) \in \mathcal{B}^{\leqq}$ such that*

$$h^{\leqq}(\overline{v}^*, \overline{y}^*, \overline{v}) \leq_K h^{\leqq}(v^*, y^*, v).$$

As stated above, for a primal-dual pair of optimization problems, weak duality must always hold, under general assumptions. This is the case for our problems, as it is proved in the following theorem.

Theorem 1 (Weak Duality for $(P)-(D^{\leqq})$)**.** *There exist no $x \in X$ and no $(v^*, y^*, v) \in \mathcal{B}^{\leqq}$ such that*

$$(f + g \circ A)(x) \leq_K h^{\leqq}(v^*, y^*, v).$$

Proof. We proceed by contradiction, assuming that there exist $\overline{x} \in X$ and $(\overline{v}^*, \overline{y}^*, \overline{v}) \in \mathcal{B}^{\leqq}$ such that $(f + g \circ A)(\overline{x}) \leq_K h^{\leqq}(\overline{v}^*, \overline{y}^*, \overline{v})$. This implies obviously that $\overline{x} \in (\operatorname{dom} f) \cap A^{-1}(\operatorname{dom} g)$. Due to the fact that $\overline{v}^* \in K^{+0}$ it follows that

$$\langle \overline{v}^*, \overline{v} \rangle > \langle \overline{v}^*, (f + g \circ A)(\overline{x}) \rangle \geq \inf_{x \in X} \left\{ \langle \overline{v}^*, f(x) \rangle + \langle \overline{v}^*, (g \circ A)(x)) \rangle \right\}.$$

Moreover, from the weak duality theorem for the scalarized optimization problem on the right hand side of the inequality above and its Fenchel dual, we have

$$\inf_{x \in X} \left\{ \langle \overline{v}^*, f(x) \rangle + \langle \overline{v}^*, (g \circ A)(x)) \rangle \right\} \geq \sup_{y^* \in Y^*} \{ -(\overline{v}^* f)^*(-A^* y^*) - (\overline{v}^* g)^*(y^*) \}.$$

Combining the relations above, we obtain

$$\langle \overline{v}^*, \overline{v} \rangle > -(\overline{v}^* f)^*(-A^* \overline{y}^*) - (\overline{v}^* g)^*(\overline{y}^*),$$

which contradicts the fact that $(\overline{v}^*, \overline{y}^*, \overline{v}) \in \mathcal{B}^{\leqq}$. Hence the conclusion of the theorem holds. $\square$

In order to ensure strong duality between the previously mentioned problems, a regularity condition has to be fulfilled. It actually ensures the existence of strong duality for the scalar optimization problem

$$(P_{\overline{v}^*})\ \inf_{x\in X}\{(\overline{v}^* f)(x) + (\overline{v}^* g)(Ax)\}$$

and its Fenchel dual problem

$$(D_{\overline{v}^*})\ \sup_{y^*\in Y^*}\{-(\overline{v}^* f)^*(-A^* y^*) - (\overline{v}^* g)^*(y^*)\}$$

for all $\overline{v}^* \in K^{+0}$. So, we are looking for sufficient conditions that are independent from the choice of $\overline{v}^* \in K^{+0}$.

The first regularity condition, which we mention at this point, is derived from [14]. In the particular case of our problem it has the following formulation:

$$(RC_1)\ \|\ \exists x_0 \in \operatorname{dom} f \cap A^{-1}(\operatorname{dom} g) \text{ such that } g \text{ is continuous at } A(x_0).$$

When $M \subset Y$ is a given set, we use the following notation:

$$A^{-1}(M) := \{x \in X : Ax \in M\}.$$

In Fréchet spaces one can state the following regularity conditions for the primal-dual pair $(P_{\overline{v}^*}) - (D_{\overline{v}^*})$:

$$(RC_2) \left\| \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces,} \\ f \text{ and } g \text{ are star-}K \text{ lower-semicontinuous,} \\ \text{ and } 0 \in \operatorname{sqri}(\operatorname{dom} g - A(\operatorname{dom} f)) \end{array} \right.$$

along with its stronger versions

$$(RC_{2'}) \left\| \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces,} \\ f \text{ and } g \text{ are star-}K \text{ lower-semicontinuous,} \\ \text{ and } 0 \in \operatorname{core}(\operatorname{dom} g - A(\operatorname{dom} f)) \end{array} \right.$$

and

$$(RC_{2''}) \left\| \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces,} \\ f \text{ and } g \text{ are star-}K \text{ lower-semicontinuous,} \\ \text{ and } 0 \in \operatorname{int}(\operatorname{dom} g - A(\operatorname{dom} f)). \end{array} \right.$$

For more details with respect to these regularity conditions we refer the reader to [2]. In the finite dimensional setting one can use the following regularity condition:

$$(RC_3) \left\| \begin{array}{c} \dim(\operatorname{lin}(\operatorname{dom} g - A(\operatorname{dom} f))) < +\infty \text{ and} \\ \operatorname{ri}(\operatorname{dom} g) \cap \operatorname{ri}(A(\operatorname{dom} f)) \neq \emptyset \end{array} \right.$$

which becomes in case $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$

$$(RC_4) \parallel \exists x' \in \operatorname{ri}(\operatorname{dom} f) \text{ s.t. } Ax' \in \operatorname{ri}(\operatorname{dom} g)\ .$$

The condition (RC_4) is the classical regularity condition for the scalar Fenchel duality in finite dimensional spaces and has been stated by Rockafellar in [19].

A newly studied approach in giving sufficient conditions for strong duality is the one employing closed cone constraint qualifications which turn out to be weaker than the interior-type ones. For such conditions and their comparison to the interior-type ones specified above, and others, we refer the reader to the paper by Boţ and Wanka [5].

Theorem 2 (Strong Duality Theorem for $(P) - (D^{\leqq})$)**.** *Assume that one of the regularity conditions $(RC_1) - (RC_3)$ is satisfied. If $\overline{x} \in X$ is a properly efficient solution to (P), then there exists an efficient solution $(\overline{v}^*, \overline{y}^*, \overline{v}) \in \mathcal{B}^{\leqq}$ to $(D^{\leqq})$ such that $(f + g \circ A)(\overline{x}) = h^{\leqq}(\overline{v}^*, \overline{y}^*, \overline{v}) = \overline{v}$.*

Proof. Due to the fact that $\overline{x}$ is a properly efficient solution to $(D^{\leqq})$ we obtain that $\overline{x} \in \operatorname{dom}(f) \cap A^{-1}(\operatorname{dom} g)$ and that there exists a $\overline{v}^* \in K^{+0}$ such that

$$\langle \overline{v}^*, (f + g \circ A)(\overline{x}) \rangle = \inf_{x \in X} \{ (\overline{v}^* f)(x) + (\overline{v}^* g)(Ax) \}.$$

The functions $(\overline{v}^* f)$ and $(\overline{v}^* g)$ are proper and convex. The regularity assumption guarantees the existence of strong duality for the scalarized optimization problem

$\inf_{x\in X}\{(\overline{v}^* f)(x) + (\overline{v}^* g)(Ax)\}$ and its Fenchel dual. Thus there exists $\overline{y}^* \in Y^*$ such that

$$\inf_{x\in X}\{(\overline{v}^* f)(x) + (\overline{v}^* g)(Ax)\} = \sup_{y^*\in Y^*}\{-(\overline{v}^* f)^*(-A^* y^*) - (\overline{v}^* g)^*(y^*)\} =$$
$$= -(\overline{v}^* f)^*(-A^* \overline{y}^*) - (\overline{v}^* g)^*(\overline{y}^*).$$

By defining $\overline{v} := (f + g \circ A)(\overline{x}) \in V$ we obtain that $(\overline{v}^*, \overline{y}^*, \overline{v}) \in \mathcal{B}^{\leq}$. Now we prove that it is an efficient solution to $(D^{\leq})$.

Let us assume by contradiction that this is not the case. This implies the existence of $(v^*, y^*, v) \in \mathcal{B}^{\leq}$ such that $\overline{v} = (f + g \circ A)(\overline{x}) \leq_K v = h^{\leq}(v^*, y^*, v)$ which is a contradiction to the weak duality theorem, Theorem 1. $\square$

The forthcoming result plays a crucial role in proving the converse duality theorem.

Theorem 3. *Assume that one of the regularity conditions* $(RC_1) - (RC_3)$ *is satisfied and that* $B^{\leq} \neq \emptyset$. *Then*

$$V \setminus \operatorname{cl}\left\{(f + g \circ A)\left(\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)\right) + K\right\} \subseteq \operatorname{core} h^{\leq}(B^{\leq}).$$

Proof. Let $\eta \in V \setminus \operatorname{cl}\left\{(f + g \circ A)\left((\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)\right) + K\right\}$ be arbitrarily chosen. Due to the fact that f and g are K-convex functions, A is a linear continuous operator and K is a convex cone, we see that the set

$$(f + g \circ A)\left(\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)\right) + K$$

is convex, thus $\operatorname{cl}\left\{(f + g \circ A)\left(\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g) + K\right)\right\}$ is a closed and convex set. According to a separation theorem (see [25]), we obtain the existence of $\eta^* \in V^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\langle \eta^*, \eta \rangle < \alpha < \langle \eta^*, b \rangle, \forall b \in \operatorname{cl}\left\{(f + g \circ A)\left(\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)\right) + K\right\}. \qquad (1)$$

We prove that $\eta^* \in K^+ \setminus \{0\}$. Let us suppose by contradiction that there exists a $k \in K$ such that $\langle \eta^*, k \rangle < 0$. This means that for a fixed $x_0 \in \operatorname{dom}(f) \cap A^{-1}(\operatorname{dom} g)$ the inequality

$$\alpha < \langle \eta^*, (f + g \circ A)(x_0) \rangle + \langle \eta^*, tk \rangle$$

holds for all $t \geq 0$. Allowing now $t \to +\infty$, we obtain that $\alpha < -\infty$, which is obviously a contradiction. Therefore, $\eta^* \in K^+ \setminus \{0\}$.

Due to the fact that $\mathcal{B}^{\leq} \neq \emptyset$, there exists $(v^*, y^*, v) \in \mathcal{B}^{\leq}$, hence

$$\langle v^*, v\rangle \leq -(v^*f)^*(-A^*y^*) - (v^*g)^*(y^*).$$

Applying the weak duality theorem for the scalarized problem

$$(P_{v^*}) \inf_{x\in X} \langle v^*, (f + g \circ A)(x)\rangle$$

and its Fenchel dual, we have that

$$-(v^*f)^*(-A^*y^*) - (v^*g)^*(y^*) \leq \inf_{x\in X} \langle v^*, (f + g \circ A)(x)\rangle,$$

and hence

$$\langle v^*, v\rangle \leq \inf_{x\in X} \langle v^*, (f + g \circ A)(x)\rangle. \tag{2}$$

For each $s \in (0, 1)$ we have that

$$\langle sv^* + (1 - s)\eta^*, \eta\rangle = \langle \eta^*, \eta\rangle + s(\langle v^*, \eta\rangle - \langle \eta^*, \eta\rangle) = \alpha - \gamma + s(\langle v^*, v\rangle - \alpha + \gamma) \tag{3}$$

with $\gamma := \alpha - \langle \eta^*, \eta\rangle > 0$. Furthermore, from (1) and (2) we obtain

$$\langle sv^* + (1 - s)\eta^*, b\rangle \geq s\langle v^*, v\rangle + (1 - s)\alpha = \alpha + s(\langle v^*, v\rangle - \alpha) \tag{4}$$

for all $b \in (f + g \circ A)(\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g))$. Thus there exists $\overline{s} \in (0, 1)$, close enough to 0, such that $\overline{s}(\langle v^*, v\rangle - \alpha + \gamma) < \frac{1}{2}\gamma$ and $\overline{s}(\langle v^*, v\rangle - \alpha) > -\frac{1}{2}\gamma$. For the convex combination obtained with the help of $\overline{s}$ it holds

$$v^*_{\overline{s}} := sv^* + (1 - s)\eta^* \in sK^{+0} + (1 - s)(K^+ \setminus \{0\}) \subseteq K^{+0} + K^+ \subseteq K^{+0}.$$

Thus, using (3) and (4), we obtain

$$\langle v^*_{\overline{s}}, \eta\rangle < \alpha - \frac{1}{2}\gamma < \langle v^*_{\overline{s}}, b\rangle, \forall b \in (f + g \circ A)(\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)).$$

From the hypothesis we know that one of the regularity conditions holds. Thus, from the strong duality for the scalar optimization problems $(P_{v^*}) - (D_{v^*})$ there exists an optimal solution y^* of the dual (D_{v^*}), therefore

$$\begin{aligned}\langle v_{\bar{s}}^*, \eta\rangle < \inf_{x\in X}\langle v_{\bar{s}}^*, (f+g\circ A)(x)\rangle &= \sup_{z^*\in Y^*}\{-(v_{\bar{s}}^* f)^*(-A^*z^*) - (v_{\bar{s}}^* g)^*(z^*)\} \\ &= -(v_{\bar{s}}^* f)^*(-A^*y^*) - (v_{\bar{s}}^* g)^*(y^*).\end{aligned}$$

This means that there exists $\varepsilon > 0$ such that

$$\langle v_{\bar{s}}^*, \eta\rangle + \varepsilon < -(v_{\bar{s}}^* f)^*(-A^*y^*) - (v_{\bar{s}}^* g)^*(y^*).$$

For all $p \in V$ there exists $\delta_p > 0$ such that $\langle v_{\bar{s}}^*, \delta_p p\rangle < \varepsilon$, and thus

$$\langle v_{\bar{s}}^*, \eta + \lambda p\rangle \leq \langle v_{\bar{s}}^*, \eta\rangle + \varepsilon < \{-(v_{\bar{s}}^* f)^*(-A^*y^*) - (v_{\bar{s}}^* g)^*(y^*)\}, \forall\lambda \in [0, \delta_p].$$

Hence $(v^*, y^*, \eta + \lambda p) \in \mathcal{B}^{\leq}$ for all $\lambda \in [0, \delta_p]$, and further $\eta + \lambda p \in h^{\leq}(\mathcal{B}^{\leq})$, guaranteeing that $\eta \in \operatorname{core}(\mathcal{B}^{\leq})$. $\square$

Theorem 4 (Converse Duality Theorem for $(P) - (D^{\leq})$)**.** *Assume that one of the regularity conditions $(RC_1) - (RC_3)$ is satisfied and the set*

$$(f + g\circ A)(\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)) + K$$

is closed. Then for each efficient solution $(\overline{v}^*, \overline{y}^*, \overline{v}) \in \mathcal{B}^{\leq}$ to $(D^{\leq})$ there exists a properly efficient solution $\overline{x} \in X$ to (P), such that

$$(f + g\circ A)(\overline{x}) = h^{\leq}(\overline{v}^*, \overline{y}^*, \overline{v}) = \overline{v}.$$

Proof. First we show that $\overline{v} \in (f + g\circ A)(\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)) + K$. Let us proceed by contradiction. This would mean, by using Theorem 3, that $\overline{v} \in \operatorname{core} h^{\leq}(\mathcal{B}^{\leq})$. Thus for a $k \in K\setminus\{0\}$ there exists $\lambda > 0$ such that $\overline{v}_\lambda := \overline{v} + \lambda k \in h^{\leq}(B^{\leq})$. Furthermore, $\overline{v}_\lambda - \overline{v} = \lambda k \in K\setminus\{0\}$ and hence $\overline{v}_\lambda \geq_K \overline{v}$, a contradiction to the efficiency of $\overline{v} \in h^{\leq}(\mathcal{B}^{\leq})$.

Thus $\overline{v} \in (f + g\circ A)(\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)) + K$. But this means that there exist $\overline{x} \in (\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)$ and $\overline{k} \in K$ such that

$$\overline{v} = (f + g\circ A)(\overline{x}) + \overline{k}.$$

By assuming that $\overline{k} \neq 0$ we would obtain that $h^{\leqq}(\overline{v}^*, \overline{y}^*, \overline{v}) = \overline{v} \geq_K (f + g \circ A)(\overline{x})$, a contradiction to the weak duality statement of Theorem 1. Hence $\overline{k} = 0$ and thus $\overline{v} = (f + g \circ A)(\overline{x})$. Employing now the definition of $\mathcal{B}^{\leqq}$ and the weak duality theorem which holds for the scalarized optimization problem $(P_{\overline{v}^*})$, we obtain

$$\begin{aligned}\langle \overline{v}^*, (f + g \circ A)(\overline{x}) \rangle = \langle \overline{v}^*, \overline{v} \rangle &\leq -(v^* f)^*(-A^* y^*) - (v^* g)^*(y^*) \\ &\leq \inf_{x \in X} \langle \overline{v}^*, (f + g \circ A)(x) \rangle.\end{aligned}$$

Therefore, $\overline{x}$ is a properly efficient solution to (P). $\square$

The scalar Fenchel duality was involved for the first time in the definition of a vector dual problem by Breckner and Kolumbán, in [10], in a very general framework. Inspired by the approach introduced in this work, one gets the following dual vector optimization problem associated with (P)

$$(D^{BK}) \underset{(v^*, y^*, v) \in \mathcal{B}^{BK}}{\text{v-max}} h^{BK}(v^*, y^*, v),$$

where

$$\mathcal{B}^{BK} := \{(v^*, y^*, v) \in K^{+0} \times Y^* \times V : \langle v^*, v \rangle = -(v^* f)^*(-A^* y^*) - (v^* g)^(y^*)\}$$

and

$$h^{BK}(v^*, y^*, v) = v.$$

Remark 1. *As it can be easily observed from the definition, without any other additional assumptions, the following inclusion holds:*

$$h^{BK}(\mathcal{B}^{BK}) \subseteq h^{\leqq}(\mathcal{B}^{\leqq}).$$

Theorem 5. *The following equality holds:*

$$\text{v-max}\, h^{BK}(\mathcal{B}^{BK}) = \text{v-max}\, h^{\leqq}(\mathcal{B}^{\leqq}).$$

Proof. " $\subseteq$ " Let $(v^*, y^*, v) \in \mathcal{B}^{BK}$ be such that $v \in \text{v-max}(h^{BK}(\mathcal{B}^{BK}))$. Then $v \in h^{\leqq}(\mathcal{B}^{\leqq})$. We suppose that $v \notin \text{v-max}(h^{\leqq}(\mathcal{B}^{\leqq}))$. This means that there exists

$(v_0^*, y_0^*, v_0) \in \mathcal{B}^{\leq}$ such that $v_0 \geq_K v$. Due to the maximality of v in $h^{BK}(\mathcal{B}^{BK})$ we have that $(v_0^*, y_0^*, v_0) \notin \mathcal{B}^{BK}$, therefore

$$\langle v_0^*, v_0 \rangle < -(v_0^* f)^*(-A^* y_0^*) - (v_0^* g)^*(y_0^*).$$

Consequently there exists a $k \in K \setminus \{0\}$ and $v_k := v_0 + k$ such that

$$\langle v_0^*, v_k \rangle = -(v_0^* f)^*(-A^* y_0^*) - (v_0^* g)^*(y_0^*),$$

which means that $(v_0^*, y_0^*, v_k) \in \mathcal{B}^{BK}$ and $v_k \geq_K v_0$. Since this is a contradiction to the maximality of v_0, $\text{v-max}\, h^{BK}(\mathcal{B}^{BK}) \subseteq \text{v-max}\, h^{\leq}(\mathcal{B}^{\leq})$.

" $\supseteq$ " By taking $(v^*, y^*, v) \in \mathcal{B}^{\leq}$ such that $v \in \text{v-max}\, h^{\leq}(\mathcal{B}^{\leq})$ we prove that it belongs to $\text{v-max}\, h^{BK}(\mathcal{B}^{BK})$. The first step is to prove that $(v^*, y^*, v) \in \mathcal{B}^{BK}$. Assuming the contrary, one has

$$\langle v^*, v \rangle < -(v^* f)^*(-A^* y^*) - (v^* g)^*(y^*)$$

and there exists $k \in K \setminus \{0\}$ such that $v_k := v + k$ satisfies

$$\langle v^*, v \rangle < \langle v^*, v_k \rangle = -(v^* f)^*(-A^* y^*) - (v^* g)^*(y^*).$$

Since $(v^*, y^*, v) \in \mathcal{B}^{\leq}$ and $v_k \geq_K v$, we have a contradiction to the maximality of v. Hence $(v^*, y^*, v) \in \mathcal{B}^{BK}$.

We further suppose that $v \notin \text{v-max}\, h^{BK}(\mathcal{B}^{BK})$. This means actually that there exists $(\overline{v}^*, \overline{y}^*, \overline{v}) \in \mathcal{B}^{BK} \subseteq \mathcal{B}^{\leq}$ such that $\overline{v} \geq_K v$, which is actually a contradiction to the maximality of $v \in h^{\leq}(\mathcal{B}^{\leq})$. Therefore, $v \in \text{v-max}\, h^{BK}(\mathcal{B}^{BK})$. $\square$

Remark 2. *In the proof of the previous theorem, no assumptions regarding the nature of the functions and sets involved in the formulation of* (P) *were made. This means that the sets of efficient elements of* $h^{\leq}(\mathcal{B}^{\leq})$ *and* $h^{BK}(\mathcal{B}^{BK})$ *are always identical.*

Using the weak, strong and converse duality theorems between the dual pair of vector optimization problems (P) and $(D^{\leq})$, similar results can be proved for the primal-dual pair $(P) - (D^{BK})$.

Theorem 6. *The following statements are true:*

a) *(Weak duality) There exist no* $x \in X$ *and no* $(v^*, y^*, v) \in \mathcal{B}^{BK}$ *such that* $(f + g \circ A)(x) \leq_K h^{BK}(v^*, y^*, v)$.

b) *(Strong duality) Let one of the regularity conditions* $(RC_1) - (RC_3)$ *be satisfied. If* $\overline{x} \in X$ *is a properly efficient solution to* (P)*, then there exists an efficient solution* $(\overline{v}^*, \overline{y}^*, \overline{v}) \in \mathcal{B}^{BK}$ *to* (D^{BK}) *such that*

$$(f + g \circ A)(\overline{x}) = h^{BK}(\overline{v}^*, \overline{y}^*, \overline{v}) = \overline{v}.$$

c) *(Converse duality) Let one of the regularity conditions* $(RC_1) - (RC_3)$ *be satisfied and let* $(f + g \circ A)(\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)) + K$ *be a closed set. Then for each efficient solution* $(\overline{v}^*, \overline{y}^*, \overline{v}) \in \mathcal{B}^{BK}$ *to* (D^{BK})*, there exists a properly efficient solution* $\overline{x} \in X$ *to* (P)*, such that*

$$(f + g \circ A)(\overline{x}) = h^{BK}(\overline{v}^*, \overline{y}^*, \overline{v}) = \overline{v}.$$

Proof. a) It follows from Remark 1 and Theorem 1.

b) It follows from Theorem 5 and Theorem 2.

c) It follows from Theorem 5 and Theorem 4. □

When $V := \mathbb{R}$ and $K := \mathbb{R}_+$, one can identify $V^\bullet$ with $\mathbb{R} \cup \{+\infty\}$. Assuming that $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ are proper and convex functions, the primal problem becomes

$$(P) \inf_{x \in X} (f + g \circ A)(x).$$

An element $(v^*, y^*, v) \in \mathcal{B}^{\leq}$ if and only if $v^* > 0$, $y^* \in Y^*$ and $v \in \mathbb{R}$ fulfill

$$v^* v \leq -(v^* f)^*(-A^* y^*) - (v^* g)(y^*).$$

Using the characterization of the conjugate functions, we get that

$$(v^* f)^*(-A^* y^*) = v^* f^*(-\frac{1}{v^*} A^* y^*) \text{ and } (v^* g)^*(y^*) = v^* g^*(\frac{1}{v^*} y^*).$$

Thus

$$v^* v \leq -v^* f^*(-\frac{1}{v^*} A^* y^*) - v^* g^*(\frac{1}{v^*} y^*) \Longleftrightarrow v \leq f^*(-\frac{1}{v^*} A^* y^*) - g^*(\frac{1}{v^*} y^*).$$

The dual problem becomes

$$(D_{\leq})\sup_{v^*>0,y^*\in Y^*}\left\{-f^*(-\frac{1}{v^*}A^*y^*)-g^*(\frac{1}{v^*}y^*)\right\}=\sup_{y^*\in Y^*}\{f^*(-A^*y^*)-g^*(y^*)\}$$

which is exactly the classical scalar Fenchel dual problem to (PV). The same conclusion applies when particularizing in an analogous manner the vector dual problem (DV_{BK}).

4. **The case when** $V := \mathbb{R}^m$

In this section we focus our attention on the special case when $V := \mathbb{R}^m$ and $K := \mathbb{R}^m_+$. In addition to the two dual problems studied before, we introduce a new one, whose formulation was inspired from [9]. Nevertheless, a more particular case was treated there, namely the one when $X := \mathbb{R}^n$ and $Y := \mathbb{R}^k$.

The primal problem turns into

$$(P)\ \underset{x\in X}{\text{v-min}}(f(x)+(g\circ A)(x)),$$

where f and g are two vector functions such that

$$f=(f_1,f_2,...f_m)^T \quad \text{and} \quad g=(g_1,g_2,...,g_m)^T$$

with $f_i : X\to\overline{\mathbb{R}}$, $g_i : Y\to\overline{\mathbb{R}}$ proper and convex functions for each $i\in\{1,...,m\}$, and $A : X\to Y$ is a linear continuous operator.

Furthermore, we assume that the following regularity condition is satisfied

$$(RC_m)\left\|\begin{array}{l}\exists x'\in\bigcap\limits_{i=1}^{m}\operatorname{dom}f_i\cap A^{-1}\left(\bigcap\limits_{i=1}^{m}\operatorname{dom}g_i\right)\text{ such that}\\ f_i\text{ and }g_i\text{ are continuous at }x'\text{ for all }i\in\{1,...,m\}.\end{array}\right.$$

We consider the following dual optimization problem associated with (P):

$$(D^{BGW})\qquad \underset{(p,q,\lambda,t)\in B^{BGW}}{\text{v-max}}\ h^{BGW}(p,q,\lambda,t),$$

where

$$\mathcal{B}^{\mathcal{BGW}} = \left\{ \begin{array}{ll} (p,q,\lambda,t): & p=(p_1,...,p_m)\in (X^*)^m, \\ & q=(q_1,...,q_m)\in (Y^*)^m \\ & \lambda=(\lambda_1,...,\lambda_m)\in \operatorname{int}\mathbb{R}^m_+, \\ & t=(t_1,...,t_m)\in \mathbb{R}^m, \\ & \sum\limits_{i=1}^m \lambda_i\left(p_i+A^*q_i\right)=0, \sum\limits_{i=1}^m \lambda_i t_i = 0 \end{array} \right\},$$

and h is defined by

$$h\left(p,q,\lambda,t\right)=\left(h_1\left(p,q,\lambda,t\right),...,h_m\left(p,q,\lambda,t\right)\right),$$

with

$$h_i\left(p,q,\lambda,t\right)=-f_i^*\left(p_i\right)-g_i^*\left(q_i\right)+t_i \text{ for all } i\in\{1,...,m\}.$$

Proposition 7. *The following relations referring to the image sets of the three dual problems hold:*

a) $h^{BK}\left(\mathcal{B}^{BK}\right)\subseteq h^{BGW}\left(\mathcal{B}^{BGW}\right)\cap\mathbb{R}^m$;

b) $h^{BGW}\left(\mathcal{B}^{BGW}\right)\cap\mathbb{R}^m\subseteq h^{\leq}\left(\mathcal{B}^{\leq}\right)$.

Proof. a) Let $v\in h^{BK}\left(\mathcal{B}^{BK}\right)$. Then there exist $v^*\in\operatorname{int}\mathbb{R}^m_+$ and $y^*\in Y^*$ such that $(v^*,y^*,v)\in\mathcal{B}^{BK}$. Furthermore,

$$\sum_{i=1}^m v_i^*v_i=-\left(\sum_{i=1}^m v_i^*f_i\right)^*\left(-A^*y^*\right)-\left(\sum_{i=1}^m v_i^*g_i\right)^*\left(y^*\right).$$

Since (RC_m) is fulfilled, we can apply the infimal convolution formula and obtain the existence of $p_i\in X^*, q_i\in Y^*$, $i\in\{1,...,m\}$, such that

$$\sum_{i=1}^m v_i^*p_i=-A^*y^*,\sum_{i=1}^m v_i^*q_i=y^*,$$

$$\left(\sum_{i=1}^m v_i^*f_i\right)^*\left(-A^*y^*\right)=\sum_{i=1}^m v_i^*f_i^*\left(p_i\right) \text{ and } \left(\sum_{i=1}^m v_i^*g_i\right)^*\left(y^*\right)=\sum_{i=1}^m v_i^*g_i^*\left(q_i\right).$$

Moreover, $\sum\limits_{i=1}^m v_i^*\left(p_i+A^*q_i\right)=0$. For more details on the the infimal convolution formula and on the regularity conditions that ensure the equalities above we refer the reader to [19] and [4]. Returning to our problem, we have that

$$\sum_{i=1}^{m} v_i^* v_i = -\sum_{i=1}^{m} v_i^* f_i^* (p_i) - \sum_{i=1}^{m} v_i^* g_i^* (q_i) .$$

For

$$t_i := v_i + f_i^*(p_i) + g_i^*(q_i) \quad \forall i \in \{1, ..., m\},$$

it holds

$$\sum_{i=1}^{m} v_i^* t_i = \sum_{i=1}^{m} v_i^* v_i + \sum_{i=1}^{m} v_i^* f_i^* (p_i) + \sum_{i=1}^{m} v_i^* g_i^* (q_i) = 0.$$

Then $(p, q, v^*, t) \in \mathcal{B}^{BGW}$ and for all $i \in \{1, ..., m\}$, $h_i (p, q, v^*, t) = v_i$, thus $v = h (p, q, v^*, t) \in h \left(\mathcal{B}^{BGW}\right) \cap \mathbb{R}^m$. Hence

$$h^{BK} \left(\mathcal{B}^{BK}\right) \subseteq h^{BGW} \left(\mathcal{B}^{BGW}\right) \cap \mathbb{R}^m.$$

b) Let $(p, q, \lambda, t) \in \mathcal{B}^{BGW}$ be such that $h(p, q, \lambda, t) \in h(\mathcal{B}^{BGW}) \cap \mathbb{R}^m$. For $y^* := \sum\limits_{i=1}^{m} \lambda_i q_i$ and $v := h^{BGW} (p, q, \lambda, t)$ we have

$$\begin{aligned} \sum_{i=1}^{m} \lambda_i v_i &= \sum_{i=1}^{m} \lambda_i h_i (p, q, \lambda, t) = \sum_{i=1}^{m} \lambda_i \Big(- f_i^* (p_i) - g_i^* (q_i) + t_i \Big) \\ &= \sum_{i=1}^{m} \lambda_i \left(-f_i^* (p_i) - g_i^* (q_i)\right) \\ &\leq \sup \left\{ -\sum_{i=1}^{m} \lambda_i f_i^* (p_i) : \sum_{i=1}^{m} \lambda_i p_i = -A^* y^* \right\} \\ &\quad + \sup \left\{ -\sum_{i=1}^{m} \lambda_i g_i^* (q_i) : \sum_{i=1}^{m} \lambda_i q_i = y^* \right\} \\ &\leq -\left(\sum_{i=1}^{m} \lambda_i f_i \right)^* (-A^* y^*) - \left(\sum_{i=1}^{m} \lambda_i g_i \right)^* (y^*) . \end{aligned}$$

Hence $(\lambda, y^*, v) \in \mathcal{B}^{\leq}$ and $h^{BGW} (p, q, \lambda, t) = v \in h^{\leq} \left(\mathcal{B}^{\leq}\right)$. Thus

$$h^{BGW} \left(\mathcal{B}^{BGW}\right) \cap \mathbb{R}^m \subseteq h^{\leq} \left(\mathcal{B}^{\leq}\right).$$

$\square$

In the following we give some examples which prove that the inclusions in Proposition 7 are in general strict, i.e.

$$h^{BK}\left(\mathcal{B}^{BK}\right) \underset{\neq}{\subseteq} h^{BGW}\left(\mathcal{B}^{BGW}\right) \cap \mathbb{R}^m \underset{\neq}{\subseteq} h^{\leq}\left(\mathcal{B}^{\leq}\right).$$

Example 8. *Consider $X = Y = \mathbb{R}$, $A(x) = x$ for all $x \in \mathbb{R}$, and the functions $f, g : \mathbb{R} \to \mathbb{R}^2$ given by*

$$f(x) = (x-1, -x-1)^T \quad \textit{and} \quad g(x) = (x, -x)^T \textit{ for all } x \in \mathbb{R}.$$

We prove that $h^{BGW}\left(\mathcal{B}^{BGW}\right) \cap \mathbb{R}^m \underset{\neq}{\subseteq} h^{\leq}\left(\mathcal{B}^{\leq}\right)$.

Since

$$f_1^*(p) = \begin{cases} 1, & \text{if } p = 1, \\ +\infty, & \text{otherwise,} \end{cases}, \quad f_2^*(p) = \begin{cases} 1, & \text{if } p = -1, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$g_1^*(p) = \begin{cases} 0, & \text{if } p = 1, \\ +\infty, & \text{otherwise,} \end{cases}, \quad g_2^*(p) = \begin{cases} 0, & \text{if } p = -1, \\ +\infty, & \text{otherwise,} \end{cases}$$

one has

$$(f_1 + f_2)^*(p) = \inf\left\{f_1^*(p_1) + f_2^*(p_2) : p_1 + p_2 = p\right\} = \begin{cases} 2, & \text{if } p = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$(g_1 + g_2)^*(p) = \inf\left\{g_1^*(p_1) + g_2^*(p_2) : p_1 + p_2 = p\right\} = \begin{cases} 0, & \text{if } p = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

For $\lambda = (1,1)^T$, $p = 0$ and $d = (-2,-2)^T$ we have

$$(\lambda, p, d) \in \mathcal{B}^{\leq} \text{ and } d \in h^{\leq}\left(\mathcal{B}^{\leq}\right)$$

due to the fact that

$$\lambda^T d = -2 - 2 = -4 < -2 = -(f_1 + f_2)^*(p) - (g_1 + g_2)^*(p).$$

Next we show that $d \notin h^{BGW}\left(\mathcal{B}^{BGW}\right)$. Let us suppose by contradiction that there exists $(p', q', \lambda', t') \in \mathcal{B}^{BGW}$ such that $h^{BGW}(p', q', \lambda', t') = d$. This means

$$-f_i^*(p_i') - g_i^*(q_i') + t_i' = 0 \quad \text{for } i \in \{1, 2\}.$$

Taking into account the values we got for the conjugate of the functions involved, the equalities above hold only if

$$p_1' = 1,\ p_2' = -1,\ q_1' = 1 \text{ and } q_2' = -1.$$

In this case, $\sum\limits_{i=1}^{2} \lambda_i' \left(p_i' + q_i'\right) = 0$, which meas that with this choice, we are still within the set $\mathcal{B}^{BGW}$. We obtain thus

$$-1 + t_i' = 0 \text{ for } i \in \{1, 2\}\,,\ \text{meaning that } t_1' = t_2' = 1.$$

Since we have supposed that $(p', q', \lambda', t') \in \mathcal{B}^{BGW}$, $\lambda_1' + \lambda_2' = 0$ must hold. This is a contradiction due to the fact that $\lambda' \in \operatorname{int} \mathbb{R}^2_+$.

Thus, for $d = (-2, -2)^T \in h^{\leqq}\left(\mathcal{B}^{\leqq}\right)$, there exists no $(p', q', \lambda', t') \in \mathcal{B}^{BGW}$ such that $h^{BGW}(p', q', \lambda', t') = d$, which shows that $h\left(\mathcal{B}^{BGW}\right) \cap \mathbb{R}^m \underset{\neq}{\subset} h^{\leqq}\left(\mathcal{B}^{\leqq}\right)$.

Example 9. *Consider $X = Y = \mathbb{R}$, $A(x) = x$ for all $x \in X$, and the functions $f, g : \mathbb{R} \to \mathbb{R}^2$ given by*

$$f(x) = \left(2x^2 - 1, x^2\right)^T \quad \textit{and} \quad g(x) = (-2x, -x + 1)^T \textit{ for all } x \in \mathbb{R}.$$

We prove that $h^{BK}\left(\mathcal{B}^{BK}\right) \underset{\neq}{\subset} h^{BGW}\left(\mathcal{B}^{BGW}\right) \cap \mathbb{R}^m$.

For $p = (3, 0)\,, q = (-2, -1)\,,$ we have $\lambda = (1, 1)^T\,,\ t = \left(\frac{3}{8}, -\frac{3}{8}\right)^T$, $\sum\limits_{i=1}^{2} \lambda_i (p_i + q_i) = 0$, and $\sum\limits_{i=1}^{2} \lambda_i t_i = 0$. Thus $(p, q, \lambda, t) \in \mathcal{B}^{BGW}$. Applying the definition of the conjugate function, we calculate the following values:

$$f_1^*(3) = \sup_{x \in \mathbb{R}} \left\{3x - 2x^2 + 1\right\} = \frac{17}{8},\ f_2^*(0) = \sup_{x \in \mathbb{R}} \left\{-x^2\right\} = 0,$$

$$g_1^*(-2) = \sup_{x \in \mathbb{R}} \{-2x + 2x\} = 0,\ g_2^*(-1) = \sup_{x \in \mathbb{R}} \{-x + x - 1\} = -1.$$

Hence

$$h_1^{BGW}(p, q, \lambda, t) = -\frac{17}{8} - 0 + \frac{3}{8} = -\frac{14}{8},\ h_2^{BGW}(p, q, \lambda, t) = 0 + 1 - \frac{3}{8} = \frac{5}{8}.$$

Now suppose that there exists $(\lambda', p', d') \in \mathcal{B}^{BK}$ such that

$$d' = h^{BGW}(p, q, \lambda, t) = \left(-\frac{14}{8}, \frac{5}{8}\right)^T.$$

Then

$$\lambda'^T d' = -\left(\sum_{i=1}^{2} \lambda'_i f_i\right)^* (p') - \left(\sum_{i=1}^{2} \lambda'_i g_i\right)^* (-p'). \tag{5}$$

But calculating the values of the conjugate functions we reach the conclusion that

$$-\left(\sum_{i=1}^{2} \lambda'_i f_i\right)^* (p') - \left(\sum_{i=1}^{2} \lambda'_i g_i\right)^* (-p') =$$

$$= \inf_{x\in\mathbb{R}} \left\{-p'x + x^2 (2\lambda'_1 + \lambda'_2) - \lambda_1\right\} + \inf_{x\in\mathbb{R}} \left\{x (p' - 2\lambda'_1 - \lambda'_2) + \lambda'_2\right\}$$

$$= \inf_{x\in\mathbb{R}} \left\{-(2\lambda'_1 + \lambda'_2) x + x^2 (2\lambda'_1 + \lambda'_2)\right\} - \lambda'_1 + \lambda'_2$$
$$= -\tfrac{2\lambda'_1 + \lambda'_2}{4} - \lambda'_1 + \lambda'_2.$$

By (3) we obtain that

$$-\frac{14}{8}\lambda'_1 + \frac{5}{8}\lambda'_2 = -\frac{2\lambda'_1 + \lambda'_2}{4} - \lambda'_1 + \lambda'_2$$

which is equivalent to

$$-\frac{3(2\lambda'_1 + \lambda'_2)}{8} = -\frac{2\lambda'_1 + \lambda'_2}{4}, \text{ i.e. } 2\lambda'_1 + \lambda'_2 = 0,$$

obviously a contradiction to $\lambda' \in \operatorname{int} \mathbb{R}^2_+$. Therefore, for (p, q, λ, t) chosen as in the beginning of the example, there exists no $(\lambda', p', d') \in \mathcal{B}^{BK}$ such that $d' = h^{BGW}(p, q, \lambda, t)$. Hence $h^{BK}(\mathcal{B}_{BK}) \cap \mathbb{R}^m \underset{\neq}{\subset} h^{BGW}(\mathcal{B}^{BGW})$.

Below we prove that the sets of optimal solutions to (D^{BGW}) and $(D^{\leq})$ coincide.

Theorem 10. *The following equality holds:*

$$\text{v-max}\, h^{BGW}(\mathcal{B}^{BGW}) = \text{v-max}\, h^{\leq}(\mathcal{B}^{\leq}).$$

Proof. "v-max $h^{BGW}(\mathcal{B}^{BGW}) \subseteq$ v-max $h^{\leq}(\mathcal{B}^{\leq})$" Let $\overline{v} \in$ v-max $h^{BGW}(\mathcal{B}^{BGW})$. Since $h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^m \subseteq h^{\leq}(\mathcal{B}^{\leq})$, one has $\overline{v} \in h^{\leq}(\mathcal{B}^{\leq})$. Let us suppose by contradiction, that $\overline{v} \notin$ v-max $h^{\leq}(\mathcal{B}^{\leq})$. Then there exists $v \in h^{\leq}(\mathcal{B}^{\leq})$, with $(v^*, y^*, v) \in \mathcal{B}^{\leq}$, such that $\overline{v} \leq_{\mathbb{R}^m_+} v$. Then we have

$$\langle v^*, \overline{v}\rangle < \langle v^*, v\rangle \leq -(v^* f)^* (-A^* y^*) - (v^* g)^* (y^*).$$

So, there exists $\widetilde{v}$ such that $v \leq_{\mathbb{R}^m_+} \widetilde{v}$ (obviously, $\overline{v} \leq_{\mathbb{R}^m_+} \widetilde{v}$) for which

$$\langle v^*, \widetilde{v}\rangle = -\left(v^* f\right)^* \left(-A^* y^*\right) - \left(v^* g\right)^* \left(y^*\right).$$

Thus we have obtained an element $(v^*, y^*, \widetilde{v}) \in \mathcal{B}^{BK}$. Since

$$h^{BK}\left(\mathcal{B}^{BK}\right) \subseteq h^{BGW}\left(\mathcal{B}^{BGW}\right) \cap \mathbb{R}^m,$$

it follows that $\widetilde{v} \in h^{BGW}\left(\mathcal{B}^{BGW}\right)$, which contradicts the maximality of $\overline{v}$ in $h^{BGW}(\mathcal{B}^{BGW})$. Therefore,

$$\text{v-max}\, h^{BGW}\left(\mathcal{B}^{BGW}\right) \subseteq \text{v-max}\, h^{\leq}\left(\mathcal{B}^{\leq}\right).$$

"$\text{v-max}\, h^{\leq}\left(\mathcal{B}^{\leq}\right) \subseteq \text{v-max}\, h^{BGW}\left(\mathcal{B}^{BGW}\right)$" Let $\overline{v} \in \text{v-max}\, h^{\leq}\left(\mathcal{B}^{\leq}\right)$. By Theorem 5 it follows that $\overline{v} \in \text{v-max}\, h^{BK}\left(\mathcal{B}^{BK}\right)$. Since $h^{BK}\left(\mathcal{B}^{BK}\right) \subseteq h^{BGW}\left(\mathcal{B}^{BGW}\right) \cap \mathbb{R}^m$, we have further $\overline{v} \in h^{BGW}\left(\mathcal{B}^{BGW}\right)$. Let us suppose by contradiction that there exists $(p, q, \lambda, t) \in \mathcal{B}^{BGW}$ such that $\overline{v} \leq_{\mathbb{R}^m_+} d := h^{BGW}(p, q, \lambda, t)$. Since $h^{BGW}\left(\mathcal{B}^{BGW}\right) \subseteq h^{\leq}\left(\mathcal{B}^{\leq}\right)$, one has $d \in h^{\leq}\left(\mathcal{B}^{\leq}\right)$, but $\overline{v} \leq_{\mathbb{R}^m_+} d$ which is a contradiction to the maximality of $\overline{v}$. Therefore

$$\text{v-max}\, h^{\leq}\left(\mathcal{B}^{\leq}\right) \subseteq \text{v-max}\, h^{BGW}\left(\mathcal{B}^{BGW}\right).$$

□

As one can easily notice from Theorems 5 and 10 along with examples 8 and 9, the following equalities hold:

$$\text{v-max}\, h^{BK}(\mathcal{B}^{BK}) = \text{v-max}\, h^{BGW}\left(\mathcal{B}^{BGW}\right) = \text{v-max}\, h^{\leq}\left(\mathcal{B}^{\leq}\right),$$

even though

$$h^{BK}\left(\mathcal{B}^{BK}\right) \underset{\neq}{\subset} h^{BGW}\left(\mathcal{B}^{BGW}\right) \cap \mathbb{R}^m \underset{\neq}{\subset} h^{\leq}\left(\mathcal{B}^{\leq}\right).$$

Using the weak, strong and converse duality theorems between the dual pair of the vector optimization problems (P) and $(D^{\leq})$, similar results can be proved for the dual pair (P) and (D^{BGW}). Thus

Theorem 11. *The following statements are true:*

a) *(Weak duality) There exist no* $x \in X$ *and no* $(p, q, \lambda, t) \in \mathcal{B}^{BGW}$ *such that* $(f + g \circ A)(x) \leq_K h^{BGW}(p, q, \lambda, t)$.

b) *(Strong duality) If* $\overline{x} \in X$ *is a properly efficient solution to* (P)*, then there exists an efficient solution* $(\overline{p}, \overline{q}, \overline{\lambda}, \overline{t}) \in \mathcal{B}^{BGW}$ *to* (D^{BGW}) *such that*

$$(f + g \circ A)(\overline{x}) = h^{BGW}(\overline{p}, \overline{q}, \overline{\lambda}, \overline{t}).$$

c) *(Converse duality) If the set*

$$(f + g \circ A)(\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)) + K$$

is closed, then for each efficient solution $(\overline{p}, \overline{q}, \overline{\lambda}, \overline{t}) \in \mathcal{B}^{BGW}$ *to* (D^{BGW}) *there exists a properly efficient solution* $\overline{x} \in X$ *to* (P) *such that*

$$(f + g \circ A)(\overline{x}) = h^{BGW}(\overline{p}, \overline{q}, \overline{\lambda}, \overline{t}).$$

Proof. a) It follows from Proposition 7 b) and Theorem 1.

b) It follows from Theorem 10 and Theorem 2.

c) It follows from Theorem 10 and Theorem 4. □

As it will be seen in the following example, Theorem 4, which was important in the proof of the converse duality for dual $(D^{\leqq})$, does not hold for the more particular dual problems (D^{BK}) and (D^{BGW}).

Example 12. *Let* $X = Y = \mathbb{R}$ *and* $V := \mathbb{R}^2$*. Put* $A(x) = x$ *for all* $x \in \mathbb{R}$ *and define the functions* $f, g : \mathbb{R} \to \mathbb{R}^2$ *by*

$$f(x) = (-3x + 7, 2x) \text{ and } g(x) = (3x - 7, -2x), \forall x \in \mathbb{R}.$$

We show that $\mathbb{R}^2 \setminus \operatorname{cl}\big((f + g)(\mathbb{R}) + \mathbb{R}^2_+\big) \not\subseteq \operatorname{core} h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^2$.

Under the above specified framework, $\operatorname{dom} f = \operatorname{dom} g = \mathbb{R}$ and the feasible solution set of (D^{BGW}) is

$$\mathcal{B}^{BGW} = \left\{ \begin{array}{l} (p, q, \lambda, d) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \operatorname{int} \mathbb{R}^2_+ \times \mathbb{R}^2 : \\ \lambda_1(p_1 + q_1) + \lambda_2(p_2 + q_2) = 0, \lambda_1 t_1 + \lambda_2 t_2 = 0 \end{array} \right\}$$

and

$$h^{BGW}(\mathcal{B}^{BGW}) = \left\{ \begin{array}{l} (-f_1^*(p_1) - g_1^*(q_1) + t_1, -f_2^*(p_2) - g_2^*(q_2) + t_2) : \\ (\lambda_1, \lambda_2) \in \operatorname{int} \mathbb{R}^2_+, \\ \lambda_1(p_1 + q_1) + \lambda_2(p_2 + q_2) = 0, \lambda_1 t_1 + \lambda_2 t_2 = 0. \end{array} \right\}$$

We start by noticing that

$$(f+g)(\mathbb{R}) + \mathbb{R}^2_+ = \mathbb{R}^2_+ = \operatorname{cl}((f+g)(\mathbb{R}) + \mathbb{R}^2_+).$$

Furthermore

$$f_1^*(p) = \begin{cases} -7, & \text{if } p = 3, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{and } f_2^*(p) = \begin{cases} 0, & \text{if } p = 2, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$g_1^*(q) = \begin{cases} 7, & \text{if } q = 3, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{and } g_2^*(p) = \begin{cases} 0, & \text{if } q = -2, \\ +\infty, & \text{otherwise.} \end{cases}$$

Therefore

$$-f_1^*(p) - g_1^*(q) + t = \begin{cases} t, & \text{if } p = -3, q = 3, \\ -\infty, & \text{otherwise,} \end{cases}$$

and

$$-f_2^*(p) - g_2^*(q) + t = \begin{cases} t, & \text{if } p = 2, q = -21, \\ -\infty, & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} h^{BGW}(\mathcal{B}^{BGW}) &= \left\{ \begin{array}{l} (t_1, t_2) \in \mathbb{R}^2 : (\lambda_1, \lambda_2) \in \operatorname{int} \mathbb{R}^2, \\ \lambda_1(-3+3) + \lambda_2(2-2) = 0, \lambda_1 t_1 + \lambda_2 t_2 = 0 \end{array} \right\} \\ &= \left\{ (t_1, t_2) \in \mathbb{R}^2 : \lambda_1 t_1 + \lambda_2 t_2 = 0, \lambda_1 > 0, \lambda_2 > 0 \right\}. \end{aligned}$$

Now let us fix $\overline{v} := (-1, -1)$, for which we have that $\overline{v} \in \mathbb{R}^2 \setminus \operatorname{cl}((f+g)(\mathbb{R}) + \mathbb{R}^2)$. We notice that $\overline{v} \notin h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^2$. We prove this by contradiction. Assuming that $\overline{v} \in h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^2$ it follows that $\lambda_1(-1) + \lambda_2(-1) = 0$ with $\lambda_1 > 0, \lambda_2 > 0$, which is obviously a contradiction.

So $\overline{v} \notin h^{BGW}(\mathcal{B}^{BGW}) \cap \mathbb{R}^2$, and hence it follows from Proposition 7 a) that $v \notin h^{BK}(\mathcal{B}^{BK})$. Nevertheless, from Theorem 3 it follows that $\overline{v} \in \operatorname{core} h^{\leq}(\mathcal{B}^{\leq})$.

The conclusion is that a direct converse duality proof for the case of problem (D^{BGW}) would be more difficult, unless embedded in $(D^{\leq})$.

Acknowledgment. The author would like to gratefully thank Dr. Radu Ioan Boţ for his help, comments and for the improvements suggested, which have essentially upgraded the quality of the paper.

References

[1] Borwein, J.M., Lewis, A.S., *Partially finite convex programming, Part I: Quasi relative interiors and duality theory*, Mathematical Programming, **57(1)**(1992), 15-48.

[2] Boţ, R.I., *Conjugate Duality in Convex Optimization*, Habilitation Thesis, Faculty of Mathematics, Chemnitz University of Technology, 2008.

[3] Boţ, R.I., *Duality and Optimality in Multiobjective Optimization*, Dissertation, Faculty of Mathematics, Chemnitz University of Technology, 2003.

[4] Boţ, R.I., Grad, S.M., Wanka, G., *New Regularity Conditions for Strong and Total Fenchel-Lagrange Duality in Infinite Dimensional Spaces*, Nonlinear Analysis, Theory, Methods and Applications, **69(1)**(2008), 323-336.

[5] Boţ, R.I., Wanka, G., *An alternative formulation of a closed cone constraint qualification*, Nonlinear Analysis, Theory, Methods and Applications, **64(6)**(2006), 1367-1381.

[6] Boţ, R.I., Wanka, G., *An analysis of some dual problems in multiobjective optimization (I)*, Optimization, **53(3)**(2004), 281-300.

[7] Boţ, R.I., Wanka, G., *An analysis of some dual problems in multiobjective optimization (II)*, Optimization, **53(3)**(2004), 301-324.

[8] Boţ, R.I., Wanka, G., *A new duality approach for multiobjetive convex optimization problems*, Journal of Nonlinear and Convex Analysis, **3(1)**(2002), 41-57.

[9] Boţ, R.I., Dumitru (Grad), A., Wanka, G., *A new Fenchel dual problem in vector optimization*, Preprint 18, Faculty of Mathematics, Chemnitz University of Technology, Germany, 2006.

[10] Breckner, W.W., Kolumbán, I., *Dualität bei Optimierungsaufgaben in topologischen Vektorräumen*, Mathematica, **10(33)**(1968), 229-244.

[11] Breckner, W.W., Kolumbán, I., *Konvexe Optimierungsaufgaben in topologischen Vektorräumen*, Mathematica Scandinavica, **25**(1969), 227-247.

[12] Breckner, W.W., *Dualität bei Optimierungsaufgaben in halbgeordneten topologischen Vektorräumen (I)*, Revue d'Analyse Numérique et de la Théorie de l'Approximation, **1**(1972), 5-35.

[13] Breckner, W.W., *Dualität bei Optimierungsaufgaben in halbgeordneten topologischen Vektorräumen (II)*, Revue d'Analyse Numérique et de la Théorie de l'Approximation, **2**(1973), 27-35.

[14] Ekeland, I., Temam, R., *Convex Analysis and Variational Problems*, North-Holland Publishing Company, Amsterdam, 1976.

[15] Gerstewitz, C., Göpfert, A., *Zur Dualität in der Vektoroptimierung*, Seminarbericht **39**(1981), Sektion Mathematik, Humboldt-Universität Berlin, 67-84.

[16] Jahn, J., *Duality in vector optimization*, Mathematical Programming, **25**(1983), 343-353.

[17] Jahn, J., *Vector Optimization - Theory, Applications, and Extensions,* Springer Verlag, Berlin, 2004.

[18] Malivert, C., *Fenchel duality in vector optimization,* in: W. Oettli, D. Pallaschke (Eds.): Advances in optimization, Lecture notes in Economics and Mathematial Systems 382, Springer Verlag, Berlin, Heidelberg, 1992, 420-438.

[19] Rockafellar, R.T., *Convex Analysis*, Princeton University Press, Princeton, 1970.

[20] Rockafellar, R.T., *Conjugate Duality and Optimization*, Regional Conference Series in Applied Mathematics 16, Society for Industrial and Applied Mathematics, Philadelplhia, 1974.

[21] Sawaragi, Y., Nakayama, H., Tanino, T., *Theory of Multiobjective Optimization*, Academic Press, New York, 1985.

[22] Wanka, G., Boţ, R.I., *On the relations between different dual problems in convex mathematical programming*, in: P. Chamoni, R. Leisten, A. Martin, J. Minnermann, H. Stadtler (Eds.), Operations Research Proceedings 2001, Springer Verlag, Berlin, 255-262.

[23] Weir, T., Mond, B., *Generalised convexity and duality in multiple objetive programming*, Bulletin of Australian Mathematical Society, **39**(1989), 287-299.

[24] Weir, T., Mond, B., *Multiple objective programming duality without a constraint qualification*, Utilitas Mathematica, **39**(1991), 41-55.

[25] Zălinescu, C., *Convex Analysis in General Vector Spaces*, World Scientific, Singapore, 2002.

Str. M. Kogălniceanu Nr.1, RO-400084 Cluj-Napoca, România

E-mail address: `dumitruanca@math.ubbcluj.ro`

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 3, September 2009

EXISTENCE AND DATA DEPENDENCE FOR MULTIVALUED WEAKLY CONTRACTIVE OPERATORS

LILIANA GURAN

Abstract. The purpose of this paper is to study the data dependence for the fixed point set of a multivalued weakly contractive operator with respect to a w-distance in the sense of T. Suzuki and W. Takahashi. We also give a fixed point result for a multivalued weakly φ-contraction on a metric space endowed with a w-distance.

1. **Introduction**

Let (X, d) be a metric space. A singlevalued operator T from X into itself is called r-contractive (see [2]) if there exists a real number $r \in [0, 1)$ such that $d(T(x), T(y)) \leq rd(x, y)$ for every $x, y \in X$. It is well know that if X is a complete metric space then a contractive operator from x into itself has a unique fixed point in X.

In 1996, the Japanese mathematicians O. Kada, T. Suzuki and W. Takahashi introduced the concept of w-distance (see[2]) and discussed some properties of this functional. Later on, T. Suzuki and W. Takahashi gave some fixed points results for a new class of nonlinear operators, namely the so-called weakly contractive operators (see[3]).

The purpose of this paper is to study the data dependence for the fixed point set of a multivalued weakly contractive operator with respect to a w-distance in the sense of T. Suzuki and W. Takahashi, see [3]. We also give a fixed point result for a

Received by the editors: 01.10.2008.

2000 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. w-distance, weakly φ-contraction, fixed point, multivalued operator.

This paper was presented at the 7-th Joint Conference on Mathematics and Computer Science, July 3-6, 2008, Cluj-Napoca, Romania.

multivalued weakly φ-contraction on a metric space endowed with a w-distance. For connected results see [6], [4].

2. **Preliminaries**

Let (X, d) be a complete metric space. We will use the following notations (see also [1], [5]).

$P(X)$ - *the set of all nonempty subsets of* X;

$\mathcal{P}(X) = P(X) \bigcup \emptyset$

$P_{cl}(X)$ - *the set of all nonempty closed subsets of* X;

$P_b(X)$ - *the set of all nonempty bounded subsets of* X;

$P_{b,cl}(X)$ - *the set of all nonempty bounded and closed subsets of* X;

We introduce now the following generalized functionals on a b-metric space (X, d).

The gap functional:

$$(1)\ \ D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$D(A,B) = \begin{cases} \inf\{d(a,b) |\ a \in A,\ b \in B\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset = B \\ +\infty, & \text{otherwise} \end{cases}$$

In particular, if $x_0 \in X$ then $D(x_0, B) := D(\{x_0\}, B)$.

The excess generalized functional:

$$(2)\ \ \rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$\rho(A,B) = \begin{cases} \sup\{D(a,B) |\ a \in A\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset \\ +\infty, & B = \emptyset \neq A \end{cases}$$

Pompeiu-Hausdorff generalized functional:

$$(3)\ \ H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$H(A,B) = \begin{cases} \max\{\rho(A,B), \rho(B,A)\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset = B \\ +\infty, & \text{othewise} \end{cases}$$

Delta functional:

$$(4)\ \ \delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$\delta(A,B) = \begin{cases} \sup\{d(a,b) : a \in A, b \in B\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset = B \\ +\infty, & \text{othewise} \end{cases}$$

In particular, $\delta(A) := \delta(A,A)$ is the diameter of the set A.

It is known that $(P_{b,cl}(X), H)$ is a complete metric space provided (X,d) is a complete metric space.

We will denote by $FixF := \{x \in X \mid x \in F(x)\}$, the set of the fixed points of F.

The concept of w-distance was introduced by O. Kada, T. Suzuki and W. Takahashi (see[2]) as follows:

Let (X,d) be a metric space. Then, the functional $w : X \times X \to [0,\infty)$ is called w-distance on X if the following axioms are satisfied :

1. $w(x,z) \leq w(x,y) + w(y,z)$, for any $x,y,z \in X$;
2. for any $x \in X : w(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous;
3. for any $\varepsilon > 0$, exists $\delta > 0$ such that $w(z,x) \leq \delta$ and $w(z,y) \leq \delta$ implies $d(x,y) \leq \varepsilon$.

Let us give some examples of w-distance (see [2])

Example 2.1. Let (X,d) be a metric space . Then the metric "d" is a w-distance on X.

Example 2.2. Let X be a normed linear space with norm $||\cdot||$. Then the function $w : X \times X \to [0,\infty)$ defined by $w(x,y) = ||x|| + ||y||$ for every $x,y \in X$ is a w-distance.

Example 2.3. Let (X,d) be a metric space and let $g : X \to X$ a continuous mapping. Then the function $w : X \times Y \to [0, \infty)$ defined by:

$$w(x, y) = max\{d(g(x), y), d(g(x), g(y))\}$$

for every $x, y \in X$ is a w-distance.

For the proof of the main results we need the following crucial result for w-distance (see[3]).

Lemma 2.4.*Let (X, d) be a metric space, and let w be a w-distance on X. Let (x_n) and (y_n) be two sequences in X, let (α_n), (β_n) be sequences in $[0, +\infty[$ converging to zero and let $x, y, z \in X$. Then the following hold:*

1. *If $w(x_n, y) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$.*
2. *If $w(x_n, y_n) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z.*
3. *If $w(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence.*
4. *If $w(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence.*

3. **Data dependence for w-contractive multivalued operators**

In [3]. the definition of a weakly contractive multivalued operator is given, as follows.

Definition 3.1. Let X be a metric space with metric d. A multivalued operator $T : X \to P(X)$ is called weakly contractive or w-contractive if there exists a w-distance w on X and $r \in [0, 1)$ such that for any $x_1, x_2 \in X$ and $y_1 \in T(x_1)$ there is $y_2 \in T(x_2)$ with $w(y_1, y_2) \leq rw(x_1, x_2)$.

Then, in the same paper, T. Suzuki and W. Takahashi gave the following fixed point result for a multivalued weakly contractive operator (see Theorem 1, [3]).

Theorem 3.2. *Let X be a complete metric space and let $T : X \to P(X)$ be a w-contractive multivalued operator such that for any $x \in X$, $T(x)$ is a nonempty closed subset of X. Then there exists $x_0 \in X$ such that $x_0 \in T(x_0)$ and $w(x_0, x_0) = 0$.*

The main result of this section is the following data dependence theorem with respect to the fixed point set of the above class of operators.

Theorem 3.3. *Let (X, d) be a complete metric space, $T_1, T_2 : X \to P_{cl}(X)$ be two w-contractive multivalued operators with $r_i \in [0, 1)$ with $i = \{1, 2\}$. Then the following are true:*

1. $FixT_1 \neq \emptyset \neq FixT_2$;
2. *We suppose that there exists $\eta > 0$ such that for every $u \in T_1(x)$ there exists $v \in T_2(x)$ such that $w(u, v) \leq \eta$, (respectively for every $v \in T_2(x)$ there exists $u \in T_1(x)$ such that $w(v, u) \leq \eta$).*

Then for every $u^* \in FixT_1$ there exists $v^* \in FixT_2$ such that

$$w(u^*, v^*) \leq \tfrac{\eta}{1-r}, \text{ where } r = r_i \text{ for } i = \{1, 2\};$$

(respectively for every $v^* \in FixT_2$ there exists $u^* \in FixT_1$ such that

$$w(v^*, u^*) \leq \tfrac{\eta}{1-r}, \text{ where } r = r_i \text{ for } i = \{1, 2\})$$

Proof. Let $u_0 \in FixT_1$, then $u_0 \in T_1(u_0)$. Using the hypothesis 2. we have that there exists $u_1 \in T_2(u_0)$ such that $w(u_0, u_1) \leq \eta$.

Since T_1, T_2 are weakly contractive with $r_i \in [0, 1)$ and $i = \{1, 2\}$ we have that for every $u_0, u_1 \in X$ with $u_1 \in T_2(u_0)$ there exists $u_2 \in T_2(u_1)$ such that

$$w(u_1, u_2) \leq rw(u_0, u_1)$$

For $u_1 \in X$ and $u_2 \in T_2(u_1)$ there exists $u_3 \in T_2(u_2)$ such that

$$w(u_2, u_3) \leq rw(u_1, u_2) \leq r^2 w(u_0, u_1)$$

By induction we obtain a sequence $(u_n)_{n\in\mathbb{N}} \in X$ such that

(1) $u_{n+1} \in T_2(u_n)$, for every $n \in \mathbb{N}$;

(2) $w(u_n, u_{n+1}) \leq r^n w(u_0, u_1)$

For $n, p \in \mathbb{N}$ we have the inequality

$$\begin{aligned} w(u_n, u_{n+p}) &\leq w(u_n, u_{n+1}) + w(u_{n+1}, u_{n+2}) + \cdots + w(u_{n+p-1}, u_{n+p}) \leq \\ &< r^n w(u_0, u_1) + r^{n+1} w(u_0, u_1) + \cdots + r^{n+p-1} w(u_0, u_1) \leq \\ &\leq \tfrac{r^n}{1-r} w(u_0, u_1) \end{aligned}$$

By the Lemma 2.4.(3) we have that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X,d) is a complete metric space we have that there exists $v^* \in X$ such that $u_n \xrightarrow{d} v^*$.

By the lower semicontinuity of $w(x,\cdot) : X \to [0,\infty)$ we have

$$w(u_n, v^*) \leq \lim_{p\to\infty} \inf w(u_n, u_{n+p}) \leq \frac{r^n}{1-r} w(u_0, u_1) \tag{1}$$

For $u_{n-1}, v^* \in X$ and $u_n \in T_2(u_{n-1})$ there exists $z_n \in T_2(v^*)$ such that, using relation (1), we have

$$w(u_n, z_n) \leq r w(u_{n-1}, v^*) \leq \frac{r^{n-1}}{1-r} w(u_0, u_1) \tag{2}$$

Applying Lemma 2.4.(2), from relations (1) and (2) we have that $z_n \xrightarrow{d} v^*$.

Then, we know that $z_n \in T_2(v^*)$ and $z_n \xrightarrow{d} v^*$. In this case, by the closure of T_2 result that $v^* \in T_2(v^*)$. Then, by $w(u_n, v^*) \leq \frac{r^n}{1-r} w(u_0, u_1)$, with $n \in \mathbb{N}$, for $n = 0$ we obtain

$$w(u_0, v^*) \leq \frac{1}{1-r} w(u_0, u_1) \leq \frac{\eta}{1-r}$$

which completes the proof. □

4. **Existence of fixed points for multivalued weakly φ-contractive operators**

Let us define first, the notion of multivalued weakly φ-contractive operator.

Definition 4.1. Let (X,d) be a metric space and $T : X \to P(X)$ be a multivalued operator. Then T is called weakly φ-contractive if there exists a w-distance on X and a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every x_1, x_2 and $y_1 \in T(x_1)$ there is $y_2 \in T(x_2)$ with $w(y_1, y_2) \leq \varphi(w(x_1, x_2))$.

The main result is the following result for weakly φ-contractive operators.

Theorem 4.2. *Let (X,d) be a complete metric space, $w : X \times X \to \mathbb{R}_+$ a w-distance on X, $T : X \to P_{cl}(X)$ be a multivalued operator and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a function such that are accomplish the following conditions:*

1. *T are weakly φ-contractive operator;*

2. *The function φ is a monotone increasing function such that*
$$\sigma(t) := \sum_{n=0}^{\infty} \varphi^n(t) < \infty, \textit{ for every } t \in \mathbb{R}_+ \setminus \{0\}.$$
Then there exists $x^* \in X$ such that $x^* \in T(x^*)$ and $w(x^*, x^*) = 0$.

Proof. First, we remark that condition (2) from hypothesis implies that $\varphi(t) < t$ for $t < 0$.

Fix $x_0 \in x$; for $x_1 \in T(x_0)$ there exists $x_2 \in T(x_1)$ such that

$$w(x_1, x_2) \leq \varphi(w(x_0, x_1)).$$

For $x_1 \in X$ and $x_2 \in T(x_1)$ there exists $x_3 \in T(x_2)$ such that

$$w(x_2, x_3) \leq \varphi(w(x_1, x_2)) \leq \varphi(\varphi(w(x_0, x_1))) = \varphi^2(w(x_0, x_1)).$$

By induction we obtain a sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that

(i) $x_{n+1} \in T(x_n)$, for $n \in \mathbb{N}$;

(ii) $w(x_n, x_{n+1}) \leq \varphi^n(w(x_0, x_1))$, for $n \in \mathbb{N}$.

For $n, p \in \mathbb{N}$ we have

$$\begin{aligned} w(x_n, x_{n+p}) \leq w(x_n, x_{n+1}) + w(x_{n+1}, x_{n+2}) + \cdots + w(x_{n+p-1}, x_{n+p}) \leq \\ < \varphi^n(w(x_0, x_1)) + \varphi^{n+1}(w(x_0, x_1)) + \cdots + \varphi^{n+p-1}(w(x_0, x_1)) \leq \\ \leq \sum_{n=k}^{\infty} \varphi^k(w(x_0, x_1)) \leq \sigma(w(x_0, x_1)). \end{aligned}$$

Letting $n \to \infty$ we have

$$\lim_{n\to\infty} w(x_n, x_{n+p}) \leq \lim_{n\to\infty} \sigma(\varphi^n(w(x_0, x_1))) = 0.$$

By the Lemma 2.4.(3) we have that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is a complete metric space then there exists $x^* \in X$ such that $\lim\limits_{n\to\infty} x_n = x^*$.

For $n, m \in \mathbb{N}$ with $m > n$ from the above inequality we have

$$w(x_n, x_m) \leq \sigma(\varphi^n(w(x_0, x_1))).$$

Since $(x_m)_{m\in\mathbb{N}}$ converge to x^* and $w(x_n, \cdot)$ is lower semicontinuous we have

$$w(x_n, x^*) \leq \lim_{m\to\infty} \inf w(x_n, x_m) \leq \lim_{m\to\infty} \sigma(\varphi^n(w(x_0, x_1))) \leq \sigma(\varphi^n(w(x_0, x_1))).$$

So, for every $n \in \mathbb{N}$, $w(x_n, x^*) \le \sigma(\varphi^n(w(x_0, x_1)))$

For $x^* \in X$ and $x_n \in T(x_{n-1})$ there exists $u_n \in T(x^*)$ such that

$$w(x_n, u_n) \le \varphi(w(x_{n-1}, x^*)) \le \varphi(\sigma(\varphi^{n-1}(w(x_0, x_1)))) < \sigma(\varphi^{n-1}(w(x_0, x_1)))$$

So, we know that:

$$w(x_n, u_n) \le \sigma(\varphi^{n-1}(w(x_0, x_1)))$$

$$w(x_n, x^*) \le \sigma(\varphi^n(w(x_0, x_1)))$$

Then, by the Lemma 2.4.(2), we obtain that $u_n \stackrel{d}{\to} x^*$. As $u_n \in T(x^*)$ and using the closure of T result that $x^* \in T(x^*)$.

For $x^* \in X$ and $x^* \in T(x^*)$, using the hypothesis (1), there exists $z_1 \in T(x^*)$ such that

$$w(x^*, z_1) \le \varphi(w(x^*, x^*)).$$

For $x^*, z_1 \in X$ and $x^* \in T(x^*)$ there exists $z_2 \in T(z_1)$ such that

$$w(x^*, z_2) \le \varphi(x^*, z_1).$$

By induction we get a sequence $(z_n)_{n\in\mathbb{N}} \in X$ such that

(i) $z_{n+1} \in T(z_n)$, for every $n \in \mathbb{N}$;

(ii) $w(x^*, z_n) \le \varphi(w(x^*, z_{n-1}))$, for every $n \in \mathbb{N} \setminus \{0\}$.

Therefore we have

$$w(x^*, z_n) \le \varphi(w(x^*, z_{n-1})) \le \varphi(\varphi(w(x^*, z_{n-2}))) = \varphi^2(w(x^*, z_{n-2})) \le \cdots \le$$

$$\le \varphi^n(w(x^*, z_1)) \le \varphi^n(w(x^*, x^*)).$$

Thus $w(x^*, z_n) \le \varphi^n(w(x^*, x^*))$.

When $n \to \infty$, $\varphi^n(w(x^*, x^*))$ converge to 0. Thus, by the Lemma 2.4.(4) we obtain that $(z_n)_{n\in\mathbb{N}} \in X$ is a Cauchy sequence in (X, d) and there exists $z^* \in X$ such that $z_n \stackrel{d}{\to} z^*$.

Since $w(x^*, \cdot)$ is lower semicontinuous we have

$$0 \le w(x^*, z^*) \le \lim_{n\to\infty} \inf w(x^*, z_n) \le \lim_{n\to\infty} \varphi^n(w(x^*, x^*)) = 0.$$

Then $w(x^*, z^*) = 0$.

So, by triangle inequality we have

$$w(x_n, z^*) \leq w(x_n, x^*) + w(x^*, z^*) \leq \sigma(\varphi^n(w(x_0, x_1))).$$

Since $\sigma(\varphi^n(w(x_0, x_1)))$ converge to 0 when $n \to \infty$ we have

$$w(x_n, z^*) \leq \sigma(\varphi^n(w(x_0, x_1)))$$

$$w(x_n, x^*) \leq \sigma(\varphi^n(w(x_0, x_1)))$$

Using Lemma 2.4.(1) result that $z^* = x^*$, then $w(x^*, x^*) = 0$. $\square$

References

[1] Dugundji, J., Granas, A., *Fixed Point Theory*, Berlin, Springer-Verlag, 2003.

[2] Kada, O., Suzuki, T., Takahashi, W., *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Math. Japonica, **44**(1996), 381-391.

[3] Suzuki, T., Takahashi, W., *Fixed points theorems and characterizations of metric completeness*, Topological Methods in Nonlinear Analysis, Journal of Juliusz Schauder Center, **8**(1996), 371-382.

[4] Proinov, P.D., *A generalization of the Banach contraction principle with high order of convergence of the successive approximations*, **67**(2007), 2361-2369.

[5] Rus, I.A., *Generalized Contractions and Applications*, Presa Clujeană Universitară, Cluj-Napoca, 2001.

[6] Rus, I.A., *The theory of a metrical fixed point theorem: theoretical and applicative relevance*, Fixed Point Theory, **9**(2008), 541-559.

Department of Applied Mathematics,
Babeş-Bolyai University,
Cluj-Napoca, Romania
E-mail address: gliliana.math@gmail.com

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 3, September 2009

A NOTE ON A GEOMETRIC CONSTRUCTION OF LARGE CAYLEY GRAPHS OF GIVEN DEGREE AND DIAMETER

GYÖRGY KISS, ISTVÁN KOVÁCS, KLAVDIJA KUTNAR, JÁNOS RUFF, AND PRIMOŽ ŠPARL

Abstract. An infinite series and some sporadic examples of large Cayley graphs with given degree and diameter are constructed. The graphs arise from arcs, caps and other objects of finite projective spaces.

A simple finite graph Γ is a (Δ, D)*-graph* if it has maximum degree Δ, and diameter at most D. The (Δ, D)*-problem* (or *degree/diameter problem*) is to determine the largest possible number of vertices that Γ can have. Denoted this number by $n(\Delta, D)$, the well-known *Moore bound* states that $n(\Delta, D) \leq \frac{\Delta(\Delta-1)^D-2}{\Delta-2}$. This is known to be attained only if either $D = 1$ and the graph is $K_{\Delta+1}$, or $D = 2$ and $\Delta = 1, 2, 3, 7$ and perhaps 57. If in addition Γ is required to be vertex-transitive, then the only known general lower bound is given as

$$n(\Delta, 2) \geq \left\lfloor \frac{\Delta+2}{2} \right\rfloor \cdot \left\lceil \frac{\Delta+2}{2} \right\rceil. \tag{1}$$

This is obtained by choosing Γ to be the Cayley graph $\mathrm{Cay}(\mathbb{Z}_a \times \mathbb{Z}_b, S)$, where $a = \lfloor \frac{\Delta+2}{2} \rfloor$, $b = \lceil \frac{\Delta+2}{2} \rceil$, and $S = \{\, (x, 0), (0, y) \mid x \in \mathbb{Z}_a \setminus \{0\}, y \in \mathbb{Z}_b \setminus \{0\} \,\}$. If $\Delta = kD+m$, where k, m are integers and $0 \leq m < D$, then a straightforward generalization of this construction results in a Cayley (Δ, D)-graph of order

$$\left\lfloor \frac{\Delta+D}{D} \right\rfloor^{D-m} \cdot \left\lceil \frac{\Delta+D}{D} \right\rceil^{m}. \tag{2}$$

Received by the editors: 05.11.2008.
2000 *Mathematics Subject Classification.* 05C62, 51E21.
Key words and phrases. degree, diameter problem, Moore bound, finite projective spaces.
This paper was presented at the 7-th Joint Conference on Mathematics and Computer Science, July 3-6, 2008, Cluj-Napoca, Romania.

Throughout this note we will refer these graphs as GCCG-graphs (General Construction from Cyclic Groups). For special values of the parameters, (1) and (2) have been improved using various constructions. For more on the topic, we refer to [1, 8].

In this note we restrict our attention to the class of linear Cayley graphs. We present some constructions where the resulting graphs improve the lower bounds (1) and (2). For small number of vertices these are also compared to the known largest vertex transitive graphs having the same degree and diameter.

Let V denote the n-dimensional vector space over the finite field $\mathbb{F}_q$ of q elements, where $q = p^e$ for a prime p. For $S \subseteq V$ such that $0 \notin S$, and $S = -S := \{-x \mid x \in S\}$, the *Cayley graph* $\mathrm{Cay}(V, S)$ is the graph having vertex-set V, and edges $\{x, x+s\}$, $x \in V$, $s \in S$. To S we also refer as the connection set of the graph. A Cayley graph $\mathrm{Cay}(V, S)$ is said to be *linear*, [6, pp. 243] if $S = \alpha S := \{\alpha x \mid x \in S\}$ for all nonzero scalars $\alpha \in \mathbb{F}_q$. In this case $S \cup \{0\}$ is a union of 1-dimensional subspaces, and therefore, it can also be regarded as a point set in the projective space $\mathrm{PG}(n-1, q)$. Conversely, any point set $\mathcal{P}$ in $\mathrm{PG}(n-1, q)$ gives rise to a linear Cayley graph, namely the one having connection set $\{x \in V \setminus \{0\} \mid \langle x \rangle \in \mathcal{P}\}$. We denote this graph by $\Gamma(\mathcal{P})$. Given an arbitrary point set $\mathcal{P}$ in $\mathrm{PG}(n, q)$, $\langle \mathcal{P} \rangle$ denotes the projective subspace generated by the points in $\mathcal{P}$, and $\binom{\mathcal{P}}{k}$ ($k \in \mathbb{N}$) is the set of all subsets of $\mathcal{P}$ having cardinality k. The degree and diameter of linear Cayley graphs are given in the next proposition.

Proposition 1. *Let $\mathcal{P}$ be a set of k points in $\mathrm{PG}(n, q)$ with $\langle \mathcal{P} \rangle = \mathrm{PG}(n, q)$. Then $\Gamma(\mathcal{P})$ has q^{n+1} vertices, with degree $k(q-1)$, and with diameter*

$$D = \min \left\{ \, d \mid \cup_{\mathcal{X} \in \binom{\mathcal{P}}{d}} \langle \mathcal{X} \rangle = \mathrm{PG}(n, q) \, \right\}. \tag{3}$$

Proof. Let $\Gamma = \Gamma(\mathcal{P})$. It is immediate from its definition that Γ has q^{n+1} vertices and that its degree is equal to $k(q-1)$. Now let V denote the $(n+1)$-dimensional vector space over $\mathbb{F}_q$. Being a Cayley graph, Γ is automatically vertex-transitive, and so its diameter is the maximal distance $\delta_\Gamma(0, x)$ where $0 \in V$, and x runs over V. By δ_Γ we denote the usual distance function of Γ.

Let $x \in V \setminus \{0\}$, and let $P = \langle x \rangle$ be the corresponding point in $\mathrm{PG}(n,q)$. It can be seen that $\delta_\Gamma(0,x) = k$ where k is the minimal number of independent points $P_1, \ldots, P_k \in \mathcal{P}$ such that $P \in \langle P_1, \ldots, P_k \rangle$. Now, (3) shows that $\delta_\Gamma(0,x) \leq D$ for every $x \in V$, in particular, the diameter of Γ is at most D.

On the other hand, by (3), there exists a $Q \in \mathrm{PG}(n,q)$ for which $Q \notin \langle P_1, \ldots, P_{D-1} \rangle$ for any $P_1, \ldots, P_{D-1} \in \mathcal{P}$. Thus if y is an element of V with $\langle y \rangle = Q$, then $\delta_\Gamma(0,y) \geq D$. Therefore, the diameter of Γ cannot be less than D, which completes the proof. $\square$

Once the number of vertices and the diameter for $\Gamma(\mathcal{P})$ are fixed to be q^{n+1} and D, respectively, our task becomes to search for the smallest possible point set $\mathcal{P}$ for which

$$\cup_{\mathcal{X} \in \binom{\mathcal{P}}{D}} \langle \mathcal{X} \rangle = \mathrm{PG}(n,q).$$

A point set having this property is called a *(D-1)-saturating set*.

The constructions

If $D = 2$, then a 1-saturating set $\mathcal{P}$ is a set of points of $\mathrm{PG}(n,q)$ such that the union of lines joining pairs of points of $\mathcal{P}$ covers the whole space. Assume that $n = 2$. If $\mathcal{P}$ contains k points, then the graph has degree $k(q-1)$ and the number of vertices is q^3. Hence this is better than the general lower bound (1) if and only if $q^3 > (k(q-1)+2)^2/4$, which is equivalent to

$$2\sqrt{q} + \frac{2}{\sqrt{q}+1} > k. \tag{4}$$

There are two known general constructions for 1-saturating sets in the plane: complete arcs and double blocking sets of Baer subplanes.

If q is a square, and $\Pi_{\sqrt{q}}$ is a Baer subplane of $\mathrm{PG}(2,q)$, of order $\sqrt{q}$, then each point of $\mathrm{PG}(2,q) \setminus \Pi_{\sqrt{q}}$ is incident with exactly one line of $\Pi_{\sqrt{q}}$. A double blocking set of a plane meets each line of the plane in at least two points. Hence a double blocking set of $\Pi_{\sqrt{q}}$ is a 1-saturating set of $\mathrm{PG}(2,q)$. The cardinality of a double blocking set of $\Pi_{\sqrt{q}}$ is at least $2(\sqrt{q}+\sqrt[4]{q}+1)$. This is greater than the bound given in (4), hence we cannot construct good graphs from these sets.

A complete k-arc $\mathcal{K}$ is a set of k points such that no three of them are collinear, and there is no $(k+1)$-arc containing $\mathcal{K}$. Thus $\mathcal{K}$ is a 1-saturating set, because if a point P would not be covered by the secants of $\mathcal{K}$, then $\mathcal{K}\cup\{P\}$ would be a $(k+1)$-arc. The cardinality of the smallest complete arc in $\mathrm{PG}(2,q)$ is denoted by $t_2(2,q)$. For the known values of $t_2(2,q)$ we refer to [3]. The general lower bounds are $t_2(2,q) > \sqrt{2q}+1$ for arbitrary q and $t_2(2,q) > \sqrt{3q}+1/2$ for $q=p^i$, $i=1,2,3$. But unfortunately the known complete arcs have bigger cardinality. The inequality

$$t_2(2,q) < 2\sqrt{q} + \frac{2}{\sqrt{q}+1}$$

is satisfied only for $q = 8,9,11$ and 13. Table 1 gives the corresponding values of $t_2(2,q)$ and the parameters of the graphs arising from these arcs.

q	$t_2(2,q)$	D	Δ	number of vertices of Γ	$\left\lfloor \frac{\Delta+2}{2} \right\rfloor \cdot \left\lceil \frac{\Delta+2}{2} \right\rceil$.
8	6	2	42	512	484
9	6	2	48	729	625
11	7	2	70	1331	1296
13	8	2	96	2197	2116

Table 1

Besides complete arcs and double blocking sets of Baer subplanes another class of small 1-saturating sets in $\mathrm{PG}(2,p)$ was examined by computer. These point sets are contained in 3 concurrent lines. For small prime orders $p = 11,13,17,19$, using a simple back-track algorithm we found 1-saturating sets of this type with cardinality $10,11,13$ and 14, respectively. The corresponding graphs do not improve the bound in (1).

Now let $n > 2$. Then a set of k points such that no three of them are collinear is called k-cap. A k-cap is complete, if it is not contained in any $(k+1)$-cap. Hence complete caps in $\mathrm{PG}(n,q)$ are 1-saturating sets. For the sizes of the known complete caps we refer to [7]. There is one infinite series which gives better graphs than the GCCG-graphs. Due to Davydov and Drozhzhina-Labinskaya [5], for $n = 2m-1 > 7$ there is a complete $(27 \cdot 2^{m-4} - 1)$-cap in $\mathrm{PG}(n,2)$. This gives a graph of degree $27 \cdot 2^{m-4} - 1$ and of order 2^{2m}. It has much more vertices than the corresponding GCCG-graph, because

$$2^{2m} = 1024 \cdot 2^{2m-10} > 729 \cdot 2^{2m-10} + 27 \cdot 2^{m-5} = \left\lfloor \frac{27 \cdot 2^{m-4}+1}{2} \right\rfloor \cdot \left\lceil \frac{27 \cdot 2^{m-4}+1}{2} \right\rceil.$$

Hence we proved the following theorem.

Theorem 1. *Let $\Delta = 27 \cdot 2^{m-4} - 1$ and $m > 7$. Then*

$$n(\Delta, 2) \geq \frac{256}{729}(\Delta+1)^2.$$

There are sporadic examples, too. For $n = 3$ and $q = 2$ there is a complete 5-cap in $\mathrm{PG}(3,2)$. The corresponding graph has degree $\Delta = 5$ and the number of vertices is $n = 16$. The best known graph of degree 5 and diameter 2 has 24 vertices, and the best known Cayley graph has 18 vertices [2], so in this case there are bigger graphs. For $q = 3, 4$ and 5 the smallest complete caps in $\mathrm{PG}(3,q)$ have $2(q+1)$ points. The corresponding graphs have the same parameters as the GCCG-graphs.

For $n = 4$ and $q = 2, 3, 4$ there are complete caps in $\mathrm{PG}(4,q)$ with cardinalities 9, 11 and 20, respectively. For $n = 5$ and $q = 2, 3$ there are complete caps in $\mathrm{PG}(5,q)$ with cardinalities 13 and 22. The corresponding graphs have more vertices than the previously known examples. Table 2 gives the parameters of the graphs arising from these caps.

projective space	size of the complete cap	D	Δ	number of vertices of Γ	$\left\lfloor \frac{\Delta+2}{2} \right\rfloor \cdot \left\lceil \frac{\Delta+2}{2} \right\rceil$.
PG(4, 2)	9	2	9	32	30
PG(4, 3)	11	2	22	243	144
PG(4, 4)	20	2	60	1024	961
PG(5, 2)	13	2	13	64	56
PG(5, 3)	22	2	44	729	529

Table 2

In $\mathrm{PG}(3, q)$, $q > 3$, the smallest known 1-saturating set has $2q + 1$ points [4]. Let π be a plane, Ω be an oval in π, P be a point of Ω, for q even let $N \in \pi$ be the nucleus of Ω, for q odd let $N \in \pi$ be a point such that the line NP is the tangent to Ω at P, and finally let ℓ be a line such that $\ell \cap \pi = \{P\}$. Then it is easy to check that $(\Omega \cup \ell \cup \{N\}) \setminus \{P\}$ is a 1-saturating set in $\mathrm{PG}(3, q)$. The corresponding graph has degree $\Delta = 2q^2 - q - 1$, and the number of its vertices is $q^4 > (\Delta + \sqrt{\Delta/2} + 5/4)^2/4$. Hence we proved the following theorem.

Theorem 2. *Let $q > 3$ be a prime power and let $\Delta = 2q^2 - q - 1$. Then*

$$n(\Delta, 2) > \frac{1}{4}\left(\Delta + \sqrt{\frac{\Delta}{2}} + \frac{5}{4}\right)^2.$$

Let ℓ_1 and ℓ_2 be two skew lines in $\mathrm{PG}(3, q)$. If P is any point not on $\ell_1 \cup \ell_2$, then the plane generated by P and ℓ_1 meets ℓ_2 in a unique point T_2, and the line PT_2 meets ℓ_1 in a unique point T_1. Hence the line T_1T_2 contains P, so the set of points of $\ell_1 \cup \ell_2$ is a 1-saturating set in $\mathrm{PG}(3, q)$. The corresponding graph has degree $\Delta = 2(q^2 - 1)$, and the number of its vertices is $q^4 = \big((\Delta + 2)/2\big)^2$. Hence this construction gives graphs having the same parameters as the GCCG-graphs.

A straightforward generalization of the skew line construction is the following. Let $\ell_1, \ell_2, \ldots, \ell_m$ be a set of m lines whose union spans $\mathrm{PG}(2m - 1, q)$. Then the set of points of $\cup_{i=1}^{m} \ell_i$ is an $(m - 1)$-saturating set and the corresponding graph has parameters $D = m$, $\Delta = 2m(q^2 - 1)$, and the number of its vertices is q^{2m}. These parameters are the same as the parameters of the GCCG-graphs.

Another class of examples for $(D-1)$-saturating sets in $\mathrm{PG}(D,q)$ is the class of complete arcs. These objects are generalizations of the planar arcs. A point set $\mathcal{K}$ is a complete k-arc in $\mathrm{PG}(D,q)$ if no D points of $\mathcal{K}$ lie in a hyperplane, and there is no $(k+1)$-arc containing $\mathcal{K}$. The corresponding graph has degree $k(q-1)$ and the number of vertices is q^{D+1}. Hence this is better than the known general lower bound if and only if

$$q^{D+1} > \left(\frac{k(q-1)+D}{D}\right)^{D}, \quad \text{that is} \quad k < \frac{D(q\sqrt[D]{q}-1)}{q-1}. \tag{5}$$

The typical examples for complete arcs are the normal rational curves, and almost all of the known complete arcs are normal rational curves, or subsets of these curves. There is only one known complete k-arc which satisfies (5). This is a normal rational curve in $\mathrm{PG}(4,3)$. The corresponding graph has degree $\Delta = 15$, diameter $D = 3$ and the number of its vertices is 256.

Acknowledgements

This research was supported in part by the Slovenian-Hungarian Intergovernmental Scientific and Technological Cooperation Project, Grant No. SLO-9/05, and by "ARRS - Agencija za raziskovalno dejavnost Republike Slovenije", program no. P1-0285. The first author was supported by the Hungarian National Office of Research and Technology, in the framework of "Öveges József" programme, and by the Hungarian National Foundation for Scientific Research, Grant No. T043758.

References

[1] *The (Degree, Diameter) Problem for Graphs,* A World Combinatorics Exhange resource at http://www-mat.upc.es/grup_de_grafs/oldpage.html.

[2] Araujo, G., Noy, M. and Serra, O., *A geometric construction of large vertex transitive graphs of diameter two,* J. Combin. Math. Combin. Comput., **57**(2006), 97-102.

[3] Davydov, A.A., Faina, G., Marcugini, S. and Pambianco, F., *Computer search in projective planes for the sizes of complete arcs,* J. Geom., **82**(2005), 50-62.

[4] Davydov, A.A., *Constructions and families of covering codes and saturating sets of points in projective geometry,* IEEE Transactions on Information Theory **41**(1995), 2071-2080.

[5] Davidov, A.A., Drozhzhina-Labinskaya, A. Yu., *Constructions, families and tables of binary linear covering codes,* IEEE, **40**(1994), 1270-1279.

[6] Godsil, C.D., *Algebraic Combinatorics,* Chapman & Hall, New York, 1993.

[7] Hirschfeld, J.W.P., Storme, S., *The packing problem in statistics, coding theory and finite projective spaces,* in: Finite Geometries, Development of Mathematics, Kluwer, 2001, 201-246.

[8] Miller, M., Širaň, J., *Moore graphs and beyond: A survey of the degree/diameter problem,* Electron. J. Combin., DS14 (2005) 61 pp.

Department of Geometry, Eötvös Loránd University
1117 Budapest, Pázmány s 1/c, Hungary
and
Bolyai Institute, University of Szeged
H-6720 Szeged, Aradi vértanúk tere 1, Hungary
E-mail address: kissgy@cs.elte.hu

Department of Mathematics and Computer Science
University of Primorska
Cankarjeva 5, 6000 Koper, Slovenia
E-mail address: kovacs@pef.upr.si

Department of Mathematics and Computer Science
University of Primorska
Cankarjeva 5, 6000 Koper, Slovenia
E-mail address: klavdijak@pef.upr.si

Institute of Mathematics and Informatics
University of Pécs
7624 Pécs, Ifjúság útja 6, Hungary
E-mail address: ruffjanos@gmail.com

Institute of Mathematics, Physics and Mechanics
IMFM, University of Ljubljana
Jadranska 19, 1000 Ljubljana, Slovenia
E-mail address: Primoz.Sparl@fmf.uni-lj.si

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 3, September 2009

EXTRACTING FUZZY IF-THEN RULE BY USING THE INFORMATION MATRIX TECHNIQUE WITH QUASI-TRIANGULAR FUZZY NUMBERS

ZOLTÁN MAKÓ

Abstract. In the paper [7] C. Huang and C. Moraga suggested a new method to extract fuzzy if-then rules from training data based on information matrix technique with Gaussian membership function. In this paper, we extend this method to the Archimedean t-normed space of quasi-triangular fuzzy numbers.

1. **Introduction**

The core of a fuzzy controller is its set of fuzzy if-then rules. Today, fuzzy control is increasingly seen as a universal approximator (H. B. Verbruggen and P. M. Brujin, 1997) by the control community, and thus is strongly used for approximating functions (D. Dubois and H. Prade, 1997).

A fuzzy system is a set of if-then fuzzy rules that maps inputs to outputs. Each fuzzy rule defines a fuzzy patch in the input-output state space of the function. A fuzzy patch is a fuzzy Cartesian product of if-part fuzzy set and then-part fuzzy set. An additive fuzzy system approximates the function by covering its graph with fuzzy patches (see Figure 1). C. Huang and C. Moraga in 2005 suggested a new method to extract fuzzy if-then rules from training data based on information matrix technique with Gaussian membership function. In this paper, we extend this method to the

Received by the editors: 21.10.2008.
2000 *Mathematics Subject Classification.* 93C42, 03E72.
Key words and phrases. fuzzy rule, information matrix, additive fuzzy system, quasi-triangular fuzzy number, triangular norm.
This paper was presented at the 7-th Joint Conference on Mathematics and Computer Science, July 3-6, 2008, Cluj-Napoca, Romania.

Archimedean t-normed space of quasi-triangular fuzzy numbers and we show that the additive fuzzy system with quasi triangular fuzzy numbers is a function approximator.

2. The Archimedean fuzzy normed space of quasi-triangular fuzzy numbers

Triangular norms and co-norms were introduced by K. Menger (1942) and studied first by B. Schweizer and A. Sklar (1961, 1963, 1983) to model distances in probabilistic metric spaces. In fuzzy sets theory triangular norms and co-norms are extensively used to model logical connection *and* and *or*. In fuzzy literatures, these concepts were studied e. g. by E. Creţu (2001), J. Dombi (1982), D. Dubois and H. Prade (1985), J. Fodor (1991, 1999), S. Jenei (1998, 1999, 2000, 2001, 2004), V. Radu (1974, 1984, 1992).

Definition 2.1. *The function* $N : [0,1] \to [0,1]$ *is a* negation operation *if:*

(i) $N(1) = 0$ *and* $N(0) = 1$*;*

(ii) N *is continuous and strictly decreasing;*

(iii) $N(N(x)) = x$*, for all* $x \in [0,1]$.

Definition 2.2. *Let* N *be a negation operation. The mapping* $T : [0,1] \times [0,1] \to [0,1]$ *is a* triangular norm *(briefly* t-norm*) if satisfies the properties:*

$$\begin{aligned} \mathit{Symmetry} &: T(x,y) = T(y,x),\ \forall x,y \in [0,1]; \\ \mathit{Associativity} &: T(T(x,y),z) = T(x,T(y,z)),\ \forall x,y,z \in [0,1]; \\ \mathit{Monotonicity} &: T(x_1,y_1) \le T(x_2,y_2)\ \ if\, x_1 \le x_2 \ \ and\ y_1 \le y_2; \\ \mathit{One\ identity} &: \ T(x,1) = x,\ \forall x \in [0,1] \end{aligned}$$

and the mapping $S : [0,1] \times [0,1] \to [0,1]$,

$$S(x,y) = N(T(N(x), N(y)))$$

is a triangular co-norm (the dual of T *given by* N*).*

Definition 2.3. *The t-norm* T *is* Archimedean *if* T *is continuous and* $T(x,x) < x$*, for all* $x \in (0,1)$.

Definition 2.4. *The t-norm T is called* strict *if T is strictly increasing in both arguments.*

Theorem 2.5 (C. H. Ling, 1965)**.** *Every Archimedean t-norm T is representable by a continuous and decreasing function $g : [0,1] \to [0,+\infty]$ with $g(1) = 0$ and*

$$T(x,y) = g^{[-1]}(g(x) + g(y)),$$

where

$$g^{[-1]}(x) = \begin{cases} g^{-1}(x) & \text{if} \quad 0 \leq x < g(0), \\ 0 & \text{if} \quad x \geq g(0). \end{cases}$$

If g_1 and g_2 are the generator function of T, then there exist $c > 0$ such that $g_1 = cg_2$.

Remark 2.6. *If the Archimedean t-norm T is strict, then $g(0) = +\infty$ otherwise $g(0) = g_0 < \infty$.*

Theorem 2.7 (E. Trillas, 1979)**.** *An application $N : [0,1] \to [0,1]$ is a negation if and only if an increasing and continuous function $e : [0,1] \to [0,1]$ exists, with $e(0) = 0$, $e(1) = 1$ such that $N(x) = e^{-1}(1 - e(x))$, for all $x \in [0,1]$.*

Remark 2.8. *The generator function of negation $N(x) = 1 - x$ is $e(x) = x$. Another negation generator function is*

$$e_\lambda(x) = \frac{\ln(1 + \lambda x)}{\ln(1 + \lambda)},$$

where $\lambda > -1$, $\lambda \neq 0$.

Remark 2.9. *Examples to t-norm are following:*

- *minim:* $\min(x,y) = \min\{x,y\}$;
- *product:* $P(x,y) = xy$, *the generator function is* $g(x) = -\ln x$;
- *weak:* $W(x,y) = \begin{cases} \min\{x,y\} & \text{if} \quad \max\{x,y\} = 1, \\ 0 & \text{otherwise.} \end{cases}$

If the negation operation is $N(x) = 1 - x$, then the dual of these t-norms are:

- *maxim:* $\max(x,y) = \max\{x,y\}$;
- *probability:* $S_P(x,y) = x + y - xy$;
- *strong:* $S_W(x,y) = \begin{cases} \max\{x,y\} & \text{if} \quad \min\{x,y\} = 0, \\ 1 & \text{otherwise.} \end{cases}$

Proposition 2.10. *If T is a t-norm and S is the dual of T, then*

$$W(x,y) \leq T(x,y) \leq \min\{x,y\},$$
$$\max\{x,y\} \leq S(x,y) \leq S_W(x,y),$$

for all $x, y \in [0,1]$.

The fuzzy set concept was introduced in mathematics by K. Menger in 1942 and reintroduced in the system theory by L. A. Zadeh in 1965. L. A. Zadeh has introduced this notion to measure quantitatively the vagueness of the linguistic variable. The basic idea was: if X is a set, then all A subsets of X can be identified with its characteristic function $\chi_A : X \to \{0,1\}$, $\chi_A(x) = 1 \Leftrightarrow x \in A$ and $\chi_A(x) = 0 \Leftrightarrow x \notin A$.

The notion of fuzzy set is another approach of the subset notion. There exist continue and transitory situations in which we have to sugest that an element belongs to a set at different levels. We indicate this fact with membership degree.

Definition 2.11. *Let X be a set. A mapping $\mu : X \to [0,1]$ is called* membership function*, and the set $A = \{(x, \mu(x)) \ / x \in X\}$ is called* fuzzy set *on X. The membership function of A is denoted by μ_A. The collection of all fuzzy sets on X we will denote by $\mathcal{F}(X)$.*

In order to use fuzzy sets and relations in any intelligent system we must be able to perform set and arithmetic operations. In fuzzy theory the extension of arithmetic operations to fuzzy sets was formulated by L.A. Zadeh in 1965.

The operations on $\mathcal{F}(X)$ are uniquely determined by T, N and the corresponding operations of X by using the generalized t-norm based extension principle (Z. Makó, 2006).

Definition 2.12. *The triplet $(\mathcal{F}(X), T, N)$ will be called* fuzzy t-normed space.

By using t-norm based extension principle the Cartesian product of fuzzy sets may be defined in the following way.

Definition 2.13. *The* T-Cartesian product*'s membership function of fuzzy sets* $A_i \in \mathcal{F}(X_i)$, $i = 1, ..., n$ *is defined as*

$$\mu_{\bar{A}}(x_1, x_2, ..., x_n) = \\ T\left(\mu_{A_1}(x_1), T\left(\mu_{A_2}(x_2), T\left(...T\left(\mu_{A_{n-1}}(x_{n-1}), \mu_{A_n}(x_n)\right)...\right)\right)\right),$$

for all $(x_1, x_2, ..., x_n) \in X_1 \times X_2 \times ... \times X_n$.

The construction of membership function of fuzzy sets is an important problem in vagueness modeling. Theoretically, the shape of fuzzy sets must depend on the applied triangular norm.

We noticed that, if the model constructed on the computer does not comply the requests of the given problem, then we choose another norm. The membership function must be defined in such a way that the change of the t-norm modifies the shape of the fuzzy sets, but the calculus with them remains valid. This desideratum is satisfied, for instance if the quasi-triangular fuzzy numbers introduced by M. Kovacs in 1992 are used.

Let $p \in [1, +\infty]$ and $g : [0, 1] \to [0, \infty]$ be a continuous, strictly decreasing function with the boundary properties $g(1) = 0$ and $\lim_{t \to 0} g(t) = g_0 \leq \infty$. We define the quasi-triangular fuzzy number in fuzzy t-normed space $(\mathcal{F}(\mathbb{R}), T_{gp}, N)$, where

$$T_{gp}(x, y) = g^{[-1]}\left((g^p(x) + g^p(y))^{\frac{1}{p}}\right) \tag{1}$$

is an Archimedean t-norm generated by g and

$$N(x) = \begin{cases} 1 - x & \text{if} \quad g_0 = +\infty, \\ g^{-1}(g_0 - g(x)) & \text{if} \quad g_0 \in \mathbb{R}. \end{cases} \tag{2}$$

is a negation operation.

Definition 2.14. *The* set of quasi-triangular fuzzy numbers *is*

$$\mathcal{N}_g = \{A \in \mathcal{F}(\mathbb{R}) \ / \textit{ there is } a \in \mathbb{R}, d > 0 \textit{ such that} \tag{3}$$
$$\left.\mu_A(x) = g^{[-1]}\left(\frac{|x-a|}{d}\right) \textit{ for all } x \in \mathbb{R}\right\} \bigcup$$
$$\{A \in \mathcal{F}(\mathbb{R}) \ / \textit{ there is } a \in \mathbb{R} \textit{ such that}$$
$$\left.\mu_A(x) = \chi_{\{a\}}(x) \textit{ for all } x \in \mathbb{R}\right\},$$

where χ_A *is characteristic function of the set* A*. The element of* $\mathcal{N}_g$ *will be called* quasi-triangular fuzzy number *generated by* g *with center* λ *and spread* d *and we will denote them with* $<\lambda, d>$ *. The triplet* $(\mathcal{N}_g, T_{gp}, N)$ *is the* Archimedean t-normed space of quasi-triangular fuzzy numbers.

3. **The information matrix**

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be observations of a given sample X. Let A_j, $j = 1, 2, \ldots p$ and B_k, $k = 1, 2, \ldots, q$ be fuzzy sets with membership functions μ_{A_j} and μ_{B_k}, respectively. Let $U = \{A_j | \ j = 1, 2, \ldots p\}$; $V = \{B_k | \ k = 1, 2, \ldots, q\}$. By using the definition of Cartesian product (2.13) we get, that the membership value of sample (x_i, y_i) in fuzzy set $A_j \times B_k$ is

$$r_{j,k}(x_i, y_i) = T\left(\mu_{A_j}(x_i), \mu_{B_k}(y_i)\right).$$

This value is called information gain of (x_i, y_i) at $A_j \times B_k$ with respect to the t-norm T.

The $R = (R_{j,k})_{j=1,2,\ldots\ p;k=1,2,\ldots,q}$ is called an information matrix of X on $U \times V$, where

$$R_{j,k} = \sum_{i=1}^{n} r_{j,k}(x_i, y_i).$$

4. **Extracting fuzzy if–then rules**

We extract fuzzy if–then rules according to the centre of the rows of an information matrix. The method consists of the following steps:

Step 1. We choose a number $p \in [1, +\infty]$ and a derivable generator function $g : [0,1] \to [0, +\infty]$ of the T_{gp} norm with $\int\limits_0^1 g(x)\,dx = \omega \in \mathbb{R}$.

Step 2. We divide the illustrating space $[a, b] \times [m, M]$ in $p \times q$ square with grid points (u_j, v_k).

Step 3. Let $A_j =< u_j, h_u >$ and $B_k =< v_k, h_v >$ be quasi-triangular fuzzy numbers with spread $h_u = 2 * \frac{b-a}{p-1}$ and $h_v = 2 * \frac{M-m}{q-1}$, for all $j = 0, 1, ..., p-1$, $k = 0, 1, ..., q-1$.

Step 4. We calculate the information gains:

$$r'_{j,k,i} = T_{gp}\left(\mu_{A_j}(x_i), \mu_{B_k}(y_i)\right) = g^{[-1]}\left(\left[\left(\frac{|x_i - u_j|}{h_u}\right)^p + \left(\frac{|y_i - v_k|}{h_v}\right)^p\right]^{1/p}\right).$$

Step 5. We calculate the normalized information gains:

$$r_{j,k,i} = \frac{r'_{j,k,i}}{\sum\limits_{j=1}^{p}\sum\limits_{k=1}^{q} r'_{j,k,i}}. \tag{4}$$

Step 6. We calculate the normalized information matrix:

$$R = \begin{bmatrix} R_{1,1} & R_{1,2} & \cdots & R_{1,q} \\ R_{2,1} & R_{2,2} & \cdots & R_{2,q} \\ \vdots & \vdots & \ddots & \vdots \\ R_{p,1} & R_{p,2} & \cdots & R_{p,q} \end{bmatrix}, \tag{5}$$

where

$$R_{j,k} = \sum_{i=1}^{n} r_{j,k,i}. \tag{6}$$

Step 7. We determine the centre of all rows in the normalized information matrix R:

$$c_j = \frac{\sum\limits_{k=1}^{q} R_{j,k} \cdot v_k}{\sum\limits_{k=1}^{q} R_{j,k}}. \tag{7}$$

Step 8. Because each B_k is quasi-triangular fuzzy number, the then-part of fuzzy if-then rules would have the same shape, hence the rule consequent is $\bar{B}_j =<$

$c_j, h_v >$ with membership function

$$\mu_{\bar{B}_j}(v) = g^{[-1]}\left(\frac{|v - c_j|}{h_v}\right).$$

Therefore, we obtain following fuzzy if-then rules:

If x is $< u_1, h_u >$ then y is $< c_1, h_v >$,

If x is $< u_2, h_u >$ then y is $< c_2, h_v >$,

$\vdots$

If x is $< u_p, h_u >$ then y is $< c_p, h_v >$.

An "If x is $< u_j, h_u >$ then y is $< c_j, h_v >$" rule is equivalent to the fuzzy set $A_j \times \bar{B}_j$ with membership function

$$\begin{aligned}
\mu_{A_j \times \bar{B}_j}(x, y) &= T_{gp}\left(\mu_{A_j}(x), \mu_{\bar{B}_j}(y)\right) \\
&= g^{[-1]}\left(\left[\left(\frac{|x - u_j|}{h_u}\right)^p + \left(\frac{|y - c_j|}{h_v}\right)^p\right]^{1/p}\right) \text{ for all } (x, y) \in \mathbb{R}^2.
\end{aligned}$$

The graph of an $A_j \times \bar{B}_j$ fuzzy set is a fuzzy patch. The size of the patch reflects the rule's vagueness or uncertainty and cover the graph of the approximand function f (See figure 1).

Step 9.The approximator function is

$$F(u) = \frac{\sum_{j=1}^{p} c_j \cdot g^{[-1]}\left(\frac{|u - u_j|}{h_u}\right)}{\sum_{j=1}^{p} g^{[-1]}\left(\frac{|u - u_j|}{h_v}\right)}.$$

Theorem 4.1. *If $f : [a, b] \to \mathbb{R}$ is continuous then F uniformly approximates the f on $[a, b]$.*

Proof. A standard additive system is a function system $G : \mathbb{R}^t \to \mathbb{R}^l$ with p fuzzy rules "If x is A_j then y is B_j" or the patch form $A_j \times B_j$. The if-part fuzzy sets A_j and the then-part fuzzy sets have membership function μ_{A_j} and μ_{B_j}. B. Kosko in

1994 proof that the function

$$G(u) = \frac{\sum_{j=1}^{p} w_j \mu_{A_j}(u) S_j c_j}{\sum_{j=1}^{p} w_j \mu_{A_j}(u) S_j}$$

uniformly approximates the function $f : X \to \mathbb{R}^l$ if $X \subset \mathbb{R}^t$ is compact and f continuous, where w_j is the rule weight, S_j is the volume (or area) of subgraph of B_j, and c_j is the centroid of B_j.

Generally, the centroid of the quasi-triangular fuzzy number $< \lambda, \delta >$ is

$$c = \frac{\int_{-\infty}^{\infty} y \mu_{<\lambda,\delta>}(y)\, dy}{\int_{-\infty}^{\infty} \mu_{<\lambda,\delta>}(y)\, dy} = \lambda,$$

where

$$\int_{-\infty}^{\infty} y \mu_{<\lambda,\delta>}(y)\, dy = 2\lambda\delta\omega \text{ and area is } S = \int_{-\infty}^{\infty} \mu_{<\lambda,\delta>}(y)\, dy = 2\delta\omega.$$

An additive fuzzy system with the same rule weight ($w_1 = w_2 = ... = w_p$) and with fuzzy sets $A_j =< u_j, h_u >$, $\bar{B}_j =< c_j, h_v >$ is the following function approximator:

$$\begin{aligned} G(u) &= \frac{\sum_{j=1}^{p} w_j g^{[-1]}\left(\frac{|u-u_j|}{h_X}\right) 2h_v \omega c_j}{\sum_{j=1}^{p} w_j g^{[-1]}\left(\frac{|u-u_j|}{h_X}\right) 2h_v \omega} \\ &= \frac{\sum_{j=1}^{p} c_j g^{[-1]}\left(\frac{|u-u_j|}{h_X}\right)}{\sum_{j=1}^{p} g^{[-1]}\left(\frac{|u-u_j|}{h_X}\right)} = F(u). \end{aligned}$$

□

Remark 4.2. 1. *If we choose* $g : (0,1] \to [0,\infty)$, $g(t) = \sqrt{-\ln t}$ *and* $p = 2$, *then the membership function of quasi-triangular fuzzy numbers* $< a, d >$ *is*

$$\mu(t) = e^{-\frac{(t-a)^2}{d^2}} \quad \text{if } d > 0, \quad \text{and} \quad \mu(t) = \begin{cases} 1 \text{ if } t = a, \\ 0 \text{ if } t \neq a \end{cases} \quad \text{if } d = 0.$$

and $r'_{j,k,i} = \mu_{A_j}(x_i) \cdot \mu_{B_k}(y_i)$. *This is the information matrix technique model with the normal diffusion function elaborated by C. Huang - C. Moraga in 2005.*

2. *If we choose* $g : (0,1] \to [0,\infty)$, $g(t) = -\ln t$ *and* $p = 1$, *then the membership functions of quasi-triangular fuzzy numbers* $< a, d >$ *is*

$$\mu(t) = e^{-\frac{|t-a|}{d}} \quad \text{if } d > 0, \quad \text{and} \quad \mu(t) = \begin{cases} 1 \text{ if } \ t = a, \\ 0 \text{ if } \ t \neq a \end{cases} \quad \text{if } d = 0.$$

and $r'_{j,k,i} = \mu_{A_j}(x_i) \cdot \mu_{B_k}(y_i)$. *This is the model with Laplace membership function studied by S. Mitaim and B. Kosko in 2001.*

Example 4.3. *Let us use the information matrix technique to approach the following nonlinear function:*

$$f : [-6, 6] \to \mathbb{R}, \ f(x) = x \sin x$$

by consider a sample with 121 values from $[-6, 6] \times [-6, 6]$ *with uniform distribution to be input values:*

$$X = \{(x_i, y_i) : \ x_i = -6 + 0.1 * i, \ y_i = f(x_i), \ i = 0, 1, ..., 120\}.$$

Step 1. Let $g : [0,1] \to [0,1]$, $g(t) = 1 - t^2$ be the generator function and $p = 3$ and

$$g^{[-1]}(t) = \begin{cases} \sqrt{1-t} \text{ if } \ t \in [0,1], \\ 0 \text{ else.} \end{cases}$$

Step 2. We divide the illustrating space $[-6, 6] \times [-6, 6]$ in 50×100 square with grid points (u_j, v_k), where $u_j = -6 + j \cdot \frac{h_u}{2}$ and $v_k = -6 + k \cdot \frac{h_v}{2}$, $h_u = 0.488$, $h_v = 0.242$, $j = 0, ..., 49$, $k = 0, ..., 99$.

Step 3. In this case $A_j =< u_j, h_u >$ and $B_k =< v_k, h_v >$ with membership functions

$$\mu_{A_j}(t) = \begin{cases} \sqrt{1 - \frac{|t-u_j|}{h_u}} \text{ if } \ t \in [u_j - h_u, u_j + h_u], \\ 0 \text{ else,} \end{cases}$$

$$\mu_{B_k}(t) = \begin{cases} \sqrt{1 - \frac{|t-v_k|}{h_v}} \text{ if } \ t \in [v_k - h_v, v_k + h_v], \\ 0 \text{ else.} \end{cases}$$

Step 4. The information gains are:

$$r'_{j,k,i} = \begin{cases} \sqrt{1-\left[\left(\frac{|x_i-u_j|}{h_u}\right)^3+\left(\frac{|y_i-v_k|}{h_v}\right)^3\right]^{1/3}} \text{ if } \left(\frac{|x_i-u_j|}{h_u}\right)^3+\left(\frac{|y_i-v_k|}{h_v}\right)^3 \leq 1, \\ 0 \text{ else.} \end{cases}$$

We calculate the normalized information matrix by using the formulas (4), (5) and (6).

Step 7. We calculate the centres of all rows in the normalized information matrix by formula (7):

$$\begin{aligned} c = {} & (-2.37, -2.90, -3.69, -4.33, -4.65, -4.64, -4.34, -3.83, -3.11, -2.28, \\ & -1.42, -0.56, 0.19, 0.83, 0.30, 0.61, 0.74, 0.71, 0.55, 0.28, \\ & 0.97, 0.67, 0.38, 0.17, 0.06, 0.064, 0.17, 0.38, 0.67, 0.97, \\ & 1.28, 1.55, 1.71, 1.74, 1.61, 1.30, 0.83, 0.19, -0.56, -1.42, \\ & -2.28, -3.11, -3.83, -4.34, -4.64, -4.65, -4.33, -3.69, -2.90, -2.37)^T . \end{aligned}$$

Step 8. The membership function of the then-part of fuzzy if-then rules are

$$\mu_{\bar{B}_j}(v) = \begin{cases} \sqrt{1-\frac{|v-c_j|}{h_v}} \text{ if } t \in [c_j - h_v, c_j + h_v], \\ 0 \text{ else.} \end{cases}$$

The membership function of $A_j \times \bar{B}_j$ fuzzy sets are

$$\mu_{A_j \times \bar{B}_j}(x,y) = \begin{cases} \sqrt{1-\left[\left(\frac{|x-u_j|}{h_u}\right)^3+\left(\frac{|y-v_k|}{h_v}\right)^3\right]^{1/3}} \text{ if } \left(\frac{|x-u_j|}{h_u}\right)^3+\left(\frac{|y-v_k|}{h_v}\right)^3 \leq 1, \\ 0 \text{ else.} \end{cases}$$

for all $(x,y) \in \mathbb{R}^2$. The graphs of these functions are patches on the figure 1.

Step 9. The approximator function of f is

$$F(x) = \frac{\sum\limits_{j=1}^{p} c_j \cdot g^{[-1]}\left(\frac{|x-u_j|}{h_u}\right)}{\sum\limits_{j=1}^{p} g^{[-1]}\left(\frac{|x-u_j|}{h_u}\right)},$$

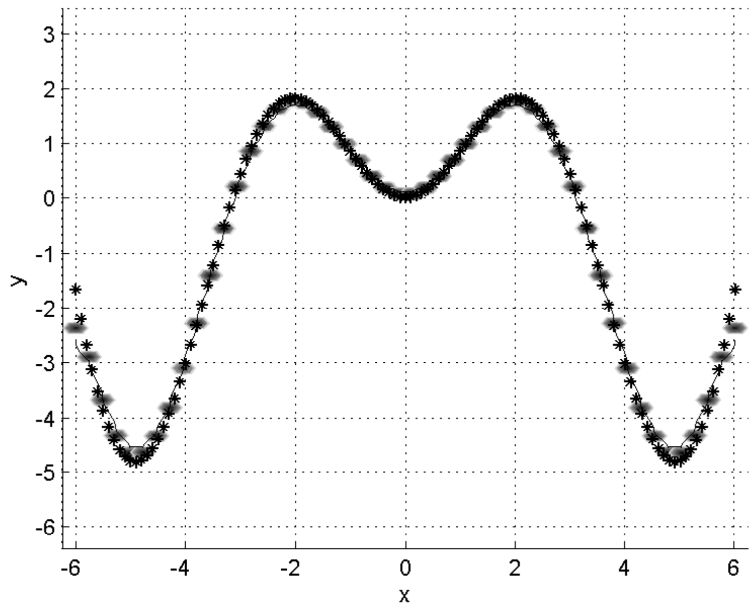


Figure 1. The additive fuzzy system F approximates the function f given in the example 4.3 by covering its graph with fuzzy patches. On the figure the stars are the elemets of sample, the curve is the graph of F and the pathces are the graph of fuzzy sets $A_j \times \bar{B}_j$.

for all $x \in [-6, 6]$, where

$$g^{[-1]}\left(\frac{|x-u_j|}{h_u}\right) = \begin{cases} \sqrt{1-\frac{|x-u_j|}{h_u}} \text{ if } x \in [u_j - h_u, u_j + h_u], \\ 0 \text{ else.} \end{cases}$$

References

[1] Creţu, E., *A triangular norm hierarchy*, Fuzzy Sets and Systems, **120**(2001), 371-383.

[2] Dombi, J., *Basic concepts for theory of evaluation: The aggregative operator*, European J. of Operational Research, **10**(1982), 282-294.

[3] Dubois, D., Prade, H., *A review of fuzzy set aggregation connectives*, Inform. Sci., **36**(1985), 85-121.

[4] Dubois, D., Prade, H., *The three semantics of fuzzy sets*, Fuzzy Sets and Systems, **90(2)**(1997), 141-150.

[5] Fodor, J., *A remark on constructing t-norms*, Fuzzy Sets and Systems, **41**(1991), 181-188.

[6] Fodor, J., Jenei, S., *On reversible triangular norms*, Fuzzy Sets and Systems, **104**(1999), 43-51.

[7] Huang, C., Moraga, C., *Extracting fuzzy if-then rules by using the information matrix technique*, Journal of Computer and System Science, **70**(2005), 26-52.

[8] Jenei, S., *Fibred triangular norms*, Fuzzy Sets and Systems, **103**(1999), 67-82.

[9] Jenei, S., *New family of triangular norms via contrapositive symmetrization of residuated implications*, Fuzzy Sets and Systems, **110**(2000), 157-174.

[10] Jenei, S., *Continuity of left-continuous triangular norms with strong induced negations and their boundary condition*, Fuzzy Sets and Systems, **124**(2001), 35-41.

[11] Jenei, S., *How to construct left-continuous triangular norms-state ofthe art*, Fuzzy Sets and Systems, **143**(2004), 27-45.

[12] Jenei, S., Fodor, J.C., *On continuous triangular norms*, Fuzzy Sets and Systems, **100**(1998), 273-282.

[13] Kosko, B., *Fuzzy systems as universal approximators*, IEEE Transactions on Computers, **43(11)**(1994), 1329-1333.

[14] Kovács, M., *A stable embedding of ill-posed linear systems into fuzzy systems*, Fuzzy Sets and Systems, **45**(1992), 305-312.

[15] Ling, C.H., *Representation of associative functions*, Publ. Math. Debrecen, **12**(1965), 189-212.

[16] Makó, Z., *Quasi-triangular fuzzy numbers. Theory and applications*, Scientia Publishing House, Cluj-Napoca, 2006.

[17] Menger, K., *Statistical metrics*, Proc. Nat. Acad. Sci. USA, **28**(1942), 535-537.

[18] Mitaim, S., Kosko, B., *The shape of fuzzy sets in adaptive function approximation*, IEEE Transactions on Fuzzy Systems, **9(4)**(2001), 637-656.

[19] Radu, V., *On the metrizability of Menger spaces*, Math. Balcanica, **4**(1974), 497-498.

[20] Radu, V., *On the t-norm of Hadzic type and locally convex random normed spaces*, Seminarul de Teoria Probabilităţilor şi Aplicaţii, Univ. Timişoara, No. 70, 1984.

[21] Radu, V., *Some remarks on the representation of triangular norms*, Anale Univ. Timisoara, **30(1)**(1992), 119-130.

[22] Schweizer, B., Sklar, A., *Associative functions and statistical triangle inequalities*, Publ. Math. Debrecen, **8**(1961), 169-186.

[23] Schweizer, B., Sklar, A., *Associative functions and abstract semigroups*, Publ. Math. Debrecen, **10**(1963), 69-81.

[24] Schweizer, B., Sklar, A., *Probabilistic Metric Spaces*, North-Holland, Amsterdam, 1983.

[25] Trillas, E., *Sobre functiones de negación en la teoria de conjuntos difusos*, Stochastica, **3**(1979), 47-60.

[26] Verbruggen, H.B., Brujin, P.M., *Fuzzy control and conventional control: what is (and can be) the real contribution of fuzzy systems?* Fuzzy Sets and Systems, **90(2)**(1997), 151-160.

[27] Zadeh, L.A., *Fuzzy Sets*, Information and Control, **8**(1965), 338-353.

Department of Mathematics and Computer Science
Sapientia University, Miercurea Ciuc, Romania
E-mail address: `makozoltan@sapientia.siculorum.ro`

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 3, September 2009

MULTIPLE SOLUTIONS FOR A HOMOGENEOUS SEMILINEAR ELLIPTIC PROBLEM IN DOUBLE WEIGHTED SOBOLEV SPACES

ILDIKÓ ILONA MEZEI AND TÜNDE KOVÁCS

Abstract. In this paper we obtain multiple solutions in double weighted Sobolev spaces for an elliptic semilinear eigenvalue problem on unbounded domain, with sublinear growth of the nonlinear term. In the proofs of the main results we use variational methods and some recent theorems from the theory of best approximation in Banach spaces, established by Ricceri in [11] and Tsar'kov in [12].

1. **Introduction**

A link between the critical point theory and the theory of best approximation was established recently by Ricceri in [11] and Tsar'kov in [12]. In the latter it is proved that, given a continuously Gâteaux differentiable functional J defined over a real Hilbert space X, for each real σ within the range of J and $x_0 \in J^{-1}(]-\infty, \sigma[)$ either there exists $\lambda > 0$ such that the energy functional $\mathcal{E}_\lambda(x) = \frac{||x-x_0||^2}{2} - \lambda J(x)$ admits at least three critical points, or the set $J^{-1}([\sigma, +\infty[)$ has a unique point minimizing the distance from x_0. The alternative is then resolved. Supposing that J admits non-convex superlevel set, and applying the results of [12], yields that the energy functional $\mathcal{E}_\lambda$ has at least three critical points for suitable $x_0 \in X$ and $\lambda > 0$. This abstract result has a natural application in the field of differential equations.

Received by the editors: 11.01.2007.

2000 *Mathematics Subject Classification.* 35J20, 46E35.

Key words and phrases. semilinear elliptic equation, eigenvalue problem, variational methods, unbounded domain, weighted Sobolev space.

This paper was presented at the 7-th Joint Conference on Mathematics and Computer Science, July 3-6, 2008, Cluj-Napoca, Romania.

The research of the author was supported by MEdC-ANCS, Grant PN. II, ID_527/2007.

The result of Ricceri was applied and extended by several authors: Kristály in [4] study a Schrödinger equation in $\mathbb{R}^N$, Faracci and Iannizzotto in [2] study boundary value problems involving the p-Laplacian on unbounded domain, Faracci, Iannizzotto, Lisei, Varga in [3] give a multiplicity result in alternative form for a class of locally Lipschitz functionals, defined on Banach spaces and applied to hemivariational inequalities on unbounded domain.

In this paper we consider a semilinear elliptic eigenvalue problem on unbounded domain and we apply a topological minimax result of Ricceri [10] to obtain a similar theorem (in alternative form) with the result of Ricceri presented above. Then, as a consequence of the obtained theorem, using the results of Tsar'kov [12], we obtain three different solutions of the considered problem.

The main problem we are confronting, is the lack of compact embeddings of Sobolev spaces. In general, if Ω is unbounded, $W^{1,p}(\Omega)$ (the space of all functions $u \in L^p(\Omega)$, such that $|\nabla u| \in L^p(\Omega)$) is not compactly embedded in any $L^r(\Omega)$. We will overcome this difficulty by using the double weighted Sobolev space $W^{1,2}(\Omega; v_0, v_1)$ with such weight functions $v_0,\ v_1,\ w$ that $W^{1,2}(\Omega; v_0, v_1)$ can be embedded compactly in $L^p(\Omega; w)$ (for $p \in [2, 2^*[$).

2. **The problem and preliminaries**

Let $\Omega \subset \mathbb{R}^N,\ (N \geq 2)$ be an unbounded domain with smooth boundary $\partial\Omega$. For the positive measurable functions u and w, both defined in Ω, we define the weighted p-norm $(1 \leq p < \infty)$ as

$$||u||_{p,\Omega,w} = \left(\int_\Omega |u(x)|^p w(x) dx \right)^{\frac{1}{p}}$$

and denote by $L^p(\Omega; w)$ the space of all measurable functions u such that $||u||_{p,\Omega,w}$ is finite. If $p = +\infty$ we consider the Sobolev space

$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \text{ is measurable, } \exists C > 0 \text{ such that } |u(x)| \leq C \text{ a.e. in } \Omega\}$$

endowed with the norm

$$||u||_\infty = \inf\{C : |u(x)| \leq C \text{ for a.e. } x \in \Omega\}.$$

The double weighted Sobolev space

$$W^{1,p}(\Omega; v_0, v_1)$$

is defined as the space of all functions $u \in L^p(\Omega; v_0)$ such that all derivatives $\frac{\partial u}{\partial x_i}$ belong to $L^p(\Omega; v_1)$. The corresponding norm is defined by

$$||u||_{p,\Omega,v_0,v_1} = \left(\int_\Omega |\nabla u(x)|^p v_1(x) + |u(x)|^p v_0(x) dx \right)^{\frac{1}{p}}.$$

We are choosing our weight functions from the so-called Muckenhoupt class A_p, which is defined as the set of all positive functions v in $\mathbb{R}^N$ satisfying

$$\frac{1}{|Q|} \left(\int_\Omega v \; dx \right)^{\frac{1}{p}} \left(\int_\Omega v^{-\frac{1}{p-1}} \; dx \right)^{\frac{p-1}{p}} \leq \bar{C}, \text{ if } 1 < p < \infty$$

$$\frac{1}{|Q|} \int_\Omega v \; dx \leq \bar{C} \text{ ess} \inf_{x \in Q} v(x), \text{ if } p = 1,$$

for all cubes $Q \in \mathbb{R}^N$ and some $\bar{C} > 0$.

In this paper we always assume that the weight functions $v_0,\ v_1,\ w$ are defined on Ω, belong to A_p and are chosen such that the following condition holds:
(E) for $p \in [2, 2^*[$ the embedding $W^{1,2}(\Omega; v_0, v_1) \hookrightarrow L^p(\Omega; w)$ is compact.

Such weight functions there exist, see for example [7], [8].

The best embedding constant is denoted by $C_{p,\Omega}$, i.e. we have the inequality

$$||u||_{p,\Omega,w} \leq C_{p,\Omega} ||u||_{v_0,v_1}, \quad \text{for all } u \in W^{1,2}(\Omega; v_0, v_1) \tag{1}$$

where we used the abbreviation $||u||_{v_0,v_1} = ||u||_{2,\Omega,v_0,v_1}$.
We define on $W^{1,2}(\Omega; v_0, v_1)$ a continuous bilinear form associated with the operator $A(u) = -\Delta u + b(x)u$ as

$$\langle u, v \rangle_A = \int_\Omega (\nabla u \nabla v + b(x)uv) dx \tag{2}$$

and the corresponding norm with

$$||u||_A^2 = \langle u, u \rangle_A = \int_\Omega (|\nabla u(x)|^2 + b(x)|u(x)|^2) dx. \tag{3}$$

Now, we define the Banach space

$$X_A = \{u \in W^{1,2}(\Omega; v_0, v_1) : ||u||_A < \infty\}, \tag{4}$$

endowed with the norm $||\cdot||_A$.

We consider the following problem

For a given $u_0 \in X_A$ *and* $\lambda > 0$ *find* $u \in X_A$ *such that*

$$(P_\lambda) \qquad \begin{cases} -\Delta(u-u_0)+b(x)(u-u_0)=\lambda\alpha(x)f(u) \ \text{in } \Omega \\ \\ \hfill u=0 \qquad\quad \text{on } \partial\Omega, \end{cases}$$

where $f:\mathbb{R}\to\mathbb{R}$ *is a continuous function,* $\alpha:\Omega\to\mathbb{R}$ *and* $b:\Omega\to\mathbb{R}$ *are a positive and measurable functions.*

By the *weak solution* to this problem we mean a function $u \in X_A$, such that for every $v \in X_A$ we have

$$\langle u-u_0, v\rangle_A - \lambda \int_\Omega \alpha(x) f(u(x)) v(x) dx = 0.$$

We will study the problem (P_λ) assuming that f is *sublinear* at the origin, that is

(f) $f(0)=0$ and there is a positive measurable function $f_0:\Omega\to\mathbb{R}$ satisfying $f_0 \in L^{\frac{p}{p-1}}(\Omega, w^{\frac{1}{1-p}})$, $f_0(x) \le C_f w(x)$ for a.e. $x \in \Omega$, where C_f is a positive constant and there exists $q \in]0,1[$ such that

$$|f(s)| \le f_0(x)|s|^q, \ \text{for every } s\in\mathbb{R} \text{ and every } x\in\Omega;$$

Furthermore we consider the following assumptions:

(K) ellipticity condition: there is a positive constant K, such that

$$||u||_A^2 \ge 2K||u||^2_{v_0,v_1}, \ \text{for every } u \in W^{1,2}(\Omega; v_0, v_1);$$

(α) $\alpha \in L^1(\Omega, w)\cap L^\infty(\Omega)$.

In the sequel we prove several lemmas needed later.

Lemma 2.1. $L^1(\Omega; w)\cap L^\infty(\Omega) \subseteq L^r(\Omega; w)$, *for every* $r \ge 1$.

Proof. Let $u \in L^1(\Omega; w) \cap L^\infty(\Omega)$. Then, for every $r \geq 1$ we have

$$\begin{aligned} ||u||_{r,\Omega,w} &= \left(\int_\Omega |u(x)|^r w(x)dx\right)^{\frac{1}{r}} = \left(\int_\Omega |u(x)|^{r-1}|u(x)|w(x)dx\right)^{\frac{1}{r}} \leq \\ &\leq \left(\int_\Omega ||u||_\infty^{r-1}|u(x)|w(x)dx\right)^{\frac{1}{r}} = ||u||_\infty^{\frac{r-1}{r}}||u||_{1,w}^{\frac{1}{r}}, \end{aligned}$$

which means that $||u||_{r,\Omega,w}$ is finite, so $u \in L^r(\Omega; w)$. □

Notation. Let $\nu = \dfrac{p}{p-(q+1)}$ and we denote by $\nu' = \frac{p}{q+1}$ its conjugate, that is $\frac{1}{\nu} + \frac{1}{\nu'} = 1$. Using the Lemma 2.1 we have that

$$L^1(\Omega; w) \cap L^\infty(\Omega) \subseteq L^\nu(\Omega; w),$$

so $\alpha \in L^\nu(\Omega; w)$.

We define the funtional $J : X_A \to \mathbb{R}$ by

$$J(u) = \int_\Omega \alpha(x)F(u(x))dx,$$

where $F(t) = \int_0^t f(s)ds$.

The next lemma summarize the properties of the functional J.

Lemma 2.2. *Let conditions (f), (K), (α) be satisfied.Then, the functional J is well defined and it is sequentially weakly continuous.*

Proof. From the assumption (f) we have

$$|F(u(x))| \leq \int_0^{u(x)} |f(s)|ds \leq f_0(x)\int_0^{u(x)} |s|^q ds \leq f_0(x)|u(x)|^{q+1}. \tag{5}$$

Then, using the conditions (f), (E) and the Hölder's inequality, we get

$$\begin{gathered} |J(u)| = \left|\int_\Omega \alpha(x)F(u(x))dx\right| \leq \int_\Omega \alpha(x)f_0(x)|u(x)|^{q+1}dx \leq \\ \leq C_f\int_\Omega \alpha(x)|u(x)|^{q+1}w(x)dx \leq C_f\int_\Omega \alpha(x)w(x)^{\frac{1}{\nu}}|u(x)|^{q+1}w(x)^{\frac{1}{\nu'}}dx \leq \\ \leq C_f\left(\int_\Omega \alpha(x)^\nu w(x)dx\right)^{\frac{1}{\nu}}\left(\int_\Omega |u(x)|^{\nu'(q+1)}w(x)dx\right)^{\frac{1}{\nu'}} = \\ = C_f||\alpha||_{\nu,\Omega,w}\left(\int_\Omega |u(x)|^p w(x)dx\right)^{\frac{q+1}{p}} = C_f||\alpha||_{\nu,\Omega,w}||u||_{p,\Omega,w}^{q+1} \leq \\ \leq C_f||\alpha||_{\nu,\Omega,w}C_{p,w}^{q+1}||u||_{v_0,v_1}^{q+1} \leq C_f||\alpha||_{\nu,\Omega,w}C_{p,w}^{q+1}(2K)^{-\frac{q+1}{2}}||u||_A^{q+1} = \end{gathered}$$

$$= C||u||_A^{q+1},$$

which means that the functional J is well defined over X_A.

We prove now, that J is sequentially weakly continuous. Let $\{u_n\}$ be a sequence in X_A, weakly convergent to some $u \in X_A$. Then, by the embedding (E), it follows that $||u_n - u||_{p,\Omega,w} \to 0$.

We use the following result: for all $s \in (0, \infty)$ there is a constant $C_s > 0$ such that

$$(x + y)^s \leq C_s(x^s + y^s), \quad \text{for any } x, y \in (0, \infty). \tag{6}$$

Applying the (6), the Hölder inequalities and the Mean Value Theorem, we obtain

$$|J(u_n) - J(u)| = \left| \int_\Omega \alpha(x)F(u_n(x))dx - \int_\Omega \alpha(x)F(u(x))dx \right| \leq$$

$$\leq \int_\Omega \alpha(x)|F(u_n(x)) - F(u(x))|dx =$$

$$= \int_\Omega \alpha(x)|f((1-\theta)u_n(x) + \theta u(x))||u_n(x) - u(x)|dx \leq$$

$$\leq \int_\Omega \alpha(x)f_0(x)\,|(1-\theta)u_n(x) + \theta u(x)|^q\,|u_n(x) - u(x)|dx \leq$$

$$\leq \int_\Omega \alpha(x)f_0(x)\,((1-\theta)|u_n(x)|^q + \theta|u(x)|^q)\,|u_n(x) - u(x)|dx \leq$$

$$\leq C_f \int_\Omega \alpha(x)w(x)^{\frac{1}{\nu}}\,(|u_n(x)|^q + |u(x)|^q)\,|u_n(x) - u(x)|w(x)^{\frac{1}{\nu'}}dx \leq$$

$$\leq C_f||\alpha||_{\nu,\Omega,w} \left(\int_\Omega (|u_n(x)|^q + |u(x)|^q)^{\nu'}\,|u_n(x) - u(x)|^{\nu'} w(x)dx \right)^{\frac{1}{\nu'}} =$$

$$= C_f||\alpha||_{\nu,\Omega,w}C_1 \cdot$$

$$\cdot \left[\int_\Omega \left(|u_n(x)|^{\frac{pq}{q+1}} + |u(x)|^{\frac{pq}{q+1}} \right) w(x)^{\frac{q}{q+1}}\,(|u_n(x) - u(x)|^p)^{\frac{1}{q+1}}\,w(x)^{\frac{1}{q+1}}dx \right]^{\frac{1}{\nu'}} \leq$$

$$\leq C_f||\alpha||_{\nu,\Omega,w}C_1 \left[\left(\int_\Omega |u_n(x)|^p w(x)dx \right)^{\frac{q}{q+1}} + \left(\int_\Omega |u(x)|^p w(x)dx \right)^{\frac{q}{q+1}} \right]^{\frac{1}{\nu'}} \cdot$$

$$\cdot \left(\int_\Omega |u_n(x) - u(x)|^p w(x)dx \right)^{\frac{1}{q+1}\cdot\frac{1}{\nu'}} =$$

$$= C_f||\alpha||_{\nu,\Omega,w}C_1 \left(||u_n||_{p,\Omega,w}^{\frac{pq}{q+1}} + ||u||_{p,\Omega,w}^{\frac{pq}{q+1}} \right)^{\frac{q+1}{p}} ||u_n - u||_{p,\Omega,w} \leq$$

$$\leq C_f||\alpha||_{\nu,\Omega,w}C_1C_2 \left(||u_n||_{p,\Omega,w}^q + ||u||_{p,\Omega,w}^q \right) ||u_n - u||_{p,\Omega,w} \leq$$

$$\leq C_f||\alpha||_{\nu,\Omega,w}C_1C_2C^q_{p,w}\left(||u_n||^q_{v_0,v_1}+||u||^q_{v_0,v_1}\right)||u_n-u||_{p,\Omega,w},$$

$$\leq C_f||\alpha||_{\nu,\Omega,w}C_1C_2C^q_{p,w}(2K)^{\frac{-q}{2}}\left(||u_n||^q_A+||u||^q_A\right)||u_n-u||_{p,\Omega,w},$$

where $\theta\in]0,1[$ is the constant from the Mean Value Theorem, C_1, C_2 are the constants from the inequality (6) and K is the constant from the ellipticity condition (K).

Since $\{u_n\}$ is weakly convergent to $u\in X_A$, we can assume without loss of generality that there exist a constant $M>0$ such that

$$||u_n||_A\leq M \text{ and } ||u_n-u||_A\leq M, \text{ for all } n\in\mathbb{N}.$$

Then we have

$$|J(u_n)-J(u)|\leq||\alpha||_{\nu,\Omega,w}C_fC_1C_2C^q_{p,w}(2K)^{\frac{-q}{2}}2M^q\cdot||u_n-u||_{p,\Omega,w},$$

concluding that $J(u_n)\to J(u)$, whenever $n\to\infty$. □

Now, for a given $u_0\in X_A$ and for $\lambda>0$, we can define the energy functional $\mathcal{E}_\lambda:X_A\to\mathbb{R}$ related to the problem (P_λ) by

$$\mathcal{E}_\lambda(u)=\frac{1}{2}||u-u_0||^2_A-\lambda J(u).$$

We observe, that for every $v\in X_A$, we have

$$\langle\mathcal{E}'_\lambda(u),v\rangle_A=\langle u-u_0,v\rangle_A-\lambda\int_\Omega\alpha(x)f(u(x))v(x)dx. \tag{7}$$

Hence the critical points of the energy functional $\mathcal{E}_\lambda$ are exactly the weak solutions of the problem (P_λ). Therefore, instead of looking for solutions of the problem (P_λ), we are seeking for the critical points of $\mathcal{E}_\lambda$.

In the next lemmas we prove two properties of the energy functional, namely that $\mathcal{E}_\lambda$ is coercive and it satisfies the Palais-Smale condition, for every $\lambda>0$.

Lemma 2.3. *Let the conditions (f), (K), (α) be satisfied. Then the functional $\mathcal{E}_\lambda$ is coercive, for every $\lambda>0$.*

Proof. Using again the Hölder's inequality combined with the conditions (f) and (E), we obtain

$$\begin{aligned}
&\mathcal{E}_\lambda(u) = \frac{1}{2}||u-u_0||_A^2 - \lambda\int_\Omega \alpha(x)F(u(x))v(x)dx \geq \\
\geq\ & \frac{1}{2}||u-u_0||_A^2 - \lambda\int_\Omega \alpha(x)f_0(x)|u(x)|^{q+1}dx \geq \\
\geq\ & \frac{1}{2}||u-u_0||_A^2 - \lambda C_f\int_\Omega \alpha(x)w(x)^{\frac{1}{\nu}}|u(x)|^{q+1}w(x)^{\frac{1}{\nu'}}dx \geq \\
\geq\ & \frac{1}{2}||u-u_0||_A^2 - \lambda C_f\left(\int_\Omega \alpha(x)^\nu w(x)dx\right)^{\frac{1}{\nu}}\left(\int_\Omega |u(x)|^{(q+1)\nu'}w(x)dx\right)^{\frac{1}{\nu'}} = \\
=\ & \frac{1}{2}||u-u_0||_A^2 - \lambda C_f||\alpha||_{\nu,\Omega,w}||u||_{p,\Omega,w}^{q+1} \geq \\
\geq\ & \frac{1}{2}||u-u_0||_A^2 - \lambda C_f C_{p,w}^{q+1}||\alpha||_{\nu,\Omega,w}||u||_{v_0,v_1}^{q+1} \geq \\
\geq\ & \frac{1}{2}||u-u_0||_A^2 - \lambda C_f C_{p,w}^{q+1}||\alpha||_{\nu,\Omega,w}(2K)^{\frac{-q-1}{2}}||u||_A^{q+1}.
\end{aligned}$$

Therefore $\mathcal{E}_\lambda(u) \to \infty$, whenever $||u||_A \to \infty$, since $q+1 < 2$. □

Lemma 2.4. *Assume that (f), (K), (α) are satisfied. Then $\mathcal{E}_\lambda$ satisfies the Palais-Smale condition for every $\lambda > 0$.*

Proof. Let $\{u_n\} \subset X_A$ be an arbitrary Palais-Smale sequence for $\mathcal{E}_\lambda$, i.e.

(a) $\{\mathcal{E}_\lambda(u_n)\}$ is bounded;

(b) $\mathcal{E}'_\lambda(u_n) \to 0$, as $n \to \infty$.

We will prove that $\{u_n\}$ contains a strongly convergent subsequence in X_A. From the coercivity of $\mathcal{E}_\lambda$, it follows that $\{u_n\}$ is bounded, hence we can find a subsequence, which we still denote by $\{u_n\}$, weakly convergent to a point $u \in X_A$. Then by the embedding condition (E), $\{u_n\}$ tends strongly to u in $L^p(\Omega; w)$, so $||u_n - u||_{p,\Omega,w} \to 0$, as $n \to \infty$.

Since the sequence from (b) tends to 0, for $n \in \mathbb{N}$ big enough, we have

$$\left|\langle \mathcal{E}'_\lambda(u_n), \frac{u_n}{||u_n||_A}\rangle_A\right| \leq \varepsilon,$$

or equivalently

$$|\langle \mathcal{E}'_\lambda(u_n), u_n\rangle_A| \leq \varepsilon||u_n||_A.$$

Then, by (7) we get

$$\langle u_n - u_0, u_n\rangle_A - \lambda \int_\Omega \alpha(x) f(u_n(x)) u_n(x) dx \le \varepsilon ||u_n||_A.$$

Rearranging the inequality and taking the absolute value, we obtain

$$|\langle u_n - u_0, u_n\rangle_A| \le \varepsilon ||u_n||_A + \lambda \int_\Omega \alpha(x) |f(u_n(x)) u_n(x)| dx.$$

After simple computations this inequality gives us

$$||u_n - u||_A^2 \le |\langle u_n - u_0, u_n - u\rangle_A| + |\langle u_0 - u, u_n - u\rangle_A| \le$$

$$\le |\langle u_n, u_n - u\rangle_A| + |\langle u, u_n - u\rangle_A| + 2|\langle u_0, u_n - u\rangle_A| \le$$

$$\le 4\varepsilon ||u_n - u||_A + \lambda \int_\Omega \alpha(x) |f(u_n(x))(u_n(x) - u(x))| dx +$$

$$+\lambda \int_\Omega \alpha(x) |f(u(x))(u_n(x) - u(x))| dx + \lambda \int_\Omega \alpha(x) |f(u_0(x))(u_n(x) - u(x))| dx.$$

Now, we will estimate the integrals from the above inequality using the inequalities of Hölder, the ellipticity condition (K) and the embedding condition (E). The first integral can be estimated as follows

$$\begin{aligned}
& \int_\Omega \alpha(x) |f(u_n(x))(u_n(x) - u(x))| dx \le \\
\le \quad & C_f \int_\Omega \alpha(x) w(x)^{\frac{1}{\nu}} |u_n(x)|^q |u_n(x) - u(x)| w(x)^{\frac{1}{\nu'}} dx \le \\
\le \quad & C_f ||\alpha||_{\nu,\Omega,w} \left(\int_\Omega |u_n(x)|^{q\nu'} |u_n(x) - u(x)|^{\nu'} w(x) dx \right)^{\frac{1}{\nu'}} = \\
= \quad & C_f ||\alpha||_{\nu,\Omega,w} \left(\int_\Omega (|u_n(x)|^p w(x))^{\frac{q}{q+1}} (|u_n(x) - u(x)|^p w(x))^{\frac{1}{q+1}} dx \right)^{\frac{1}{\nu'}} \le \\
\le \quad & C_f ||\alpha||_{\nu,\Omega,w} \left[\left(\int_\Omega |u_n(x)|^p w(x) dx \right)^{\frac{q}{q+1}} \left(\int_\Omega |u_n(x) - u(x)|^p w(x) dx \right)^{\frac{1}{q+1}} \right]^{\frac{q+1}{p}} = \\
= \quad & C_f ||\alpha||_{\nu,\Omega,w} ||u_n||^q_{p,\Omega,w} ||u_n - u||_{p,\Omega,w} \le \\
\le \quad & C_f ||\alpha||_{\nu,\Omega,w} C^q_{p,w} ||u_n||^q_{v_0,v_1} ||u_n - u||_{p,\Omega,w} \le \\
\le \quad & C_f ||\alpha||_{\nu,\Omega,w} C^q_{p,w} (2K)^{\frac{-q}{2}} M^q ||u_n - u||_{p,\Omega,w}
\end{aligned}$$

where in the last inequality we used that $\{u_n\}$ is bounded, hence there is a constant $M > 0$ such that $||u_n||_A < M$, $||u||_A < M$.

Proceeding in the same manner for the other two integrals, we obtain:

$$\int_\Omega \alpha(x)|f(u(x))(u_n(x)-u(x))|dx \le C_f||\alpha||_{\nu,\Omega,w}C_{p,w}^q(2K)^{\frac{-q}{2}}M^q||u_n-u||_{p,\Omega,w}$$

$$\int_\Omega \alpha(x)|f(u_0(x))(u_n(x)-u(x))|dx \le C_f||\alpha||_{\nu,\Omega,w}C_{p,w}^q(2K)^{\frac{-q}{2}}||u_0||_A^q||u_n-u||_{p,\Omega,w}.$$

Then, we have

$$\begin{aligned}||u_n-u||_A^2 \quad &\le 4\varepsilon||u_n-u||_A+ \\ &+\lambda C_f||\alpha||_{\nu,\Omega,w}C_{p,w}^q(2K)^{\frac{-q}{2}}(2M^q+||u_0||_A^q)||u_n-u||_{p,\Omega,w}.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily choosen, $||u_n-u||_A$ is bounded, $||u_0||_{p,\Omega,w}$ is finite (u_0 being given) and $||u_n-u||_{p,\Omega,w}$ tends to 0 as $n\to\infty$, we conclude that $||u_n-u||_A\to 0$, whenever $n\to\infty$. □

We conclude this section by recalling two results which will be used in proofs of the next section. The first one is a topological minimax theorem due to B. Ricceri:

Theorem 2.1. [10, Theorem 1 and Remark 1] *Let X be a topological space, Γ a real interval, and $f : X\times\Gamma\to\mathbb{R}$ a function satisfying the following conditions:*

($A1$) *for every $x\in X$, the function $f(x,\cdot)$ is quasi-concave and continuous;*

($A2$) *for every $\lambda\in\Gamma$, the function $f(\cdot,\lambda)$ is lower semicontinuous and each of its local minima is a global minimum;*

($A3$) *there exist $\rho_0 > \sup_\Gamma \inf_X f$ and $\lambda_0\in\Gamma$ such that $\{x\in X : f(x,\lambda_0)\le\rho_0\}$ is compact.*

Then,

$$\sup_\Gamma \inf_X f = \inf_X \sup_\Gamma f.$$

The next result of Tsar'kov is from the theory of best approximation in Banach spaces.

Theorem 2.2. [12, Theorem 2] *Let X be an uniformly convex Banach space, with strictly convex topological dual, M a sequentially weakly closed, non-convex subset of X. Then, for any convex, dense subset S of X, there exists $x_0\in S$ such that the set*

$$\{y\in M : ||y-x_0|| = d(x_0,M)\}$$

contains at least two distinct points.

3. Main result

The main theorem of our paper is the following

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an unbounded domain with smooth boundary $\partial\Omega$ or $\Omega = \mathbb{R}^N (N \geq 2)$. Suppose that $W^{1,2}(\Omega; v_0, v_1)$ satisfies the embedding property (E) and X_A is the space defined by (4). Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the condition (f) and let $\alpha : \Omega \to \mathbb{R}$ be a strictly positive function satisfying (α).*

Then for every $\sigma \in]\inf_{X_A} J, \sup_{X_A} J[$ and every $u_0 \in J^{-1}(]-\infty, \sigma[)$, one of the following assertions is true:

$(B1)$ there exists $\lambda > 0$ such that the problem (P_λ) has at least three solutions in X_A;

$(B2)$ there exists $v \in J^{-1}(\sigma)$ such that for all $u \in J^{-1}([\sigma, \infty[)$, $u \neq v$,

$$||u - u_0||_A > ||v - u_0||_A.$$

Proof. Fix λ and u_0 as in the statement of the theorem and assume that (B1) does not hold. We shall prove that (B2) is true.

Choosing $\Lambda = [0, \infty)$ and endowing X_A with the weak topology, we define the function $g : X_A \times \Lambda \to \mathbb{R}$ by

$$g(u, \lambda) = \frac{||u - u_0||_A^2}{2} + \lambda(\sigma - J(u)).$$

We show that all the hypotheses of Theorem 2.1 are satisfied.

(A1): It is trivial.

(A2): Let $\lambda > 0$ be fixed. By Lemma 2.2, the functional $g(\cdot, \lambda)$ is sequentially weakly continuous. Moreover, $g(\cdot, \lambda)$ is coercive. Indeed, using Lemma 2.3, we have the following inequality for all $u \in X_A$

$$g(u, \lambda) \geq \frac{1}{2}||u - u_0||_A^2 - \lambda C_f C_{p,w}^{q+1} (2K)^{\frac{-q-1}{2}} ||\alpha||_{\nu,\Omega,w} ||u||_A^{q+1} + \lambda\sigma.$$

Since $q+1 < 2$, the right-hand side of the above inequality goes to $+\infty$ as $||u||_A \to \infty$.

Then, as a consequence of the Eberlain-Smulian theorem, $g(\cdot, \lambda)$ is weakly continuous.

It remains to check that every local minima of $g(\cdot, \lambda)$ is a global minimum. Arguing by contradiction, we suppose that $g(\cdot, \lambda)$ has a local minimum, which is not global minimum. Besides, $g(\cdot,\lambda)$ being coercive and satisfying the Palais-Smale condition (which results from Lemma 2.4), it has a global minimum too. Then using the Eberlain-Smulian theorem, it follows that it has two strong local minima. Hence, by the Mountain-Pass theorem (see [9]) results that $g(\cdot,\lambda)$ (or equivalently the energy functional $\mathcal{E}_\lambda$) admits a third critical point. Therefore the problem (P_λ) should have at least three solutions in X_A, against our assumption, that $(B1)$ does not hold. Thus, the condition (A2) is fulfilled.

(A3): We observe that there exists some $u_1 \in X_A$ such that $J(u_1) > \sigma$, so

$$\sup_{\lambda\in\Lambda}\inf_{u\in X_A} g(u,\lambda) \le \sup_{\lambda\in\Lambda} g(u_1,\lambda) = \frac{||u_1-u_0||_A}{2} < \infty,$$

hence (A3) is satisfied.

Now, Theorem 2.1 assures that

$$\sup_{\lambda\in\Lambda}\inf_{u\in X_A} g(u,\lambda) = \inf_{u\in X_A}\sup_{\lambda\in\Lambda} g(u,\lambda) := \alpha. \tag{8}$$

We observe, that the function $\lambda \mapsto \inf_{u\in X_A} g(u,\lambda)$ tends to $-\infty$ as $\lambda\to\infty$ (since $\sigma < \sup_{u\in X_A} J(u)$) and it is upper semicontinuous in Λ. Hence, it attains its supremum in some $\bar\lambda \in \Lambda$, that is,

$$\alpha = \inf_{u\in X_A} g(u,\bar\lambda) = \inf_{u\in X_A}\left(\frac{||u-u_0||_A^2}{2} + \bar\lambda(\sigma - J(u))\right). \tag{9}$$

We will determine the infimum in the right-hand side of (8). Since for any $u \in J^{-1}(]-\infty,\sigma[)$ we have $\sup_{\lambda\in\Lambda} g(u,\lambda) = \infty$, it follows that

$$\alpha = \inf_{u\in J^{-1}([\sigma,\infty[)} \frac{||u-u_0||_A^2}{2}.$$

Then, since the functional $u \mapsto \frac{||u-u_0||_A^2}{2}$ is coercive and sequentially weakly lower semicontinuous while the set $J^{-1}([\sigma,\infty[)$ is sequentially weakly closed, there exists $v \in J^{-1}([\sigma,\infty[)$ such that it attains its infimum in v, that is

$$\alpha = \frac{||v-u_0||_A^2}{2}.$$

We can observe that v is actually belonging to $J^{-1}(\sigma)$, so we can write

$$\alpha = \inf_{u\in J^{-1}(\sigma)} \frac{||u-u_0||_A^2}{2} > 0, \tag{10}$$

where the inequality is motivated by the choice of u_0 in the assertion of the theorem.

Combining (9) and (10) yields that

$$\inf_{u\in X_A}\left(\frac{||u-u_0||_A^2}{2}+\bar{\lambda}(\sigma-J(u))\right)=\inf_{u\in J^{-1}(\sigma)}\frac{||u-u_0||_A^2}{2}, \tag{11}$$

which became after a rearrangment of the equation

$$\inf_{u\in X_A}\left(\frac{||u-u_0||_A^2}{2}-J(u)\right)=\inf_{u\in J^{-1}(\sigma)}\left(\frac{||u-u_0||_A^2}{2}-\bar{\lambda}\sigma\right). \tag{12}$$

Now, we prove that $\bar{\lambda}>0$. Arguing by contradiction, we suppose that $\bar{\lambda}=0$. Then by (9) we get, that $\alpha=0$, against (10).

Finally, we prove $(B2)$, namely we prove that v defined above is the only point of $J^{-1}([\sigma,+\infty[)$ minimizing the distance from u_0. We argue by contradiction.

Let $w\in J^{-1}([\sigma,+\infty[)$ be such that $||w-u_0||_A=||v-u_0||_A$ and w is different from v. As above, we have that $w\in J^{-1}(\sigma)$, so w and v are global minima of the functional $\mathcal{E}_\lambda$ over $J^{-1}(\sigma)$ for $\lambda=\bar{\lambda}$. Hence, by (12) both w and v are global minima for $\mathcal{E}_\lambda$ over the all space X_A. Thus, applying the mountain pass theorem again (see [9]), we obtain that $\mathcal{E}_\lambda$ has at least three critical points, against the assumption that (B1) does not hold (recall $\bar{\lambda}$ is positive). This concludes the proof. □

In the next corollary the alternative of Theorem 3.1 is resolved, so we obtain a multiplicity result for the problem (P_λ).

Corollary 1. *Let Ω, f, α, X_A be as in the Theorem 3.1 and let S be a convex, dense subset of X_A. Moreover, let $J^{-1}([\sigma,+\infty[)$ be not convex for some $\sigma\in]\inf_{X_A}J,\sup_{X_A}J[$.*

Then there exist $u_0\in J^{-1}(]-\infty,\sigma[)\cap S$ and $\lambda>0$ such that the problem (P_λ) admits at least tree solutions.

Proof. From Lemma 2.2, it follows that J is sequentially weakly continuous, hence the set $M=J^{-1}(]\sigma,+\infty[)$ is sequentially weakly closed. Since M is not convex, we can apply the Theorem 2.2, which assures the existence of some $u_0\in S$, such that the set $\{y\in M : ||y-u_0||_A=d(u_0,M)\}$ contains at least two distinct points. So, there exist two different points $v_1,v_2\in M$ such that

$$||v_1-u_0||_A=||v_2-u_0||_A=d(u_0,M).$$

Clearly $u_0 \notin M$, so $u_0 \in J^{-1}(]-\infty, \sigma[)$. Then the condition (B2) in Theorem 3.1 is false, so (B1) must be true, which means that there exist $\lambda > 0$ such that (P_λ) has at least three solutions in X_A. $\square$

References

[1] Brézis, H., *Analyse fonctionelle. Theorie et applications*, Masson, Paris, 1983.

[2] Faraci, F., Ianizotto, A., *An extension of a multiplicity theorem by Ricceri with an application to a class of quasilinear equations*, Studia Math., **172**(2006), 275-287.

[3] Faraci, F., Iannizzotto, A., Lisei, H., Varga, Cs., *A Multiplicity Result for Hemivariational Inequalities*, Journal of Math. Anal. And Appl., **330(1)**(2007), 683-698.

[4] Kristály, A., *Multiple solutions of a sublinear Schrodinger equation*, NoDEA-Nonlinear Differential Equations and applications, **14(3-4)**(2007), 291-302.

[5] Mezei, I.I., *Multiple solutions for a double eigenvalue semilinear problem in double weighted Sobolev spaces*, Studia Univ. "Babeş-Bolyai", LIII, **3**(2008), 33-48.

[6] Motreanu, D., Panagiotopoulos, P.D., *Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities, in: Nonconvex Optimization and Its Appliction*, vol.29, Kluwer Academic Publishers, Dordrecht, 1999.

[7] Pflüger, K., *Semilinear Elliptic Problems in Unbounded Domains: Solutions in weighted Sobolev Spaces*, Institüt for Mathematik I, Freie Universität Berlin, Prepint nr. 21, 1995.

[8] Pflüger, K., *Compact traces in weighted Sobolev space.* Analysis, **18**(1998), 65-83.

[9] Pucci, P., Serrin, J., *A mountain pass theorem*, J. Differential Equations, **60**(1985).

[10] Ricceri, B., *A further improvement of a minimax theorem of Borenshtein and Shul'man*, J. Nonlinear Convex Anal., **2**(2001), 279-283.

[11] Ricceri, B., *A general multiplicity theorem for certain nonlinear equations in Hilbert spaces*, Proc. Amer. Math. Soc., **133**(2005), 3255-3261.

[12] Tsar'kov, I.G., *Nonunique solvability of certain differential equations andtheir connection with geometric approximation theory*, Math. Notes, **75**(2004), 259-271.

Faculty of Mathematics and Computer Science
University of Babeş-Bolyai
str. M. Kogalniceanu 1, 400084 Cluj Napoca, Romania
E-mail address: `ildiko.mezei@math.ubbcluj.ro, ktunde84@yahoo.com`

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 3, September 2009

ABOUT CANONICAL FORMS OF THE NOMOGRAPHIC FUNCTIONS

MARIA MIHOC

Abstract. This paper carries on the study of the functions of four variables in order to find the canonical forms (analogous to those of three variables) as well as nomograms in space with coplanar points on which the functions can be nomographically represented. We also build the nomograms in space for the canonical forms found out by Kazangapov respectively Wojtowicz. The factors of anamorphosis are examined.

The data in many physical, chemical, biological, tehnical experiments presented in tables with several entrances can by analyzed by performed computers. However the reading of the data and the subsequent results can be difficult. Therefore the use some simple ,,drawings", in order to analyses the relations between these data renders efficient employment of nomograms.

In [5]and [6] we are concerned with the study of nomographic functions of four variables $F(z_1, z_2, z_3, z_4)$, $F : D \subset \mathbb{R}^4 \to \mathbb{R}, D = D_1 \times D_2 \times D_3 \times D_4$, $D_i : a_i \le z_i \le b_i$, $i = \overline{1, 4}$.

In the first one we have in view a classification of these functions according to their rank with respect to the variables they depend on. In the second one we analyze the nomograms in space with coplanar points, on which the functions can be nomographically represented. These functions of four variables are of rank two with respect to each of their variables.

Received by the editors: 22.10.2008.

2000 *Mathematics Subject Classification.* 65S05.

Key words and phrases. nomogram, nomographic functions, canonical forms.

This paper was presented at the 7-th Joint Conference on Mathematics and Computer Science, July 3-6, 2008, Cluj-Napoca, Romania.

The study of these functions and of their corresponding equations (i.e. of the forms $F(z_1, z_2, z_3, z_4) = 0$) is further performed by looking for their canonical forms (analogous to those with three variables). We also have in view the nomographic representation of these canonical forms, both by the compound plane nomograms and the nomograms in space with coplanar points. We provide a classification of these functions according to the genus of nomogram on which the equation (and also the corresponding function) can be nomographically represented.

This study is also extended to the case of the functions of several variables (of five-eight variables).We also attempt a study of the canonical forms for the equations with four variables, founded by N.Kazangapov [3] and J. Wojtowics [8]. We will study the anamorphosis factors which permit the writing of the function of several variables by a determinant Massau of fourth order. We shell also analyze the number of these determinants and, consequently, the corresponding nomograms in space with coplanar points.

Definition 1. [5] *The function $F \equiv F(z_1, z_2, z_3, z_4)$ is called nomographic in space if:*

a) the rank of the function F with respect to each of its variables is at least two;

b) there exist the functions $X_i(z_1), Y_i(z_2), Z_i(z_3), T_i(z_4)$, $i = \overline{1, 4}$ so that:

$$F(z_1, z_2, z_3, z_4) \equiv \begin{vmatrix} X_1(z_1) & X_2(z_1) & X_3(z_1) & X_4(z_1) \\ Y_1(z_2) & Y_2(z_2) & Y_3(z_2) & Y_4(z_2) \\ Z_1(z_3) & Z_2(z_3) & Z_3(z_3) & Z_4(z_3) \\ T_1(z_4) & T_2(z_4) & T_3(z_4) & T_4(z_4) \end{vmatrix}, \tag{1}$$

(i.e. F may be written in the form of determinant Massau of fourth order).

The definition of the rank of a function of four variables [5] is further generalized for the functions of eight variables in order to nomographically represent this function (as well of the equation which is attached to it) by a nomogram in space with coplanar points and with binary nets.

Let us consider the function of eight variables $F(z_1, z_2, z_3, ..., z_8)$ where $F : E \subset \mathbb{R}^8 \to \mathbb{R}$, $E = E_1 \times E_2 \times ... \times E_8$, $a_i \leq z_i \leq b_i$, $i = \overline{1, 8}.$

Definition 2. *The function with eighth variables $F = F(z_1, z_2, ..., z_8)$ is of rank n with respect to the variables z_1 and z_5 if there exist the real functions of two variables, $U_i(z_1, z_5)$, $i = \overline{1, n}$, as well as, the real functions of six variables $V_i(z_2, z_3, z_4, z_6, z_7, z_8)$, $i = \overline{1, n}$ so as to have:*

$$F(z_1, z_2, ..., z_8) \equiv \sum_{i=1}^{n} U_i(z_1, z_5) V_i(z_2, z_3, z_4, z_6, z_7, z_8), \qquad (2)$$

where n is the greatest possible natural number for which the relation (2) occurs.

In a similar way, we can define the rank of the function $F(z_1, z_2, ..., z_8)$ with respect to any pair of two variables z_i, z_j; $i, j = \overline{1, 8}$, $i < j$.

The nomographic function of eight variables can also be defined according to Definition l.

Definition 3. *The function of eight variables $F(z_1, z_2, ..., z_8)$ is called nomographic in space if:*

1) the rank of the function F with respect to each pair of two variables z_i, z_j; $i, j = \overline{1, 8}$, $i < j$, is at least two;

2) there exist the functions $X_i(z_1, z_5), Y_i(z_2, z_6), Z_i(z_3, z_7), T_i(z_4, z_8)$, $i = \overline{1, 4}$, so that:

$$F(z_1, z_2, ..., z_8) \equiv \begin{vmatrix} X_1(z_1, z_5) & X_2(z_1, z_5) & X_3(z_1, z_5) & X_4(z_1, z_5) \\ Y_1(z_2, z_6) & Y_2(z_2, z_6) & Y_3(z_2, z_6) & Y_4(z_2, z_6) \\ Z_1(z_3, z_7) & Z_2(z_3, z_7) & Z_3(z_3, z_7) & Z_4(z_3, z_7) \\ T_1(z_4, z_8) & T_2(z_4, z_8) & T_3(z_4, z_8) & T_4(z_4, z_8) \end{vmatrix}. \qquad (3)$$

This definition calls for writing the function F as a determinant Massau of fourth order. This determinant contains only the functions of two variables in each of its lines.

In particular, if z_5, z_6, z_7, z_8 are real constants, we obtain from (3) the form (1). It is obvious that, if the number of variables in the pairs z_i, z_j; $i, j = \overline{1, 8}$, $i < j$

from (2) is reduced by a unit we can obtain the rank of function under study with respect only to one variable (the remaining variable). Considering that one, two or three from variables z_5, z_6, z_7, z_8 become real constants we can also obtain other particular cases of the determinant (3).

The equation of Soreau can be solved by a nomogram, thanks to geometrically imposed conditions; i.e. the condition of coplanarity of four points in space.

Definition 4. *By a nomogram associated to a nomographic function we understand the equation's nomogram which is obtained by the equalization of respective function with zero.*

According to this definition, the nomographic representation of the function F (which has been brought to the form (3)) is equivalent to the nomographic representation of the Soreau equation associated to this function.

The equation Soreau associated to the function (3)

$$\begin{vmatrix} X_1(z_1,z_5) & X_2(z_1,z_5) & X_3(z_1,z_5) & X_4(z_1,z_5) \\ Y_1(z_2,z_6) & Y_2(z_2,z_6) & Y_3(z_2,z_6) & Y_4(z_2,z_6) \\ Z_1(z_3,z_7) & Z_2(z_3,z_7) & Z_3(z_3,z_7) & Z_4(z_3,z_7) \\ T_1(z_4,z_8) & T_2(z_4,z_8) & T_3(z_4,z_8) & T_4(z_4,z_8) \end{vmatrix} = 0 \tag{4}$$

represents (after elementary transformations) the condition that the four points, from the space $\mathbb{R}^3$, are situated in the same plane. This is possible, since according to the Definition 2, a nomographic function of eight variables has at least the rank two with respect to any from the pairs of the variables z_i, z_j; $i, j = \overline{1,8}$, $i < j$.

From the relation (4) we obtain:

$$\begin{vmatrix} \frac{X_1(z_1,z_5)}{X_5(z_1,z_5)} & \frac{X_2(z_1,z_5)}{X_5(z_1,z_5)} & \frac{X_3(z_1,z_5)}{X_5(z_1,z_5)} & 1 \\ \frac{Y_1(z_2,z_6)}{Y_5(z_2,z_6)} & \frac{Y_2(z_2,z_6)}{Y_5(z_2,z_6)} & \frac{Y_3(z_2,z_6)}{Y_5(z_2,z_6)} & 1 \\ \frac{Z_1(z_3,z_7)}{Z_5(z_3,z_7)} & \frac{Z_2(z_3,z_7)}{Z_5(z_3,z_7)} & \frac{Z_3(z_3,z_7)}{Z_5(z_3,z_7)} & 1 \\ \frac{T_1(z_4,z_8)}{T_5(z_4,z_8)} & \frac{T_2(z_4,z_8)}{T_5(z_4,z_8)} & \frac{T_3(z_4,z_8)}{T_5(z_4,z_8)} & 1 \end{vmatrix} = 0, \tag{5}$$

where the functions $X_5(z_1,z_5), Y_5(z_2,z_6), Z_5(z_3,z_7), T_5(z_4,z_8)$ are the linear combinations of the functions X_i, Y_i, Z_i, T_i, $i=\overline{1,4}$ and $a,b,c,d \in \mathbb{R}$, i.e.:

$$\begin{aligned} X_5(z_1,z_5) \equiv\ & aX_1(z_1,z_5)+bX_2(z_1,z_5)+cX_3(z_1,z_5)+dX_4(z_1,z_5)\\ Y_5(z_2,z_6) \equiv\ & aY_1(z_2,z_6)+bY_2(z_2,z_6)+cY_3(z_2,z_6)+dY_4(z_2,z_6)\\ Z_5(z_3,z_7) \equiv\ & aZ_1(z_3,z_7)+bZ_2(z_3,z_7)+cZ_3(z_3,z_7)+dZ_4(z_3,z_7)\\ T_5(z_4,z_8) \equiv\ & aT_1(z_4,z_8)+bT_2(z_4,z_8)+cT_3(z_4,z_8)+dT_4(z_4,z_8). \end{aligned}$$

According to Definition 3 each of the above linear combinations has at least two terms. Subsequently those four coplanar points will have the Cartesian coordinates equal to:

$$(P_i):\ \ x=\frac{A_1(z_i,z_{i+4})}{A_5(z_i,z_{i+4})};\quad y=\frac{A_2(z_i,z_{i+4})}{A_5(z_i,z_{i+4})};\quad z=\frac{A_3(z_i,z_{i+4})}{A_5(z_i,z_{i+4})} \tag{6}$$

for $i=\overline{1,4}$; and $A_j(z_i,z_{i+4})$ for $j=1,2,3,5$ successively take the values $X_j(z_1,z_5)$, $Y_j(z_2,z_6)$, $Z_j(z_3,z_7)$, $T_j(z_4,z_8)$.

The points P_i are situated on the binary nets (z_i,z_{i+4}) for $i=\overline{1,4}$ (i.e. on a net consisting of two families of marked curves in space; one of them depends on the parameter z_i; the other on the parameters z_{i+4} (see Fig.1)

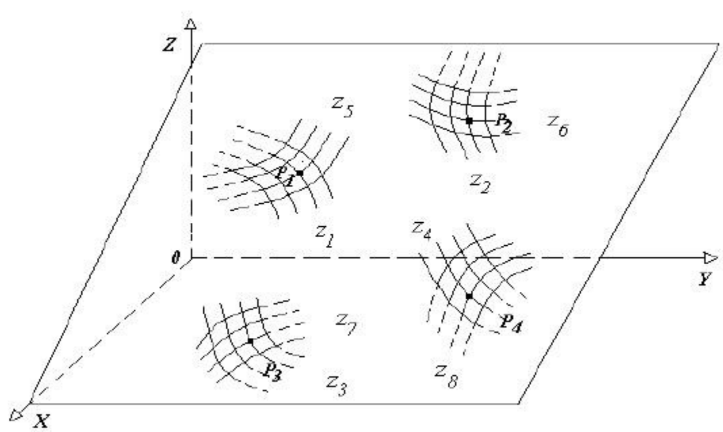


FIGURE 1

These families are obtained by the elimination, for all points P_i, $i=\overline{1,4}$, of the parameters z_{i+4}, respectively z_i,from the equation (6). This way we found the equations of two pairs of cylindrical surfaces for each $i=\overline{1,4}$:

$$S_1^i(x,y,z_i)=0, \quad S_2^i(x,z,z_i)=0, \tag{7}$$

$$S_3^i(x,y,z_{i+4})=0, \quad S_4^i(x,z,z_{i+4})=0. \tag{8}$$

The cylindrical surfaces (7) and(8) provide families of marked distorted curves in space with the parameter z_i (respectively z_{i+4}) from the binary net (z_i, z_{i+4}).

Thus, a function of eight variables $F(z_1,z_2,...,z_8)$, which can be written in the form (3), is nomographically represented (like the equation associated to it, $F(z_1,z_2,...,z_8)=0$) by a nomogram in space with coplanar points. This nomogram contains four binary nets (i.e. both of them consisting of two families of distorted marked curves in space $\mathbb{R}^3$ (see Fig. 1).

The usage (or the ,,key" to its usage) of the nomogram from Fig.1 is simple: given the values of the first seven variables of the equation (4) we can find with the help of the first six of them, the coordinates of the three points in space, P_1, P_2, P_3, situated in three binary nets of the nomogram . The plane determinate by these three points, intersects the curves marked z_7 in a point P_4 from the binary net (z_7, z_8). The mark of the curve from the family having parameter z_8, which passes through last point P_4, will given the value of the eighth variable of the equation.

If we written the function of four variables $F(z_1,z_2,z_3,z_4)$ in the form (1) (and also the equation $F(z_1,z_2,z_3,z_4)=0$) than it can be represented by a nomogram with coplanar points. The scales of the variables $z_i, i=\overline{1,4}$ are situated on the distorted curves C_i, $i=\overline{1,4}$ in $\mathbb{R}^3$, [4]. The usage of this nomogram is almost the same as the usage of the one above, if we replace the binary net with the marked scales.

If the number of variables of the nomographic function varies between five and eight, we also obtain a nomographic representation by a nomogram in space with coplanar points. The nomogram has four marked elements and it consists of the combinations including both marked scales and binary marked nets.

M. d'Ocagne and R. Soreau have found the canonical forms for the equation with three variables. M.Warmus [7] asserted seven main case for the nomographic functions of three variables. In [1] we studied the connection between the canonical forms and the main cases of Warmus.

A.N.Kazangapov [3] analyzed the canonical forms for the equations of four variables of four nomographic order

$$\begin{aligned}&A_0f_1f_2f_3f_4+A_1f_2f_3f_4+A_2f_1f_3f_4+A_3f_1f_2f_4+A_4f_1f_2f_3+\\&+B_{12}f_1f_2+B_{13}f_1f_3+B_{14}f_1f_4+B_{23}f_2f_3+B_{24}f_2f_4+B_{34}f_3f_4+\\&+C_1f_1+C_2f_2+C_3f_3+C_4f_4+D=0,\end{aligned} \tag{9}$$

where $f_i = f_i(z_i)$, $i = \overline{1,4}$ and its coefficients are real numbers.

He found three canonical equations:

$$f_1f_2f_3f_4 - 1 = 0 \tag{10}$$

$$f_1 + f_2 + f_3 + f_4 = 0 \tag{11}$$

$$f_1f_2f_3 + f_1f_2f_4 + f_1f_3f_4 + f_2f_3f_4 = f_1 + f_2 + f_3 + f_4. \tag{12}$$

Consequently for the functions of four variables of the rank two with respect to each variable we have the following canonical forms :

$$F(z_1,z_2,z_3,z_4) \equiv X_1Y_1Z_1T_1 - X_2Y_2Z_2T_2, \tag{13}$$

$$F(z_1,z_2,z_3,z_4) \equiv X_1Y_2Z_2T_2 + X_2Y_1Z_2T_2 + X_2Y_2Z_1T_2 + X_2Y_2Z_2T_1, \tag{14}$$

$$\begin{aligned}F(z_1,z_2,z_3,z_4) \equiv\ & X_1Y_1Z_1T_2 + X_1Y_1Z_2T_1 + X_1Y_2Z_1T_1 + X_2Y_1Z_1T_1\\ -\ & X_1Y_2Z_2T_2 - X_2Y_1Z_2T_2 - X_2Y_2Z_1T_2 - X_2Y_2Z_2T_1,\end{aligned} \tag{15}$$

where $X_i = X_i(z_1)$, $Y_i = Y_i(z_2)$, $Z_i = Z_i(z_3)$, $T_i = T_i(z_4)$, $i = 1, 2$.

We will study the nomograms in space by which the canonical equations (10)-(12) (as well as the functions (13)-(15)) are represented.

1. a) In [6] we analysed the six distinct projective nomograms that correspond to the equation (10) (and to function (13)). These nomograms are of genus zero (all its scales are rectilinear). We also built the nomograms in space, which is a compound nomogram of two plane nomograms with alignment points. One of them is situated in the plane X0Y, and another in Y0Z.

Since the variables of (10) are separated (so the Goursat condition is satisfied) for this equation we can build a plane compound nomogram from two nomograms with alignment points [4].

b) We can increase the genus of a nomogram that corresponds to (10) with two-four units if we multiply this equation with an anamorphosis factor. In this way, if the factor is $f_i - f_j$, $i, j = \overline{1,4}$, $i < j$, we can build a space nomogram with coplanar points of genus two, whose scales of variables z_i and z_j are situated on a quadratic curve (a conic) and the other two scales on the straight lines (see Fig. 2).

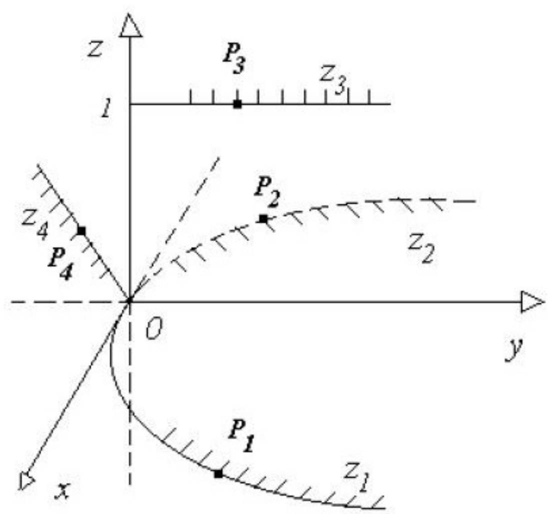


FIGURE 2

For example, if $i = 1$ and $j = 2$, we have:

$$(f_1 - f_2)(f_1 f_2 f_3 f_4 - 1) = 0 \tag{16}$$

or

$$f_1 \left[f_2 + \frac{1}{f_2 f_3 f_4} \right] - \frac{1}{f_3 f_4} - f_1^2 = 0. \tag{17}$$

With the notations

$$x = f_2 + \frac{1}{f_2 f_3 f_4}, \quad y = \frac{1}{f_3 f_4}, \quad z = \frac{1}{f_4}, \tag{18}$$

we obtain the disjunction equations

$$\begin{cases} f_i x \quad -y \qquad\quad -f_i^2 & = 0, \quad i = 1,2 \\ \qquad f_3 y \quad -z & = 0 \\ \qquad\qquad f_4 z \quad -1 & = 0, \end{cases} \tag{19}$$

and, after the elementary transformations, the equation Soreau

$$\begin{vmatrix} \frac{1}{f_1} & \frac{1}{f_1^2} & 0 & 1 \\ \frac{1}{f_2} & \frac{1}{f_2^2} & 0 & 1 \\ 0 & -f_3 & 1 & 1 \\ 0 & 0 & \frac{-f_4}{1-f_4} & 1 \end{vmatrix} = 0 \tag{20}$$

In the case of building the nomogram for a concrete equation the modulus of the scales and the dimensions of nomogram must necessarily appear.

Remark 1. *By a convenient permutation of the variables in the anamorphosis factor we can build other two scales on the curvilinear support.*

In this case the canonical form of the function of four variables is

$$F(z_1, z_2, z_3, z_4) \equiv X_1^2 Y_1 Y_2 Z_1 T_1 + X_1 X_2 [Y_2^2 Z_2 T_2 - Y_1^2 Z_1 T_1] - X_2^2 Y_1 Y_2 Z_2 T_2. \tag{21}$$

The function is of rank three with respect to variables z_1 and z_2, and respectively of rank two with z_3 and z_4.

c) For obtain the nomogram in space with coplanar points of genus three subsequently, we must multiply the equation (10) by the anamorphosis factor $(f_1 - f_2)(f_1 - f_3)(f_2 - f_3)$ and get

$$f_1^3 - f_1^2 \left[f_2 + f_3 + \frac{1}{f_2 f_3 f_4} \right] + f_1 \left[f_2 f_3 + \frac{1}{f_3 f_4} + \frac{1}{f_2 f_4} \right] - \frac{1}{f_4} = 0 \tag{22}$$

By notations

$$x = f_2 + f_3 + \frac{z}{f_2 f_3}; \; y = f_2 f_3 + \frac{f_2 + f_3}{f_2 f_3} z; \; z = \frac{1}{f_4}, \tag{23}$$

we find, from (22), the equation of disjunction of the variables

$$f_1^3 - f_1^2 x + f_1 y - z = 0. \tag{24}$$

By removing the f_3 and f_2 from (23), we find other two equations of disjunction, which together with the last one from (23) and with (24) give:

$$\begin{vmatrix} f_1 & f_1^2 & f_1^3 & 1 \\ f_2 & f_2^2 & f_2^3 & 1 \\ f_3 & f_3^2 & f_3^3 & 1 \\ 0 & 0 & \dfrac{1}{f_4} & 1 \end{vmatrix} = 0, \tag{25}$$

where we multiply by -1 the second and fourth columns, then interchange the first and third columns, and finally divide each of its lines by the elements of the last column.

In this way the equation (10) (respectively (22)) is nomographically represented by a nomogram in space with coplanar points of genus three. The scales of the variables z_i, $i = \overline{1,3}$ are on the curve (curve distorted in space) of the equations: $Y = X^2$ and $Z = X^3$; while for the variable z_4 there exists a straight line support.

Remark 2. *We can choose scales on the curved support for any three variables of (10) by a convenient change of the anamorphosis factor.*

Therefore another canonical form for the function of four variables is:

$$\begin{aligned} F(z_1, z_2, z_3, z_4) \quad & \equiv X_1 X_2^2 [Y_1^2 Y_2 (Z_1^3 T_1 - Z_2^3 T_2) + Z_1^2 Z_2 (Y_2^3 T_2 - Y_1^3 T_1)] - \\ & - X_1^2 X_2 [Y_1 Y_2^2 (Z_1^3 T_1 - Z_2^3 T_2) + Z_1 Z_2^2 (Y_2^3 T_2 - Y_1^3 T_1)] + \\ & + Y_1 Y_2 Z_1 Z_2 (Y_2 Z_1 - Y_1 Z_2)(X_1^3 T_1 - X_2^3 T_2) \end{aligned} \tag{26}$$

This function is of rank four with respect to variables z_i, $i = \overline{1,3}$ and of rank two with respect to z_4.

d) The nomogram in space with coplanar points of genus four for the equation (10) is obtained by multiplying it by the factor $(f_1 - f_2)(f_1 - f_3)(f_1 - f_4)(f_2 - f_3)(f_2 - f_4)(f_3 - f_4)$.

We obtain

$$f_1^4 + 1 - \left[f_2 + f_3 + f_4 + \frac{1}{f_2 f_3 f_4}\right] f_1^3 + [f_2 f_3 + f_2 f_4 + f_3 f_4 + \\ + \frac{1}{f_2 f_3} + \frac{1}{f_2 f_4} + \frac{1}{f_3 f_4}\right] f_1^2 - \left[\frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4} + f_2 f_3 f_4\right] f_1 = 0. \tag{27}$$

With the substitutions

$$\begin{aligned} x = & \ \frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4} + f_2 f_3 f_4, \ z = f_2 + f_3 + f_4 + \frac{1}{f_2 f_3 f_4} \\ y = & \ f_2 f_3 + f_2 f_4 + f_3 f_4 + \frac{1}{f_2 f_3} + \frac{1}{f_2 f_4} + \frac{1}{f_3 f_4}, \end{aligned} \tag{28}$$

we find the disjunction equations:

$$-f_i \ x + f_i^2 \ y - f_i^3 \ z + f_i^4 + 1 = 0, \ i = \overline{1,4} \tag{29}$$

and the equation Soreau

$$\begin{vmatrix} f_1 & f_1^2 & f_1^3 & 1 + f_1^4 \\ f_2 & f_2^2 & f_2^3 & 1 + f_2^4 \\ f_3 & f_3^2 & f_3^3 & 1 + f_3^4 \\ f_4 & f_4^2 & f_4^3 & 1 + f_4^4 \end{vmatrix} = 0. \tag{30}$$

Therefore the equation (10) (respectively (27)) is nomographically represented by a nomogram in space with all its scales situated on a distorted curve in space $X^4 + Y^4 - X^2 Y = 0$ and $Z(X^4 + Y^4) - XY^3 = 0$.

The corresponding nomographic function is of rank four with respect to all its variables and have the canonical form

$$F(z_1, z_2, z_3, z_4) \equiv \begin{vmatrix} X_1 X_2^3 & X_1^2 X_2^2 & X_1^3 X_2 & X_1^4 + X_2^4 \\ Y_1 Y_2^3 & Y_1^2 Y_2^2 & Y_1^3 Y_2 & Y_1^4 + Y_2^4 \\ Z_1 Z_2^3 & Z_1^2 Z_2^2 & Z_1^3 Z_2 & Z_1^4 + Z_2^4 \\ T_1 T_2^3 & T_1^2 T_2^2 & T_1^3 T_2 & T_1^4 + T_2^4 \end{vmatrix} \tag{31}$$

2. The equation (11) is obtain from (10) using the logarithmic function. The function (14) will be nomographically represented by the same kind of nomograms in space like as equation (10) (respectively the function (11)).

3. For the equation (12) brought to the form

$$(f_1 f_2 + f_1 f_3 + f_2 f_3 - 1) f_4 + f_1 f_2 f_3 - f_1 - f_2 - f_3 = 0 \tag{32}$$

and with $x = f_1 + f_2 + f_3$, $y = f_1 f_2 + f_1 f_3 + f_2 f_3$, $z = f_1 f_2 f_3$, we obtain the equation Soreau

$$\begin{vmatrix} f_1^2 & f_1 & \frac{1}{f_1} & 1 \\ f_2^2 & f_2 & \frac{1}{f_2} & 1 \\ f_3^2 & f_3 & \frac{1}{f_3} & 1 \\ -1 & \frac{1}{f_4} & -\frac{1}{f_4} & 1 \end{vmatrix} = 0 \tag{33}$$

and the nomogram in space with three scales which lie on the distorted curve and one scale on the straight line support.

The corresponding nomographic function is of rank four with respect to the variable z_i, $i = \overline{1,3}$ and two with respect to z_4. Its canonical form can be obtained from (33)

4. J. Wojtovicz [8] found another canonical form

$$f_i + f_j = f_k f_m, \quad i,\ j,\ k,\ m = \overline{1,4}, \quad i \neq j \neq k \neq m. \tag{34}$$

With notation $f_i = x$; $f_j = y$; $f_k = z$, we obtain the equation Soreau

$$\begin{vmatrix} 0 & 0 & f_i & 1 \\ 1 & 0 & f_j & 1 \\ 0 & 1 & f_k & 1 \\ \frac{1}{2 - f_m} & \frac{f_m}{f_m - 2} & 0 & 1 \end{vmatrix} = 0 \tag{35}$$

and also the corresponding nomogram in space of genus zero (in fact the scale of the variable z_m has as support a degenerate quadratic curve: two straight lines of equations $X = 0$ and $2X + Y - 1 = 0$; only the last one of these being considerate as a support).

The canonical forms is of rank two with respect to all its variables

$$F(z_i, z_j, z_k, z_m) \equiv X_1 Y_2 Z_2 T_2 - X_2 (Y_2 Z_1 T_1 - Y_1 Z_2 T_2). \tag{36}$$

We can obtain other canonical forms for the equations (respectively functions) of several variables by generalization those of four variables i.e.

$$F_{12}(z_1, z_5)F_{26}(z_2, z_6)F_{37}(z_3, z_7)F_{48}(z_4, z_8) - 1 = 0, \tag{37}$$

$$\begin{aligned} F(z_1, z_2, ..., z_8) \quad &\equiv X_1(z_1, z_5)Y_1(z_2, z_6)Z_1(z_3, z_7)T_1(z_4, z_8) - \\ &- X_2(z_1, z_5)Y_2(z_2, z_6)Z_2(z_3, z_7)T_2(z_4, z_8). \end{aligned} \tag{38}$$

They are nomographically represented by a nomogram in space with coplanar points. These points are situated in four binary marked nets.

References

[1] Bal, L., Mihoc, M., *Asupra reprezentării funcţiilor nomografice*, Stud. Cerc. Mat. 23, **4**(1971), 511-522.

[2] Glagolev, A., *Kurs nomografiĭ*, Izd. 2-e, Moskva, Vysš. Škola, 1961.

[3] Kazangapov, A.N, *O postroeniah nomogramm v prostranstve*, Teoria codirovanie i slojnostĭ vycisleniĭ, Kazanskiĭ SSSR, 1980, 152-168.

[4] Mihoc, M., *Reprezentarea nomografică a diferitelor tipuri de ecuaţii prin nomograme care conţin scări rectilinii*, (Ph.D. Thesis), Cluj-Napoca, 1981.

[5] Mihoc, M., *On the classification of the nomographic functions of four variables* (I), Pu.M.A., 9, **1-2**(1998), 129-137.

[6] Mihoc, M., *On the classification of the nomographic functions of four variables* (II), Studia Univ. "Babeş-Bolyai", Mathematica, XLVII, **4**(2002), 85-94.

[7] Warmus, M., *Nomographic functions*, Rozpravy Matematyczne, **16**(1959), 1-151.

[8] Wojtowicz, J., *Sprowadzanie równań do postaci kanonicznych równania czwartego rzedu nomograficznego z czterema zmiennymi*, Zastosowania Matematyki, **V**(1960), 261-269.

Faculty of Economics and Business Administration Sciences
Babeş-Bolyai University
Cluj-Napoca, Romania
E-mail address: maria.mihoc@econ.ubbcluj.ro

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 3, September 2009

ON THE DIVERGENCE OF THE PRODUCT QUADRATURE PROCEDURES

ALEXANDRU IOAN MITREA

Abstract. The main result of this paper emphasizes the phenomenon of the double condensation of singularities with respect to the product-quadrature procedures associated to the spaces C and L^1; some estimations concerning the error of these procedures are given, too.

1. **Introduction**

Let us consider the Banach space C of all continuous functions $f : [-1, 1] \to \mathbb{R}$, endowed with the uniform norm $\|\cdot\|$. Denote by L^1 the Banach space of all measurable functions (classes of functions) $g : [-1, 1] \to \mathbb{R}$ so that $|g|$ is Lebesgue integrable on $[-1, 1]$, endowed with the norm:

$$\|g\|_1 = \int_{-1}^{1} |g(x)|dx, \quad g \in L^1.$$

Let $\mathcal{M} = \{x_n^k : \ n \geq 1; \ 1 \leq k \leq n\}$ be a triangular node matrix, with $-1 \leq x_n^1 < x_n^2 < x_n^3 < \cdots < x_n^n \leq 1$, $\forall\ n \geq 1$. For each integer $n \geq 1$, denote by $\Lambda_n : [-1, 1] \to \mathbb{R}$ the *Lebesgue function* associated to the n-th row of $\mathcal{M}$, i.e.

$$\Lambda_n(x) = \Lambda_n(\mathcal{M}; x) = \sum_{k=1}^{n} |l_n^k(x)|, \quad |x| \leq 1,$$

Received by the editors: 01.09.2008.

2000 *Mathematics Subject Classification.* 41A05.

Key words and phrases. product-quadrature procedures, condensation of the singularities, error estimations, Lebesgue functions, Lebesgue constants, superdense unbounded divergence.

This paper was presented at the 7-th Joint Conference on Mathematics and Computer Science, July 3-6, 2008, Cluj-Napoca, Romania.

where $l_n^k = l_n^k(\mathcal{M};\cdot)$, $1 \le k \le n$, are the fundamental polynomials of Lagrange interpolation with respect to the nodes x_n^k, $1 \le k \le n$. The real numbers

$$\lambda_n = \lambda_n(\mathcal{M}) = \|\Lambda_n\|, \quad n \ge 1$$

are known as *Lebesgue constants*.

Starting from these data, let us consider the *product-quadrature procedures* described by the formulas

$$\int_{-1}^{1} g(x)f(x)dx = \int_{-1}^{1} g(x)L_n(f;x)dx + R_n(f;g), \quad f \in C,\ g \in L^1,\ n \ge 1 \tag{1.1}$$

where

$$L_n(f;x) = L_n(\mathcal{M}, f; x) = \sum_{k=1}^{n} f(x_n^k) l_n^k(x), \quad n \ge 1 \tag{1.2}$$

are the *Lagrange interpolation polynomials* associated to the node matrix $\mathcal{M}$ and to the function f, while $R_n(f;g)$, $n \ge 1$, will be refered to as the *errors* of the product-quadrature procedures described by (1.1).

Denoting by

$$a_n^k : L^1 \to \mathbb{R}, \quad a_n^k(g) = \int_{-1}^{1} g(x) l_n^k(x)dx, \quad n \ge 1,\ 1 \le k \le n \tag{1.3}$$

$$D_n : C \times L^1 \to \mathbb{R}, \quad D_n(f;g) = \sum_{k=1}^{n} f(x_n^k) a_n^k(g), \quad n \ge 1 \tag{1.4}$$

$$A : C \times L^1 \to \mathbb{R}, \quad A(f;g) = \int_{-1}^{1} g(x)f(x)dx, \tag{1.5}$$

the product quadrature formulas (1.1) become:

$$A(f;g) = D_n(f;g) + R_n(f;g), \quad f \in C,\ g \in L^1,\ n \ge 1. \tag{1.6}$$

Remark that the product-quadrature procedures described by (1.1) or (1.6) are of interpolatory type with respect to the space C, i.e.:

$$A(P,g) = D_n(P,g), \quad n \ge 1,\ P \in \mathcal{P}_{n-1},\ g \in L^1 \tag{1.7}$$

where $\mathcal{P}_m$ is the space of all polynomials of degree at most $m \in \mathbb{N}$.

I.H. Sloan and W.E. Smith, [7], have established important results concerning the convergence of the product-quadrature procedures (1.6), for some node matrices

$\mathcal{M}$ whose n-th rows consist of the roots of the orthogonal polynomials associated to a weight-function $w(x)$ satisfying given integral inequalities, particularly for some Jacobi matrices $\mathcal{M}^{(\alpha,\beta)}$, $\alpha > -1$, $\beta > -1$. Moreover, these authors proved the existence of a pair $(f_0, g_0) \in C \times L^1$ so that the sequence $(D_n(f_0; g_0))_{n\geq 1}$ does not converge to $A(f_0, g_0)$ in (1.6).

The aim of this paper is to establish the topological structure of the sets of unbounded divergence in C and L^1, corresponding to the product-quadrature procedures described by (1.6). On this subject, remark the results obtained by I. Muntean and S. Cobzaş for $g(x) = 1$, [1], [2].

2. Estimations concerning the norm of the functionals and operators involved in the product quadrature procedures

2.1. Firstly, let us consider the functionals a_n^k given by (1.3). It is clear that a_n^k are linear functionals for each $n \geq 1$ and $k \in \{1, 2, 3, \dots, n\}$. On the other hand, the inequality

$$|a_n^k(g)| \leq \|l_n^k\| \cdot \|g\|_1 \tag{2.1}$$

proves the continuity of a_n^k and leads to the inequality

$$\|a_n^k\| \leq \|l_n^k\| \tag{2.2}$$

Conversely, let $u \in [-1, 1]$ and $h > 0$ be given real numbers so that $u + h \in [-1, 1]$. Defining the function $g_0 \in L^1$ with $\|g_0\|_1 = 1$ by:

$$g_0(x) = \begin{cases} 1/h; & u \leq x \leq u + h \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

we deduce:

$$\|a_n^k\| = \sup\{|a_n^k(g)| : \ g \in L^1, \ \|g\|_1 \leq 1\} \geq |a_n^k(g_0)|$$
$$= \left| \frac{1}{h} \int_u^{u+h} l_n^k(x) dx \right|, \ \forall \ h > 0, \ \forall \ u \in [-1, 1] \text{ with } u + h \in [-1, 1],$$

which implies:

$$\|a_n^k\| \geq \lim_{h \searrow 0} \left| \frac{1}{h} \int_u^{u+h} l_n^x(x) dx \right| = |l_n^k(u)|, \ \forall \ u \in [-1, 1],$$

so:

$$\|a_n^k\| \geq \|l_n^k\| \tag{2.4}$$

The relations (2.2) and (2.4) give:

$$\|a_n^k\| = \|l_n^k\| \tag{2.5}$$

2.2. Further, let C^* be the Banach space of all linear and continuous functionals defined on C. Let us introduce the operators $T_n : L^1 \to C^*$, $g \mapsto T_n g$, $g \in L^1$, $n \geq 1$, where

$$(T_n g)(f) = \sum_{k=1}^{n} a_n^k(g) f(x_n^k), \quad f \in C \tag{2.6}$$

The linearity of the operators T_n, $n \geq 1$, follows from the corresponding property of the functionals a_n^k, $1 \leq k \leq n$. For each given $n \geq 1$, T_n is a continuous operator, too; indeed, the inequality

$$|(T_n g)(f)| \leq \left(\sum_{k=1}^{n} |a_n^k(g)| \right) \|f\|$$

is valid for all $f \in C$ and it implies:

$$\|T_n g\| \leq \sum_{k=1}^{n} |a_n^k(g)|, \ \forall\, n \geq 1, \ \forall\, g \in L^1 \tag{2.7}$$

Now, the relations (2.7) and (2.5) give:

$$\|T_n g\| \leq \left(\sum_{k=1}^{n} \|l_n^k\| \right) \|g\|_1,$$

which proves the continuity of T_n, $n \geq 1$.

Now, let us establish the equality:

$$\|T_n g\| = \sum_{k=1}^{n} |a_n^k(g)|, \quad n \geq 1. \tag{2.8}$$

It remains to prove the converse inequality of (2.7). To this end, let consider for each $n \geq 1$, the function $f_n \in C$, $\|f_n\| = 1$, defined by:

$$f_n(x) = \begin{cases} \operatorname{sign} a_n^k(g), & \text{if } x \in \{x_n^k : \ 1 \leq k \leq n\} \\ 1, & \text{if } x \in \{-1, 1\} \setminus \{x_n^k : \ 1 \leq k \leq n\} \\ \text{linear}, & \text{otherwise} \end{cases}$$

We obtain, in accordance with (2.6):

$$\|T_n g\| = \sup\{|(T_n g)(f)| : \ f \in C, \ \|f\| \le 1\} \ge |(T_n g)(f_n)| = \sum_{k=1}^{n} |a_n^k(g)|;$$

so, the equality (2.8) is true.

2.3. Finally, let us deduce the norm of the operator T_n, $n \ge 1$. Taking into account the relations (2.8) and (2.3), we have:

$$\|T_n\| = \sup\left\{\sum_{k=1}^{n} |a_n^k(g)| : \ g \in L^1, \ \|g\|_1 \le 1\right\} \ge \sum_{k=1}^{n} |a_n^k(g_0)|$$

$$= \sum_{k=1}^{n} \left| \frac{1}{h} \int_u^{u+h} l_n^k(x)dx \right|, \ \forall \ h > 0,$$

therefore:

$$\|T_n\| \ge \lim_{h \searrow 0} \sum_{k=1}^{n} \left| \frac{1}{h} \int_u^{u+h} l_n^k(x)dx \right| = \sum_{k=1}^{n} |l_n^k(u)|, \ \forall \ u \in [-1, 1]$$

which leads to the inequality

$$\|T_n\| \ge \lambda_n, \ \forall \ n \ge 1 \tag{2.9}$$

Conversely, we obtain from (2.6) and (1.3), by using the classic equality

$$\lambda_n = \sup\{\|L_n(f;\cdot)\| : \ f \in C, \ \|f\| \le 1\}, \quad n \ge 1, \quad [6], [8], [3] :$$

$$\|T_n g\| = \sup\left\{\left|\int_{-1}^{1} g(x) L_n(f;x)dx\right| : \ f \in C, \ \|f\| \le 1\right\}$$
$$\le \|g\|_1 \cdot \sup\{\|L_n(f;\cdot)\| : \ f \in C, \ \|f\| \le 1\} = \lambda_n \|g\|_1,$$

which shows that the opposite inequality of (2.9) is also true; so, we have:

$$\|T_n\| = \lambda_n, \ \forall \ n \ge 1. \tag{2.10}$$

A lower bound of the Lebesgue constants λ_n, $n \ge 1$, is given by Theorem of Lozinski-Harsiladze, [6], [8], [3]:

$$\lambda_n \ge \frac{2}{\pi^2} \ln n, \ \forall \ n \ge 1. \tag{2.11}$$

3. Superdense unbounded divergence of the product quadrature procedures

The main result of this paper is the following:

Theorem 3.1. *Given a node matrix $\mathcal{M}$ in the interval $[-1,1]$, there exists a superdense set X_0 in C so that for each f in X_0 the set*

$$Y_0(f) = \{g \in L^1 : \ \sup\{|D_n(f;g)| : \ n \geq 1\} = \infty\}$$

is superdense in L^1.

Proof. Firstly, we shall use the following *principle of condensation of the singularities*, deduced from [1, Theorem 5.4]:

If X is a Banach space, Y is a normed space and $(A_n)_{n\geq 1}$ is a sequence of continuous linear operators from X into Y so that the set of norms $\{\|A_n\| : \ n \geq 1\}$ is unbounded, then the set of singularities of the family $\{A_n : \ n \geq 1\}$, i.e.

$$\mathcal{S}(A_n) = \{x \in X : \ \sup\{\|A_n(x)\| : \ n \geq 1\} = \infty\},$$

is superdense in X.

Take $X = L^1$, $Y = C^*$ and $A_n = T_n : L^1 \to C^*$. The set $\{\|T_n\| : \ n \geq 1\}$ is unbounded, in accordance with (2.10) and (2.11); consequently, the set

$$\mathcal{S}(T_n) = \{g \in L^1 : \ \sup\{\|T_n g\| : \ n \geq 1\} = \infty\} \tag{3.1}$$

is superdense in L^1.

Next, let us apply the following *principle of the double condensation of singularities* [1], [2]:

Suppose that X is a Banach space, Y is a normed space and T is a nonvoid separable complete metric space without isolated points.

Let $\{A_n : \ n \geq 1\}$ be a family of mappings of $X \times T$ into Y satisfying the following conditions:

(i) For each $t \in T$ and $n \geq 1$, the operator $A_n^t : X \to Y$, $A_n^t(x) = A_n(x,t)$, is linear and continuous.

(ii) For each $x \in X$ and $n \geq 1$, the operator $A_n^x : T \to Y$, $A_n^x(t) = A_n(x,t)$, is continuous.

(iii) There exists a dense set $\mathcal{T}_0$ in T so that

$$\sup\{\|A_n^t\| : \ n \geq 1\} = \infty, \ \forall \ t \in \mathcal{T}_0.$$

Then, there exists a superdense set X_0 in X so that for each $x \in X$ the set

$$Y_0(x) = \{t \in T : \ \sup\{\|A_n(x,t)\| : \ n \geq 1\} = \infty\}$$

is superdense in T.

Take $X = (C, \|\cdot\|)$, $T = (L^1, \|g\|_1)$, $Y = \mathbb{R}$ and $A_n = D_n : C \times L^1 \to \mathbb{R}$, $n \geq 1$, see (1.4). Let us verify the validity of the previous hypotheses.

(i) We have:

$$D_n^g = T_n g, \quad g \in L^1, \ n \geq 1 \tag{3.2}$$

The linearity of D_n^g follows from (2.6) and (1.3), while its continuity is a consequence of (2.7).

(ii) Taking into account (2.1), we deduce

$$|D_n^f| = \left| \sum_{k=1}^{n} a_n^k(g) f(x_n^k) \right| \leq \|g\|_1 \cdot \|f\| \cdot \sum_{k=1}^{n} \|l_n^k\|,$$

which proves the continuity of the linear functional D_n^f.

(iii) In accordance with (3.1) and (3.2) and taking $\mathcal{T}_0 = \mathcal{S}(T_n)$ we have:

$$\sup\{\|D_n^g\| : \ n \geq 1\} = \sup\{\|T_n g\| : \ n \geq 1\} = \infty, \ \forall \ g \in \mathcal{T}_0.$$

Now, let us apply the previous principle of the double condensation of singularities, which completes the proof of this theorem.

Remark 3.2. A dual-type result with respect to the Theorem 3.1 is also true [3]:

Given a node matrix $\mathcal{M}$ in the interval $[-1,1]$, there exists a superdense set X_1 in L^1 so that for each $g \in X_1$ there exists a superdense set $Y_1(g)$ in C satisfying the equality

$$\limsup_{n\to\infty} |D_n(f;g)| = \infty, \ \textit{for each } g \in X_1 \textit{ and } f \in Y_1(g).$$

4. Estimations for the error of the product-quadrature procedures

In accordance with (1.6) and (1.7), writing

$$|R_n(f;g) = A(f-P;g) + D_n(P-f;g),$$

with an arbitrary $P \in \mathcal{P}_{n-1}$, we deduce:

$$|R_n(f;g)| \leq |A(f-P;g)| + |D_n(f-P;g)|. \tag{4.1}$$

Let $s \geq 0$ be an integer and denote by C^s the Banach space of all functions $f : [-1,1] \to \mathbb{R}$ which are continuous together with their derivatives up to the order s, endowed with the norm:

$$\|f\|^{(s)} = \|f^{(s)}\| + \sum_{i=0}^{s-1} \|f^{(i)}(0)|, \text{ if } s \geq 1$$

and $\|f\|^{(0)} = \|f\|$.

It follows from the Theorem of Jackson [6], [8], [9], that there exist a polynomial $P \in \mathcal{P}_{n-1}$ and a positive number M which does not depend on n so that:

$$\|f^{(j)} - P^{(j)}\| \leq \frac{M}{n^{s-j}} \omega\left(f^{(s)}; \frac{1}{n}\right), \quad 0 \leq j \leq s, \tag{4.2}$$

for sufficient large $n \geq 1$, where $\omega(h;\cdot)$ is the modulus of continuity of a function $h \in C$.

Now, we deduce for each $i \in \{0,1,2,3,\ldots,s\}$, see also [4]:

$$\|f-P\|^{(i)} \leq \sum_{j=0}^{i} \|f^{(j)} - P^{(j)}\| \leq 2Mn^{i-s}\omega\left(f^{(s)}; \frac{1}{n}\right) \tag{4.3}$$

Taking $A_g = A(\cdot, g) : C^s \to \mathbb{R}$, with a given $g \in L^1$, we deduce from (4.1):

$$|R_n(f;g)| \leq \|A_g\| \cdot \|f-P\|^{(s)} + \left(\sum_{k=1}^{n} |a_n^k(g)|\right) \cdot \|f-P\| \tag{4.4}$$

Next, combining (4.2), (4.3) and (4.4), we obtain:

$$|R_n(f,g)| \leq M\left(2\|A_g\| + n^{-s}\sum_{k=1}^{n} |a_n^k(g)|\right)\omega\left(f^{(s)}; \frac{1}{n}\right) \tag{4.5}$$

A simple exercise leads to the inequalities:

$$\|A_g\| \leq \|g\|_1, \quad \sum_{k=1}^{n} |a_n^k(g)| \leq \lambda_n \|g\|_1,$$

which, together with (4.5), give for each $f \in C^s$ and $g \in L^1$:

$$|R_n(f;g)| \leq M(2 + \lambda_n \cdot n^{-s})\|g\|_1 \cdot \omega\left(f^{(s)}; \frac{1}{n}\right). \tag{4.6}$$

Denote by $DL(C)$ the subset of C which consists of all functions $f \in C$ satisfying a Dini-Lipschitz condition

$$\lim_{\delta \searrow 0} \omega(f;\delta) \ln \delta = 0.$$

We are in a position to prove the following statement.

Theorem 4.1. *Suppose that $\mathcal{M} = \mathcal{M}^T$ is the Chebyshev node matrix, namely its n-th row consists of the roots of the Chebyshev polynomial*

$$P_n(x) = \cos(n \arccos x), \quad n \geq 1.$$

The product-quadrature procedures described by (1.6) are convergent for each pair $(f,g) \in DL(C) \times L^1$ and for each pair $(f,g) \in C^s \times L^1$, if $s \geq 1$.

Proof. If $s = 0$, we obtain from (4.6) and $\lambda_n \sim \ln n$, [5], [8], [3]:

$$|R_n(f;g)| \leq M(2 + \ln n)\|g\|_1 \cdot \omega\left(f; \frac{1}{n}\right),$$

so

$$\lim_{n \to \infty} R_n(f;g) = 0$$

for each $f \in DL(C)$ and $g \in L^1$. If $s \geq 1$, remark that $\lambda_n n^{-s} \sim n^{-s} \ln n$ and use again (4.6).

References

[1] Cobzaş, S. and Muntean, I., *Condensation of singularities and divergence results in Approximation Theory*, J. Approx. Theory, **31**(1981), 138-153.

[2] Cobzaş, S. and Muntean, I., *Superdense A.E. Unbounded Divergence of some Approximation Process of Analysis*, Real Analysis Exchange, **25**(1999/2000), 501-512.

[3] Mitrea, A.I., *Convergence and Superdense Unbounded Divergence in Approximation Theory*, Transilvania Press (Cluj-Napoca), 1998.

[4] Mitrea, A.I., *On the convergence of a class of approximation procedures*, PU.M.A., vol. 15, no.2-3(2005), 225-234.

[5] Natanson, G.I., *Two-sided estimates for Lebesgue function of Lagrange interpolation processes based on Jacobi nodes* (Russian), Izv. Vyss. Ucebn. Zaved. Matematika, **11**(1967), 67-74.

[6] Schönhage, A., *Approximationstheorie*, Berlin, Walter de Gruyter, 1971.

[7] Sloan, I.H., Smith W.E., *Properties of interpolatory product integration rules*, SIAM J. Numer. Anal., **19**(1982), 427-442.

[8] Szabados, J. and Vertesi, P., *Interpolation of Functions*, World Sci. Publ. Co., Singapore, 1990.

[9] Szegö, G., *Orthogonal Polynomials*, Amer. Math. Soc. Providence, R.I., 1975.

Technical University, Department of Mathematics,
Str. C. Daicoviciu Nr.15, 400020 Cluj-Napoca, Romania
E-mail address: alexandru.ioan.mitrea@math.utcluj.ro

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 3, September 2009

CLOSEDNESS OF THE SOLUTION MAP FOR PARAMETRIC VECTOR EQUILIBRIUM PROBLEMS

JÚLIA SALAMON

Abstract. The objective of this paper is to study the parametric vector equilibrium problems governed by vector topologically pseudomonotone maps. The main result gives sufficient conditions for closedness of the solution map defined on the set of parameters.

1. Introduction

M. Bogdan and J. Kolumbán [5] gave sufficient conditions for closedness of the solution map. They considered the parametric equilibrium problems governed by topologically pseudomonotone maps depending on a parameter. In this paper we generalize their result for parametric vector equilibrium problems.

Let (X, σ) be a Hausdorff topological space and let P (the set of parameters) be another Hausdorff topological space. Let $\mathcal{Z}$ be a real topological vector space with an ordering cone C, where C is a closed convex cone in $\mathcal{Z}$ with $\operatorname{Int} C \neq \emptyset$ and $C \neq \mathcal{Z}$.

We consider the following parametric vector equilibrium problem, in short $(VEP)_p$:

Find $a_p \in D_p$ such that

$$f_p\left(a_p, b\right) \in \left(-\operatorname{Int} C\right)^c, \ \forall b \in D_p,$$

where D_p is a nonempty subset of X and $f_p : X \times X \to \mathcal{Z}$ is a given function.

Received by the editors: 01.09.2008.

2000 *Mathematics Subject Classification.* 49N60, 90C31.

Key words and phrases. parametric vector equilibrium problems, vector topological pseudomonotonicity, Mosco convergence, generalized Hadamard well-posedness.

This paper was presented at the 7-th Joint Conference on Mathematics and Computer Science, July 3-6, 2008, Cluj-Napoca, Romania.

Denote by $S(p)$ the set of the solutions for a fixed p. Suppose that $S(p) \neq \emptyset$, for all $p \in P$. (For sufficient conditions for the existence of solutions see [6].)

The paper is organized as follows. In Section 2, we recall the notions of the vector topological pseudomonotonicity and the Mosco convergence of the sets. Section 3 is devoted to the closedness of the solution map for parametric vector equilibrium problems. In the final section, we investigate the generalized Hadamard well-posedness of parametric vector equilibrium problems.

2. **Preliminaries**

In this section, the notion of vector topologically pseudomonotone bifunctions with values in $\mathcal{Z}$ is used. First the definition of the suprema and the infima of subsets of $\mathcal{Z}$ are given. Following [1], for a subset A of $\mathcal{Z}$ the superior of A with respect to C is defined by

$$\operatorname{Sup} A = \left\{ z \in \bar{A} : A \cap (z + \operatorname{Int} C) = \emptyset \right\}$$

and the inferior of A with respect to C is defined by

$$\operatorname{Inf} A = \left\{ z \in \bar{A} : A \cap (z - \operatorname{Int} C) = \emptyset \right\}.$$

Let $(z_i)_{i \in I}$ be a net in $\mathcal{Z}$. Let $A_i = \{z_j : j \geq i\}$ for every i in the index set I. The limit inferior of (z_i) is given by

$$\operatorname{Liminf} z_i := \operatorname{Sup} \left(\bigcup_{i \in I} \operatorname{Inf} A_i \right).$$

Similarly, the limit superior of (z_i) is defined as

$$\operatorname{Limsup} z_i := \operatorname{Inf} \left(\bigcup_{i \in I} \operatorname{Sup} A_i \right).$$

The next definition is a generalization of the vector topological pseudomonotonicity given by Chadli, Chiang and Huang in [6].

Definition 2.1. *Let (X, σ) be a Hausdorff topological space, and let D be a nonempty subset of X. A function $f : D \times D \to \mathcal{Z}$ is called vector topologically pseudomonotone if for every $b \in D$, $v \in \operatorname{Int} C$ and for each net $(a_i)_{i \in I}$ in D satisfying*

$$a_i \xrightarrow{\sigma} a \in D \text{ and } \operatorname{Liminf} f(a_i, a) \cap (-\operatorname{Int} C) = \emptyset,$$

there is i_0 in the index set I such that

$$\overline{\{f(a_j, b) : j \geq i\}} \subset f(a,b) + v - \operatorname{Int} C$$

for all $i \geq i_0$.

Let us consider σ and τ two topologies on X. Suppose that τ is stronger than σ on X.

For the parametric domains in $(VEP)_p$ we shall use the following type of convergence, which is a slight generalization of Mosco's convergence in [11].

Definition 2.2 ([5], Definition 2.2)**.** *Let D_p be subsets of X for all $p \in P$. The sets D_p converge to D_{p_0} in the Mosco sense ($D_p \overset{M}{\to} D_{p_0}$) as $p \to p_0$ if:*

a) *for every subnet $(a_{p_i})_{i \in I}$ with $a_{p_i} \in D_{p_i}$, $p_i \to p_0$ and $a_{p_i} \overset{\sigma}{\to} a$ imply $a \in D_{p_0}$;*

b) *for every $a \in D_{p_0}$, there exist $a_p \in D_p$ such that $a_p \overset{\tau}{\to} a$ as $p \to p_0$.*

3. Closedness of the solution map

This section is devoted to prove the closedness of the solution map for parametric vector equilibrium problems.

Theorem 3.1. *Let X be a Hausdorff topological space with σ and τ two topologies, where τ is stronger than σ. Let D_p be nonempty sets of X, and let $p_0 \in P$ be fixed. Suppose that $S(p) \neq \emptyset$ for each $p \in P$ and the following conditions hold:*

i) $D_p \overset{M}{\to} D_{p_0}$;

ii) *For each net of elements $(p_i, a_{p_i}) \in GraphS$, if $p_i \to p_0$, $a_{p_i} \overset{\sigma}{\to} a$, $b_{p_i} \in D_{p_i}$, $b \in D_{p_0}$, and $b_{p_i} \overset{\tau}{\to} b$ there exists a subnet of $(p_i, a_{p_i})_{i \in I}$, denoted by the same indexes, such that one of the following conditions applies*

$(C1)$ $\left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)\right)_{i \in I}$ *converge to an element belonging to* $-\operatorname{Int} C$, *when* $p_i \to p_0$

or

$(C2)$ $\left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)\right)_{i \in I}$ *converge to an element belonging to* $-\partial C$, *when* $p_i \to p_0$ *and* $\left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)\right)_{i \in I} \in -C$;

iii) $f_{p_0} : X \times X \to \mathcal{Z}$ *is vector topologically pseudomonotone.*

Then the solution map $p \longmapsto S(p)$ *is closed at* p_0*, i.e. for each net of elements* $(p_i, a_{p_i}) \in GraphS$*,* $p_i \to p_0$ *and* $a_{p_i} \xrightarrow{\sigma} a$ *imply* $(p_0, a) \in GraphS$*.*

Proof. Let $(p_i, a_{p_i})_{i\in I}$ be a net of elements $(p_i, a_{p_i}) \in GraphS$ i.e.

$$f_{p_i}(a_{p_i}, b) \in (-\operatorname{Int} C)^c, \ \forall b \in D_{p_i} \tag{1}$$

with $p_i \to p_0$ and $a_{p_i} \xrightarrow{\sigma} a$. By the Mosco convergence of the sets D_p we get $a \in D_{p_0}$. Moreover there exists a net $(b_{p_i})_{i\in I}$, $b_{p_i} \in D_{p_i}$ such that $b_{p_i} \xrightarrow{\tau} a$. From the assumption *ii)* we obtain that there exists a subnet of $(p_i, a_{p_i})_{i\in I}$, denoted by the same indexes, such that

$$\begin{aligned} &(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, a))_{i\in I} \text{ converge to an element} \\ &\text{belonging to } -\operatorname{Int} C, \text{ when } p_i \to p_0 \end{aligned} \tag{2}$$

or

$$\begin{aligned} &(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, a))_{i\in I} \text{ converge to an element} \\ &\text{belonging to } -\partial C, \text{ when } p_i \to p_0 \text{ and} \\ &(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, a))_{i\in I} \in -C. \end{aligned} \tag{3}$$

Since $-\operatorname{Int} C$ is an open cone, from (2) follows that there exists an index $j_0 \in I$ such that

$$f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, a) \in -\operatorname{Int} C \subset -C, \ i \geq j_0. \tag{4}$$

By replacing b with b_{p_i} in (1) we get

$$f_{p_i}(a_{p_i}, b_{p_i}) \in (-\operatorname{Int} C)^c. \tag{5}$$

From (5), (3) and (4) we obtain that

$$f_{p_0}(a_{p_i}, a) \in (-\operatorname{Int} C)^c, \text{ for } i \geq j_0,$$

since $(-\operatorname{Int} C)^c$ is closed, in both cases we have

$$\operatorname{Liminf} f_{p_0}(a_{p_i}, a) \subset (-\operatorname{Int} C)^c \text{ for } i \geq j_0.$$

Now we can apply $iii)$ and we obtain that for every $b \in D$ and $v \in \operatorname{Int} C$, there exists $j_1 \in I$ such that

$$\overline{\{f_{p_0}(a_{p_i}, b) : i \geq j\}} \subset f_{p_0}(a, b) + v - \operatorname{Int} C,\ \forall j \geq j_1. \tag{6}$$

We have to prove that

$$f_{p_0}(a, b) \in (-\operatorname{Int} C)^c,\ \forall b \in D_{p_0}.$$

Assume the contrary, that there exists $\overline{b} \in D_{p_0}$ such that

$$f_{p_0}\left(a, \overline{b}\right) \in -\operatorname{Int} C.$$

Let be $f_{p_0}\left(a, \overline{b}\right) = -v$ where $v \in \operatorname{Int} C$. From (6) we obtain that there exists $j_1 \in I$ such that

$$\overline{\left\{f_{p_0}\left(a_{p_i}, \overline{b}\right) : i \geq j\right\}} \subset -v + v - \operatorname{Int} C = -\operatorname{Int} C,\ \forall j \geq j_1. \tag{7}$$

Since $\overline{b} \in D_{p_0}$ from the Mosco convergence of the sets D_p, we have that there exists $\left(\overline{b}_{p_i}\right)_{i \in I} \subset D_{p_i}$ such that $\overline{b}_{p_i} \xrightarrow{\tau} \overline{b}$. By using again $ii)$, it follows that there exists a subnet of $(p_i, a_{p_i})_{i \in I}$, denoted by the same indexes, such that

$$f_{p_i}\left(a_{p_i}, \overline{b}_{p_i}\right) - f_{p_0}\left(a_{p_i}, \overline{b}\right) \in -\operatorname{Int} C \subset -C,\ i \geq j_2, \tag{8}$$

where we have used the same reasoning as before.

From (7) and (8) it follows

$$f_{p_i}\left(a_{p_i}, \overline{b}_{p_i}\right) \in -\operatorname{Int} C,\ i \geq \sup\{j_1, j_2\}, \tag{9}$$

but on other side $(p_i, a_{p_i}) \in GraphS$, and

$$f_{p_i}\left(a_{p_i}, \overline{b}_{p_i}\right) \in (-\operatorname{Int} C)^c$$

which is a contradiction. Hence $(p_0, a) \in GraphS$. $\square$

Remark 3.2. *The assumption $ii)$ of the Theorem 3.1 is weaker then the following statement*

$ii')$ For each net of elements $(p_i, a_{p_i}) \in GraphS$, if $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, $b_{p_i} \in D_{p_i}$, $b \in D_{p_0}$, and $b_{p_i} \xrightarrow{\tau} b$ then

$$\operatorname{Liminf}\left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)\right) \cap (-\operatorname{Int} C) \neq \emptyset.$$

Indeed, first we prove that $ii^{'}) \Rightarrow ii)$.

For simplicity, we introduce the following notation

$$u_{p_i} = f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b).$$

From $ii')$ *we obtain that for every* $i_0 \in I$ *we have*

$$\operatorname{Liminf} u_{p_i} \cap (-\operatorname{Int} C) \neq \emptyset \text{ where } i \geq i_0.$$

Wherefrom it follows that there exists a point u *from the limit points of net* $(u_{p_i})_{i\in I}$ *such that for every neighborhood* U *of* u *we have*

$$U \cap [\operatorname{Liminf} u_{p_i} \cap (-\operatorname{Int} C)] \neq \emptyset. \tag{10}$$

There are two cases to be distinguished:

Case 1. $u \in \operatorname{Liminf} u_{p_i} \cap (-\operatorname{Int} C)$. *Since* u *is a limit point of* (u_{p_i}) *there exists a subnet* (u_{p_j}) *converging to* u. *So we have that* $u \in -\operatorname{Int} C$ *then the condition* $(C1)$ *in assumption* $ii)$ *holds.*

Case 2. $u \notin \operatorname{Liminf} u_{p_i} \cap (-\operatorname{Int} C)$. *In this case we must have that* $u \in -\partial C$. *From (10) it follows that for every neighborhood* U *of* u *there exists an* $u_{p_i} \in -\operatorname{Int} C \subset -C$ *such that* $u_{p_i} \in U$. *This leads to the condition* $(C2)$ *of the assumption* $ii)$.

These two assumptions are not equivalent, because there exist nets which satisfy only the assumption $ii)$. *For example, let the net* $(u_{p_i})_{i\in I}$ *be defined by* $u_{p_i} = (2, 4 + 1/p_i)$ *for* $i \in I$, *where* $p_i \to \infty$ *and the cone* C *is given by*

$$C = \left\{(a, b) \in \mathbb{R}^2 : b \geq |2a|\right\}.$$

This net has only one limit inferior point in the $(2, 4)$ *which is located on the boundary of the* C *cone. Hence the assumption* $ii)$ *holds, but the assumption* $ii')$ *fails.*

Remark 3.3. *The Theorem 3.1 does not imply the scalar case. The only exception represents the following condition:*

For each net of elements $(p_i, a_{p_i}) \in Graph S$, *if* $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, $b_{p_i} \in D_{p_i}$, $b \in D_{p_0}$, *and* $b_{p_i} \xrightarrow{\tau} b$ *there exists a subnet of* $(p_i, a_{p_i})_{i\in I}$, *denoted by the same indexes,* *such that*

$$(C3)\quad \begin{array}{l} \left(f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, b\right)\right)_{i\in I} \text{ converge to } 0 \text{ and} \\ \left(f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, b\right)\right)_{i\in I} \notin -C. \end{array}$$

The following example confirms this statement.

Example 3.4. *Let $P = \mathbb{N} \cup \{\infty\}$, $p_0 = \infty$ (∞ means $+\infty$ from real analysis), where we consider the topology induced by the metric given by $d(m,n) = |1/m - 1/n|$, $d(n,\infty) = d(\infty,n) = 1/n$, for $m,n \in \mathbb{N}$, and $d(\infty,\infty) = 0$. Let $X = [0,1]$ where σ, τ are the natural topology, $\mathcal{Z} = \mathbb{R}^2$, $D_p = [0,1]$, $p \in P$, the real vector functions $f_p : [0,1] \times [0,1] \to \mathbb{R}^2$. The ordering cone C is the third quadrant i.e. $C = \left\{(a,b) \in \mathbb{R}^2 : a \leq 0, b \leq 0\right\}$.*

Let $f_n(a,b) = (a - b - 1/n, -2a + 1)$, $n \in \mathbb{N}$ and

$$f_\infty(a,b) = \begin{cases} (a-b, -a+1) & \text{if } \ a > 0 \\ (b,1) & \text{if } \ a = 0 \end{cases}.$$

The f_∞ is vector topologically pseudomonotone, and the condition $(C3)$ holds. We have $(n, 1/n) \in GraphS$ for each $n \in \mathbb{N}$, $S(\infty) = \{1\}$ so $0 \notin S(\infty)$. Hence S is not closed at ∞.

M. Bogdan and J. Kolumbán [5] showed that the topological pseudomonotonicity and the assumption $ii)$ are essential in scalar case.

If the $(VEP)_p$ is defined on constant domains, $D_p = X$ for all $p \in P$, we can omit the Mosco convergence. In this case condition $ii)$ can be weakened to:

(C) For each net of elements $(p_i, a_{p_i}) \in GraphS$, if $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, and $b \in X$, there exists a subnet of $(p_i, a_{p_i})_{i\in I}$, denoted by the same indexes, such that

$$\begin{array}{l} \left(f_{p_i}\left(a_{p_i}, b\right) - f_{p_0}\left(a_{p_i}, b\right)\right)_{i\in I} \text{ converge to an element} \\ \text{belonging to } -\operatorname{Int} C, \text{ when } p_i \to p_0 \end{array}$$

or

$$\begin{array}{l} \left(f_{p_i}\left(a_{p_i}, b\right) - f_{p_0}\left(a_{p_i}, b\right)\right)_{i\in I} \text{ converge to an element} \\ \text{belonging to } -\partial C, \text{ when } p_i \to p_0 \text{ and} \\ \left(f_{p_i}\left(a_{p_i}, b\right) - f_{p_0}\left(a_{p_i}, b\right)\right)_{i\in I} \in -C. \end{array}$$

Theorem 3.5. *Let (X, σ) be a Hausdorff topological space and let $p_0 \in P$ be fixed. Suppose that $S(p) \neq \emptyset$, for each $p \in P$, and*

i) *f_p satisfies condition (C) at p_0;*

ii) *$f_{p_0} : X \times X \to \mathcal{Z}$ is vector topologically pseudomonotone.*

Then the solution map $p \longmapsto S(p)$ is closed at p_0.

Proof. The proof is similar to the proof of the Theorem 3.1. □

4. **Hadamard well-posedness**

Let us recall some classical definitions from set-valued analysis. Let X, Y be topological spaces. The map $T : X \to 2^Y$ is said to be *upper semi-continuous* at $u_0 \in domT := \{u \in X | T(u) \neq \emptyset\}$ if for each neighborhood V of $T(u_0)$, there exists a neighborhood U of u_0 such that $T(U) \subset V$. The map T is considered to be *closed* at $u \in domT$ if for each net $(u_i)_{i \in I}$ in $domT$, $u_i \to u$ and each net $(y_i)_{i \in I}$, $y_i \in T(u_i)$, with $y_i \to y$ one has $y \in T(u)$. The map T is said to be *closed* if its graph $GraphT = \{(u, y) \in X \times Y | y \in T(u)\}$ is closed, namely if $(u_i, y_i) \in GraphT$, $(u_i, y_i) \to (u, y)$ then $(u, y) \in GraphT$.

Closedness and upper semi-continuity of a multifunction are closely related.

Proposition 4.1 ([3] Proposition 1.4.8, 1.4.9)**.** i) *If $T : Y \to 2^X$ has closed values and is upper semi-continuous then T is closed;*

ii) *If X is compact and T is closed at $y \in Y$ then T is upper semi-continuous at $y \in Y$.*

Now we recall the notion of generalized Hadamard well-posedness.

Definition 4.2. *The problem $(VEP)_p$ is said to be Hadamard well-posed (briefly H-wp) at $p_0 \in P$ if $S(p_0) = \{a_{p_0}\}$ and for any $a_p \in S(p)$ one has $a_p \xrightarrow{\sigma} a_{p_0}$, as $p \to p_0$. The problem $(VEP)_p$ is said to be generalized Hadamard well-posed (briefly gH-wp) at $p_0 \in P$ if $S(p_0) \neq \emptyset$ and for any $a_p \in S(p)$, if $p \to p_0$, (a_p) must have a subsequence $\sigma-$converging to an element of $S(p_0)$.*

With the help of the next result we are able to establish the relationship between upper semi-continuity and Hadamard well-posedness.

Proposition 4.3 ([13] Theorem 2.2). *Let X and Y be Hausdorff topological spaces and $T : Y \to 2^X$ be a set valued map. If T is upper semi-continuous at $y \in Y$ and $T(y)$ is compact, then T is gH-wp at y. If more, $T(y) = \{x^*\}$, then T is H-wp at y.*

In the following we prove that the solution map of $(VEP)_p$ has closed value at p_0.

Proposition 4.4. *If D_{p_0} is closed with respect to the σ topology and $f_{p_0} : X \times X \to \mathcal{Z}$ is vector topologically pseudomonotone, then $S(p_0)$ is closed with respect to the σ topology.*

Proof. Let $a_i \in S(p_0)$, with $a_i \overset{\sigma}{\to} a$. Since D_{p_0} is closed with respect to the σ topology, we have $a \in D_{p_0}$. From $a_i \in S(p_0)$ it follows that

$$f_{p_0}(a_i, a) \in (-\operatorname{Int} C)^c, \ \forall i \in I,$$

since $(-\operatorname{Int} C)^c$ is closed, we get

$$\operatorname{Liminf} f_{p_0}(a_i, a) \subset (-\operatorname{Int} C)^c.$$

By using the vector topological pseudomonotonicity we obtain that for every $b \in D$ and $v \in \operatorname{Int} C$ there is j_1 in the index set I such that

$$\overline{\{f_{p_0}(a_i, b) : i \geq j\}} \subset f_{p_0}(a, b) + v - \operatorname{Int} C, \ \forall j \geq j_1. \tag{11}$$

We have to prove that $a \in S(p_0)$, i.e.

$$f_{p_0}(a, b) \in (-\operatorname{Int} C)^c, \ \forall b \in D_{p_0}.$$

Assume the contrary, that there exists $\bar{b} \in D_{p_0}$ such that

$$f_{p_0}\left(a, \bar{b}\right) \in -\operatorname{Int} C.$$

Let $f_{p_0}\left(a, \bar{b}\right) = -v$ where $v \in \operatorname{Int} C$. From (11) we obtain that

$$\overline{\left\{f_{p_0}\left(a_i, \bar{b}\right) : i \geq j\right\}} \subset -v + v - \operatorname{Int} C = -\operatorname{Int} C, \ \forall j \geq j_1$$

which is a contradiction to $a_i \in S(p_0)$. Thus $a \in S(p_0)$. $\square$

Now we can formulate the following result.

Corollary 4.5. *Let (X, σ) be a compact Hausdorff topological space and P be a Hausdorff topological space. Let D_p be nonempty sets of X, and D_{p_0} be a closed subset of X. Suppose that $S(p) \neq \emptyset$ for each $p \in P$ and the following conditions hold:*

i) $D_p \xrightarrow{M} D_{p_0}$;

ii) *For each net of elements $(p_i, a_{p_i}) \in GraphS$, if $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, $b_{p_i} \in D_{p_i}$, $b \in D_{p_0}$, and $b_{p_i} \xrightarrow{\tau} b$ there exists a subnet of $(p_i, a_{p_i})_{i \in I}$, denoted by the same indexes, such that*

$$\begin{gathered}(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b))_{i \in I} \text{ converge to an element} \\ \text{belonging to } -\operatorname{Int} C, \text{ when } p_i \to p_0\end{gathered}$$

or

$$\begin{gathered}(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b))_{i \in I} \text{ converge to an element} \\ \text{belonging to } -\partial C, \text{ when } p_i \to p_0 \text{ and} \\ (f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b))_{i \in I} \in -C;\end{gathered}$$

iii) *$f_{p_0} : X \times X \to \mathcal{Z}$ is vector topologically pseudomonotone.*

Then $(VEP)_p$ is generalized Hadamard well-posed at p_0. Furthermore, if $S(p_0) = \{a_{p_0}\}$ (a singleton), then $(VEP)_p$ is Hadamard well-posed at p_0.

Proof. By Theorem 3.1 it follows that the solution map S is closed at p_0. We may use Proposition 4.1 *ii*) to state that S is upper semi-continuous at p_0. The set $S(p_0)$ is closed by Proposition 4.4, hence it is compact. The conclusion follows by Proposition 4.3. □

We can obtain similar result in the case of constant domains.

Corollary 4.6. *Let (X, σ) be a compact Hausdorff topological space. Let $p_0 \in P$ be fixed and $S(p) \neq \emptyset$, for each $p \in P$. If the hypotheses of Theorem 3.5 are satisfied then $(VEP)_p$ is generalized Hadamard well-posed at p_0. Furthermore, if $S(p_0) = \{a_{p_0}\}$ (a singleton), then $(VEP)_p$ is Hadamard well-posed at p_0.*

References

[1] Ansari, Q.H., Yang, X.C., Yao, J.C., *Existence and duality of implicit vector variational problems*, Numer. Funct. Anal. Optim., **22**(2001), 815-829.

[2] Ansari, Q.H., Yao, J.C., *An existence result for the generalized vector equilibrium problem*, Appl. Math. Lett., **12**(1999), 53-56.

[3] Aubin, J.P., Frankowska, H., *Set-Valued Analysis*, Birkhäuser, Boston, Massachusetts, 1990.

[4] Avriel, M., Diewert, W.E., Schaible, S., Zang, I., *Generalized concavity*, Plenum Press, New York, 1988.

[5] Bogdan, M., Kolumbán, J., *Some regularities for parametric equilibrium problems*, J. Glob. Optim. (to appear, DOI 10.1007/s10898-008-9345-3).

[6] Chadli, O., Chiang, Y., Huang, S., *Topological pseudomonotonicity and vector equilibrium problems*, J. Math. Anal. Appl., **270**(2002), 435-450.

[7] Chiang, Y., *Vector Superior and Inferior*, Taiwanese Journal of Mathematics, **8**(2004), 477-487.

[8] Chiang, Y., Yao, J.C., *Vector variational inequalities and the $(S)_+$ condition*, J. Optim. Theory Appl., **123**(2004), 271-290.

[9] Dontchev, A.L., Zolezzi, T., *Well-posed optimization problems*, Lecture notes in Mathematics, vol. 1543, Springer-Verlag, Berlin, 1993.

[10] Konnov, I., *Generalized Monotone Equilibrium Problems and Variational Inequalities.* In: Hadjisavvas, N., Komlosi, S., and Schaible, S. (eds.) Handbook of Generalized Convexity and Generalized Monotonicity, Springer, vol. 76, 2000.

[11] Mosco, U., *Convergence of convex sets and of solutions of variational inequalities*, Advances in Mathematics, **3**(1969), 510-585.

[12] Tanaka, T., *Generalized semicontinuity and existence theorems for cone saddle points*, Appl. Math. Optim., **36**(1997), 313-322.

[13] Yang, H., Yu, J., *Unified Approaches to Well-posed with Some Applications*, J. Glob. Optim., **31**(2005), 371-381.

Department of Mathematics and Computer Science
Sapientia University, Miercurea Ciuc, Romania
E-mail address: salamonjulia@sapientia.siculorum.ro

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 3, September 2009

THE CHARACTERS OF THE BLASCHKE-GROUP OF THE ARITHMETIC FIELD

ILONA SIMON

Abstract. We consider a locally compact metric space, $\mathbb{B}$ with arithmetic addition and multiplication, which is closely related to the usual multiplication of real numbers in the dyadic system. This results a non-Archimedian local field, the so-called 2-adic local field. Some orthogonal series are studied with respect the inner product defined with the Haar-measure μ. The Blaschke-functions defined on the 2-adic field, $B_a(x) = \frac{x \dot{+} a}{e \dot{+} a \bullet x}$ form a commutative group with respect to the function composition, the so-called Blaschke-group. We shall determine the characters of this group. By means of the exponential and tangent functions on the 2-adic field and the characters of its additive group we can identify the desired characters. We consider Fourier-series with respect to these characters and summability questions are examined. A simple recursion leads to the FFT-algorithm, the so-called Fast-Fourier Transform.

1. Introduction

According to Volovich[4] some non-Archimedean normed fields must be used for a global space-time theory in order to unify both microscopic and macroscopic physics. Some problems occured with the practical applications of the classical fields $\mathbb{R}$ and $\mathbb{C}$, because in sciences there are absolute limitations on measurements like Plank time, Plank length, Plank mass, and also there is a problem with the Archimedean axiom on the microscopic level. Volovich proposes to base physics on a coalition of

Received by the editors: 11.01.2007.

2000 *Mathematics Subject Classification.* 11F85, 43A40.

Key words and phrases. p-adic theory, local fields, character groups, (C,1)-summability, Fast-Fourier Transform.

This paper was presented at the 7-th Joint Conference on Mathematics and Computer Science, July 3-6, 2008, Cluj-Napoca, Romania.

non-Archimedean normed fields and classical fields as $\mathbb{R}$ or $\mathbb{C}$. The so-called p-adic field is a suitable non-Archimedean normed field. As $p \to \infty$, many of the fundamental functions of p-adic analysis approach their counterparts in classical analysis. Thus p-adic analysis could provide a bridge from microscopic to macroscopic physics. The simplest example of a p-adic field is the 2-adic field used in this paper.

Characters are very useful in numerous branches of mathematics, for example in many cases are used Fourier-series with respect to characters.

Denote by $\mathbb{A} := \{0, 1\}$ the set of bits and by

$$\mathbb{B} := \{a = (a_j, j \in \mathbb{Z}) \mid a_j \in \mathbb{A} \text{ and } \lim_{j \to -\infty} a_j = 0\} \tag{1}$$

the set of bytes. The numbers a_j are called the additive digits of $a \in \mathbb{B}$. The zero element of $\mathbb{B}$ is $\theta := (x_j, j \in \mathbb{Z})$ where $x_j = 0$ for $j \in \mathbb{Z}$, that is, $\theta = (\cdots, 0, 0, 0, \cdots)$. The order of a byte $x \in \mathbb{B}$ is defined in the following way: For $x \neq \theta$ let $\pi(x) = n$ if and only if $x_n = 1$ and $x_j = 0$ for all $j < n$, furthermore set $\pi(\theta) = +\infty$. The norm of a byte x is defined by

$$\|x\| := 2^{-\pi(x)} \text{ for } x \in \mathbb{B} \setminus \{\theta\}, \qquad \text{and } \|\theta\| := 0. \tag{2}$$

The sets $I_n(x) := \{y \in \mathbb{B} : y_k = x_k \text{ for } k < n\}$, the so-called intervals in $\mathbb{B}$ of rank $n \in \mathbb{Z}$ and center x are of basic importance. Set $\mathbb{I}_n := I_n(\theta) = \{x \in \mathbb{B} : \|x\| \leqq 2^{-n}\}$ for any $n \in \mathbb{Z}$. The unit ball $\mathbb{I} := \mathbb{I}_0$ can be identified with the set of sequences $\mathbb{I} = \{a = (a_j, j \in \mathbb{N}) | \; a_j \in \mathbb{A}\}$ via the map $(\ldots, 0, 0, a_0, a_1, \ldots) \mapsto (a_0, a_1, \ldots)$. Furthermore $\mathbb{S} := \{x \in \mathbb{B} : \|x\| = 1\} = \{x \in \mathbb{B} : \pi(x) = 0\} = \{x \in \mathbb{I} : x_0 = 1\}$ is the unit sphere of the field.

We will use the normalized Haar-measure on $\mathbb{B}$, which satisfies $\mu(I_n(a)) := 2^{-n}$. Some orthogonal series are studied with respect the inner product defined with the Haar-measure μ by

$$\langle f, g \rangle := \int_{\mathbb{I}} f(x)\overline{g(x)} d\mu(x).$$

Now, consider the 2-adic (or arithmetical) sum $a \overset{\bullet}{+} b$ of elements $a = (a_n, n \in \mathbb{Z}), b = (b_n, n \in \mathbb{Z}) \in \mathbb{B}$, defined by

$$a \overset{\bullet}{+} b := (s_n, n \in \mathbb{Z})$$

where the bits $q_n, s_n \in \mathbb{A}$ $(n \in \mathbb{Z})$ are obtained recursively as follows:

$$\begin{aligned} &q_n = s_n = 0 \quad \text{ for } n < m := \min\{\pi(a), \pi(b)\}, \\ &\text{and } \ a_n + b_n + q_{n-1} = 2q_n + s_n \quad \text{ for } n \geq m. \end{aligned} \tag{3}$$

The 2-adic (or arithmetical) product of $a, b \in \mathbb{B}$ is $a \bullet b := (p_n, n \in \mathbb{Z})$, where the sequences $q_n \in \mathbb{N}$ and $p_n \in \mathbb{A}$ $\ (n \in \mathbb{Z})$ are defined recursively by

$$\begin{aligned} &q_n = p_n = 0 \quad (n < m := \pi(a) + \pi(b)) \\ &\text{and } \sum_{j=-\infty}^{\infty} a_j b_{n-j} + q_{n-1} = 2q_n + p_n \quad (n \geq m). \end{aligned} \tag{4}$$

The reflection x^- of a byte $x = (x_j, j \in \mathbb{Z})$ is defined by its additive digits:

$$(x^-)_j = \begin{cases} x_j, & \text{for } j \leqq \pi(x) \\ 1 - x_j, & \text{for } j > \pi(x). \end{cases}$$

Note, that x^- is the additive inverse of an $x \in \mathbb{B}$.

The operations $\overset{\bullet}{+}, \bullet$ are commutative. Notice, that

$$\pi(a \bullet b) = \pi(a) + \pi(b). \tag{5}$$

Moreover, $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ is a non-Archimedian normed field with respect the (2) norm, that is,

$$\|x \overset{\bullet}{+} y\| \leq \max\{\|x\|, \|y\|\}, \qquad \|x \bullet y\| = \|x\| \cdot \|y\| \tag{6}$$

with equality if and only if $\|x\| \neq \|y\|$ See [2]. The operations $\overset{\bullet}{+}, \bullet$ are continuous with respect the metric introduced by the norm (2), that is, $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ is a topological field. $(\mathbb{S}, \bullet)$ is a subgroup of $(\mathbb{B}, \bullet)$.

We will use the following notation: $a \overset{\bullet}{-} b := a \overset{\bullet}{+} b^-$.

The multiplicative identity of $\mathbb{B}$ is the element $e = e_0 = (\delta_{n0}, n \in \mathbb{Z})$, where δ_{nk} is the Kronecker-symbol. Furthermore we will use the elements $e_k := (\delta_{nk}, n \in \mathbb{Z})$

for some $k \in \mathbb{Z}$. We can observe, that $e_k \bullet e_m = e_{k+m}$ for all $k, m \in \mathbb{Z}$. In general, multiplication by e_k shifts bytes: $e_k \bullet a = (a_{n-k}, n \in \mathbb{Z})$. We will represent infinite products on this field by $\prod_{n=1}^{\infty} \bullet\, \alpha_j := \lim_{n\to\infty}(\alpha_1 \bullet \alpha_2 \bullet \cdots \bullet \alpha_n)$.

A character of a topological group $(\mathbb{G}, +)$ is a continuous function $\phi : \mathbb{G} \to \mathbb{C}$ which satisfies $|\phi(x)| = 1$ and $\phi(x + y) = \phi(x)\phi(y)$ for all $x, y \in \mathbb{G}$.

2. **The characters of the Blaschke-group**

For $x \in \mathbb{I}$ and $a \in \mathbb{I}_1$ we have by (6) and (5) that $e \overset{\bullet}{-} a \bullet x \neq \theta$, thus $e \overset{\bullet}{-} a \bullet x$ has a multiplicative inverse in $\mathbb{B}$. For $a \in \mathbb{I}_1$ define the Blaschke function on $\mathbb{I}$:

$$B_a(x) := (x \overset{\bullet}{-} a) \bullet (e \overset{\bullet}{-} a \bullet x)^{-1} = \frac{x \overset{\bullet}{-} a}{e \overset{\bullet}{-} a \bullet x}. \qquad (x \in \mathbb{I}) \tag{7}$$

The Blaschke function $B_a : \mathbb{I} \to \mathbb{I}$ is a bijection for any $a \in \mathbb{I}_1$. The composition of two Blaschke-functions is also a Blaschke-function: $B_a \circ B_b = B_c$ where $c = \frac{a \overset{\bullet}{+} b}{e \overset{\bullet}{+} a \bullet b}$ is also in $\mathbb{I}_1$ for $a, b \in \mathbb{I}_1$. Thus the maps B_a $(a \in \mathbb{I}_1)$ form a commutative group with respect to the function composition. See[3]. We will call

$$\mathcal{B} := \{B_a, a \in \mathbb{I}_1\} \tag{8}$$

the Blashke-group of the field $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$.

We will determine the characters of the Blashke-group $(\mathcal{B}, \circ)$, where $\circ$ denotes the function composition.

Using the notation $x \lhd y := \frac{x \overset{\bullet}{+} y}{e \overset{\bullet}{+} x \bullet y}$ $(x, y \in \mathbb{I}_1)$, the map

$$B : (\mathbb{I}_1, \lhd) \to (\mathcal{B}, \circ),\ a \mapsto B_a$$

is an isomorphism, which is continuous, consequently it is useful if we define the character group of $(\mathbb{I}_1, \lhd)$.

We already know the characters of $(\mathbb{I}_1, \overset{\bullet}{+})$ and for this reason it is suitable to find a continuous isomorphism from $(\mathbb{I}_1, \overset{\bullet}{+})$ onto $(\mathbb{I}_1, \lhd)$, that is a function γ satisfying the equation

$$\gamma(x \stackrel{\bullet}{+} y) = \frac{\gamma(x) \stackrel{\bullet}{+} \gamma(y)}{e \stackrel{\bullet}{+} \gamma(x) \bullet \gamma(y)}. \ (x, y \in \mathbb{I}_1) \tag{9}$$

This equation is the analogue of the function equation of the tangent function on $\mathbb{C}$, where the tangent function can be expressed by the exponential function in the following way:

$$\tan(x) = \frac{\exp(ix) - \exp(-ix)}{i(\exp(ix) + \exp(-ix))} = \frac{\exp(2ix) - 1}{i(\exp(2ix) + 1)}. \quad (x \in \mathbb{C})$$

Furthermore, we will use the function ζ, expressed in the following infinite product form:

$$\zeta(x) := \prod_{j=1}^{\infty} \bullet \, b_j^{x_j} \qquad (x = (x_j, j \in \mathbb{Z}) \in \mathbb{I}_1) \tag{10}$$

where

$$b_1 := e \stackrel{\bullet}{+} e_2, \ b_n := b_{n-1} \bullet b_{n-1} \qquad (n \geq 2). \tag{11}$$

We will call the function ζ the $(\mathbb{S}, \bullet)$-valued exponential function on $\mathbb{I}_1$, which is a continuous function satisfying the function-equation

$$\zeta(x \stackrel{\bullet}{+} y) = \zeta(x) \bullet \zeta(y) \qquad (x, y \in \mathbb{I}_1). \tag{12}$$

This function ζ satisfies indeed (12) on $\mathbb{I}_1$, which can be easily seen analogous to [2], pp 59-60, where we find in a way different basis $(b_n, n \geq 1)$. Since $b_n = e \stackrel{\bullet}{+} c_n$ $(n \geq 1)$ with $\pi(c_n) = n + 1$, the function ζ has the following representation:

$$\zeta(x) = \prod_{j=1}^{\infty} \bullet \, (e \stackrel{\bullet}{+} c_j)^{x_j} = \prod_{j=1}^{\infty} \bullet \, (e \stackrel{\bullet}{+} x_j c_j). \tag{13}$$

Let us denote $\tilde{\mathbb{S}} := \{x \in \mathbb{S} : x_1 = 0\}$. We can see as in Theorem 2 in [2] that ζ is 1-1 and continuous from $\mathbb{I}_1$ onto $\tilde{\mathbb{S}}$.

Now, we will call the function

$$\gamma(x) := \frac{\zeta(x) \stackrel{\bullet}{-} e}{\zeta(x) \stackrel{\bullet}{+} e} \quad (x \in \mathbb{I}_1) \tag{14}$$

the tangent-like function on $(\mathbb{I}_1, \overset{\bullet}{+})$ and

$$\tan(x) := \frac{\zeta^2(x) \overset{\bullet}{-} e}{\zeta^2(x) \overset{\bullet}{+} e} \quad (x \in \mathbb{I}_1) \tag{15}$$

the tangent function on $(\mathbb{I}_1, \overset{\bullet}{+})$.

Lemma 1. *For any $a, b \in \mathbb{B}, x \in \mathbb{I}_1$, and $y \in \mathbb{I}_1$ holds*

$$\begin{aligned}
&a)\ \frac{a \overset{\bullet}{+} a}{b \overset{\bullet}{+} b} = \frac{a}{b}\\
&b)\ a \overset{\bullet}{+} a = e_1 \bullet a\\
&c)\ \zeta^2(x) = \zeta(e_1 \bullet x)\\
&d)\ \frac{e \overset{\bullet}{+} y}{e \overset{\bullet}{-} y} \in \tilde{\mathbb{S}},
\end{aligned} \tag{16}$$

where $\zeta^2(x) = \zeta(x) \bullet \zeta(x)$.

Proof. a) The relation holds, because $a \bullet (b \overset{\bullet}{+} b) = b \bullet (a \overset{\bullet}{+} a)$ is satisfied by the commutativity and distributivity of the operations.

b) Using the notations of the recursive definition for the addition $\overset{\bullet}{+}$, we have $(a \overset{\bullet}{+} a)_n = 0$ if and only if $q_{n-1} = 0$. But $q_{n-1} = 0$ is equivalent with $a_{n-1} = 0$, which holds exactly when $(e_1 \bullet a)_n = 0$, because multiplication by e_1 shifts a. Similarly $(a \overset{\bullet}{+} a)_n = 1 \Leftrightarrow q_{n-1} = 1 \Leftrightarrow a_{n-1} = 1 \Leftrightarrow (e_1 \bullet a)_n = 1$.

c) It is a simple consequence of b) or directly: $b_j \bullet b_j = b_{j+1}$ $(j \geq 1)$, thus using the commutativity and associativity of $\bullet$ we have $\zeta^2(x) = \left(\prod_{j=1}^{\infty} \bullet\, b_j^{x_j}\right) \bullet \left(\prod_{j=1}^{\infty} \bullet\, b_j^{x_j}\right) = \prod_{j=1}^{\infty} \bullet\, b_{j+1}^{x_j} = \zeta(e_1 \bullet x)$ $(x \in \mathbb{I}_1)$

d) It can be easily established, that for $y = (0, y_1, y_2 \ldots) \in \mathbb{I}_1$ holds:

$$e \overset{\bullet}{+} y = (1, y_1, y_2, y_3, \ldots)$$

and

$$e \overset{\bullet}{-} y = (1, y_1, (y^-)_2, \ldots) = e \overset{\bullet}{+} y^-.$$

Applying the notation

$$\frac{e \overset{\bullet}{+} y}{e \overset{\bullet}{-} y} = z,$$

we can state first, that $\pi(z) = \pi(e \overset{\bullet}{+} y) - \pi(e \overset{\bullet}{-} y) = 0$ that is, $z \in \mathbb{S}$, and then

$$e \overset{\bullet}{+} y = z \bullet (e \overset{\bullet}{-} y).$$

Now, examining the 0th and the 1-st digits of the right and left side, we find that:

$$\begin{cases} 1 = z_0 \cdot 1 \\ y_1 = z_0 \cdot y_1 + z_1 \cdot 1 \pmod 2 \end{cases}$$

which means, that $z_0 = 1$ and $z_1 = 0$, and so $z \in \tilde{\mathbb{S}}$. Note, that $z = e \Leftrightarrow y = \theta$.

$\square$

With Lemma 1 c) we can see, that the the tangent-like function γ is closely related to tan: namely $\gamma(x) = \tan(e_{-1} \bullet x)$ $(x \in \mathbb{I}_1)$.

Theorem 1. *The function γ is a continuous isomorphism from $(\mathbb{I}_1, \overset{\bullet}{+})$ onto $(\mathbb{I}_1, \lhd)$.*

Proof.

$$\gamma(x) \lhd \gamma(y) = \frac{\gamma(x) \overset{\bullet}{+} \gamma(y)}{e \overset{\bullet}{+} \gamma(x) \bullet \gamma(y)} = \frac{\frac{\zeta(x) \overset{\bullet}{-} e}{\zeta(x) \overset{\bullet}{+} e} \overset{\bullet}{+} \frac{\zeta(y) \overset{\bullet}{-} e}{\zeta(y) \overset{\bullet}{+} e}}{e \overset{\bullet}{+} \frac{\zeta(x) \overset{\bullet}{-} e}{\zeta(x) \overset{\bullet}{+} e} \bullet \frac{\zeta(y) \overset{\bullet}{-} e}{\zeta(y) \overset{\bullet}{+} e}}$$

$$= \frac{\zeta(x) \bullet \zeta(y) \overset{\bullet}{+} \zeta(x) \bullet \zeta(y) \overset{\bullet}{-} e \overset{\bullet}{-} e}{\zeta(x) \bullet \zeta(y) \overset{\bullet}{+} \zeta(x) \bullet \zeta(y) \overset{\bullet}{+} e \overset{\bullet}{+} e} = \frac{\zeta(x) \bullet \zeta(y) \overset{\bullet}{-} e}{\zeta(x) \bullet \zeta(y) \overset{\bullet}{+} e} = \gamma(x \overset{\bullet}{+} y)$$

where we used Lemma 1 a).

The function γ is a 1-1 map from $(\mathbb{I}_1, \overset{\bullet}{+})$ onto $(\mathbb{I}_1, \lhd)$. To see, that γ is a 1-1 map, we have from

$$\frac{\zeta(x) \overset{\bullet}{-} e}{\zeta(x) \overset{\bullet}{+} e} = \frac{\zeta(y) \overset{\bullet}{-} e}{\zeta(y) \overset{\bullet}{+} e}$$

the equation

$$\zeta(x) \overset{\bullet}{+} \zeta(x) = \zeta(y) \overset{\bullet}{+} \zeta(y).$$

Taking in consideration, that $f(a) := a \overset{\bullet}{+} a$ is 1-1, satisfying $a \overset{\bullet}{+} a = e_1 \bullet a$, we have

$$\zeta(x) = \zeta(y),$$

which gives that $x = y$.

To see, that for any $y \in \mathbb{I}_1$ there is an $x \in \mathbb{I}_1$ such that $\gamma(x) = y$, we have to solve in x the equation:

$$\frac{\zeta(x) \overset{\bullet}{-} e}{\zeta(x) \overset{\bullet}{+} e} = y,$$

thus

$$\zeta(x) = \frac{e \overset{\bullet}{+} y}{e \overset{\bullet}{-} y}.$$

Now,

$$x = \zeta^{-1}\left(\frac{e \overset{\bullet}{+} y}{e \overset{\bullet}{-} y}\right).$$

Thus we proved that γ is onto if $\zeta^{-1}\left(\frac{e \overset{\bullet}{+} y}{e \overset{\bullet}{-} y}\right) \in \mathbb{I}_1$ which holds in consequence of Lemma 1 d). Thus we proved that γ is an isomorphism from $(\mathbb{I}_1, \overset{\bullet}{+})$ onto $(\mathbb{I}_1, \triangleleft)$.

$\square$

We consider $\varepsilon(t) := \exp(2\pi i t)$ $(t \in \mathbb{R})$. The characters of the group $(\mathbb{I}_1, \overset{\bullet}{+})$ are given by the product system $(v_m, m \in \mathbb{P})$ generated by the functions

$$v_{2^n}(x) := \varepsilon\left(\frac{x_n}{2} + \frac{x_{n-1}}{2^2} + \cdots + \frac{x_1}{2^n}\right) \qquad (x = (0, x_1, x_2 \ldots) \in \mathbb{I}_1, n \in \mathbb{P}),$$

that is, the functions $v_m(x) = \prod_{j=1}^{\infty} (v_{2^j}(x))^{m_j}$ $(m \in \mathbb{P})$. [2] Recall, that $\mathbb{P}$ is the set of positive numbers, $\mathbb{P} := \mathbb{N} \setminus \{0\}$.

Theorem 2. *The characters of the group* $(\mathbb{I}_1, \triangleleft)$ *are the functions*

$$v_n \circ \gamma^{-1} (n \in \mathbb{P}).$$

Corollary 1. *The characters of* $(\mathcal{B}, \circ)$ *are the functions*

$$v_n \circ \gamma^{-1} \circ B^{-1} (n \in \mathbb{P}),$$

where $(\mathcal{B}, \circ)$ *denotes the Blaschke-group of the arithmetic field* $(\mathbb{I}, \stackrel{\bullet}{+}, \bullet)$*, and* $B : (\mathbb{I}_1, \triangleleft) \to (\mathcal{B}, \circ)$ *is the function* $a \mapsto B_a$.

3. **Recursion**

In (13) we used the notation $b_n = e \stackrel{\bullet}{+} c_n \ (n \geq 1)$ where $\pi(c_n) = n+1$, now consider $b_n = e \stackrel{\bullet}{+} e_{n+1} \stackrel{\bullet}{+} d_n \ (n \geq 1)$ where $\pi(d_n) \geq n+2$. Now the function ζ has the following representation:

$$\zeta(x) = \prod_{j=1}^{\infty} \bullet \, (e \stackrel{\bullet}{+} e_{j+1} \stackrel{\bullet}{+} d_j)^{x_j} = \prod_{j=1}^{\infty} \bullet \, (e \stackrel{\bullet}{+} x_j e_{j+1} \stackrel{\bullet}{+} x_j d_j). \ (d_j \in \mathbb{I}_{n+2})$$

Easy inductive arguments establish that $\zeta(x)$ is a simple recursion:

$$(\zeta(x))_n = x_{n-1} + f(x_1, \ldots, x_{n-2}) \qquad (n \geq 1) \tag{17}$$

and $(\zeta(x))_0 = 1$. Thus $z := \zeta(x) \stackrel{\bullet}{-} e = (\zeta(x) \stackrel{\bullet}{+} e^-) = (1, 0, \zeta_2, \zeta_3, \ldots) \stackrel{\bullet}{+} (1, 1, 1, 1, \ldots) = (0, 0, \zeta_2, \zeta_3, \zeta_4, \ldots)$ can also be written as a simple recursion:

$$z_n = x_{n-1} + f(x_1, \ldots, x_{n-2}) \qquad (n \geq 2).$$

Analogous, $t := \zeta(x) \stackrel{\bullet}{+} e = (1, 0, \zeta_2, \zeta_3, \ldots) \stackrel{\bullet}{+} (1, 0, 0, 0, \ldots) = (0, 1, \zeta_2, \zeta_3, \zeta_4, \ldots)$ as a simple recursion:

$$t_n = x_{n-1} + f(x_1, \ldots, x_{n-2}) \qquad (n \geq 2).$$

The multiplicative inverse element of $t \in \mathbb{I}_1$ is also a simple recursion:

$$(t^{-1})_n = x_{n+1} + f(x_1, \ldots, x_n)$$

for some function f. See[[2], pp. 39-40.]

Using $(t^{-1})_{-1} = 1$ and $(\gamma(x))_n = z_2(t^{-1})_{n-2} + \ldots + z_{n+1}(t^{-1})_{-1} + q_{n-1} \pmod 2$, follows that

$$(\gamma(x))_n = x_n + f(x_1, \ldots, x_{n-1}). \tag{18}$$

Denote with $\mathcal{A}$ the σ-algebra generated by the intervals $I_n(a)$ $(a \in \mathbb{I}, n \in \mathbb{N})$. Let $\mu(I_n(a)) \doteq 2^{-n}$ be the measure of $I_n(a)$. Extending this measure to $\mathcal{A}$ we get a probability measure space $(\mathbb{I}, \mathcal{A}, \mu)$. Let $\mathcal{A}_n$ be the sub-σ-algebra of $\mathcal{A}$ generated by the intervals $I_n(a)$ $(a \in \mathbb{I})$. Let $L(\mathcal{A}_n)$ denote the set of $\mathcal{A}_n$-measurable functions on $\mathbb{I}$. The conditional expectation of an $f \in L^1(\mathbb{I})$ with respect to $\mathcal{A}_n$ is of the form

$$(\mathcal{E}_n f)(x) = \frac{1}{\mu(I_n(x))} \int_{I_n(x)} f d\mu.$$

A sequence of functions $(f_n, n \in \mathbb{N})$ is called a dyadic martingale if each f_n is $\mathcal{A}_n$-measurable and

$$(\mathcal{E}_n f_{n+1}) = f_n \qquad (n \in \mathbb{N}).$$

The sequence of martingale differences of f_n $(n \in \mathbb{N})$ is the sequence

$$\phi_n \doteq f_{n+1} - f_n \qquad (n \in \mathbb{N}).$$

We notice that every dyadic martingale difference sequence has the form $\phi_n = r_n g_n$ $(n \in \mathbb{N})$ where $(g_n, n \in \mathbb{N})$ is a sequence of functions such that each g_n is $\mathcal{A}_n$-measurable and $(r_n, n \in \mathbb{N})$ denotes the Rademacher system on $\mathbb{I}$:

$$r_n(x) \doteq (-1)^{x_n} \ (n \in \mathbb{N}).$$

The martingale difference sequence $(\phi_n, n \in \mathbb{N})$ is called a unitary dyadic martingale difference sequence or a UDMD sequence if $|\phi_n(x)| = 1$ $(n \in \mathbb{N})$. Thus $(\phi_n, n \in \mathbb{N})$ is a UDMD sequence if and only if

$$\phi_n = r_n g_n, \ g_n \in L(\mathcal{A}_n), \ |g_n| = 1 \ \ (n \in \mathbb{N}). \tag{19}$$

Let us call a system $\psi = (\psi_m, m \in \mathbb{N})$ a UDMD product system if it is a product system generated by a UDMD system, i.e., there is a UDMD system $(\phi_n, n \in \mathbb{N})$ such that for each $m \in \mathbb{N}$, with binary expansion is given by $m = \sum_{j=0}^{\infty} m_j 2^j$ $(m_j \in \mathbb{A}, j \in \mathbb{N})$, the function ψ_m satisfies

$$\psi_m = \prod_{j=0}^{\infty} \phi_j^{m_j} \qquad (m \in \mathbb{N}).$$

By (18) the byte $\gamma^{-1}(x)$ can also be written by a simple recursion for any $x \in \mathbb{I}_1$, therefore we have the following:

Corollary 2. *The functions* $v_n \circ \gamma^{-1} (n \in \mathbb{P})$*, the characters of* $(\mathbb{I}_1, \triangleleft)$ *form a UDMD product system.*

Proof. The $(v_{2^n} \circ \gamma^{-1}, n \in \mathbb{P})$ functions satisfy the requirements of a UDMD-system: $v_{2^n}(\gamma(x)) = \varepsilon\left(\frac{x_n}{2}\right) g(x_1, \ldots, x_{n-1}) = (-1)^{x_n} g(x_1, \ldots, x_{n-1})$, with some $g \in L(\mathcal{A}_n)$, and $|g(x_1, \ldots, x_{n-1})| = 1$.

□

As $(v_n \circ \gamma^{-1}, n \in \mathbb{P})$ is a UDMD product system, the discrete Fourier coefficients with respect this system can be computed with the Fast Fourier Algorithm.

4. (C,1) summability

By (18) $\gamma : I_n(x) \to I_n(\gamma(x))$ is a bijection $(x \in \mathbb{I}_1, n \in \mathbb{N})$, thus for any dyadic interval E holds $\mu(t \in \mathbb{I}_1 : \gamma(t) \in E) = \mu(E)$ and this follows for any E measurable sets also. Therefore the variable transformation $\gamma(x)$ is measure preserving. Consequently, it holds

$$\int_{\mathbb{I}_1} f \circ \gamma d\mu = \int_{\mathbb{I}_1} f d\mu. \tag{20}$$

The Gamma-Fourier coefficients of an $f \in L^1(\mathbb{I}_1)$ are defined by

$$\widehat{f^\gamma}(n) \doteq \int_{\mathbb{I}_1} f(x) v_n(\gamma(x)^{-1}) d\mu(x) \qquad (n \in \mathbb{P}).$$

We have by (20):

$$\widehat{f^\gamma}(n) = \widehat{f \circ \gamma}(n), \tag{21}$$

where $\widehat{f}(n)$ are the well-known Fourier coefficients of an $f \in L^1(\mathbb{I})$. [1]

The Gamma-Fourier series of an $f \in L^1(\mathbb{I}_1)$ is the series

$$S^\gamma f \doteq \sum_{k=0}^{\infty} \widehat{f^\gamma}(k) v_k \circ \gamma^{-1},$$

and the n-th partial sums of the Gamma-Fourier series S^γ is

$$S_n^\gamma f \doteq \sum_{k=0}^{n-1} \widehat{f^\gamma}(k) v_k \circ \gamma^{-1} \ (n \in \mathbb{P}).$$

It follows by (21) that

$$S_n^\gamma f = [S_n(f \circ \gamma)] \circ \gamma^{-1} \tag{22}$$

where S_n is the well-known n-th partial sum of the Walsh-Fourier series. See[1] .

If the Gamma-Cesaro (or $(G-C,1)$) means of $S^\gamma f$ are defined by $\sigma_0 f \doteq 0$ and

$$\sigma_n^\gamma f \doteq \frac{1}{n}\sum_{k=1}^{n} S_k^\gamma f, \qquad (n\in\mathbb{P})$$

then it follows by (22) that

$$\sigma_n^\gamma f(x) = \frac{1}{n}\sum_{k=1}^{n}\left[S_k(f\circ\gamma)\right](\gamma^{-1}(x)) = \sigma_n(f\circ\gamma)(\gamma^{-1}(x)). \tag{23}$$

where σ_n means the well known n-th Cesaro mean of Sf. [1]

Now, we use the theorem of the $(C,1)$-summability of the Walsh-Fourier series on the field $(\mathbb{I},\overset{\bullet}{+},\bullet)$ due to Gy. Gát [5]: $\lim\limits_{m\to\infty}(\sigma_m f)(x) = f(x)$ a.e. for any $f\in L^1(\mathbb{I})$.

Thus with (23) we have $\lim\limits_{n\to\infty}\sigma_n^\gamma f(x) = \lim\limits_{n\to\infty}\sigma_n(f\circ\gamma)(\gamma^{-1}(x)) = (f\circ\gamma\circ\gamma^{-1})(x) = f(x)$ a.e. for any $f\in L^1(\mathbb{I}_1)$.

Theorem 3. *On the field* $(\mathbb{I}_1,\overset{\bullet}{+},\bullet)$ *holds* $\lim\limits_{n\to\infty}\sigma_n^\gamma f(x) = f(x)$ *a.e. for any* $f\in L^1(\mathbb{I}_1)$.

References

[1] Schipp, F., Wade, W.R., Simon, P. Pál, J., *Walsh Series, An Introduction to Dyadic Harmonic Analysis*, Adam Hilger, Ltd., Bristol and New York, 1990.

[2] Schipp, F., Wade, W.R., *Transforms on normed fields*, Budapest (1995), available also at *http://numanal.inf.elte.hu/~schipp/TrNFields.pdf*.

[3] Simon, I., *Discrete Laguerre functions on the dyadic fields*, PU.M.A, **17(3-4)**(2006), 459-468.

[4] Vladimirov, V.S., Volovich, I.V., Zelnov, E.I., *p-adic Analysis and Mathematical Physics*, Series in Soviet and East European Mathematics, Vol. **10**, World Scientific Publishers, Singapore-New Jersey-London-Hong Kong, 1994.

[5] Gát, Gy., *On the almost everywhere convergence of Fejér means of functions on the group of 2-adic integers*, Journal of Approx. Theory, **90(1)**(1997), 88-96.

University of Pécs
Institute of Mathematics and Informatics
Hungary, 7624 Pécs, Ifjúság u. 6
E-mail address: `simoni@gamma.ttk.pte.hu`

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 3, September 2009

GRONWALL LEMMAS AND COMPARISON THEOREMS FOR THE CAUCHY PROBLEM ASSOCIATED TO A SET DIFFERENTIAL EQUATION

IOANA CAMELIA TIŞE

Abstract. Let $P_{cp,cv}(\mathbb{R}^n)$ be the family of all nonempty compact, convex subset of $\mathbb{R}^n$. We consider the following Cauchy problem:

$$(1) \quad \begin{cases} D_H U = F(t,U), \ t \in J \\ U(t_0) = U^0 \end{cases}$$

where $U^0 \in P_{cp,cv}(\mathbb{R}^n), \ t_0 \geq 0, \ J = [t_0, t_0 + a], \ a > 0$, and

$$F : J \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n).$$

The purpose of the paper is to study the existence of a solution as well as some comparison theorems and Gronwall type lemmas for the above Cauchy problem.

1. **Introduction**

Let $\mathbb{R}^n$ be the real n-dimensional space and $P_{cp,cv}(\mathbb{R}^n)$ the family of all nonempty compact, convex subset of $\mathbb{R}^n$ endowed with the Pompeiu-Hausdorff metric H.

We consider the following Cauchy problem with respect to a set differential equation:

$$(1) \quad \begin{cases} D_H U = F(t,U), \ t \in J \\ U(t_0) = U^0 \end{cases}$$

Received by the editors: 24.10.2008.

2000 *Mathematics Subject Classification.* 47H10.

Key words and phrases. L-space, fixed point, Cauchy problem, lower fixed points, upper fixed points, Gronwall lemmas.

This paper was presented at the 7-th Joint Conference on Mathematics and Computer Science, July 3-6, 2008, Cluj-Napoca, Romania.

where $U^0 \in P_{cp,cv}(\mathbb{R}^n),\ t_0 \geq 0,\ J = [t_0, t_0 + a],\ a > 0,$
$F \in C(J \times P_{cp,cv}(\mathbb{R}^n), P_{cp,cv}(\mathbb{R}^n))$ and D_H is the Hukuhara derivative of U.

A solution of (1) is a continuous function $U : J \to P_{cp,cv}(\mathbb{R}^n)$ which satisfies (1) for each $t \in J$.

The aim of the article is to study the existence of a solution as well as some comparison theorems and Gronwall type lemmas for the above Cauchy problem.

The paper is organized as follows. The next section, Preliminaries, contains some basic notations and notions used throughout the paper. The third section presents some comparison theorems and Gronwall type lemmas for the above Cauchy problem (1).

2. **Preliminaries**

The aim of this section is to present some notions and symbols used in the paper.

Definition 1. *$U \in C^1(J, P_{cp,cv}(\mathbb{R}^n))$ is a solution of the problem (1)$\Longleftrightarrow$ U satisfies (1) for all $t \in J$.*

Let us consider the following equations:

$$(2)\ \ U(t) = U^0 + \int_{t_0}^{t} D_H(U(s))ds,\ \ t \in J,$$

$$(3)\ \ U(t) = U^0 + \int_{t_0}^{t} F(s, U(s))ds,\ \ t \in J.$$

Lemma 2. *If $U \in C^1(J, P_{cp,cv}(\mathbb{R}^n))$, then* (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3).

We consider on $C(J, P_{cp,cv}(\mathbb{R}^n))$ the metric H_*^B defined by:

$$H_*^B(U, V) := \max_{t \in [t_0, t_0+a]} [H(U(t), V(t))e^{-\tau(t-t_0)}],\ \ \tau > 0.$$

The pair $(C(J, P_{cp,cv}(\mathbb{R}^n)), H_*^B)$ forms a complete metric space.

We consider on $P_{cp,cv}(\mathbb{R}^n)$ the order relation $\leq_m$ defined by:

$$U, V \in P_{cp,cv}(\mathbb{R}^n) : U \leq_m V \Longleftrightarrow U \subseteq V.$$

Definition 3. *The operator $F(t,\cdot) : J \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$, is called increasing if*

$$A, B \in P_{cp,cv}(\mathbb{R}^n),\ A \leq_m B \Rightarrow F(t,A) \leq_m F(t,B),\ \textit{for all } t \in J.$$

Define on $C(J, P_{cp,cv}(\mathbb{R}^n))$ an order relation " $\leq$ " defined by:

$$X, Y \in C(J, P_{cp,cv}(\mathbb{R}^n)),\ X \leq Y \Leftrightarrow X(t) \leq_m Y(t),\ \text{for all } t \in J.$$

The space $(C(J, P_{cp,cv}(\mathbb{R}^n)), H_*^B, \leq)$ being an ordered and complete metric space is also an L-space (see[3]).

Let $(X, d, \leq)$ be a ordered metric space and $T : X \to X$ an operator.

We note: $F_T := \{x \in X | Tx = x\}$the fixed point set of T;

$$(UF)_T := \{x \in X | Tx \leq x\} \text{ the upper fixed point set for T;}$$

$$(LF)_T := \{x \in X | Tx \geq x\} \text{ the lower fixed point set for T.}$$

Definition 4. *([4]) Let X be an L-space. Then, the operator $T : X \to X$ is a Picard operator (PO) if*

(i) $F_T = \{x_T^*\}$;

(ii) $T^n x \to x_T^*$ as $n \to \infty$, for all $x \in X$.

Definition 5. *([4]) Let X be an L-space. Then, the operator $T : X \to X$ is a weakly Picard operator (WPO) if the sequence $(T^n x)_{n\in\mathbb{N}}$ converges for all $x \in X$ and the limit (witch may depend on x) is a fixed point of T.*

3. **Main results**

Theorem 6. *We consider the problem (1) and $F : J \times P_{cp,cv}(\mathbb{R}^n) \longrightarrow P_{cp,cv}(\mathbb{R}^n)$ be an operator.*

Suppose that:

i) F is continuous on $J \times P_{cp,cv}(\mathbb{R}^n)$ and $U^0 \in P_{cp,cv}(\mathbb{R}^n)$;

ii) $F(t,\cdot)$ is Lipschitz, i.e. there exists $L \geq 0$

such that $H(F(t,U), F(t,V)) \leq LH(U,V)$ for all $U, V \in P_{cp,cv}(\mathbb{R}^n)$ and $t \in J$.

Then the problem (1) *has a unique solution* U^* *and* $U^*(t) = \lim_{n\to\infty} U_n(t)$, *where* $U_n \in C(J, P_{cp,cv}(\mathbb{R}^n))$ *is recurrently defined by the relation:*

$$\begin{cases} U_{n+1}(t) = U^0 + \int_{t_0}^t F(s, U_n(s))ds, \ n \in \mathbb{N} \\ \hfill U_0 \in P_{cp,cv}(\mathbb{R}^n). \end{cases}$$

Proof. Consider the operator: $\Gamma : C(J, P_{cp,cv}(\mathbb{R}^n)) \to C(J, P_{cp,cv}(\mathbb{R}^n))$ where

$$\Gamma U(t) = U^0 + \int_{t_0}^t F(s, U(s))ds, \ t \in J.$$

We will verify the contraction condition for Γ.

$$H(\Gamma(U)(t), \Gamma(V)(t)) = H(U^0 + \int_{t_0}^t F(s, U(s))ds, U^0$$

$$+ \int_{t_0}^t F(s, V(s))ds) \leq H(U^0, U^0) + H(\int_{t_0}^t F(s, U(s))ds, \int_{t_0}^t F(s, V(s))ds)$$

$$\leq \int_{t_0}^t H(F(s, U(s)), F(s, V(s)))ds \leq \int_{t_0}^t LH(U(s), V(s))ds =$$

$$= L \int_{t_0}^t H(U(s), V(s))e^{-\tau(s-t_0)}e^{\tau(s-t_0)}ds \leq LH_*^B(U, V) \int_{t_0}^t e^{\tau(s-t_0)}ds =$$

$$= \frac{L}{\tau} H_*^B(U, V)(e^{\tau(t-t_0)} - 1) \leq \frac{L}{\tau} H_*^B(U, V)e^{\tau(t-t_0)},$$

then we have:

$$H(\Gamma(U)(t), \Gamma(V)(t))e^{-\tau(t-t_0)} \leq \frac{L}{\tau} H_*^B(U, V), \text{ for all } t \in J.$$

Taking the maximum for $t \in J$, then we have:

$$H_*^B(\Gamma(U), \Gamma(V)) \leq \frac{L}{\tau} H_*^B(U, V), \text{ for all } U, V \in C(J, P_{cp,cv}(\mathbb{R}^n)), \ \tau > 0.$$

Thus, the integral operator Γ is Lipschitz with constant $L_\Gamma = \frac{L}{\tau}$, $\tau > 0$.

Choosing τ such as $\frac{L}{\tau} < 1$, then Γ is an contraction and by contraction principle the operator Γ has unique fixed point U^*. According to Lemma 2 then U^* is the unique solution for the Cauchy problem. □

In what follows we will present the Abstract Gronwall Lemma:

Lemma 7. *(Abstract Gronwall Lemma [3]) Let $(X, d, \leq)$ be an ordered L-space and $T : X \to X$ an operator. We suppose that:*

(i) T is PO;

(ii) T is increasing.

Then $(LF)_T \leq x_T^* \leq (UF)_T$, where x_T^* is the unique fixed point of the operator T.

We will apply this abstract lemma to the Cauchy problem (1).

Theorem 8. *Let the Cauchy problem*

$$(1) \quad \begin{cases} D_H U = F(t, U), \ t \in J \\ \hfill U(t_0) = U^0 \end{cases}$$

where $U^0 \in P_{cp,cv}(\mathbb{R}^n), \ t_0 \geq 0, \ J = [t_0, t_0 + a], \ a > 0.$

Suppose that $F(t, \cdot)$ is an L-Lipschitz increasing monotone operator for all $t \in J$. Then we have:

$$(LS)_{(1)} \leq U^* \leq (US)_{(1)}$$

where U^* is the unique solution for problem (1) and $(LS)_{(1)}$ respectively $(US)_{(1)}$ represents the set of lower solution respectively the set of upper solution for the problem (1).

Proof. Let $\Gamma : C(J, P_{cp,cv}(\mathbb{R}^n)) \longrightarrow C(J, P_{cp,cv}(\mathbb{R}^n))$

$$\Gamma U(t) := U^0 + \int_{t_0}^{t} F(s, U(s))ds, \ t \in J.$$

Then we have:

i) By Theorem 6 we have that Γ as a contraction. We denote by $U^* \in C(J, P_{cp,cv}(\mathbb{R}^n))$ the unique fixed point. According to Lemma 2 we have that U^* is the unique solution for the Cauchy problem.

ii) We proved that Γ is increasing. Let $U, V \in C(J, P_{cp,cv}(\mathbb{R}^n))$ with $U \leq V \Rightarrow U(t) \leq_m V(t), \ \text{for all } t \in J.$

Since $F(t, \cdot)$ is monotone we have $F(t, U(t)) \leq F(t, V(t)), \ \text{for all } t \in J.$

Then

$$U^0 + \int_{t_0}^{t} F(s, U(s)) \leq U^0 + \int_{t_0}^{t} F(s, V(s))ds, \quad \text{for all } t \in J$$

$$\Rightarrow \Gamma U(t) \leq \Gamma V(t), \quad \text{for all } t \in J \Rightarrow \ \Gamma U \leq \Gamma V.$$

So Γ is monotonously increasing and Picard. Be applying Lemma 7 we have:

$$(LF)_\Gamma \leq U^* \leq (UF)_\Gamma.$$

Consequently $(LF)_\Gamma,\ (UF)_\Gamma$ coincide to the set of the lower and upper solutions for problem (1). □

In what follows an abstract comparison lemma will be presented.

Lemma 9. *(Abstract Gronwall- comparison lemma* [3]*) Let $(X, d, \leq)$ be an ordered L-space and $T, \Gamma : X \to X$ two operators. We suppose that:*

(i) T and Γ are POs;

(ii) T is increasing;

(iii) $T \leq \Gamma$.

Then $x \leq Tx \Rightarrow x \leq x^*_\Gamma$.

We have the folowing theorem.

Theorem 10. *Let as consider the following two Cauchy problems:*

$$(1) \quad \begin{cases} D_H U = F(t, U),\ t \in J \\ \hfill U(t_0) = U^0 \end{cases}$$

$$(2) \quad \begin{cases} D_H V = G(t, V),\ t \in J \\ \hfill V(t_0) = V^0 \end{cases}$$

where $U^0, V^0 \in P_{cp,cv}(\mathbb{R}^n),\ t_0 \geq 0,\ J = [t_0, t_0 + a],\ a > 0$.

Suppose that:

i) F is continuous on $J \times P_{cp,cv}(\mathbb{R}^n)$ and $F(t, \cdot)$ is Lipschitz;

ii) G is continuous on $J \times P_{cp,cv}(\mathbb{R}^n)$, $V^0 \in P_{cp,cv}(\mathbb{R}^n)$ and $G(t, \cdot)$ is Lipschitz;

iii) $F(t, \cdot)$ is increasing for all $t \in J$.

Then $U \leq \Gamma U \Longrightarrow U \leq V^*$ where V^* is the unique solution for the problem (2).

Proof. Since $F(t, \cdot)$ is Lipschitz, there exists $L \geq 0$ such that

$$H(F(t, U), F(t, V)) \leq LH(U, V), \text{ for all } U, V \in P_{cp,cv}(\mathbb{R}^n),\ t \in J.$$

Since $G(t,\cdot)$ is Lipschitz, there exists $L_G \geq 0$ such that

$$H(G(t,U),G(t,V)) \leq L_G H(U,V), \text{ for all } U,V \in P_{cp,cv}(\mathbb{R}^n),\ t \in J.$$

By Theorem 8, Γ and T satisfy the contraction principle and we have that Γ and T are Picard operators.

By iii) we have $F(t,U) \subset G(t,U),$ for all $U \in P_{cp,cv}(\mathbb{R}^n),\ t \in J,$

$$\text{then } U^0 + \int_{t_0}^{t} F(s,U(s))ds \leq V^0 + \int_{t_0}^{t} G(s,U(s))ds,$$

$$\text{thus } \Gamma U(t) \subseteq TU(t) \Longrightarrow \Gamma U \leq TU \Longrightarrow \Gamma \leq T.$$

By Lemma 9 the proof is complete. □

We recall the following abstract Gronwall lemma for the case of WPO.

Lemma 11. *(Abstract Gronwall lemma* [3]*) Let* $(X,d,\leq)$ *be an ordered L-space and* $T : X \to X$ *an operator. We suppose that*

(i) T *is WPO;*

(ii) T *is increasing.*

Then

a) $x \leq Tx \Rightarrow x \leq T^{\infty}x$*;*

b) $x \geq Tx \Rightarrow x \geq T^{\infty}x$*.*

The basic result in the WPOs theory is the following:

Theorem 12. *(Characterization theorem* [4]*) Let* (X,d) *be an L-space and* $f : X \to X$ *be an operator. The operator* f *is WPO if and only if there exists a partition of* X, $X = \bigcup_{\gamma\in\Gamma} X_\gamma$ *such that:*

(a) $X_\gamma \in I(A),$ *for all* $\gamma \in \Gamma$;

(b) f *to* X_γ, $f_{|X_\gamma} : X_\gamma \to X_\gamma$ *is PO for all* $\gamma \in \Gamma$.

We will apply the above lemma to the Cauchy problem (1).

Theorem 13. *Let us consider the Cauchy (1)*

We suppose that:

i) $F(t,.)$ *is Lipschitz, for all* $t \in J$;

ii) $F(t,.)$ *is increasing, for all* $t \in J$;

iii) F is continuous on $J \times P_{cp,cv}(\mathbb{R}^n)$ and $U^0 \in P_{cp,cv}(\mathbb{R}^n)$.

Then

i) U is a lower solution of the problem (1) $\Rightarrow U \leq U_U^*$;

ii) U is a upper solution of the problem (1) $\Rightarrow U \geq U_U^*$

where U_U^* is a solution for the problem (1) and $U_U^*(t) = \lim\limits_{n\to\infty} U_n(t)$ where $U_n \in C(J, P_{cp,cv}(\mathbb{R}^n))$ is recurrently defined by the relation:

$$\begin{cases} U_{n+1}(t) = U_n(t_0) + \int_{t_0}^t F(s, U_n(s))ds, \ n \in \mathbb{N} \\ \hfill U^0 = U. \end{cases}$$

Proof. Let $T : C(J, P_{cp,cv}(\mathbb{R}^n)) \to C(J, P_{cp,cv}(\mathbb{R}^n))$ defined by

$$TU(t) = U(t_0) + \int_{t_0}^t F(s, U(s))ds, \quad \text{for all } t \in J.$$

According to Lemma 2 we have (1) $\Leftrightarrow$ (2) $\Leftrightarrow U = TU$. Thus $S_{(1)} = FixT$.

Let $Z = C(J, P_{cp,cv}(\mathbb{R}^n))$ and $Z_\gamma = \{U \in C(J, P_{cp,cv}(\mathbb{R}^n)) | U(t_0) = \gamma\}, \ \gamma \in \mathbb{R}$. Then $Z = \bigcup\limits_{\gamma\in\mathbb{R}} Z_\gamma$ is a partition of $C(J, P_{cp,cv}(\mathbb{R}^n))$. Moreover $Z_\gamma \in I(T)$ and Z is a closed subset of $C(J, P_{cp,cv}(\mathbb{R}^n))$ for all $\gamma \in \mathbb{R}$.

Since $F(t,.) : Z \to Z$ is a L-Lipschitz for all $t \in J$. By Theorem 6 the operator $T_{|Z_\gamma}$ is Picard for all $\gamma \in \mathbb{R}$. Hence T is WPO (by the characterization Theorem 12).

In the above conditions the Cauchy problem (1) is equivalent with the fixed point equation, $TU = U$, where the operator T is WPO.

Since $F(t,\cdot)$ is monotone we have $F(t, U(t)) \leq F(t, V(t)), \text{ for all } t \in J$.

Then

$$U^0 + \int_{t_0}^t F(s, U(s)) \leq U^0 + \int_{t_0}^t F(s, V(s))ds, \quad \text{for all } t \in J$$

$$\Rightarrow TU(t) \leq TV(t), \quad \text{for all } t \in J \Rightarrow \ TU \leq TV.$$

Thus T is monotonously increasing and WPO. By applying Lemma 11 the proof is complete. $\square$

References

[1] Lakshmikantham, V., Gnana Bhaskar, T., Devi J. Vasundhara, *Theory of Set Differential Equations in Metric Spaces*, Cambridge Scientific Publishers, 2006.

[2] Petruşel, A., Petruşel, G., Moţ, G., *Topics in Nonlinear Analysis and Applications to Mathematical Economics*, House of the Book Science, Cluj-Napoca, 2007.

[3] Rus, I.A., *Fixed points, upper and lower fixed points: abstract Gronwall lemmas*, Carpathian J. Math., **20**(2004), No.1, 125-134.

[4] Rus, I.A., *Picard operators and applications*, Scientiae Mathematicae Japonicae, **58**(2003), No.1, 191-219.

[5] Rus, I.A., *Generalized Contractions and Applications*, Cluj Univ. Press, Cluj-Napoca, 2001.

Department of Applied Mathematics
Babeş-Bolyai University
Cluj-Napoca, Romania
E-mail address: ti_camelia@yahoo.com

BOOK REVIEWS

Advanced Courses in Mathematical Analysis, III, Proceedings of the Third International School, JUan M. Delgado Sánchez and Tomás Domínguez Benavides (Editors), World Scientific Publishers, London - Singapore 2008, xiv+194 pp, ISBN:13 978-981-281-844-7 and 10-981-281-844-8.

Several Andalusian universities decided to organize an International Course on Mathematical Analysis in Andalusia, a task that was achieved by the friendly cooperation of several research groups in analysis and by the support of Spanish National Government, of several universities and from several private companies as well. The first course took place in Cádiz (2002), followed by the second in Granada (2004). The success these courses had determined the organization of the third one in La Rábida (Huelva) from 3 to 7 September, 2007. The aim of these courses is to provide an extensive overview, by leading experts, of the research in various areas of analysis -real analysis, complex analysis, functional analysis.

The present course was attended by more that 70 participants from various countries, who had the opportunity to here eleven plenary lectures and to participate to three seminars, delivered by invited distinguished mathematicians from all over the world. The volume contains the written (and usually expanded) versions of the talks and covers a lot of topics in various areas of mathematical analysis, applications to economy, or history of mathematics (a nice paper by Beckenstein and Narici on the life of E. Helly and the Hahn-Banach theorem). The eleven survey papers included in this volume deal with topics as: Dynamics in one complex variable (M. Abate, Univ. di Pisa), Bilinear Hilbert transform and multipliers (O. Blasco, Univ. de Valencia), Functions whose translations generate $L^1(\mathbb{R})$ (J. Bruna, Univ. Autònoma de Barcelona), Compactness and distances to function spaces (C. Angosto and B. Cascales, Univ. de Murcia), Spaces of smooth functions (E. Harboure, Wayne State Univ., Detroit), Domination by positive operators and singularity (F. L. Hernández, Univ. Complut. de Madrid), The Hahn-Banach theorem and the sad life of E. Helly (L. Narici and E. Beckenstein, St. John's Univ., NY), Small subspaces of the space L_p (E. Odell, Univ. of Texas, Austin), Hypercyclic operators (H. N. Salas, Univ. de Puerto Rico), Operator spaces (B. M. Schreiber, Wayne State Univ., Detroit), Mathematics and markets - competitive equilibrium (A. Villar, Univ. Pablo de Olavide, Sevilla), Ideals in F-algebras (W. Želazko, Mathematical Institute, Warszawa).

By the survey papers on topics of current interest, written by mathematicians with substantial contributions to the subject, this collection of papers will be very useful to graduate students (post-graduate as well) desiring to learn about topics of high research interest from leading experts.

S. Cobzaş

Ben Klemens, ***Modeling with Data. Tools and Techniques for Scientific Computing,*** Princeton University Press, Princeton and Oxford, 2009, XIV+454 pp; ISBN: 978-0-691-13414-0.

This book presents an original, cheap and powerful solution to the problem of analysis of large data sets. The solution combines C language, data base query and management, statistics and data visualization. The book intends to be an alternative to classical statistical books: it does not separate descriptive and inferential statistics, simple models are combined into more complex model in a hierarchical way and it is computer oriented.

All software tools used by author are *free* and reliable: GNU C Compiler, SQLite and MySQL, GNU Scientific Library (GSL), Gnuplot and Graphviz. Each of them may be considered ugly for nonprofessional users. The way the author uses them and the accessibility of presentation endows the user with a set of open and unlimited tools to solve the difficult tasks of statistical data analysis.

The first part part of the book (six chapters) is devoted to computing. Chapter 2 introduces the basics of C programming language. The next chapter is on data bases and SQL query language, since working with large data sets is now more necessary than ever. Chapter 4 presents matrices and vectors from GSL ad Apophenia library, built upon the GSL. Computer graphics is the topic of Chapter 5. Gnuplot interpreter assures a simple way to plot and portability. The last chapter of the first part emphasizes the features of C language already presented and introduces data structures like linked lists and binary trees.

The second part, Statistics, does not deals with very advanced concepts, but their combination into a creative manner allow the modeling and handling of situations of arbitrary complexity. Chapter 7 treats numerical characteristics of samples and classical probability distributions. Principal component analysis, ordinary least squares and related methods, and multilevel modeling are the topics of Chapter 8. The next chapter is devoted to Central Limit Theorem and hypothesis testing. Chapter 10 introduce Maximum Likelihood Estimation and related statistical inferential procedures. The last Chapter, 11, is devoted to Monte Carlo techniques, and related subject like random number generation, bootstrapping and resampling.

Three appendices increase the readability of the book.

The vision of the author is to present the things as a pipeline going from raw data to a final publishable output; the pipe sections and filters assure different level of abstractions which reach to a full program.

This book includes more than 80 working programs; they allow the readers to explore the data, find out to what changes the procedure is robust and freely modify the code.

The programs, and moreover, the ability to combine the tools into a fully-functional pipeline are intended to be a natural alternative to sophisticated and expensive statistical softwares and packages.

The book is devoted mainly to the practitioner of Statistics, but is also useful to mathematicians, computer scientists, researchers and students in the biology, economics and social sciences.

Radu Trîmbiţaş

Walter Roth, ***Operator-Valued Measures and Integrals of Cone-Valued Functions***, Lectures Notes in Mathematics, Vol. 1964, Springer-Verlag, Berlin - Heidelberg, 2009, x+356 pp; ISBN: 978-3-540-87564-1, e-ISBN: 978-3-540-87565-1.

The theory of locally convex cones was developed by the author and K. Keimel in a previous book, *Ordered Cones and Approximation*, Lecture Notes in Mathematics, Vol. 1517, Springer-Verlag, Berlin 1992, as a general framework for Korovkin type linear approximation theory. The aim of the present book is to use the theory of locally convex cones for developing a very general and unified theory of integration for extended real-valued, vector-valued, operator-valued and cone-valued countably additive measures and functions.

A full locally convex cone is a cone $\mathbf{P}$ endowed with an order $\leq$ compatible with the algebraic operations and a downward directed subset $\mathbf{V}$ of positive elements, closed for addition and multiplication by strictly positive scalars, called an abstract neighborhood system. An element $v \in \mathbf{V}$ determines upper and a lower neighborhoods of any element $a \in \mathbf{P}$ given by $v(a) = \{b \in \mathbf{P} : b \leq a + v\}$ and $(a)v = \{b \in \mathbf{P} : a \leq b + v\}$, respectively, and a symmetric neighborhood $v^s(a) = v(a) \cap (a)v$. A locally convex cone is a subcone $(\mathbf{Q}, \mathbf{V})$ of a full locally convex cone, not necessarily containing the abstract neighborhood system $\mathbf{V}$, equipped with the induced topologies (upper, lower and symmetric). The cancellation law $a + c = b + c \Rightarrow a = b$ need not hold in the cone $\mathbf{P}$. If it holds then $\mathbf{P}$ can be embedded in a vector space and, conversely, every cone in a vector space is cancellative.

Based on powerful Hahn-Banach type extension and separation theorems for additive and positively homogeneous functionals with respect to sublinear or superlinear functionals, one can develop a theory of locally convex cones similar to that of locally convex spaces: duality theory, weak topologies and a Mackey-Arens type theorem, Uniform Boundedness Principle and Open Mapping Theorem. This theory is exposed in the first chapter of the book, *Locally convex spaces*, which partly has a survey character, the proofs of some theorem being referred to the above mentioned book of the author and K. Keimel.

The integration theory for cone-valued measures and functions is developed in the second chapter of the book, *Measures and integrals. The general theory.* The framework is that of two locally convex cones $\mathbf{P}$ and $\mathbf{Q}$ (the latter being supposed to be a complete lattice cone), the cone $\mathcal{L}(\mathbf{P}, \mathbf{Q})$ of additive and positively homogeneous operators from $\mathbf{P}$ to $\mathbf{Q}$, and a $\mathcal{L}(\mathbf{P}, \mathbf{Q})$-valued measure defined on a σ-field (or a σ-ring) on a set X. The lattice completeness of $\mathbf{Q}$ allows to define integrals of measurable $\mathbf{P}$-valued functions as suprema of the integrals of measurable step $\mathbf{P}$-valued functions.

The central result of the last chapter of the book, III, *Measures on locally compact spaces,* is a very general Riesz type integral representation theorem for continuous linear operators from function cones over a locally compact space X into a locally convex complete lattice cone $\mathbf{Q}$, which contains as particular cases a lot of known integral representations for compact and weakly compact operators on Banach space-valued functions, as well as some new general cases. As a very special case, one obtains also the classical spectral representation theorem for normal linear operators on a complex Hilbert space.

As the author points out, a demanding topic, of great interest, but not included in the book, is that of a Choquet-type representation theory within the general framework of locally convex cones, which could be a subject for further investigation.

Providing a very general and nontrivial approach to integration theory, the book is of interest for researchers in functional analysis, abstract integration theory and its applications to integral representations of linear operators. It can be used also for advanced post-graduate courses in functional analysis.

S. Cobzaş