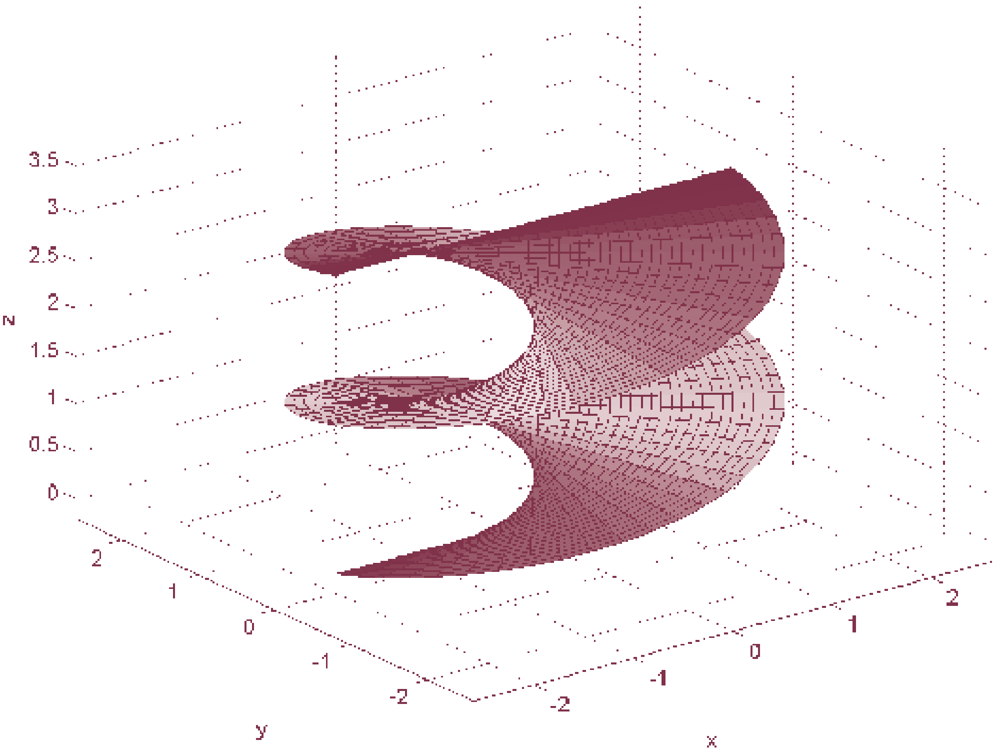
STUDIA UNIVERSITATIS
BABEŞ-BOLYAI



MATHEMATICA

2/2009

Anul LIV 2009

S T U D I A

UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA

2

Redacția: 400084 Cluj-Napoca, Str. M. Kogălniceanu nr. 1 Tel: 405300

SUMAR CONTENTS SOMMAIRE

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 2, June 2009

THE GLOBAL BEHAVIOR OF THE DIFFERENCE EQUATION

F. BOZKURT, I. OZTURK, AND S. OZEN

Abstract. In this paper, we investigate the global stability and the periodic nature of the positive solutions of the difference equation

$$y_{n+1} = \alpha + \frac{y_{n-1}}{y_n} - \frac{y_n}{y_{n-1}}, n = 0, 1, 2, ...,$$

where $\alpha > 0$ and the initial conditions y_0, y_{-1} are arbitrary positive real numbers.

1. **Introduction**

Consider the difference equation

$$y_{n+1} = \alpha + \frac{y_{n-1}}{y_n} - \frac{y_n}{y_{n-1}}, \ n = 0, 1, 2, ... \tag{1.1}$$

where $\alpha > 0$ and the initial conditions y_0, y_{-1} are arbitrary positive real numbers.We investigate the asymptotic stability and the periodic character of the solutions of Eq. (1.1).

We prove that the positive equilibrium point of Eq. (1.1) is local asymptotic stable or a saddle point under specified conditions of the parameter and show that the solution of the subtraction of two difference equations in [1] and [3], which solutions are globally asymptotically stable, are also asymptotically stable.

The global asymptotic stability, the boundedness character and the periodic nature of the positive solutions of the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, \ n = 0, 1, 2, ... \tag{1.2}$$

Received by the editors: 30.10.2006.

2000 *Mathematics Subject Classification.* 39A10, 39A11.

Key words and phrases. Difference equations, period two solutions, global stability.

was investigated in [1], where $\alpha \in [0,\infty)$ and the initial conditions x_{-1} and x_0 are arbitrary positive real numbers. H. M. El- Owaidy et al. [2] studied the global stability and the periodic character of positive solutions of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}}{x_n},\ n = 0, 1, 2, ... \tag{1.3}$$

where $\alpha \in [1,\infty)$, $k \in \{1, 2, ...\}$ and the initial conditions $x_{-k}, ..., x_0, x_{-1}$ are arbitrary positive real numbers.

R. M. Abu-Saris and R. De Vault find conditions for the global asymptotic stability of the unique positive equilibrium

$$\bar{y} = A + 1$$

of the equation

$$x_{n+1} = A + \frac{y_n}{y_{n-k}},\ n = 0, 1, 2, ... \tag{1.4}$$

where A,$y_{-k}, ..., y_0, y_{-1} \in (0,\infty)$ and $k \in \{2, 3, ...\}$ [3].

Here, we recall some definitions and results which will be useful in the sequel.

Let $I \subset R$ and let $f : I \times I \to I$ be a continuous function. Consider the difference equation

$$y_{n+1} = f(y_n, y_{n-1}), n = 0, 1, 2, ... \tag{1.5}$$

where the initial conditions $y_0, y_{-1} \in I$. We say that $\bar{y}$ is an equilibrium of Eq. (1.5) if

$$y_{n+1} = f(\bar{y}, \bar{y}),\ n = 0, 1, 2, ... \tag{1.6}$$

Let

$$s = \frac{\partial f}{\partial u}(\bar{y}, \bar{y})\ \ and\ \ t = \frac{\partial f}{\partial v}(\bar{y}, \bar{y})$$

denote the partial derivatives of $f(u, v)$ evaluated at an equilibrium $\bar{y}$ of Eq. (1.5). Then the equation

$$x_{n+1} = sx_n + tx_{n-1}$$

is called the linearized equation associated with Eq. (1.5) about the equilibrium point ȳ [4].

The sequence $\{y_n\}$ is said to be periodic with period p if

$$y_{n+p} = y_n$$

for $n = 0, 1, ...$ [5].

Theorem 1.1. [4] **(Linearized Stability)**

$$x_{n+1} = sx_n + tx_{n-1} \tag{1.7}$$

is the linearized equation associated with the difference equation

$$y_{n+1} = f(y_n, y_{n-1}), n = 0, 1, 2, ... \tag{1.8}$$

about the equilibrium point $\bar{y}$. The characteristic equation associated with (1.7) is

$$\lambda^2 - s\lambda - t = 0. \tag{1.9}$$

(i) If both roots of the quadratic equation (1.9) lie in the unit disk $|\lambda| < 1$, then the equilibrium $\bar{y}$ of Eq. (1.8) is locally asymptotically stable.

(ii) If at least one of the roots of Eq. (1.9) has absolute value greater than one, then the equilibrium of Eq. (1.8) is unstable.

(iii) A necessary and sufficient condition for both roots of Eq. (1.9) to lie in the open unit disk $|\lambda| < 1$, is

$$|s| < 1 - t < 2.$$

In this case the locally asymptotically stable equilibrium point $\bar{y}$ is also called a sink.

(iv) A necessary and sufficient condition for both roots of Eq. (1.9) to have absolute value greater than one is

$$|t| > 1 \textit{ and } |s| < |1 - t|\,.$$

In this case $\bar{y}$ is called a repeller.

(v) A necessary and sufficient condition for one root of Eq. (1.9) to have absolute value greater than one and for the other to have absolute value less than one is

$$s^2 + 4t > 0 \textit{ and } |s| > |1 - t|\,.$$

In this case the unstable equilibrium point is called a saddle point.

Theorem 1.2 [6] Assume that p,q$\in R$ and $k \in \{0,1,...\}$. Then

$$|p| + |q| \ < 1$$

is a sufficient condition for asymptotic stability of the difference equation

$$x_{n+1} - px_n + qx_{n-k} = 0. \tag{1.10}$$

Suppose in addition that one of the following two cases holds:

(i) k odd and $q < 0$

(ii) k even and $pq < 0$.

Then (1.10) is also a necessary condition for asymptotic stability of Eq. (1.10).

Theorem 1.3. [7] Consider the difference equation

$$y_{n+1} = f(y_n, y_{n-k}), n = 0, 1, 2, ... \tag{1.12}$$

where $k \in \{1, 2, ...\}$. Let I=[a,b] be some interval of real numbers, and assume that f:[a,b]x[a,b]$\rightarrow [a,b]$ is a continuous function satisfying the following properties:

(i) f(u,v) is non-increasing in each arguments.

(ii) If (m,M)$\in [a,b]x[a,b]$ is a solution of the system

$$M = f(m,m), m = f(M,M) \tag{1.13}$$

then m=M. From this, Eq. (1.12) has a unique positive equilibrium point and every solution of Eq. (1.12) converges to $\bar{y}$.

2. Linearized stability and period two solutions

In this section, we consider Eq. (1.1) and show that unique positive equilibrium point $\bar{y}$ =α of Eq. (1.1) is asymptotically stable with basin which depends on certain conditions posed on the coefficient.

The linearized equation associated with Eq. (1.1) about the equilibrium $\bar{y}$ is

$$x_{n+1} + \frac{2}{\alpha}x_n - \frac{2}{\alpha}x_{n-1} = 0. \tag{2.1}$$

Its characteristic equation is

$$\lambda^2 + \frac{2}{\alpha}\lambda - \frac{2}{\alpha} = 0. \tag{2.2}$$

By Theorem 1.1. and Theorem 1.2. we have the following results.

Theorem 2.1. **(i)** The equilibrium point $\bar{y}$ of Eq. (1.1) is locally asymptotically stable iff $\alpha > 4$.

(ii) The equilibrium point $\bar{y}$ of Eq. (1.1) is unstable (and in fact is a saddle point) if $0 < \alpha < 4$.

Proof. (i) The inequality (1.10) can be written as

$$\left|\frac{2}{\alpha}\right| + \left|\frac{-2}{\alpha}\right| < 1 \ . \tag{2.3}$$

This inequality holds if $\alpha > 4$. By using Theorem 1.2., we can also see that q$=\frac{-2}{\alpha} < 0$. These results give us necessary and sufficient conditions for the asymptotic stability of Eq. (2.1) .

(ii) From Theorem 1.1./(v) we have,

$$\left(\frac{-2}{\alpha}\right)^2 + 4\left(\frac{2}{\alpha}\right) > 0 \ \ and \ \ \left|\frac{-2}{\alpha}\right| > \left|1 - \frac{2}{\alpha}\right|.$$

Easy computations give

$$\left(\frac{-2}{\alpha}\right)^2 + 4\left(\frac{2}{\alpha}\right) = \frac{4}{\alpha^2} + \frac{8}{\alpha} > 0$$

and

$$\left|\frac{-2}{\alpha}\right| > \left|1 - \frac{2}{\alpha}\right|.$$

Then we have the inequality

$$2 > |\alpha - 2| \ .$$

This implies that by Theorem 1.1./(v), the equilibrium point is unstable (and is a saddle point).

Theorem 2.2. Suppose that $\{y_n\}_{n=-1}^{\infty} \neq 2$ is a solution of Eq. (1.1). The following statements are true.

(i) If $0 < \alpha \leq 4$, then Eq. (1.1) has no real period two solutions. Suppose k is odd.

(ii) If $\alpha > 4$, then Eq. (1.1) has real period two solutions.

Proof. Let

$$..., \phi, \psi, \phi, \psi, ...$$

be a period-2 solution of Eq. (1.1). Then,

$$\phi = \alpha + \frac{\phi}{\psi} - \frac{\psi}{\phi} \tag{2.4}$$

$$\psi = \alpha + \frac{\psi}{\phi} - \frac{\phi}{\psi}. \tag{2.5}$$

Subtracting above two statements, we get

$$\psi = \frac{2\phi}{\phi - 2}. \tag{2.6}$$

From (2.6), we have

$$\phi^2 - 2\alpha\phi + 4\alpha = 0. \tag{2.7}$$

We consider (2.7) under two cases, where Δ indicates the discriminant of (2.7).

(i) Let $\Delta = 0$. Under this condition we have $\alpha = 0$ and $\alpha = 4$.If Eq. (1.1) has period 2 solutions then it must be $\Delta \neq 0$. This implies that if $\alpha \in (0, 4]$, then Eq. (1.1) has no period 2 solutions.

(ii) Let $\Delta > 0$. In this case we have $\alpha > 4$. While $\alpha > 4$, Eq. (1.1) has period 2 solutions. These solutions are

$$\phi_1 = \alpha + \sqrt{\alpha\left(\alpha - 4\right)} \text{ and } \phi_2 = \alpha - \sqrt{\alpha\left(\alpha - 4\right)}$$

and they must be of the form

$$..., \alpha - \sqrt{\alpha\left(\alpha - 4\right)}, \alpha + \sqrt{\alpha\left(\alpha - 4\right)}, ...$$

Theorem 2.3. Suppose $\alpha > 4$. Let be $\{y_n\}_{n=-1}^{\infty} \neq 2$ be a solution of Eq. (1.1). If $\{y_n\}_{n=-1}^{\infty} \neq 2$ is periodic with period-2, then y_0 is

$$y_0 = \frac{-\left(y_{-1} - \alpha\right) y_{-1} \pm y_{-1}\sqrt{\left(y_{-1} - \alpha\right)^2 + 4}}{2}. \tag{2.8}$$

Proof. If the solution of Eq. (1.1) is periodic with period- 2, we can write Eq.(1.1) as

$$y_{-1} = \alpha + \frac{y_{-1}}{y_0} - \frac{y_0}{y_{-1}}.$$

Computations give

$$y_0^2 + y_{-1}y_0\left(y_{-1} - \alpha\right) - y_{-1}^2 = 0,$$

and we have $\Delta = y_{-1}^2\left[\left(y_{-1} - \alpha\right)^2 + 4\right] > 0$. So, we obtain

$$y_0 = \frac{-(y_{-1}-\alpha)y_{-1} \pm y_{-1}\sqrt{(y_{-1}-\alpha)^2+4}}{2}.$$

Theorem 2.4. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of Eq. (1.1). Then the following statements are true.

1. Let $\alpha = 2\sqrt{L-1}$ and $L > 1$.

(i) If $\lim\limits_{n\to\infty} y_{2n} = L$, then $\lim\limits_{n\to\infty} y_{2n+1} = \frac{L}{\sqrt{L-1}}$.

(ii) If $\lim\limits_{n\to\infty} y_{2n+1} = L$, then $\lim\limits_{n\to\infty} y_{2n} = \frac{L}{\sqrt{L-1}}$.

2. Let $\alpha > 2\sqrt{L-1}$ and $L > 1$.

(i) If $\lim\limits_{n\to\infty} y_{2n} = L$, then $\lim\limits_{n\to\infty} y_{2n+1} = \frac{L\left[\alpha \pm \sqrt{\alpha^2 - 4(L-1)}\right]}{2(L-1)}$.

(ii) If $\lim\limits_{n\to\infty} y_{2n+1} = L$, then $\lim\limits_{n\to\infty} y_{2n} = \frac{L\left[\alpha \pm \sqrt{\alpha^2 - 4(L-1)}\right]}{2(L-1)}$.

Proof. **1.** **(i)** Let $\lim\limits_{n\to\infty} y_{2n} = L$ and $\lim\limits_{n\to\infty} y_{2n+1} = x$. By Eq (1.1) we have

$$x = \alpha + \frac{x}{L} - \frac{L}{x}$$

and so we get

$$\left(\frac{L-1}{L}\right)x^2 - \alpha x + L = 0. \tag{2.9}$$

Since $\Delta = \alpha^2 - 4\left(L-1\right)$, the discriminant is $\Delta = 0$. So, (2.9) has only one root, and that is

$$x = \lim_{n\to\infty} y_{2n+1} = \frac{L}{\sqrt{L-1}}.$$

(ii) The proof is similar and will be omitted.

2. **(i)** Let $\lim\limits_{n\to\infty} y_{2n} = L$ and $\lim\limits_{n\to\infty} y_{2n+1} = x$. While $\alpha > 2\sqrt{L-1}$, then from (2.9) we have $\Delta > 0$. So,

$$x = \lim_{n\to\infty} y_{2n+1} = \frac{L\left[\alpha \pm \sqrt{\alpha^2 - 4\left(L-1\right)}\right]}{2\left(L-1\right)}.$$

(ii) The proof follows in the same way.

3. **Analysis of the semi-cycles of eq.(1.1)**

In this section, we give some results about the semi-cycles of Eq. (1.1).

Let $\{y_n\}_{n=-1}^{\infty}$ be a positive solution of Eq. (1.1). A positive semi-cycle of $\{y_n\}_{n=-1}^{\infty}$ consists of a "string" of terms $\{y_p, y_{p+1}, ..., y_m\}$, all greater than or equal to $\bar{y}$, with p≥ -1 and m$\leq \infty$ and such that either p=-1 or $p > -1$ and $y_{p-1} < \bar{y}$ and either m=∞ or m<∞ and $y_{m+1} < \bar{y}$.

A negative semi-cycle of consists of $\{y_n\}_{n=-1}^{\infty}$ consists of a "string" of terms $\{y_q, y_{q+1}, ..., y_l\}$, all less than with and and such that either q=-1 or q>-1 and $y_{q-1} \geq \bar{y}$ and either l=∞ or l<∞ and $y_{l+1} \geq \bar{y}$.

A solution $\{y_n\}_{n=-1}^{\infty}$ of Eq. (1.1) is non-oscillatory if there exists N≥ -1 such that either

$y_n > \bar{y}$ for all n$\geq N$ or

$y_n < \bar{y}$ for all n$\geq N$.

$\{y_n\}_{n=-1}^{\infty}$ is called oscillatory if it is not non-oscillatory.

Theorem 3.1. Let $\{y_n\}_{n=-1}^{\infty}$ be a positive solution of Eq. (1.1) which consists of a single semi-cycle. Then $\{y_n\}_{n=-1}^{\infty}$converges monotonically to $\bar{y} = \alpha$.

Proof. Suppose 0<$y_{n-1} < \alpha$ for all n$\geq$ 0. Note that for all n$\geq$ 0,

$$0 < \alpha + \frac{y_{n-1}}{y_n} - \frac{y_n}{y_{n-1}} < \alpha$$

and so

$$0 < y_{n-1} < y_n < \alpha.$$

From this it is clear that the positive solutions converge monotonically to $\bar{y}$.

Theorem 3.2. Let be $\{y_n\}_{n=-1}^{\infty}$a positive solution of Eq. (1.1) which consists at least two semi-cycles. Then $\{y_n\}_{n=-1}^{\infty}$is oscillatory.

Proof. We consider the following two cases.

Case I. Suppose that $y_{-1} < \alpha \leq y_0$. Then

$$y_1 = \alpha + \frac{y_{-1}}{y_0} - \frac{y_0}{y_{-1}} < \alpha$$

and

$$y_2 = \alpha + \frac{y_0}{y_1} - \frac{y_1}{y_0} > \alpha.$$

Case II. Suppose that $y_0 < \alpha \leq y_{-1}$. Then

$$y_1 = \alpha + \frac{y_{-1}}{y_0} - \frac{y_0}{y_{-1}} > \alpha$$

and

$$y_2 = \alpha + \frac{y_0}{y_1} - \frac{y_1}{y_0} < \alpha.$$

Hence the proof is complete.

4. **Global asymptotically stability of eq. (1.1)**

In this section, we find a global asymptotic stability result for Eq. (1.1).

Lemma 4.1. Let $\alpha \in (0, \infty)$ and $f(u, v) = \alpha + \frac{v}{u} - \frac{u}{v}$. If u,v$\in (0, \infty)$, then f(u,v) is nonincreasing in each arguments.

Proof. The proof is simple and will be omitted.

Theorem 4.1. Let $\alpha > 4$. Then the unique positive equilibrium $\bar{y}$ of Eq. (1.1) is globally asymptotically stable.

Proof. For u,v$\in (0, \infty)$, set $f(u, v) = \alpha + \frac{v}{u} - \frac{u}{v}$. Then f:IxI$\rightarrow I$ is a continuous function and is non-increasing in each arguments. Let (m,M)$\in IxI$ is a solution of the system

$$M = f(m, m)$$

$$m = f(M, M),$$

then

$$M = \alpha + \frac{m}{m} - \frac{m}{m}$$

and

$$m = \alpha + \frac{M}{M} - \frac{M}{M}.$$

Since M-m=0, we get m=M. By using Theorem 1.3, we have which shows that is globally asymptotically stable equilibrium point of Eq. (1.1).

$$\lim_{n\to\infty} y_n = \bar{y}$$

which shows that $\bar{y} = \alpha$ is globally asymptotically stable equilibrium point of Eq. (1.1).

Acknowledgement. The first-named author would like to thank the Scientific and Technical research Council of Turkey (TUBITAK) for financial helps (code:2211).

References

[1] Amleh, A.M., Grove, E.A., Ladas, G., *On the Recursive Sequence* $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$, Journal of Mathematical Analysis and Applications, **233**(1999), 790-798.

[2] El-Owaidy, H.M., Ahmed, A.M., Mousa, M.S., *On asymptotic behavior of the difference equation*$x_{n+1} = \alpha + \frac{x_{n-k}}{x_n}$, Applied Mathematics and Computation, **147**(2004), 163-167.

[3] Abu-Saris, R.M., De Vault, R., *Global stability of* $x_{n+1} = A + \frac{y_n}{y_{n-k}}$, Applied Mathematics Letters, **16**(2003), 173-178.

[4] Gibbons, C.H., Kulenović, M.R.S., Ladas, G., Voulov, H.D., *On the Trichotomy Character of* $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + x_n)$, J. Difference Equations and Applications, **8(1)**(2002), 75-92.

[5] El-Afifi, M.M., Ahmed, A.M., *On the difference equation* $x_{n+1} = \frac{\alpha + \alpha x_n + \ldots + \alpha x_{n-k+2}}{x_{n-k+1}}$, Applied Mathematics and Computation, **144**(2003), 537-542.

[6] Fan, Y., Wang, L., Li, W., *Global behavior of a higher order nonlinear difference equation*, J. Math. Anal. Appl., **299**(2004), 113-126.

[7] De Vault, R., Kosmala, W., Ladas, G., Schultz, S.W., *Global Behavior of* $y_{n+1} = \frac{p + y_{n-k}}{q y_n + y_{n-k}}$, Nonlinear Analysis, **47**(2001), 4743-4751.

Department of Mathematics, Faculty of Education
Erciyes University
E-mail address: fbozkurt@erciyes.edu.tr

Department of Mathematics Erciyes University (KMYO)

Department of Mathematics, Faculty of Science and Arts
Erciyes University,Kayseri, Turkey, 38039

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 2, June 2009

A DIFFERENTIAL SANDWICH THEOREM FOR ANALYTIC FUNCTIONS DEFINED BY THE INTEGRAL OPERATOR

LUMINIŢA-IOANA COTÎRLĂ

Abstract. Let q_1 and q_2 be univalent in the unit disk U, with $q_1(0) = q_2(0) = 1$. We give an application of first order differential subordination to obtain sufficient condition for normalized analytic functions $f \in \mathcal{A}$ to satisfy

$$q_1(z) \prec \left(\frac{I^n f(z)}{z}\right)^{\delta} \prec q_2(z),$$

where I^n is an integral operator.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : \ f(z) = a + a_n z^n + \dots\}.$$

We also consider the class

$$\mathcal{A} = \{f \in \mathcal{H} : \ f(z) = z + a_2 z^2 + \dots\}.$$

We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial U : \ \lim_{z \to \zeta} f(z) = \infty\right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Received by the editors: 01.10.2008.

2000 *Mathematics Subject Classification.* 30C80, 30C45.

Key words and phrases. integral operator, differential superordination, differential subordination.

Since we use the terms of subordination and superordination, we review here those definitions.

Let $f, F \in \mathcal{H}$. The function f is said to be subordinate to F or F is said to be superordinate to f, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case we write $f \prec F$ or $f(z) \prec F(z)$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Since most of the functions considered in this paper and conditions on them are defined uniformly in the unit disk U, we shall omit the requirement "$z \in U$".

Let $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let h be univalent in U and $q \in Q$. In [3] the authors considered the problem of determining conditions on admissible function ψ such that

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \tag{1.1}$$

implies $p(z) \prec q(z)$, for all functions $p \in \mathcal{H}[a, n]$ that satisfy the differential subordination (1.1).

Moreover, they found conditions so that the function q is the "smallest" function with this property, called the best dominant of the subordination (1.1).

Let $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let $h \in \mathcal{H}$ and $q \in \mathcal{H}[a, n]$. Recently, in [4] the authors studied the dual problem and determined conditions on φ such that

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \tag{1.2}$$

implies $q(z) \prec p(z)$, for all functions $p \in Q$ that satisfy the above differential superordination.

Moreover, they found conditions so that the function q is the "largest" function with this property, called the best subordinant of the superordination (1.2).

For two functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

The integral operator I^n of a function f is defined in [6] by

$$I^0 f(z) = f(z),$$
$$I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt,$$
$$I^n f(z) = I(I^{n-1} f(z)), \quad z \in U.$$

In this paper we will determine some properties on admissible functions defined with the integral operator.

2. **Preliminaries**

Theorem 2.1. [3] *Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$, when $w \in q(U)$. Set*

$$Q(z) = zq'(z) \cdot \phi[q(z)], \quad h(z) = \theta[q(z)] + Q(z)$$

and suppose that either h is convex or Q is starlike. In addition, assume that

$$\mathrm{Re}\ \frac{zh'(z)}{Q(z)} > 0.$$

If p is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta[p(z)] + zp'(z) \cdot \phi[p(z)] \prec \theta[q(z)] + zp'(z) \cdot \phi[q(z)] = h(z),$$

then $p \prec q$, and q is the best dominant.

By taking $\theta(w) := w$ and $\phi(w) := \gamma$ in Theorem 2.1, we get

Corollary 2.2. *Let q be univalent in U, $\gamma \in \mathbb{C}^*$ and suppose*

$$\mathrm{Re}\ \left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\mathrm{Re}\ \frac{1}{\gamma}\right\}.$$

If p is analytic in U, with $p(0) = q(0)$ and

$$p(z) + \gamma zp'(z) \prec q(z) + \gamma zq'(z),$$

then $p \prec q$, and q is the best dominant.

Theorem 2.3. ([4]) *Let θ and ϕ be analytic in a domain D and let q be univalent in U, with $q(0) = a$, $q(U) \subset D$. Set*

$$Q(z) = zq'(z) \cdot \phi[q(z)], \quad h(z) = \theta[q(z)] + Q(z)$$

and suppose that

(i) $\operatorname{Re}\left\{\dfrac{\theta'[q(z)]}{\phi[q(z)]}\right\} > 0$ *and*

(ii) $Q(z)$ is starlike.

If $p \in \mathcal{H}[a,1] \cap Q$, $p(U) \subset D$ and $\theta[p(z)] + zp'(z) \cdot \phi[p(z)]$ is univalent in U, then

$$\theta[q(z)] + zp'(z)\phi[q(z)] \prec \theta[p(z)] + zp'(z)\phi[p(z)] \;\Rightarrow\; q \prec p$$

and q is the best subordinant.

By taking $\theta(w) := w$ and $\phi(w) := \gamma$ in Theorem 2.3, we get

Corollary 2.4. ([2]) *Let q be convex in U, $q(0) = a$ and $\gamma \in \mathbb{C}$,* $\operatorname{Re}\gamma > 0$. *If $p \in \mathcal{H}[a,1] \cap Q$ and $p(z) + \gamma zp'(z)$ is univalent in U, then*

$$q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z) \;\Rightarrow\; q \prec p$$

and q is the best subordinant.

3. Main results

Theorem 3.1. *Let q be univalent in U with $q(0) = 1$, $\alpha \in \mathbb{C}^*$, $\delta > 0$ and suppose*

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\operatorname{Re}\frac{\delta}{\alpha}\right\}.$$

If $f \in \mathcal{A}$ satisfies the subordination

$$(1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \cdot \frac{I^n f(z)}{I^{n+1}f(z)} \prec q(z) + \frac{\alpha}{\delta}zq'(z), \tag{3.1}$$

then

$$\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \prec q(z)$$

and q is the best dominant.

Proof. We define the function

$$p(z) := \left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}.$$

By calculating the logarithmic derivative of p, we obtain

$$\frac{zp'(z)}{p(z)} = \delta\left(\frac{z(I^{n+1}f(z))'}{I^{n+1}f(z)} - 1\right). \tag{3.2}$$

Because the integral operator I^n satisfies the identity:

$$z[I^{n+1}f(z)]' = I^n f(z), \tag{3.3}$$

equation (3.2) becomes

$$\frac{zp'(z)}{p(z)} = \delta\left(\frac{I^n f(z)}{I^{n+1}f(z)} - 1\right)$$

and, therefore,

$$\frac{zp'(z)}{\delta} = \left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}\left(\frac{I^n f(z)}{I^{n+1}f(z)} - 1\right).$$

The subordination (3.1) from the hypothesis becomes

$$p(z) + \frac{\alpha}{\delta}zp'(z) \prec q(z) + \frac{\alpha}{\delta}zq'(z).$$

We apply now Corollary 2.4 with $\gamma = \dfrac{\alpha}{\delta}$ to obtain the conclusion of our theorem. □

If we consider $n = 0$ in Theorem 3.1, we obtain the following result.

Corollary 3.2. *Let q be univalent in U with $q(0) = 1$, $\alpha \in \mathbb{C}^*$, $\delta > 0$ and suppose*

$$\mathrm{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\mathrm{Re}\,\frac{\delta}{\alpha}\right\}.$$

If $f \in \mathcal{A}$ satisfies the subordination

$$(1-\alpha)\left(\frac{If(z)}{z}\right)^{\delta} + \alpha\left(\frac{If(z)}{z}\right)^{\delta}\cdot\frac{f(z)}{If(z)} \prec q(z) + \frac{\alpha}{\delta}zq'(z) \tag{3.4}$$

then

$$\left(\frac{If(z)}{z}\right)^{\delta} \prec q(z)$$

and q is the best dominant.

We consider a particular convex function

$$q(z) = \frac{1 + Az}{1 + Bz}$$

to give the following application to Theorem 3.1.

Corollary 3.3. *Let $A, B, \alpha \in \mathbb{C}$, $A \neq B$ be such that $|B| \leq 1$, $\operatorname{Re} \alpha > 0$ and let $\delta > 0$. If $f \in \mathcal{A}$ satisfies the subordination*

$$(1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \cdot \frac{I^{n}f(z)}{I^{n+1}f(z)}$$

$$\prec \frac{1+Az}{1+Bz} + \frac{\alpha}{\delta} \cdot \frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \dfrac{1+Az}{1+Bz}$ is the best dominant.

Theorem 3.4. *Let q be convex in U with $q(0) = 1$, $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha > 0$, $\delta > 0$. If $f \in \mathcal{A}$ such that*

$$\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \in \mathcal{H}[1,1] \cap Q,$$

$$(1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \cdot \frac{I^{n}f(z)}{I^{n+1}f(z)}$$

is univalent in U and satisfies the superordination

$$q(z) + \frac{\alpha}{\delta} zq'(z) \prec (1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \cdot \frac{I^{n}f(z)}{I^{n+1}f(z)}, \tag{3.5}$$

then

$$q(z) \prec \left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}$$

and q is the best subordinant.

Proof. Let

$$p(z) := \left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}.$$

If we proceed as in the proof of Theorem 3.1, the subordination (3.5) become

$$q(z) + \frac{\alpha}{\delta} zq'(z) \prec p(z) + \frac{\alpha}{\delta} zp'(z).$$

The conclusion of this theorem follows by applying the Corollary 2.4. □

If $n = 0$, then we obtain

Corollary 3.5. *Let q be convex in U, with $q(0) = 1$, $\alpha \in \mathbb{C}$, with* $\mathrm{Re}\,\alpha > 0$ *and $\delta > 0$. If $f \in \mathcal{A}$ such that*

$$\left(\frac{If(z)}{z}\right)^{\delta} \in \mathcal{H}[1,1] \cap Q$$

$$(1-\alpha)\left(\frac{If(z)}{z}\right)^{\delta} + \alpha\left(\frac{If(z)}{z}\right)^{\delta} \cdot \frac{f(z)}{If(z)}$$

is univalent in U and satisfies the superordination

$$q(z) + \frac{\alpha}{\delta} zq'(z) \prec (1-\alpha)\left(\frac{If(z)}{z}\right)^{\delta} + \alpha\left(\frac{If(z)}{z}\right)^{\delta} \cdot \frac{f(z)}{If(z)},$$

then $q(z) \prec \left(\frac{If(z)}{z}\right)^{\delta}$ and q is the best subordinant.

Corollary 3.6. *Let q be convex in U with $q(0) = 1$, $\alpha \in \mathbb{C}$ with* $\mathrm{Re}\,\alpha > 0$, *$\alpha > 0$. If $f \in \mathcal{A}$ such that*

$$\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \in \mathcal{H}[1,1] \cap Q,$$

$$(1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \cdot \frac{I^{n}f(z)}{I^{n+1}f(z)}$$

is univalent in U and satisfies the superordination

$$q(z) + \frac{\alpha}{\delta} zq'(z) \prec (1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \cdot \frac{I^{n}f(z)}{I^{n+1}f(z)},$$

then

$$q(z) \prec \left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}$$

and q is the best subordinant.

Concluding the results of differential subordination and superordination we state the following sandwich result.

Theorem 3.7. *Let q_1, q_2 be convex in U with $q_1(0) = q_2(0) = 1$, $\alpha \in \mathbb{C}$,* $\mathrm{Re}\,\alpha > 0$, *$\delta > 0$. If $f \in \mathcal{A}$ such that*

$$\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta} \in \mathcal{H}[1,1] \cap Q$$

$$(1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}+\alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}\cdot\frac{I^{n}f(z)}{I^{n+1}f(z)}$$

is univalent in U and satisfies

$$q_1(z)+\frac{\alpha}{\delta}zq_1'(z)\prec(1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}+\alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}\cdot\frac{I^{n}f(z)}{I^{n+1}f(z)}$$

$$\prec q_2(z)+\frac{\alpha}{\delta}zq_2'(z),$$

then

$$q_1(z)\prec\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}\prec q_2(z)$$

and q_1, q_2 are the best subordinant and the best dominant respectively.

Corollary 3.8. *Let q_1, q_2 be convex in U with $q_1(0)=q_2(0)=1$, $\alpha\in\mathbb{C}$ with* $\operatorname{Re}\alpha>0$*, $\delta>0$. If $f\in\mathcal{A}$ such that*

$$\left(\frac{If(z)}{z}\right)^{\delta}\in\mathcal{H}[1,1]\cap Q,$$

$$(1-\alpha)\left(\frac{If(z)}{z}\right)^{\delta}+\alpha\left(\frac{If(z)}{z}\right)^{\delta}\cdot\frac{f(z)}{If(z)}$$

is univalent in U and satisfies

$$q_1(z)+\frac{\alpha}{\delta}zq_1'(z)\prec(1-\alpha)\left(\frac{If(z)}{z}\right)^{\delta}+\alpha\left(\frac{If(z)}{z}\right)^{\delta}\cdot\frac{f(z)}{If(z)}$$

$$\prec q_2(z)+\frac{\alpha}{\delta}zq_2'(z),$$

then

$$q_1(z)\prec\left(\frac{If(z)}{z}\right)^{\delta}\prec q_2(z)$$

and q_1, q_2 are the best subordinant and the best dominant respectively.

Corollary 3.9. *Let q_1, q_2 be convex in U with $q_1(0)=q_2(0)=1$, $\alpha\in\mathbb{C}$,* $\operatorname{Re}\alpha>0$*, $\delta>0$. If $f\in\mathcal{A}$ such that*

$$\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}\in\mathcal{H}[1,1]\cap Q,$$

$$(1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}+\alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}\cdot\frac{I^{n}f(z)}{I^{n+1}f(z)}$$

is univalent in U and satisfies

$$q_1(z)+\frac{\alpha}{\delta}zq_1'(z)\prec(1-\alpha)\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}+\alpha\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}\cdot\frac{I^nf(z)}{I^{n+1}f(z)}$$
$$\prec q_2(z)+\frac{\alpha}{\delta}zq_2'(z),$$

then

$$q_1(z)\prec\left(\frac{I^{n+1}f(z)}{z}\right)^{\delta}\prec q_2(z)$$

and q_1, q_2 are the best subordinant and the best dominant respectively.

Similar results was obtained by D. Răducanu and V.O. Nechita in [5] for differential Sălăgean operator defined in [6].

References

[1] Acu, M., *Some preserving properties of the generalized Alexander operator*, General Mathematics, Vol. 10, No. 3-4(2002), 37-46.

[2] Bulboacă, T., *Classes of first order differential superordinations*, Demonstr. Math., **35**(2002), No. 2, 287-292.

[3] Miller, S.S., Mocanu, P.T., *Differential Subordinations. Theory and Applications*, Marcel Dekker, Inc., New York, Basel, 2000.

[4] S.S. Miller, P.T. Mocanu, *Briot-Bouquet differential superordinations and sandwich theorems*, J. Math. Anal. Appl., **329**(2007), No. 1, 327-335.

[5] Răducanu, D., Nechita, V.O., *A differential sandwich theorem for analytic functions defined by the generalized Sălăgean derivative*, (to appear).

[6] Şt. Sălăgean, Gr., *Subclasses of univalent functions*, Complex Analysis, Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., vol. 1013, Springer, Berlin, 1983, 362-372.

Babeş-Bolyai University
Faculty of Mathematics and Computer Science
400084 Cluj-Napoca, Romania
E-mail address: `uluminita@math.ubbcluj.ro`

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 2, June 2009

ASSOCIATED CLASSES OF MODULES

IULIU CRIVEI AND SEPTIMIU CRIVEI

Abstract. Let $\mathcal{C}$ be a non-empty class of modules closed under isomorphic copies. We consider some classes of modules associated to $\mathcal{C}$. Among them, we study two important classes in the theory of natural and conatural classes of modules, namely the class consisting of all modules having no non-zero submodule in $\mathcal{C}$, as well as its dual.

1. Introduction

Throughout R is an associative ring with non-zero identity and all modules are unitary right R-modules. Also, $\mathcal{C}$ is a class of modules, always non-empty and closed under isomorphic copies. For modules A, B, C, we denote by $A \leq B$ (respectively $A < B$, $A \trianglelefteq B$, $A << B$) the fact that A is a submodule (respectively proper, essential, superfluous submodule) of B. Also, we denote by $A \hookrightarrow B$ a monomorphism from A to B and by $B \twoheadrightarrow C$ an epimorphism from B to C.

Consider the following classes associated to $\mathcal{C}$:

$\mathcal{F}(\mathcal{C}) = \{A \mid 0 \neq B \leq A \Longrightarrow B \notin \mathcal{C}\}$,

$\mathcal{T}(\mathcal{C}) = \{A \mid B < A \Longrightarrow A/B \notin \mathcal{C}\}$,

$\mathcal{F}'(\mathcal{C}) = \{B \mid A \text{ submodule of } B, M \in \mathcal{C}, A \hookrightarrow M \Longrightarrow A = 0\}$,

$\mathcal{T}'(\mathcal{C}) = \{B \mid C \text{ homomorphic image of } B, M \in \mathcal{C}, M \twoheadrightarrow C \Longrightarrow C = 0\}$,

$\mathcal{H}(\mathcal{C}) = \{A \mid A \notin \mathcal{C}, \text{ but } 0 \neq B \leq A \Longrightarrow A/B \in \mathcal{C}\}$

$\mathcal{S}(\mathcal{C}) = \{A \mid A \notin \mathcal{C}, \text{ but } B < A \Longrightarrow B \in \mathcal{C}\}$

$\mathcal{H}'(\mathcal{C}) = \{A \mid 0 \neq B \leq A \Longrightarrow A/B \in \mathcal{C}\}$

Received by the editors: 28.07.2008.

2000 *Mathematics Subject Classification.* 16D80.

Key words and phrases. Class of modules, natural class, conatural class.

Partially supported by the Romanian grant PN-II-ID-PCE-2008-2 project ID 2271.

$\mathcal{S}'(\mathcal{C}) = \{A \mid B < A \Longrightarrow B \in \mathcal{C}\}$

The first four classes are important bricks in the theory of hereditary and cohereditary classes and, in particular, in the theory of natural and conatural classes [7]. The last four ones arise naturally and include various examples, such as simple modules or almost finitely generated modules [9]. We shall establish some properties of these classes.

2. **Classes related to natural and conatural classes**

The terminology of natural classes appeared in the beginning of the 1990s, allowing unification and simplification of some previous results in ring and module theory. A *natural class* is defined as a class of modules closed under isomorphic copies, submodules, direct sums and injective hulls. They have been studied by J. Dauns [5], S.S. Page and Y. Zhou [8] in the 1990s. In recent years, in a series of articles further developed in a recent monograph [7], J. Dauns and Y. Zhou have created a powerful theory of what is thought to be the new generation of ring and module theory.

Note that since projective covers of modules do not exist in general, the notion of natural class previously defined cannot be always dualized (this is possible for instance in the case of modules over perfect rings). The class $\mathcal{C}$ is called a *conatural class* if the condition

$$(*) \qquad \forall M \twoheadrightarrow N \neq 0, \text{there exist } C \in \mathcal{C},\ K \neq 0 \text{ and } N \twoheadrightarrow K \twoheadleftarrow C$$

implies $M \in \mathcal{C}$ [1]. In general one only has that, if $\mathcal{C}$ is a conatural class, then $\mathcal{C}$ is closed under homomorphic images and superfluous epimorphisms [1, Theorem 24].

Alternatively, natural classes in Mod-R may be seen as the skeleton of the class of all hereditary classes (closed under submodules) in Mod-R. This point of view allows one to introduce conatural classes, as the skeleton of the class of all cohereditary classes (closed under homomorphic images) in Mod-R. This was the approach of A. Alvarado García, H. Rincón and J. Ríos Montes [1].

It is known that natural classes and conatural classes form complete Boolean lattices. If $\mathcal{C}$ is a natural class, then its complement is $\mathcal{F}(\mathcal{C})$, whereas if $\mathcal{C}$ is a conatural class, then its complement is $\mathcal{T}(\mathcal{C})$ [1]. Moreover, natural classes and conatural classes are characterized as follows, properties that show the strong relationship between them and the considered associated classes and motivates the interest in their study.

Theorem 2.1. [8, Proposition 4] *A hereditary class $\mathcal{C}$ is a natural class if and only if $\mathcal{C} = \mathcal{F}(\mathcal{F}(\mathcal{C}))$ if and only if $\mathcal{F}(\mathcal{F}(\mathcal{C})) \subseteq \mathcal{C}$.*

Theorem 2.2. [1, Theorem 23] *A cohereditary class $\mathcal{C}$ is a conatural class if and only if $\mathcal{C} = \mathcal{T}(\mathcal{T}(\mathcal{C}))$ if and only if $\mathcal{T}(\mathcal{T}(\mathcal{C})) \subseteq \mathcal{C}$.*

Now we establish several properties of our classes, giving proofs only for the classes $\mathcal{T}'(\mathcal{C})$ and $\mathcal{T}(\mathcal{C})$. Note first that if $\mathcal{C}$ is hereditary, then $\mathcal{F}'(\mathcal{C}) = \mathcal{F}(\mathcal{C})$ and, if $\mathcal{C}$ is cohereditary, then $\mathcal{T}'(\mathcal{C}) = \mathcal{T}(\mathcal{C})$.

Denote $\mathcal{C}^{\perp} = \{Y \mid \mathrm{Hom}_R(\mathcal{C}, Y) = 0\}$ and ${}^{\perp}\mathcal{C} = \{X \mid \mathrm{Hom}_R(X, \mathcal{C}) = 0\}$. The final part of the following result completes [6, Theorem 3.1].

Theorem 2.3. *(i) $\mathcal{T}'(\mathcal{C})$ is closed under homomorphic images, extensions and superfluous epimorphisms.*

(ii) Let $B \in \mathcal{T}'(\mathcal{C})$ and $A \leq B$ with $A \in \mathcal{C}$. Then $A << B$.

(iii) If $\mathcal{C}$ is hereditary, then ${}^{\perp}\mathcal{C} = \mathcal{T}(\mathcal{C})$ and $\mathcal{C} \subseteq \mathcal{T}(\mathcal{C})^{\perp}$.

(iv) If $\mathcal{C}$ is hereditary, then $\mathcal{T}(\mathcal{C})$ is a torsion class. If $\mathcal{C}$ is also closed under essential extensions, then $\mathcal{T}(\mathcal{C})$ is a hereditary torsion class.

(v) Let $\mathcal{C}$ be a natural class. Then $\mathcal{C}$ cogenerates a hereditary torsion theory, namely $(\mathcal{T}(\mathcal{C}), \mathcal{F}(\mathcal{T}(\mathcal{C})))$, and $\mathcal{F}(\mathcal{T}(\mathcal{C}))$ is a natural class.

Proof. (i) Clearly, $\mathcal{T}'(\mathcal{C})$ is cohereditary.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence with $A, C \in \mathcal{T}'(\mathcal{C})$. We may assume that $A \leq B$. Let D' be a homomorphic image of B, say B/D, and let $M \in \mathcal{C}$ and $M \twoheadrightarrow D'$. Then there exists $M \twoheadrightarrow B/D \twoheadrightarrow B/(A + D) \cong (B/A)/((A + D)/A)$, the last one being a homomorphic image of C. Since $C \in \mathcal{T}'(\mathcal{C})$, it follows that $B/(A + D) = 0$, hence $A + D = B$. Now $D' \cong B/D = (A + D)/D \cong A/(A \cap D)$ is a homomorphic image of $A \in \mathcal{T}'(\mathcal{C})$. Then $D' = 0$. Thus $\mathcal{T}'(\mathcal{C})$ is closed under extensions.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence with $f(A) << B$ and $C \in \mathcal{T}'(\mathcal{C})$. We may assume that $A \leq B$. Let D' be a homomorphic image of B, say B/D, $M \in \mathcal{C}$ and $M \twoheadrightarrow D'$. As above, it follows that $A + D = B$. But since $A << B$, we get $D = B$, hence $D' = 0$. Thus $B \in \mathcal{T}'(\mathcal{C})$. Hence $\mathcal{T}'(\mathcal{C})$ is closed under superfluous epimorphisms.

(ii) Let D be such that $A + D = B$. Then $A/(A \cap D) \cong (A + D)/D = B/D$, hence there exists $A \twoheadrightarrow B/D$. Since $A \in \mathcal{C}$ and $B \in \mathcal{T}'(\mathcal{C})$, it follows that $D = B$. Hence $A << B$.

(iii) Let $A \in \mathcal{T}(\mathcal{C})$, $B \in \mathcal{C}$ and $0 \neq f \in \mathrm{Hom}_R(A, B)$. Since $\mathcal{C}$ is closed under submodules, $\mathrm{Im}\, f \in \mathcal{C}$. But since $\mathrm{Ker}\, f \neq A \in \mathcal{T}(\mathcal{C})$, we have $\mathrm{Im}\, f \cong A/\mathrm{Ker}\, f \notin \mathcal{C}$, a contradiction. Thus $\mathrm{Hom}_R(\mathcal{T}(\mathcal{C}), \mathcal{C}) = 0$. Hence we have $\mathcal{T}(\mathcal{C}) \subseteq^{\perp} \mathcal{C}$ and $\mathcal{C} \subseteq \mathcal{T}(\mathcal{C})^{\perp}$.

Now let $A \in^{\perp} \mathcal{C}$. If $A = 0$, we are done, so that assume $A \neq 0$. Let $B < A$ and suppose that $A/B \in \mathcal{C}$. Then $\mathrm{Hom}_R(A, A/B) = 0$. But the natural homomorphism $p : A \to A/B$ is non-zero, a contradiction. Hence $A/B \notin \mathcal{C}$, so that $A \in \mathcal{T}(\mathcal{C})$.

(iv) The first part follows by (iii). In order to show that $\mathcal{T}(\mathcal{C})$ is hereditary, let $A \in \mathcal{T}(\mathcal{C})$ and $B \leq A$. Suppose that $B \notin \mathcal{T}(\mathcal{C})$, hence there exists $C \in \mathcal{C}$ and a non-zero homomorphism $f : B \to C$. Taking the injective hull $j : C \to E$ of C, it follows that there is a non-zero homomorphism $h : A \to E$ extending jf. But this contradicts the fact that $A \in \mathcal{T}(\mathcal{C})$ and $E \in \mathcal{C}$.

(v) By (iii) and (iv), $\mathcal{T}(\mathcal{C})$ is the torsion class of the torsion theory cogenerated by $\mathcal{C}$, while $\mathcal{F}(\mathcal{T}(\mathcal{C}))$ is its torsionfree class. Now $\mathcal{F}(\mathcal{T}(\mathcal{C}))$ is a natural class. $\square$

In a dual manner one obtains the following result. Note that in case R is right perfect we have a characterization of conatural classes as follows: $\mathcal{C}$ is a conatural class if and only if $\mathcal{C}$ is closed under homomorphic images, projective covers and direct sums of simple modules [2, Theorem 17].

Theorem 2.4. *(i) $\mathcal{F}'(\mathcal{C})$ is closed under submodules, extensions and essential extensions.*

(ii) Let $B \in \mathcal{F}'(\mathcal{C})$ and $A \leq B$ be such that $B/A \in \mathcal{C}$. Then $A \trianglelefteq B$.

(iii) If $\mathcal{C}$ is cohereditary, then $\mathcal{C}^{\perp} = \mathcal{F}(\mathcal{C})$ and $\mathcal{C} \subseteq^{\perp} \mathcal{F}(\mathcal{C})$.

(iv) If $\mathcal{C}$ is cohereditary, then $\mathcal{F}(\mathcal{C})$ is a torsionfree class. If R is right perfect and $\mathcal{C}$ is also closed under superfluous epimorphisms, then $\mathcal{F}(\mathcal{C})$ is a cohereditary torsionfree class.

(v) Let R be right perfect and $\mathcal{C}$ a conatural class. Then $\mathcal{C}$ generates a cohereditary torsion theory, namely $(\mathcal{T}(\mathcal{F}(\mathcal{C})), \mathcal{F}(\mathcal{C}))$, and $\mathcal{T}(\mathcal{F}(\mathcal{C}))$ is a conatural class.

In what follows, let us see other connections between our classes. The second part of the next result was established for a natural class $\mathcal{C}$ in [6, Lemma 3.3].

Theorem 2.5. *If either $\mathcal{T}(\mathcal{C})$ is hereditary or $\mathcal{C}$ is closed under essential extensions, then*

$$\mathcal{T}(\mathcal{C}) = \{A \mid B < A \Longrightarrow A/B \in \mathcal{F}(\mathcal{C})\} \subseteq \mathcal{F}(\mathcal{C}).$$

Proof. Denote $\mathcal{A} = \{A \mid B < A \Longrightarrow A/B \in \mathcal{F}(\mathcal{C})\}$. Note that the inclusion $\mathcal{A} \subseteq \mathcal{T}(\mathcal{C})$ holds for any class $\mathcal{C}$. Indeed, let $A \in \mathcal{A}$ and $B < A$. Then $A/B \in \mathcal{F}(\mathcal{C})$, hence $A/B \notin \mathcal{C}$. Thus $A \in \mathcal{T}(\mathcal{C})$. Now we show the converse inclusion.

(i) Suppose that $\mathcal{T}(\mathcal{C})$ is hereditary. Let $A \in \mathcal{T}(\mathcal{C})$ and $B < A$. Let us prove that $A/B \in \mathcal{F}(\mathcal{C})$. Let $0 \neq D/B \leq A/B$. By hypothesis, we have $D \in \mathcal{T}(\mathcal{C})$. Since $B < D$, it follows that $D/B \notin \mathcal{C}$. Thus $A/B \in \mathcal{F}(\mathcal{C})$, whence $A \in \mathcal{A}$.

(ii) Suppose that $\mathcal{C}$ is closed under essential extensions. Let $A \in \mathcal{T}(\mathcal{C}) \setminus \mathcal{A}$. Then there exists $B < A$ such that $A/B \notin \mathcal{F}(\mathcal{C})$, whence there exists $0 \neq D/B \leq A/B$ such that $D/B \in \mathcal{C}$. Let D'/B be a complement of D/B in A/B. Then $D/B \cap D'/B = 0$ and $D/B + D'/B \trianglelefteq A/B$. It follows that $(D/B + D'/B)/(D'/B) \trianglelefteq (A/B)/(D'/B)$, that is, $D/B \trianglelefteq A/D'$. Since $D/B \in \mathcal{C}$, we get $A/D' \in \mathcal{C}$ by the hypothesis on $\mathcal{C}$. Having noted that $D' < A$, we have $A \notin \mathcal{T}(\mathcal{C})$, a contradiction. Hence $\mathcal{T}(\mathcal{C}) \subseteq \mathcal{A}$. $\square$

Dually, one has the following result. Note that if the ring is right perfect, then every submodule of a module has a supplement.

Theorem 2.6. *If either $\mathcal{F}(\mathcal{C})$ is cohereditary or R is right perfect and $\mathcal{C}$ is closed under superfluous epimorphisms, then*

$$\mathcal{F}(\mathcal{C}) = \{A \mid 0 \neq B \leq A \Longrightarrow B \in \mathcal{T}(\mathcal{C})\} \subseteq \mathcal{T}(\mathcal{C}).$$

3. **Other associated classes**

As before, let $\mathcal{C}$ be a class of modules. We have considered in the introduction the classes $\mathcal{H}(\mathcal{C})$, $\mathcal{S}(\mathcal{C})$, $\mathcal{H}'(\mathcal{C})$ and $\mathcal{S}'(\mathcal{C})$. Now we mention some examples covered by such general associated classes.

Example 3.1. (1) If $\mathcal{C} = \{0\}$, then $\mathcal{H}(\mathcal{C})$ consists of the simple modules.

(2) If $\mathcal{C}$ is the torsion class for a hereditary torsion theory τ in Mod-R, then $\mathcal{H}(\mathcal{C})$ consists of the τ-cocritical modules.

(3) If $\mathcal{C}$ is the class of commutative perfect rings, then $\mathcal{H}'(\mathcal{C})$ consists of the almost perfect rings [3].

(4) If $\mathcal{C} = \{0\}$, then $\mathcal{S}(\mathcal{C})$ consists of the simple modules.

(5) If $\mathcal{C}$ is the class of finitely generated modules, then $\mathcal{S}(\mathcal{C})$ consists of the almost finitely generated (a.f.g.) modules [9].

(6) If $\mathcal{C}$ is the class of modules having maximal submodules, then $\mathcal{S}(\mathcal{C})$ consists of the a.m.s. modules [4].

Let us give some properties of these classes. We will give proofs only for the classes $\mathcal{H}(\mathcal{C})$ and $\mathcal{H}'(\mathcal{C})$, the other ones being dual.

Theorem 3.2. *(i) If $0 \in \mathcal{C}$, then every simple module belongs either to $\mathcal{C}$ or to $\mathcal{H}(\mathcal{C})$.*

(ii) $\mathcal{H}(\mathcal{C}) \subseteq \mathcal{H}'(\mathcal{C})$. If $\mathcal{C}$ is closed under extensions, then $\mathcal{H}(\mathcal{C}) \subseteq \mathcal{F}(\mathcal{C})$.

(iii) If $\mathcal{C}$ is closed under submodules and extensions, then $\mathcal{H}(\mathcal{C})$ is closed under non-zero submodules and every module in $\mathcal{H}(\mathcal{C})$ is uniform.

(iv) If $\mathcal{C}$ is hereditary, then $\mathcal{H}'(\mathcal{C})$ is closed under non-zero submodules.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of modules and assume that $\mathcal{C}$ is closed under extensions.

(v) If $A \in \mathcal{H}'(\mathcal{C})$, $f(A) \trianglelefteq B$ and every homomorphic image of C belongs to $\mathcal{C}$, then $B \in \mathcal{H}'(\mathcal{C})$.

Proof. (i) and (ii) Straightforward.

(iii) Let $A \in \mathcal{H}(\mathcal{C})$ and $0 \neq B \leq A$. Suppose that $B \in \mathcal{C}$. Since $A/B \in \mathcal{C}$ and $\mathcal{C}$ is closed under extensions, we have $A \in \mathcal{C}$, a contradiction. Hence $B \notin \mathcal{C}$. Now let $0 \neq D \leq B$. Then $B/D \leq A/D \in \mathcal{C}$, hence $B/D \in \mathcal{C}$. Therefore $B \in \mathcal{H}(\mathcal{C})$.

Let $A \in \mathcal{H}(\mathcal{C})$ and suppose that it is not uniform. Then there exist non-zero submodules B and D of A such that $B \cap D = 0$. Since $\mathcal{H}(\mathcal{C})$ is closed under non-zero submodules, we have $B, B + D \in \mathcal{H}(\mathcal{C})$. Hence $B \cong (B + D)/D \in \mathcal{C}$, a contradiction. Therefore A is uniform.

(iv) See (iii).

(v) We may assume that A is a submodule of B. Let D be a non-zero proper submodule of B. Then $A \cap D \neq 0$, hence $(A + D)/D \cong A/(A \cap D) \in \mathcal{C}$. Consider the exact sequence of modules $0 \to (A + D)/D \to B/D \to B/(A + D) \to 0$. Since $\mathcal{C}$ is closed under extensions, it follows that $B/D \in \mathcal{C}$. Therefore $B \in \mathcal{H}'(\mathcal{C})$. $\square$

Theorem 3.3. *(i) If $0 \in \mathcal{C}$, then every simple module belongs either to $\mathcal{C}$ or to $\mathcal{S}(\mathcal{C})$.*

(ii) $\mathcal{S}(\mathcal{C}) \subseteq \mathcal{S}'(\mathcal{C})$. If $\mathcal{C}$ is closed under extensions, then $\mathcal{S}(\mathcal{C}) \subseteq \mathcal{T}(\mathcal{C})$.

(iii) If $\mathcal{C}$ is closed under homomorphic images and extensions, then $\mathcal{S}(\mathcal{C})$ is closed under proper homomorphic images and every module in $\mathcal{S}(\mathcal{C})$ is hollow.

(iv) If $\mathcal{C}$ is cohereditary, then $\mathcal{S}'(\mathcal{C})$ is closed under proper homomorphic images.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of modules and assume that $\mathcal{C}$ is closed under extensions.

(v) If $C \in \mathcal{S}'(\mathcal{C})$, $f(A) << B$ and every submodule of A belongs to $\mathcal{C}$, then $B \in \mathcal{S}'(\mathcal{C})$.

Theorem 3.4. *(i) Let $A \in \mathcal{H}'(\mathcal{C})$, $B \in \mathcal{F}'(\mathcal{C})$ and let $f : A \to B$ be a non-zero homomorphism. Then f is a monomorphism.*

(ii) Let $M \in \mathcal{H}'(\mathcal{C})$, let $N \in \mathcal{F}(\mathcal{C})$ be quasi-injective and let $S = \mathrm{End}_R(N)$. Then $\mathrm{Hom}_R(M, N)$ is a simple left S-module.

Proof. (i) We have $\mathrm{Ker}\, f \neq A$. Suppose that $\mathrm{Ker}\, f \neq 0$. Then $\mathrm{Im}\, f \cong A/\mathrm{Ker}\, f \in \mathcal{C}$, because $A \in \mathcal{H}'(\mathcal{C})$. Since $\mathrm{Im}\, f \hookrightarrow A/\mathrm{Ker}\, f \in \mathcal{C}$ and $B \in \mathcal{F}'(\mathcal{C})$, it follows that $\mathrm{Im}\, f = 0$, a contradiction. Hence f is a monomorphism.

(ii) Let $0 \neq f \in \mathrm{Hom}_R(M, N)$. By (i), f is a monomorphism. Let $g \in \mathrm{Hom}_R(M, N)$. Since N is quasi-injective, there exists $h \in S$ such that $hf = g$. Hence

$g \in Sf$, so that $\mathrm{Hom}_R(M,N) = Sf$. Thus $\mathrm{Hom}_R(M,N)$ is a simple left S-module. $\square$

Theorem 3.5. *(i) Let $A \in \mathcal{T}'(\mathcal{C})$, $B \in \mathcal{S}'(\mathcal{C})$ and let $f : A \to B$ be a non-zero homomorphism. Then f is an epimorphism.*

(ii) Let $M \in \mathcal{S}'(\mathcal{C})$, let $N \in \mathcal{T}(\mathcal{C})$ be quasi-projective and let $S = \mathrm{End}_R(N)$. Then $\mathrm{Hom}_R(N,M)$ is a simple left S-module.

In the sequel, let us see some other properties of the class $\mathcal{H}(\mathcal{C})$, when $\mathcal{C}$ is closed under submodules and extensions.

Theorem 3.6. *Let $\mathcal{C}$ be closed under submodules and extensions. Let A be a non-zero uniform module that has a submodule $B \in \mathcal{H}(\mathcal{C})$. Then A has a unique maximal submodule that belongs to $\mathcal{H}(\mathcal{C})$.*

Proof. Denote by D_i the submodules of A that belong to $\mathcal{H}(\mathcal{C})$, where $1 \leq i \leq \omega$ and ω is some ordinal. We show that $D = \sum_{i \leq \omega} D_i \in \mathcal{H}(\mathcal{C})$ by transfinite induction on ω. For $\omega = 1$, the result is trivial. Suppose that $\omega > 1$ and that $E = \sum_{i<\omega} D_i \in \mathcal{H}(\mathcal{C})$. If $D_\omega \subseteq E$, then $D = E \in \mathcal{H}(\mathcal{C})$. Now suppose that $D_\omega \nsubseteq E$. By Theorem 3.2, we have $A \in \mathcal{F}(\mathcal{C})$, hence $D \notin \mathcal{C}$. Let $0 \neq F \leq D$. Then $(E+F)/F \cong E/(F \cap E) \in \mathcal{C}$, because $F \cap E \neq 0$. We also have

$$D/(E+F) = (E+F+D_\omega)/(E+F) \cong D_\omega/((E+F) \cap D_\omega) \in \mathcal{C}\,,$$

because $(E+F) \cap D_\omega \neq 0$. By the exactness of the sequence $0 \to (E+F)/F \to D/F \to D/(E+F) \to 0$ and the fact that the class $\mathcal{C}$ is closed under extensions, it follows that $D/F \in \mathcal{C}$. Hence $D \in \mathcal{H}(\mathcal{C})$. Clearly, D is the unique maximal submodule of A that belongs to $\mathcal{H}(\mathcal{C})$. $\square$

A module satisfying the hypothesis of the above theorem does exist by the following result.

Proposition 3.7. *Let $\mathcal{C}$ be closed under submodules and extensions. Let A be a noetherian module such that $A \notin \mathcal{C}$. Then there exists a proper submodule D of A such that $A/D \in \mathcal{H}(\mathcal{C})$.*

Proof. Let $\mathcal{M}$ be the family of all submodules B of A such that $A/B \notin \mathcal{C}$. Clearly $\mathcal{M} \neq \emptyset$ since $0 \in \mathcal{M}$. Since A is noetherian, $\mathcal{M}$ has a maximal element, say D, that is a proper submodule of A. Hence $A/D \notin \mathcal{C}$. Now let $D < F \leq A$. Then $(A/D)/(F/D) \cong A/F \in \mathcal{C}$ by the maximality of D. Therefore $A/D \in \mathcal{H}(\mathcal{C})$. $\square$

Lemma 3.8. *Let $\mathcal{C}$ be closed under submodules and extensions. Let B be a uniform module that contains a submodule $A \in \mathcal{H}(\mathcal{C})$ such that $B/A \in \mathcal{F}(\mathcal{C})$. Then A is the maximal submodule of B that belongs to $\mathcal{H}(\mathcal{C})$.*

Proof. Suppose the contrary. Then there exists $A < D \leq B$ such that $D \in \mathcal{H}(\mathcal{C})$. Hence $D/A \in \mathcal{C}$, therefore $B/A \notin \mathcal{F}(\mathcal{C})$, a contradiction. $\square$

With the same assumption on $\mathcal{C}$ to be closed under submodules and extensions, for $A \in \mathcal{H}(\mathcal{C})$, let us denote by $M_{\mathcal{C}}(A)$ the maximal submodule of the injective hull $E(A)$ of A that belongs to $\mathcal{H}(\mathcal{C})$. Also, denote by $\mathcal{M}_{\mathcal{C}}$ the class consisting of all modules $M_{\mathcal{C}}(A)$ for $A \in \mathcal{H}(\mathcal{C})$.

Theorem 3.9. *Let $\mathcal{C}$ be closed under submodules and extensions. Let $A, B \in \mathcal{M}_{\mathcal{C}}$ and let $f : A \to B$ be a non-zero homomorphism. Then f is an isomorphism.*

Proof. By Theorem 3.4, f is a monomorphism. There exists a homomorphism $g : E(A) \to E(B)$ that extends jf, where $j : B \to E(B)$ is the inclusion homomorphism. Since $A \trianglelefteq E(A)$, g is a monomorphism. But $E(B)$ is indecomposable, hence g is an isomorphism. Clearly $g(A) \subseteq j(B)$, whence $A \subseteq g^{-1}(B)$. We also have $g^{-1}(B) \in \mathcal{H}(\mathcal{C})$ and by the maximality of A it follows that $A = g^{-1}(B)$. Thus $g(A) = B$, whence $f(A) = B$. Therefore f is an isomorphism. $\square$

Theorem 3.10. *Let $\mathcal{C}$ be closed under submodules and extensions. Let $D \in \mathcal{M}_{\mathcal{C}}$ and let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of modules with $B \in \mathcal{H}(\mathcal{C})$. Then D is injective with respect to the above sequence.*

Proof. Let $u : A \to D$ be a homomorphism. We may assume that $u \neq 0$. By Theorem 3.4, u is a monomorphism, because $A \in \mathcal{H}(\mathcal{C})$. Let $v : D \to E(D)$ be the inclusion homomorphism. Then there exists a homomorphism $w : B \to E(D)$ such that $wf = vu$. Since $f(A) \trianglelefteq B$, w is a monomorphism. But $w(B) \in \mathcal{H}(\mathcal{C})$. By the maximality of D, it follows that $w(B) \subseteq D$. Now let $h : B \to D$ be the homomorphism defined by $h(b) = w(b)$ for every $b \in B$. Then $hf = u$, showing that D is injective with respect to the above sequence. $\square$

Corollary 3.11. *Let $\mathcal{C}$ be closed under submodules and extensions. Then every module in $\mathcal{H}(\mathcal{C})$ is quasi-injective.*

References

[1] Alvarado García, A., Rincón, H., Ríos Montes, J., *On the lattice of natural and conatural classes in R-mod*, Comm. Algebra, **29**(2001), 541-556.

[2] Alvarado García, A., Rincón, H., Ríos Montes, J., *On some lattices of module classes*, J. Algebra Appl., **5**(2006), 105-117.

[3] Bazzoni, S., Salce, L., *Almost perfect domains*, Colloq. Math., **95**(2003), 285-301.

[4] Crivei, I., *On a class of modules having no maximal submodule*, Pure Math. Appl., **13**(2002), 111-116.

[5] Dauns, J., *Classes of modules*, Forum Math., **3**(1991), 327-338.

[6] Dauns, J., *Natural classes and torsion theories*, J. Algebra Appl., **2**(2003), 85-99.

[7] Dauns, J., Zhou, Y., *Classes of modules*, Chapman and Hall/CRC, Boca Raton, 2006.

[8] Page, S.S., Zhou, Y., *On direct sums of injective modules and chain conditions*, Canad. J. Math., **46**(1994), 634-647.

[9] Weakley, W.D., *Modules whose proper submodules are finitely generated*, J. Algebra, **84**(1983), 189-219.

Department of Mathematics
Technical University
Str. C. Daicoviciu 15
400020 Cluj-Napoca, Romania
E-mail address: crivei@math.utcluj.ro

Faculty of Mathematics and Computer Science
"Babeş-Bolyai" University
Str. M. Kogălniceanu 1
400084 Cluj-Napoca, Romania
E-mail address: crivei@math.ubbcluj.ro

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 2, June 2009

VARIATIONAL ANALYSIS OF A ELASTIC-VISCOPLASTIC CONTACT PROBLEM WITH FRICTION AND ADHESION

SALAH DRABLA AND ZHOR LERGUET

Abstract. The aim of this paper is to study the process of frictional contact with adhesion between a body and an obstacle. The material's behavior is assumed to be elastic-viscoplastic, the process is quasistatic, the contact is modeled by the Signorini condition and the friction is described by a non local Coulomb law coupled with adhesion. The adhesion process is modelled by a bonding field on the contact surface. We derive a variational formulation of the problem, then, under a smallness assumption on the coefficient of friction, we prove an existence and uniqueness result of a weak solution for the model. The proof is based on arguments of time-dependent variational inequalities, differential equations and Banach fixed point theorem.

1. Introduction

The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature. Basic modelling can be found in [10], [12], [14] and [6]. Analysis of models for adhesive contact can be found in [2]-[4], [13] and in the recent monographs [17],[18]. An application of the theory of adhesive contact in the medical field of prosthetic limbs was considered in [15], [16]; there, the importance of the bonding between the bone-implant and the tissue was outlined, since debonding may lead to decrease in the persons ability to use the artificial limb or joint.

Received by the editors: 15.05.2008.

2000 *Mathematics Subject Classification.* 74H10, 74H10, 74M15, 74F25, 49J40.

Key words and phrases. Elastic-viscoplastic material, non local Coulomb friction law, adhesion, variational inequality, weak solution, fixed point.

Contact problems for elastic and elastic-viscoelastic bodies with adhesion and friction appear in many applications of solids mechanics such as the fiber-matrix interface of composite materials. A consistent model coupling unilateral contact, adhesion and friction is proposed by Raous, Cangémi and Cocu in [14]. Adhesive problems have been the subject of some recent publications (see for instance [14], [9], [1], [3], [6]). The novelty in all the above papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by β; it describes the pointwise fractional density of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [10], [11], the bonding field satisfies the restrictions $0 \leq \beta \leq 1$; when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. We refer the reader to the extensive bibliography on the subject in [12], [14], [15]. Such models contain a new internal variable β which represents the adhesion intensity over the contact surface, it takes values between 0 and 1, and describes the fractional density of active bonds on the contact surface.

Elastic quasistatic contact problems with *Signorini* conditions and local *Coulomb* friction law were recently studied by Cocu and Rocca in [5]. Other elastic-viscoplastic contact models with *Signorini* conditions and non local *Coulomb* friction law were variationally analyzed in [7], [8] There exists at least one solution to such problems if the friction coefficient is sufficiently small.

The aims of this paper is to extend the result when non local Coulomb friction law coupled with adhesion are taken into account at the interface and the material behavior is assumed to be elastic-viscoplastic.

The paper is structured as follows. In Section 2 we present the elastic-viscoplastic contact model with fiction and adhesion and provide comments on the contact boundary conditions. In Section 3 we list the assumptions on the data, derive the variational formulation. In Sections 4, we present our main existence and uniqueness results, Theorems 4.1, which state the unique weak solvability of the Signorini adhesive contact problem with non local Coulomb friction law conditions.

2. Problem statement

We consider an elastic-viscoplastic body, which occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$, with a smooth boundary $\partial\Omega = \Gamma$ divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $meas(\Gamma_1) > 0$. Let $[0,T]$ be the time interval of interest, where $T > 0$. The body is clamped on $\Gamma_1 \times (0,T)$ and therefore the displacement field vanishes there, it is also submitted to the action of volume forces of density f_0 in $\Omega \times (0,T)$ and surface tractions of density f_2 on $\Gamma_2 \times (0,T)$. On $\Gamma_3 \times (0,T)$, the body is in adhesive contact with friction with an obstacle the so-called foundation. The friction is modelled by a non local Coulomb law. We denote by ν the outward normal unit vector on Γ.

With these assumptions, the classical formulation of the elastic-viscoplastic contact problem with friction and adhesion is the following.

Problem $\mathcal{P}$. *Find a displacement field $u : \Omega \times [0,T] \to R^d$, a stress field $\sigma : \Omega \times [0,T] \to S^d$, and a bonding field $\beta : \Omega \times [0,T] \to R$ such that*

$$\dot{\sigma} = \mathcal{E}\varepsilon(\dot{u}) + \mathcal{G}(\sigma, \varepsilon(u)) \qquad \text{in } \Omega \times (0,T), \tag{2.1}$$

$$\mathrm{Div}\sigma + f_0 = 0 \qquad \text{in } \Omega \times (0,T), \tag{2.2}$$

$$u = 0 \qquad \text{on } \Gamma_1 \times (0,T), \tag{2.3}$$

$$\sigma\nu = f_2 \qquad \text{on } \Gamma_2 \times (0,T), \tag{2.4}$$

$$u_\nu \le 0,\ \sigma_\nu - \gamma_\nu\beta^2 R_\nu(u_\nu) \le 0,\ u_\nu(\sigma_\nu - \gamma_\nu\beta^2 R_\nu(u_\nu)) = 0 \text{ on } \Gamma_3 \times (0,T), \tag{2.5}$$

$$\left\{\begin{array}{l} |\sigma_\tau + \gamma_\tau\beta^2 R_\tau(u_\tau)| \le \mu p(|R(\sigma_\nu) - \gamma_\nu\beta^2 R_\nu(u_\nu)|), \\ |\sigma_\tau + \gamma_\tau\beta^2 R_\tau(u_\tau)| < \mu p(|R(\sigma_\nu) - \gamma_\nu\beta^2 R_\nu(u_\nu)|) \Rightarrow u_\tau = 0, \\ |\sigma_\tau + \gamma_\tau\beta^2 R_\tau(u_\tau) = \mu p(|R(\sigma_\nu) - \gamma_\nu\beta^2 R_\nu(u_\nu)|) \Rightarrow \exists\lambda \ge 0, \\ \text{such that } \sigma_\tau + \gamma_\tau\beta^2 R_\tau(u_\tau) = -\lambda u_\tau. \end{array}\right. \quad \text{on } \Gamma_3 \times (0,T), \tag{2.6}$$

$$\dot{\beta} = -(\beta(\gamma_\nu R_\nu(u_\nu)^2 + \gamma_\tau\|R_\tau(u_\tau)\|^2) - \epsilon_a)_+ \qquad \text{on } \Gamma_3 \times (0,T), \tag{2.7}$$

$$u(0) = u_0,\ \sigma(0) = \sigma_0 \text{ in } \Omega,\ \beta(0) = \beta_0 \text{ on } \Gamma_3 \tag{2.8}$$

We now provide some comments on equations and conditions (2.1)-(2.8). The material is assumed to be elastic-viscoplastic with a constitutive law of the form (2.1), where $\mathcal{E}$ and $\mathcal{G}$ are constitutive functions which will be described below. We denote by $\varepsilon(u)$ the linearized strain tensor. The equilibrium equation is given by (2.2), where "*Div*" denotes the divergence operator for tensor valued functions. Equations (2.3) and (2.4) represent the displacement and traction boundary conditions.

Conditions (2.5) represent he Signorini contact condition with adhesion where u_ν is the normal displacement $\sigma_\nu$represents the normal stress, γ_ν denote a given adhesion coefficient and R_ν is the truncation operator define by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \leq s \leq 0, \\ 0 & \text{if } s > 0, \end{cases}$$

where $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of operator R_ν, together with the operator R_τ defined below , is motivated by the mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows. Thus, by choosing L very large, we can assume that $R_\nu(u_\nu) = u_\nu$ and, therefore, from (2.5) we recover the contact conditions

$$u_\nu \leq 0,\ \sigma_\nu - \gamma_\nu \beta^2 u_\nu \leq 0,\ u_\nu(\sigma_\nu - \gamma_\nu \beta^2 u_\nu) = 0 \text{ on } \Gamma_3 \times (0,T),$$

It follows from (2.5) that there is no penetration between the body and the foundation, since $u_\nu \leq 0$ during the process.

Conditions (2.6) are a non local Coulomb friction law conditions coupled with adhesion, where u_τ, σ_τ denote tangential components of vector u and tensor σ respectively. R_τ is the truncation operator given by

$$R_\tau(v) = \begin{cases} v & \text{if } \|v\| \leq L, \\ L\frac{v}{\|v\|} & \text{if } \|v\| > L. \end{cases}$$

This condition shows that the magnitude of the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L.

R will represent a normal regularization operator that is , linear and continues operator $R: \; H^{-\frac{1}{2}}(\Gamma) \longrightarrow L^2(\Gamma)$. We shall need it to regularize the normal trace of the stress witch is too rough on Γ. p is a non-negative function, the so-called friction bound, $\mu \geq 0$ is the coefficient of friction. The friction law was used with $p\,(r) = r_+$. A new version of *Coulomb* law consists to take

$$p(r) = r(1 - \alpha r)_+,$$

where α is a small positive coefficient related to the hardness and the wear of the contact surface and $r_+ = \max\{0, r\}$.

Also, note that when the bonding field vanishes, then the contact conditions (2.5) and (2.6) become the classic Signorini contact with a non local Coulomb friction law conditions were used in ([8]), that is

$$u_\nu \leq 0, \;\; \sigma_\nu \leq 0, \;\; u_\nu \sigma_\nu = 0 \text{ on } \Gamma_3 \times (0, T),$$

$$\begin{cases} |\sigma_\tau| \leq \mu p(|R(\sigma_\nu)|), \\ |\sigma_\tau| < \mu p(|R(\sigma_\nu)|) \Rightarrow u_\tau = 0, & \text{on } \Gamma_3 \times (0, T), \\ |\sigma_\tau| = \mu p(|R(\sigma_\nu)|) \Rightarrow \exists \lambda \geq 0, \text{ such that } \sigma_\tau = -\lambda u_\tau. \end{cases}$$

The evolution of the bonding field is governed by the differential equation (2.7) with given positive parameters γ_ν, γ_τ and ϵ_a, where $r_+ = \max\{0, r\}$. Here and below in this paper, a dot above a function represents the derivative with respect to the time variable. We note that the adhesive process is irreversible and, indeed, once debonding occurs bonding cannot be reestablished, since $\dot{\beta} \leq 0$. Finally, (2.8) is the initial condition in which β_0 is a given bonding field.

3. **Variational formulation and preliminaries**

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end we need to introduce some notation and preliminary material.

Here and below $\mathbb{S}^d$ represents the space of second order symmetric tensors on $\mathbb{R}^d$. We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$u \cdot v = u_i v_i \ , \qquad \|v\| = (v \cdot v)^{\frac{1}{2}} \qquad \forall\, u, v \in \mathbb{R}^d,$$

$$\sigma \cdot \tau = \sigma_{ij} \tau_{ij} \ , \qquad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}} \qquad \forall\, \sigma, \tau \in \mathbb{S}^d.$$

Here and everywhere in this paper, i, j, k, l run from 1 to d, summation over repeated indices is applied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \frac{\partial u_i}{\partial x_j}$. Everywhere below, we use the classical notation for L^p and Sobolev spaces associated to Ω and Γ. Moreover, we use the notation $L^2(\Omega)^d$, $H^1(\Omega)^d$ and $\mathcal{H}$ and $\mathcal{H}_1$ for the following spaces :

$$L^2(\Omega)^d = \{ \ v = (v_i) \mid v_i \in L^2(\Omega) \ \}, \quad H^1(\Omega)^d = \{ \ v = (v_i) \mid v_i \in H^1(\Omega) \ \},$$

$$\mathcal{H} = \{ \ \tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega) \ \}, \quad \mathcal{H}_1 = \{ \ \tau \in \mathcal{H} \mid \tau_{ij,j} \in L^2(\Omega) \ \}.$$

The spaces $L^2(\Omega)^d$, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(u, v)_{L^2(\Omega)^d} = \int_\Omega u \cdot v \, dx, \quad (u, v)_{H^1(\Omega)^d} = \int_\Omega u \cdot v \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx,$$

$$(\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma \cdot \tau \, dx, \quad (\sigma, \tau)_{\mathcal{H}_1} = \int_\Omega \sigma \cdot \tau \, dx + \int_\Omega \operatorname{Div} \sigma \cdot \operatorname{Div} \tau \, dx,$$

and the associated norms $\| \cdot \|_{L^2(\Omega)^d}$, $\| \cdot \|_{H^1(\Omega)^d}$, $\| \cdot \|_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}_1}$, respectively. Here and below we use the notation

$$\varepsilon(v) = (\varepsilon_{ij}(v)), \quad \varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i}) \quad \forall\, v \in H^1(\Omega)^d,$$

$$\operatorname{Div} \tau = (\tau_{ij,j}) \quad \forall\, \tau \in \mathcal{H}_1.$$

For every element $v \in H^1(\Omega)^d$ we also write v for the trace of v on Γ and we denote by v_ν and v_τ the normal and tangential components of v on Γ given by $v_\nu = v \cdot \nu$, $v_\tau = v - v_\nu \nu$.

Let now consider the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \{\ v \in H^1(\Omega)^d \mid v = 0 \text{ on } \Gamma_1\ \}.$$

Since $meas\,(\Gamma_1) > 0$, the following Korn's inequality holds:

$$\|\varepsilon(v)\|_{\mathcal{H}} \geq c_K\, \|v\|_{H^1(\Omega)^d} \quad \forall\, v \in V, \tag{3.1}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. Over the space V we consider the inner product given by

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}} \tag{3.2}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (3.1) that $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent norms on V and, therefore, $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, (3.1) and (3.2), there exists a constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq c_0 \|v\|_V \qquad \forall v \in V. \tag{3.3}$$

For every real Hilbert space X we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $1 \leq p \leq \infty$, $k \geq 1$ and we also introduce the set

$$\mathcal{Q} = \{\, \theta \in L^\infty(0,T;L^2(\Gamma_3)) \mid 0 \leq \theta(t) \leq 1 \ \ \forall\, t \in [0,T],\ \text{a.e. on } \Gamma_3 \,\}.$$

Finally, if X_1 and X_2 are two Hilbert spaces endowed with the inner products $(\cdot,\cdot)_{X_1}$ and $(\cdot,\cdot)_{X_2}$ and the associated norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$, respectively, we denote by $X_1 \times X_2$ the product space together with the canonical inner product $(\cdot,\cdot)_{X_1 \times X_2}$ and the associated norm $\|\cdot\|_{X_1 \times X_2}$.

In the study of the problem $\mathcal{P}$, we consider the following assumptions on the problem data.

$$\begin{cases} \mathcal{E}\colon \Omega\times\mathcal{S}_d \longrightarrow \mathcal{S}_d \text{ is a symmetric and positive definite tensor :} \\ (a)\ \ \mathcal{E}_{ijkl}\in L^{\infty}(\Omega) \ \text{ for every } i,j,k,l=\overline{1,d}; \\ (b)\ \ \mathcal{E}\sigma\cdot\tau=\sigma\cdot\mathcal{E}\tau \text{ for every } \sigma,\,\tau\in\mathcal{S}_d; \\ (c)\ \ \text{there exists } \alpha>0 \text{ such that } \mathcal{E}\sigma\cdot\sigma\geq\alpha\left|\sigma\right|^2\ \ \forall\sigma\in\mathcal{S}_d,\ \text{a.e. in } \Omega \end{cases} \tag{3.4}$$

$$\begin{cases} \mathcal{G}:\Omega\times\mathcal{S}_d\times\mathcal{S}_d\longrightarrow\mathcal{S}_d \text{ and} \\ (a)\ \ \text{there exists } L_{\mathcal{G}}>0 \text{ such that :} \\ \qquad \left|\mathcal{G}\left(\cdot,\sigma_1,\varepsilon_1\right)-\mathcal{G}\left(\cdot,\sigma_2,\varepsilon_2\right)\right|\leq L_{\mathcal{G}}(\left|\sigma_1-\sigma_2\right|+\left|\varepsilon_1-\varepsilon_2\right|) \\ \qquad \text{for every } \sigma_1,\sigma_2,\varepsilon_1,\varepsilon_2\in\mathcal{S}_d \text{ a.e. in } \Omega; \\ (b)\ \ \mathcal{G}\left(\cdot,\sigma,\varepsilon\right) \ \text{ is a measurable function with respect to the Lebesgue} \\ \qquad \text{measure on } \Omega \text{ for every } \varepsilon,\sigma\in\mathcal{S}_d; \\ (c)\ \ \mathcal{G}\left(\cdot,0,0\right)\in\mathcal{H}. \end{cases} \tag{3.5}$$

$$\begin{cases} \text{The friction function p : } \Gamma_3\times\mathbb{R}_+\longrightarrow\mathbb{R}_+ \text{ verifies} \\ (a)\ \ \text{there exists } M>0 \text{ such that :} \\ \qquad \left|p\left(x,r_1\right)-p\left(x,r_2\right)\right|\leq M\left|r_{1-}r_2\right| \\ \qquad \text{for every } r_1,r_2\in\mathbb{R}_+, \quad \text{a.e. } x\in\Gamma_3; \\ (b)\ \ x\mapsto p\left(x,r\right) \ \text{ is measurable on } \Gamma_3,\ \text{for every } r\in\mathbb{R}_+; \\ (c)\ \ p\left(x,0\right)=0, \quad \text{a.e. } x\in\Gamma_3. \end{cases} \tag{3.6}$$

We also suppose that the body forces and surface tractions have the regularity

$$f_0\in W^{1,\infty}(0,T;L^2(\Omega)^d), \quad f_2\in W^{1,\infty}(0,T;L^2(\Gamma_2)^d), \tag{3.7}$$

and we define the function $f:[0,T]\to V$ by

$$(f(t),v)_V=\int_{\Omega} f_0(t)\cdot v\,dx+\int_{\Gamma_2} f_2(t)\cdot v\,da, \tag{3.8}$$

for all $u, v \in V$ and $t \in [0, T]$, and we note that the condition (3.7) implies that

$$f \in W^{1,\infty}(0, T; V). \tag{3.9}$$

For the Signorini problem we use the convex subset of admissible displacements given by

$$U_{ad} = \{v \in H_1 \mid \ v = 0 \text{ on } \Gamma_1\,,\ v_\nu \leq 0 \text{ on } \Gamma_3\} \tag{3.10}$$

The adhesion coefficients γ_ν, γ_τ and the limit bound ϵ_a satisfy the conditions

$$\gamma_\nu,\ \gamma_\tau \in L^\infty(\Gamma_3), \quad \epsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu,\, \gamma_\tau,\, \epsilon_a \geq 0 \quad \text{a.e. on } \Gamma_3 \tag{3.11}$$

while the friction coefficient μ is such that

$$\mu \in L^\infty(\Gamma_3), \quad \mu(x) \geq 0 \quad \text{a.e. on } \Gamma_3. \tag{3.12}$$

Finally, we assume that the initial data verifies

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1 \ \text{ a.e. on } \Gamma_3. \tag{3.13}$$

We define the adhesion functional $j_{ad} : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ by

$$j_{ad}(\beta, u, v) = \int_{\Gamma_3} \big(-\gamma_\nu \beta^2 R_\nu(u_\nu) v_\nu + \gamma_\tau \beta^2 R_\tau(u_\tau) \cdot v_\tau \big)\, da, \tag{3.14}$$

and the friction functional $j_{fr} : L^\infty(\Gamma_3) \times \mathcal{H}_1 \times V \times V \to \mathbb{R}$ by

$$j_{fr}(\beta, \sigma, u, v) = \int_{\Gamma_3} \mu p(|R(\sigma_\nu) - \gamma_\nu \beta^2 R_\nu(u_\nu)|) \cdot |v_\tau|\, da, \tag{3.15}$$

The initial conditions u_0, σ_0 and β_0 satisfy

$$u_0 \in U_{ad}, \quad \sigma_0 \in \mathcal{H}_1, \quad \beta_0 \in L^2(\Gamma_3) \cap \mathcal{Q}, \tag{3.16}$$

and

$$(\sigma_0, \varepsilon(v) - \varepsilon(u_0))_{\mathcal{H}} + j_{ad}(\beta_0, \sigma_0, v - u_0) + j_{fr}(\beta_0, \sigma_0, \xi_0, v) - j_{fr}(\beta_0, \sigma_0, \xi_0, u_0) \geq$$
$$\geq (f_0, v - u_0)_V + (f_2, v - u_0)_{L^2(\Gamma_2)^d} \quad \forall\, v \in U_{ad}. \tag{3.17}$$

Let us remark that assumption (3.16) and (3.17) involve regularity conditions of the initial data u_0, σ_0 and β_0 and a compatibility condition between u_0, σ_0, β_0, f_0 and f_2.

By a standard procedure based on Green's formula combined with (2.2)-(2.4) and (3.8), we can derive the following variational formulation of problem $\mathcal{P}$, in terms of displacement, stress and bonding fields.

Proof.[**Problem** $\mathcal{P}^V$] *Find a displacement field* $u : [0,T] \to V$, *a stress field* $\sigma : [0,T] \to H_1$ *and a bonding field* $\beta : [0,T] \to L^2(\Gamma_3)$ *such that*

$$\dot{\sigma} = \mathcal{E}\varepsilon\,(\dot{u}) + \mathcal{G}(\sigma, \varepsilon\,(u)) \qquad \text{in } \ \Omega \times (0,T), \tag{3.18}$$

$$\begin{gathered} u(t) \in U_{ad}, \quad (\sigma(t), \varepsilon(v) - \varepsilon(u(t)))_{\mathcal{H}} + j_{ad}(\beta(t), u(t), v - u(t)) + \\ j_{fr}(\beta(t), \sigma(t), u(t), v) - j_{fr}(\beta(t), \sigma(t), u(t), u(t)) \geq (f(t), v - u(t))_V \\ \forall\, v \in U_{ad}, \ t \in [0,T], \end{gathered} \tag{3.19}$$

$$\dot{\beta}(t) = -\Big(\beta(t)\big(\gamma_\nu R_\nu(u_\nu(t))^2 + \gamma_\tau \|R_\tau(u_\tau(t))\|^2\big) - \epsilon_a\Big)_+ \text{ a.e. on } t \in (0,T),$$

$$u(0) = u_0, \qquad \sigma(0) = \sigma_0, \qquad \beta(0) = \beta_0. \tag{3.20}$$

□

In the rest of this section, we derive some inequalities involving the functionals j_{ad}, and j_{fr} which will be used in the following sections. Below in this section β, β_1, β_2 denote elements of $L^2(\Gamma_3)$ such that $0 \leq \beta$, β_1, $\beta_2 \leq 1$ a.e. on Γ_3, u_1, u_2, v_1, v_2, u and v represent elements of V; σ, σ_1, σ_2 denote elements of $\mathcal{H}_1$ and c is a generic positive constants which may depend on Ω, Γ_1, Γ_3, p, γ_ν, γ_τ and L, whose value may change from place to place. For the sake of simplicity, we suppress in what follows the explicit dependence on various functions on $\boldsymbol{x} \in \Omega \cup \Gamma_3$.

First, we remark that the j_{ad} is linear with respect to the last argument and therefore

$$j_{ad}(\beta, u, -v) = -j_{ad}(\beta, u, v). \tag{3.21}$$

Next, using (3.14) and the inequalities $|R_\nu(u_{1\nu})| \leq L$, $\|R_\tau(u_\tau)\| \leq L$, $|\beta_1| \leq 1$, $|\beta_2| \leq 1$, for the previous inequality, we deduce that

$$j_{ad}(\beta_1, u_1, u_2 - u_1) + j_{ad}(\beta_2, u_2, u_1 - u_2) \leq c \int_{\Gamma_3} |\beta_1 - \beta_2|\, \|u_1 - u_2\|\, da,$$

then, we combine this inequality with (3.3), to obtain

$$j_{ad}(\beta_1, u_1, u_2 - u_1) + j_{ad}(\beta_2, u_2, u_1 - u_2) \leq c\,\|\beta_1 - \beta_2\|_{L^2(\Gamma_3)}\|u_1 - u_2\|_V. \quad (3.22)$$

Next, we choose $\beta_1 = \beta_2 = \beta$ in (3.22) to find

$$j_{ad}(\beta, u_1, u_2 - u_1) + j_{ad}(\beta, u_2, u_1 - u_2) \leq 0. \quad (3.23)$$

Similar manipulations, based on the Lipschitz continuity of operators R_ν, R_τ show that

$$|j_{ad}(\beta, u_1, v) - j_{ad}(\beta, u_2, v)| \leq c\,\|u_1 - u_2\|_V\|v\|_V. \quad (3.24)$$

Also, we take $u_1 = v$ and $u_2 = 0$ in (3.23), then we use the equalities $R_\nu(0) = 0$, $R_\tau(0) = 0$ and (3.22) to obtain

$$j_{ad}(\beta, v, v) \geq 0. \quad (3.25)$$

Next, we use (3.15), (3.6)(a), keeping in mind (3.3), propriety of R and the inequalities $|R_\nu(u_{1\nu})| \leq L$, $\|R_\tau(u_\tau)\| \leq L$, $|\beta_1| \leq 1$, $|\beta_2| \leq 1$ we obtain

$$j_{fr}(\beta_1, \sigma_1, u_1, u_2) - j_{fr}(\beta_1, \sigma_1, u_1, u_1) + j_{fr}(\beta_2, \sigma_2, u_2, u_1) - j_{fr}(\beta_2, \sigma_2, u_2, u_2) \leq$$
$$\leq c_0^2 M\ \|\mu\|_{L^\infty(\Gamma_3)}(\|\beta_2 - \beta_1\|_{L^2(\Gamma_3)} + \|\sigma_{2-}\sigma_1||_{\mathcal{H}_1})||u_2 - u_1\|_V. \quad (3.26)$$

Now, by using (3.6)(a) and (3.12), it follows that the integral in (3.15) is well defined. Moreover, we have

$$j_{fr}(\beta, \sigma, u, v) \leq c_0^2 M\|\mu\|_{L^\infty(\Gamma_3)}\|(\|\sigma||_{\mathcal{H}_1} + \|\beta\|_{L^2(\Gamma_3)})||u\|_V\|v\|_V. \quad (3.27)$$

The inequalities (3.22)-(3.27) combined with equalities (3.21) will be used in various places in the rest of the paper.

4. **Existence and uniqueness result**

Our main result which states the unique solvability of Problem $\mathcal{P}^V$, is the following.

Theorem 4.1. *Assume that assumptions* (3.4)-(3.7) *and* (3.11)-(3.13) *hold. Then, there exists* $\mu_0 > 0$ *depending only on* Ω, Γ_1, Γ_3, $\mathcal{E}$ *and* p *such that, if* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, *then Problem* $\mathcal{P}^V$ *has a unique solution* (u, σ, β). *Moreover, the solution satisfies*

$$u \in W^{1,\infty}(0,T;V), \tag{4.1}$$

$$\sigma \in W^{1,\infty}(0,T;\mathcal{H}_1), \tag{4.2}$$

$$\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Q}. \tag{4.3}$$

A triple of functions $(u,\ \sigma,\ \beta)$ which satisfies (2.1), and (3.19)-(3.20) is called a weak solution of the frictional adhesive contact Problem $\mathcal{P}$. We conclude by Theorem 4.1. that, under the assumptions (3.4)-(3.7) and (3.11)-(3.13), if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, then there exists a unique weak solution of Problem $\mathcal{P}$ which verifies (4.1)-(4.3), that we present in what follows.

The proof of the Theorem 4.1 will be carried out in several steps. It based on fixed-point arguments. To this end, we assume in the following that (3.4)-(3.7)and (3.11)-(3.13) hold; below, c is a generic positive constants which may depend on Ω, Γ_1, Γ_3, $\mathcal{E}$ *and* p, γ_ν, γ_τ and L, whose value may change from place to place. For the sake of simplicity, we suppress in what follows the explicit dependence on various functions on $\boldsymbol{x} \in \Omega \cup \Gamma_3$.

For each $\eta = (\eta_1, \eta_2) \in L^\infty(0,T;\mathcal{H}\times L^2(\Gamma_3))$ we introduce the function $z_\eta = (z_\eta^1, z_\eta^2) \in W^{1,\infty}(0,T;\mathcal{H} \times L^2(\Gamma_3))$ defined by

$$z_\eta(t) = \int_0^t \eta(s)ds + z_0 \qquad \forall t \in [0,T], \tag{4.4}$$

where

$$z_0 = (\sigma_0 - \mathcal{E}\varepsilon\,(u_0)\,, \beta_0). \tag{4.5}$$

In the first step, we consider the following variational problem.

Proof.[Problem $\mathcal{P}^\eta$] *Find a displacement field* u_η : $[0,T]\to V$, *a stress field* σ_η : $[0,T] \to \mathcal{H}_1$ *such that*

$$\sigma_\eta\,(t) = \mathcal{E}\varepsilon(u_\eta\,(t)) + z_\eta^1\,(t) \tag{4.6}$$

$$u_\eta(t) \in U_{ad}, \quad (\sigma_\eta(t), \varepsilon(v) - \varepsilon(u_\eta(t)))_{\mathcal{H}} + j_{ad}(z_\eta^2(t), u_\eta(t), v - u_\eta(t)) +$$
$$+ j_{fr}(z_\eta^2(t), \sigma_\eta, u_\eta(t), v) - j_{fr}(z_\eta^2(t), \sigma_\eta, u_\eta(t), u_\eta(t)) \geq \tag{4.7}$$
$$\geq (f(t), v - u_\eta(t))_V \quad \forall v \in U_{ad}.$$

□

We have the following result.

Lemma 4.2. *There exists* $\mu_0 > 0$ *which depends on* Ω, Γ_1, Γ_3, $\mathcal{E}$ *and* p *such that, if* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, *then, Problem* $\mathcal{P}^\eta$ *has a unique solution having the regularity* $u_\eta \in W^{1,\infty}(0,T,V)$, $\sigma_\eta(t) \in W^{1,\infty}(0,T;\mathcal{H}_1)$. *Moreover,*

$$u_\eta(0) = u_0, \qquad \sigma_\eta(0) = \sigma_0 \tag{4.8}$$

Proof. *Using Riez's representation theorem we may define the operator* $A_\eta(t) : V \to V$ *and the element* $f_\eta(t) \in V$ *by*

$$(A_\eta(t)u_\eta(t), v)_V = (\mathcal{E}\varepsilon(u_\eta(t)), \varepsilon(v))_{\mathcal{H}} + j_{ad}(z_\eta^2(t), u_\eta(t), v)$$
$$\forall\, t \in [0,T], \quad \forall\, w,\ v \in U_{ad}, \tag{4.9}$$

$$(f_\eta(t), v)_V = (f(t), v)_V - (z_\eta^1(t), \varepsilon(v))_{\mathcal{H}}$$
$$\forall\, t \in [0,T], \quad \forall v \in U_{ad}, \tag{4.10}$$

Let $t \in [0,T]$. *We use the assumption (3.4), the equalities (3.21) and the inequalities (3.23) and (3.24) to prove that* $A_\eta(t)$ *is a strongly monotone Lipschitz continuous operator on V. Moreover,by (3.10) we have that* U_{ad} *is a closed convex non-empty set of* V. *Using (3.15), we can easily check that* $j_{fr}(z_\eta^2(t), \sigma_\eta, u_\eta(t), .)$ *is a continuous seminorm on* V *and moreover, it satisfies (3.26) and(3.27). Then by an existence and uniqueness result on elliptic quasivariational inequalities, drabla it follows that there exists a unique solution* $u_\eta(t)$ *such that*

$$u_\eta(t) \in U_{ad}.$$

$$(A_\eta(t)u_\eta(t),v)_V+j_{fr}(,z_\eta^2(t),\sigma_\eta,u_\eta(t),v)-j_{fr}(z_\eta^2(t),\sigma_\eta,u_\eta(t),u_\eta(t))\geq$$
$$\geq(f_\eta(t),v-u_\eta(t))_V\qquad\forall\, v\in U_{ad}.$$

Taking $\sigma_\eta\,(t))$, defined by (4.6) and using (4.4), we deduce that $\sigma_\eta\in W^{1,\infty}(0,T;\mathcal{H})$ and (4.7). Let us remark, that for $v=u_{g\eta}\,(t)\mp\varphi\ \ \forall\varphi\in D\,(\Omega)^d$, it comes from (4.6) and Green's formula

$$\mathrm{Div}\sigma_\eta(t)+f_0(t)=0. \tag{4.11}$$

Keeping in mind that $f_0\in W^{1,\infty}(0,T;L^2\,(\Omega)^d)$ it follows that $\sigma_\eta\in W^{1,\infty}(0,T;\mathcal{H}_1)$. Therefore, the existence and uniqueness of $(u_\eta(t),\sigma_\eta\,(t))\in V\times\mathcal{H}_1$ solution of problem $\mathcal{P}^\eta$ is established under smallness assumption. The initial conditions (4.8) follows from (3.17), (4.4) and (4.5) and the uniqueness of the problem for $t=0$.

Let now $t_1,\ t_2\in[0,T]$, Using (3.4), (3.1) and (4.4)we obtain

$$\|u_\eta(t_1)-u_\eta(t_2)\|_V\leq c(\|f(t_1)-f(t_2)\|_V+||z_\eta(t_1)-z_\eta(t_2)\|_{\mathcal{H}\times L^2(\Gamma_3)}+$$
$$+\|\sigma_\eta(t_1)-\sigma_\eta(t_2)\|), \tag{4.12}$$

and from (4.6), (4.11) and (4.12), it result that

$$\|\sigma_\eta(t_1)-\sigma_\eta(t_2)\|_V\leq c(\|f(t_1)-f(t_2)\|_V+||z_\eta^1(t_1)-z_\eta^1(t_2)\|_{\mathcal{H}}). \tag{4.13}$$

Recall that $f\in W^{1,\infty}(0,T;V)$, $z_\eta=(z_\eta^1,z_\eta^2)\in W^{1,\infty}(0,T;\mathcal{H}\times L^2(\Gamma_3))$, it follows from (4.12) and (4.13) that $u_\eta\in W^{1,\infty}(0,T;V)$ and $\sigma_\eta\in W^{1,\infty}(0,T;\mathcal{H}_1)$. □

We denote by $\beta_\eta\in W^{1,\infty}(0,T;L^2(\Gamma_3))$ the function defined by

$$\beta_\eta=z_\eta^2, \tag{4.14}$$

and consider the mapping $F:\ [0,T]\times L^2(\Gamma_3)\to L^2(\Gamma_3)$ defined by

$$F(t,\beta_\eta)=-(\beta_\eta(t)(\gamma_\nu R_\nu((u_\eta)_\gamma(t))^2+\gamma_\tau\|R_\tau((u_\eta)_\tau(t))\|^2)-\epsilon_a)_+, \tag{4.15}$$

for all $t\in[0,T]$ and $\beta_\eta\in W^{1,\infty}(0,T;L^2(\Gamma_3))$

Using the assumptions (3.4), (3.5), (4.4), and (4.5), we may consider the operator

$$\Lambda\eta:L^\infty(0,T;\mathcal{H}\times L^2(\Gamma_3))\longrightarrow L^\infty(0,T;\mathcal{H}\times L^2(\Gamma_3))$$

define by

$$\Lambda\eta = (\mathcal{G}(\sigma_\eta, \varepsilon(u_\eta)), F(t, \beta_\eta)) \quad \forall \eta \in L^\infty(0,T; \mathcal{H} \times L^2(\Gamma_3)) \tag{4.16}$$

where (σ_η, u_η) is the solution of the variational problem $\mathcal{P}^\eta$.

In the last step, we will prove the following result.

Lemma 4.3. *There exists a unique element* $\eta^* = (\eta_1^*, \eta_2^*)$ *such that* $\Lambda\eta^* = \eta^*$ *and* $\eta^* \in L^\infty(0,T; \mathcal{H} \times L^2(\Gamma_3))$

Proof. Let $\eta_1 = (\eta_1^1, \eta_1^2)$ and $\eta_2 = (\eta_2^1, \eta_2^2) \in L^\infty(0,T; \mathcal{H} \times L^2(\Gamma_3))$ and let $t \in [0,T]$. We use similar arguments to those used in the proof of (4.10) to deduce that

$$\|u_{\eta_1} - u_{\eta_2}\|_V \le c(||z_{\eta_1} - z_{\eta_2}\|_{\mathcal{H}\times L^2(\Gamma_3)} + \|\sigma_{\eta_1} - \sigma_{\eta_2}\|_{\mathcal{H}}), \tag{4.17}$$

and from (3.4), (3.5) and (4.6), we obtain that

$$\|\sigma_{\eta_1} - \sigma_{\eta_2}\|_{\mathcal{H}} \le c(\|u_{\eta_1} - u_{\eta_2}\|_V + ||z_{\eta_1} - z_{\eta_2}\|_{\mathcal{H}\times L^2(\Gamma_3)}), \tag{4.18}$$

from (4.17) and (4.18), it results that

$$\|u_{\eta_1} - u_{\eta_2}\|_V \le c||z_{\eta_1} - z_{\eta_2}\|_{\mathcal{H}\times L^2(\Gamma_3)}. \tag{4.19}$$

On the other hand, it follows from (4.15) that

$$||F_{\eta_2}(t, \beta_{\eta_2}) - F_{\eta_1}(t, \beta_{\eta_1})||_{L^2(\Gamma_3)} \le$$

$$\le c\,\|\beta_{\eta_1}(t) R_\nu(u_{\eta_1\nu}(t))^2 - \beta_{\eta_2}(t) R_\nu(u_{\eta_2\nu}(t))^2\|_{L^2(\Gamma_3)} +$$

$$+ \|\beta_{\eta_1}(t)\|R_\tau(u_{\eta_1\tau}(t))\|^2 - \beta_{\eta_2}(t)\|R_\tau(u_{\eta_2\tau}(t))\|^2\|_{L^2(\Gamma_3)}.$$

Using the definition of R_ν and R_τ and writing $\beta_{\eta_1} = \beta_{\eta_1} - \beta_{\eta_2} + \beta_{\eta_2}$, we get

$$||F(t, \beta_{\eta_2}) - F(t, \beta_{\eta_1})||_{L^2(\Gamma_3)} \le c\,\|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|_{L^2(\Gamma_3)} + c\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_{L^2(\Gamma_3)}.$$

We now use (4.17), (4.18), (4.14) and (4.5) to deduce

$$\|\Lambda\eta_2(t) - \Lambda\eta_1(t)\|_{\mathcal{H}\times L^2(\Gamma_3)} \le c\|z_{\eta_2}(t) - z_{\eta_1}(t)\|_{\mathcal{H}\times L^2(\Gamma_3)}.$$

From (3.5), (4.4), (4.16) and the last inequalities, it result that

$$\|\Lambda\eta_2(t) - \Lambda\eta_1(t)\|_{\mathcal{H}\times L^2(\Gamma_3)} \le c\int_0^t \|\eta_2(t) - \eta_1(t)\|_{\mathcal{H}\times L^2(\Gamma_3)} ds. \tag{4.20}$$

Denoting now by Λ^p the power of the operator Λ, (4.20) implies by recurrence that

$$\|\Lambda\eta_2(t)-\Lambda\eta_1(t)\|_{\mathcal{H}\times L^2(\Gamma_3)} \leq c\int_0^t\int_0^s \qquad \int_0^q \|\eta_2(t)-\eta_1(t)\|_{\mathcal{H}\times L^2(\Gamma_3)}dr \quad ds,$$

for all $t\in[0,T]$ and $p\in N$. Hence, it follows that

$$\|\Lambda^p\eta_2-\Lambda^p\eta_1\|_{L^\infty(0,T,\mathcal{H}\times L^2(\Gamma_3))} \leq \frac{c^nT^n}{n!}\|\eta_2-\eta_1\|_{L^\infty(0,T,\mathcal{H}\times L^2(\Gamma_3))}, \quad \forall p\in N. \quad (4.21)$$

and since $\lim\limits_{p\to\infty}\dfrac{c^pT^p}{p!} = 0$, inequality (4.21) shows that for p sufficiently large $\Lambda^p : L^\infty(0,T;\mathcal{H}\times L^2(\Gamma_3))\to L^\infty(0,T;\mathcal{H}\times L^2(\Gamma_3))$ is a contraction. Then, we conclude by using the Banach fixed point theorem that Λ has a unique fixed point $\eta^*\in L^\infty(0,T;\mathcal{H}\times L^2(\Gamma_3))$ such that $\Lambda\eta^*=\eta^*$. Hence, from (4.16) it results for all $t\in[0,T]$,

$$\eta^*(t)=(\eta^{*1}(t),\eta^{*2}(t))=(\mathcal{G}(\sigma_{\eta^*}(t),\varepsilon\,(u_{\eta^*}(t))),(F(t,\beta_{\eta^*}(t)))) \qquad (4.22)$$

□

Now, we have all the ingredients to provide the proof of Theorem 4.1.

Proof.[Proof of Theorem 4.1.] *Existence.* Let $\eta^*\in L^\infty(0,T;\mathcal{H}\times L^2(\Gamma_3))$ be the fixed point of Λ and let $(u_{\eta*},\sigma_{\eta*})\in W^{1,\infty}(0,T;\mathcal{H}_1\times V)$ be the solution of Problem $\mathcal{P}^{\eta^*}$. Let also $\beta_{\eta^*}\in W^{1,\infty}(0,T;L^2(\Gamma_3))$ be the solution of Problem $\mathcal{P}^\eta$ for $\eta=\eta^*$. We shall prove that $(u_{\eta^*},\sigma_{\eta^*},\beta_{\eta^*})$ is a unique solution of Problem $\mathcal{P}^V$.

The regularity expressed in (4.1) follow from Lemma 4.1, Lemma 4.3 and the fixed point of operators Λ.

The initial conditions (3.20) follow from (4.5), (4.14) and (4.8) for $\eta=\eta^*$. Moreover, the equalities (3.18) and (3.20) follow from (4.4), (4.6), Lemma 4.4, (4.12) and (4.16) for $\eta=\eta^*$ since

$$\begin{aligned}
\dot{\sigma}_{\eta^*}(t) &= \mathcal{E}\varepsilon(u_{\eta^*}(t))+\dot{z}^1_{\eta^*}(t) && a.e.\ t\in(0,T)\\
z^1_{\eta^*}(t) &= \eta^{*1}(t)=\mathcal{G}(\sigma_{\eta^*}(t),\varepsilon\,(u_{\eta^*}(t))) && a.e.\ t\in(0,T)\\
\dot{\beta}_{\eta^*}(t) &= \dot{z}^2_{\eta^*}(t)=\eta^{*2}(t)=F(t,\beta_{\eta^*}(t)) && a.e.\ t\in(0,T)
\end{aligned}$$

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operators Λ defined by (4.16). Indeed, let $(u, \sigma, \beta) \in W^{1,\infty}(0, T; V \times \mathcal{H}_1 \times L^2(\Gamma_3))$ be another solution of Problem $\mathcal{P}^V$.

We denote by $\eta \in L^\infty(0, T; \mathcal{H} \times L^2(\Gamma_3))$ the function defined by

$$\eta(t) = (\mathcal{G}(\sigma, \varepsilon(u)), F(t, \beta)), \qquad \forall t \in [0, T], \tag{4.23}$$

and let $z_\eta \in W^{1,\infty}(0, T; \mathcal{H} \times L^2(\Gamma_3))$ be the function given by (4.4) and (4.5). It results that (u, σ) is a solution to Problem $\mathcal{P}_\eta$ and since by Lemma 4.1, this problem has a unique solution denoted (u_η, σ_η), we obtain

$$u = u_\eta \text{ and } \sigma = \sigma_\eta. \tag{4.24}$$

Then, we replace $(u, \sigma) = (u_\eta, \sigma_\eta) = (u_{\eta*}, \sigma_{\eta*})$ in (3.20) and use the initial condition (3.20) to see that β is a solution to Problem $\mathcal{P}_\eta$. Since by Lemma 4.2, this last problem has a unique solution denoted β_η, we find

$$\beta = \beta_\eta. \tag{4.25}$$

We use now (4.16) and (4.25) to obtain that $\eta = (\mathcal{G}(\sigma_\eta, \varepsilon(u_\eta)), F_\eta(t, \beta_\eta))$, i.e. η is a fixed point of the operator Λ. It follows now from Lemma 4.3 that

$$\eta = \eta^*. \tag{4.26}$$

The uniqueness part of the theorem is now a consequence of (4.24), (4.25) and (4.26). □

References

[1] Andrews, K.T., Chapman, L., Fernández, J.R., Fisackerly, M., Shillor, M., Vanerian, L., Van Houten, T., *A membrane in adhesive contact*, SIAM J. Appl. Math., **64**(2003), 152-169.

[2] Andrews, K.T., Shillor, M., *Dynamic adhesive contact of a membrane*, Advances in Mathematical Sciences and Applications, **13**(2003), no. 1, 343-356.

[3] Chau, O., Fernández, J.R., Shillor, M., Sofonea, M., *Variational and numerical analysis of a quasistatic viscoelastic contact problem with adhesion*, J. Comput. Appl. Math., **159**(2003), 431-465.

[4] Chau, O., Shillor, M., Sofonea, M., *Dynamic frictionless contact with adhesion*, Journal of Applied Mathematics and Physics (ZAMP), **55**(2004), no. 1, 32-47.

[5] Cocu, M., *On A Model Coupling Adhesion and Friction: Thermodynamics Basis and Mathematical Analysis*, Proceed. of the fifth. Inter. Seminar. On Geometry, Continua and Microstructures, Romania, 2001, 37-52.

[6] Curnier, A., Talon, C., *A model of adhesion added to contact with friction*, in Contact Mechanics, JAC Martins and MDP Monteiro Marques (Eds.), Kluwer, Dordrecht, 2002, 161-168.

[7] Drabla, S., *Analyse Variationnelle de Quelques Problèmes aux Limites en Elasticité et en Viscoplasticité*, Thèse de Docorat d'Etat, Univ, Ferhat Abbas, Sétif, 1999.

[8] Drabla, S., Sofonea, M., *Analysis of a Signorini problem with friction*, IMA journal of applied mathematics, **63**(1999), 113-130.

[9] Jianu, L., Shillor, M., Sofonea, M., *A Viscoelastic Frictionless Contact problem with Adhesion*, Appl. Anal..

[10] Frémond, M., *Equilibre des structures qui adhèrent à leur support*, C. R. Acad. Sci. Paris, Série II, **295**(1982), 913-916.

[11] Frémond, M., *Adhérence des Solides*, Jounal. Mécanique Théorique et Appliquée, **6**(1987), 383-407.

[12] Frémond, M., *Non-Smooth Thermomechanics*, Springer, Berlin, 2002.

[13] Han, W., Kuttler, K.L., Shillor, M., Sofonea, M., *Elastic beam in adhesive contact*, International Journal of Solids and Structures, **39**(2002), no. 5, 1145-1164.

[14] Raous, M., Cangémi, L., Cocu, M., *A consistent model coupling adhesion, friction, and unilateral contact*, Computer Methods in Applied Mechanics and Engineering, **177**(1999), no. 3-4, 383-399.

[15] Rojek, J., Telega, J.J., *Contact problems with friction, adhesion and wear in orthopedic biomechanics. I: general developments,* Journal of Theoretical and Applied Mechanics, **39**(2001), no. 3, 655-677.

[16] Rojek, J., Telega, J.J., Stupkiewicz, S., *Contact problems with friction, adhesion and wear in orthopedic biomechanics. II: numerical implementation and application to implanted knee joints*, Journal of Theoretical and Applied Mechanics, **39**(2001), 679-706.

[17] Shillor, M., Sofonea, M., Telega, J.J., *Models and Analysis of Quasistatic Contact. Variational Methods*, Lect. Notes Phys., vol. 655, Springer, Berlin, 2004.

[18] Sofonea, M., Han, W., Shillor, M., *Analysis and Approximation of Contact Problems with Adhesion or Damage*, Pure and Applied Mathematics (Boca Raton), vol. 276, Chapman & Hall/CRC Press, Florida, 2006.

Département de Matématiques, Faculté des Sciences
Université Farhat Abbas de Sétif
Cité Maabouda, 19000 Sétif, Algérie
E-mail address: drabla_s@yahoo.fr

Département de Matématiques, Faculté des Sciences
Université Farhat Abbas de Sétif
Cité Maabouda, 19000 Sétif, Algérie
E-mail address: zhorlargor@yahoo.fr

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 2, June 2009

A CONSTRUCTION OF ADMISSIBLE STRATEGIES FOR AMERICAN OPTIONS ASSOCIATED WITH PIECEWISE CONTINUOUS PROCESSES

BOGDAN IFTIMIE AND MARINELA MARINESCU

Abstract. We provide the construction of some admissible strategies in a "feedback shape" for American Options, and where the contingent claim depends on a nontrivial solution of some possibly degenerate elliptic inequation.

1. **Setting of the problem**

Let $W(t)$ be a standard m-dimensional Wiener process over a complete probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$, $\{\lambda(t); t \geq 0\}$ and $\{y(t); t \geq 0\}$ piecewise constant adapted processes of dimension n, respectively d defined on the same probability space. $\lambda(t)$ takes values in some subset S of $\mathbb{R}^n$.

We denote $\mu(t) = (y(t), \lambda(t))$, for $t \geq 0$ and

$$\mu(t, \omega) = \mu_k(\omega) = (y_k(\omega), \lambda_k(\omega)),\ t \in [t_k(\omega), t_{k+1}(\omega)),$$

where the sequence $\{t_k; k \geq 0\}$ is increasing and it's elements are positive random variables with $t_0 = 0$, $t_k \to \infty$, $\mathbb{P}a.s.$, as $k \to \infty$ and (y_k, λ_k) are multidimensional $\mathcal{F}_{t_k}$-measurable random variables. Then we may assume $S = \{\lambda_k; k \geq 1\}$.

We make the assumption that the process $W(t)$ and the sequence $\{(t_k, \mu_k); k \geq 1\}$ are mutually independent.

Received by the editors: 28.03.2008.

2000 *Mathematics Subject Classification.* 35B40, 35J70, 60G44, 91B28.

Key words and phrases. American options, admissible strategies, piecewise continuous process, degenerate elliptic equation.

The research of the authors was supported by the *ANCS* research contract Nr. 2-CEx06-11-18/2006.

Consider a small investor acting in a financial market on which is given a *riskless asset* (for instance a *bond*) whose price evolves in time as

$$dS_0(t) = rS_0(t)dt;\ S_0(0) = 1,\ t \geq 0, \tag{1}$$

implying that $S_0(t) = e^{rt}$ and d *risky assets* (that we call *stocks*), for which the vector $S(t, x)$ collecting the prices of the assets satisfies the SDE

$$\begin{cases} dS(t) & = g_0(S(t); \lambda(t))dt + \displaystyle\sum_{j=1}^{m} g_j(S(t); \lambda(t))dW_j(t),\ t \in [t_k, t_{k+1}), \\ S(t_k) & = S_{\text{-}}(t_k) + y_k,\ \text{for any } k \geq 1, \\ S(0) & = x. \end{cases} \tag{2}$$

where the vector fields

$$g_i(y; \lambda) = a_i(\lambda) + A_i(\lambda)y,\ i = 1, \ldots, m,\ \lambda \in S,\ y \in \mathbb{R}^d, \tag{3}$$

are assumed continuous and bounded with respect to λ. We denoted $S_{\text{-}}(t_k) = \lim\limits_{t \uparrow t_k} S(t)$. $x = (x_1, x_2, \ldots, x_d)$ and x_i represents the amount of money invested at the initial time $t = 0$ in the stock i, for $i = 1, \ldots, d$. x_i may be negative and this happens if the quantity $-x_i$ is borrowed at the interest rate r.

The unique solution of the system (2) is a piecewise continuous and $\{\mathcal{F}_t\}$-adapted process $\{S(t, x); t \geq 0\}$, such that at each jump time t_k, the jump $S(t_k, x) - S_{\text{-}}(t_k, x) = y_k$ occurs. The linear shape of $g_0(y; \lambda)$ is not required and we assume that $g_0(y; \lambda)$ is global Lipschitz continuous with respect to $y \in \mathbb{R}^d$.

A portofolio problem for an American Option with maturity T and its admissible strategies can be described by a value function of the following form

$$V(t, x) = e^{rt}\theta_0(t, x) + \theta(t, x) \cdot S(t, x),\ t \in [0, T],\ x \in \mathbb{R}^d, \tag{4}$$

where $\theta_0(t, x) \in \mathbb{R}, \theta(t, x) \in \mathbb{R}^d$ are some $\mathcal{F}_t^1$-adapted processes, for each fixed $x \in \mathbb{R}^d$ representing the amount of assets form the bond, respectively the quantities of stocks possessed by the investor.

We accept only self-financing portfolios, i.e. portfolios for which the differential of the value function is given by

$$dV(t,x) = \theta_0(t,x)de^{rt} + \theta(t,x) \cdot dS(t,x),\ t \in [0,T],$$

and this formula is understood in the integral sense, i.e.

$$\begin{aligned} V(t,x) &= V(t_k,x) + r\int_{t_k}^{t} \theta_0(s,x)e^{rs}ds + \int_{t_k}^{t} \theta(s,x) \cdot dS(s,x) \\ &= \theta_0(t_k,x)e^{rt_k} + \theta(0,x) \cdot x + r\int_{t_k}^{t} \theta_0(s,x)e^{rs}ds \\ &\quad + \int_{t_k}^{t} \theta(s,x) \cdot g_0(S(s,x);\lambda_k)ds \\ &\quad + \sum_{j=1}^{m} \int_{t_k}^{t} \theta(s,x) \cdot g_j(S(s,x);\lambda_k)dW_j(s),\ t \in [t_k \wedge T, t_{k+1} \wedge T). \end{aligned} \tag{5}$$

Instead of $[t_k \wedge T, t_{k+1} \wedge T)$, we shall simply write $[t_k, t_{k+1})$.

American options, in contrast with European options may be exercised at any moment of time between 0 and T, and thus the value function for an admissible strategy has to satisfy the constraint

$$V(t,x) \geq h_\gamma(t,x),\ 0 \leq t \leq T, \tag{6}$$

where $h_\gamma(t,x)$ is a positive $\mathcal{F}_t$-measurable random variable which stands for the value of the option at the moment t, i.e. the amount of money that the investor has to be able to provide at time t.

We consider here only functionals of the form

$$h_\gamma(t,x) := e^{\gamma t}\varphi_\gamma(S(t,x),\lambda(t)), \tag{7}$$

where γ is a negative constant and $\varphi_\gamma(y,\lambda) \in \mathcal{P}_2(y;\lambda)$, the set consisting of second degree polynomials with respect to the variables $(y_1,\ldots,y_d) = y$, whose coefficients are continuous and bounded functions of λ.

$\mathcal{P}_2(y) \subseteq \mathcal{P}_2(y;\lambda)$ stands for the set of constant coefficients polynomials.

We consider functions φ_γ of a particular form, which we shall make precise later on.

In order to find such strategies, we need to emphasize those conditions which allow to get them in a "feedback shape"

$$\theta(t,x) = e^{\gamma t}\nabla_y \varphi_\gamma(S(t,x);\lambda(t)),\ t\in[0,T],\ x\in\mathbb{R}^d \tag{8}$$

and

$$\theta_0(t_k,x) = e^{(\gamma-r)t_k}\varphi_\gamma(0,\lambda_k). \tag{9}$$

Remark 1. *For the sake of simplicity, when computing admissible strategies we shall include the "feedback shape"* (8) *and* (9) *in the definition of such strategies and we look for appropriate* (γ,φ_γ), $\varphi_\gamma\in\mathcal{P}_2(y;\lambda)$, *such that the equations* (5) *and* (6) *are fulfilled. We emphasize that this approach will lead us to an admissible couple* $(\theta_0(t,x),\theta(t,x))\in\mathbb{R}^{d+1}$, *provided*

(a) $\varphi_\gamma\in\mathcal{P}_2(y;\lambda)$ *is a convex function with respect to* $y\in\mathbb{R}^d$*;*

(b) (γ,φ_γ) *is a nontrivial solution of the following elliptic inequality*

$$\gamma\varphi_\gamma(y;\lambda) + \sum_{j=1}^m \frac{1}{2}\left\langle \partial_y^2\varphi_\gamma(y;\lambda)\, g_j(y;\lambda), g_j(y;\lambda)\right\rangle \le 0,\ (y,\lambda)\in\mathbb{R}^d\times S. \tag{10}$$

The "feedback shape" (8) agrees with the constraints (5) and (6), without involving the convexity property (a) and the analysis can be reduced to the elliptic inequality (10).

2. Auxiliary results

Set $L:\mathcal{P}_2(y;\lambda)\to\mathcal{P}_2(y;\lambda)$ the second order linear operator defined as

$$L(\psi)(y;\lambda) := \sum_{j=1}^m \frac{1}{2}\langle \partial_y^2\psi(y;\lambda)g_j(y;\lambda), g_j(y;\lambda)\rangle, \text{for } \psi\in\mathcal{P}_2(y;\lambda), \tag{11}$$

where we denoted $\partial_y^2\psi(y;\lambda)$ the Hessian matrix of ψ with respect to y.

Notice that L is a *possibly degenerate* elliptic operator.

Lemma 1. *Let* $f\in\mathcal{P}_2(y)$ *such that* $f(y)\ge 0$, $\forall y\in\mathbb{R}^d$ *and* γ *a nonzero constant such that the elliptic equation*

$$L(\psi)(y;\lambda) + \gamma\psi(y;\lambda) + f(y) = 0, \text{ for any } y\in\mathbb{R}^d,\ \lambda\in S \tag{12}$$

has a nontrivial solution $\varphi_\gamma\in\mathcal{P}_2(y;\lambda)$.

Then the following estimate holds true

$$h_\gamma(t,x) \le \exp(\gamma t_k)\varphi_\gamma(S(t_k,x);\lambda_k) + \int_{t_k}^{t} \exp(\gamma s)\nabla_y \varphi_\gamma(S(s,x);\lambda_k)\cdot dS(s,x), \quad (13)$$

for any $t \in [t_k, t_{k+1})$.

Proof. Apply the Itô formula for the process $h_\gamma(t,x) = e^{\gamma t}\varphi_\gamma(S(t,x),\lambda(t))$ on the interval $[t_k, t_{k+1})$ and get

$$\begin{aligned}
h_\gamma(t,x) &:= \exp(\gamma t_k)\varphi_\gamma(S(t_k,x);\lambda_k) + \int_{t_k}^{t} \exp(\gamma s)\nabla_y \varphi_\gamma(S(s,x);\lambda_k)\cdot g_0(S(s,x);\lambda_k)ds \\
&\quad + \int_{t_k}^{t} \exp(\gamma s)\left[\gamma\varphi_\gamma + f + L(\varphi_\gamma)(S(s,x);\lambda_k)\right] ds \\
&\quad + \sum_{j=1}^{m}\int_{t_k}^{t} \exp(\gamma s)\nabla_y \varphi_\gamma(S(s,x);\lambda_k)\cdot g_j(S(s,x);\lambda_k)dW_j(s) \\
&\quad - \int_{t_k}^{t} \exp(\gamma s) f(S(s,x))ds = \exp(\gamma t_k)\varphi_\gamma(S(t_k,x);\lambda_k) \\
&\quad + \int_{t_k}^{t} \exp(\gamma s)\nabla_y \varphi_\gamma(S(s,x);\lambda_k)\cdot g_0(S(s,x);\lambda_k)ds \\
&\quad + \sum_{j=1}^{m}\int_{t_k}^{t} \exp(\gamma s)\nabla_y \varphi_\gamma(S(s,x);\lambda_k)\cdot g_j(S(s,x);\lambda_k)dW_j(s) \\
&\quad - \int_{t_k}^{t} \exp(\gamma s) f(S(s,x))ds \\
&= \exp(\gamma t_k)\varphi_\gamma(S(t_k,x);\lambda_k) + \int_{t_k}^{t} \exp(\gamma s)\nabla_y \varphi_\gamma(S(s,x);\lambda_k)\cdot dS(s,x) \\
&\quad - \int_{t_k}^{t} \exp(\gamma s) f(S(s,x))ds,
\end{aligned} \quad (14)$$

for any $t \in [t_k, t_{k+1})$, by virtue of our assumptions.

This leads us to the conclusion of the lemma, since f takes positive values. □

Lemma 2. *Let the assumptions of the Lemma 1 be in force and, in addition, we make the hypothesis that a nontrivial solution* φ_γ *of the elliptic equation* (12) *is a convex function. Define*

$$\theta(t,x) := e^{\gamma t}\nabla_y \varphi_\gamma(S(t,x);\lambda(t)),\ 0 \le t \le T,\ x \in \mathbb{R}^d \quad (15)$$

and let $\{\theta_0(t,x); t \in [0,T]\}$ *be the piecewise continuous process satisfying the integral equation* (5)*, with*

$$\theta_0(t_k, x) := e^{(\gamma - r)t_k}\varphi_\gamma(0; \lambda_k). \tag{16}$$

Moreover, we assume that

$$\theta_0(t,x) \geq 0,\ \forall t \in [0,T],\ x \in \mathbb{R}^n. \tag{17}$$

Then $(\theta_0(t,x), \theta(t,x)) \in \mathbb{R}^{d+1}$ *is an admissible strategy (see the formulas* (5) *and* (6)*) satisfying the "feedback shape"* (8) *and* (9)*.*

Proof. The value function V considered at the time t_k may be written as

$$V(t_k, x) = \theta_0(t_k, x)e^{rt_k} + e^{\gamma t_k}\nabla_y\varphi_\gamma(S(t_k, x); \lambda_k) \cdot S(t_k, x)$$

and we require that

$$V(t_k, x) \geq \exp(\gamma t_k)\varphi_\gamma(S(t_k, x); \lambda_k), \tag{18}$$

where we used the choice (15) for $\theta(t,x)$.

The equation (18) is equivalent with

$$\theta_0(t_k, x)e^{rt_k} + e^{\gamma t_k}\nabla_y\varphi_\gamma(S(t_k, x); \lambda_k) \cdot S(t_k, x) \geq e^{\gamma t_k}\varphi_\gamma(S(t_k, x), \lambda_k). \tag{19}$$

Since φ_γ is a convex function, its gradient $\partial_y\varphi_\gamma(y; \lambda)$ satisfies

$$\langle \nabla_y\varphi_\gamma(y_2; \lambda) - \nabla_y\varphi_\gamma(y_1; \lambda),\ y_2 - y_1\rangle \geq 0, \text{ for any } y_1, y_2 \in \mathbb{R}^d \text{ and } \lambda \in S. \tag{20}$$

and thus, if $\theta_0(t_k, x)$ is defined as in (16), we easily get the estimate (19) fulfilled, via the Lagrange Mean Value Theorem.

$\theta_0(t,x)$ is finally obtained as the unique solution of the integral equation

$$\begin{aligned} V(t,x) &= e^{rt}\theta_0(t,x) + e^{\gamma t}\nabla_y\varphi_\gamma(S(t,x); \lambda(t)) \cdot S(t,x) \\ &= V(t_k, x) + r\int_{t_k}^{t}\theta_0(s,x)e^{rs}ds + \int_{t_k}^{t} e^{\gamma t}\nabla_y\varphi_\gamma(S(t,x); \lambda(t)) \cdot dS(s,x), \end{aligned} \tag{21}$$

for $t \in [t_k, t_{k+1})$.

Let t be arbitrary chosen in some interval $[t_k, t_{k+1})$. Then

$$\begin{aligned}
V(t,x) &= e^{\gamma t_k}\varphi_\gamma(0;\lambda_k) + e^{\gamma t_k}\nabla_y\varphi_\gamma(S(t_k,x);\lambda_k)\cdot S(t_k,x) + r\int_{t_k}^{t}\theta_0(s,x)e^{rs}ds \\
&\quad + \int_{t_k}^{t} e^{\gamma s}\nabla_y\varphi_\gamma(S(s,x);\lambda(s))\cdot dS(s,x) \\
&\geq e^{\gamma t_k}\varphi_\gamma(0;\lambda_k) + e^{\gamma t_k}\nabla_y\varphi_\gamma(S(t_k,x);\lambda_k)\cdot S(t_k,x) \\
&\quad + \int_{t_k}^{t} e^{\gamma s}\nabla_y\varphi_\gamma(S(s,x);\lambda(s))\cdot dS(s,x) \\
&\geq e^{\gamma t_k}\varphi_\gamma(S(t_k,x);\lambda_k) + \int_{t_k}^{t} e^{\gamma s}\nabla_y\varphi_\gamma(S(s,x);\lambda(s))\cdot dS(s,x) \\
&\geq h_\gamma(t,x),
\end{aligned} \tag{22}$$

where we used the self-financing equation (5), the asumption (17), the convexity property of φ_γ with respect to y and the Lemma 1. The conclusion of the lemma is now straightforward. $\square$

Remark 2. *For a fixed $f \in \mathcal{P}_2(y)$, a solution (γ, φ_γ) of the elliptic equation* (12) *is constructed using the following series*

$$\varphi_\gamma(y;\lambda) = \frac{1}{|\gamma|}\left[\sum_{k=0}^{\infty} L_{|\gamma|}^k(f)(y;\lambda)\right], \ \textit{for } \gamma < 0, \tag{23}$$

where $L_{|\gamma|} = \dfrac{1}{|\gamma|}L$ and $L : \mathcal{P}_2(y;\lambda) \to \mathcal{P}_2(y;\lambda)$ stands for the linear operator defined in the formula (11).

As far as the linear operator $L_{|\gamma|}$ is acting on $\mathcal{P}_2(y;\lambda)$, for the sake of simplicity we shall assume that $f(y) = (\langle q, y\rangle)^2$, where $q \neq 0$ is a common eigen vector of the matrices $A_j(\lambda)$, such that $A_j^*(\lambda)q = \mu_j(\lambda)q$ and $\mu_j : S \to \mathbb{R}$ is continuous and bounded, for any $1 \leq j \leq m$.

Lemma 3. *Let $f \in \mathcal{P}_2(y)$ and $g_j(y;\lambda) = A_j(\lambda)y + a_j(\lambda)$, $j = 1, \ldots, m$, be given as above. Let $\gamma < 0$ such that $\frac{\|\mu\|}{|\gamma|} \leq 1$, where $\mu(\lambda) = \sum_{j=1}^m \mu_j^2(\lambda)$ and $\|\mu\| =$*

$\sup_{\lambda \in S} \mu(\lambda)$. *Then the function*

$$\varphi_\gamma(y;\lambda) = \frac{1}{|\gamma|}\left[\sum_{k=0}^{\infty} L^k_{|\gamma|}(f)(y;\lambda)\right] \tag{24}$$

$$= \frac{1}{|\gamma| - \mu(\lambda)}\left[f(y) + \frac{b(\lambda)}{|\gamma|}\langle q, y\rangle + \frac{a(\lambda)}{|\gamma|}\right], y \in \mathbb{R}^d, \lambda \in S \tag{25}$$

is a solution of the elliptic equation (12), *where* $b(\lambda) = 2\sum_{j=1}^m \mu_j(\lambda)\langle q, a_j(\lambda)\rangle$ *and* $a(\lambda) = \sum_{j=1}^m (\langle q, a_j(\lambda)\rangle)^2$.

Proof. By hypothesis, we easily see that

$$L(f)(y;\lambda) = \sum_{j=1}^m [A_j(\lambda)y + a_j(\lambda)]^*\, q\, q^*\, [A_j(\lambda)y + a_j(\lambda)] \tag{26}$$

$$= \sum_{j=1}^m (\langle q, A_j(\lambda)y + a_j(\lambda)\rangle)^2 = \mu(\lambda)f(y) + b(\lambda)\langle q, y\rangle + a(\lambda).$$

Hence

$$L_{|\gamma|}(f)(y;\lambda) = \frac{\mu(\lambda)}{|\gamma|}f(y) + \frac{b(\lambda)}{|\gamma|}\langle q, y\rangle + \frac{a(\lambda)}{|\gamma|}. \tag{27}$$

An induction argument leads us to

$$L^k_{|\gamma|}(f)(y;\lambda) = \left(\frac{\mu(\lambda)}{|\gamma|}\right)^k f(y) + \left(\frac{\mu(\lambda)}{|\gamma|}\right)^{k-1}\left[\frac{b(\lambda)}{|\gamma|}\langle q, y\rangle\right] \tag{28}$$

$$+ \left(\frac{\mu(\lambda)}{|\gamma|}\right)^{k-1}\left[\frac{a(\lambda)}{|\gamma|}\right], \text{ for any } k \geq 1.$$

Denote $\rho_\gamma(\lambda) = \frac{\mu(\lambda)}{|\gamma|}$ and

$$T(\lambda) = \sum_{k=0}^{\infty} [\rho_\gamma(\lambda)]^k = \frac{|\gamma|}{|\gamma| - \mu(\lambda)},$$

where $\rho_\gamma(\lambda) < 1$, for any $\lambda \in S$ (see $\frac{\|\mu\|}{|\gamma|} \leq 1$). Inserting the formula (28) in (24), we obtain

$$\varphi_\gamma(y;\lambda) = \frac{1}{|\gamma|}T(\lambda)f(y) + \frac{1}{|\gamma|}T(\lambda)\frac{b(\lambda)}{|\gamma|}\langle q, y\rangle + \frac{1}{|\gamma|}T(\lambda)\frac{a(\lambda)}{|\gamma|}$$

and substituting $T(\lambda)$ we get the conclusion fulfilled. □

Remark 3. *Notice that*

$$\theta_0(t_k, x) = e^{(\gamma - r)t_k}\varphi_\gamma(0;\lambda_k) = e^{(\gamma - r)t_k}\frac{a(\lambda)}{|\gamma|(|\gamma| - \mu(\lambda))} \geq 0.$$

Therefore, the assumption that $\theta_0(t,x) \geq 0$, for all $t \in [0,T]$, $x \in \mathbb{R}^n$ is very reasonable.

Remark 4. *The solution of the function φ_γ makes use of a special convex function $f(y) = (\langle q, y\rangle)^2$, with $q \in \mathbb{R}^d$ as a common eigen vector of the matrices $A_j(\lambda)$, $j = 1, \dots, m$.*

Assuming that there exist several eigen vectors $Q = (q_1, \dots, q_s)$, $s \leq d$, such that

$$Q^* A_j(\lambda) = \mu_j(\lambda) Q^*,\ \mu_j(\lambda) \in \mathbb{R},\ j = 1, \dots, m, \tag{29}$$

then $f(y) = \langle Q^* y, Q^* y\rangle$ agrees with the conclusion of the Lemma 3 and the computation of the convex function $\varphi_\gamma \in \mathcal{P}_2(y)$ follows the same procedure.

In addition, for an arbitrarily fixed $y_0 \in \mathbb{R}^d$, we may consider a convex function

$$f(y) = \langle Q^*(y - y_0), Q^*(y - y_0)\rangle, \tag{30}$$

where $\tilde{S}(t,x) = S(t,x) - y_0$, $t \geq 0$, satisfies the following linear system

$$\begin{cases} dz(t) & = h_0(z(t);\lambda)dt + \sum_{j=1}^{m} h_j(z(t);\lambda) dW_j(t),\ t \geq 0 \\ z(0) & = x - y_0. \end{cases} \tag{31}$$

Here $h_i(z;\lambda) = A_i(\lambda) z + d_i(\lambda)$, $d_i(\lambda) = a_i(\lambda) + A_i(\lambda) y_0$, $i = 0, 1, \dots, m$ replaces the original vector fields $g_i(y;\lambda$ of the system (2) and the function $f(z) = \langle Q^* z, Q^* z\rangle$ satisfies (29).

3. **Main results**

We conclude the above given analysis by the following

Theorem 1. *Let $g_j(y;\lambda) = A_j(\lambda) y + a_j(\lambda)$ be given such that the $(d \times d)$ matrix $A_j(\lambda)$ and the vector $a_j(\lambda) \in \mathbb{R}^d$ are continuous and bounded with respect to $\lambda \in S$, for any $j = 1, \dots, m$ and $d \leq n$. Consider a continuous vector field $g_0(y;\lambda) \in \mathbb{R}^d$ which is globally Lipschitz continuous with respect to $y \in \mathbb{R}^d$, uniformly in $\lambda \in S$.*

Define a convex function $f \in \mathcal{P}_2(y)$ by

$$f(y) = \langle Q^*(y - y_0), Q^*(y - y_0)\rangle, \tag{32}$$

where $y_0 \in \mathbb{R}^d$ is arbitrarily fixed and $Q = (q_1, \ldots, q_s),\ q_i \in \mathbb{R}^d,\ s \leq d$ stand for some common eigen vectors satisfying

$$Q^* A_j(\lambda) = \mu_j(\lambda) Q^*,\ \mu_j(\lambda) \in \mathbb{R},\ j = 1, \ldots, m. \tag{33}$$

Let $\gamma < 0$ be such that $\frac{\|\tilde{\mu}\|}{|\gamma|} < 1$, where $\mu(\lambda) = \sum_{j=1}^m \mu_j^2(\lambda)$ and $\|\tilde{\mu}\| = \sup_{k\geq 0} \mu(\tilde{\lambda}_k)$.

Then

$$\begin{aligned}\varphi_\gamma(y;\lambda) =& \frac{1}{|\gamma|}\left[\sum_{k=0}^{\infty} L_{|\gamma|}^k(f)(y;\lambda)\right] = \frac{1}{|\gamma| - \mu(\lambda)} \\ & \times \left[f(y) + \langle \frac{b(\lambda)}{|\gamma|}, Q^*(y - y_0)\rangle + \frac{a(\lambda)}{|\gamma|}\right],\ y \in \mathbb{R}^d,\ \lambda \in S,\end{aligned} \tag{34}$$

is a solution of the elliptic equation (12)*, where $b(\lambda) = 2\sum_{j=1}^m \mu_j(\lambda) Q^* d_j(\lambda)$, $a(\lambda) = \sum_{j=1}^m \|Q^* d_j(\lambda)\|^2$, $d_j(\lambda) = a_j(\lambda) + A_j(\lambda) y_0,\ j = 1, \ldots, m$.*

Proof. Using the linear mapping $z = y - y_0$, we rewrite

$$f(y) = \tilde{f}(z) = \langle Q^* z, Q^* z\rangle$$

and the solution $\{S(t,x); t \geq 0\}$ satisfying (2) is shifted into $\tilde{S}(t,x) = S(t,x) - y_0$, which satisfies the system (31). Here $h_j(z;\lambda) = A_j(z;\lambda) z + d_j(\lambda),\ j = 1, \ldots, m$ and $h_0(z;\lambda) = g_0(z + y_0;\lambda)$.

The procedure employed in the proof of the Lemma 3 is applicable here and the convex function $\varphi_\gamma \in \mathcal{P}_2(y;\lambda)$ given in (34) satisfies the equation (12). $\square$

Theorem 2. *Assume that the assumptions of the previous theorem and also the estimate* (17) *stand in force. Define*

$$\theta(t,x) = \nabla_y \varphi_\gamma(\hat{y}(t,x); \hat{\lambda}(t)),\ t \in [0,T],\ x \in \mathbb{R}^d \tag{35}$$

and let $\{\theta_0(t,x); t \geq 0\}$ be the piecewise continuous process satisfying the integral equation (5)*, where*

$$\theta_0(t_k, x) = \exp(\gamma t_k) \nabla_y \varphi_\gamma(y_0; \lambda_k),\ k \geq 0,\ x \in \mathbb{R}^d. \tag{36}$$

Then $(\theta_0(t,x), \theta(t,x)) \in \mathbb{R}^{d+1}$ is an admissible strategy corresponding to the value function

$$V(t,x) = \theta_0(t,x) e^{rt} + \theta(t,x) \cdot (S(t,x) - y_0).$$

Proof. By hypothesis, the nontrivial solution $(f, \gamma, \varphi_\gamma)$ of the equation (12) constructed in the Theorem 1 fulfills the conditions assumed in the Lemma 2. The "feedback shape" recommended by the equations (16) and (15) uses the deterministic values $\theta_0(t_k, x) = \exp(\gamma t_k)\varphi_\gamma(0; \lambda_k)$, for $k \geq 0$, which are not correlated with the special form that we obtain here for the convex functions $f \in \mathcal{P}_2(y)$, $\varphi_\gamma \in \mathcal{P}_2(y; \lambda)$.

According to the expression of φ_γ given in the formula (34), the simplest values are obtained for $y = y_0 \in \mathbb{R}^d$, i.e.

$$\varphi_\gamma(y_0, \lambda_k) = \frac{1}{|\gamma| - \mu(\lambda_k)} \frac{a(\lambda_k)}{|\gamma|},\ k \geq 0.$$

This is a slight changing in the definition of the "feedback shape" (see the formulas (8) and (9)) and it agrees with the linear mapping $z = y - y_0$ used in the proof of the Theorem 1, for which $z = 0$ corresponds to the special "feedback shape" given in (16) and (15).

As a consequence, $(\theta_0(t, x), \theta(t, x)) \in \mathbb{R}^{d+1}$ defined in (35) and (36) is an admissible strategy corresponding to the value function

$$V(t, x) = \theta_0(t, x)e^{rt} + \theta(t, x) \cdot (S(t, x) - y_0),\ t \geq 0,\ x \in \mathbb{R}^d$$

and $\tilde{S}(t, x) = S(t, x) - y_0,\ t \geq 0$, is the solution of the system (31). □

References

[1] Iftimie, B., Molnar, I., Vârsan, C., *Stochastic Differential Equations with Jumps; Liapunov Exponents, Asymptotic Behaviour of Solutions and Applications to Financial Mathematics*, Preprint IMAR Nr. 10/2007.

[2] Iftimie, B., Molnar, I., Vârsan, C., *Solutions of some elliptic equations associated with a piecewise continuous process*, accepted for publication at Revue Roumaine Math. Pures Appl., 2007.

[3] Karatzas, I., Shreve, S., *Brownian Motion and Stochastic Calculus,* 2nd Edition, Springer Verlag, 1991.

[4] Lamberton, D., Lapeyre, B., *Introduction to Stochastic Calculus Applied to Finance.* Chapman and Hall, 2007.

[5] Oksendal, B., *Stochastic Differential Equations. An Introduction with Applications,* 5th Edition. Springer, 2000.

Academy of Economic Studies
Department of Mathematics,
6 Piaţa Romană, 010374, Bucharest, Romania
E-mail address: iftimieb@csie.ase.ro

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 2, June 2009

ON SUBCLASSES OF PRESTARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

SANTOSH B. JOSHI

Abstract. The present paper is aim at defining new subclasses of prestarlike functions with negative coefficients in unit disc U and study there basic properties such as coefficient estimates, closure properties. Further distortion theorem involving generalized fractional calculus operator for functions $f(z)$ belonging to these subclasses are also established.

1. **Introduction**

Let A denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the unit disc $U = \{z : |z| < 1\}$ and let S denote the subclass of A, consisting functions of the type (1.1) which are normalized and univalent in U. A function $f \in S$, is said to be starlike of order $\mu (0 \leq \mu < 1)$ in U if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) \geq \mu. \tag{1.2}$$

We denote by $S^*(\mu)$, the class of all functions in S, which are starlike of order μ in U.

It is well-known that

$$S^*(\mu) \subseteq S^*(0) \equiv S^*.$$

The class $S^*(\mu)$ was first introduced by Robertson [7] and further it was rather extensively studied by Schild [8], MacGregor [2].

Received by the editors: 05.01.2009.

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Prestarlike function, distortion theorem, closure properties, extremal function, fractional calculus.

Also

$$S_\mu(z) = \frac{z}{(1-z)^{2(1-\mu)}} \tag{1.3}$$

is the familiar extremal function for class $S^*(\mu)$. Setting

$$C(\mu,n) = \frac{\prod_{k=2}^{n}(k-2\mu)}{(n-1)!}, n \in \mathbb{N}\backslash\{1\}, \mathbb{N} = \{1,2,3,...\}. \tag{1.4}$$

The function $S_\mu(z)$ can be written in the form

$$S_\mu(z) = z + \sum_{n=2}^{\infty} C(\mu,n)\ z^n. \tag{1.5}$$

We note that $C(\mu,n)$ is decreasing function in μ and that

$$\lim_{n\to\infty} C(\mu,n) = \begin{cases} \infty, & \mu < 1/2 \\ 1, & \mu = 1 \\ 0, & \mu > 1. \end{cases} \tag{1.6}$$

We say that $f \in S$, is in the class $S^*(\alpha,\beta,\gamma)$ if and only if it satisfies the following condition

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - 1}{\gamma \dfrac{zf'(z)}{f(z)} + 1 - (1+\gamma)\alpha} \right| < \beta, \tag{1.7}$$

where $0 \le \alpha < 1, 0 < \beta \le 1, 0 \le \gamma \le 1$.

Furthermore, a function f is said to be in the class $K(\alpha,\beta,\gamma)$ if and only if

$$zf'(z) \in S^*(\alpha,\beta,\gamma).$$

Let $f(z)$ be given by (1.1) and $g(z)$ be given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.8}$$

then the Hadamard product(or convolution) of (1.1) and (1.8) is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.9}$$

Let $R_\mu(\alpha,\beta,\gamma)$ be the subclass of A consisting functions $f(z)$ such that

$$\left|\frac{\dfrac{zh'(z)}{h(z)}-1}{\gamma\dfrac{zh'(z)}{h(z)}+1-(1+\gamma)\alpha}\right|<\beta \tag{1.10}$$

where,

$$h(z)=(f*S_\mu(z)),0\le\mu<1. \tag{1.11}$$

Also, let $C_\mu(\alpha,\beta,\gamma)$ be the subclass of A consisting functions $f(z)$, which satisfy the condition

$$zf'(z)\in R_\mu(\alpha,\beta,\gamma).$$

We note that $R_\mu(\alpha,1,1)=R_\mu(\alpha)$ is the class functions introduced by Sheil-Small *et al* [9]and such type of classes were studied by Ahuja and Silverman[1].

Finally, let T denote the subclass of S consisting of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n,a_n\ge 0. \tag{1.12}$$

We denote by $T^*(\alpha,\beta,\gamma)$, $C^*(\alpha,\beta,\gamma)$, $R_\mu[\alpha,\beta,\gamma]$ and $\mathrm{C}_\mu[\alpha,\beta,\gamma]$ the classes obtained by taking the intersection of the classes $S^*(\alpha,\beta,\gamma)$, $K(\alpha,\beta,\gamma)$, $R_\mu(\alpha,\beta,\gamma)$ and $\mathrm{C}_\mu(\alpha,\beta,\gamma)$ with the class T. In the present paper we aim at finding various interesting properties and characterization of aforementioned general classes $R_\mu[\alpha,\beta,\gamma]$ and $\mathrm{C}_\mu[\alpha,\beta,\gamma]$. Further we note that such classes were studied by Owa and Uralegaddi [6], Silverman and Silvia [10] and Owa and Ahuja [4].

2. **Basic Characterization**

Theorem 1. *A function $f(z)$ defined by* (1.12) *is in the class $R_\mu[\alpha,\beta,\gamma]$ if and only if*

$$\sum_{n=2}^{\infty}C(\mu\,,n)\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha]\right\}a_n\le\beta(1+\gamma)(1-\alpha). \tag{2.1}$$

The result (2.1) *is sharp and is given by*

$$f(z) = z - \frac{\beta(1+\gamma)(1-\alpha)}{C(\mu,n)\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha]\right\}} z^n, n \in \mathbb{N}\backslash\{1\}. \tag{2.2}$$

Proof. The proof of Theorem 1 is straightforward and hence details are omitted. □

Theorem 2. *Let $f(z) \in T$, then $f(z)$ is in the class $C_\mu[\alpha,\beta,\gamma]$ if and only if*

$$\sum_{n=2}^{\infty} C(\mu\ , n)n\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha]\right\} a_n \leq \beta(1+\gamma)(1-\alpha). \tag{2.3}$$

The result (2.3) *is sharp for the function $f(z)$ given by*

$$f(z) = z - \frac{\beta(1+\gamma)(1-\alpha)}{C(\mu,n)n\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha]\right\}} z^n, n \in \mathbb{N}\backslash\{1\}. \tag{2.4}$$

Proof. Since $f(z) \in C_\mu[\alpha,\beta,\gamma]$ if and only if $zf'(z) \in R_\mu[\alpha,\beta,\gamma]$, we have Theorem 2, by replacing a_n by na_n in Theorem 1. □

Corollary 1. *Let $f(z) \in T$, be in the class $R_\mu[\alpha,\beta,\gamma]$ then*

$$a_n \leq \frac{\beta(1+\gamma)(1-\alpha)}{C(\mu,n)\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha]\right\}}, n \in \mathbb{N}\backslash\{1\}. \tag{2.5}$$

Equality holds true for the function $f(z)$ given by (2.2).

Corollary 2. *Let $f(z) \in T$, be in the class $C_\mu[\alpha,\beta,\gamma]$ then*

$$a_n \leq \frac{\beta(1+\gamma)(1-\alpha)}{C(\mu,n)n\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha]\right\}}, n \in \mathbb{N}\backslash\{1\}. \tag{2.6}$$

Equality in (2.6) *holds true for the function $f(z)$ given by* (2.4).

3. **Closure Properties**

Theorem 3. *The class $R_\mu[\alpha,\beta,\gamma]$ is closed under convex linear combination.*

Proof. Let, each of the functions $f_1(z)$ and $f_2(z)$ be given by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, a_{n,j} \geq 0, j = 1, 2 \tag{3.1}$$

be in the class $R_\mu[\alpha,\beta,\gamma]$ .It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \lambda f_1(z) + (1-\lambda) f_2(z), 0 \leq \lambda \leq 1 \tag{3.2}$$

is also in the class $R_\mu[\alpha,\beta,\gamma]$. Since, for $0 \le \lambda \le 1$,

$$h(z) = z - \sum_{n=2}^{\infty} [\lambda a_{n,1} + (1-\lambda)a_{n,2}] z^n \tag{3.3}$$

by using Theorem 1,we have

$$\sum_{n=2}^{\infty} C(\mu\ , n) \{(n-1) + \beta[\gamma n + 1 - (1+\gamma)\alpha]\} [\lambda a_{n,1} + (1-\lambda)a_{n,2}] \le \beta(1+\gamma)(1-\alpha) \tag{3.4}$$

which proves that $h(z) \in R_\mu[\alpha,\beta,\gamma]$.

Similarly we have □

Theorem 4. *The class $C_\mu[\alpha,\beta,\gamma]$ is closed under convex linear combination.*

Theorem 5. *Let,*

$$f_1(z) = z \tag{3.5}$$

and,

$$f_n(z) = z - \frac{\beta(1-\alpha)(1+\gamma)}{C(\mu,n)\{(n-1) + \beta[\gamma n + 1 - (1+\gamma)\alpha]\}} z^n. \tag{3.6}$$

Then $f(z)$ is in the class $R_\mu[\alpha,\beta,\gamma]$ if and only if it can be expressed as

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \tag{3.7}$$

where, $\lambda_n \ge 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. Let,

$$\begin{aligned} f(z) &= \sum_{n=2}^{\infty} \lambda_n f_n(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\beta(1-\alpha)(1+\gamma)}{C(\mu,n)\{(n-1) + \beta[\gamma n + 1 - (1+\gamma)\alpha]\}} \lambda_n z^n. \end{aligned} \tag{3.8}$$

Then it follows that

$$\sum_{n=2}^{\infty} \frac{\beta(1-\alpha)(1+\gamma)}{C(\mu,n)\{(n-1) + \beta[\gamma n + 1 - (1+\gamma)\alpha]\}} \lambda_n \frac{C(\mu,n)\{(n-1) + \beta[\gamma n + 1 - (1+\gamma)\alpha]\}}{\beta(1-\alpha)(1+\gamma)}$$

$$= \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 < 1. \tag{3.9}$$

Therefore by Theorem 1, $f(z) \in R_\mu[\alpha,\beta,\gamma]$.

Conversely, assume that the function $f(z)$ defined by (1.12) belongs to the class $R_\mu[\alpha,\beta,\gamma]$, and then we have

$$a_n \leq \frac{\beta(1+\gamma)(1-\alpha)}{C(\mu,n)\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha]\right\}}, n \in \mathbb{N}\backslash\{1\}. \tag{3.10}$$

Setting

$$\lambda_n = a_n \frac{C(\mu,n)\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha\right\}}{\beta(1-\alpha)(1+\gamma)}, n \in \mathbb{N}\backslash\{1\}, \tag{3.11}$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n, \tag{3.12}$$

we see that $f(z)$ can be expressed in the form(3.7).This completes the proof of Theorem 5.

In the same manner we can prove, $\square$

Theorem 6. *Let,*

$$f_1(z) = z \tag{3.13}$$

and

$$f_n(z) = z - \frac{\beta(1-\alpha)(1+\gamma)}{C(\mu,n)n\left\{(n-1)+\beta[\gamma n+1-(1+\gamma)\alpha]\right\}} z^n, n \in \mathbb{N}\backslash\left\{1\right\}. \tag{3.14}$$

Then $f(z)$ is in the class $C_\mu[\alpha,\beta,\gamma]$ if and only it can be expressed as

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \tag{3.15}$$

where, $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

4. **Generalized Fractional Integral Operator**

Various operators of fractional calculus, that is fractional derivative operator, fractional integral operator have been studied in the literature rather extensively for *e.g.* [3, 5, 11, 12]. In the present section we shall make use of generalized fractional integral operator $I_{0,z}^{\lambda,\delta,\eta}$ given by Srivastava *et al* [13].

Definition. For real numbers $\lambda > 0, \delta$ and η the generalized fractional integral operator $I_{0,z}^{\lambda,\delta,\eta}$ is defined as

$$I_{0,z}^{\lambda,\delta,\eta} f(z) = \frac{z^{-\lambda-\delta}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} {}_2F_1(\lambda+\delta, -\eta, 1-{}^t\!/_z) f(t) dt \tag{4.1}$$

where $f(z)$ is an analytic function in a simply connected region of the z-plane containing origin with order

$$f(z) = 0(|z|)^{\varepsilon}, (z \to 0, \varepsilon > \max[0,\delta-\eta] - 1) \tag{4.2}$$

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{z^n}{n} \tag{4.3}$$

and $(\nu)_n$ is the Pochhammer symbol defined by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 \\ \nu(\nu+1)...(\nu+n+1), \nu \in \mathbb{N} \end{cases} \tag{4.4}$$

an the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

In order to prove the results for generalized fractional integral operator $I_{0,z}^{\lambda,\delta,\eta}$, we recall here the following lemma due to Srivastava *et al* [13].

Lemma 1 (Srivastava *et al* [13])**.** *If* $\lambda > 0$ and $k > \delta-\eta-1$ *then*

$$I_{0,z}^{\lambda,\delta,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\delta+\eta+1)}{\Gamma(k-\delta+1)\Gamma(k+\lambda+\eta+1)} z^{k-\delta}. \tag{4.5}$$

Theorem 7. *Let* $\lambda > 0, \delta < 2, \lambda+\eta > -2, \delta-\eta < 2$ and $\delta(\lambda+\eta) \leq 3\lambda$. *If* $f(z) \in T$ *is in the class* $R_\mu[\alpha,\beta,\gamma]$ *with* $0 \leq \mu \leq {}^1\!/_2$, $0 < \beta \leq 1, 0 \leq \alpha < 1$ *and* $0 \leq \gamma \leq 1$ *then*

$$\frac{\Gamma(2-\delta+\eta)\,|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)} \left\{1 - \frac{(2-\delta+\eta)\beta(1-\alpha)(1+\gamma)}{1+\beta\{\gamma(2-\alpha)+1-\alpha\}(1-\mu)(2-\delta)(2+\lambda+\eta)} |z|\right\}$$
$$\leq \left| I_{0,z}^{\lambda,\delta,\eta} f(z) \right| \leq$$
$$\frac{\Gamma(2-\delta+\eta)\,|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)} \left\{1 + \frac{(2-\delta+\eta)\beta(1-\alpha)(1+\gamma)}{1+\beta\{\gamma(2-\alpha)+1-\alpha\}(1-\mu)(2-\delta)(2+\lambda+\eta)} |z|\right\}, \tag{4.6}$$

when

$$U_0 = \begin{cases} U, \delta \leq 1 \\ U \backslash \{1\}, \delta > 1. \end{cases} \tag{4.7}$$

Equality in (4.6) *is attended for the function given by*

$$f(z) = z - \frac{\beta(1-\alpha)(1+\gamma)}{2\left\{1+\beta[\gamma(2-\alpha)+1-\alpha]\right\}}z^2. \tag{4.8}$$

Proof. By making use of Lemma 1, we have

$$I_{0,z}^{\lambda,\delta,\eta}f(z) = \frac{\Gamma(2-\delta+\eta)}{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)}z^{1-\delta} - \sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(n-\delta+\eta+1)}{\Gamma(n-\delta+1)\Gamma(n+\lambda+\eta+1)}a_n z^{n-\delta}. \tag{4.9}$$

Letting,

$$\begin{aligned} H(z) &= \frac{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)}{\Gamma(2-\delta+\eta)}z^{\delta}I_{0,z}^{\lambda,\delta,\eta} \\ &= z - \sum_{n=2}^{\infty}\psi(n)a_n z^n \end{aligned} \tag{4.10}$$

where,

$$\psi(n) = \frac{(2-\delta+\eta)\,(1)_n}{(2-\delta)_{n-1}(2+\lambda+\eta)}, n \in \mathbb{N}\backslash\{1\}. \tag{4.11}$$

We can see that $\psi(n)$ is non -increasing for integers $n, n \in \mathbb{N}\backslash\{1\}$, and we have

$$0 < \psi(n) \le \psi(2) = \frac{2(2-\delta+\eta)}{(2-\delta)(2+\lambda+\eta)}, n \in \mathbb{N}\backslash\{1\}. \tag{4.12}$$

Now in view of Theorem 1 and (4.12), we have

$$|H(z)| \ge |z| - \psi(2)\,|z|^2 \sum_{n=2}^{\infty} a_n$$

$$\ge |z| - \frac{(2-\delta+\eta)\beta(1-\alpha)(1+\gamma)}{1+\beta[\gamma(2-\alpha)+1-\alpha](1-\mu)(2-\delta)(2+\lambda+\eta)}\,|z|^2 \tag{4.13}$$

and

$$\begin{aligned} |H(z)| &\le |z| + \psi(2)|z|^2\sum_{n=2}^{\infty} a_n \\ &\ge |z| + \frac{(2-\delta+\eta)\beta(1-\alpha)(1+\gamma)}{1+\beta[\gamma(2-\alpha)+1-\alpha](1-\mu)(2-\delta)(2+\lambda+\eta)}\,|z|^2. \end{aligned} \tag{4.14}$$

This completes the proof of Theorem 7.

Now, by applying Theorem 2 to the functions $f(z)$ belonging to the class $C_\mu[\alpha,\beta,\gamma]$, we can derive $\square$

Theorem 8. *Let* $\lambda > 0, \delta < 2, \lambda + \eta > -2, \delta - \eta < 2$ and $\delta(\lambda + \eta) \le 3\lambda$ *.If* $f(z) \in T$ *is in the class* $C_\mu[\alpha, \beta, \gamma]$ *with* $0 \le \mu \le 1/2$, $0 < \beta \le 1, 0 \le \alpha < 1$ *and* $0 \le \gamma \le 1$ *then*

$$\frac{\Gamma(2-\delta+\eta)\,|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)}\left\{1-\frac{(2-\delta+\eta)\beta(1-\alpha)(1+\gamma)}{2[1+\beta\{\gamma(2-\alpha)+1-\alpha\}](1-\mu)(2-\delta)(2+\lambda+\eta)}\,|z|\right\} \tag{4.15}$$

$$\le \left|I_{0,z}^{\lambda,\delta,\eta} f(z)\right| \le$$

$$\frac{\Gamma(2-\delta+\eta)\,|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\lambda+\eta)}\left\{1+\frac{(2-\delta+\eta)\beta(1-\alpha)(1+\gamma)}{2[1+\beta\{\gamma(2-\alpha)+1-\alpha\}](1-\mu)(2-\delta)(2+\lambda+\eta)}\,|z|\right\} \tag{4.16}$$

where U_0 *is defined by* (4.7). *Equality in* (4.6) *is attended for the function given by*

$$f(z) = z - \frac{\beta(1-\alpha)(1+\gamma)}{2\left\{1+\beta[\gamma(2-\alpha)+1-\alpha]\right\}} z^2.$$

References

[1] Ahuja, O.P, Silverman, H., *Convolutions of prestarlike functions*, Internat J.Math Math Sci., **6**(1983), 59-68.

[2] MacGregor, T.H., *The radius of convexity for starlike functions of order1/2*, Proc. Amer. Math. Soc., **14**(1963), 71-76.

[3] Owa, S., *On the distortion theorems I*, Kyungpook Math J., **18**(1978), 53-59.

[4] Owa, S., Ahuja, O.P., *An application of fractional calculus*, *Math Japon*, **30**(1985), 947-955.

[5] Owa, S., Saigo, M., Srivastava, H.M., *Some characterization theorems for starlike and convex functions involving a certain fractional integral operator*, J. Math Anal. Appl., **140**(1989), 419-426.

[6] Owa, S., Uralegaddi, B.A., *A class of functions* α *prestarlike of order* β, Bull. Korean Math. Soc., **21**(1984), 77-85.

[7] Robertson, M.S., *On the theory of univalent functions*, Ann. Math., **37**(1936), 374-408.

[8] Schild, A., *On starlike functions of order* α, Amer. J. Math., **87**(1965), 65-70.

[9] Sheil-Small, T., Silverman, H., Silvia, E.M., *Convolutions multipliers and starlike functions*, J. Anal. Math., **41**(1982), 181-192.

[10] Silverman, H., Silvia, E.M., *Subclasses of prestarlike functions*, Math. Japon., **29**(1984), 929-936.

[11] Srivastava, H.M., Owa, S., (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[12] Srivastava, H.M., Owa, S., (Editors), *Univalent Functions, Fractional Calculus and Their Applications,* Halested Press (Ellis Horwood Limted, Chichester), John Wiley and Sons, New York, Chichester,Brisbane, and Toronto, 1989.

[13] Srivastava, H.M., Saigo, M., Owa, S., *A class of distortion theorems involving certain operators of fractional calculus*, J. Math Anal. Appl., **131**(1988), 412-420.

Department of Mathematics
Walchand College of Engineering
Sangli-416415, Maharashtra, India
E-mail address: joshisb@hotmail.com

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 2, June 2009

FUGLEDE-PUTNAM THEOREM FOR log-HYPONORMAL OR CLASS $\mathcal{Y}$ OPERATORS

SALAH MECHERI AND AISSA NASLI BAKIR

Abstract. The equation $AX = XB$ implies $A^*X = XB^*$ when A and B are normal is known as the familiar Fuglede-Putnam's theorem. In this paper we will extend Fuglede-Putnam's theorem to a more general class of operators. We show that if A is log-hyponormal and B^* is a class $\mathcal{Y}$ operator, then A, B satisfy Fuglede-Putnam's theorem. Other related results are also given.

1. **Introduction**

Let $\mathcal{H}, \mathcal{K}$ be complex Hilbert spaces and $B(\mathcal{H}), B(\mathcal{K})$ the algebras of all bounded linear operators on $\mathcal{H}, \mathcal{K}$. The familiar Fuglede-Putnam's theorem is as follows:

Theorem 1.1. *(Fuglede-Putnam) Let $A \in B(\mathcal{H}), B \in B(\mathcal{K})$ be normal operators. If $AX = XB$ for some $X \in B(\mathcal{K}, \mathcal{H})$, then $A^*X = XB^*$.*

Many authors have extented this theorem for several classes of operators, for example (see [7, 10, 11, 22, 24]). We say that A, B satisfy Fuglede-Putnam's theorem if $AX = XB$ implies $A^*X = XB^*$. In [22] A. Uchiyama proved that if A, B^* are class $\mathcal{Y}$ operators, then A, B satisfy Fuglede-Putnam's theorem. In [10] the authors showed that Fuglede-Putnam's theorem holds when A is p-hyponormal and B^* is a class A operator. The aim of this paper is to show that if A is log-hyponormal and B^* is a class $\mathcal{Y}$ operator, then A, B satisfy Fuglede-Putnam's theorem.

Received by the editors: 28.03.2008.
2000 *Mathematics Subject Classification.* Primary 47B47, 47A30, 47B20, Secondary 47B10.
Key words and phrases. Fuglede-Putnam's theorem, log-hyponormal operator, class $\mathcal{Y}$ operator.

For any operator $A \in B(\mathcal{H})$ set, as usual, $|A| = (A^*A)$ and $[A^*, A] = A^*A - AA^* = |A|^2 - |A^*|^2$ (the self commutator of A), and consider the following standard definitions: A is normal if $A^*A = AA^*$, hyponormal if $A^*A - AA^* \geq 0$. A is said to be a class $\mathcal{Y}_\alpha$ operator for $\alpha \geq 1$ (or $A \in \mathcal{Y}_\alpha$) if there exists a positive number k_α such that

$$|AA^* - A^*A|^\alpha \leq k_\alpha^2 (A - \lambda)^*(A - \lambda) \, \text{for all} \, \lambda \in \mathbb{C}.$$

It is known that $\mathcal{Y}_\alpha \subset \mathcal{Y}_\beta$ if $1 \leq \alpha \leq \beta$. Let $\mathcal{Y} = \bigsqcup_{1 \leq \alpha} \mathcal{Y}_\alpha$. We remark that a class $\mathcal{Y}_1$ operator A is M-hyponormal, i.e., there exists a positive number M such that

$$(A - \lambda I)(A - \lambda I)^* \leq M^2 (A - \lambda I)^*(A - \lambda I) \, \text{for all} \, \lambda \in \mathbb{C},$$

and M-hyponormal operators are class $\mathcal{Y}_2$ operators (see [22]). A is said to be dominant if for any $\lambda \in \mathbb{C}$ there exists a positive number M_λ such that

$$(A - \lambda I)(A - \lambda I)^* \leq M_\lambda^2 (A - \lambda I)^*(A - \lambda I).$$

It is obvious that dominant operators are M-hyponormal. But it is known that there exists a dominant operator which is not a class $\mathcal{Y}$ operator, and also there exists a class $\mathcal{Y}$ operator which is not dominant. In this paper we will extend Fuglede-Putnam's theorem for log-hyponormal operators and class $\mathcal{Y}$ operators.

A is said to be log-hyponormal if A is invertible and satisfies the following equality

$$\log(A^*A) \geq \log(AA^*).$$

It is known that invertible p-hyponormal operators are log-hyponormal operators but the converse is not true [18]. However it is very interesting that we may regard log-hyponormal operators are 0-hyponormal operators [18, 19]. The idea of log-hyponormal operator is due to Ando [3] and the first paper in which log-hyponormality appeared is [6].

2. **Results**

We will recall some known results which will be used in the sequel.

Lemma 2.1. [16] *Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$. Then the following assertions are equivalent*

(i) The pair (A, B) satisfies Fuglede-Putnam's theorem;

(ii) if $AC = CB$ for some $C \in B(\mathcal{K}, \mathcal{H})$, then $\overline{\mathrm{ran}(C)}$ reduces A, $(\ker C)^{\perp}$ reduces B and $A|_{\overline{ran(C)}}$ and $B|_{(\ker C)^{\perp}}$ are normal operators.

Lemma 2.2. *(Stampfli and Wadhwa*[15]*) Let $A \in B(\mathcal{H})$ be a dominant operator and $\mathcal{M} \subset \mathcal{H}$ invariant under A. If $A|_M$ is normal, then $\mathcal{M}$ reduces A.*

Lemma 2.3. [22] *Let $A \in B(\mathcal{H})$ be a class $\mathcal{Y}$ operator and $\mathcal{M} \subset \mathcal{H}$ invariant under A. If $A|_M$ is normal, then $\mathcal{M}$ reduces A.*

Lemma 2.4. *(Stampfli and Wadhwa*[15]*) Let $A \in B(\mathcal{H})$ be dominant. Let $\delta \subset \mathbb{C}$ be closed. If there exists a bounded function $f : \mathbb{C} \setminus \delta \mapsto \mathcal{H}$ such that $(A - \lambda)f(\lambda) = x \neq 0$ for some $x \in \mathcal{H}$, then there exists an analytic function $g : \mathbb{C} \setminus \delta \mapsto \mathcal{H}$ such that $(A - \lambda)g(\lambda) = x$.*

Lemma 2.5. [22] *Let $A \in B(\mathcal{H})$ be a class $\mathcal{Y}$ operator and $\mathcal{M} \subset \mathcal{H}$ invariant under A. Then $A|_M$ is a class $\mathcal{Y}$ operator.*

Lemma 2.6. [20] *Let $A \in B(\mathcal{H})$ be* log*-hyponormal and $\mathcal{M} \subset \mathcal{H}$ invariant under A. Then $A|_M$ is* log*-hyponormal.*

Theorem 2.1. *Let $A \in B(\mathcal{H})$ be* log*-hyponormal and $B^* \in B(\mathcal{K})$ be class $\mathcal{Y}$. If $AC = CB$ for some operator $C \in B(\mathcal{K}, \mathcal{H})$, then $A^*C = CB^*$. Moreover the closure $\overline{\mathrm{ran}C}$ of the range of C reduces A, $(\ker C)^{\perp}$ reduces B and $A|_{\overline{\mathrm{ran}C}}, B|_{(\ker C)^{\perp}}$ are unitary equivalent normal operators.*

Proof. Since B^* is class $\mathcal{Y}$, there exist positive numbers α and k_α such that

$$|BB^* - B^*B|^{\alpha} \leq k_\alpha^2 (B - \lambda)(B - \lambda)^*, \text{ for all} \, \lambda \in \mathbb{C}.$$

Hence for $x \in |BB^* - B^*B|^{\frac{\alpha}{2}}\mathcal{K}$ there exists a bounded function $f : \mathbb{C} \mapsto \mathcal{K}$ such that

$$(B - \lambda)f(\lambda) = x, \text{ for all} \, \lambda \in \mathbb{C}$$

by [4]. Let $A = U|A|$ be the polar decomposition of A and define its Aluthge transform by $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$. Let $\tilde{A} = V|\tilde{A}|$, and define the second Aluthge transform of A by

$\hat{A} = |\tilde{A}|^{\frac{1}{2}} V |\tilde{A}|^{\frac{1}{2}}$. Then $\hat{A}$ is hyponormal [7]. Therefore

$$(\hat{A} - \lambda I) f(\lambda) = |\tilde{A}|^{\frac{1}{2}} (\tilde{A} - \lambda I) C f(\lambda)$$

$$= |\tilde{A}|^{\frac{1}{2}} C (B - \lambda) f(\lambda) = |\tilde{A}|^{\frac{1}{2}} Cx, \text{ for all } \lambda \in \mathbb{C}.$$

We claim that $|\tilde{A}|^{\frac{1}{2}} Cx = 0$. Because if $|\tilde{A}|^{\frac{1}{2}} Cx \neq 0$, there exists a bounded entire analytic function $g : \mathbb{C} \mapsto \mathcal{H}$ such that $(\hat{A} - \lambda) g(\lambda) = |\tilde{A}|^{\frac{1}{2}} Cx$ by Lemma 2.4. Since

$$g(\lambda) = (\hat{A} - \lambda)^{-1} |\tilde{A}|^{\frac{1}{2}} Cx \to 0 \text{ as } \lambda \to \infty,$$

we have $g(\lambda) = 0$ by Liouville's theorem, and hence $|\tilde{A}|^{\frac{1}{2}} Cx = 0$. This is a contradiction. Thus $|\tilde{A}|^{\frac{1}{2}} C |BB^* - B^* B|^{2n-1} \mathcal{K} = \{0\}$ and hence

$$C |BB^* - B^* B|^2 \mathcal{K} = \{0\}. \tag{1}$$

It follows from $AC = CB$ that $\overline{\operatorname{ran} C}$ and $(\ker C)^{\perp}$ are invariant subspaces of A and B^* respectively. Then A and B can be written

$$A = \begin{bmatrix} A_1 & S \\ 0 & A_2 \end{bmatrix} \text{ on } \mathcal{H} = \overline{\operatorname{ran} C} \oplus \overline{\operatorname{ran} C}^{\perp}$$

$$B = \begin{bmatrix} B_1 & 0 \\ S & B_2 \end{bmatrix} \text{ on } \mathcal{K} = (\ker C)^{\perp} \oplus \ker C$$

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} : (\ker C)^{\perp} \oplus \ker C \mapsto \overline{\operatorname{ran} C} \oplus \overline{\operatorname{ran} C}^{\perp}.$$

Hence C_1 is injective, with dense range and

$$A_1 C_1 = C_1 A_1. \tag{2}$$

We have

$$BB^* - B^* B = \begin{bmatrix} B_1 & 0 \\ S & B_2 \end{bmatrix} \begin{bmatrix} B_1^* & S^* \\ 0 & B_2^* \end{bmatrix} - \begin{bmatrix} B_1^* & S^* \\ 0 & B_2^* \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ S & B_2 \end{bmatrix}$$

$$= \begin{bmatrix} B_1 B_1^* - B_1^* B_1 - S^* S & B_1 S^* - S^* B_2 \\ (B_1 S^* - S^* B_2)^* & SS^* + B_2 B_2^* - B_2^* B_2 \end{bmatrix}$$

$$\begin{bmatrix} B_1B_1^* - B_1^*B_1 - S^*S & E_1 \\ E_1^* & F_1 \end{bmatrix}.$$

Thus

$$|BB^* - B^*B|^2 = \begin{bmatrix} (B_1B_1^* - B_1^*B_1 - S^*S)^2 + E_1E_1^* & E_2 \\ E_2^* & F_2 \end{bmatrix}.$$

Since $C|BB^* - B^*B|^2(\ker C)^\perp = \{0\}$ by (1), we have

$$C[B_1B_1^* - B_1^*B_1 - S^*S)^2 + E_1E_1^*] = 0$$

and since C_1 is injective, $(B_1B_1^* - B_1^*B_1 - S^*S)^2 + E_1E_1^* = 0$. Hence $B_1B_1^* - B_1^*B_1 - S^*S = 0$, that is, B_1^* is hyponormal. Multiply the two members of (2) by $|\tilde{A}|^{\frac{1}{2}}$ and since the polar decomposition of $\tilde{A} = V|\tilde{A}|$, we get

$$\hat{A}_1(|\tilde{A}_1|^{\frac{1}{2}}C_1) = (|\tilde{A}_1|^{\frac{1}{2}}C_1)B_1.$$

Since the second Aluthge transform $\hat{A} = |\tilde{A}|^{\frac{1}{2}}V|\tilde{A}|^{\frac{1}{2}}$ is hyponormal and B_1^* is hyponormal, we have $\hat{A}_1, B_1$ satisfy Fuglede-Putnam's theorem. Thus

$$\hat{A}_1{}^*(|\tilde{A}_1|^{\frac{1}{2}}C_1) = (|\tilde{A}_1|^{\frac{1}{2}}C_1)B_1^*.$$

Hence $\hat{A}_1|_{\overline{\mathrm{ran}(|\tilde{A}|^{\frac{1}{2}})C_1}}$ and $B_1|_{\ker(|\tilde{A}|^{\frac{1}{2}}C_1)^\perp}$ are normal operators by Lemma 2.1. Since $|\tilde{A}_1|^{\frac{1}{2}}$ and C_1 are injective, $|\tilde{A}_1|^{\frac{1}{2}}C_1$ is also injective. Hence

$$[\ker(|\tilde{A}|^{\frac{1}{2}}C_1)]^\perp = 0^\perp = (\ker C_1)^\perp = (\ker C)^\perp.$$

By the same arguments as above, we have

$$\overline{\mathrm{ran}(|\tilde{A}|^{\frac{1}{2}})C_1} = C_1^*\ker(|\tilde{A}|^{\frac{1}{2}})^\perp = 0^\perp = \overline{\mathrm{ran}C_1} = \overline{\mathrm{ran}C}.$$

Hence $\hat{A}_1$ is normal. This implies that A_1 is normal by [20]. Hence $\overline{\mathrm{ran}C}$ reduces A_1 by Lemma 2.5 and $(\ker C_1)^\perp$ reduces B_1^* by [24]. Since A_1 is normal, B_1^* is hyponormal and $A_1C_1 = C_1B_1$, we obtain $A_1^*C_1 = C_1B_1^*$ by the Fuglede-Putnam's theorem, and so $A^*C = CB^*$. The rest follows from Lemma 2.1. $\square$

Corollary 2.1. *Let $A \in B(\mathcal{H})$ be log-hyponormal and $B^* \in B(\mathcal{K})$ be class $\mathcal{Y}$. If $AC = CB$ for some operator $C \in B(\mathcal{K}, \mathcal{H})$, then $A^*C = CB^*$. Moreover the closure $\overline{\mathrm{ran}C}$ of the range of C reduces A, $(\ker C)^\perp$ reduces B and $A|_{\overline{\mathrm{ran}C}}, B|_{(\ker C)^\perp}$ are unitary equivalent normal operators.*

Proof. Since $AC = CB$, we have $B^*C^* = C^*A^*$. Hence $BC^* = B^{**}C^* = C^*A^{**} = C^*A$ by the previous theorem. Hence $A^*C = CB^*$. The rest follows from Lemma 2.1. $\square$

Corollary 2.2. *Let $A \in B(\mathcal{H})$. Then A is normal if and only if A is* log-*hyponormal and A^* is class $\mathcal{Y}$.*

The following version of the Fuglede-Putnam's theorem for log-hyponormal operators is immediate from Theorem 2.1 and [9, Theorem 4].

Corollary 2.3. *Let $A \in B(\mathcal{H})$ be log-hyponormal and $B^* \in B(\mathcal{K})$ be class $\mathcal{Y}$. If $AX_n - X_nB \to 0$ for a bounded sequence $\{X_n\}$, $X_n : \mathcal{K} \mapsto \mathcal{H}$, then $A^*X_n - X_nB^* \to 0$.*

Corollary 2.4. *Let $A \in B(\mathcal{H})$ and $B^* \in B(\mathcal{K})$ be such that $AX = XB$. If either A is pure* log-*hyponormal and B^* is class $\mathcal{Y}$, or A is* log-*hyponormal and B^* is pure class $\mathcal{Y}$, then $X = 0$.*

Proof. The hypotheses imply that $AX = XB$ and $A^*X = XB^*$ similtaneously by Theorem 2.1. Therefore $A|_{\overline{\mathrm{ran}X}}$ and $B|_{(\ker X)^\perp}$ are unitarily equivalent normal operators, which contradicts the hypotheses that A or B^* is pure. Hence we must have $X = 0$. $\square$

References

[1] Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integr. Equat. Oper. Theor, **13**(1990), 307-315.

[2] Arora, S.C., Arora, P., *On p-quasihyponormal operators for* $0 < p < 1$, Yokohama Math. J, **4(1)**(1993), 25-29.

[3] Ando, T., *Operators with a norm condition*, Acta. Sci. Math. (Szeged), **33**(1972), 169-178.

[4] Douglas, R.G., *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc., **17**(1966), 413-415.

[5] Fuglede, B., *A commutativity theorem for normal operators*, Proc. Nat. Acad. Sc. USA, **36**(1959), 35-40.

[6] Fujii, M., Himeji, C., Matsumoto, A., *Theorems of Ando and Saito for p-hyponormal operators*, Math. Japonica, **39**(1994), 595-598.

[7] Jeon, I.H., Tanahashi, K., Uchiyama, A., *On quasisimilarity for* log-*hyponormal operators*, Glasgow Math. J., **46**(2004), 169-176.

[8] Kim, I.H., *The Fuglede-Putnam's theorem for* (p,k)*-quasihyponormal operators*, J. Ineq. Appl., 2006, article ID 47481, page 1-7.

[9] Moore, R.L., Rogers, D.D., Trent, T.T., *A note on* M*-hyponormal operators*, Proc. Amer. Math. Soc, **83**(1981), 514-516.

[10] Mecheri, S., Tanahashi, K., Uchiyama, A., *Fuglede-Putnam theorem for p-hyponormal or class* $\mathcal{Y}$ *operator*, Bull Korean. Math. Soc., **43**(2006), 747-753.

[11] Patal, S.M., *On intertwing p-hyponormal operators*, Indian J. Math, **38**(1996), 287-290.

[12] Putnam, C.R., *On normal operators in Hilbert spaces*, Amer. J. Math, **73**(1951), 357-362.

[13] Putnam, C.R., *Range of Normal and subnormal normal operators*, Michigan Math. J., **18**(1971), 33-36.

[14] Stampfli, J.G., Wadhwa, B.L., *An asymmetric Putnam-Fuglede theorem for dominant operators*, Indiana Univ. Math. J., **25**(1976), 359-365.

[15] Stampfli, J.G., Wadhwa, B.L., *On Dominant operators*, Monatsshefte für Math., **84**1977, 143-153.

[16] Takahashi, K., *On the converse of the Fuglede-Putnam theorem*, Acta Sci. Math. (Szeged), **43**(1981), 123-125.

[17] Sheth, I.H., *On hyponormal operators*, Proc. Amer. Math. Soc., **17**(1966), 998-1000.

[18] Tanahashi, K., Uchiyama, A., *On* log-*hyponormal operators*, Integr. Equat. operat. Theor, **34**(1999), 364-372.

[19] Tanahashi, K., *Putnam's inequality for* log-*hyponormal operators*, Integr. Equat. Operat. Theor, **43**(2004), 364-372.

[20] Uchiyama, A., Tanahashi, K., *Fuglede-Putnam's theorem for p-hyponormal or log-hyponormal operators*, Glaskow Math. J., **44**(2002), 397-410.

[21] Uchiyama, A., *Berger-Shaw's theorem for p-hyponormal operators*, Integral equations Operator theory., Inequalities in operator theory and related topics (Japanese) Kyoto, 1997. Sūrikaisekikenkyūsho Kōkyūroku No. 1027, (1998), 110-121.

[22] Uchiyama, A., Yochino, T., *On the class* $\mathcal{Y}$ *operators*, Nihonkai. Math. J., **8**(1997), 179-174.

[23] Williams, J.P., *Operators similar to their adjoints*, Proc. Amer. Math. Soc., **20**(1969), 121-123.

[24] Yoshino, T., *Remark on the Generalized Putnam-Fuglede theorem*, Proc. Amer. Math. Soc, **95**(1985), 571-572.

[25] Xia, D., *Spectral theory for hyponormal operators*, Birkhauser Verlag, Boston, 1983.

King Saud University, College of Science
Department of Mathematics B.O.Pox 2455
Riyah 11451, Saudi Arabia
E-mail address: mecherisalah@hotmail.com

Department of Mathematics
Chlef University
02000, Chlef Algeria

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 2, June 2009

DIFFERENTIAL SUBORDINATIONS AND SUPERORDINATIONS FOR ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION STRUCTURE

G. MURUGUSUNDARAMOORTHY AND N. MAGESH

Abstract. In the present investigation, we obtain some subordination and superordination results involving Hadamard product for certain normalized analytic functions in the open unit disk. Relevant connections of the results, which are presented in this paper, with various other known results also pointed out.

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathcal{U} := \{z : |z| < 1\}$ and $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots .$$

Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + a_2 z^2 + \dots .$$

Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$.

If p and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \tag{1.1}$$

Received by the editors: 15.01.2009.
2000 *Mathematics Subject Classification.* Primary 30C45, Secondary 30C80.
Key words and phrases. Univalent functions, starlike functions, convex functions, differential subordination, differential superordination, Hadamard product (convolution), Dziok-Srivastava linear operator.

then p is a solution of the differential superordination (1.1). (If f is subordinate to F, then F is superordinate to f.) An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1.1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.1) is said to be the best subordinant. Recently Miller and Mocanu [12] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

For two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z).$$

For $\alpha_j \in \mathbb{C}$ $(j = 1, 2, \ldots, l)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ $(j = 1, 2, \ldots m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$$ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \leq m+1; l, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in \mathbb{N} := \{1, 2, 3 \ldots\}). \end{cases}$$

Corresponding to the function

$$h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z \, {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

the Dziok-Srivastava operator [6] (see also [7, 22]) $H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ is defined by the Hadamard product

$$\begin{aligned} H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) &:= h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \frac{a_n z^n}{(n-1)!}. \end{aligned} \tag{1.2}$$

For brevity, we write

$$H^l_m[\alpha_1]f(z) := H^l_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z).$$

It is easy to verify from (1.2) that

$$z(H^l_m[\alpha_1]f(z))' = \alpha_1 H^l_m[\alpha_1 + 1]f(z) - (\alpha_1 - 1)H^l_m[\alpha_1]f(z). \tag{1.3}$$

Special cases of the Dziok-Srivastava linear operator includes the Hohlov linear operator [8], the Carlson-Shaffer linear operator $L(a, c)$ [5], the Ruscheweyh derivative operator D^n [17], the generalized Bernardi-Libera-Livingston linear integral operator (*cf.* [2], [9], [10]) and the Srivastava-Owa fractional derivative operators (*cf.* [15], [16]).

Using the results of Miller and Mocanu [12], Bulboacă [4] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators (see [3]). Further, using the results of Mocanu [12] and Bulboacă [4] many researchers [1, 18, 19, 20, 21] have obtained sufficient conditions on normalized analytic functions f by means of differential subordinations and superordinations.

Recently, Murugusundaramoorthy and Magesh [13, 14] obtained sufficient conditions for a normalized analytic functions f to satisfy

$$q_1(z) \prec \left(\frac{H^l_m[\alpha_1]f(z)}{(z)}\right)^\delta \prec q_2(z), \quad q_1(z) \prec \frac{(f * \Phi)(z)}{f * \Psi)(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{H^l_m[\alpha_1 + 1](f * \Phi)(z)}{H^l_m[\alpha_1](f * \Psi)(z)} \prec q_2(z)$$

where q_1, q_2 are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$.

The main object of the present paper is to find sufficient condition for certain normalized analytic functions $f(z)$ in $\mathcal{U}$ such that $(f * \Psi)(z) \neq 0$ and f to satisfy

$$q_1(z) \prec \frac{H^l_m[\alpha_1](f * \Phi)(z)}{H^l_m[\alpha_1 + 1](f * \Psi)(z)} \prec q_2(z),$$

where q_1, q_2 are given univalent functions in $\mathcal{U}$ and $\Phi(z) = z + \sum\limits_{n=2}^{\infty} \lambda_n z^n$, $\Psi(z) = z + \sum\limits_{n=2}^{\infty} \mu_n z^n$ are analytic functions in $\mathcal{U}$ with $\lambda_n \geq 0$, $\mu_n \geq 0$ and $\lambda_n \geq \mu_n$. Also, we obtain the number of known results as their special cases.

2. **Subordination and Superordination results**

For our present investigation, we shall need the following:

Definition 2.1. [12] Denote by Q, the set of all functions f that are analytic and injective on $\overline{\mathcal{U}} - E(f)$, where

$$E(f) = \{\zeta \in \partial\mathcal{U} : \lim_{z\to\zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathcal{U} - E(f)$.

Lemma 2.2. [11] *Let q be univalent in the unit disk $\mathcal{U}$ and θ and ϕ be analytic in a domain D containing $q(\mathcal{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathcal{U})$. Set*

$$\psi(z) := zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) := \theta(q(z)) + \psi(z).$$

Suppose that

1. *$\psi(z)$ is starlike univalent in $\mathcal{U}$ and*
2. *$Re\left\{\frac{zh'(z)}{\psi(z)}\right\} > 0$ for $z \in \mathcal{U}$.*

If p is analytic with $p(0) = q(0)$, $p(\mathcal{U}) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.1}$$

then

$$p(z) \prec q(z)$$

and q is the best dominant.

Lemma 2.3. [4] *Let q be convex univalent in the unit disk $\mathcal{U}$ and ϑ and φ be analytic in a domain D containing $q(\mathcal{U})$. Suppose that*

1. *$Re\{\vartheta'(q(z))/\varphi(q(z))\} > 0$ for $z \in \mathcal{U}$ and*
2. *$\psi(z) = zq'(z)\varphi(q(z))$ is starlike univalent in $\mathcal{U}$.*

If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\mathcal{U}) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in $\mathcal{U}$ and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.2}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

Using Lemma 2.2, we first prove the following theorem.

Theorem 2.4. *Let $\Phi, \Psi \in \mathcal{A}$, $\gamma_4 \neq 0$, γ_1, γ_2, γ_3 be the complex numbers and q be convex univalent in $\mathcal{U}$ with $q(0) = 1$. Further assume that*

$$Re\left\{\frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4}q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0 \quad (z \in \mathcal{U}). \tag{2.3}$$

If $f \in \mathcal{A}$ satisfies

$$\Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z), \tag{2.4}$$

where

$$\Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$$
$$:= \begin{cases} \gamma_1 + \gamma_2 \left(\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)}\right)^2 + \gamma_3 \frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \\ +\gamma_4 \left(\alpha_1 \frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Phi)(z)} - (\alpha_1+1) \frac{H_m^l[\alpha_1+2](f*\Psi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} + 1\right) \\ \times \left(\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)}\right), \end{cases} \tag{2.5}$$

then

$$\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

Proof. Define the function p by

$$p(z) := \frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \quad (z \in \mathcal{U}). \tag{2.6}$$

Then the function p is analytic in $\mathcal{U}$ and $p(0) = 1$. Therefore, by making use of (2.6), we obtain

$$\begin{aligned} &\gamma_1 + \gamma_2 \left(\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)}\right)^2 + \gamma_3 \frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \\ &+\gamma_4 \left(\alpha_1 \frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Phi)(z)} - (\alpha_1+1) \frac{H_m^l[\alpha_1+2](f*\Psi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} + 1\right) \\ &\times \left(\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)}\right) \\ &= \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z). \end{aligned} \tag{2.7}$$

By using (2.7) in (2.4), we have

$$\gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z). \tag{2.8}$$

By setting

$$\theta(w) := \gamma_1 + \gamma_2\omega^2(z) + \gamma_3\omega \quad \text{and} \quad \phi(\omega) := \gamma_4,$$

it can be easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}-\{0\}$ and that $\phi(w) \neq 0$. Hence the result now follows by an application of Lemma 2.2. $\square$

When $l = 2$, $m = 1$, $\alpha_1 = a$, $\alpha_2 = 1$ and $\beta_1 = c$ in Theorem 2.4, we state the following corollary.

Corollary 2.5. *Let $\Phi, \Psi \in \mathcal{A}$. Let $\gamma_4 \neq 0$, γ_1, γ_2, γ_3 be the complex numbers and q be convex univalent in $\mathcal{U}$ with $q(0) = 1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$\Upsilon_1(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 zq'(z)$$

where

$$\Upsilon_1(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4) := \begin{cases} \gamma_1 + \gamma_2 \left(\frac{L(a,c)(f*\Phi)(z)}{L(a+1,c)(f*\Psi)(z)}\right)^2 + \gamma_3 \frac{L(a,c)(f*\Phi)(z)}{L(a+1,c)(f*\Psi)(z)} \\ +\gamma_4 \left(a\frac{L(a+1,c)(f*\Phi)(z)}{L(a,c)(f*\Phi)(z)} - (a+1)\frac{L(a+2,c)(f*\Psi)(z)}{L(a+1,c)(f*\Psi)(z)} + 1\right) \\ \times \left(\frac{L(a,c)(f*\Phi)(z)}{L(a+1,c)(f*\Psi)(z)}\right), \end{cases} \tag{2.9}$$

then

$$\frac{L(a,c)(f*\Phi)(z)}{L(a+1,c)(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

By fixing $\Phi(z) = \frac{z}{1-z}$ and $\Psi(z) = \frac{z}{1-z}$ in Theorem 2.4, we obtain the following corollary.

Corollary 2.6. *Let $\gamma_4 \neq 0$, γ_1, γ_2, γ_3 be the complex numbers and q be convex univalent in $\mathcal{U}$ with $q(0) = 1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$\begin{aligned} &\gamma_1 + \gamma_2\left(\frac{H_m^l[\alpha_1]f(z)}{H_m^l[\alpha_1+1]f(z)}\right)^2 + \gamma_3\frac{H_m^l[\alpha_1]f(z)}{H_m^l[\alpha_1+1]f(z)} \\ &+\gamma_4\left(\alpha_1\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)} - (\alpha_1+1)\frac{H_m^l[\alpha_1+2]f(z)}{H_m^l[\alpha_1+1]f(z)} + 1\right)\left(\frac{H_m^l[\alpha_1]f(z)}{H_m^l[\alpha_1+1]f(z)}\right) \\ &\qquad \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 zq'(z), \end{aligned}$$

then

$$\frac{H_m^l[\alpha_1]f(z)}{H_m^l[\alpha_1+1]f(z)} \prec q(z)$$

and q is the best dominant.

By taking $l=2$, $m=1$, $\alpha_1=1$, $\alpha_2=1$ and $\beta_1=1$ in Theorem 2.4, we state the following corollary.

Corollary 2.7. *Let $\Phi, \Psi \in \mathcal{A}$. Let $\gamma_4 \neq 0$, γ_1, γ_2, γ_3 be the complex numbers and q be convex univalent in $\mathcal{U}$ with $q(0)=1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$\gamma_1+\gamma_2\left(\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}\right)^2+\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}\left[(\gamma_3-\gamma_4)+\gamma_4\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)}-\gamma_4\frac{z(f*\Psi)''(z)}{(f*\Psi)'(z)}\right]$$
$$\prec \gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4 zq'(z),$$

then

$$\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)} \prec q(z)$$

and q is the best dominant.

By fixing $\Phi(z)=\Psi(z)$ in Corollary 2.7, we obtain the following corollary.

Corollary 2.8. *Let $\Phi \in \mathcal{A}$. Let $\gamma_4 \neq 0$, γ_1, γ_2, γ_3 be the complex numbers and q be convex univalent in $\mathcal{U}$ with $q(0)=1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$\gamma_1+\gamma_2\left(\frac{(f*\Phi)(z)}{z(f*\Phi)'(z)}\right)^2+\frac{(f*\Phi)(z)}{z(f*\Phi)'(z)}\left[(\gamma_3-\gamma_4)+\gamma_4\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)}-\gamma_4\frac{z(f*\Phi)''(z)}{(f*\Phi)'(z)}\right]$$
$$\prec \gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4 zq'(z),$$

then

$$\frac{(f*\Phi)(z)}{z(f*\Phi)'(z)} \prec q(z)$$

and q is the best dominant.

By fixing $\Phi(z)=\frac{z}{1-z}$ in Corollary 2.8, we obtain the following corollary.

Corollary 2.9. *Let $\gamma_4 \neq 0$, γ_1, γ_2, γ_3 be the complex numbers and q be convex univalent in $\mathcal{U}$ with $q(0)=1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$\gamma_1+\gamma_2\left(\frac{f(z)}{zf'(z)}\right)^2+\frac{f(z)}{zf'(z)}\left[(\gamma_3-\gamma_4)+\gamma_4\frac{zf'(z)}{f(z)}-\gamma_4\frac{zf''(z)}{f'(z)}\right]$$
$$\prec \gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4 zq'(z),$$

then

$$\frac{f(z)}{zf'(z)} \prec q(z)$$

and q is the best dominant.

Remark 2.10. For the choices of $\gamma_1 = \gamma_2 = 0$ and $\gamma_3 = 1$ in Corollary 2.9, we get the result obtained by Shanmugam et.al [19].

By taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2.4, we have the following corollary.

Corollary 2.11. *Assume that (2.3) holds. If $f \in \mathcal{A}$ and*

$$\Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \prec \gamma_1 + \gamma_2 (\frac{1+Az}{1+Bz})^2 + \gamma_3 \frac{1+Az}{1+Bz} + \gamma_4 \frac{(A-B)z}{(1+Bz)^2},$$

then

$$\frac{H_m^l[\alpha_1](f * \Phi)(z)}{H_m^l[\alpha_1+1](f * \Psi)(z)} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Now, by applying Lemma 2.3, we prove the following theorem.

Theorem 2.12. *Let $\Phi, \Psi \in \mathcal{A}$. Let γ_1, γ_2, γ_3 and $\gamma_4 \neq 0$ be the complex numbers. Let q be convex univalent in $\mathcal{U}$ with $q(0) = 1$. Assume that*

$$Re\, \left\{ \frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4} q(z) \right\} \geq 0. \tag{2.10}$$

Let $f \in \mathcal{A}$, $\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \in H[q(0), 1] \cap Q$. Let $\Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ be univalent in $\mathcal{U}$ and

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \prec \Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4), \tag{2.11}$$

where $\Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is given by (2.5), then

$$q(z) \prec \frac{H_m^l[\alpha_1](f * \Phi)(z)}{H_m^l[\alpha_1+1](f * \Psi)(z)}$$

and q is the best subordinant.

Proof. Define the function p by

$$p(z) := \frac{H_m^l[\alpha_1](f * \Phi)(z)}{H_m^l[\alpha_1+1](f * \Psi)(z)}. \tag{2.12}$$

Simple computation from (2.12), we get,

$$\Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4) = \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z),$$

then

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \prec \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 z p'(z).$$

By setting $\vartheta(w) = \gamma_1 + \gamma_2 w^2 + \gamma_3 w$ and $\phi(w) = \gamma_4$, it is easily observed that $\vartheta(w)$ is analytic in $\mathbb{C}$. Also, $\phi(w)$ is analytic in $\mathbb{C} - \{0\}$ and that $\phi(w) \neq 0$.

Since $q(z)$ is convex univalent function, it follows that

$$\mathrm{Re}\left\{\frac{\vartheta'(q(z))}{\phi(q(z))}\right\} = \Re\left\{\frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4} q(z)\right\} > 0, \quad z \in \mathcal{U}.$$

Now Theorem 2.12 follows by applying Lemma 2.3. $\square$

When $l = 2$, $m = 1$, $\alpha_1 = a$, $\alpha_2 = 1$ and $\beta_1 = c$ in Theorem 2.12, we state the following corollary.

Corollary 2.13. *Let $\Phi, \Psi \in \mathcal{A}$. Let γ_1, γ_2, γ_3 and $\gamma_4 \neq 0$ be the complex numbers. Let q be convex univalent in $\mathcal{U}$ with $q(0) = 1$ and (2.10) holds true. If $f \in \mathcal{A}$ $\frac{L(a,c)(f*\Phi)(z)}{L(a+1,c)(f*\Psi)(z)} \in H[q(0), 1] \cap Q$. Let $\Upsilon_1(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ be univalent in $\mathcal{U}$ and*

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 z q'(z) \prec \Upsilon_1(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4),$$

where $\Upsilon_1(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is given by (2.9), then

$$q(z) \prec \frac{L(a,c)(f*\Phi)(z)}{L(a+1,c)(f*\Psi)(z)}$$

and q is the best subordinant.

When $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$ and $\beta_1 = 1$ in Theorem 2.12, we derive the following corollary.

Corollary 2.14. *Let $\Phi, \Psi \in \mathcal{A}$. Let γ_1, γ_2, γ_3 and $\gamma_4 \neq 0$ be the complex numbers. Let q be convex univalent in $\mathcal{U}$ with $q(0) = 1$ and (2.10) holds true. If $f \in \mathcal{A}$, $\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)} \in H[q(0), 1] \cap Q$. Let*

$$\gamma_1 + \gamma_2 \left(\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}\right)^2 + \frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}\left[(\gamma_3 - \gamma_4) + \gamma_4 \frac{z(f*\Phi)'(z)}{(f*\Phi)(z)} - \gamma_4 \frac{z(f*\Psi)''(z)}{(f*\Psi)'(z)}\right]$$

be univalent in $\mathcal{U}$ *and*

$$\gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4 zq'(z)$$

$$\prec \gamma_1+\gamma_2\left(\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}\right)^2+\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}\left[(\gamma_3-\gamma_4)+\gamma_4\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)}-\gamma_4\frac{z(f*\Psi)''(z)}{(f*\Psi)'(z)}\right],$$

then

$$q(z)\prec\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)}$$

and q *is the best subordinant.*

By fixing $\Phi(z)=\Psi(z)$ in Corollary 2.14, we obtain the following corollary.

Corollary 2.15. *Let* $\Phi\in\mathcal{A}$. *Let* γ_1, γ_2, γ_3 *and* $\gamma_4\neq 0$ *be the complex numbers. Let* q *be convex univalent in* $\mathcal{U}$ *with* $q(0)=1$ *and (2.10) holds true. If* $f\in\mathcal{A}$, $\frac{(f*\Phi)(z)}{z(f*\Phi)'(z)}\in H[q(0),1]\cap Q$. *Let*

$$\gamma_1+\gamma_2\left(\frac{(f*\Phi)(z)}{z(f*\Phi)'(z)}\right)^2+\frac{(f*\Phi)(z)}{z(f*\Phi)'(z)}\left[(\gamma_3-\gamma_4)+\gamma_4\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)}-\gamma_4\frac{z(f*\Phi)''(z)}{(f*\Phi)'(z)}\right]$$

be univalent in $\mathcal{U}$ *and*

$$\gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4 zq'(z)$$

$$\prec \gamma_1+\gamma_2\left(\frac{(f*\Phi)(z)}{z(f*\Phi)'(z)}\right)^2+\frac{(f*\Phi)(z)}{z(f*\Phi)'(z)}\left[(\gamma_3-\gamma_4)+\gamma_4\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)}-\gamma_4\frac{z(f*\Phi)''(z)}{(f*\Phi)'(z)}\right],$$

then

$$q(z)\prec\frac{(f*\Phi)(z)}{z(f*\Phi)'(z)}$$

and q *is the best subordinant.*

By fixing $\Phi(z)=\frac{z}{1-z}$ in Corollary 2.15, we obtain the following corollary.

Corollary 2.16. *Let* γ_1, γ_2, γ_3 *and* $\gamma_4\neq 0$ *be the complex numbers. Let* q *be convex univalent in* $\mathcal{U}$ *with* $q(0)=1$ *and (2.10) holds true. If* $f\in\mathcal{A}$, $\frac{f(z)}{zf'(z)}\in H[q(0),1]\cap Q$. *Let* $\gamma_1+\gamma_2\left(\frac{f(z)}{zf'(z)}\right)^2+\frac{f(z)}{zf'(z)}\left[(\gamma_3-\gamma_4)+\gamma_4\frac{zf'(z)}{f(z)}-\gamma_4\frac{zf''(z)}{f'(z)}\right]$ *be univalent in* $\mathcal{U}$ *and*

$$\gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4 zq'(z)\prec$$
$$\gamma_1+\gamma_2\left(\frac{f(z)}{zf'(z)}\right)^2+\frac{f(z)}{zf'(z)}\left[(\gamma_3-\gamma_4)+\gamma_4\frac{zf'(z)}{f(z)}-\gamma_4\frac{zf''(z)}{f'(z)}\right],$$

then

$$q(z) \prec \frac{f(z)}{zf'(z)}$$

and q *is the best subordinant.*

By taking $q(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$ in Theorem 2.12, we obtain the following corollary.

Corollary 2.17. *Assume that (2.10) holds true. If* $f \in \mathcal{A}$, $\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \in H[q(0), 1] \cap Q$. *Let* $\Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ *be univalent in* $\mathcal{U}$ *and*

$$\gamma_1 + \gamma_2(\frac{1 + Az}{1 + Bz})^2 + \gamma_3 \frac{1 + Az}{1 + Bz} + \gamma_4 \frac{(A - B)z}{(1 + Bz)^2} \prec \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4),$$

then

$$\frac{1 + Az}{1 + Bz} \prec \frac{H_m^l[\alpha_1](f * \Phi)(z)}{H_m^l[\alpha_1 + 1](f * \Psi)(z)}$$

and $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

3. **Sandwich results**

We conclude this paper by stating the following sandwich results.

Theorem 3.1. *Let* q_1 *and* q_2 *be convex univalent in* $\mathcal{U}$, γ_1, γ_2, γ_3 *and* $\gamma_4 \neq 0$ *be the complex numbers. Suppose* q_2 *satisfies (2.3) and* q_1 *satisfies (2.10). Let* $\Phi, \Psi \in \mathcal{A}$. *Moreover suppose* $\frac{H_m^l[\alpha_1](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Psi)(z)} \in \mathcal{H}[1, 1] \cap Q$ *and* $\Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ *is univalent in* $\mathcal{U}$. *If* $f \in \mathcal{A}$ *satisfies*

$$\gamma_1 + \gamma_2 q_1^2(z) + \gamma_3 q_1(z) + \gamma_4 z q_1'(z) \prec \Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$$

$$\prec \gamma_1 + \gamma_2 q_2^2(z) + \gamma_3 q_2(z) + \gamma_4 z q_2'(z),$$

where $\Upsilon(f, \Phi, \Psi, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ *is given by (2.5), then*

$$q_1(z) \prec \frac{H_m^l[\alpha_1](f * \Phi)(z)}{H_m^l[\alpha_1 + 1](f * \Psi)(z)} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and best dominant.*

By taking

$$q_1(z) = \frac{1 + A_1 z}{1 + B_1 z} \ (-1 \leq B_1 < A_1 \leq 1)$$

and

$$q_2(z) = \frac{1 + A_2 z}{1 + B_2 z} \ (-1 \leq B_2 < A_2 \leq 1)$$

in Theorem 3.1 we obtain the following result.

Corollary 3.2. *Let* $\Phi, \Psi \in \mathcal{A}$. *If* $f \in \mathcal{A}$,

$$\frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \in \mathcal{H}[1, 1] \cap Q$$

and $\Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ *is univalent in U. Further*

$$\gamma_1 + \gamma_2 (\frac{1 + A_1 z}{1 + B_1 z})^2 + \gamma_3 \frac{1 + A_1 z}{1 + B_1 z} + \gamma_4 \frac{(A_1 - B_1) z}{(1 + B_1 z)^2}$$

$$\prec \Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$$

$$\prec \gamma_1 + \gamma_2 (\frac{1 + A_2 z}{1 + B_2 z})^2 + \gamma_3 \frac{1 + A_2 z}{1 + B_2 z} + \gamma_4 \frac{(A_2 - B_2) z}{(1 + B_2 z)^2}$$

where $\Upsilon(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ *is given by (2.5), then*

$$\frac{1 + A_1 z}{1 + B_1 z} \prec \frac{(f * \Phi)(z)}{z(f * \Psi)'(z)} \prec \frac{1 + A_2 z}{1 + B_2 z}$$

and $\frac{1+A_1 z}{1+B_1 z}$, $\frac{1+A_2 z}{1+B_2 z}$ *are respectively the best subordinant and best dominant.*

We remark that Theorem 3.1 can easily restated, for the different choices of $\Phi(z)$, $\Psi(z)$, l, m, $\alpha_1, \alpha_2, \dots \alpha_l$, $\beta_1, \beta_2, \dots \beta_m$ and for $\gamma_1, \gamma_2, \gamma_3, \gamma_4$.

References

[1] Ali, R.M., Ravichandran, V., Hussain Khan, M., Subramanian, K.G., *Differential sandwich theorems for certain analytic functions*, Far East J. Math. Sci., **15(1)**(2005), 87-94.

[2] Bernardi, S.D., *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., **135**(1969), 429-446.

[3] Bulboacă, T., *A class of superordination-preserving integral operators*, Indag. Math., New Ser., **13(3)**(2002), 301-311.

[4] Bulboacă, T., *Classes of first order differential superordinations*, Demonstr. Math., **35**(2002), no. 2, 287-292.

[5] Carlson, B.C., Shaffer, D.B., *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal., **15(4)**(1984), 737-745.

[6] Dziok, J., Srivastava, H.M., *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput., **103(1)**(1999), 1-13.

[7] Dziok, J., Srivastava, H.M., *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, ntegral Transforms Spec. Funct., **14(1)**(2003), 7-18.

[8] Hohlov, J.E., *Operators and operations on the class of univalent functions*, Izv. Vyssh. Uchebn. Zaved. Mat., **10(197)**(1978), 83-89.

[9] Libera, R.J., *Some classes of regular univalent functions,* Proc. Amer. Math. Soc., **16**(1965), 755-758.

[10] Livingston, A.E., *On the radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc., **17**(1966), 352-357.

[11] Miller, S.S., Mocanu, P.T., *Differential Subordinations: Theory and Applications*, Marcel Dekker Inc., New York, 2000.

[12] Miller, S.S., Mocanu, P.T., *Subordinants of differential superordinations*, Complex Variables, **48(10)**(2003), 815-826.

[13] Murugusundaramoorthy, G., Magesh, N., *Differential subordinations and superordinations for analytic functions defined by Dziok-Srivastava linear operator*, J. Inequal. Pure Appl. Math., **7**(2006), no. 4, Article 152, 1-9. (`http://jipam.uv.edu.au`)

[14] Murugusundaramoorthy, G., Magesh, N., *Differential Sandwich theorems for analytic functions defined by Hadamard product*, Annales Universitatis M. Curie-Sklodowska, **LXI**, Sec. A (2007), 117-127.

[15] Owa, S., *On the distortion theorems I*, Kyungpook Math. J., **18(1)**(1978), 53-59.

[16] Owa, S., Srivastava, H.M., *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math., **39(5)**(1987), 1057-1077.

[17] Ruscheweyh, S., *New criteria for univalent functions,* Proc. Amer. Math. Soc., **49**(1975), 109-115.

[18] Ravichandran, V., Sivaprasad Kumar, S., Taneja, H.C., *Classes of multivalent functions defined by Dziok-Srivastava linear operator and multiplier transformation*, Kyungpook Math. J., **46**(2006), 97-109.

[19] Shanmugam, T.N., Ravichandran, V., Sivasubramanian, S., *Differential sandwich theorems for some subclasses of analytic functions*, Aust. J. Math. Anal. Appl., **3**(2006), Issue 1, Article 8, 1-11.

[20] Shanmugam, T.N., Sivasubramanian, S., Darus, M., *On certain sublcasses of functions involving a linear operator,* Far East J. Math. Sci., **23(3)**(2006), 329-339.

[21] Shanmugam, T.N., Ramachandran, C., Darus, M., Sivasubramanian, S., *Differential sandwich theorems for some subclasses of analytic functions involving a linear operator*, Acta. Math. Univ. Comenianae, **LXXVI(2)**(2007), 287-294.

[22] Srivastava, H.M., *Some families of fractional derivative and other linear operators associated with analytic, univalent and multivalent functions*, Proc. International Conf. Analysis and its Applications, Allied Publishers Ltd, New Delhi, 2001, 209-243.

School of Science and Humanities
VIT University
Vellore - 632014, India
E-mail address: gmsmoorthy@yahoo.com

Department of Mathematics
Adhiyamaan College of Engineering
Hosur - 635109, India
E-mail address: nmagi_2000@yahoo.co.in

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 2, June 2009

A PIEZOELECTRIC FRICTIONLESS CONTACT PROBLEM WITH ADHESION

MOHAMED SELMANI

Abstract. We consider a quasistatic frictionless contact problem for a piezoelectric body. The contact is modelled with normal compliance. The adhesion of the contact surfaces is taken into account and modelled by a surface variable, the bonding field. We provide variational formulation for the mechanical problem and prove the existence of a unique weak solution to the problem. The proofs are based on arguments of time-dependent variational inequalities, differential equations and fixed point.

1. Introduction

A deformable material which undergoes piezoelectric effects is called a piezoelectric material. However, there are very few mathematical results concerning contact problems involving piezoelectric materials and therefore there is a need to extend the results on models for contact with deformable bodies which include coupling between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be found in [8, 9, 10, 18, 19] and more recently in [1, 17]. The importance of this paper is to make the coupling of the piezoelectric problem and a frictionless contact problem with adhesion. The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [3, 4, 6, 7, 12, 13, 14] and recently in the monographs

Received by the editors: 12.06.2007.
2000 *Mathematics Subject Classification.* 74M15, 74F99, 74M99.
Key words and phrases. Quasistatic process, frictionless contact, normal compliance, adhesion, piezoelectric material, existence and uniqueness, monotone operator, fixed point, weak solution.

[15, 16] . The novelty in all these papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by α, it describes the pointwise fractional density of adhesion of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [6, 7], the bonding field satisfies the restriction $0 \leq \alpha \leq 1$, when $\alpha = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\alpha = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \alpha < 1$ the adhesion is partial and only a fraction α of the bonds is active.

In this paper we describe a model of frictionless, adhesive contact between a piezoelectric body and a foundation. We provide a variational formulation of the model and, using arguments of evolutionary equations in Banach spaces, we prove that the model has a unique weak solution.

The paper is structured as follows. In section 2 we present notations and some preliminaries. The model is described in section 3 where the variational formulation is given. In section 4, we present our main result stated in Theorem 4.1 and its proof which is based on the construction of mappings between appropriate Banach spaces and a fixed point arguments.

2. **Notation and preliminaries**

In this short section, we present the notation we shall use and some preliminary material. For more details, we refer the reader to [2, 5, 11] . We denote by S^d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while "." and $|\,.\,|$ represent the inner product and the Euclidean norm on S^d and $\mathbb{R}^d$, respectively. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary Γ and let ν denote the unit outer normal on Γ. We shall use the notation

$$H = L^2(\Omega)^d = \left\{ \mathbf{u} = (u_i) \ / \ u_i \in L^2(\Omega) \ \right\},$$

$$H^1(\Omega)^d = \left\{ \mathbf{u} = (u_i) \ / \ u_i \in H^1(\Omega) \ \right\},$$

$$\mathcal{H} = \left\{ \sigma = (\sigma_{ij}) \ / \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \ \right\},$$

$$\mathcal{H}_1 = \left\{ \sigma \in \mathcal{H} \ / \ Div \ \sigma \in H \right\},$$

where $\varepsilon : H^1(\Omega)^d \to \mathcal{H}$ and $Div : \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \ Div\,\sigma = (\sigma_{i\,j,\,j}).$$

Here and below, the indices i and j run between 1 to d, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The spaces H, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u}, \mathbf{v})_H = \int_\Omega \mathbf{u} \,.\, \mathbf{v}\; dx \qquad \forall\, \mathbf{u}, \mathbf{v} \in H,$$

$$(\mathbf{u}, \mathbf{v})_{H^1(\Omega)^d} = \int_\Omega \mathbf{u} \,.\, \mathbf{v}\; dx + \int_\Omega \nabla\mathbf{u} \,.\, \nabla\mathbf{v}\; dx \qquad \forall\, \mathbf{u}, \mathbf{v} \in H^1(\Omega)^d,$$

where

$$\nabla\mathbf{v} = (v_{i,j}) \ \forall\, \mathbf{v} \in H^1(\Omega)^d,$$

$$(\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma \,.\, \tau\; dx \qquad \forall\, \sigma, \tau \in \mathcal{H},$$

$$(\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (Div\ \sigma, Div\ \tau)_H \ \ \forall\ \sigma, \tau \in \mathcal{H}_1.$$

The associated norms on the spaces H, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are denoted by $|\ .\ |_H$, $|\ .\ |_{H^1(\Omega)^d}$, $|\ .\ |_{\mathcal{H}}$ and $|\ .\ |_{\mathcal{H}_1}$respectively. Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and let $\gamma : H^1(\Omega)^d \to H_\Gamma$ be the trace map. For every element $\mathbf{v} \in H^1(\Omega)^d$, we also use the notation $\mathbf{v}$ to denote the trace $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ and we denote by v_ν and $\mathbf{v}_\tau$ the normal and the tangential components of $\mathbf{v}$ on the boundary Γ given by

$$v_\nu = \mathbf{v} \,.\, \nu, \ \mathbf{v}_\tau = \mathbf{v} - v_\nu \nu. \tag{2.1}$$

Similarly, for a regular (say C^1) tensor field $\sigma : \Omega \to S^d$ we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu) \,.\, \nu, \ \sigma_\tau = \sigma\nu - \sigma_\nu \nu, \tag{2.2}$$

and we recall that the following Green's formulas hold:

$$(\sigma, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (Div\ \sigma, \mathbf{v})_H = \int_\Gamma \sigma\nu \,.\, \mathbf{v}\; da \qquad \forall\, \mathbf{v} \in H^1(\Omega)^d. \tag{2.3}$$

$$(\mathbf{D}, \nabla\varphi)_H + (div\ \mathbf{D}, \varphi)_{L^2(\Omega)} = \int_\Gamma \mathbf{D}\ .\ \nu\ \varphi\ da \qquad \forall\,\varphi \in H^1(\Omega). \tag{2.4}$$

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \leq p \leq +\infty$ and $k \geq 1$. We denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$|\ \mathbf{f}\ |_{C(0,T;X)} = \max_{t\in[0,T]} |\ \mathbf{f}(t)\ |_X,$$

$$|\ \mathbf{f}\ |_{C^1(0,T;X)} = \max_{t\in[0,T]} |\ \mathbf{f}(t)\ |_X + \max_{t\in[0,T]} |\ \dot{\mathbf{f}}(t)\ |_X,$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable and, for a real number r, we use r_+ to represent its positive part, that is $r_+ = \max\{0, r\}$. For the convenience of the reader, we recall the following version of the classical theorem of Cauchy-Lipschitz (see, e.g., [20]).

Theorem 2.1. *Assume that* $(X, |\ .\ |_X)$ *is a real Banach space and* $T > 0$. *Let* $F(t,.) : X \to X$ *be an operator defined a.e. on* $(0,T)$ *satisfying the following conditions:*

1- There exists a constant $L_F > 0$ such that

$$|\ F(t,x) - F(t,y)\ |_X \leq L_F\ |\ x - y\ |_X\ \forall\, x,y \in X,\ \text{a.e. } t \in (0,T)\,.$$

2- There exists $p \geq 1$ such that $t \longmapsto F(t,x) \in L^p(0,T;X) \quad \forall\, x \in X$.

Then for any $x_0 \in X$, there exists a unique function $x \in W^{1,\,p}(0,T;X)$ such that

$$\dot{x}(t) = F(t, x(t)) \qquad \text{a.e. } t \in (0,T)\,,$$

$$x(0) = x_0.$$

Theorem 2.1 will be used in section 4 to prove the unique solvability of the intermediate problem involving the bonding field.

Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1\times X_2}$.

3. Mechanical and variational formulations

We describe the model for the process, we present its variational formulation. The physical setting is the following. An electro-elastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with outer Lipschitz surface Γ. The body is submitted to the action of body forces of density $\mathbf{f}_0$ and volume electric charges of density q_0. It is also submitted to mechanical and electric constraint on the boundary. We consider a partition of Γ into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, on one hand, and on two measurable parts Γ_a and Γ_b, on the other hand, such that $meas\ (\Gamma_1) > 0$ and $meas\ (\Gamma_a) > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0, T)$, so the displacement field vanishes there. A surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (0, T)$ and a body force of density $\mathbf{f}_0$ acts in $\Omega \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0, T)$ and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (0, T)$. The body is in adhesive contact with an obstacle, or foundation, over the contact surface Γ_3. We suppose that the body forces and tractions vary slowly in time, and therefore, the accelerations in the system may be neglected. Neglecting the inertial terms in the equation of motion leads to a quasistatic approach of the process. We denote by $\mathbf{u}$ the displacement field, by σ the stress tensor field and by $\varepsilon(\mathbf{u})$ the linearized strain tensor. We use a piezoelectric constitutive law given by

$$\sigma = \mathcal{A}(\varepsilon(\mathbf{u})) - \mathcal{E}^*\mathbf{E}(\varphi),$$

$$\mathbf{D} = \mathcal{E}\varepsilon(\mathbf{u}) + B\mathbf{E}(\varphi),$$

these relations represent the electro-viscoelastic constitutive law of the material which $\mathcal{A}$ is a given nonlinear function, $\mathbf{E}(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represents the third order piezoelectric tensor, $\mathcal{E}^*$ is its transposed and is given by $\mathcal{E}^* = (e^*_{ijk})$, where $e^*_{ijk} = e_{kij}$ and B denotes the electric permitivitty tensor.

To simplify the notation, we do not indicate explicitely the dependence of various functions on the variables $\mathbf{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$. Then, the classical formulation of the mechanical problem of piezoelectric material, frictionless, adhesive contact may be stated as follows.

Problem P. Find a displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\sigma : \Omega \times [0,T] \to S^d$ *, an electric potential field* $\varphi : \Omega \times [0,T] \to \mathbb{R}$*, an electric displacement field* $\mathbf{D} : \Omega \times [0,T] \to \mathbb{R}^d$ and *a bonding field* $\alpha : \Gamma_3 \times [0,T] \to \mathbb{R}$ *such that*

$$\sigma = \mathcal{A}(\varepsilon(\mathbf{u})) + \mathcal{E}^* \nabla\varphi \text{ in } \Omega \times (0,T), \tag{3.1}$$

$$\mathbf{D} = \mathcal{E}\varepsilon(\mathbf{u}) - B\nabla\varphi \text{ in } \Omega \times (0,T), \tag{3.2}$$

$$Div\ \sigma + \mathbf{f}_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{3.3}$$

$$div\ \mathbf{D} = q_0 \text{ in } \Omega \times (0,T), \tag{3.4}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{3.5}$$

$$\sigma\nu = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{3.6}$$

$$-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu \alpha^2 R_\nu(u_\nu) \quad \text{on } \Gamma_3 \times (0,T), \tag{3.7}$$

$$-\sigma_\tau = p_\tau(\alpha)\mathbf{R}_\tau(\mathbf{u}_\tau) \text{ on } \Gamma_3 \times (0,T), \tag{3.8}$$

$$\dot{\alpha} = -(\alpha(\gamma_\nu(R_\nu(u_\nu))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_\tau) \mid^2) - \varepsilon_a)_+ \text{ on } \Gamma_3 \times (0,T), \tag{3.9}$$

$$\alpha(0) = \alpha_0 \text{ on } \Gamma_3, \tag{3.10}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), \tag{3.11}$$

$$\mathbf{D} \,.\, \nu = q_2 \quad \text{on } \Gamma_b \times (0,T). \tag{3.12}$$

First, (3.1) and (3.2) represent the electro-elastic constitutive law described above. Equations (3.3) and (3.4) represent the equilibrium equations for the stress and electric-displacement fields while (3.5) and (3.6) are the displacement and traction boundary condition, respectively. Condition (3.7) represents the normal compliance conditions with adhesion where γ_ν is a given adhesion coefficient and p_ν is a given positive function which will be described below. In this condition the interpenetrability between the body and the foundation is allowed, that is u_ν can be positive on Γ_3. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu \alpha^2 R_\nu(u_\nu)$, the adhesive traction is tensile and is proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu L$. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of the operator R_ν, together with the operator $\mathbf{R}_\tau$ defined below, is motivated by mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows. Condition (3.8) represents the adhesive contact condition on the tangential plane, in which p_τ is a given function and $\mathbf{R}_\tau$ is the truncation operator given by

$$\mathbf{R}_\tau(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{if} \quad |\,\mathbf{v}\,| \le L, \\ L\frac{\mathbf{v}}{|\mathbf{v}|} & \text{if} \quad |\,\mathbf{v}\,| > L. \end{cases}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L. The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted.

Next, the equation (3.9) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [3], see also [15, 16] for more details. Here, besides γ_ν, two new adhesion coefficients are involved, γ_τ and ε_a. Notice that in this model once debonding occurs bonding cannot be reestablished since, as it follows from (3.9), $\dot{\alpha} \le 0$. Finally, (3.10) represents the initial condition in which α_0 is the given initial bonding field, (3.11) and (3.12) represent the electric boundary conditions. To obtain the variational formulation of the problem (3.1)-(3.12), we introduce for the bonding field the set

$$Z = \{\theta \in L^\infty(\Gamma_3) \;/\; 0 \le \theta \le 1 \text{ a.e. on } \Gamma_3\}\,,$$

and for the displacement field we need the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \left\{ \mathbf{v} \in H^1(\Omega)^d \ / \ \mathbf{v} = 0 \text{ on } \Gamma_1 \right\}.$$

Since $meas\ (\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $C_k > 0$, that depends only on Ω and Γ_1, such that

$$\mid \varepsilon(\mathbf{v}) \mid_{\mathcal{H}} \geq C_k \mid \mathbf{v} \mid_{H^1(\Omega)^d} \quad \forall \mathbf{v} \in V.$$

A proof of Korn's inequality may be found in [11, p.79]. On the space V we consider the inner product and the associated norm given by

$$(\mathbf{u}, \mathbf{v})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \mid \mathbf{v} \mid_V = \mid \varepsilon(\mathbf{v}) \mid_{\mathcal{H}} \quad \forall \mathbf{u}, \mathbf{v} \in V. \tag{3.13}$$

It follows that $\mid . \mid_{H^1(\Omega)^d}$ and $\mid . \mid_V$ are equivalent norms on V and therefore $(V, \mid . \mid_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3.13), there exists a constant $C_0 > 0$, depending only on Ω, Γ_1 and Γ_3 such that

$$\mid \mathbf{v} \mid_{L^2(\Gamma_3)^d} \leq C_0 \mid \mathbf{v} \mid_V \qquad \forall \mathbf{v} \in V. \tag{3.14}$$

We also introduce the spaces

$$W = \left\{ \phi \in H^1(\Omega) \ / \ \phi = 0 \text{ on } \Gamma_a \right\},$$

$$\mathcal{W} = \left\{ \mathbf{D} = (D_i) \ / \ D_i \in L^2(\Omega),\ div\ \mathbf{D} \in L^2(\Omega) \right\},$$

where $div\ \mathbf{D} = (D_{i,i})$. The spaces W and $\mathcal{W}$ are real Hilbert spaces with the inner products

$$(\varphi, \phi)_W = \int_\Omega \nabla\varphi . \nabla\phi \ dx,$$

$$(\mathbf{D}, \mathbf{E})_{\mathcal{W}} = \int_\Omega \mathbf{D}.\mathbf{E} \ dx + \int_\Omega div\ \mathbf{D}. div\ \mathbf{E} \ dx.$$

The associated norms will be denoted by $\mid . \mid_W$ and $\mid . \mid_{\mathcal{W}}$, respectively. Notice also that, since $meas(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$\mid \nabla\phi \mid_{L^2(\Omega)^d} \geq C_F \mid \phi \mid_{H^1(\Omega)} \quad \forall \phi \in W, \tag{3.15}$$

where $C_F > 0$ is a constant which depends only on Ω and Γ_a.

In the study of the mechanical problem (3.1)-(3.12), we assume that the constitutive function $\mathcal{A} : \Omega \times S^d \to S^d$ satisfies

$$\left\{\begin{array}{l}(a)\text{ There exists a constant } L_{\mathcal{A}} > 0 \text{ Such that}\\ \mid \mathcal{A}(\mathbf{x},\varepsilon_1) - \mathcal{A}(\mathbf{x},\varepsilon_2) \mid \leq L_{\mathcal{A}} \mid \varepsilon_1 - \varepsilon_2 \mid \ \ \forall \varepsilon_1, \varepsilon_2 \in S^d, \text{ a.e. } \mathbf{x} \in \Omega.\\ (b)\text{ There exists a constant } m_{\mathcal{A}} > 0 \text{ Such that}\\ (\mathcal{A}(\mathbf{x},\varepsilon_1) - \mathcal{A}(\mathbf{x},\varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}} \mid \varepsilon_1 - \varepsilon_2 \mid^2 \ \ \forall \varepsilon_1, \varepsilon_2 \in S^d, \text{ a.e. } \mathbf{x} \in \Omega.\\ (c)\text{ The mapping } \mathbf{x} \to \mathcal{A}(\mathbf{x},\varepsilon) \text{ is Lebesgue measurable on } \Omega \text{ for any } \varepsilon \in S^d.\\ (d)\text{ The mapping } \mathbf{x} \to \mathcal{A}(\mathbf{x},\mathbf{0}) \text{ belongs to } \mathcal{H}.\end{array}\right. \tag{3.16}$$

The operator $B = (B_{ij})$: $\Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\left\{\begin{array}{l}(a)\ B(\mathbf{x},\mathbf{E}) = (b_{ij}(\mathbf{x})E_j)\ \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e.} \ \mathbf{x} \in \Omega.\\ (b)\ b_{ij} = b_{ji}\,, \ \ b_{ij} \in L^\infty(\Omega), \quad 1 \leq i, j \leq d.\\ (c)\ \text{ There exists a constant } m_B > 0 \ \text{ such that } B\mathbf{E}.\mathbf{E} \geq m_B \mid \mathbf{E} \mid^2\\ \ \forall\, \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. in } \Omega.\end{array}\right. \tag{3.17}$$

The operator $\mathcal{E} : \Omega \times S^d \to \mathbb{R}^d$ satisfies

$$\left\{\begin{array}{l}(a)\ \mathcal{E} = (e_{ijk}), e_{ijk} \in L^\infty(\Omega), \quad 1 \leq i, j, k \leq d.\\ (b)\ \mathcal{E}(\mathbf{x})\sigma.\tau = \sigma.\ \mathcal{E}^*(\mathbf{x})\tau \quad \forall \sigma, \tau \in S^d, \text{ a.e. in } \Omega.\end{array}\right. \tag{3.18}$$

The normal compliance function $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\left\{\begin{array}{l}(a)\text{ There exists a constant } L_\nu > 0 \text{ such that}\\ \mid p_\nu(\mathbf{x},r_1) - p_\nu(\mathbf{x},r_2) \mid \leq L_\tau \mid r_1 - r_2 \mid \ \ \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3.\\ (b)\ (p_\nu(\mathbf{x},r_1) - p_\nu(\mathbf{x},r_2))(r_1 - r_2) \geq 0 \ \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3.\\ (c)\text{ The mapping } \mathbf{x} \to p_\nu(\mathbf{x},r) \text{ is measurable on } \Gamma_3, \text{ for any } r \in \mathbb{R}.\\ (d)\ p_\nu(\mathbf{x},r) = 0 \text{ for all } r \leq 0, \text{ a.e. } \mathbf{x} \in \Gamma_3.\end{array}\right. \tag{3.19}$$

The tangential contact function $p_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_\tau > 0 \text{ such that} \\ \mid p_\tau(\mathbf{x}, d_1) - p_\tau(\mathbf{x}, d_2) \mid \leq L_\tau \mid d_1 - d_2 \mid \ \forall d_1, d_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ (b) \text{ There exists } M_\tau > 0 \text{ such that } \mid p_\tau(\mathbf{x}, d) \mid \leq M_\tau \ \forall d \in \mathbb{R}, \text{a.e. } \mathbf{x} \in \Gamma_3. \\ (c) \text{ The mapping } \mathbf{x} \to p_\tau(\mathbf{x}, d) \text{ is measurable on } \Gamma_3, \text{ for any } d \in \mathbb{R}. \\ (d) \text{ The mapping } \mathbf{x} \to p_\tau(\mathbf{x}, 0) \in L^2(\Gamma_3). \end{cases} \tag{3.20}$$

We also suppose that the body forces and surface tractions have the regularity

$$\mathbf{f}_0 \in W^{1,\infty}(0,T; L^2(\Omega)^d), \quad \mathbf{f}_2 \in W^{1,\infty}(0,T; L^2(\Gamma_2)^d), \tag{3.21}$$

$$q_0 \in W^{1,\infty}(0,T; L^2(\Omega)), \quad q_2 \in W^{1,\infty}(0,T; L^2(\Gamma_b)). \tag{3.22}$$

The adhesion coefficients satisfy

$$\gamma_\nu, \gamma_\tau, \varepsilon_a \in L^\infty(\Gamma_3), \gamma_\nu, \gamma_\tau, \varepsilon_a \geq 0 \text{ a.e. on } \Gamma_3. \tag{3.23}$$

The initial bonding field satisfies

$$\alpha_0 \in Z. \tag{3.24}$$

Next, we denote by $\mathbf{f} : [0,T] \to V$ the function defined by

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t) \, . \, \mathbf{v} \; dx \; + \; \int_{\Gamma_2} \mathbf{f}_2(t) \, . \, \mathbf{v} \; da \ \ \forall \mathbf{v} \in V, \ t \in [0,T] \,, \tag{3.25}$$

and we denote by q: $[0,T] \to W$ the function defined by

$$(q(t), \phi)_W = \int_\Omega q_0(t) \, . \, \phi \; dx \; - \; \int_{\Gamma_b} q_2(t) \, . \, \phi \; da \ \ \forall \phi \in W, \ t \in [0,T] \,. \tag{3.26}$$

Next, we denote by $j_{ad} : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ the adhesion functional defined by

$$j_{ad}(\alpha, \mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} (-\gamma_\nu \alpha^2 R_{\,\nu}(u_\nu) \, v_\nu + p_\tau(\alpha) \mathbf{R}_\tau(\mathbf{u}_{\,\tau}) \, .\mathbf{v}_\tau) \; da. \tag{3.27}$$

In addition to the functional (3.27), we need the normal compliance functional

$$j_{nc}(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\nu(u_\nu) v_\nu \; da. \tag{3.28}$$

Keeping in mind (3.19)-(3.20), we observe that the integrals (3.27) and (3.28) are well defined and we note that conditions (3.21)-(3.22) imply

$$\mathbf{f} \in W^{1,\infty}(0,T;V), \;\; q \in W^{1,\infty}(0,T;W). \tag{3.29}$$

Using standard arguments we obtain the variational formulation of the mechanical problem (3.1)-(3.12).

Problem PV. Find a displacement field $\mathbf{u} : [0,T] \to V$, an electric potential field $\varphi : [0,T] \to W$ and a bonding field $\alpha : [0,T] \to L^\infty\ (\Gamma_3)$ such that

$$(\mathcal{A}\varepsilon(\mathbf{u}(t)), \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j_{ad}(\alpha(t), \mathbf{u}(t), \mathbf{v})$$

$$+ j_{nc}(\mathbf{u}(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_V \;\; \forall \mathbf{v} \in V, t \in (0,T)\,, \tag{3.30}$$

$$(B\nabla\varphi(t), \nabla\phi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(\mathbf{u}(t)), \nabla\phi)_{L^2(\Omega)^d} = (q(t), \phi)_W \quad \forall \phi \in W, t \in (0,T)\,, \tag{3.31}$$

$$\dot{\alpha}(t) = -(\alpha(t)(\gamma_\nu(R_\nu(u_\nu(t)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_\tau(t)) \mid^2 \;\;) - \varepsilon_a)_+ \text{ a.e. } t \in (0,T)\,, \tag{3.32}$$

$$\alpha(0) = \alpha_0. \tag{3.33}$$

The existence of the unique solution of problem PV is stated and proved in the next section.

Remark 3.1. *We note that, in problem P and in problem PV, we do not need to impose explicitly the restriction $0 \leq \alpha \leq 1$. Indeed, equation (3.32) guarantees that $\alpha(\mathbf{x}, t) \leq \alpha_0(\mathbf{x})$ and, therefore, assumption (3.24) shows that $\alpha(\mathbf{x}, t) \leq 1$ for $t \geq 0$, a.e. $\mathbf{x} \in \Gamma_3$. On the other hand, if $\alpha(\mathbf{x}, t_0) = 0$ at time t_0, then it follows from (3.32) that $\dot{\alpha}(\mathbf{x}, t) = 0$ for all $t \geq t_0$ and therefore, $\alpha(\mathbf{x}, t) = 0$ for all $t \geq t_0$, a.e. $\mathbf{x} \in \Gamma_3$. We conclude that $0 \leq \alpha(\mathbf{x}, t) \leq 1$ for all $t \in [0,T]$, a.e. $\mathbf{x} \in \Gamma_3$.*

Below in this section $\alpha, \alpha_1, \alpha_2$ denote elements of $L^2(\Gamma_3)$ such that $0 \leq \alpha, \alpha_1, \alpha_2 \leq 1$ a.e. $\mathbf{x} \in \Gamma_3$, $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{v}$ represent elements of V and $C > 0$ represents generic constants which may depend on $\Omega, \Gamma_3, \Gamma_3, p_\nu, p_\tau, \gamma_\nu, \gamma_\tau$ and L.

First, we note that the functional j_{ad} and j_{nc} are linear with respect to the last argument and, therefore,

$$j_{ad}(\alpha, \mathbf{u}, -\mathbf{v}) = -j_{ad}(\alpha, \mathbf{u}, \mathbf{v}),\; j_{nc}(\mathbf{u}, -\mathbf{v}) = -j_{nc}(\mathbf{u}, \mathbf{v}). \tag{3.34}$$

Next, using (3.27), the properties of the truncation operators R_ν and $\mathbf{R}_\tau$ as well as assumption (3.20) on the function p_τ, after some calculus we find

$$j_{ad}(\alpha_1, \mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{ad}(\alpha_2, \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)$$
$$\leq C \int_{\Gamma_3} |\alpha_1 - \alpha_2 || \mathbf{u}_1 - \mathbf{u}_2 | \, da,$$

and, by (3.14), we obtain

$$j_{ad}(\alpha_1, \mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{ad}(\alpha_2, \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)$$
$$\leq C |\alpha_1 - \alpha_2 |_{L^2(\Gamma_3)} | \mathbf{u}_1 - \mathbf{u}_2 |_V \, . \tag{3.35}$$

Similar computations, based on the Lipschitz continuity of R_ν, $\mathbf{R}_\tau$ and p_τ show that the following inequality also holds

$$| \, j_{ad}(\alpha, \mathbf{u}_1, \mathbf{v}) - j_{ad}(\alpha, \mathbf{u}_2, \mathbf{v}) \, | \leq C \mid \mathbf{u}_1 - \mathbf{u}_2 |_V | \, \mathbf{v} \, |_V \, . \tag{3.36}$$

We take now $\alpha_1 = \alpha_2 = \alpha$ in (3.35) to deduce

$$j_{ad}(\alpha, \mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{ad}(\alpha, \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \leq 0. \tag{3.37}$$

Also, we take $\mathbf{u}_1 = \mathbf{v}$ and $\mathbf{u}_2 = 0$ in (3.36) then we use the equalities $R_\nu(0) = 0$, $\mathbf{R}_\tau(\mathbf{0}) = \mathbf{0}$ and (3.34) to obtain

$$j_{ad}(\alpha, \mathbf{v}, \mathbf{v}) \geq 0. \tag{3.38}$$

Now, we use (3.28) to see that

$$| \, j_{nc}(\mathbf{u}_1, \mathbf{v}) - j_{nc}(\mathbf{u}_2, \mathbf{v}) \, | \leq \int_{\Gamma_3} | \, p_\nu(u_{1\nu}) - p_\nu(u_{2\nu}) \, || \, v_\nu \, | \, da,$$

and therefore (3.19) (a) and (3.14) imply

$$| \, j_{nc}(\mathbf{u}_1, \mathbf{v}) - j_{nc}(\mathbf{u}_2, \mathbf{v}) \, | \leq C \mid \mathbf{u}_1 - \mathbf{u}_2 |_V | \, \mathbf{v} \, |_V \, . \tag{3.39}$$

We use again (3.28) to see that

$$j_{nc}(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{nc}(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \leq \int_{\Gamma_3} (p_\nu(u_{1\nu}) - p_\nu(u_{2\nu}))(u_{2\nu} - u_{1\nu}) da,$$

and therefore (3.19) (b) implies

$$j_{nc}(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{nc}(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \leq 0. \tag{3.40}$$

We take $\mathbf{u}_1 = \mathbf{v}$ and $\mathbf{u}_2 = 0$ in the previous inequality and use (3.19) (d) and (3.40) to obtain

$$j_{nc}(\mathbf{v}, \mathbf{v}) \geq 0. \tag{3.41}$$

Inequalities (3.35)-(3.41) and equality (3.34) will be used in various places in the rest of the paper.

4. **An existence and uniqueness result**

Now, we propose our existence and uniqueness result.

Theorem 4.1. *Assume that (3.16)-(3.24) hold. Then there exists a unique solution* $\{\mathbf{u},\varphi, \alpha\}$ *to problem* PV*. Moreover, the solution satisfies*

$$\mathbf{u} \in W^{1,\infty}(0, T; V), \tag{4.1}$$

$$\varphi \in W^{1,\infty}(0, T; W), \tag{4.2}$$

$$\alpha \in W^{1,\infty}(0, T; L^\infty(\Gamma_3)). \tag{4.3}$$

The functions $\mathbf{u}$,$\varphi, \sigma, \mathbf{D}$ and α which satisfy (3.1)-(3.2) and (3.30)-(3.33) are called a weak solution of the contact problem P.

We conclude that, under the assumptions (3.16)-(3.24), the mechanical problem (3.1)-(3.12) has a unique weak solution satisfying (4.1)-(4.3). The regularity of the weak solution is given by (4.1)-(4.3) and, in term of stresses,

$$\sigma \in W^{1,\infty}(0, T; \mathcal{H}_1), \tag{4.4}$$

$$\mathbf{D} \in W^{1,\infty}(0, T; \mathcal{W}). \tag{4.5}$$

Indeed, it follows from (3.30) and (3.31) that $Div\ \sigma(t) + \mathbf{f}_0(t) = 0$, $div\ \mathbf{D} = q_0(t)$ for all $t \in [0, T]$ and therefore the regularity (4.1) and (4.2) of $\mathbf{u}$ and φ, combined with (3.16)-(3.22) implies (4.4) and (4.5).

The proof of Theorem 4.1 is carried out in several steps that we prove in what follows, everywhere in this section we suppose that assumptions of Theorem 4.1 hold, and we consider that C is a generic positive constant which depends on

$\Omega, \Gamma_1, \Gamma_3, p_\nu, p_\tau, \gamma_\nu, \gamma_\tau$ and L and may change from place to place. Let $\mathcal{Z}$ denote the closed subset of $C(0,T;L^2(\Gamma_3))$ defined by

$$\mathcal{Z} = \left\{\theta \in C(0,T;L^2(\Gamma_3)) \ / \ \theta(t) \in Z \ \forall t \in [0,T], \theta(0) = \alpha_0\right\}. \tag{4.6}$$

Let $\alpha \in \mathcal{Z}$ be given. In the first step we consider the following variational problem.

Problem PV_α. Find a displacement field $\mathbf{u}_\alpha : [0,T] \to V$, an electric potential field $\varphi_\alpha : [0,T] \to W$ *such that*

$$(\mathcal{A}\varepsilon(\mathbf{u}_\alpha(t)), \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_\alpha(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j_{ad}(\alpha(t), \mathbf{u}_\alpha(t), \mathbf{v})$$
$$+ j_{nc}(\mathbf{u}_\alpha(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_V \quad \forall \mathbf{v} \in V, \ t \in [0,T], \tag{4.7}$$
$$(B\nabla\varphi_\alpha(t), \nabla\phi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(\mathbf{u}_\alpha(t)), \nabla\phi)_{L^2(\Omega)^d}$$
$$= (q(t), \phi)_W \quad \forall \phi \in W, t \in (0,T). \tag{4.8}$$

We have the following result for the problem.

Lemma 4.2. *There exists a unique solution to problem PV_α. The solution satisfies* $(\mathbf{u}_\alpha, \varphi_\alpha) \in C(0,T;V) \times C(0,T;W)$.

Proof. Let $t \in [0,T]$ we consider the product space $X = V \times W$ with the inner product:

$$(x,y)_X = (\mathbf{u}, \mathbf{v})_V + (\varphi, \phi)_W \quad \forall x = (\mathbf{u}, \varphi), \ y = (\mathbf{v}, \phi) \in X, \tag{4.9}$$

and the associated norm $| \, . \, |_X$. Let $A_t : X \to X$ be the operator given by

$$(A_t x, y)_X = (\mathcal{A}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} + (B\nabla\varphi, \nabla\phi)_{L^2(\Omega)^d} + (\mathcal{E}^*\nabla\varphi, \varepsilon(\mathbf{v}))_{\mathcal{H}}$$
$$-(\mathcal{E}\varepsilon(\mathbf{u}), \nabla\phi)_{L^2(\Omega)^d} + j_{nc}(\mathbf{u}, \mathbf{v}) + j_{ad}(\alpha(t), \mathbf{u}, \mathbf{v})$$
$$\forall x = (\mathbf{u}, \varphi), \ y = (\mathbf{v}, \phi) \in X. \tag{4.10}$$

We consider the element $F \in X$ given by

$$F = (\mathbf{f}, q) \in X. \tag{4.11}$$

We consider the following equivalence result the couple $x_\alpha = (\mathbf{u}_\alpha, \varphi_\alpha)$ is a solution to problem PV_α if and only if

$$(A_t x_\alpha(t), y)_X = (F(t), y)_X, \ \forall y \in X, \ t \in [0,T]. \tag{4.12}$$

Indeed, let $x_\alpha(t) = (\mathbf{u}_\alpha(t), \varphi_\alpha(t)) \in X$ be a solution to problem PV_α and let $y = (\mathbf{v}, \phi) \in X$. We add the equality (4.7) to (4.8) and we use (4.9)-(4.11) to obtain (4.12). Conversely, let $x_\alpha(t) = (\mathbf{u}_\alpha(t), \varphi_\alpha(t)) \in X$ be a solution to the quasivariational inequality (4.12). We take $y = (\mathbf{v}, 0) \in X$ in (4.12) where $\mathbf{v}$ is an arbitrary element of V and obtain (4.7), then we take $y = (\mathbf{0}, \phi)$ in (4.12), where ϕ is an arbitrary element of W, as a result we obtain (4.8). We use (3.13), (3.15), (3.16)-(3.18), (3.36) and (3.39) to see that the operator A_t is strongly monotone and Lipschitz continuous, it follows by standard results on elliptic variational inequalities that there exists a unique element $(\mathbf{u}_\alpha(t), \varphi_\alpha(t)) \in X$ which solves (4.7)-(4.8).

Now let us show that $(\mathbf{u}_\alpha, \varphi_\alpha) \in C(0,T;V) \times C(0,T;W)$. We let $t_1, t_2 \in [0,T]$ and use the notation $\mathbf{u}_\alpha(t_i) = \mathbf{u}_i$, $\alpha(t_i) = \alpha_i$, $\varphi_\alpha(t_i) = \varphi_i$, $\mathbf{f}(t_i) = \mathbf{f}_i$, $q(t_i) = q_i$ and $x_\alpha(t_i) = (\mathbf{u}_\alpha(t_i), \varphi_\alpha(t_i)) = x_i$ for $i = 1, 2$. We use standard arguments in (4.7) and (4.8) to find

$$\begin{aligned}
&(\mathcal{A}\varepsilon(\mathbf{u}_1 - \mathbf{u}_2), \varepsilon(\mathbf{u}_1 - \mathbf{u}_2))_{\mathcal{H}} + (\mathcal{E}^*\nabla(\varphi_1 - \varphi_2), \varepsilon(\mathbf{u}_1 - \mathbf{u}_2))_{\mathcal{H}} \\
&= (\mathbf{f}_1 - \mathbf{f}_2, \mathbf{u}_1 - \mathbf{u}_2)_V + j_{nc}(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{nc}(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \\
&\quad + j_{ad}(\alpha_1, \mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{ad}(\alpha_2, \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2),
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&(B\nabla(\varphi_1 - \varphi_2), \nabla(\varphi_1 - \varphi_2))_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(\mathbf{u}_1 - \mathbf{u}_2), \nabla(\varphi_1 - \varphi_2))_{L^2(\Omega)^d} \\
&= (q_1 - q_2, \varphi_1 - \varphi_2)_W,
\end{aligned} \tag{4.14}$$

and, by using the assumption (3.16)-(3.18) on $\mathcal{A}$, B and $\mathcal{E}$, the properties (3.35) and (3.39) on the functional j_{ad} and j_{nc} respectively and (3.14)-(3.15), we obtain

$$\mid \mathbf{u}_1 - \mathbf{u}_2 \mid_V \leq C(\mid \mathbf{f}_1 - \mathbf{f}_2 \mid_V + \mid q_1 - q_2 \mid_W + \mid \alpha_1 - \alpha_2 \mid_{L^2(\Gamma_3)}). \tag{4.15}$$

$$\mid \varphi_1 - \varphi_2 \mid_W \leq C(\mid \mathbf{u}_1 - \mathbf{u}_2 \mid_V + \mid q_1 - q_2 \mid_W). \tag{4.16}$$

The inequality (4.15) and the regularity of the functions $\mathbf{f}$, q and α show that $\mathbf{u}_\alpha \in C(0,T;V)$. We use (4.16) and the regularity of the functions $\mathbf{u}_\alpha$, q to show that $\varphi_\alpha \in C(0,T;W)$. Thus we conclude the existence part in lemma 4.2 and we note that the uniqueness of the solution follows from the unique solvability of (4.7) and (4.8) at any $t \in [0,T]$. □

In the next step, we use the displacement field $\mathbf{u}_\alpha$ obtained in lemma 4.2 and we consider the following initial-value problem.

Problem PV_θ. Find the adhesion field $\theta_\alpha : [0,T] \to L^\infty(\Gamma_3)$ such that for a.e. $t \in (0,T)$

$$\dot{\theta}_\alpha(t) = -(\theta_\alpha(t)(\gamma_\nu(R_\nu(u_{\alpha\nu}(t)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_{\alpha\tau}(t)) \mid^2 \ ) - \varepsilon_a)_+, \tag{4.17}$$

$$\theta_\alpha(0) = \alpha_0. \tag{4.18}$$

We have the following result.

Lemma 4.3. *There exists a unique solution $\theta_\alpha \in W^{1,\infty}(0,T; L^\infty(\Gamma_3))$ to problem PV_θ. Moreover, $\theta_\alpha(t) \in Z$ for all $t \in [0,T]$.*

Proof. For the simplicity we suppress the dependence of various functions on Γ_3, and note that the equalities and inequalities below are valid a.e. on Γ_3. Consider the mapping $F_\alpha : [0,T] \times L^\infty(\Gamma_3) \to L^\infty(\Gamma_3)$ defined by

$$F_\alpha(t,\theta) = -(\theta(\gamma_\nu(R_\nu(u_{\alpha\nu}(t)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_{\alpha\tau}(t)) \mid^2 \ ) - \varepsilon_a)_+, \tag{4.19}$$

for all $t \in [0,T]$ and $\theta \in L^\infty(\Gamma_3)$. It follows from the properties of the truncation operator R_ν and $\mathbf{R}_\tau$ that F_α is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\theta \in L^\infty(\Gamma_3)$, the mapping $t \to F_\alpha(t,\theta)$ belongs to $L^\infty(0,T; L^\infty(\Gamma_3))$. Thus using a version of Cauchy-Lipschitz theorem given in Theorem 2.1 we deduce that there exists a unique function $\theta_\alpha \in W^{1,\infty}(0,T; L^\infty(\Gamma_3))$ solution to the problem PV_θ. Also, the arguments used in Remark 3.1 show that $0 \le \theta_\alpha(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set Z, we find that $\theta_\alpha(t) \in Z$, which concludes the proof of the lemma. $\square$

It follows from lemma 4.3 that for all $\alpha \in \mathcal{Z}$ the solution θ_α of problem PV_θ belongs to $\mathcal{Z}$. Therefore, we may consider the operator $\Lambda : \mathcal{Z} \to \mathcal{Z}$ given by

$$\Lambda\alpha = \theta_\alpha. \tag{4.20}$$

We have the following result.

Lemma 4.4. *There exists a unique element $\alpha^* \in \mathcal{Z}$ such that $\Lambda\alpha^* = \alpha^*$.*

Proof. We show that, for a positive integer m, the mapping Λ^m is a contraction on $\mathcal{Z}$. To this end, we suppose that α_1 and α_2 are two functions in $\mathcal{Z}$ and denote $\mathbf{u}_{\alpha_i} = \mathbf{u}_i$, and $\theta_{\alpha_i} = \theta_i$ the functions obtained in lemmas 4.4 and 4.5, respectively, for $\alpha = \alpha_i$, $i = 1, 2$. Let $t \in [0, T]$. We use (4.7) and (4.8) and arguments similar to those used in the proof of (4.15) to deduce that

$$| \mathbf{u}_1(t) - \mathbf{u}_2(t) |_V \leq C \mid \alpha_1(t) - \alpha_2(t) |_{L^2(\Gamma_3)}, \tag{4.21}$$

which implies

$$\int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_V \, ds \leq C \int_0^t | \alpha_1(s) - \alpha_2(s) |_{L^2(\Gamma_3)} \, ds. \tag{4.22}$$

On the other hand, from the Cauchy problem (4.17)-(4.18) we can write

$$\theta_i(t) = \alpha_0 - \int_0^t (\theta_i(s)(\gamma_\nu (R_\nu(u_{i\nu}(s)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_{i\tau}(s)) \mid^2) - \varepsilon_a)_+ \, ds, \tag{4.23}$$

and then

$$\begin{aligned} &| \theta_1(t) - \theta_2(t) |_{L^2(\Gamma_3)} \\ &\leq C \int_0^t | \theta_1(s)(R_\nu(u_{1\nu}(s)))^2 - \theta_2(s)(R_\nu(u_{2\nu}(s)))^2 |_{L^2(\Gamma_3)} \, ds \\ &\leq C \int_0^t | \theta_1(s) \mid \mathbf{R}_\tau(\mathbf{u}_{1\tau}(s)) \mid^2 - \theta_2(s) \mid \mathbf{R}_\tau(\mathbf{u}_{2\tau}(s)) \mid^2 |_{L^2(\Gamma_3)} \, ds. \end{aligned}$$

Using the definition of R_ν and $\mathbf{R}_\tau$ and writing $\theta_1 = \theta_1 - \theta_2 + \theta_2$, we get

$$\begin{aligned} &| \theta_1(t) - \theta_2(t) |_{L^2(\Gamma_3)} \\ &\leq C(\int_0^t | \theta_1(s) - \theta_2(s) |_{L^2(\Gamma_3)} \, ds + \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_{L^2(\Gamma_3)^d} \, ds). \end{aligned} \tag{4.24}$$

Next, we apply Gronwall's inequality to deduce

$$| \theta_1(t) - \theta_2(t) |_{L^2(\Gamma_3)} \leq C \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_{L^2(\Gamma_3)^d} \, ds. \tag{4.25}$$

The relation (4.20), the estimate (4.25) and the relation (3.14) lead to

$$| \Lambda\alpha_1(t) - \Lambda\alpha_2(t) |_{L^2(\Gamma_3)} \leq C \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_V \, ds. \tag{4.26}$$

We now combine (4.22) and (4.26) and see that

$$| \Lambda\alpha_1(t) - \Lambda\alpha_2(t) |_{L^2(\Gamma_3)} \leq C \int_0^t | \alpha_1(s) - \alpha_2(s) |_{L^2(\Gamma_3)} \, ds, \tag{4.27}$$

and reiterating this inequality m times we obtain

$$\mid \Lambda^m \alpha_1 - \Lambda^m \alpha_2 \mid_{C(0,T;L^2(\Gamma_3))} \leq \frac{C^m T^m}{m!} \mid \alpha_1 - \alpha_2 \mid_{C(0,T;L^2(\Gamma_3))} . \tag{4.28}$$

Recall that $\mathcal{Z}$ is a nonempty closed set in the Banach space $C(0,T;L^2(\Gamma_3))$ and note that (4.28) shows that for m sufficiently large the operator $\Lambda^m : \mathcal{Z} \to \mathcal{Z}$ is a contraction. Then by the Banach fixed point theorem (see [16]) it follows that Λ has a fixed point $\alpha^* \in \mathcal{Z}$. □

Now, we have all the ingredients to prove Theorem 4.1.

Proof. Existence. Let $\alpha^* \in \mathcal{Z}$ be the fixed point of Λ and let $(\mathbf{u}^*, \varphi^*)$ be the solution of problem PV_α for $\alpha = \alpha^*$, i.e. $\mathbf{u}^* = \mathbf{u}_{\alpha^*}$, $\varphi^* = \varphi_{\alpha^*}$. Arguments similar to those used in the proof of (4.15) lead to

$$\mid \mathbf{u}^*(t_1) - \mathbf{u}^*(t_2) \mid_V \leq C(\mid q(t_1) - q(t_2) \mid_W$$

$$+ \mid \mathbf{f}(t_1) - \mathbf{f}(t_2) \mid_V + \mid \alpha^*(t_1) - \alpha^*(t_2) \mid_{L^2(\Gamma_3)}), \tag{4.29}$$

$$\mid \varphi^*(t_1) - \varphi^*(t_2) \mid_W \leq C(\mid \mathbf{u}^*(t_1) - \mathbf{u}^*(t_2) \mid_V + \mid q(t_1) - q(t_2) \mid_W), \tag{4.30}$$

for all $t_1, t_2 \in [0,T]$. Since $\alpha^* = \theta_{\alpha^*}$ it follows from lemma 4.3 that $\alpha^* \in W^{1,\infty}(0,T;L^\infty(\Gamma_3))$, the regularity of $\mathbf{f}$ and q given by (3.29) and the estimate (4.29) imply that $\mathbf{u}^* \in W^{1,\infty}(0,T;V)$ and (4.30) implies that $\varphi^* \in W^{1,\infty}(0,T;W)$. We conclude by (4.7), (4.8), (4.17) and (4.18) that $(\mathbf{u}^*, \varphi^*, \alpha^*)$ is a solution of problem PV and it satisfies (4.1)-(4.3).

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.20).

Let $(\mathbf{u}, \varphi, \alpha)$ be a solution of problem PV which satisfies (4.1)-(4.3). Using arguments in remark 3.1 we deduce that $\alpha \in \mathcal{Z}$, moreover, it follows from (3.30)-(3.31) that $(\mathbf{u}, \varphi)$ is a solution to problem PV_α and since by lemma 4.2 this problem has a unique solution denoted $(\mathbf{u}_\alpha, \varphi_\alpha)$, we obtain

$$\mathbf{u} = \mathbf{u}_\alpha \text{ and } \varphi = \varphi_\alpha. \tag{4.31}$$

We replace $\mathbf{u}$ by $\mathbf{u}_\alpha$ in (3.32) and use the initial condition (3.33) to see that α is a solution to problem PV_θ. Since by Lemma 4.3 this last problem has a unique solution denoted θ_α, we find

$$\alpha = \theta_\alpha. \tag{4.32}$$

We use now (4.20) and (4.32) to see that $\Lambda\alpha = \alpha$, i.e. α is a fixed point of the operator Λ. It follows now from lemma 4.4 that

$$\alpha = \alpha^*. \tag{4.33}$$

The uniqueness part of the theorem is now a consequence of equalities (4.31), (4.32) and (4.33). $\square$

References

[1] Batra, R.C., Yang, J.S., *Saint Venant's principle in linear piezoelectricity*, Journal of Elasticity, **38**(1995), 209-218.

[2] Brézis, H., *Equations et inequations non linéaires dans les espaces vectoriels en dualité*, Ann. Inst. Fourier, **18**(1968), 115-175.

[3] Chau, O., Fernandez, J.R., Shillor, M., Sofonea, M., *Variational and numerical analysis of a quasistatic viscoelastic contact problem with adhesion*, Journal of Computational and Applied Mathematics, **159**(2003), 431-465.

[4] Chau, O., Shillor, M., Sofonea, M., *Dynamic frictionless contact with adhesion*, Journal of Applied Mathematics and Physics (ZAMP), **55**(2004), 32-47.

[5] Duvaut, G., Lions, J.L., *Les Inéquations en Mécanique et en Physique*, Springer-Verlag, Berlin, 1976.

[6] Frémond, M., *Equilibre des structures qui adhèrent à leur support*, C. R. Acad. Sci. Paris, 295, Série II (1982), 913-916.

[7] Frémond, M., *Adhérence des solides*, J. Mécanique Théorique et Appliquée, **6(3)**(1987), 383-407.

[8] Mindlin, R.D., *Polarisation gradient in elastic dielectrics,* Int. J. Solids Structures, **4**(1968), 637-663.

[9] Mindlin, R.D., *Continuum and lattice theories of influence of electromechanical coupling on capacitance of thin dielectric films*, Int. J. Solids, **4**(1969), 1197-1213.

[10] Mindlin, R.D., *Elasticity, Piezoelectricity and Cristal lattice dynamics*, J. of Elasticity, **4**(1972), 217-280.

[11] Nečas, J., Hlavaček, I., *Mathematical Theory of Elastic and Elastoplastic Bodies: An Introduction*, Elsevier, Amsterdam, 1981.

[12] Raous, M., Cangémi, L., Cocu, M., *A consistent model coupling adhesion, friction, and unilateral contact*, Comput. Meth. Appl. Mech. Engng., **177**(1999), 383-399.

[13] Rojek, J., Telega, J.J., *Contact problems with friction, adhesion and wear in orthopaedic biomechanics. I: General developments*, J. Theoretical and Applied Mechanics, **39**(2001).

[14] Rojek, J., Telega, J.J., Stupkiewicz, S., *Contact problems with friction, adherence and wear in orthopaedic biomechanics. II: Numerical implementation and application to implanted knee joints*, J. Theoretical and Applied Mechanics, **39**(2001).

[15] Shillor, M., Sofonea, M., Telega, J.J., *Models and Variational Analysis of Quasistatic Contact*, Lect. Notes Phys. 655, Springer, Berlin Heidelberg, 2004.

[16] Sofonea, M., Han, W., Shillor, M., *Analysis and Approximation of Contact Problems with Adhesion or Damage*, Pure and Applied Mathematics 276, Chapman-Hall / CRC Press, New york, 2006.

[17] Tengiz, B., Tengiz, G., *Some Dynamic of the Theory of Electroelasticity*, Memoirs on Differential Equations and Mathematical Physics, **10**(1997), 1-53.

[18] Toupin, R.A., *The elastic dielectrics*, J. Rat. Mech. Analysis, **5**(1956), 849-915.

[19] Toupin, R.A., *A dynamical theory of elastic dielectrics*, Int. J. Engrg. Sci., **1**(1963).

[20] Suquet, P., *Plasticité et homogénéisation,* Thèse de doctorat d'Etat, Université Pierre et Marie Curie, Paris 6 1982.

Department of Mathematics
University of Setif
19000 Setif Algeria
E-mail address: s_elmanih@yahoo.fr

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 2, June 2009

ON THE STANCU TYPE LINEAR POSITIVE OPERATORS OF APPROXIMATION CONSTRUCTED BY USING THE BETA AND THE GAMMA FUNCTIONS

ELENA IULIA STOICA-LAZE

Abstract. The objective of this paper is to present some extensions of several classes of Stancu type linear positive operators of approximation by using the beta and the gamma functions.

Section 1 is devoted to consideration of the Stancu-Bernstein type operator S_m^α, introduced in 1968 by D.D. Stancu in the paper [28] starting from the Markov-Polya probability distribution.

In section 2 is considered the Stancu-Baskakov operator V_m^α, defined at (2.1) and (2.2). If $\alpha = 0$ then we obtain the Baskakov operator defined at (2.3).

Section 3 is devoted to the operator W_m^α of Stancu, Meyer-König and Zeller operator defined at (3.1) and (3.2) introduced γ in [22] by these authors starting from the Pascal probability distribution.

In section 4 is presented and discussed the Stancu beta operators of second-kind L_m, defined at (4.2), which was obtained by using Karl Pearson type VI, $b_{p,q}$, with positive parameters p and q.

1. The Stancu-Bernstein operator S_m^α

In the previous papers [32], [33], [34], professor D.D. Stancu has considered probabilistic methods for construction and investigation of some linear positive operators useful in approximation theory of functions.

First we mention that in 1968 he has introduced and investigated in [28] a new parameter-dependent linear polynomial operator S_m^α of Bernstein type associated

Received by the editors: 01.12.2008.
2000 *Mathematics Subject Classification.* 41A10, 41A36, 65D10, 65Q05.
Key words and phrases. Stancu linear operators, Beta and Gamma functions.

to a function $f \in C[0,1]$, defined by the formula

$$(S_m^\alpha f)(x) = \sum_{k=0}^{m} p_{m,k}^\alpha(x) f\left(\frac{k}{m}\right), \tag{1.1}$$

where α is a non negative parameter and $w_{m,k}^\alpha$ is a polynomial which can be expressed by means of the factorial power $u^{(n,k)}$ of the non-negative order n and increment h, given by the formula

$$u^{(n,h)} = u(u-h)\dots(u-(n-1)h),$$

namely

$$p_{m,k}^\alpha(x) = \frac{1}{1^{(m,-\alpha)}}\binom{m}{k} x^{(k,-\alpha)}(1-x)^{(m-k,-\alpha)}. \tag{1.2}$$

If $\alpha = 0$ then this operator reduces to the classical operator B_m, introduced by Bernstein in 1912 in the paper [7] by starting from the binomial Bernoulli distribution.

By using the Markov-Pólya probability distribution (introduced by A. Markov [19] in 1917 and encountered in 1930 by G. Pólya [25] studying the contageous diseases. We mention that at this distribution we can arrive by using the following urn model. An urn contains a white balls and b black balls. One ball is drawn at random from this urn and then it is returned together with a constant number of c identical balls of the same color. This process is repeated m times. Denoting by Z_j the one-zero random variable, according as the j-th trial results in white or black, the probability that the total number of white balls $Z_1 + \dots + Z_m$ be equal with k $(0 \le k \le m)$ is given by

$$P(k;m,a,b,c) = \binom{m}{k}\frac{a(a+c)\dots(a+(k-1)c)b(b+c)\dots(b+(m-k-1)c)}{(a+b)(a+b+c)\dots(a+b+(m-1)c)}.$$

If we adopt the notations: $x = a/(a+b)$, $\alpha = c/(a+b)$, and we hold α a constant, allowing x to vary, we obtain the discrete probability distribution of Markov-Pólya. We can see that the probability to have

$$Y_m = \frac{1}{m}(Z_1 + \dots + Z_m) = \frac{k}{m}$$

is given just by the formula

$$p_{m,k}^\alpha(x) = \binom{m}{k}\frac{x^{(k,-\alpha)}(1-x)^{(m-k,-\alpha)}}{1^{(m,-\alpha)}}.$$

If we assume that $\alpha > 0$, then the operator S_m^α can be represented by means of the formula

$$(S_m^\alpha f)(x)\frac{1}{B\left(\dfrac{x}{\alpha}, \dfrac{1-x}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha}} (1-t)^{\frac{1-x}{\alpha}} (B_m f)(t)dt, \tag{1.3}$$

where B is the Euler (1707-1783) Beta function of the first kind, where

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad (x,y>0).$$

This is a function of two variables x and y from $\mathbb{R}_+$.

Putting $x = t/(1+t)$ we obtain

$$B(x,y) = \int_0^\infty \frac{t^{x-1}dt}{(1+t)^{x+y}}.$$

The Beta function can be expressed by the Euler Gamma function $\Gamma(u)$, where $u > 0$ and

$$\Gamma(u) = \int_0^\infty t^{u-1}e^{-u}dt.$$

Now let us make the remarks that

$$B(m,n) = (m-1)!(n-1)!/(m+n-1)!, \quad B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

as can be seen in the books [9], [13].

Concerning the remainder of the approximation formula of the function f by the operator of D.D. Stancu (1.3), we should mention that it can be represented under the form

$$(R_m^\alpha f)(x) = (R_m^\alpha e_2)(x)(D_m^\alpha f)(x),$$

where

$$(R_m^\alpha e_2)(x) = \frac{1+\alpha m}{1+\alpha} \cdot \frac{x(1-x)}{m}$$

and

$$(D_m^\alpha f)(x) = \sum_{k=0}^{m-1} p_{m-1,k}^\alpha (x+\alpha, 1-x+\alpha)\left[x, \frac{k}{m}, \frac{k+1}{m}; f\right],$$

the brackets representing the symbol for divided differences.

2. The Stancu-Baskakov operator V_m^α

If one uses the generalization given by D.D. Stancu in the paper [29] for the Fisher probability distribution

$$P(\chi = k) = q(k; n, x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}, \tag{2.1}$$

namely

$$P(\chi = k) = \binom{m+k-1}{k} \frac{B\left(\frac{1}{\alpha} + m, \frac{x}{\alpha} + k\right)}{B\left(\frac{1}{\alpha}, \frac{x}{\alpha}\right)},$$

where α and x are positive numbers and B is the Euler beta function, then we obtain the generalized Baskakov operator V_m^α considered by D.D. Stancu [29], defined by the formula

$$(V_m^\alpha f)(x) = \sum_{k=0}^{\infty} v_{m,k}^\alpha(x) f\left(\frac{k}{m}\right),$$

where $x \in [0, \infty)$ and

$$v_{m,k}^\alpha(x) = \binom{m+k-1}{k} \frac{(1+\alpha)(1+2\alpha)\dots(1+(m-1)\alpha)(x(x+\alpha))\dots(x+(k-1)\alpha)}{(1+x)(1+x+\alpha)\dots(1+x+(m+k-1)\alpha)}. \tag{2.2}$$

Since $v_{m,0}^\alpha(0) = 1$ and $v_{m,k}^\alpha(0) = 0$ if $k \geq 1$ one observes that we always have $(V_m^\alpha f)(0) = f(0)$.

We notice that the fundamental polynomials $v_{m,k}^\alpha$ can be expressed in a more compact form by means of the notion of factorial powers, namely

$$v_{m,k}^\alpha(x) = \binom{m+k-1}{k} \frac{1^{(m,-\alpha)} x^{(k,-\alpha)}}{(1+x)^{(m+k,-\alpha)}}.$$

The operator V_m^α includes as a special case the Baskakov operator Q_m, defined by the following formula

$$(Q_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k-1}{k} \frac{x^k}{(1+x)^{m+k}} f\left(\frac{k}{m}\right), \tag{2.3}$$

which can be constructed if one uses the form considered by Fisher [10] for the negative binomial distribution.

We further note that if $\alpha > 0$ and $x > 0$ then one verifies directly that the fundamental polynomials $v^{\alpha}_{m,k}$ can also be represented in the following form

$$v^{\alpha}_{m,\alpha} = \binom{m+k-1}{k} \frac{B\left(\frac{1}{\alpha}+m, \frac{x}{\alpha}+k\right)}{B\left(\frac{1}{\alpha}, \frac{x}{\alpha}\right)} \tag{2.4}$$

in terms of the Euler beta function B.

3. The Stancu Meyer-König and Zeller operator W^{α}_{m}

If we use a generalization given by D.D. Stancu [29] for the Pascal probability distribution, then we can obtain the following general linear operator W^{α}_{m}, defined for a function f bounded on the interval $[0,1]$ namely

$$(W^{\alpha}_{m} f)(x) = \sum_{k=0}^{\infty} w^{\alpha}_{m,k}(x) f\left(\frac{k}{m+k}\right), \tag{3.1}$$

where for $0 \leq x < 1$ we have

$$w^{\alpha}_{m,k}(x) = \binom{m+k}{k} \frac{x^{(k,-\alpha)}(1-x)^{(m+1,-\alpha)}}{1^{(m+k+1,-\alpha)}}$$

$$= \binom{m+k}{k} \frac{x(x+\alpha)\dots(x+(k-1)\alpha)(1-x)(1-x+\alpha)\dots(1-x+m\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+(m+k)\alpha)}.$$

One observes that if $x = 0$ then $W^{\alpha}_{m} f(0) = f(0)$, while if $x = 1$ it is convenient to take

$$(W^{\alpha}_{m} f) = \lim_{x \nearrow 1} (W^{\alpha}_{m} f)(x) = f(1).$$

It is obvious to see that W^{α}_{m} includes as a special case ($\alpha = 0$) the operator of Meyer-König and Zeller [22] defined by

$$(M_m f)(x) = \sum_{k=0}^{\infty} w_{m,k}(x) f\left(\frac{k}{m+k}\right), \tag{3.2}$$

where

$$w_{m,k}(x) = \binom{m+k}{k} x^k (1-x)^{m+1}$$

obtained by these authors by using the negative binomial probability distribution of Pascal.

In the paper [29] D.D. Stancu has given an integral representation of $W_m^\alpha f$ by using the Beta transformation of the operator M_m defined at (3.2), which is valid for $\alpha > 0$ and $0 < x < 1$ namely,

$$(W_m^\alpha f)(x) = \frac{1}{B\left(\dfrac{x}{\alpha}, \dfrac{1-x}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}}(M_m f)(t)dt. \tag{3.3}$$

Concerning this operator we want to mention that the paper [37] J. Swetits and B. Wood referring to a probabilistic method in connection with the Markov-Polya urn scheme, used by D.D. Stancu in [28] for constructing the operator S_m^α, have presented a variation of the Pascal urn scheme.

It permits to give a probabilistic interpretation of the operator M_m^α.

By using the notation

$$w_{m,k}^\alpha(u,v) = \binom{m+k}{k} \frac{u^{(k,-\alpha)}v^{(m+1,-\alpha)}}{(u+v)^{(m+1+k,-\alpha)}}$$

and the second order divided differences of the function f, D.D. Stancu has evaluated in the paper [33] the remainder term in the approximation formula of the function f:

$$f(x) = (M_m^\alpha f)(x) + (R_m^\alpha f)(x) \tag{3.4}$$

by means of $M_m^\alpha f$, namely

$$(R_m^\alpha f)(x) = -x(1-x)\sum_{k=0}^{\infty}(m+1+k)^{-1} w_{m-1,k}^\alpha(x+\alpha, 1-x+\alpha)\left[x, \frac{k}{m+k}, \frac{k+1}{m+1+k}; f\right],$$

where the brackets represent the symbol for divided differences.

4. **The Stancu beta operators of second kind**

By starting from the beta distribution of second kind $b_{p,q}$ (with positive parameters), which belongs to Karl Pearson's Type VI, one defines the beta second kind transformation $T_{p,q}$ of a function $g : [0,\infty) \to \mathbb{R}$ bounded and Lebesgue measurable in every interval $[a,b]$, where $0 < a < b < \infty$ such that $T_{p,q}|q| < \infty$. The moment of order r $(1 \le r < q)$ of the functional $T_{p,q}$ has the following value

$$\nu_r(p,q) = T_{p,q}e_r = \frac{p(p+1)\dots(p+r-1)}{(q-1)(q-2)\dots(q-r)}. \tag{4.1}$$

If one applies this transformation to the image of a function $f : [a, \infty) \to \mathbb{R}$ by the operator defined at (2.3) then we obtain the functional

$$F_m^f(p,q) = T_{p,q}(Q_m f),$$

given explicitly under the following form

$$F_m^f(p,q) = T_{p,q}(Q_m f)$$

$$= \sum_{k=0}^{m} \binom{m+k-1}{k} \frac{p(p+1)\dots(p+q-1)q(q+1)\dots(q+m-1)}{(p+q)(p+q+1)\dots(p+q+m+k-1)} f\left(\frac{k}{m}\right).$$

If we select $p = \dfrac{x}{\alpha}$ and $q = \dfrac{1}{\alpha}$, where $\alpha > 0$ then the preceding formulas leads us to the parameter dependent operator S_m^α introduced in 1970 in the paper of D.D. Stancu [29] (see also the paper [31] of the same author) as a generalization of the Baskakov operator [6]. By using the factorial powers, with the step $h = -\alpha$, it can be expressed under the following compact form

$$(L_m^\alpha f)(x) = \sum_{k=0}^{\infty} \binom{m+k-1}{k} \frac{x^{[k,-\alpha]} 1^{[m,-\alpha]}}{(1+x)^{[m+k,-\alpha]}} f\left(\frac{k}{m}\right).$$

It should be noticed that the operator $T_{\frac{x}{\alpha},\frac{1}{\alpha}} = T^\alpha$ was used in the paper [5] for obtaining the operator L_m^α by Adell and De la Cal.

Professor D.D. Stancu has introduced the new beta second kind approximation operator, defined by the formula

$$(L_m f)(x) = (T_{mx,m+1} f)(x) = \frac{1}{B(mx, m+1)} \int_0^\infty f(t) \frac{t^{mx-1} dt}{(1+t)^{mx+m+1}}. \tag{4.2}$$

Because

$$(L_m e_0)(x) = \int_0^\infty b_{mx,m+1}(t)dt = 1$$

it follows that the operator L_m reproduces the linear functions.

It is easily seen that this operator is of Feller's type but it is not an averaging operator.

If we use an inequality established by D.D. Stancu in [35], one can find an inequality given the order of approximation of the function f by means of $L_m f$, namely

$$|f(x) - (L_m f)(x)| \leq (1 + \sqrt{x(x+1)})\omega_1 \left(f; \frac{1}{\sqrt{m-1}}\right),$$

respectively

$$|f(x) - (L_m f)(x)| \leq (1 + x(x+1))\omega_2 \left(f; \frac{1}{\sqrt{m-1}}\right)$$

where $\omega_k(f, \delta)$ represents the modulus of continuity of order k $(k = 1, 2)$ of the function f.

According to the Bohman-Korovkin convergence criterion we can deduce that the sequence $(L_m f)$ converges uniformly in $[a, b]$ to the function f when m tends to infinity.

In the paper [29] D.D. Stancu has established an inequality of Lorentz type and an asymptotic formula of Voronovskaja type.

Ending this paper we mention that the operator of beta type of second kind of D.D. Stancu is distinct from other beta operators considered earlier by Mülbach [24], Lupaş [22], Upreti [38], Khan [14] and Adell [3].

References

[1] Abel, U., *Asymptotic approximation with Stancu Beta operators*, Rev. Anal. Numer. Theor. Approx., **27**(1988), 5-13.

[2] Abel, U., Gupta, V., *Rate of convergence for Stancu beta operators for functions of bounded variation*, Rev. Anal. Numer. Theor. Approx., **33**(2004), 3-9.

[3] Adell, J.A., *On the property of monotonic convergence for beta operators*, J. Approx. Theory, **84**(1996), 61-73.

[4] Adell, A., Badia, F.G., De la Cal, J., *Beta-type operators preserve shape properties*, Stochastic Process, **48**(1993), 1-8.

[5] Adell, J.A., De la Cal, J., *On a Bernstein-type operator associated with the inverse Polya-Eggenberger distribution*, Rend. Circolo Matem. Palermo, Ser. II, nr. **33**(1993), 143-154.

[6] Baskakov, V.A., *An example of a linear positive operators in the space of continuous function*, Dokl. Akad. Nauk SSSR, **113**(1957), 249-251.

[7] Bernstein, S.N., *Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités*, Comm. Soc. Math. Krakov, (2)**13**(1912-1913), 1-2.

[8] Euler, L., *Institutionum calculi integralis*, v.s., sec. 2, ch. 7, Impenfis Academial Imperials Scientiarum Petropoli (Opera Omnia, ser. 1, V. 11, 1768, 424-434).

[9] Fichtenholz, G.M., *Course on differential and integral calculus*, vol. II (in Russian), Goz. Izd. Fiz.-Mat. Lit., Moscow, 1964, 689-725.

[10] Fisher, R.A., *The negative binomial distribution*, Ann. Eugenics, **11**(1941), 182-187.

[11] Gupta, V., Abel, U., Ivan, M., *Rate of convergence of Stancu Beta operators for functions with derivatives of bounded variation*, J. Math. and Math. Sciences (in print).

[12] Johnson, N.L., Kotz, S., Kemp, A.W., *Univariate Discrete Distribution*, Wiley, New York, 1992.

[13] Jordan, C., *Calculus of finite differences*, Chelsea, New York, 1947, 53-56 and 80-83.

[14] Khan, M.K., *Approximation properties of beta operators*, Progress in Approx. Theory, Academic Press, New York, 1991, 483-495.

[15] Kirov, G.H., *A generalization of the Bernstein polynomials*, Math. Balkanica, **2**(1992), 147-153.

[16] Loha, R.G., *On a characterization of the Gamma distribution*, Annals of Mathematical Statistics, **25**(1954), 784.

[17] Lupaş, A., Müller, M., *Approximations eigenschaften der Gamma operatoren*, Math. Z., **98**(1967), 208-226.

[18] Mamedov, R.G., *The asymptotic value of the approximation of differentiable functions by linear positive operators*, Dokl. Akad. Nauk SSR, **128**(1959), 471-474.

[19] Markov, A.A., *On some limit formulas of calculus of probability* (Russian), Izv. Akad. Nauk, seria VI, **11**(1917), no. 3, 177-186.

[20] Müller, M.W., *On asymptotic approximation theorems for sequences of linear positive operators* (Proc. Symp. Lancester, 1969, ed. A. Talbot), 315-320, Acad. Press, London, 1970.

[21] Lupaş, A., *Die Folge der Beta operators*, Dissertation, Universität Stuttgart, 1972.

[22] Meyer-König, Zeller, A., *Bernsteinsche Potenzreihen*, Studia Math., **19**(1960), 89-94.

[23] Müller, M.W., *Die Folge der Gammaoperatoren*, Dissertation, Stuttgart, 1967.

[24] Mühlbach, G., *Verallgemeinerungen der Bernstein und der Lagrange Polynome Bemerkungen zur einer Klasse linearer Polynomoperatoren von D.D. Stancu*, Rev. Roumaine Math. Pures Appl., **15**(1970), 1235-1252.

[25] Pólya, G., *Sur quelques points de la théorie des probabilutés*, Annales de l'Institut H. Poincaré, **1**(1930), 117-161.

[26] Rempulska, L., Skorupka, M., *Approximation properties of modified Stancu beta operators*, Rev. Anal. Numer. Theor. Approx., **35**(2006), 189-197.

[27] Rempulska, L., Skorupka, M., *The Voronovskaja-type theorems for Post-Widder and Stancu operators*, Intern. J. Pure Appl. Math., **38**(2007), 467-480.

[28] Stancu, D.D., *Approximation of functions by a new class of linear polynomial operators*, Rev. Rom. Math. Pures Appl., **13**(1968), 1173-1194.

[29] Stancu, D.D., *Two classes of positive linear operators*, Anal. Univ. Timişoara, Ser. Sti. Matem., **8**(1970), 213-220.

[30] Stancu, D.D., *On the monotonicity of a sequence of operators of Meyer-König and Zeller type*, Studia Univ. Babeş-Bolyai, Mathematica, **39**(1994), 97-106.

[31] Stancu, D.D., *On the Beta approximating operators of second kind*, Rev. Anal. Numer. Theor. Approx., **24**(1995), 231-239.

[32] Stancu, D.D., *Use of probabilistic methods in the theory of uniform approximation of continuous functions*, Rev. Rom. Math. Pures Appl., **14**(1969), 675-691.

[33] Stancu, D.D., *Probabilistic methods in the theory of approximation of functions of several variables by linear positive operators*, In: Approximation Theory (Proc. Sympos. Lancaster, 1969, ed. A. Talbot), 329-342.

[34] Stancu, D.D., *Probabilistic approach to a class of generalized Bernstein approximating operators*, Rev. Anal. Numer. Theor. Approx., **14**(1985), 83-89.

[35] Stancu, D.D., *Representation of remainders in approximation formula by some discrete type linear positive operators*, Rend. Circolo Matematica di Palermo, Ser. II, Suppl., **52**(1998), 781-791.

[36] Steffens, Karl-Georg, *The History of Approximation Theory from Euler to Bernstein*, Birkhäuser Verlag, Basel-Boston-Berlin, 2006, XIX + 219 pp.

[37] Swetits, J., Wood, B., *Generalized Bernstein power series*, Rev. Roumaine Math. Pures Appl., **18**(1973), 461-471.

[38] Upreti, R., *Approximation properties of Beta operators*, J. Approx. Theory, **45**(1985), 85-89.

Babeş-Bolyai University
Doctoral Department of Mathematics
400370 Str. Kogălniceanu nr. 1
Cluj-Napoca, Romania
E-mail address: `lazeln@yahoo.com`