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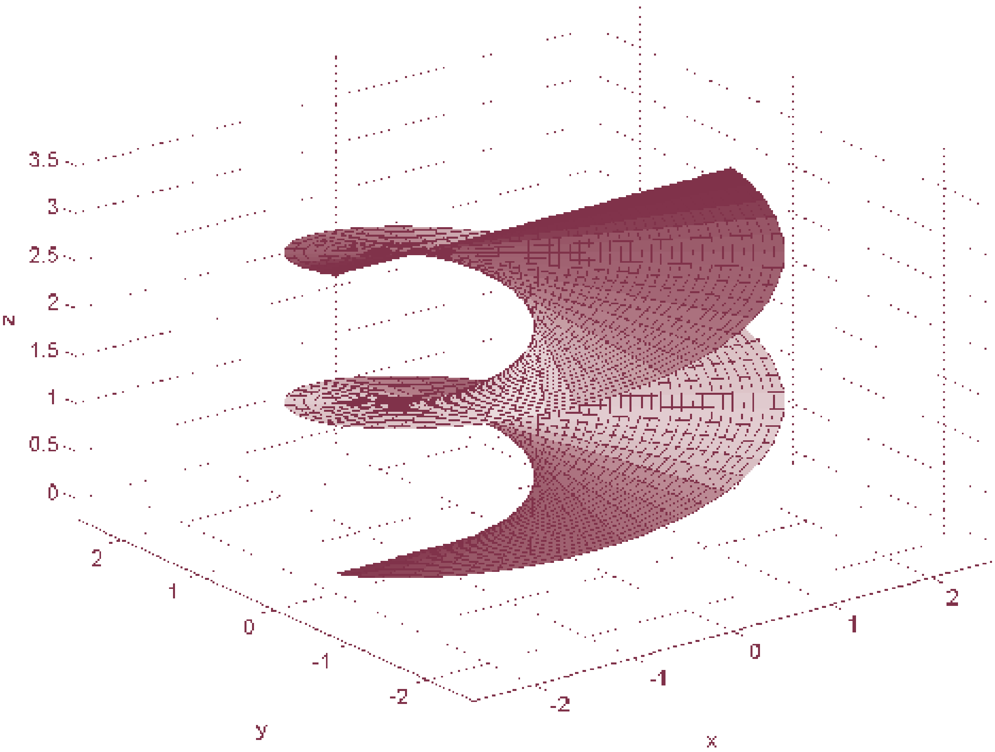

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ENCOURAGING THE GRAND COALITION IN CONVEX COOPERATIVE GAMES

TITU ANDREESCU AND ZORAN ŠUNIĆ

Abstract. A solution function for convex transferable utility games encourages the grand coalition if no player prefers (in a precise sense defined in the text) any coalition to the grand coalition. We show that the Shapley value encourages the grand coalition in all convex games and the τ-value encourages the grand coalitions in convex games up to three (but not more than three) players. Solution functions that encourage the grand coalition in convex games always produce allocations in the core, but the converse is not necessarily true.

1. **Cooperative games**

We begin by recalling the main concepts and their basic properties. The notation mostly follows [3] and/or [2].

Let $N = \{1, \dots, n\}$. The elements of N are called *players*, its subsets are called *coalitions*, and the set N is called the *grand coalition*. A *cooperative transferable utility game* with n-players is a function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$, where 2^N is the set of all subsets of N.

For a given game v, we often denote $v(\{i\})$ by $v(i)$ or v_i. More generally, for any function $x : N \to \mathbb{R}$ and $i \in N$, we denote $x(i) = x_i$. Thus we (sometimes) think of functions $x : N \to \mathbb{R}$ as vectors in $\mathbb{R}^n$. For a function $x : N \to \mathbb{R}$ and a coalition $A \subseteq N$, we write $x(A) = \sum_{j \in A} x_j$.

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A game $v : 2^N \to \mathbb{R}$ is called *super-additive* if, for all disjoint coalitions $A, B \subseteq N$,

$$v(A) + v(B) \leq v(A \cup B)$$

and is called *convex* if, for all coalitions $A, B \subseteq N$,

$$v(A) + v(B) \leq v(A \cup B) + v(A \cap B).$$

Example 1. Define a 4-player game v on $N = \{1, 2, 3, 4\}$ by the diagram in Figure 1 (the value of each coalition is provided at the vertex representing the coalition). The

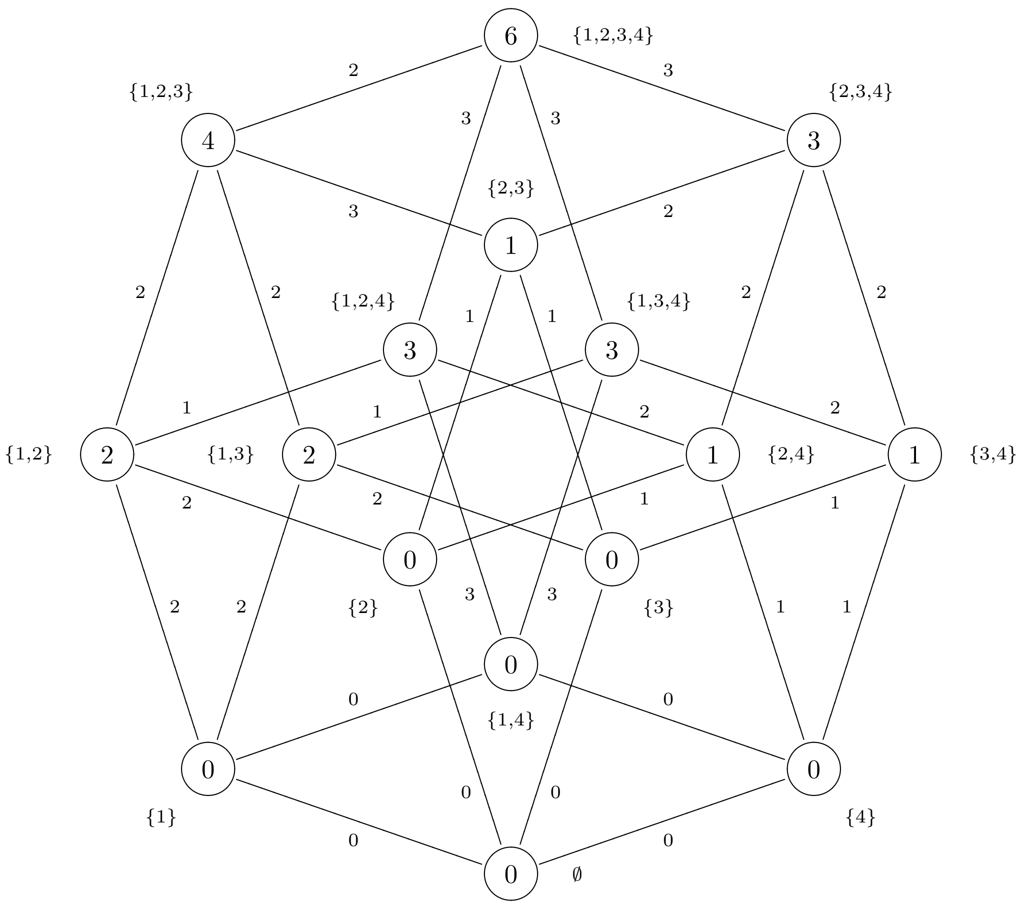

FIGURE 1. A convex 4-player game

same game is given in a tabular form in Table 1.

A	$v(A)$	A	$v(A)$	A	$v(A)$	A	$v(A)$
$\{1\}$	0	$\{2\}$	0	$\{3\}$	0	$\{4\}$	0
$\{1,2\}$	2	$\{1,3\}$	2	$\{1,4\}$	0	$\{2,3\}$	1
$\{2,4\}$	1	$\{3,4\}$	1	$\{1,2,3\}$	4	$\{1,2,4\}$	3
$\{1,3,4\}$	3	$\{2,3,4\}$	3	N	6	$\emptyset$	0

Table 1. A convex 4-player game

It is straightforward to check that the game v is convex.

A way to interpret cooperative games is as follows. Assume that the players in the set N can form various coalitions each of which has value prescribed by v (say $v(A)$ represents the amount the coalition A can earn by cooperating). The super-additivity condition implies that "the whole is larger than the sum of its parts", i.e., forming larger coalitions positively affects the value. The convexity condition is just a stronger form of the super-additivity condition. It says that it is more (or at least equally) beneficial to add a coalition to a larger coalition than to a smaller one.

Assume that i is not a member of some coalition A. The *marginal contribution* $m_i(A)$ of i to the coalition A is the quantity

$$m_i(A) = v(A \cup i) - v(A),$$

where $A \cup i$ denotes the coalition $A \cup \{i\}$. Therefore, the marginal contribution of i to A measures the added value obtained by bringing player i into the coalition A.

A game is convex if and only if, for every player i, and all coalitions $A \subseteq B$ that do not contain i,

$$m_i(A) \leq m_i(B),$$

i.e., it is more beneficial to add a player to a larger coalition than to a smaller one (this is a well known fact; see for instance [3, Theorem 1.4.2] or [2, Theorem 4.9]).

Example 2. The marginal contributions in the game from Example 1 are written on the edges of the lattice of coalitions. For instance, the fact that $m_2(\{1,3\}) =$

$v(\{1,2,3\}) - v(\{1,3\}) = 4 - 2 = 2$ is indicated by the label 2 on the edge between $\{1,3\}$ and $\{1,3\} \cup \{2\} = \{1,2,3\}$.

The top marginal contributions $m_1(N - \{1\}), \ldots, m_n(N - \{n\})$ are often denoted by $m_1, \ldots, m_n$. Further, we denote

$$M = \sum_{i \in N} m_i, \qquad T = v(N), \qquad V = \sum_{i \in N} v_i.$$

Note that, in a convex game, $M \geq T \geq V$.

Example 3. We provide a diagram for a general example of a game on three players. The marginal contributions are indicated on the edges, Note that, for $i, j \in N = \{1,2,3\}$, $m_i(\emptyset) = v_i$, and whenever $i \neq j$, we denote $m_i(\{j\}) = m_{ij}$.

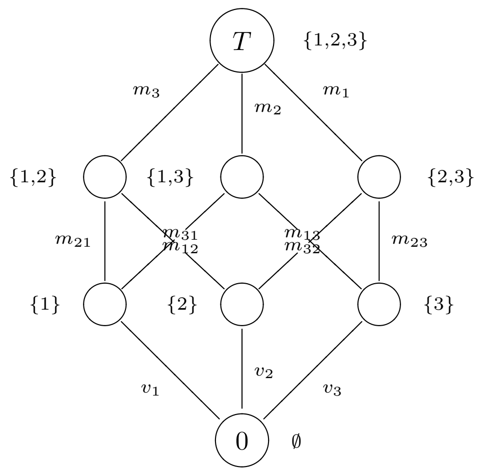

FIGURE 2. A game on 3 players

Note that, if the triple (i, j, k) is a permutation of N then

$$v_i + m_{ji} + m_k = T.$$

The convexity of the game is equivalent to the system of inequalities

$$v_i + v_j + m_k \leq T \leq m_i + m_j + v_k, \tag{1}$$

where (i, j, k) ranges over the permutations of N (see the appendix for details).

An *efficient allocation* is a function $x : N \to \mathbb{R}$ such that $x(N) = v(N)$. If in addition $x_i \geq v_i$, for $i \in N$, the allocation is called *individually rational*.

An efficient allocation assigns revenue to each player in the game in such a way that the total revenue shared among the players is exactly the value of the grand coalition N. The individual rationality of an allocation then just means that each player should be assigned revenue that is not below the individual value of that player (otherwise that player would choose not to cooperate).

A convex game is *essential* if $T > V$. In an inessential game, $m_i(A) = v_i$, for all coalitions A not containing the player i, and there exists a unique efficient and individually rational allocation, namely $x_i = v_i$, for all $i \in N$.

For a permutation π of N, and a player i in N, denote by $P_i(\pi)$ the set of *predecessors* of i in π. This is the set of players that appear before i (to the left of i) in the one-line representation of the permutation π. For instance, if $n = 6$ and $\pi = 142536$, then $P_3(\pi) = \{1, 4, 2, 5\}$.

The set of permutations of N, denoted Π_n, represents all possible orders in which the grand coalition can be formed by adding the players one by one to the coalition. For each such order, the players have different marginal contributions depending on the set of players that has already joined. The *marginal contribution* of the player i to the permutation π, denoted $m_i(\pi)$, is the marginal contribution of the player i to the coalition $P_i(\pi)$ consisting of the predecessors of i in π. In a convex game, for any permutation $\pi \in \Pi_n$, we have $m_i(\pi) \geq m_i(\emptyset) = v_i$ and $\sum_{i \in N} m_i(\pi) = T$. Thus the marginal contribution vector along π represents an efficient and individually rational allocation for v.

An *efficient solution function* f is a function that assigns an efficient allocation f^v to every convex game (we emphasize that we are not concerned with non-convex games).

Recall the definition of a well known efficient solution function introduced by Shapley [4].

Definition 1. The *Shapley value* of a convex game $v : 2^N \to \mathbb{R}$ is the allocation s given by

$$s_i = \frac{1}{n!} \sum_{\pi \in \Pi_n} m_i(\pi).$$

Thus the Shapley value is the average of all marginal contribution vectors along all permutations of N.

We also recall the definition of τ-value, introduced by Tijs [6].

Definition 2. The τ-*value* of an essential convex game $v : 2^N \to \mathbb{R}$ is the allocation given by

$$\tau_i = \frac{M-T}{M-V}\, v_i + \frac{T-V}{M-V}\, m_i.$$

In the case of an inessential game, the τ value is the unique efficient and individually rational allocation.

Note that, for an essential game, $\tau_i = \lambda v_i + (1-\lambda) m_i$, where $\lambda = \frac{M-T}{M-V}$ is the unique real number in $[0,1]$ making the allocation efficient. For an inessential game, $v_i = m_i$, for all i, and therefore the formula $\tau_i = \lambda v_i + (1-\lambda) m_i$ gives the correct τ-value for all λ in the interval $[0,1]$, i.e., the normalizing coefficient λ is not unique.

Both the Shapley value and the τ-value are efficient solution functions that assign an individually rational allocation to every convex game.

Example 4. For any convex 2-player game,

$$s_1 = \tau_1 = \frac{1}{2}(T + v_1 - v_2), \qquad s_2 = \tau_2 = \frac{1}{2}(T + v_2 - v_1).$$

2. **Encouraging the grand coalition**

We come to our main definition.

Definition 3. An efficient solution function f *encourages the grand coalition* if for every convex game $v : 2^N \to \mathbb{R}$ and every coalition $A \subseteq N$,

$$f_i^v \geq f_i^{v_A},$$

where $v_A : 2^A \to \mathbb{R}$ is the convex sub-game of v obtained by restriction on the coalition A.

Thus an efficient solution functions encourages the grand coalitions if no player in any convex game would prefer any coalition (and its associated allocation) over the grand coalition. If f is an efficient solution that encourages the grand coalition and if all players were to vote for all coalitions they like (based on maximizing the revenue they would obtain by applying the proposed solution function f) the grand coalition would be chosen by each player (even though some players may like some additional choices).

Note that the property of encouraging the grand coalition is a global property of solution functions and not of individual allocations (the property requires that we compare allocations in different games).

Theorem 1. *The Shapley value encourages the grand coalition in convex games.*

Proof. Without loss of generality, it is sufficient to show that player 1 does not prefer any coalition $M = \{1, \ldots, m\}$ to the grand coalition, i.e., it is sufficient to show that

$$\frac{1}{n!} \sum_{\pi \in \Pi_n} m_1(\pi) \geq \frac{1}{m!} \sum_{\sigma \in \Pi_m} m_1(\sigma),$$

for $1 \leq m \leq n$.

Define a map $\bar{}: \Pi_n \to \Pi_m$ by flattening the permutations of N to permutations of M. Namely, for a permutation $\pi \in \Pi_n$ define the permutation $\bar{\pi} \in \Pi_m$ by deleting the symbols $m+1, \ldots, n$ from π and keeping the relative order of the symbols $1, \ldots, m$ the same as in π (for instance, if $n = 6$, $m = 4$ and $\pi = 153462$, then $\bar{\pi} = 1342$). Every permutation in Π_m is the image of exactly $n!/m!$ permutations in Π_n under the flattening map.

Note that the set of predecessors $P_1(\pi)$ of 1 in the permutation π contains the set of predecessors $P_1(\bar{\pi})$ of 1 in the flattened permutation $\bar{\pi}$. Therefore, by the convexity of the game, $m_1(\pi) = m_1(P_1(\pi)) \geq m_1(P_1(\bar{\pi})) = m_1(\bar{\pi})$.

It follows that

$$\frac{1}{n!} \sum_{\pi \in \Pi_n} m_1(\pi) \geq \frac{1}{n!} \sum_{\pi \in \Pi_n} m_1(\bar{\pi}) = \frac{1}{n!} \cdot \frac{n!}{m!} \sum_{\sigma \in \Pi_n} m_1(\sigma) = \frac{1}{m!} \sum_{\sigma \in \Pi_n} m_1(\sigma),$$

which is what we needed to prove. □

Theorem 2. *The τ-value encourages the grand coalition in all convex games of up to three players.*

Proof. Since the τ-value coincides with the Shapley value for 2-player convex games and Shapley value encourages the grand coalition, it suffices to consider only 3-player games.

Further, the τ-value always produces individually rational allocations. Thus, it suffices to consider only 3-player games and their 2-player sub-games.

By symmetry, it suffices to show that the convexity of a 3-player game v on $N = \{1, 2, 3\}$ implies the inequality

$$\tau_1^v \geq \tau_1^{v'},$$

where v' is the sub-game corresponding to the coalition $A = \{1, 2\}$.

If the game v is inessential then so is its sub-game v' and the τ-values for v and v' agree on A.

Thus we may assume that v is essential and we need to show that the convexity of v implies

$$\frac{M - T}{M - V}\ v_1 + \frac{T - V}{M - V}\ m_1 \geq \frac{1}{2}(T' + v_1 - v_2), \tag{2}$$

where $T' = v(A) = T - m_3$ is the value of the coalition $A = \{1, 2\}$.

Denote $m_{12} = T' - v_2 = T - m_3 - v_2$ (Figure 2 may be useful for visualization; the marginal contribution vector along the permutation $\pi = 213$ is important in our considerations). Taking into account that $M > V$ (from the fact that v is convex and essential) the inequality (2) takes the form

$$(M - T)v_1 + (T - V)m_1 \geq \frac{1}{2}(T' + v_1 - v_2)(M - V),$$

which is equivalent to

$$v_1(m_1 + m_3 + v_2 - v_1 - v_3 - m_2) + m_{12}(m_2 + m_3 + v_1 - v_2 - v_3 - m_1) \leq 2m_1(m_3 - v_3),$$

after substituting $V = v_1 + v_2 + v_3$, $M = m_1 + m_2 + m_3$, $T = v_2 + m_{12} + m_3$, and $T' = v_2 + m_{12}$, and performing simple algebraic manipulations. The convexity implies that $v_1 \leq m_{12} \leq m_1$, as well as that $m_1 + m_3 + v_2 - v_1 - v_3 - m_2 \geq 0$ and $m_2 + m_3 + v_1 - v_2 - v_3 - m_1 \geq 0$ (see the inequalities in (1)). Thus

$$v_1(m_1 + m_3 + v_2 - v_1 - v_3 - m_2) + m_{12}(m_2 + m_3 + v_1 - v_2 - v_3 - m_1) \leq$$
$$\leq m_1(m_1 + m_3 + v_2 - v_1 - v_3 - m_2 + m_2 + m_3 + v_1 - v_2 - v_3 - m_1) = 2m_1(m_3 - v_3),$$

which is what we needed to prove. □

Example 5. Consider again the convex game in Example 1. This example shows that the τ-value does not necessarily encourage the grand coalition for convex 4-player games.

Indeed, we have $T = 6$, $V = 0$, $M = 11$, which shows that the normalizing coefficient λ in the formula for the τ-value is $\lambda = (M - T)/(M - V) = 5/11$. Direct calculation then gives the τ-values for v

$$\tau_1^v = \frac{18}{11} \approx 1.64, \qquad \tau_2^v = \frac{18}{11} \approx 1.64, \qquad \tau_3^v = \frac{18}{11} \approx 1.64, \qquad \tau_4^v = \frac{12}{11} \approx 1.09 \ .$$

On the other hand, for the 3-player sub-game v' determined by the coalition $A = \{1, 2, 3\}$, we have $T' = 4$, $V' = 0$, $M' = 7$, $\lambda' = 3/7$ and the τ-values for v' are

$$\tau_1^{v'} = \frac{12}{7} \approx 1.71. \qquad \tau_2^{v'} = \frac{8}{7} \approx 1.14, \qquad \tau_3^{v'} = \frac{8}{7} \approx 1.14 \ .$$

Thus player 1 would prefer the coalition A to the grand coalition, showing that the τ-value does not necessarily encourage the grand coalition.

3. Relation to the core

We consider the relation between efficient solution functions that encourage the grand coalition and the core of a convex game.

Definition 4. The *core* of a convex game $v : 2^N \to \mathbb{R}$ is the set of all efficient allocations $x : N \to \mathbb{R}$ such that, for every coalition $A \subseteq N$,

$$x(A) \geq v(A).$$

Note that every allocation in the core is individually rational and we may say that the core allocations are rational with respect to any coalition.

Proposition 1. *Let f be an efficient solution function that encourages the grand coalition in convex games. Then, for every convex game v, the allocation f^v is in the core of v.*

Proof. Let f be an efficient solution function that encourages the grand coalition in convex games and let v be a convex game. Then, for any coalition A,

$$f^v(A) = \sum_{i \in A} f_i^v \geq \sum_{i \in A} f_i^{v_A} = v(A).$$

Thus f^v is in the core of v. □

Since the τ-value does not always produce allocations in the core of a convex function, we could immediately see that it cannot encourage the grand coalition in general. However, in Example 5 the τ-value of the game v on N is in the core, as is the τ-value of all of its sub-games, but this was still not sufficient to encourage the grand coalition.

The proof of Proposition 1 indicates that, for essential solution functions, the property of encouraging the grand coalition is a refinement of the property of producing solutions in the core. Indeed, the core condition requires that, for each coalition $A \subseteq N$, the sum $f^v(A) = \sum_{i \in A} f_i^v$ is at least as large as the sum $\sum_{i \in A} f_i^{v_A} = v(A)$. On the other hand, for solution functions that encourage the grand coalition, each term in the former sum must be at least as large as the corresponding term in the latter sum. In order to see that this refinement is proper, we provide an example of an efficient solution function that always produces allocations in the core of convex games, but nevertheless fails to encourage the grand coalition.

Example 6. For any convex game v and any permutation π of N, the vector of marginal contributions along π is an efficient solution in the core of v. By convexity of the core, any convex linear combination of marginal contributions along several permutations is also in the core. Therefore, we may define an efficient solution function f as follows. Among all permutations of N select those that give the largest vectors (in the usual sense in $\mathbb{R}^n$) of marginal contributions and calculate their average. Thus, if

$$L = \left\{ \pi \in \Pi_n \mid \sum_{i \in N} m_i^v(\pi)^2 \geq \sum_{i \in N} m_i^v(\sigma)^2, \text{ for all } \sigma \in \Pi_n \right\},$$

define

$$f_i^v = \frac{1}{|L|} \sum_{\pi \in L} m_i^v(\pi).$$

To see that f does not encourage the grand coalition in convex games, even though it always produces allocations in the core, consider the game in Example 1 restricted to $N = \{1, 2, 3\}$ (completely ignore player 4).

In this game, the largest marginal vectors are the two vectors along $\pi_1 = 231$ and $\pi_2 = 321$ giving

$$f_1^v = 3, \qquad f_2^v = \frac{1}{2}, \qquad f_3^v = \frac{1}{2}.$$

On the other hand, if we restrict to the sub-game v' defined by the coalition $A = \{1, 2\}$ we obtain

$$f_1^{v'} = 1, \qquad f_2^{v'} = 1.$$

Thus player 2 would prefer the coalition A to the grand coalition.

4. **Relation to population monotone allocation schemes**

The notion of a population monotone allocation scheme was introduced in [5].

Given a game v, a monotone allocation scheme is a set of efficient allocations $\{x^{v_A} \mid A \subseteq N\}$ associated to the sub-games of v, in such a way that, for every player i and all coalitions A and B with $i \in A \subseteq B \subseteq N$,

$$x_i^{v_A} \leq x_i^{v_B}.$$

This definition is close in spirit to our definition of solution functions that encourage the grand coalition. However, the emphasis goes in different direction. We study efficient solution functions that behave well on convex games, while Sprumont studies games for which well behaved allocation schemes exist. More precisely, the main thrust of Sprumont's work is a characterization of games for which population monotone allocation schemes exist (this includes all convex games, but not all games with non-empty core). For us, on the other hand, the important question is which solution functions always produce (or fail to produce) population monotone allocation schemes in all convex games.

Sprumont shows that every 3-player game that is totally balanced (see the appendix for a definition) always has a population monotone allocation scheme. Nevertheless, Theorem 2 does not follow directly from this observation (we still need to prove that the specific scheme induced by the τ-value solution function provides such an allocation scheme).

Further, Sprumont shows that the glove game on 4 players fails to have a population monotonic allocation scheme. Again, this example is not helpful in our considerations, since the glove game is not convex (the main point of Example 5 is that the τ-value fails to provide a population monotone allocation scheme on a convex game; on the other hand this game certainly has a population monotone allocation scheme, namely the one induced by the Shapley value).

5. **On necessity versus desirability**

Observe that even if a solution function that does not encourage the grand coalition is used and, for a concrete game v, there exists a player that prefers some smaller coalition over the grand one, this does not mean that the grand coalition will not be formed. For instance, in Example 5 player 1 prefers $A = \{1, 2, 3\}$ to N, but will have difficulties convincing player 2 and player 3 to form this coalition, since they certainly prefer the payout provided to them by the grand coalition. Therefore player 1 would perhaps choose to join the grand coalition (however grudgingly), since it is still offering a better payoff than going-it-alone (which would bring a payoff of 0 to player 1). However, even if player 1 joins the grand coalition, it would be unsatisfied with the situation and may show its discontent by actively and visibly (or covertly and by using inappropriate means) working to undermine the grand coalition and exclude player 4.

Thus encouraging the grand coalition is not necessary to coalescence all players into the grand coalition, but may be desirable in practice.

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Appendix A. Remarks on 3-player games

In Example 3 we provided a quick remark on a condition on 3-player games that is equivalent to convexity and we used this condition in the course of the proof of Theorem 2. We provide a brief justification.

Proposition 2. *A 3-player game v is convex if and only if, for every permutation (i, j, k) of N,*

$$v_i + v_j + m_k \leq T \leq m_i + m_j + v_k. \tag{3}$$

Proof. As we already remarked, a game is convex if and only if, for every player i, and all coalitions $A \subseteq B$ that do not contain i, $m_i(A) \leq m_i(B)$.

Therefore, in the context of a 3-player game the convexity is equivalent to the system of inequalities

$$v_i \leq m_{ij} \leq m_i, \tag{4}$$

for $i, j \in N$, $i \neq j$.

The inequality (4) is equivalent to

$$v_i + v_j + m_k \leq v_j + m_{ij} + m_k \leq m_i + v_j + m_k,$$

where k is the third player (different from i and j). Since $T = v_j + m_{ij} + m_k$ we obtain

$$v_i + v_j + m_k \leq T \leq m_i + v_j + m_k.$$

Thus, when looked as systems of inequalities, (3) and (4) are equivalent. □

The games with non-empty core were characterized by Bondareva [1]. Namely, a game has a non-empty core if and only if it is balanced. A game v is balanced if, for every sequence of non-empty subsets $A_1, \dots, A_s$ of N and every sequence of positive real numbers $\lambda_1, \dots, \lambda_s$ such that

$$\sum_{\ell=1}^{s} \lambda_\ell \chi_{A_\ell} = \chi_N, \tag{5}$$

where χ_{A_ℓ} and χ_N denote the characteristic function of the sets A_ℓ and N, we have

$$\sum_{\ell=1}^{s} \lambda_\ell v(A_\ell) \le v(N).$$

A game is totally balanced if all of its sub-games are balanced.

The following modification of the balancing condition is also valid.

Proposition 3. *A game v has non-empty core if and only if, for every sequence of non-empty subsets $A_1, \dots, A_s$ of N and every sequence of positive real numbers $\lambda_1, \dots, \lambda_s$ such that*

$$\sum_{\ell=1}^{s} \lambda_\ell \chi_{A_\ell} \le \chi_N, \tag{6}$$

where the inequality is considered pointwise, we have

$$\sum_{\ell=1}^{s} \lambda_\ell v(A_\ell) \le v(N). \tag{7}$$

Proof. Let x be an efficient allocation in the core of v, and let $\sum_{\ell=1}^{s} \lambda_\ell \chi_{A_\ell} \le \chi_N$, for some positive real numbers $\lambda_1, \dots, \lambda_s$ and a sequence of non-empty subsets $A_1, \dots, A_s$ of N. We have

$$\sum_{\ell=1}^{s} \lambda_\ell v(A_\ell) \le \sum_{\ell=1}^{s} \lambda_\ell x(A_\ell) = \sum_{\ell=1}^{s} \lambda_\ell \sum_{i \in A_\ell} x_i = \sum_{\ell=1}^{s} \lambda_\ell \sum_{i=1}^{n} \chi_{A_\ell}(i) x_i =$$
$$= \sum_{i=1}^{n} \left(\sum_{\ell=1}^{s} \lambda_\ell \chi_{A_\ell}(i) \right) x_i \le \sum_{i=1}^{n} x_i = v(N).$$

The other direction follows from the result of Bondareva. Namely, if (7) holds whenever (6) does, then (7) also holds whenever (5) does. Therefore the core of v is non-empty. □

It is easy to see that for a 2-player game, convexity, super-additivity, and the existence of core allocations are equivalent properties and it is well known that these properties are not equivalent for more than 2 players.

Proposition 4. *Let v be a 3-player super-additive game. Define $M_{12} = v_{12} - v_1 - v_2$, $M_{13} = v_{13} - v_1 - v_3$, $M_{23} = v_{23} - v_2 - v_3$, and $S = v(N) - v_1 - v_2 - v_3$, where v_{ij} is the value of the coalition $\{i, j\}$.*

(a) The game v has a non-empty core if and only if

$$S \geq \frac{1}{2}(M_{12} + M_{13} + M_{23}). \tag{8}$$

(b) The game v is convex if and only if

$$S \geq \max\{M_{12} + M_{13},\ M_{12} + M_{23},\ M_{13} + M_{23}\ \}. \tag{9}$$

Proof. (a) Assume v has a non-empty core. By Proposition 3 (or directly by the argument used in the proof), since $\chi_{12} + \chi_{23} + \chi_{23} = 2\chi_N$, we obtain that $v_{12}+v_{13}+v_{23} \leq 2v(N)$. Therefore, $M_{12}+M_{13}+M_{23} = v_{12}+v_{13}+v_{23}-2(v_1+v_2+v_3) \leq 2v(N) - 2(v_1 + v_2 + v_3) = 2S$.

Conversely, assume that (8) holds. Instead of trying to use Proposition 3, we construct explicitly an element in the core.

Assume that the sum of every pair of numbers from $\{M_{12}, M_{13}, M_{23}\}$ is no smaller than the third one (triangle-like inequalities hold). Set $a_1 = \frac{M_{12}+M_{13}-M_{23}}{2}$, $a_2 = \frac{M_{12}+M_{23}-M_{13}}{2}$, $a_3 = \frac{M_{13}+M_{23}-M_{12}}{2}$, and $t = \frac{1}{3}\left(S - \frac{1}{2}\left(M_{12} + M_{13} + M_{23}\right)\right)$. Then a_1, a_2, a_3, and t are non-negative. Set $x_1 = v_1 + a_1 + t$, $x_2 = v_2 + a_2 + t$, and $x_3 = v_3 + a_3 + t$. Since $x_1 + x_2 + x_3 = v(N)$, the allocation x is efficient. The allocation x is individually rational (by the non-negativity of a_1, a_2, a_3, and t). We also have

$$x_1 + x_2 = v_1 + v_2 + M_{12} + 2t = v_{12} + 2t \geq v_{12}.$$

Thus the allocation x is rational for the coalition $\{1, 2\}$. By symmetry, x is rational for the other two 2-element coalitions as well. Note that we have not used yet the super-additivity property.

Assume that the sum of two of the numbers M_{12}, M_{13}, M_{23} is smaller than the third, say $M_{12} > M_{13} + M_{23}$ and set $t = \frac{1}{2}(S - M_{13} - M_{23})$. The super-additivity implies that $v(N) \geq v_{12}+v_3$. Therefore $S = v(N)-v_1-v_2-v_3 \geq v_{12}+v_3-v_1-v_2-v_3 = M_{12}$. Since $S \geq M_{12} > M_{13} + M_{23}$, we have that $t > 0$. Set $x_1 = v_1 + M_{13} + t$,

$x_2 = v_2 + M_{23} + t$, and $x_3 = v_3$. Since $x_1 + x_2 + x_3 = v(N)$, the allocation x is efficient. The allocation x is individually rational by the non-negativity of M_{12}, M_{13}, M_{23}, and t (for $i \neq j$, M_{ij} is non-negative by the super-additivity property). Further,

$$x_1 + x_3 = v_1 + v_3 + M_{13} + t = v_{13} + t \geq v_{13}$$

and, by symmetry,

$$x_2 + x_3 \geq v_{23}.$$

We also have

$$x_1 + x_2 = v_1 + M_{13} + t + v_2 + M_{23} + t = v_1 + v_2 + S = v_{12} - M_{12} + S \geq v_{12}.$$

Thus the allocation x is rational for all 2-element coalitions.

(b) Note that the convexity needs to be checked only for coalitions that are not comparable (the convexity condition is trivially satisfied when one of the coalitions is included in the other). Therefore, given the super-additivity of the game, v is convex if and only if, for every permutation (i, j, k) of N

$$v_{ij} + v_{ik} \leq v(N) + v_i.$$

The last inequality is equivalent to

$$M_{ij} + M_{ik} \leq S.$$

□

Therefore, we see that the convexity and the existence of the core are not equivalent for 3-player games even in the presence of super-additivity. For instance, if $v_1 = v_2 = v_3 = 0$, $v_{12} = v_{13} = v_{23} = 1$ and $v_N = 3/2$, we have a super-additive, non-convex game with non-empty core.

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ON CERTAIN SUBCLASS OF p-VALENTLY BAZILEVIC FUNCTIONS

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Abstract. A certain subclass $B_1(p,n,\alpha,\beta)$ with $p,n \in N = \{1,2,\dots\}$, $\alpha > 0$ and $0 \le \beta < p$, of p-valently Bazilevic functions in the unit disc $U = \{z : |z| < 1\}$ is introduced. The object of the present paper is to derive some properties of the class $B_1(p,n,\alpha,\beta)$.

1. Introduction

Let $A(p,n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k (p,n \in N = \{1,2,..\}, \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in A(p,n)$ is said to be in the class $S(p,n,\beta)$ of p-valently starlike functions of order $\beta(0 \le \beta < p)$ if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \text{ and } \int_0^{2\pi} \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta = 2\pi p\ . \tag{1.2}$$

The class $S(p,n,\beta)$ was studied recently by Owa [8] and Aouf et al. [1]. Also, we note that $S(p,n,0) = S^*(p,n)$ and $S(p,1,0) = S^*(p)$.

A function $f(z) \in A(p,n)$ is said to be in the class $B(p,n,\alpha,\beta)$ if it satisfies

$$\mathrm{Re}\left\{\frac{z\,f'(z)}{f^{1-\alpha}(z)\,g^{\alpha}(z)}\right\} > \beta \tag{1.3}$$

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for some $\alpha(\alpha > 0)$, $\beta(0 \leq \beta < p)$, $g(z) \in S^*(p, n)$ and for all $z \in U$. Further, let $B_1(p, n, \alpha, \beta)$ be the subclass of $B(p, n, \alpha, \beta)$ for $g(z) = z^p \in S^*(p)$. Also we say that $f(z)$ in the class $B(p, n, \alpha, \beta)$ is a Bazilevic function of order β and type α (see [5]).

Remark 1. *(i) The classes $B(p, n, \alpha, \beta)$ and $B_1(p, n, \alpha, \beta)$ are the subclasses of p-valently Bazilevic functions in U.*

(ii) The classes $B(p, 1, \alpha, \beta) = B(p, \alpha, \beta)$ and $B_1(p, 1, \alpha, \beta) = B_1(p, \alpha, \beta)$ when $p = 1$ were studied by Owa [10] *and the class $B(p, \alpha, \beta)$ was studied by Nunokawa et al.* [5].

(iii) The class $B_1(1, n, \alpha, \beta)$ was studied by Owa [9].

(iv) The classes $B(1, 1, \alpha, \beta) = B(\alpha, \beta)$ and $B_1(1, 1, \alpha, \beta) = B_1(\alpha, \beta)$ when $p = n = 1$ were studied by Owa and Obradovic [11].

(v) The classes $B(1, 1, \alpha, 0) = B(\alpha)$ and $B_1(1, 1, \alpha, 0) = B_1(\alpha)$ when $p = n = 1$ and $\beta = 0$ were studied by Singh [12].

2. **Properties of the class** $B_1(p, n, \alpha, \beta)$

In order to establish our main result, we have to recall here the following lemma due to Miller and Mocanu [4].

Lemma 1. *Let $\varphi(u, v)$ be a complex valued function,*

$$\varphi : D \to C,\ D \subset C \times C = C^2 \qquad (C \text{ is the complex plane}),$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\varphi(u, v)$ satisfies

(i) $\varphi(u, v)$ is continuous in D;

(ii) $(1, 0) \in D$ and $\mathrm{Re}\{\varphi(1, 0)\} > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{n}{2}(1 + u_2^2)$, $\mathrm{Re}\{\varphi(iu_2, v_1)\} \leq 0$.

Let $q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + ...$ be regular in the unit disc U such that $(q(z), z q'(z)) \in D$ for all $z \in U$. If

$$\mathrm{Re}\{\varphi(iu_2, v_1)\} > 0 \quad (z \in U),$$

then

$$\mathrm{Re}\{q(z)\} > 0 \qquad (z \in U).$$

Using the above lemma, we prove the following result.

Theorem 2. *If* $f(z) \in B_1(p,n,\alpha,\beta)$,*with* $p,n \in N, \alpha > 0$ *and* $0 \leq \beta < p$, *then*

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\}^{\alpha} > \frac{n+2\alpha\beta}{n+2\alpha\, p} \quad (z \in U). \tag{2.1}$$

Proof. We define the function $q(z)$ by

$$\left\{\frac{f(z)}{z^p}\right\}^{\alpha} = \delta + (1-\delta)\, q(z) \tag{2.2}$$

with

$$\delta = \frac{n+2\alpha\beta}{n+2\alpha p} \quad . \tag{2.3}$$

Then, we see that $q(z) = 1 + q_n\, z^n + q_{n+1}\, z^{n+1} + ...$ is regular in U. It follows from (2.2) that

$$\frac{f^{'}(z)\, f^{\alpha-1}(z)}{z^{p\,\alpha-1}} - \beta = [p\delta + (p-p\delta)q(z)] + \frac{(1-\delta)\, z\, q^{'}(z)}{\alpha}, \tag{2.4}$$

or

$$\operatorname{Re}\left\{\frac{f^{'}(z)\, f^{\alpha-1}(z)}{z^{p\,\alpha-1}} - \beta\right\}$$

$$= \operatorname{Re}\left\{p\delta - \beta + (p-p\delta)\, q(z) + \frac{(1-\delta)\, z\, q^{'}(z)}{\alpha}\right\} > 0\ . \tag{2.5}$$

Now, setting $q(z) = u = u_1 + iu_2$, $z\, q^{'}(z) = v = v_1 + iv_2$, and

$$\varphi(u,v) = p\delta - \beta + (p-p\delta)u + \frac{(1-\delta)v}{\alpha}\ , \tag{2.6}$$

it is easily seen that

(i) $\varphi(u,v)$ is continuous in $D = C \times C$

(ii) $(1,0) \in D$ and $\operatorname{Re}\{\varphi(1,0)\} = p - \beta > 0$, and

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq \frac{-1}{2}\, n(1+u_2^2)$,

$$\begin{aligned} \operatorname{Re}\{\varphi(iu_2,v_1)\} &= p\delta - \beta + \frac{(1-\delta)\, v_1}{\alpha} \\ &\leq p\delta - \beta - \frac{n(1-\delta)(1+u_2^2)}{2\alpha} \leq 0, \end{aligned}$$

for δ given by (2.3). Therefore the function $\varphi(u,v)$ satisfies the condition in Lemma 1. This implies that $\operatorname{Re}\{q(z)\} > 0$ $(z \in U)$, that is, that

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\}^{\alpha} > \delta = \frac{n+2\alpha\beta}{n+2\alpha p} \quad (z \in U). \tag{2.7}$$

This completes the proof of Theorem 1.

Putting $\beta = 0$ in Theorem 1, we have

Corollary 3. *If* $f(z) \in B_1(p, n, \alpha, 0)$, *with* $p, n \in N$ *and* $\alpha > 0$, *then*

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\}^{\alpha} > \frac{n}{n + 2\alpha p} \quad (z \in U). \tag{2.8}$$

Further, making $\alpha = \frac{1}{p}$ in Corollary 1, we have

Corollary 4. *If* $f(z) \in B_1(p, n, \frac{1}{p}, 0)$, *then*

$$\operatorname{Re}\left\{\frac{f^{\frac{1}{p}}(z)}{z}\right\} > \frac{n}{n + 2} \quad (z \in U). \tag{2.9}$$

Putting $\alpha = \frac{1}{2}$ in Theorem 1, we have

Corollary 5. *If* $f(z) \in B_1(p, n, \frac{1}{2}, \beta)$, *with* p, $n \in N$ *and* $0 \leq \beta < p$, *then*

$$\operatorname{Re}\sqrt{\frac{f(z)}{z^p}} > \frac{n + \beta}{n + p} \, (z \in U)\,. \tag{2.10}$$

Remark 2. *(1) Putting* $p = 1$ *in Theorem 1, Corollary 1 and Corollary 2, respectively, we obtain the results obtained by Owa [*9*, Theorem1, Corollary1 and Corollary2, respectively].*

(2) Putting $p = \alpha = 1$ *and* $\beta = 0$ *in Theorem1, we obtain the result obtained by Cho [*2*, Theorem2].*

(3)Putting $n = 1$ *in Theorem1, we obtain the result obtained by Owa [*10*, Lemma 4]. Owa* [10] *obtained this result by different method.*

(4) Putting $n = 1$ *in Corollary1 and Corollary2, respectively, we obtain the results obtained by Owa [*10*, Corollary3 and Corollary4, respectively].*

(5) Putting $n = p = 1$ *in Theorem1, we obtain the result obtained by Owa and Obradovic [*11*, Theorem4].*

(6) Putting $n = p = 1$ *in Corollary1, we obtain the result obtained by Owa and Obradovic [*11 *Corollary3] and Obradovic [*7*, Theorem3].*

(7) Putting $n = p = 1$ *and* $\alpha = 1$ *in Theorem1, we obtain the result obtained by Owa and Obradovic [*11*, Corollary4].*

(8) Putting $n = p = \alpha = 1$ in Theorem1 and $\beta = 0$, we obtain the result obtained by Obradovic [6, Theorem2].

Theorem 6. *If $f(z) \in B_1(p,n,\alpha,\beta)$, with $p,n \in N, \alpha > 0$ and $0 \le \beta < p$, then*

$$\mathrm{Re}\left\{\frac{f(z)}{z^p}\right\}^{\frac{\alpha}{2}} = \frac{n+\sqrt{n^2+4\alpha\beta(n+p\alpha)}}{2(n+p\alpha)} \quad (z \in U). \tag{2.11}$$

Proof. Defining the function $q(z)$ by

$$\left\{\frac{f(z)}{z^p}\right\}^{\frac{\alpha}{2}} = \delta + (1-\delta)\,q(z) \tag{2.12}$$

with

$$\delta = \frac{n+\sqrt{n^2+4\alpha\beta(n+p\alpha)}}{2(n+p\alpha)}\ , \tag{2.13}$$

we easily see that $q(z) = 1 + q_n\, z^n + q_{n+1}\, z^{n+1} + ...$ is regular in U. Taking the differentiations of both sides in (2.12), we obtain that

$$\frac{f^{'}(z) f^{\alpha-1}(z)}{z^{p\,\alpha-1}} = p[\delta+(1-\delta)q(z)]^2 + \frac{2}{\alpha}(1-\delta)[\delta+(1-\delta)q(z)]\, z\, q^{'}(z), \tag{2.14}$$

that is, that

$$\mathrm{Re}\left\{\frac{f^{'}(z) f^{\alpha-1}(z)}{z^{p\,\alpha-1}} - \beta\right\} = \mathrm{Re}\{p[\delta+(1-\delta)q(z)]^2 + \frac{2}{\alpha}(1-\delta)[1+(1-\delta)q(z)]\, z\, q^{'}(z) - \beta\} > 0 \quad (z \in U). \tag{2.15}$$

Taking $q(z) = u = u_1 + iu_2$ and $z\, q^{'}(z) = v = v_1 + iv_2$, we define the function $\varphi(u,v)$ by

$$\varphi(u,v) = p[\delta+(1-\delta)u]^2 + \frac{2}{\alpha}(1-\delta)[\delta+(1-\delta)u]v - \beta. \tag{2.16}$$

Then $\varphi(u,v)$ satisfies

(i) $\varphi(u,v)$ is continuous in $D = C \times C$;

(ii) $(1,0) \in D$ and $\mathrm{Re}\{\varphi(1,0)\} = p - \beta > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \le \frac{-n}{2}(1+u_2^2)$,

$$\begin{aligned}\mathrm{Re}\{\varphi(iu_2,v_1)\} &= p[\delta^2-(1-\delta)^2\ u_2^2] + \frac{2}{\alpha}(1-\delta)\delta\ v_1 - \beta \\ &\le p[\delta^2-(1-\delta)^2\ u_2^2] - \beta - \frac{n}{\alpha}\,\delta(1-\delta)(1+u_2^2) \le 0\ .\end{aligned}$$

Thus the function $\varphi(u,v)$ satisfies the conditions in Lemma 1. Applying Lemma 1, we conclude that

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\}^{\frac{\alpha}{2}} > \delta = \frac{n+\sqrt{n^2+4\alpha\beta(n+p\alpha)}}{2(n+p\alpha)} \quad (z \in U). \tag{2.17}$$

This completes the proof of Theorem 2.

Putting $\beta = 0$ in Theorem 2, we have

Corollary 7. *If $f(z) \in B_1(p,n,\alpha,0)$, with p, $n \in N$ and $\alpha > 0$, then*

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\}^{\frac{\alpha}{2}} > \frac{n}{n+p\alpha} \quad (z \in U). \tag{2.18}$$

Putting $\alpha = 1$ in Theorem 2, we have

Corollary 8. ***[3, Theorem 2].*** *If $f(z) \in B_1(p,n,1,\beta)$, with p, $n \in N$ and $0 \le \beta < p$, then*

$$\operatorname{Re}\left\{\sqrt{\frac{f(z)}{z^p}}\right\} > \frac{n+\sqrt{n^2+4\beta(n+p)}}{2(n+p)} \quad (z \in U). \tag{2.19}$$

Putting $\alpha = 1$ and $\beta = 0$ in Theorem 2, we have

Corollary 9. ***[3, Corollary 3].*** *If $f(z) \in B_1(p,n,1,0)$, with $p, n \in N$, then*

$$\operatorname{Re}\left\{\sqrt{\frac{f(z)}{z^p}}\right\} > \frac{n}{n+p} \quad (z \in U). \tag{2.20}$$

Remark 3. *(i) Putting $p = 1$ in Theorem 2, we obtain the result obtained by Owa [9, Theorem 2].*

(ii) Putting $\alpha = p = 1$ in Theorem 2, we obtain the result obtained by Owa [9, Corollary 3].

(iii) Putting $\alpha = 2$, $p = 1$ and $\beta = 0$ in Theorem 2, we obtain the result obtained by Cho [2, Theorem 3].

Theorem 10. *If $f(z) \in B_1(p,n,\alpha,\beta)$, with p, $n \in N, \alpha > 0$ and $0 \le \beta < p$, then the function $G_1(z)$ defined by*

$$G_1^{\alpha+\gamma}(z) = z^{p\,\gamma} f^{\alpha}(z) \quad (\gamma \ge 0) \tag{2.21}$$

is in the class $B_1(p, n, \alpha+\gamma, \delta)$, where

$$\delta = \frac{1}{\alpha+\gamma}\,(\frac{p\,\gamma(n+2\alpha\,\beta)}{n+2p\,\alpha} + \alpha\;\beta). \tag{2.22}$$

Proof. Noting that

$$\frac{(\alpha+\gamma)G_1^{'}(z)}{1-(\alpha+\gamma)} = p\,\gamma\,z^{p\gamma-1}f^{\alpha}(z) + \alpha\,z^{p\gamma}f^{'}(z)\,f^{\alpha-1}(z),$$

that is, that

$$(\alpha+\gamma)\,\frac{z\,G_1^{'}(z)\,G_1^{(\alpha+\gamma)-1}(z)}{z^{p(\alpha+\gamma)}} = p\,\gamma\,(\frac{f(z)}{z^p})^{\alpha} + \frac{\alpha\;zf^{'}(z)f^{\alpha-1}(z)}{z^{p\,\alpha}}. \tag{2.23}$$

Therefore, it follows from Theorem 1 that

$$\begin{aligned} \mathrm{Re}\left\{\frac{z\,G_1^{'}(z)G_1^{(\alpha+\gamma)-1}(z)}{z^{p(\alpha+\gamma)}}\right\} &= \frac{1}{\alpha+\gamma}\;\mathrm{Re}\left\{p\,\gamma(\frac{f(z)}{z^p})^{\alpha} + \frac{\alpha zf^{'}(z)\,f^{\alpha-1}(z)}{z^{p\,\alpha}}\right\} \\ &> \frac{1}{\alpha+\gamma}\left\{p\,\gamma(\frac{n+2\alpha\,\beta}{n+2\alpha\,p}) + \alpha\,\beta\right\}\;. \end{aligned}$$

This completes the proof of Theorem 3.

Taking $\beta = 0$ in Theorem 3, we have

Corollary 11. *If $f(z) \in B_1(p, n, \alpha, 0)$, with p, $n \in N$ and $\alpha > 0$, then the function $G_1(z)$ defined by* (2.21) *is in the class $B_1(p, n, \alpha+\gamma, \delta)$, where*

$$\delta = \frac{p\,\gamma}{(\alpha+\gamma)(n+2p\,\alpha)}\;. \tag{2.24}$$

Taking $p = 1$ in Theorem 3, we have

Corollary 12. *If $f(z) \in B_1(1, n, \alpha, \beta)$ with $n \in N$, $\alpha > 0$ and $0 \le \beta < 1$, then the function $G_2(z)$ defined by*

$$G_2^{\alpha+\gamma}(z) = z^{\,\gamma}f^{\alpha}(z) \quad (\gamma \ge 0) \tag{2.25}$$

is in the class $B_1(1, n, \alpha+\gamma, \delta)$, where

$$\delta = \frac{1}{\alpha+\gamma}\,(\frac{\gamma(n+2\alpha\,\beta)}{n+2\alpha} + \alpha\,\beta). \tag{2.26}$$

Remark 4. *Putting $n = 1$ in Theorem 3, Corollary 7 and Corollary 8, respectively, we obtain the results obtained by Owa [10, Theorem 2, Corollary 5 and Corollary 6, respectively].*

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ON MIXED NONLINEAR INTEGRAL EQUATIONS OF VOLTERRA-FREDHOLM TYPE WITH MODIFIED ARGUMENT

CLAUDIA BACOŢIU

Abstract. In the present paper we consider the following mixed Volterra-Fredholm nonlinear integral equation with modified argument:

$$u(t,x) = g\big(t,x,u(t,x)\big) + \int_0^t H\left(t,x,s,u(s,x)\right) ds$$
$$+ \int_0^t \int_a^b K\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big) dyds$$

For this equation, we will study: the existence and the uniqueness of the solution, the data dependence of the solution and the differentiability of the solution with respect to parameters.

1. **Introduction**

Let $(X, \|\cdot\|_X)$ be a Banach space.

In this paper we consider the following nonlinear integral equation of Volterra-Fredholm type:

$$u(t,x) = g\big(t,x,u(t,x)\big) + \int_0^t H\left(t,x,s,u(s,x)\right) ds$$
$$+ \int_0^t \int_a^b K\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big) dyds \qquad (1)$$

for all $(t,x) \in [0,T] \times [a,b] := \overline{D}$; $u \in C(\overline{D}, \mathbb{R}^m)$, where $b > a > 0$ and $T > 0$.

Volterra-Fredholm (VF on short) integral equations often arise from the mathematical modelling of the spreading, in space and time, of some contagious diseases, in the theory of nonlinear parabolic boundary value problem and in many physical and

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biological models.

Most results for VF equation establish numerical approximation of the solutions; e.g. [8], [9], [22], [2], [11], [3], [7].

In [21] H. R. Thieme considered a model for the spatial spread of an epidemic consisting of a nonlinear integral equation of Volterra-Fredholm type which has an unique solution. The author showed that this solution has a temporally asymptotic limit which describes the final state of the epidemic and is the minimal solution of another nonlinear integral equation.

In [4] O. Diekmann described, derived and analysed a model of spatio-temporal development of an epidemic. The model considered leads (see [13]) to the following nonlinear integral equation of Volterra-Fredholm type:

$$u(t,x) = g(t,x) + \int_0^t \int_\Omega g(u(t-\tau,\xi))S_0(\xi)A(\tau,x,\xi)d\xi d\tau \tag{2}$$

for all $(t,x) \in [0,\infty] \times \Omega$, where Ω is a bounded domain in $\mathbb{R}^n$.

In [13] B. G. Pachpatte considered the integral equation

$$u(t,x) = g(t,x) + \int_0^t \int_\Omega g(t,x,s,y,u(s,y))dyds \tag{3}$$

for all $(t,x) \in [0,T] \times \Omega = D$, where Ω is a bounded domain in $\mathbb{R}^n$. Using Contraction Principle, the author proved that, under appropriate assumptions, (3) has a unique solution in a subset S of $C(D,\mathbb{R}^n)$. The result was then applied to show the existence and uniqueness of solutions to certain nonlinear parabolic differential equations and mixed Volterra-Fredholm integral equations occurring in specific physical and biological problems (e.g. a reliable treatment of the Diekmann's model mentioned above is given).

In [10], D. Mangeron and L. E. Krivošein obtained existence, uniqueness and stability conditions for the solutions of a class of boundary problems for linear and nonlinear heat equation with delay. Under certain conditions, this problem is equivalent with the following nonlinear VF equation:

$$u(t,x) = n(t,x) + \int_0^t \int_0^a \Big[G(x,\xi,t-\alpha)g\big(\xi,\alpha,u(\xi,\alpha),u\big(\xi,\alpha-r_1(\alpha)\big)\Big)$$

$$+\int_0^a \int_0^\alpha K(\xi,\alpha,s,y)g\big(s,y,u(s,y),u(s,y-r_2(s))\big)dyds\bigg]d\xi d\alpha$$

where

$$n(t,x)=\int_0^a \left[\frac{2}{a}\sum_{i=1}^{\infty} e^{-(\frac{\pi i}{a})^2 t}\cdot sin\frac{\pi ix}{a}\cdot sin\frac{\pi i\xi}{a}\cdot \varphi_0(\xi)\right]d\xi$$

Applying Contraction Principle, an existence and uniqueness theorem is obtained.
In [14], the following problem is considered:

$$\begin{cases} u_t(t,x)=a^2u_{xx}(t,x)+g\big(u(t,x),u(x,[t])\big) \\ u(x,0)=\varphi(x) \quad t\in\mathbb{R} \end{cases}$$

where $[t]$ means the integer part of t. Using integration by parts twice for the equation above, in appropriate conditions, the problem is equivalent with a VF equation and the successive approximation method is applied.

The purpose of the present paper is to give results concerning the following problems related to equation (1): the existence and the uniqueness of the solution, the data dependence of the solution and the differentiability of the solution with respect to parameters.

Because the tool used in the present paper is the Picard operators theory, for the convenient of the reader, we present some basic notions and results concerning this important class of operators.

2. **Picard operators**

Let (X,d) be a metric space and $A : X \to X$ an operator. In this paper we will use the following notations:

$$F_A := \{x \in X : A(x) = x\};$$

$$A^0 := 1_X\ ,\ A^{n+1} := A\circ A^n \text{ for all } n\in\mathbb{N}.$$

Definition 2.1. *(Rus* [15]*) The operator A is said to be:*
(i) ***weakly Picard operator (wPo)*** *if* $A^n(x_0)\to x_0^*$ *for any* $x_0\in X$ *and the limit* x_0^* *is a fixed point of A, which may depend on* x_0*.*

(ii) ***Picard operator (Po)*** *if* $F_A = \{x^*\}$ *and* $A^n(x_0) \to x^*$ *for any* $x_0 \in X$ *.*

For a weakly Picard operator A, the operator A^∞ *is defined as follows:*

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Notice that $A^\infty(X) = F_A$*.*

If A is Picard operator, then $A^\infty(x) = x^*$ *for all* $x \in X$*, where* x^* *is the unique fixed point of A.*

Example 2.1. *Any* α*-contraction on a complete metric space* (X, d) *is a Picard operator.*

The following abstract theorem is needed in the study of data dependence of the solution:

Theorem 2.1. *(Rus* [17]*) Let* (X, d) *a complete metric space and* $A, B : X \to X$ *two operators. Assume that:*

(i) there exists $\alpha \in [0, 1[$ *such that A is* α*-contraction; let* $F_A = \{x_A^*\}$

(ii) $F_B \neq \emptyset$*; let* $x_B^* \in F_B$*;*

(iii) there exists $\eta > 0$ *such that* $d(A(x), B(x)) \leq \eta$ *for all* $x \in X$*.*

Then

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\alpha}.$$

In order to study the differentiability of the solution with respect to a parameter, we need the following theorem, due to I. A. Rus:

Theorem 2.2. *(Fiber Contraction Principle, Rus* [16]*) Let* (X, d)*,* (Y, ρ) *be two metric spaces and* $B : X \to X$*,* $C : X \times Y \to Y$ *two operators such that:*

(i) (Y, ρ) *is complete;*

(ii) B is a Picard operator, $F_B = \{x^*\}$*;*

(iii) $C(\cdot, y) : X \to Y$ *is continuous for all* $y \in Y$*;*

(iv) there exists $\alpha \in]0, 1[$ *such that the operator* $C(x, \cdot) : Y \to Y$ *is* α*-contraction for all* $x \in X$*; let* y^* *be the unique fixed point of* $C(x^*, \cdot)$*.*

Then

$$A : X \times Y \to X \times Y, \quad A(x, y) := (B(x), C(x, y))$$

is a Picard operator and $F_A = \{(x^*, y^*)\}$.

For Picard operators theory applied in the study of differential or integral equations see [19], [18], [17], [12], [20], [6], [5].

3. **Existence and uniqueness theorem**

Consider the equation (1).

Theorem 3.1. *If the following conditions are satisfied:*

(c1) $g \in C(\overline{D} \times X, X)$, $H \in C(\overline{D} \times [0,T] \times X, X)$ $K \in C(\overline{D}^2 \times X^2, X)$, $\varphi_1 \in C(\overline{D}, [0,T])$ *and* $\varphi_2 \in C(\overline{D}, [a,b])$;

(c2) *there exists* $L_g > 0$ *such that:*

$$\|g(t,x,u) - g(t,x,v)\|_X \le L_g \|u - v\|_X \tag{4}$$

for all $(t,x) \in \overline{D}$ *and* $u, v \in X$

(c3) *there exists* $L_H > 0$ *such that:*

$$\|H(t,x,s,u) - H(t,x,s,v)\|_X \le L_H \|u - v\|_X \tag{5}$$

for all $(t,x,s) \in \overline{D} \times [0,T]$ *and* $u, v \in X$

(c4) *there exists* $L_K > 0$ *such that:*

$$\|K(t,x,s,y,u,\overline{u}) - K(t,x,s,y,v,\overline{v})\|_X \le L_K\big(\|u - v\|_X + |\overline{u} - \overline{v}\|_X\big) \tag{6}$$

for all $(t,x,s,y) \in \overline{D}^2$ *and* $u, v, \overline{u}, \overline{v} \in X$

(c5) $L_g < 1$

(c6) *there exists* $\tau > 0$ *such that:*

$$\alpha := L_g + \frac{1}{\tau} L_H + \frac{b-a}{\tau} L_K + \max\Big\{ \int_0^t \int_a^b e^{\tau[\varphi_1(s,y) - t]} dyds : t \in [0,T] \Big\} L_K < 1 \tag{7}$$

Then (1) has an unique solution $u^* \in C(\overline{D}, X)$.

Proof. Let the space $C(\overline{D}, X)$ be endowed with a Bielecki-Chebysev suitable norm

$$\|u\|_{BC} := \sup\{\|u(t,x)\|_X \, e^{-\tau t} : t \in [0,T], x \in [a,b]\}, \quad \tau > 0 \tag{8}$$

Consider the operator $A : C(\overline{D}, X) \to C(\overline{D}, X)$ defined by:

$$A(u)(t,x) := g(t,x) + \int_0^t \int_a^b K\Big(t,x,s,y,u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big)dyds \qquad (9)$$

for all $u \in C(\overline{D})$, for all $(t,x) \in \overline{D}$.

For any $u, v \in C(\overline{D}, X)$ we have (see [1]):

$$\|A(u)(t,x) - A(v)(t,x)\|_X$$

$$\le L_g\|u(t,x) - v(t,x)\|_X + L_H \int_0^t \|u(s,x) - v(s,x)\|_X ds + L_K \frac{b-a}{\tau} \|u - v\|_{BC} \cdot e^{\tau t}$$

$$+L_K \max\Big\{ \int_0^t \int_a^b e^{\tau[\varphi_1(s,y)-t]} dyds : t \in [0,T]\Big\} \cdot \|u - v\|_{BC} \cdot e^{\tau t}$$

so:

$$\|A(u) - A(v)\|_{BC} \le \alpha \|u - v\|_{BC}.$$

From **(c6)** there exists $\tau > 0$ such that $A : C(\overline{D}, X) \to C(\overline{D}, X)$ is α-contraction and, by Contraction Principle, A is a Picard operator, i.e. the equation has a unique solution in $C(\overline{D}, X)$.

Remark 3.1. *Condition **(c6)** from Theorem 3.1 can be replaced by the next simpler condition:*

(c7) $\quad \varphi_1(t,x) \le t \quad$ *for all* $(t,x) \in \overline{D}$

In this case the operator A given by (9) is $\overline{\alpha}$-contraction, with

$$\overline{\alpha} = L_g + \frac{L_H + 2(b-a)L_K}{\tau} < 1 \qquad (10)$$

for a suitable chosen τ.

4. **Data dependence of the solution**

In order to prove the dependence of the solution on data, let us consider two mixed VF equations:

$$u(t,x) = g_i\big(t,x,u(t,x)\big) + \int_0^t H_i\left(t,x,s,u(s,x)\right) ds$$

$$+ \int_0^t \int_a^b K_i\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big)dyds \qquad (11)$$

for all $u \in C(\overline{D}, X)$ and $(t,x) \in \overline{D}$, with $g_i \in C(\overline{D} \times X, X)$, $H_i \in C(\overline{D} \times [0,T] \times X, X)$ and $K_i \in C(\overline{D}^2 \times X^2, X)$ for $i = \overline{1,2}$.

Theorem 4.1. *Assume that the first equation from (11) satisfies conditions **(c1)**-**(c5)** and **(c7)**; let u^* be its unique solution. Assume that the second equation from (11) has at least one solution; let v^* be a such solution.*
If there exist $\eta_1, \eta_2, \eta_3 > 0$ such that:

$$\|g_1(t,x,u) - g_2(t,x,u)\|_X \le \eta_1 \quad \textit{for all } (t,x,u) \in \overline{D} \times X$$

$$\|H_1(t,x,s,u) - H_2(t,x,s,u)\|_X \le \eta_2 \quad \textit{for all } (t,x,s,u) \in \overline{D} \times [0,T] \times X$$

$$\|K_1(t,x,s,y,u) - K_2(t,x,s,y,u)\|_X \le \eta_3 \quad \textit{for all } (t,x,s,y,u) \in \overline{D}^2 \times X$$

Then:

$$\|u^* - v^*\|_{BC} \le \frac{\eta_1 + T\eta_2 + T(b-a)\eta_3}{1 - \overline{\alpha}}$$

where $\overline{\alpha} = L_g + \dfrac{L_H + 2(b-a)L_K}{\tau} < 1$ *for suitable chosen* τ.

Proof. Consider the operators $A_1, A_2 : C(\overline{D}, X) \to C(\overline{D}, X)$ given by:

$$A_i(u)(t,x) := g_i\big(t,x,u(t,x)\big) + \int_0^t H_i\,(t,x,s,u(s,x))\,ds$$

$$+ \int_0^t \int_a^b K_i\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big)dyds$$

for all $u \in C(\overline{D})$ and $(t,x) \in \overline{D}$, $i = \overline{1,2}$.
For any $u \in C(\overline{D})$ we have:

$$\|A_1(u)(t,x) - A_2(u)(t,x)\|_X \le \eta_1 + T\eta_2 + T(b-a)\eta_3 \quad \text{for all } (t,x) \in \overline{D}$$

Applying $sup_{(t,x)\in\overline{D}}$, we obtain:

$$\|A_1(u) - A_2(u)\|_C \le \eta_1 + T\eta_2 + T(b-a)\eta_3$$

where $\|\cdot\|_C$ is Chebysev norm:

$$\|u\|_C := \sup\{\|u(t,x)\|_X : (t,x) \in \overline{D}\}, \quad \text{for all } u \in C(\overline{D}, X)$$

But $\|\cdot\|_{BC} \le \|\cdot\|_C$, so:

$$\|A_1(u) - A_2(u)\|_{BC} \le \eta_1 + T\eta_2 + T(b-a)\eta_3 \tag{12}$$

Consider the operators A_1 and A_2 defined above, on the space $\big(C(\overline{D}, X), \|\cdot\|_{BC}\big)$. By Theorem 3.1, A_1 is $\overline{\alpha}$-contraction for suitable chosen τ , so $F_{A_1} = \{u^*\}$. Taking account of (12), we are in the conditions of Theorem 2.1 and the conclusion follows.

5. **Differentiability of the solution with respect to parameters**

In order to study the differentiability of the solution with respect to parameters a and b, let us consider the same equation (1):

$$u(t,x) = g\big(t,x,u(t,x)\big) + \int_0^t H\left(t,x,s,u(s,x)\right) ds$$
$$+\int_0^t \int_a^b K\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big) dyds$$

for all $t \in [0,T]$, for all $x \in [\alpha,\beta]$, where $0 < \alpha < a < b < \beta$.

Theorem 5.1. *Assume that:*

(i) $g \in C([0,T] \times [\alpha,\beta] \times \mathbb{R})$,
$H \in C([0,T] \times [\alpha,\beta] \times [0,T] \times \mathbb{R})$, $K \in C([0,T] \times [\alpha,\beta] \times [0,T] \times [\alpha,\beta] \times \mathbb{R}^2)$,
$\varphi_1 \in C([0,T] \times [\alpha,\beta], [0,T])$ *and* $\varphi_2 \in C([0,T] \times [\alpha,\beta], [\alpha,\beta])$;

(ii) $g(t,x,\cdot) \in C^1(\mathbb{R})$ *for all* $(t,x) \in [0,T] \times [\alpha,\beta]$ *and there exists* $M_g > 0$ *such that:*

$$\left|\frac{\partial g(t,x,u)}{\partial u}\right| \le M_g \tag{13}$$

for all $(t,x,u) \in [0,T] \times [\alpha,\beta] \times \mathbb{R}$;

(iii) $H(t,x,s,\cdot) \in C^1(\mathbb{R})$ *for all* $(t,x,s) \in [0,T] \times [\alpha,\beta] \times [0,T]$ *and there exists* $M_H > 0$ *such that:*

$$\left|\frac{\partial H(t,x,s,u)}{\partial u}\right| \le M_H \tag{14}$$

for all $(t,x,s,u) \in [0,T] \times [\alpha,\beta] \times [0,T] \times \mathbb{R}$;

(iv) $K(t,x,s,y,\cdot,\cdot) \in C^1(\mathbb{R}^2)$ *for all* $(t,x,s,y) \in [0,T] \times [\alpha,\beta] \times [0,T] \times [\alpha,\beta]$ *and there exists* $M_K > 0$ *such that:*

$$\left|\frac{\partial K(t,x,s,y,u,\overline{u})}{\partial u}\right| \le M_K \quad \text{and} \quad \left|\frac{\partial K(t,x,s,y,u,\overline{u})}{\partial \overline{u}}\right| \le M_K \tag{15}$$

for all $(t,x,s,y,u,\overline{u}) \in [0,T] \times [\alpha,\beta] \times [0,T] \times [\alpha,\beta] \times \mathbb{R}^2$;

(v) $M_g < 1$;

(vi) $\varphi_1(t,x) \le t$ *for all* $(t,x) \in [0,T] \times [\alpha,\beta]$.

Then:

a) for all $a < b \in [\alpha, \beta]$*, the equation (1) has a unique solution* $u^*(\cdot, \cdot, a, b) \in C([0,T] \times [\alpha, \beta])$*;*

b) for all $u_0 \in C([0,T] \times [\alpha, \beta])$*, the sequence* $(u_n)_{n \geq 0}$ *defined by:*

$$u_n(t,x,a,b) = g(t,x,u_{n-1}(t,x,a,b)) + \int_0^t H\left(t,x,s,u_{n-1}(s,x)\right) ds$$

$$+ \int_0^t \int_a^b K\left(t,x,s,y,u_{n-1}(s,y,a,b), u_{n-1}\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\right) dyds$$

converges uniformly to u^* *on* $[0,T] \times [\alpha, \beta] \times [\alpha, \beta] \times [\alpha, \beta]$*;*

c) $u^* \in C([0,T] \times [\alpha, \beta] \times [\alpha, \beta] \times [\alpha, \beta])$*;*

d) $u^*(t,x,\cdot,\cdot) \in C^1([\alpha, \beta] \times [\alpha, \beta])$*, for all* $(t,x) \in [0,T] \times [\alpha, \beta]$*.*

Proof. Let $X := C([0,T] \times [\alpha, \beta] \times [0,T] \times [\alpha, \beta])$ and $B : X \to X$ defined by:

$$B(u)(t,x,a,b) := g(t,x,u(t,x,a,b)) + \int_0^t H\left(t,x,s,u(s,x)\right) ds$$

$$+ \int_0^t \int_a^b K\Big(t,x,s,y,u(s,y,a,b), u\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\Big) dyds.$$

The boundedness conditions (13) and (15) implies that f and K are Lipschitz, with Lipschitz constants M_g and M_K. B satisfies **(c1)-(c5)** and **(c7)**, so a), b) and c) result. Let $u^* \in C(X)$ be the unique fixed point of B.

Obviously we have:

$$u^*(t,x,a,b) = g(t,x,u^*(t,x,a,b)) + \int_0^t H\left(t,x,s,u^*(s,x,a,b)\right) ds$$

$$+ \int_0^t \int_a^b K\Big(t,x,s,y,u^*\big(s,y,a,b\big), u^*\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\Big) dyds. \qquad (16)$$

Let us prove that $\dfrac{\partial u^*(t,x,a,b)}{\partial a}$ and $\dfrac{\partial u^*(t,x,a,b)}{\partial b}$ exist and are continuous.

1. Assume that $\dfrac{\partial u^*(t,x,a,b)}{\partial a}$ exists. Differentiate (16) with respect to a we have:

$$\frac{\partial u^*(t,x,a,b)}{\partial a} = \frac{\partial g\big(t,x,u^*(t,x,a,b)\big)}{\partial u} \cdot \frac{\partial u^*(t,x,a,b)}{\partial a}$$

$$+ \int_0^t \frac{\partial H\big(t,x,s,u^*(s,x,a,b)\big)}{\partial u} \cdot \frac{\partial u^*(s,x,a,b)}{\partial a} ds$$

$$
\begin{aligned}
&-\int_0^t K(t,x,s,a,u^*(s,a,a,b),u^*\big(\varphi_1(s,a),\varphi_2(s,a),a,b\big))ds\\
&+\int_0^t\int_a^b \frac{\partial K\Big(t,x,s,y,u^*\big(s,y,a,b\big),u^*\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\Big)}{\partial u}\cdot\frac{\partial u^*(s,y,a,b)}{\partial a}dyds\\
&+\int_0^t\int_a^b \frac{\partial K\Big(t,x,s,y,u^*\big(s,y,a,b\big),u^*\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\Big)}{\partial \overline{u}}\\
&\cdot\frac{\partial u^*((\varphi_1(s,y),\varphi_2(s,y),a,b)\Big)}{\partial a}dyds.
\end{aligned}
$$

This last relationship suggests us to consider the operator $C : X \times X \to X$ defined by:

$$
\begin{aligned}
&C(u,v)(t,x,a,b) := \frac{\partial g\big(t,x,u(t,x,a,b)\big)}{\partial u}\cdot v(t,x,a,b)\\
&+\int_0^t \frac{\partial H\big(t,x,s,u(s,x,a,b)\big)}{\partial u}\cdot v(s,x,a,b)ds\\
&-\int_0^t K\Big(t,x,s,a,u(s,a,a,b),u\big(\varphi_1(s,a),\varphi_2(s,a),a,b\big)\Big)ds\\
&+\int_0^t\int_a^b \frac{\partial K\Big(t,x,s,y,u(s,y,a,b),u\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\Big)}{\partial u}\cdot v(s,y,a,b)dyds\\
&+\int_0^t\int_a^b \frac{\partial K\Big(t,x,s,y,u(s,y,a,b),u\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\Big)}{\partial \overline{u}}\\
&\cdot v\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)dyds.
\end{aligned}
$$

From the hypotheses, the operator $C(u,\cdot)$ is a contraction, for any $u \in X$. Let v^* be the unique fixed point of $C(u^*,\cdot)$.

Now consider the operator $A : X \times X \to X \times X$ defined by

$$A(u,v)(t,x,a,b) := (B(u)(t,x,a,b), C(u,v)(t,x,a,b))\,,$$

which is in the hypotheses of Theorem 2.2. So A is a Picard operator and $F_A = \{(u^*,v^*)\}$.

Consider the sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ defined by:

$$u_n(t,x,a,b) := B(u_{n-1}(t,x,a,b))$$

$$= g(t,x,u_{n-1}(t,x,a,b)) + \int_0^t H\left(t,x,s,u_{n-1}(s,x)\right) ds$$

$$+ \int_0^t \int_a^b K\Big(t,x,s,y,u_{n-1}(s,y,a,b), u_{n-1}\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\Big) dyds$$

for all $n \geq 1$ and

$$v_n(t,x,a,b) := C(u_{n-1}(t,x,a,b), v_{n-1}(t,x,a,b))$$

$$= \frac{\partial g\big(t,x,u_{n-1}(t,x,a,b)\big)}{\partial u} \cdot v_{n-1}(t,x,a,b)$$

$$+ \int_0^t \frac{\partial H\big(t,x,s,u_{n-1}(s,x,a,b)\big)}{\partial u} \cdot v_{n-1}(s,x,a,b) ds$$

$$- \int_0^t K\Big(t,x,s,a,u_{n-1}(s,a,a,b), u_{n-1}\big(\varphi_1(s,a),\varphi_2(s,a),a,b\big)\Big) ds$$

$$+ \int_0^t \int_a^b \frac{\partial K\Big(t,x,s,y,u_{n-1}(s,y,a,b), u_{n-1}\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\Big)}{\partial u} v_{n-1}(s,y,a,b) dyds$$

$$+ \int_0^t \int_a^b \frac{\partial K\Big(t,x,s,y,u_{n-1}(s,y,a,b), u_{n-1}\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\Big)}{\partial \overline{u}}$$

$$\cdot\, v_{n-1}\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big) dyds,$$

for all $n \geq 1$.

Obviously, we have:

$$u_n \to u^* \text{ for } n \to \infty \quad \text{and} \quad v_n \to v^* \text{ for } n \to \infty$$

uniformly with respect to $(t,x,a,b) \in [0,T] \times [\alpha,\beta] \times [\alpha,\beta] \times [\alpha,\beta]$, for any $u_0, v_0 \in C([0,T] \times [\alpha,\beta] \times [\alpha,\beta] \times [\alpha,\beta])$.

Choosing $u_0 = v_0 := 0$ we have $v_1 = \dfrac{\partial u_1}{\partial a}$.

By induction we can prove that $v_n = \dfrac{\partial u_n}{\partial a}$ for any positive integer n, so

$$\frac{\partial u_n}{\partial\, a} \to v^* \text{ for } n \to \infty$$

From Weierstrass theorem, it follows that $\dfrac{\partial u^*}{\partial\, a}$ exists and

$$\frac{\partial u^*(t,x,a,b)}{\partial a} = v^*(t,x,a,b).$$

2. The differentiability with respect to b can be proved in the same way.

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A FRICTIONLESS ELASTIC-VISCOPLASTIC CONTACT PROBLEM WITH NORMAL COMPLIANCE, ADHESION AND DAMAGE

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Abstract. We study a quasistatic frictionless contact problem with normal compliance, adhesion and damage for elastic-viscoplastic material.The adhesion of the contact surfaces is modeled with a surface variable, the bonding field, whose evolution is described by a first order differential equation. The mechanical damage of the material, caused by excessive stess or strains, is described by a damage function whose evolution is modeled by an inclusion of parabolic type. We provide a variational formulation of the problem and prove the existence and uniqueness of a weak solution. The proofs are based on time-dependent variational equalities, classical results on elliptic and parabolic variational inequalities, differential equations and fixed point arguments.

1. **Introduction**

We consider a mathematical model for a quasistatic process of frictionless contact between an elastic-viscoplastic body and an obstacle, within the framework of small deformation theory. The contact is modeled with normal compliance. The effect of damage due to the mechanical stress or strain is included in the model. Such situation is common in many engeneering applications where the forces acting on the system very periodically leading to the appearence and growth of microcracks which may deteriorate the mechanism of the system. Because of the safety issue

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of mechanical equipments, considerable efforts were been devoted to modeling and numerically simulating damage.

Early models for mechanical damage derived from the termodyamical considerations appeared in [9, 10], where numerical simulations were included. Mathematical analysis of one-dimensional problems can be found in [11]. In all these papers the damage of the material is described with a damage function α, restricted to have values between zero and one. When $\alpha = 1$ there is no damage in the material, when $\alpha = 0$, the material is completely damaged, when $0 < \alpha < 1$ there is partial damage and the system has a reduced load carrying capacity. Quasistatic contact problems with damage have been investigated in [13, 14, 17]. In this paper, the inclusion used for the evolution of the damage field is

$$\dot{\alpha} - k \triangle \alpha + \partial\varphi_K(\alpha) \ni \Phi(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}), \alpha),$$

where K denotes the set of admissible damage functions defined by

$$K = \left\{\xi \in H^1(\Omega) \,/\, 0 \le \xi \le 1 \text{ a.e. in } \Omega\right\},$$

k is a positive coefficient, $\partial\varphi_K$ represents the subdifferential of the indicator function of the set K and Φ is a given constitutive function which describes the sources of the damage in the system. In the present paper we consider a rate type elastic-viscoplastic material with constitutive relation

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}), \alpha),$$

where $\mathcal{E}$ is a fourth order tensor, $\mathcal{G}$ is a nonlinear constitutive function and α is the damage field and the adhesion between the body and the obstacle is taken into account during the conact. The adhesive contact between bodies, when a glue is added to keep surfaces from relative motion, is receiving increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [2, 3, 4, 6, 12, 15, 20]. The novelty in all the above papers is the introduction of a surface internal variable, the *bonding field*, denoted in the paper by β; it describes the pointwise fractional density of active bonds on the contact surface, and sometimes referred to as the *intensity of adhesion.* Following [7, 8], the bonding field satisfies the restrictions $0 \leq \beta \leq 1$; when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. We refer the reader to the extensive bibliography on the subject in [16,18,19].

The paper is structured as follows. In section 2 we present the notation and some preliminaries. In section 3 we present the mechanical problem, we list the assumptions and in section 4 we give and prove our main existence and uniqueness result, Theorem 4.1. The proof is based on monotone operator theory, classical results on parabolic inequalities and Banach fixed point arguments.

2. **Notation and preliminaries**

In this short section, we present the notation we shall use and some preliminary material. For more details, we refer the reader to [5]. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$,($d = 2, 3$), while (.) and |.| represent the inner product and the Euclidean norm on S_d and $\mathbb{R}^d$, respectively. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary Γ and let ν denote the unit outer normal on Γ. we shall use the notation

$$H = L^2(\Omega)^d = \{\mathbf{u} = (u_i) \ / \ u_i \in L^2(\Omega)\},$$

$$\mathcal{H} = \{\boldsymbol{\sigma} = (\sigma_{ij}) \ / \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$

$$H_1 = \{\mathbf{u} = (u_i) \in H \ / \ \varepsilon(\mathbf{u}) \in \mathcal{H}\},$$

$$\mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \ / \ Div \ \boldsymbol{\sigma} \in H\},$$

where $\varepsilon : H_1 \to \mathcal{H}$ and Div : $\mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \qquad \boldsymbol{\varepsilon}_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \ Div \ \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

Here and below, the indices i and j run between 1 to d, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u},\mathbf{v})_H = \int_\Omega u_i v_i dx \qquad \forall \mathbf{u},\mathbf{v} \in H,$$

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\, \tau_{ij} dx \qquad \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathcal{H},$$

$$(\mathbf{u},\mathbf{v})_{H_1} = (\mathbf{u},\mathbf{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} \qquad \forall \mathbf{u},\mathbf{v} \in H_1,$$

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (Div\ \boldsymbol{\sigma}, Div\ \boldsymbol{\tau})_H \qquad \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathcal{H}_1.$$

The associated norms on the spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are denoted by $|\ \ |_H$, $|\ \ |_{\mathcal{H}}$, $|\ \ |_{H_1}$ and $|\ \ |_{\mathcal{H}_1}$, respectively. Let $H_\Gamma = H^{1/2}(\Gamma)^d$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $\mathbf{v} \in H_1$ we also use the notation $\mathbf{v}$ to denote the trace $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ and we denote by $\mathfrak{v}_\nu$ and $\mathbf{v}_\tau$ the *normal* and *tangential* components of $\mathbf{v}$ on the boundary Γ given by

$$v_\nu = \mathbf{v}.\boldsymbol{\nu}, \qquad \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}. \tag{2.1}$$

Similarly, for a regular (say C^1) tensor field $\boldsymbol{\sigma} : \Omega \to S_d$, we define its *normal* and *tangential* components by

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}).\boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}, \tag{2.2}$$

and we recall that the following Green's formula holds

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\mathrm{Div}\ \boldsymbol{\sigma}, \mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}.\mathbf{v}\ da \qquad \forall \mathbf{v} \in H_1. \tag{2.3}$$

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \le p \le +\infty$, and $k \ge 1$. We denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$|f|_{C(0,T;X)} = \max_{t\in[0,T]} |f(t)|_X,$$

$$|f|_{C^1(0,T;X)} = \max_{t\in[0,T]} |f(t)|_X + \max_{t\in[0,T]} \left|\dot{f}(t)\right|_X,$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable and, for a real number r, we use r_+ to present its positive part, that is $r_+ = \max\{0, r\}$. Finally, for the convenience of the reader, we recall the following version of the classical theorem of Cauchy-Lipschitz (see, e.g., [21, p. 60]).

Theorem 1. *Assume that* $(X, |.|_X)$ *is a real Banach space and* $T > 0$. Let $F(t,.) : X \to X$ *be an operator defined a.e. on* $(0,T)$ *satisfying the following conditions:* 1- $\exists$ $L_F > 0$ *such that* $|F(t,x) - F(t,y)|_X \leq L_F\ |x-y|_X \quad \forall x, y \in X,$ *a.e.* $t \in (0,T)$. 2- $\exists\ p \geq 1$ *such that* $t \longmapsto F(t,x) \in L^p(0,T;X) \quad \forall x \in X$. *Then for any* $x_0 \in X$, *there exists a unique function* $x \in W^{1,p}(0,T;X)$ *such that*

$$\dot{x}(t) = F(t, x(t)) \qquad a.e.\ t \in (0,T),$$

$$x(0) = x_0.$$

Theorem 2.1 will be used in section 4 to prove the unique solvability of the intermediate problem involving the bonding field.

Moreover, if X_1 and X_2 are real Hilbert spaces, then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1 \times X_2}$.

3. **Problem statement**

A viscoplastic body occupies the domain $\Omega \subset \mathbb{R}^d$ with the boundary Γ divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that *meas* $(\Gamma_1) > 0$. The time interval of interest is $[0,T]$ where $T > 0$. The body is clamped on Γ_1 and so the displacement field vanishes there. A volume force of density $\mathbf{f}_0$ acts in $\Omega \times (0,T)$ and surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (0,T)$. We assume that the body is in adhesive frictionless contact with an obstacle, the so called foundation, over the potential contact surface Γ_3. Moreover, the process is quasistatic, i.e. the inertial terms are neglected in the equation of motion. We use an elasto-viscoplastic constitutive law with damage to model the material's behavior and an ordinary differential equation to describe the evolution of the bonding field. The mechanical formulation of the frictionless problem with normal compliance is as follows.

Problem P. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to S_d$, *a damage field* $\alpha : \Omega \times [0,T] \to \mathbb{R}$ *and a bonding field* $\beta : \Gamma_3 \times [0,T] \to [0,1]$ *such that*

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}\left(\dot{\mathbf{u}}\right) + \mathcal{G}\left(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}\left(\mathbf{u}\right), \alpha\right), \tag{3.1}$$

$$\dot{\alpha} - k \triangle \alpha + \partial\varphi_K\left(\alpha\right) \ni \Phi\left(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}\left(\mathbf{u}\right), \alpha\right), \tag{3.2}$$

$$Div\ \boldsymbol{\sigma} + \mathbf{f}_0 = 0 \text{ in } \Omega \times (0,T), \tag{3.3}$$

$$\mathbf{u} = 0 \text{ on } \Gamma_1 \times (0,T), \tag{3.4}$$

$$\boldsymbol{\sigma\nu} = \mathbf{f}_2 \text{ on } \Gamma_2 \times (0,T), \tag{3.5}$$

$$-\sigma_\nu = p_\nu\left(u_\nu\right) - \gamma_\nu\ \beta^2\left(-R\left(u_\nu\right)\right)_+ \text{ on } \Gamma_3 \times (0,T), \tag{3.6}$$

$$\boldsymbol{\sigma}_\tau = 0 \text{ on } \Gamma_3 \times (0,T), \tag{3.7}$$

$$\frac{\partial\alpha}{\partial\nu} = 0 \text{ on } \Gamma \times (0,T), \tag{3.8}$$

$$\dot{\beta} = -\left[\gamma_\nu\beta\left[\left(-R\left(u_\nu\right)\right)_+\right]^2 - \epsilon_a\right]_+ \text{ on } \Gamma_3 \times (0,T), \tag{3.9}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0,\ \alpha(0) = \alpha_0 \text{ in } \Omega, \tag{3.10}$$

$$\beta(0) = \beta_0 \text{ on } \Gamma_3. \tag{3.11}$$

The relation (3.1) represents the viscoplastic constitutive law with damage, the evolution of the damage field is governed by the inclusion given by the relation (3.2), k is a constant, $\partial\varphi_K$ denotes the subdifferential of the indicator function φ_K of K which represents the set of admissible damage functions satisfying $0 \le \alpha \le 1$ and Φ is a given constitutive function which describes damage sources in the system. (3.3) represents the equilibrium equation, (3.4) and (3.5) are the displacement and traction boundary conditions, respectively. (3.6) represents the normal compliance contact condition with adhesion in which γ_ν and ϵ_a are given adhesion coefficients and R is the truncation operator defined by

$$R\left(s\right) = \begin{cases} -L & \text{if } s \le -L, \\ s & |s| < L, \\ L & \text{if } s \ge L. \end{cases} \tag{3.12}$$

Here $L > 0$ is the caracteristic length of the bond, beyonding which it does not offer any additional traction. The introduction of R is motivated by the mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows. Also, p_ν is a given positive function which will be decribed below. In this condition the interpenetrability between the body and the foundation is allowed, that is u_ν may be positive on Γ_3. The contribution of the adhesive to normal traction is represented by the term $\gamma_\nu \beta \left(-R\left(u_\nu\right)\right)_+$, the adhesive traction is tensile, and is proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion, and to the normal displacement, but as in various papers see e.g. [2, 3] and the references threin. Condition (3.7) represents the frictionless contact condition and shows that the tangential stress vanishes on the contact surface during the process. (3.8) represents a homogeneous Newmann boundary condition where $\frac{\partial \alpha}{\partial \nu}$ represents the normal derivative of α. Next, equation (3.9) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [2], see also [19] for more details. Here, γ_ν and ϵ_a are given adhesion coefficients which may depend on $\mathbf{x} \in \Gamma_3$ and R is the truncation operator given by (3.12). Notice that in this model once debonding occurs bonding connot be reestablished since, as it follows from (3.9), $\dot{\beta} \leq 0$. In (3.10), we consider the initial conditions where $\mathbf{u}_0$ is the initial displacement, $\boldsymbol{\sigma}_0$ is the initial stress and α_0 is the initial damage. Finally, (3.11) is the initial condition, in which β_0 denotes the initial bonding field. Let Z denote the bonding fields set

$$Z = \left\{\beta \in L^2\left(\Gamma_3\right) \ / \ 0 \leq \beta \leq 1 \text{ a.e. on } \Gamma_3\right\},$$

and for displacement field we need the closed subspace of H_1 defined by

$$V = \{\mathbf{v} \in H_1 | \mathbf{v} = 0 \text{ on } \Gamma_1\}.$$

Since meas $(\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $C_K > 0$, that depends only on Ω and Γ_1 such that

$$|\boldsymbol{\varepsilon}(\mathbf{v})|_{\mathcal{H}} \geq C_K |\mathbf{v}|_{H_1} \quad \forall \mathbf{v} \in V.$$

On V we consider the inner product and the associated norm given by

$$(\mathbf{u},\mathbf{v})=(\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, \qquad |\mathbf{v}|_V=|\boldsymbol{\varepsilon}(\mathbf{v})|_{\mathcal{H}} \qquad \forall \mathbf{u},\mathbf{v}\in V.$$

It follows from Korn's inequality that $|.|_{H_1}$ and $|.|_V$ are equivalent norms on V and therefore $(V,|.|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant C_0, depending only on Ω, Γ_1 and Γ_3 such that

$$|\mathbf{v}|_{L^2(\Gamma_3)^d}\le C_0\,|\mathbf{v}|_V \qquad \forall \mathbf{v}\in V. \tag{3.13}$$

In the study of the mechanical problem (3.1)-(3.11), we make the following assumptions. The operator $\mathcal{E}:\Omega\times S_d\to S_d$ satisfies

$$\begin{cases} \text{(a) } \mathcal{E}=(e_{ijkh})\ /\ e_{ijkh}\in L^\infty(\Omega), \\ \text{(b) } \mathcal{E}\,\boldsymbol{\sigma}\,.\,\boldsymbol{\tau}=\boldsymbol{\sigma}\,.\,\mathcal{A}\,.\,\boldsymbol{\tau} \quad \forall \boldsymbol{\sigma},\,\boldsymbol{\tau}\in S_d, \text{ a.e. in } \Omega, \\ \text{(c) } \mathcal{E}\,\boldsymbol{\sigma}\,.\,\boldsymbol{\sigma}\ge m_{\mathcal{E}}\ |\boldsymbol{\sigma}|^2\ \ \forall\boldsymbol{\sigma}\in S_d, \text{ for some } m_{\mathcal{E}}>0. \end{cases} \tag{3.14}$$

The operator $\mathcal{G}:\Omega\times S_d\times S_d\times\mathbb{R}\to S_d$ satisfies

$$\begin{cases} \text{(a) There exists a constant } L_{\mathcal{G}}>0 \text{ such that} \\ |\mathcal{G}(\mathbf{x},\boldsymbol{\sigma}_1,\ \boldsymbol{\varepsilon}_1,\alpha_1)-\mathcal{G}(\mathbf{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2,\alpha_2)|\le L_{\mathcal{G}}\ (|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2|+|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|+|\alpha_1-\alpha_2|) \\ \forall\boldsymbol{\sigma}_1,\ \boldsymbol{\sigma}_2,\ \boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2\in S_d,\ \alpha_1,\ \alpha_2\in\mathbb{R}, \text{ a.e. } \mathbf{x}\in\Omega; \\ \text{(b) } \mathbf{x}\longmapsto\mathcal{G}(\mathbf{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\alpha) \text{ is a Lebesgue measurable function on } \Omega \\ \forall\boldsymbol{\sigma},\boldsymbol{\varepsilon}\in S_d,\ \forall\alpha\in\mathbb{R}; \\ \text{(c) } \mathbf{x}\longmapsto\mathcal{G}(\mathbf{x},\mathbf{0},\mathbf{0},0)\in\mathcal{H}. \end{cases} \tag{3.15}$$

The damage function $\Phi:\Omega\times S_d\times S_d\times\mathbb{R}\to\mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There exists a constant } L > 0 \text{ such that} \\ |\Phi(\mathbf{x}, \boldsymbol{\sigma}_1,\ \boldsymbol{\varepsilon}_1, \alpha_1) - \Phi\left(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \alpha_2\right)| \leq L\left(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\alpha_1 - \alpha_2|\right) \\ \forall \boldsymbol{\sigma}_1,\ \boldsymbol{\sigma}_2,\ \boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2 \in S_d,\ \alpha_1,\ \alpha_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Omega; \\ \text{(b) } \mathbf{x} \longmapsto \Phi\left(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \alpha\right) \text{ is a Lebesgue measurable function on } \Omega \\ \forall \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in S_d,\ \forall \alpha \in \mathbb{R}; \\ \text{(c) } \mathbf{x} \longmapsto \Phi\left(\mathbf{x}, \mathbf{0}, \mathbf{0}, 0\right) \in \mathcal{H}. \end{cases} \tag{3.16}$$

The normal compliance function $p_\nu : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}_+$ satisfies

$$\begin{cases} \text{(a) There exists } L_\nu > 0 \text{ such that} \\ |p_\nu\left(\mathbf{x}, \mathbf{r}_1\right) - p_\nu\left(\mathbf{x}, \mathbf{r}_2\right)| \leq L_\nu\left|\mathbf{r}_1 - \mathbf{r}_2\right| \quad \forall \mathbf{r}_1, \mathbf{r}_2 \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(b) } \left(p_\nu\left(\mathbf{x}, \mathbf{r}_1\right) - p_\nu\left(\mathbf{x}, \mathbf{r}_2\right)\right).\left(\mathbf{r}_1 - \mathbf{r}_2\right) \geq 0 \quad \forall \mathbf{r}_1, \mathbf{r}_2 \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(c) } \mathbf{r} \mapsto p_\nu\left(., \mathbf{r}\right) \text{ is Lebesgue measurable on } \Gamma_3,\ \forall \mathbf{r} \in \mathbb{R}^d. \\ \text{(d) The mapping } p_\nu\left(., \mathbf{r}\right) = 0 \quad \text{for all } \mathbf{r} \leq 0. \end{cases} \tag{3.17}$$

The adhesion coefficients satisfy

$$\gamma_\nu \in L^\infty\left(\Gamma_3\right),\ \gamma_\nu \geq 0, \epsilon_a \in L^\infty\left(\Gamma_3\right),\ \epsilon_a \geq 0. \tag{3.18}$$

We also suppose that the body forces and surface traction have the regularity

$$\mathbf{f}_0 \in C\left(0, T; H\right), \qquad \mathbf{f}_2 \in C(0, T; L^2\left(\Gamma_2\right)^d). \tag{3.19}$$

Finally we assume that the initial data satisfy the following conditions

$$\mathbf{u}_0 \in V,\ \boldsymbol{\sigma}_0 \in \mathcal{H}_1, \tag{3.20}$$

$$\alpha_0 \in K, \tag{3.21}$$

$$\beta_0 \in Z. \tag{3.22}$$

We define the bilinear form $a : H^1\left(\Omega\right) \times H^1\left(\Omega\right) \to \mathbb{R}$ by

$$a(\xi, \varphi) = k \int_\Omega \nabla\xi\ .\ \nabla\varphi\ dx. \tag{3.23}$$

Next, we denote $\mathbf{f} : [0, T] \to V$ the function defined by

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t).\mathbf{v}\, dx + \int_{\Gamma_2} \mathbf{f}_2(t).\mathbf{v}\, da \qquad \forall \mathbf{v} \in V, \text{ a.e. } t \in (0, T). \tag{3.24}$$

The adhesion functional $j_{ad} : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ defined by

$$j_{ad}(\beta, \mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} -\gamma_\nu\ \beta^2 \left(-R(u_\nu)\right)_+ v_\nu\ da. \tag{3.25}$$

In addition to the functional (3.25), we need the normal compliance functional j_{nc} : $V \times V \to \mathbb{R}$ given by

$$j_{nc}(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\nu(u_\nu)\, v_\nu\ da. \tag{3.26}$$

Keeping in mind (3.17)-(3.18), we observe that the integrals in (3.25) and (3.26) are well defined and we note that conditions (3.19) imply

$$\mathbf{f} \in C(0, T; V). \tag{3.27}$$

Finally we assume the following condition of compatibility

$$(\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + j_{ad}(\beta_0, \mathbf{u}_0, \mathbf{v}) + j_{nc}(\mathbf{u}_0, \mathbf{v}) = (\mathbf{f}(0), \mathbf{v})_V \qquad \forall \mathbf{v} \in V. \tag{3.28}$$

Using standard arguments based on green's formula (2.3) we can derive the following variational formulation of the frictionless problem with normal compliance (3.1)-(3.11) as follows.

Problem PV. Find a displacement field $\mathbf{u} : [0, T] \to V$, *a stress field* $\boldsymbol{\sigma}$: $[0, T] \to \mathcal{H}$ *a damage field* $\alpha : [0, T] \to H^1(\Omega)$ *and a bonding field* $\beta : [0, T] \to L^2(\Gamma_3)$ *such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}\left(\dot{\mathbf{u}}(t)\right) + \mathcal{G}\left(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \alpha(t)\right), \qquad \text{a.e. } t \in (0, T), \tag{3.29}$$

$$\begin{aligned} &\alpha(t) \in K \text{ for all } t \in [0, T], \ (\dot{\alpha}(t), \xi - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \xi - \alpha(t)) \\ &\qquad \geq (\Phi(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \alpha(t)), \xi - \alpha(t))_{L^2(\Omega)}\ \forall \xi \in K, \end{aligned} \tag{3.30}$$

$$\begin{aligned} &(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + j_{ad}(\beta(t), \mathbf{u}(t), \mathbf{v}) + j_{nc}(\mathbf{u}(t), \mathbf{v}) \\ &\qquad = (\mathbf{f}(t), \mathbf{v})_V \qquad \forall \mathbf{v} \in V,\ \forall t \in [0, T], \end{aligned} \tag{3.31}$$

$$\dot{\beta}(t) = -\left[\gamma_\nu\ \beta(t)\ \ \left[(-R(u_\nu(t)))_+\right]^2 - \epsilon_a\right]_+ \qquad \text{a.e. } t \in (0, T), \tag{3.32}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \alpha(0) = \alpha_0, \beta(0) = \beta_0. \tag{3.33}$$

We notice that the variational problem PV is formulated in terms of displacement, stress field, damage field and bonding field. The existence of the unique solution of problem PV is stated and proved in the next section. To this end, we consider the following remark whose estimates will be used in different places of the paper.

Remark 1. *From* (3.32) *we obtain that* $\beta(\mathbf{x}, t) \leq \beta_0(\mathbf{x})$, *since* $\beta_0(\mathbf{x}) \in Z$ *then* $\beta(\mathbf{x}, t) \leq 1$ *for all* $t \geq 0$, *a.e. on* Γ_3. *If* $\beta(\mathbf{x}, t_0) = 0$ *for all* $t = t_0$ *it follows from* (3.32) *that* $\dot{\beta}(\mathbf{x}, t) = 0$ *for all* $t \geq t_0$, *therefore,* $\beta(\mathbf{x}, t) = 0$ *for all* $t \geq t_0$. *We conclude that* $0 \leq \beta(\mathbf{x}, t) \leq 1$ $\forall t \in [0, T]$, *a.e.* $\mathbf{x} \in \Gamma_3$.

In the sequel we consider that C is a generic positive constant which depends on $\Omega, \Gamma_1, \Gamma_3, \gamma_\nu, L$ and may change from place to place. First, we remark that j_{ad} and j_{nc} are linear with respect to the last argument and therefore

$$j_{ad}(\beta, \mathbf{u}, -\mathbf{v}) = -j_{ad}(\beta, \mathbf{u}, \mathbf{v}), \qquad j_{nc}(\mathbf{u}, -\mathbf{v}) = -j_{nc}(\mathbf{u}, \mathbf{v}). \tag{3.34}$$

Next, using (3.25) as well as the properties of the operator R , (3.12), we find

$$j_{ad}(\beta_1, \mathbf{u}_1, \mathbf{v}) - j_{ad}(\beta_2, \mathbf{u}_2, \mathbf{v}) = \int_{\Gamma_3} \gamma_\nu \ \beta_1^2[(-R(u_{2\nu}))_+ - (-R(u_{1\nu}))_+] \ v_\nu \ da$$

$$+ \int_{\Gamma_3} \gamma_\nu \ (\beta_2^2 - \beta_1^2)(-R(u_{2\nu}))_+ \ v_\nu \ da \leq C \int_{\Gamma_3} |\beta_1 - \beta_2| \, |\mathbf{v}| \ da,$$

and from (3.13) we obtain

$$j_{ad}(\beta_1, \mathbf{u}_1, \mathbf{v}) - j_{ad}(\beta_2, \mathbf{u}_2, \mathbf{v}) \leq c \ |\beta_1 - \beta_2|_{L^2(\Gamma_3)} \ |\mathbf{v}|_V \ . \tag{3.35}$$

Now, we use (3.26) to see that

$$|j_{nc}(\mathbf{u}_1, \mathbf{v}) - j_{nc}(\mathbf{u}_2, \mathbf{v})| \leq \int_{\Gamma_3} |p_\nu(u_{1\nu}) - p_\nu(u_{2\nu})| \ |v_\nu| \ da,$$

and therefore (3.17) (a) and (3.13) imply

$$|j_{nc}(\mathbf{u}_1, \mathbf{v}) - j_{nc}(\mathbf{u}_2, \mathbf{v})| \leq C \, | \, \mathbf{u}_1 - \mathbf{u}_2|_V \ |\mathbf{v}|_V \,. \tag{3.36}$$

We use again (3.26) to see that

$$j_{nc}(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{nc}(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) = \int_{\Gamma_3} (p_\nu(\mathbf{u}_{1\nu}) - p_\nu(\mathbf{u}_{2\nu}))\ (u_{2\nu} - u_{1\nu})\ da,$$

and therefore (3.17) (b) implies

$$j_{nc}(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{nc}(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \leq 0. \tag{3.37}$$

The inequalities (3.35)-(3.37) combined with equalities (3.34) will be used in various places in the rest of the paper.

4. **Well posedness of the problem**

The main result in this section is the following existence and uniqueness result.

Theorem 2. *Assume that* (3.14)-(3.22) *and* (3.28) *hold. Then, problem PV has a unique solution* $\{\mathbf{u}, \boldsymbol{\sigma}, \beta, \alpha\}$ which satisfies

$$\mathbf{u} \in C(0, T; V),$$

$$\boldsymbol{\sigma} \in C(0, T; \mathcal{H}_1),$$

$$\beta \in W^{1,\infty}\left(0, T; L^2(\Gamma_3)\right),$$

$$\alpha \in W^{1,2}\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H^1(\Omega)\right). \tag{4.1}$$

A quadruplet $(\mathbf{u}, \boldsymbol{\sigma}, \beta, \alpha)$ which satisfies (3.29)-(3.33) is called a weak solution to the compliance contact problem P. We conclude that, under the stated assumptions, problem (3.1)-(3.11) has a unique weak solution satisfying (4.1). We turn now to the proof of Theorem 4.1 which is carried out in several steps. To this end, we assume in the following that (3.14)-(3.22) and (3.28) hold. Below, C denotes a generic positive constant which may depend on $\Omega, \Gamma_1, \Gamma_3, \mathcal{E}, \gamma_\nu, L$ and T but does not depend on t nor of the rest of input data, and whose value may change from place to place. Moreover, for the sake of simplicity, we supress, in what follows, the explicit dependence of various functions on $\mathbf{x} \in \Omega \cup \Gamma$. The proof of Theorem 4.1 will be carried out in several steps. In the first step we solve the differential equation in (3.32) for the adhesion field, where **u** is given, and study the continuous dependence of the adhesion solution with respect to **u**.

Lemma 3. *For every* $\mathbf{u} \in C(0,T;V)$, there exists a unique solution

$$\beta_u \in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right)$$

satisfying

$$\dot{\beta}_u(t) = -\left[\gamma_\nu\ \beta_u(t)\ \ \left[(-R(u_\nu(t)))_+\right]^2 - \epsilon_a\right]_+ \qquad \text{a.e. } t \in (0,T),$$

$$\beta_u(0) = \beta_0.$$

Moreover, $\beta_u(t) \in Z$ *for* $t \in [0,T]$, *a.e. on* Γ_3, *and there exists a constant* $C > 0$, *such that, for all* $\mathbf{u}_1$, $\mathbf{u}_2 \in C(0,T;V)$,

$$\left|\beta_{u_1}(t) - \beta_{u_2}(t)\right|^2_{L^2(\Gamma_3)} \leq C\int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|^2_V\ ds \qquad \forall t \in [0,T].$$

Proof. Consider the mapping $F : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F(t,\beta) = -\left[\gamma_\nu \beta(t)\left[(-R(\mathbf{u}_\nu))_+\right]^2 - \epsilon_a\right]_+,$$

$\forall t \in [0,T]$ and $\beta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R that F is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, for any $\beta \in L^2(\Gamma_3)$, the mapping $t \longmapsto F(t,\beta)$ belongs to $L^\infty\left(0,T,L^2(\Gamma_3)\right)$. Thus, the existence and the uniqueness of the solution β_u follows from the classical theorem of Cauchy-Lipschitz given in Theorem 2.1. Notice also that the argument used in Remark 3.1 shows that $0 \leq \beta_u(t) \leq 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set Z, we find that $\beta_u(t) \in Z$ for all $t \in [0,T]$, which concludes the proof of the Lemma. Now let $\mathbf{u}_1$, $\mathbf{u}_2 \in C(0,T;V)$ and let $t \in [0,T]$. We have, for $i = 1,2$,

$$\beta_{u_i}(t) = \beta_0 - \int_0^t \left[\gamma_\nu\ \beta_{u_i}(t)\ \ \left[(-R(u_{i\nu}(t)))_+\right]^2 - \epsilon_a\right]_+\ ds,$$

and then

$$\left|\beta_{u_1}(t) - \beta_{u_2}(t)\right|_{L^2(\Gamma_3)}$$

$$\leq C\int_0^t \left|\beta_{u_1}(s)\ \ \left[(-R(u_{1\nu}(s)))_+\right]^2 - \beta_{u_2}(s)\ \ \left[(-R(u_{2\nu}(s)))_+\right]^2\right|_{L^2(\Gamma_3)}\ ds.$$

Using the definition of the truncation operator R given by (3.12) and considering $\beta_{u_1} = \beta_{u_1} - \beta_{u_2} + \beta_{u_2}$ we find

$$\left|\beta_{u_1}(t) - \beta_{u_2}(t)\right|_{L^2(\Gamma_3)}$$

$$\leq C\left(\int_0^t \left|\beta_{u_1}(s) - \beta_{u_2}(s)\right|_{L^2(\Gamma_3)}\ ds + \int_0^t \left|\mathbf{u}_1(s) - \mathbf{u}_2(s)\right|_{L^2(\Gamma_3)^d}\ ds\right).$$

Applying Gronwall's inequality, it follows that

$$\left|\beta_{u_1}(t) - \beta_{u_2}(t)\right|^2_{L^2(\Gamma_3)} \leq C\int_0^t \left|\mathbf{u}_1(s) - \mathbf{u}_2(s)\right|^2_{L^2(\Gamma_3)^d}\ ds,$$

and using (3.13) we obtain the second part of Lemma 4.2. □

Now we consider the following viscoplastic problem and we prove an existence and uniqueness result for (3.29), (3.31) and (3.33) with the corresponding initial condition.

Problem QV. Find a displacement field $\mathbf{u} : [0,T] \to V$, a damage field $\alpha : [0,T] \to H^1(\Omega)$ and a stress field $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}$ satisfying (3.29) and

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + j_{ad}(\beta_{\mathbf{u}}(t), \mathbf{u}(t), \mathbf{v}) + j_{nc}(\mathbf{u}(t), \mathbf{v})$$

$$= (\mathbf{f}(t), \mathbf{v})_V \qquad \forall \mathbf{v} \in V,\ \forall t \in [0,T], \tag{4.2}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \alpha(0) = \alpha_0. \tag{4.3}$$

Let $(\boldsymbol{\eta},\omega) \in C(0,T;\mathcal{H} \times L^2(\Omega))$ and let $\mathfrak{Z}_\eta(t) = \int_0^t \boldsymbol{\eta}(s)\ ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_0)$, then

$$\mathfrak{Z}_\eta \in C^1(0,T;\mathcal{H}),$$

and consider the following variational problem.

Problem QV_η. Find a displacement field $\mathbf{u}_\eta : [0,T] \to V$ and a stress field $\boldsymbol{\sigma}_\eta : [0,T] \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)) + \mathfrak{Z}_\eta(t),\ \forall t \in [0,T], \tag{4.4}$$

$$(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + j_{ad}(\beta_{u_\eta}(t), \mathbf{u}_\eta(t), \mathbf{v}) + j_{nc}(\mathbf{u}_\eta(t), \mathbf{v})$$

$$= (\mathbf{f}(t), \mathbf{v})_V \qquad \forall \mathbf{v} \in V,\ \forall t \in [0,T], \tag{4.5}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0, \boldsymbol{\sigma}_\eta(0) = \boldsymbol{\sigma}_0. \tag{4.6}$$

To solve problem QV_η we consider $\boldsymbol{\theta} \in C(0,T;V)$ and we construct the following intermediate problem.

Problem $QV_{\eta\theta}$. Find a displacement field $\mathbf{u}_{\eta\theta} : [0,T] \to V$ and a stress field $\boldsymbol{\sigma}_{\eta\theta} : [0,T] \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}_{\eta\theta}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_{\eta\theta}(t)) + \mathfrak{z}_\eta(t), \tag{4.7}$$

$$(\boldsymbol{\sigma}_{\eta\theta}(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\boldsymbol{\theta}(t), \mathbf{v})_V = (\mathbf{f}(t), \mathbf{v})_V \qquad \forall \mathbf{v} \in V,\ \forall t \in [0,T], \tag{4.8}$$

$$\mathbf{u}_{\eta\theta}(0) = \mathbf{u}_0, \boldsymbol{\sigma}_{\eta\theta}(0) = \boldsymbol{\sigma}_0. \tag{4.9}$$

Lemma 4. There exists a unique solution $(\mathbf{u}_{\eta\theta}, \boldsymbol{\sigma}_{\eta\theta})$ of the problem $QV_{\eta\theta}$ which satisfies $\mathbf{u}_{\eta\theta} \in C(0,T;V)$, $\boldsymbol{\sigma}_{\eta\theta} \in C(0,T;\mathcal{H}_1)$.

Proof. We define the operator $A : V \to V$ by

$$(A\,\mathbf{u}, \mathbf{v})_V = (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}},\ \forall \mathbf{u}, \mathbf{v} \in V. \tag{4.10}$$

Using (3.14), it follows that A is a strongly monotone Lipschitz operator, thus A is invertible and $A^{-1} : V \to V$ is also a strongly monotone Lipschitz operator. It follows that there exists a unique function $\mathbf{u}_{\eta\theta}$ which satisfies

$$\mathbf{u}_{\eta\theta} \in C(0,T;V), \tag{4.11}$$

$$A\,\mathbf{u}_{\eta\theta}(t) = \mathbf{h}_{\eta\theta}(t), \tag{4.12}$$

where $\mathbf{h}_{\eta\theta} \in C(0,T;V)$ is such that

$$(\mathbf{h}_{\eta\theta}(t), \mathbf{v})_V = (\mathbf{f}(t), \mathbf{v})_V - (\mathfrak{z}_\eta(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} - (\boldsymbol{\theta}(t), \mathbf{v})_V\ \ \forall \mathbf{v} \in V,\ \forall t \in [0,T]. \tag{4.13}$$

It follows from, (4.12) that $\mathbf{u}_{\eta\theta} \in C(0,T;V)$. Consider $\boldsymbol{\sigma}_{\eta\theta}$ defined in (4.7), since, $\mathfrak{z}_\eta \in C^1(0,T;\mathcal{H})$, $\mathbf{u}_{\eta\theta} \in C(0,T;V)$ we deduce that $\boldsymbol{\sigma}_{\eta\theta} \in C(0,T;\mathcal{H})$. Since $\mathrm{Div}\boldsymbol{\sigma}_{\eta\theta} = -\mathbf{f}_0 \in C(0,T;H)$, we further have $\boldsymbol{\sigma}_{\eta\theta} \in C(0,T;\mathcal{H}_1)$. This concludes the existence part of Lemma 4.3. The uniqueness of the solution follows from the unique solvability of the time-dependent equation (4.12). Finally $(\mathbf{u}_{\eta\theta}, \boldsymbol{\sigma}_{\eta\theta})$ is the unique solution of problem $QV_{\eta\theta}$ obtained in Lemma 4.3, which concludes the proof. □

Let $\Lambda\boldsymbol{\theta}\,(t)$ denote the element of V defined by

$$(\Lambda\boldsymbol{\theta}\,(t)\,,\mathbf{v})_V = j_{ad}(\beta_{u_{\eta\theta}}\,(t)\,,\mathbf{u}_{\eta\theta}\,(t)\,,\mathbf{v}) + j_{nc}(\mathbf{u}_{\eta\theta}\,(t)\,,\mathbf{v}) \quad \forall\mathbf{v}\in V,\ \forall t\in[0,T]\,. \tag{4.14}$$

We have the following result.

Lemma 5. For each $\boldsymbol{\theta} \in C(0,T;V)$ the function $\Lambda\boldsymbol{\theta} : [0,T] \to V$ belongs to $C(0,T;V)$. Moreover, there exists a unique element $\boldsymbol{\theta}^* \in C(0,T;V)$ such that $\Lambda\boldsymbol{\theta}^* = \boldsymbol{\theta}^*$.

Proof. Let $\boldsymbol{\theta} \in C(0,T;V)$ and let $t_1,\ t_2 \in [0,T]$. Using (3.35), (3.36) and (4.14) we obtain

$$|\Lambda\boldsymbol{\theta}\,(t_1) - \Lambda\boldsymbol{\theta}\,(t_2)|_V \le C\left(\left|\beta_{u_{\eta\theta}}\,(t_1) - \beta_{u_{\eta\theta}}\,(t_2)\right|_{L^2(\Gamma_3)} + |\mathbf{u}_{\eta\theta}\,(t_1) - \mathbf{u}_{\eta\theta}\,(t_2)|_V\right). \tag{4.15}$$

By Lemma 4.3, $\mathbf{u}_{\eta\theta} \in C(0,T;V)$ and, by Lemma 4.2, $\beta_{u_{\eta\theta}} \in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right)$, then we deduce from inequality (4.15) that $\Lambda\boldsymbol{\theta} \in C(0,T;V)$. Let now $\boldsymbol{\theta}_1,\boldsymbol{\theta}_2 \in C(0,T;V)$ and denote $\mathbf{u}_{\eta\theta_i} = \mathbf{u}_i$ and $\beta_{u_{\eta\theta_i}} = \beta_{u_i}$ for $i = 1,2$. Using again the relations (3.35), (3.36) and (4.14) we find

$$|\Lambda\boldsymbol{\theta}_1\,(t) - \Lambda\boldsymbol{\theta}_2\,(t)|_V^2 \le C\left(\left|\beta_{u_1}\,(t) - \beta_{u_2}\,(t)\right|_{L^2(\Gamma_3)}^2 + |\mathbf{u}_1\,(t) - \mathbf{u}_2\,(t)|_V^2\right). \tag{4.16}$$

Then by Lemma 4.2, we have

$$\left|\beta_{u_1}\,(t) - \beta_{u_2}\,(t)\right|_{L^2(\Gamma_3)}^2 \le C\int_0^t |\mathbf{u}_1\,(s) - \mathbf{u}_2\,(s)|_{L^2(\Gamma_3)}^2\ ds,$$

and by (3.13) we get

$$\left|\beta_{u_1}\,(t) - \beta_{u_2}\,(t)\right|_{L^2(\Gamma_3)}^2 \le C\int_0^t |\mathbf{u}_1\,(s) - \mathbf{u}_2\,(s)|_V^2\ ds.$$

Use the previous inequality in (4.16) to obtain

$$|\Lambda\boldsymbol{\theta}_1\,(t) - \Lambda\boldsymbol{\theta}_2\,(t)|_V^2 \le C\left(|\mathbf{u}_1\,(t) - \mathbf{u}_2\,(t)|_V^2 + \int_0^t |\mathbf{u}_1\,(s) - \mathbf{u}_2\,(s)|_V^2\ ds\right). \tag{4.17}$$

Moreover, from (4.8) it follows that

$$(\mathcal{E}\boldsymbol{\varepsilon}\,(\mathbf{u}_1) - \mathcal{E}\boldsymbol{\varepsilon}\,(\mathbf{u}_2)\,,\boldsymbol{\varepsilon}\,(\mathbf{u}_1) - \boldsymbol{\varepsilon}\,(\mathbf{u}_2))_{\mathcal{H}} + (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2,\mathbf{u}_1 - \mathbf{u}_2)_V = 0 \text{ on } (0,T)\,. \tag{4.18}$$

Hence

$$|\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V \leq C \ |\boldsymbol{\theta}_1(t) - \boldsymbol{\theta}_2(t)|_V \ \forall t \in [0,T]. \tag{4.19}$$

Now from the inequalities (4.17) and (4.19) we have

$$|\Lambda\boldsymbol{\theta}_1(t) - \Lambda\boldsymbol{\theta}_2(t)|_V^2 \leq C \left(|\boldsymbol{\theta}_1(t) - \boldsymbol{\theta}_2(t)|_V^2 + \int_0^t |\boldsymbol{\theta}_1(s) - \boldsymbol{\theta}_2(s)|_V^2 \ ds \right) \ \forall t \in [0,T].$$

Applying Gronwall's inequality we obtain

$$|\Lambda\boldsymbol{\theta}_1(t) - \Lambda\boldsymbol{\theta}_2(t)|_V^2 \leq C \int_0^t |\boldsymbol{\theta}_1(s) - \boldsymbol{\theta}_2(s)|_V^2 \ ds \ \forall t \in [0,T].$$

Reiterating this inequality n times yields

$$|\Lambda^n\boldsymbol{\theta}_1 - \Lambda^n\boldsymbol{\theta}_2|_{C(0,T;V)}^2 \leq \frac{(CT)^n}{n!} \ |\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2|_{C(0,T;V)}^2,$$

which implies that for n sufficiently large a power Λ^n of Λ is a contraction in the Hilbert space $C(0,T;V)$. Then, there exists a unique $\boldsymbol{\theta}^* \in C(0,T;V)$ such that $\Lambda^n\boldsymbol{\theta}^* = \boldsymbol{\theta}^*$ and $\boldsymbol{\theta}^*$ is also the unique fixed point of Λ. □

Lemma 6. *There exists a unique solution of problem QV_η satisfying $\mathbf{u}_\eta \in C(0,T;V)$, $\boldsymbol{\sigma}_\eta \in C(0,T;\mathcal{H}_1)$.*

Proof. Let $\boldsymbol{\theta}^* \in C(0,T;V)$ be the fixed point of Λ, Lemma 4.3 implies that $(\mathbf{u}_{\eta\theta^*}, \boldsymbol{\sigma}_{\eta\theta^*}) \in C(0,T;V) \times C(0,T;\mathcal{H}_1)$ is the unique solution of $QV_{\eta\theta}$ for $\boldsymbol{\theta} = \boldsymbol{\theta}^*$. since $\Lambda\boldsymbol{\theta}^* = \boldsymbol{\theta}^*$ and from the relations (4.14), (4.7), (4.8) and (4.9), we obtain that $(\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta) = (\mathbf{u}_{\eta\theta^*}, \boldsymbol{\sigma}_{\eta\theta^*})$ is the unique solution of QV_η. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ given in (4.14). □

Now for $(\boldsymbol{\eta},\omega) \in C(0,T;\mathcal{H} \times L^2(\Omega))$, we suppose that the assumptions of Theorem 4.1 hold and we consider the following intermediate problem for the damage field.

Probem PV_ω. Find a a damage field $\alpha_\omega : [0,T] \to H^1(\Omega)$ such that $\alpha_\omega(t) \in$ K, for all $t \in [0,T]$ and

$$(\dot{\alpha_\omega}(t), \xi - \alpha_\omega(t))_{L^2(\Omega)} + a(\alpha_\omega(t), \xi - \alpha_\omega(t))$$

$$\geq \left(\omega\left(t\right), \xi - \alpha_{\omega}\left(t\right)\right)_{L^2(\Omega)}\ \forall \xi \in K, \text{ a.e. } t \in (0,T) \tag{4.20}$$

$$\alpha_{\omega}\left(0\right) = \alpha_0 \tag{4.21}$$

Lemma 7. *Problem PV_{ω} has a unique soltion α_{ω} such that*

$$\alpha_{\omega} \in W^{1,2}\left(0,T;L^2\left(\Omega\right)\right) \cap L^2\left(0,T;H^1\left(\Omega\right)\right). \tag{4.22}$$

Proof. We use (3.21), (3.23) and a classical existence and uniqueness result on parabolic inequalities (see for instance [1 p. 124]). □

As a consequence of the problems QV_{η} and PV_{ω}, we may define the operator $\mathcal{L} : C(0,T;\mathcal{H} \times L^2\left(\Omega\right)) \to C(0,T;\mathcal{H} \times L^2\left(\Omega\right))$ by

$$\mathcal{L}(\boldsymbol{\eta},\omega) = (\mathcal{G}(\boldsymbol{\sigma}_{\eta}, \varepsilon(\mathbf{u}_{\eta}), \alpha_{\omega}), \Phi(\boldsymbol{\sigma}_{\eta}, \varepsilon(\mathbf{u}_{\eta}), \alpha_{\omega})), \tag{4.23}$$

for all $(\boldsymbol{\eta},\omega) \in C(0,T;\mathcal{H} \times L^2\left(\Omega\right))$. Then we have.

Lemma 8. *The operator $\mathcal{L}$ has a unique fixed point*

$$(\boldsymbol{\eta}^*,\omega^*) \in C(0,T;\mathcal{H} \times L^2\left(\Omega\right)).$$

Proof. Let $(\boldsymbol{\eta}_1,\omega_1)$, $(\boldsymbol{\eta}_2,\omega_2) \in C(0,T;\mathcal{H} \times L^2\left(\Omega\right))$, let $t \in [0,T]$ and use the notation $\mathbf{u}_{\eta i} = \mathbf{u}_i$, $\boldsymbol{\sigma}_{\eta i} = \boldsymbol{\sigma}_i$, $\mathfrak{z}_{\eta i} = \mathfrak{z}_i$ and $\alpha_{\omega_i} = \alpha_i$ for $i = 1,2$. Taking into account the relations (3.15), (3.16) and (4.23), we deduce that

$$\left|\mathcal{L}(\boldsymbol{\eta}_1,\omega_1) - \mathcal{L}(\boldsymbol{\eta}_2,\omega_2)\right|_{\mathcal{H}\times L^2(\Omega)}$$

$$\leq C\left(\left|\mathbf{u}_1\left(t\right) - \mathbf{u}_2\left(t\right)\right|_V + \left|\alpha_1\left(t\right) - \alpha_2\left(t\right)\right|_{L^2(\Omega)} + \left|\boldsymbol{\sigma}_1\left(t\right) - \boldsymbol{\sigma}_2\left(t\right)\right|_{\mathcal{H}}\right). \tag{4.24}$$

Using (4.5) we obtain

$$\left(\mathcal{E}\varepsilon\left(\mathbf{u}_1\right) - \mathcal{E}\varepsilon\left(\mathbf{u}_2\right), \varepsilon\left(\mathbf{u}_1\right) - \varepsilon\left(\mathbf{u}_2\right)\right)_{\mathcal{H}} = j_{ad}(\beta_{u_2}, \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) - j_{ad}(\beta_{u_1}, \mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2)$$

$$+ j_{nc}(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) - j_{nc}(\mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) + \left(\mathfrak{z}_2 - \mathfrak{z}_1, \varepsilon\left(\mathbf{u}_1\right) - \varepsilon\left(\mathbf{u}_2\right)\right)_{\mathcal{H}} \text{ a.e. } t \in (0,T). \tag{4.25}$$

Keeping in mind (3.35), (3.37) and (3.14) we find

$$\left|\mathbf{u}_1\left(t\right) - \mathbf{u}_2\left(t\right)\right|_V \leq C\left(\left|\beta_{u_1}\left(t\right) - \beta_{u_2}\left(t\right)\right|_{L^2(\Gamma_3)} + \left|\mathfrak{z}_1\left(t\right) - \mathfrak{z}_2\left(t\right)\right|_{\mathcal{H}}\right), \tag{4.26}$$

and

$$|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2 \le C\left(\left|\beta_{u_1}(t)-\beta_{u_2}(t)\right|^2_{L^2(\Gamma_3)}+|\mathfrak{z}_1(t)-\mathfrak{z}_2(t)|^2_{\mathcal{H}}\right).$$

By Lemma 4.2, we obtain

$$|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2 \le C\left(|\mathfrak{z}_1(t)-\mathfrak{z}_2(t)|^2_{\mathcal{H}}+\int_0^t |\mathbf{u}_1(s)-\mathbf{u}_2(s)|_V^2\ ds\right)$$

$$\le C(\int_0^t |\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)|^2_{\mathcal{H}}\ ds+\int_0^t |\mathbf{u}_1(s)-\mathbf{u}_2(s)|_V^2\ ds). \tag{4.27}$$

Applying Gronwall's inequality yields

$$|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2 \le C\int_0^t |\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)|^2_{\mathcal{H}}\ ds, \tag{4.28}$$

which implies

$$|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V \le C\int_0^t |\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)|_{\mathcal{H}}\ ds. \tag{4.29}$$

Moreover, by (4.4) we find

$$|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)|_{\mathcal{H}} \le C\left(|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V+|\mathfrak{z}_1(t)-\mathfrak{z}_2(t)|_{\mathcal{H}}\right).$$

Substituting (4.29) in the previous inequality we obtain

$$|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)|_{\mathcal{H}} \le C\int_0^t |\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)|_{\mathcal{H}}\ ds. \tag{4.30}$$

From (4.20) we deduce that

$$\left(\dot{\alpha_1},\alpha_2-\alpha_1\right)_{L^2(\Omega)}+a(\alpha_1,\alpha_2-\alpha_1)$$
$$\ge(\omega_1,\alpha_2-\alpha_1)_{L^2(\Omega)}\ \text{ a.e. } t\in(0,T),$$

and

$$\left(\dot{\alpha_2},\alpha_1-\alpha_2\right)_{L^2(\Omega)}+a(\alpha_2,\alpha_1-\alpha_2)$$
$$\ge(\omega_2,\alpha_1-\alpha_2)_{L^2(\Omega)}\ \text{ a.e. } t\in(0,T).$$

Adding the previous inequalities we obtain

$$\left(\dot{\alpha_1}-\dot{\alpha_2},\alpha_1-\alpha_2\right)_{L^2(\Omega)}+a(\alpha_1-\alpha_2,\alpha_1-\alpha_2)$$
$$\le|\omega_1-\omega_2|_{L^2(\Omega)}|\alpha_1-\alpha_2|_{L^2(\Omega)}\ \text{ a.e. } t\in(0,T).$$

Integrating the previous inequality on $[0, t]$, after some manipulations we obtain

$$\frac{1}{2}\left|\alpha_1(t)-\alpha_2(t)\right|_{L^2(\Omega)}^2 \leq C\int_0^t \left|\omega_1(s)-\omega_2(s)\right|_{L^2(\Omega)}\left|\alpha_1(s)-\alpha_2(s)\right|_{L^2(\Omega)}\ ds$$

$$+C\int_0^t \left|\alpha_1(s)-\alpha_2(s)\right|_{L^2(\Omega)}^2\ ds.$$

Applying Gronwall's inequality to the previous inequality yields

$$\left|\alpha_1(t)-\alpha_2(t)\right|_{L^2(\Omega)} \leq C\int_0^t \left|\omega_1(s)-\omega_2(s)\right|_{L^2(\Omega)}\ ds. \tag{4.31}$$

Substituting (4.29), (4.30) and (4.31) in (4.24), we obtain

$$|\mathcal{L}(\boldsymbol{\eta}_1,\omega_1)-\mathcal{L}(\boldsymbol{\eta}_2,\omega_2)|_{\mathcal{H}\times L^2(\Omega)}$$

$$\leq C\int_0^t |(\boldsymbol{\eta}_1,\omega_1)(s)-(\boldsymbol{\eta}_2,\omega_2)(s)|_{\mathcal{H}\times L^2(\Omega)}\ ds. \tag{4.32}$$

Lemma 4.7 is a consequence of the result (4.32) and Banach's fixed point Theorem. □

Now, we have all ingredients to solve QV.

Lemma 9. *There exists a unique solution* $(\mathbf{u},\boldsymbol{\sigma},\alpha)$ *of problem PV satisfying* $\mathbf{u}\in C(0,T;V)$, $\boldsymbol{\sigma}\in C(0,T;\mathcal{H}_1)$, $\alpha\in W^{1,2}\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;H^1(\Omega)\right)$.

Proof. Let $(\boldsymbol{\eta}^*,\omega^*)\in L^2(0,T;\mathcal{H}\times L^2(\Omega))$ be the fixed point of $\mathcal{L}$ given by (4.24), by Lemma 4.5, we deduce that $(\mathbf{u}_\eta,\boldsymbol{\sigma}_\eta)=(\mathbf{u}_{\eta\theta^*},\boldsymbol{\sigma}_{\eta\theta^*})\in C(0,T;V)\times C(0,T;\mathcal{H}_1)$ is the unique solution of QV_η. Since $\mathcal{L}(\boldsymbol{\eta}^*,\omega^*)=(\boldsymbol{\eta}^*,\omega^*)$, from the relations (4.4), (4.5), (4.6) and Lemma 4.6 we obtain that $(\mathbf{u},\boldsymbol{\sigma},\alpha)=(\mathbf{u}_{\eta^*\theta^*},\boldsymbol{\sigma}_{\eta^*\theta^*},\alpha_{\omega^*})$ is the unique solution of QV. The regularity of the solution follows from Lemma 4.6. The uniqueness of the solution results from the uniqueness of the fixed point of the operator $\mathcal{L}$. □

Theorem 4.1 is now a consequence of Lemma 4.2 and Lemma 4.8.

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HARMONIC MULTIVALENT FUNCTIONS DEFINED BY INTEGRAL OPERATOR

LUMINIŢA-IOANA COTÎRLĂ

Abstract. We define and investigate a new class of harmonic multivalent functions defined by integral operator. We obtain coefficient inequalities, extreme points and distortion bounds for the functions in our classes.

1. Introduction

A continuous complex-valued function $f = u+iv$ defined in a complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$. (See Clunie and Sheil-Small [2]).

Denote by H the class of functions $f = h+\overline{g}$ that are harmonic univalent and sense preserving in the unit disc $U = \{z : \ |z| < 1\}$ so that $f = h + \overline{g}$ is normalized by $f(0) = h(0) = f'_z(0) - 1 = 0$.

Recently, Ahuja and Jahangiri [5] defined the class $H_p(n)$ $(p, n \in \mathbb{N})$, consisting of all p-valent harmonic functions $f = h + \overline{g}$ that are sense preserving in U and h and g are of the form

$$h(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \quad g(z) = \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}, \quad |b_p| < 1. \tag{1.1}$$

The integral operator I^n is defined (see [4], for $p = 1$) by:

(i) $I^0 f(z) = f(z)$;

(ii) $I^1 f(z) = If(z) = p\int_0^z f(t)t^{-1}dt$;

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(iii) $I^n f(z) = I(I^{n-1} f(z))$, $n \in \mathbb{N}$, $f \in \mathcal{A}$,

where $\mathcal{A} = \{f \in \mathcal{H} : \ f(z) = z + a_2 z^2 + \dots\}$ and $\mathcal{H} = \mathcal{H}(U)$.

For $f = h + \overline{g}$ given by (1.1) the integral operator of f is defined as

$$I^n f(z) = I^n h(z) + (-1)^n \overline{I^n g(z)}, \quad p > n \tag{1.2}$$

where

$$I^n h(z) = z^p + \sum_{k=2}^{\infty} \left(\frac{p}{k+p-1} \right)^n a_{k+p-1} z^{k+p-1}$$

and

$$I^n g(z) = \sum_{k=1}^{\infty} \left(\frac{p}{k+p-1} \right)^n b_{k+p-1} z^{k+p-1}.$$

For $0 \le \alpha < 1$, $n \in \mathbb{N}$, $z \in U$, let $H_p(n, \alpha)$ denote the family of harmonic functions f of the form (1.1) such that

$$\text{Re} \left(\frac{I^n f(z)}{I^{n+1} f(z)} \right) > \alpha, \tag{1.3}$$

where I^n is defined by (1.2).

The families $H_p(m, n, \alpha)$ and $H_p^-(m, n, \alpha)$ include a variety of well-known classes of harmonic functions as well as many new ones. For example $HS(\alpha) = \overline{H_1}(1, 0, \alpha)$ is the class of sense-preserving,harmonic univalent functions f which are starlike of order $\alpha \in U$, and $HK(\alpha) = \overline{H_1}(2, 1, \alpha)$ is the class of sense-preserving,harmonic univalent functions f which are convex of order α in U, and $\overline{H_1}(n+1, n, \alpha) = \overline{H}(n, \alpha)$ is the class of Sălăgean-type harmonic univalent functions.

Let we denote the subclass $H_p^-(n, \alpha)$ consists of harmonic functions $f_n = h + \overline{g}_n$ in $H_p^-(n, \alpha)$ so that h and g_n are of the form

$$h(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} \quad \text{and} \quad g_n(z) = (-1)^{n-1} \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1} \tag{1.4}$$

where $a_{k+p-1}, b_{k+p-1} \ge 0$, $|b_p| < 1$.

For the harmonic functions f of the form (1.1) with $b_1 = 0$, Avei and Zlotkiewich in [1] show that if

$$\sum_{k=2}^{\infty} k(|a_k| + |b_k|) \le 1,$$

then $f \in SH(0)$, where $HS(0) = \overline{H_1}(1,0,0)$ and if

$$\sum_{k=2}^{\infty} k^2(|a_k| + |b_k|) \leq 1$$

then $f \in HK(0)$, where $HK(0) = \overline{H_1}(2,1,0)$.

For the harmonic functions f of the form (1.4) with $n = 0$, Jahongiri in [3] showed that $f \in HS(\alpha)$ if and only if

$$\sum_{k=2}^{\infty}(k-\alpha)|a_k| + \sum_{k=1}^{\infty}(k+\alpha)|b_k| \leq 1-\alpha$$

and $f \in \overline{H_1}(2,1,\alpha)$ if and only if

$$\sum_{k=2}^{\infty} k(k-\alpha)|a_k| + \sum_{k=1}^{\infty} k(k+\alpha)|b_k| \leq 1-\alpha.$$

2. **Main results**

In our first theorem, we deduce a sufficient coefficient bound for harmonic functions in $H_p(n,\alpha)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be given by (1.1). If*

$$\sum_{k=1}^{\infty}\{\psi(n,p,k,\alpha)|a_{k+p-1}| + \theta(n,p,k,\alpha)|b_{k+p-1}|\} \leq 2 \tag{2.1}$$

where

$$\psi(n,p,k,\alpha) = \frac{\left(\frac{p}{k+p-1}\right)^n - \alpha\left(\frac{p}{k+p-1}\right)^{n+1}}{1-\alpha}$$

$$\theta(n,p,k,\alpha) = \frac{\left(\frac{p}{k+p-1}\right)^n + \alpha\left(\frac{p}{k+p-1}\right)^{n+1}}{1-\alpha},$$

$$a_p = 1, \quad 0 \leq \alpha < 1, \quad n \in \mathbb{N}.$$

Then f is sense preserving in U and $f \in H_p(n,\alpha)$.

Proof. According to (1.2) and (1.3) we only need to show that

$$\operatorname{Re}\left(\frac{I^n f(z) - \alpha I^{n+1} f(z)}{I^{n+1} f(z)}\right) \geq 0.$$

The case $r = 0$ is obvious.

For $0 < r < 1$, it follows that

$$\operatorname{Re}\left(\frac{I^n f(z) - \alpha I^{n+1} f(z)}{I^{n+1} f(z)}\right)$$

$$=\operatorname{Re}\left\{\frac{z^p(1-\alpha)+\sum_{k=2}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n-\alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right]a_{k+p-1}z^{k+p-1}}{z^p+\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}a_{k+p-1}z^{k+p-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}\overline{b}_{k+p-1}\overline{z}^{k+p-1}}\right.$$

$$\left.+\frac{(-1)^n\sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n+\alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right]\overline{b}_{k+p-1}\overline{z}^{k+p-1}}{z^p+\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}a_{k+p-1}z^{k+p-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}\overline{b}_{k+p-1}\overline{z}^{k+p-1}}\right\}$$

$$=\operatorname{Re}\left\{\frac{(1-\alpha)+\sum_{k=2}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n-\alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right]a_{k+p-1}z^{k-1}}{1+\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}a_{k+p-1}z^{k-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}\overline{b}_{k+p-1}\overline{z}^{k+p-1}z^{-p}}\right.$$

$$\left.+\frac{(-1)^n\sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n+\alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right]\overline{b}_{k+p-1}\overline{z}^{k+p-1}z^{-p}}{1+\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}a_{k+p-1}z^{k-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}\overline{b}_{k+p-1}\overline{z}^{k+p-1}z^{-p}}\right\}$$

$$=\operatorname{Re}\left[\frac{(1-\alpha)+A(z)}{1+B(z)}\right].$$

For $z = re^{i\theta}$ we have

$$A(re^{i\theta})=\sum_{k=2}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n-\alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right]a_{k+p-1}r^{k-1}e^{(k-1)\theta i}$$

$$+(-1)^n\sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n+\alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right]\overline{b}_{k+p-1}r^{k-1}e^{-(k+2p-1)\theta i};$$

$$B(re^{i\theta})=\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}a_{k+p-1}r^{k-1}e^{(k-1)\theta i}$$

$$+(-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}\overline{b}_{k+p-1}r^{k-1}e^{-(k+2p-1)\theta i}.$$

Setting

$$\frac{(1-\alpha)+A(z)}{1+B(z)}=(1-\alpha)\frac{1+w(z)}{1-w(z)},$$

the proof will be complete if we can show that $|w(z)| \leq 1$. This is the case since, by the condition (2.1), we can write

$$|w(z)| = \left| \frac{A(z) - (1-\alpha)B(z)}{A(z) + (1-\alpha)B(z) + 2(1-\alpha)} \right|$$

$$= \left| \frac{\sum_{k=2}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n - \left(\frac{p}{k+p-1}\right)^{n+1}\right] a_{k+p-1} r^{k-1} e^{(k-1)\theta i}}{2(1-\alpha) + \sum_{k=2}^{\infty} C(n,p,k,\alpha) a_{k+p-1} r^{k-1} e^{(k-1)\theta i} + (-1)^n \sum_{k=1}^{\infty} D(n,p,k,\alpha) \overline{b}_{k+p-1} r^{k-1} e^{-(k+2p-1)\theta i}} \right.$$

$$\left. + \frac{(-1)^n \sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n + \left(\frac{p}{k+p-1}\right)^{n+1}\right] \overline{b}_{k+p-1} r^{k-1} e^{-(k+2p-1)\theta i}}{2(1-\alpha) + \sum_{k=2}^{\infty} C(n,p,k,\alpha) a_{k+p-1} r^{k-1} e^{(k-1)\theta i} + (-1)^n \sum_{k=1}^{\infty} D(n,p,k,\alpha) \overline{b}_{k+p-1} r^{k-1} e^{-(k+2p-1)\theta i}} \right|$$

$$\leq \frac{\sum_{k=2}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n - \left(\frac{p}{k+p-1}\right)^{n+1}\right] |a_{k+p-1}| r^{k-1}}{2(1-\alpha) - \sum_{k=2}^{\infty} C(n,p,k,\alpha) |a_{k+p-1}| r^{k-1} - \sum_{k=1}^{\infty} D(n,p,k,\alpha) |b_{k+p-1}| r^{k-1}}$$

$$+ \frac{\sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n + \left(\frac{p}{k+p-1}\right)^{n+1}\right] |b_{k+p-1}| r^{k-1}}{2(1-\alpha) - \sum_{k=2}^{\infty} C(n,p,k,\alpha) |a_{k+p-1}| r^{k-1} - \sum_{k=1}^{\infty} D(n,p,k,\alpha) |b_{k+p-1}| r^{k-1}}$$

$$= \frac{\sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n - \left(\frac{p}{k+p-1}\right)^{n+1}\right] |a_{k+p-1}| r^{k-1}}{4(1-\alpha) - \sum_{k=1}^{\infty} \{C(n,p,k,\alpha)|a_{k+p-1}| + D(n,p,k,\alpha)|b_{k+p-1}|\} r^{k-1}}$$

$$+ \frac{\sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n + \left(\frac{p}{k+p-1}\right)^{n+1}\right] |b_{k+p-1}| r^{k-1}}{4(1-\alpha) - \sum_{k=1}^{\infty} \{C(n,p,k,\alpha)|a_{k+p-1}| + D(n,p,k,\alpha)|b_{k+p-1}|\} r^{k-1}}$$

$$< \frac{\sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n - \left(\frac{p}{k+p-1}\right)^{n+1}\right] |a_{k+p-1}|}{4(1-\alpha) - \sum_{k=1}^{\infty} \{C(n,p,k,\alpha)|a_{k+p-1}| + D(n,p,k,\alpha)|b_{k+p-1}|\}}$$

$$+\frac{\sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^{n}+\left(\frac{p}{k+p-1}\right)^{n+1}\right]|b_{k+p-1}|}{4(1-\alpha)-\sum_{k=1}^{\infty}\{C(n,p,k,\alpha)|a_{k+p-1}|+D(n,p,k,\alpha)|b_{k+p-1}|\}}\leq 1.$$

where

$$C(n,p,k,\alpha)=\left(\frac{p}{k+p-1}\right)^{n}+(1-2\alpha)\left(\frac{p}{k+p-1}\right)^{n+1}$$

and

$$D(n,p,k,\alpha)=\left(\frac{p}{k+p-1}\right)^{n}+(-1)(1-2\alpha)\left(\frac{p}{k+p-1}\right)^{n+1}$$

The harmonic univalent functions

$$f(z)=z^{p}+\sum_{k=2}^{\infty}\frac{1}{\psi(n,p,k,\alpha)}x_k z^{k+p-1}+\sum_{k=1}^{\infty}\frac{1}{\theta(n,p,k,\alpha)}\overline{y_k z^{k+p-1}}, \tag{2.2}$$

where $n\in\mathbb{N}$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.2) are in $H_p(n,\alpha)$ because

$$\sum_{k=1}^{\infty}\{\psi(n,p,k,\alpha)|a_{k+p-1}|+\theta(n,p,k,\alpha)|b_{k+p-1}|\}=1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=2.$$

In the following theorem it is show that the condition (2.1) is also necessary for functions $f_n=h+\overline{g}_n$, where h and g_n are of the form (1.4).

Theorem 2.2. *Let $f_n=h+\overline{g}_n$ be given by (1.4). Then $f_n\in H_p^{-}(n,\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}\{\psi(n,p,k,\alpha)a_{k+p-1}+\theta(n,p,k,\alpha)b_{k+p-1}\}\leq 2, \tag{2.3}$$

where $a_p=1$, $0\leq\alpha<1$, $n\in\mathbb{N}$.

Proof. Since $H_p^{-}(n,\alpha)\subset H_p(n,\alpha)$, we only need to prove the "only if" part of the theorem. For functions f_n of the form (1.4), we note that the condition

$$\text{Re}\left\{\frac{I^{n}f_n(z)}{I^{n+1}f_n(z)}\right\}>\alpha$$

is equivalent to

$$\operatorname{Re}\left\{\frac{(1-\alpha)z^p-\sum_{k=2}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n-\alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right]a_{k+p-1}z^{k+p-1}}{z^p-\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}a_{k+p-1}z^{k+p-1}+(-1)^{2n}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}b_{k+p}\overline{z}^{k+p-1}}\right. \tag{2.4}$$

$$\left.+\frac{(-1)^{2n-1}\sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n+\alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right]b_{k+p-1}\overline{z}^{k+p-1}}{z^p-\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}a_{k+p-1}z^{k+p-1}+(-1)^{2n}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}b_{k+p-1}\overline{z}^{k+p-1}}\right\}\geq 0.$$

The above required condition (2.4) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$\frac{(1-\alpha)-\sum_{k=2}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n-\alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right]a_{k+p-1}r^{k-1}}{1-\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}a_{k+p-1}r^{k-1}+\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}b_{k+p-1}r^{k-1}}$$

$$+\frac{-\sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n+\alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right]b_{k+p-1}r^{k-1}}{1-\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}a_{k+p-1}r^{k-1}+\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}b_{k+p-1}r^{k-1}}\geq 0. \tag{2.5}$$

If the condition (2.3) does not hold, then the expression in (2.5) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0,1)$ for which the quotient in (2.5) is negative.

This contradicts the required condition for $f_n \in H_p^-(n,\alpha)$. So the proof is complete.

Next we determine the extreme points of the closed convex hull of $H_p^-(n,\alpha)$, denoted by $clcoH_p^-(n,\alpha)$.

Theorem 2.3. *Let f_n be given by (1.4). Then $f_n \in H_p^-(n,\alpha)$ if and only if*

$$f_n(z)=\sum_{k=1}^{\infty}[x_{k+p-1}h_{k+p-1}(z)+y_{k+p-1}g_{n_{k+p-1}}(z)],$$

where

$$h_p(z) = z^p, \quad h_{k+p-1}(z) = z^p - \frac{1}{\psi(n,p,k,\alpha)} z^{k+p-1}, \quad k = 2, 3, \dots$$

and

$$g_{n_{k+p-1}}(z) = z^p + (-1)^{n-1} \cdot \frac{1}{\theta(n,p,k,\alpha)} \overline{z}^{k+p-1}, \quad k = 1, 2, 3, \dots$$

$$x_{k+p-1} \geq 0, \quad y_{k+p-1} \geq 0, \quad x_p = 1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1}.$$

In particular, the extreme points of $H_p^-(n,\alpha)$ *are* $\{h_{k+p-1}\}$ *and* $\{g_{n_{k+p-1}}\}$.

Proof. For functions f_n of the form (2.1),

$$f_n(z) = \sum_{k=1}^{\infty} [x_{k+p-1} h_{k+p-1}(z) + y_{k+p-1} g_{n_{k+p-1}}(z)]$$

$$= \sum_{k=1}^{\infty} (x_{k+p-1} + y_{k+p-1}) z^p - \sum_{k=2}^{\infty} \frac{1}{\psi(n,p,k,\alpha)} x_{k+p-1} z^{k+p-1}$$

$$+ (-1)^{n-1} \sum_{k=1}^{\infty} \frac{1}{\theta(n,p,k,\alpha)} y_{k+p-1} \overline{z}^{k+p-1}.$$

Then

$$\sum_{k=2}^{\infty} \psi(n,p,k,\alpha) \left(\frac{1}{\psi(n,p,k,\alpha)} x_{k+p-1} \right) + \sum_{k=1}^{\infty} \theta(n,p,k,\alpha) \left(\frac{1}{\theta(n,p,k,\alpha)} y_{k+p-1} \right)$$

$$= \sum_{k=2}^{\infty} x_{k+p-1} + \sum_{k=1}^{\infty} y_{k+p-1} = 1 - x_p \leq 1,$$

and so $f_n(z) \in clcoH_p^-(n,\alpha)$.

Conversely, suppose $f_n(z) \in clcoH_p^-(n,\alpha,\beta)$. Letting

$$x_p = 1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1},$$

let

$$x_{k+p-1} = \psi(n,p,k,\alpha) a_{k+p-1}$$

and

$$y_{k+p-1} = \theta(n,p,k,\alpha) b_{k+p-1}, \quad k = 2, 3, \dots$$

We obtain the required representation, since

$$f_n(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + (-1)^{n-1} \sum_{k=1}^{\infty} b_{k+p-1} \overline{z}^{k+p-1}$$

$$= z^p - \sum_{k=2}^{\infty} \frac{1}{\psi(n,p,k,\alpha)} x_{k+p-1} z^{k+p-1} + (-1)^{n-1} \sum_{k=1}^{\infty} \frac{1}{\theta(n,p,k,\alpha)} y_{k+p-1} \overline{z}^{k+p-1}$$

$$= z^p - \sum_{k=2}^{\infty} [z^p - h_{k+p-1}(z)] x_{k+p-1} - \sum_{k=1}^{\infty} [z^p - g_{n_{k+p-1}}(z)] y_{k+p-1}$$

$$= \left[1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1}\right] z^p + \sum_{k=2}^{\infty} x_{k+p-1} h_{k+p-1}(z)$$

$$+ \sum_{k=1}^{\infty} y_{k+p-1} g_{n_{k+p-1}}(z) = \sum_{k=1}^{\infty} [x_{k+p-1} h_{k+p-1}(z) + y_{k+p-1} g_{n_{k+p-1}}(z)].$$

The following theorem gives the distortion bounds for functions in $H_p^-(n,\alpha)$ which yields a covering results for this class.

Theorem 2.4. *Let* $f_n \in H_p^-(n,\alpha)$. *Then for* $|z| = r < 1$ *we have*

$$|f_n(z)| \le (1 + b_p) r^p + \{\phi(n,p,k,\alpha) - \Omega(n,p,k,\alpha) b_p\} r^{p+1}$$

and

$$|f_n(z)| \ge (1 - b_p) r^p - \{\phi(n,p,k,\alpha) - \Omega(n,p,k,\alpha) b_p\} r^{p+1},$$

where

$$\phi(n,p,k,\alpha) = \frac{1-\alpha}{\left(\frac{p}{p+1}\right)^n - \alpha \left(\frac{p}{p+1}\right)^{n+1}},$$

$$\Omega(n,p,k,\alpha) = \frac{1+\alpha}{\left(\frac{p}{p+1}\right)^n - \alpha \left(\frac{p}{p+1}\right)^{n+1}}.$$

Proof. We prove the right hand side inequality for $|f_n|$. The proof for the left hand inequality can be done using similar arguments. Let $f_n \in H_p^-(n,\alpha)$. Taking the absolute value of f_n then by Theorem 2.2, we obtain:

$$|f_n(z)| = \left| z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + (-1)^{n-1} \sum_{k=1}^{\infty} b_{k+p-1} \overline{z}^{k+p-1} \right|$$

$$\le r^p + \sum_{k=2}^{\infty} a_{k+p-1} r^{k+p-1} + \sum_{k=1}^{\infty} b_{k+p-1} r^{k+p-1}$$

$$= r^p + b_p r^p + \sum_{k=2}^{\infty}(a_{k+p-1} + b_{k+p-1})r^{k+p-1}$$
$$\leq r^p + b_p r^p + \sum_{k=2}^{\infty}(a_{k+p-1} + b_{k+p-1})r^{p+1}$$
$$= (1+b_p)r^p + \phi(n,p,k,\alpha)\sum_{k=2}^{\infty}\frac{1}{\phi(n,p,k,\alpha)}(a_{k+p-1} + b_{k+p-1})r^{p+1}$$
$$\leq (1+b_p)r^p + \phi(n,p,k,\alpha)r^{p+1}\left[\sum_{k=2}^{\infty}\psi(n,p,k,\alpha)a_{k+p-1} + \theta(n,p,k,\alpha)b_{k+p-1}\right]$$
$$\leq (1+b_p)r^p + \{\phi(n,p,k,\alpha) - \Omega(n,p,k,\alpha)b_p\}r^{p-1}.$$

The following covering result follows from the left hand inequality in Theorem 2.4.

Corollary 2.1. *Let $f_n \in H_p^-(n,\alpha)$, the for $|z| = r < 1$ we have*

$$\{w:\ |w| < 1 - b_p - [\phi(n,p,k,\alpha) - \Omega(n,p,k,\alpha)b_p] \subset f_b(U)\}.$$

Similar results was obtained in [6] by Bilal Şekel and Sevtap Sümer Eker for the differential operator of Sălăgean defined in [4].

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ANALYSIS OF A ELECTRO-ELASTIC CONTACT PROBLEM WITH FRICTION AND ADHESION

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Abstract. We consider a mathematical model which describes the quasistatic frictional contact between a piezoelectric body and an obstacle, the so-called foundation. A nonlinear electro-elastic constitutive law is used to model the piezoelectric material. The contact is modelled with Signorini's conditions and the associated with a regularized Coulomb's law of dry friction in witch the adhesion of contact surfaces is taken into account. The evolution of the bonding field is described by a first order differential equation. We derive a variational formulation for the model, in the form of a coupled system for the displacements, the electric potential and the adhesion. Under a smallness assumption on the coefficient of friction, we prove the existence of a unique weak solution of the model. The proof is based on arguments of time-dependent quasi-variational inequalities, differential equations and Banach's fixed point theorem.

1. Introduction

The piezoelectric effect is characterized by the coupling between the mechanical and electrical properties of the materials. Indeed, the apparition of electric charges on some crystals submitted to the action of body forces and surface tractions was observed and their dependence on the deformation process was underlined. Conversely, it was proved experimentally that the action of electric field on the crystals may generate strain and stress. A deformable material which presents such a behavior is called a piezoelectric material. Piezoelectric materials are used extensively as switches and

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actuary in many engineering systems, in radioelectronics, electroacoustics, and measuring equipments. General models for electro-elastic materials can be found in [3], [5] and in [17]. A static frictional contact problem for electro-elastic materials was considered in [4] and in [20]. A slip-dependent frictional contact problem for electro-elastic materials was studied in [26] and a frictional problem with normal compliance for electroviscoelastic materials was considered in [27], [19] and in [18]. In the last two references the variational formulations of the corresponding problems were derived and existence and uniqueness results for the weak solutions were obtained.

The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature. Basic modelling can be found in [13], [15] and in [9]. Analysis of models for adhesive contact can be found in [2]-[7], [16] and in the recent monographs [24] and [25]. An application of the theory of adhesive contact in the medical field of prosthetic limbs was considered in [22] and in [23]; there, the importance of the bonding between the bone-implant and the tissue was outlined, since debonding may lead to decrease in the persons ability to use the artificial limb or joint.

Contact problems for elastic and elastic-viscoelastic bodies with adhesion and friction appear in many applications of solids mechanics such as the fiber-matrix interface of composite materials. A consistent model coupling unilateral contact, adhesion and friction is proposed by Raous, Cangémi and Cocu in [21]. Adhesive problems have been the subject of some recent publications (see for instance [12], [1], [6] and [9]). The novelty in all the above papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by β; it describes the pointwise fractional density of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [13], [14], the bonding field satisfies the restrictions $0 \leq \beta \leq 1$; when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. We refer the reader to the extensive bibliography on the subject in [15] and in [22]. Such models contain a new internal variable β which represents the adhesion intensity over the contact surface, it takes values between 0 and 1, and describes the fractional density of active bonds on the contact surface.

The aim of this paper is to continue the study of problems begun in [19], [27] and in [18]. The novelty of the present paper is to extend the result when the contact and friction are modelled by Signorini's conditions and a non local Coulomb's friction law, respectively. Moreover, the adhesion is taken into account at the interface and the material behavior is assumed to be electro-elastic.

The paper is structured as follows. In Section 2 we present the electro-elastic contact model with friction and adhesion and provide comments on the contact boundary conditions. In Section 3 we list the assumptions on the data and derive the variational formulation. In Sections 4, we present our main existence and uniqueness results, Theorems 4.1, which states the unique weak solvability of the Signorini's adhesive contact electro-elastic problem with non local Coulomb's friction law conditions.

2. **Problem statement**

We consider the following physical setting. An electro-elastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$. The body is submitted to the action of body forces of density f_0 and volume electric charges of density q_0. It is also submitted to mechanical and electric constraints on the boundary. To describe them, we consider a partition of Γ into three measurable parts Γ_1, Γ_2, Γ_3 on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand., such that $meas(\Gamma_1) > 0$, $meas(\Gamma a) > 0$. We assume that the body is clamped on Γ_1 and surface tractions of density f_2 act on Γ_2. On Γ_3 the body is in adhesive contact with an insulator obstacle, the so-called foundation. We also assume that the electrical potential vanishes on Γ_a and a surface electric charge of density q_2 is prescribed on Γ_b. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ and we use $\cdot$ and $||\cdot||$ for the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively. Also, below ν represents the unit outward normal on Γ. With these assumptions, the classical formulation of the electro-elastic contact problem coupling friction and adhesion is the following.

Problem 2.1 ($\mathcal{P}$). *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$*, an electric potential* $\varphi : \Omega \times [0,T] \to \mathbb{R}$*, an electric displacement field* $D : \Omega \times [0,T] \to R^d$ *and a bonding field* $\beta : \Omega \times [0,T] \to R$ *such that*

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{E}^* E(\varphi) \quad in\ \Omega \times (0,T), \tag{2.1}$$

$$D = \mathcal{B}E(\varphi) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad in\ \Omega \times (0,T), \tag{2.2}$$

$$Div\boldsymbol{\sigma} + f_0 = 0 \quad in\ \Omega \times (0,T), \tag{2.3}$$

$$div\ D = q_0 \quad on\ \Omega \times (0,T), \tag{2.4}$$

$$\boldsymbol{u} = 0 \quad on\ \Gamma_1 \times (0,T), \tag{2.5}$$

$$\boldsymbol{\sigma}\nu = f_2 \quad on\ \Gamma_2 \times (0,T), \tag{2.6}$$

$$\boldsymbol{u}_\nu \leq 0,\ \boldsymbol{\sigma}_\nu - \gamma_\nu\beta^2 R_\nu(\boldsymbol{u}_\nu) \leq 0,\ \boldsymbol{u}_\nu(\boldsymbol{\sigma}_\nu - \gamma_\nu\beta^2 R_\nu(\boldsymbol{u}_\nu)) = 0\ on\ \Gamma_3 \times (0,T), \tag{2.7}$$

$$\begin{cases} |\boldsymbol{\sigma}_\tau + \gamma_\tau\beta^2 R_\tau(\boldsymbol{u}_\tau)| \leq \mu p(|R(\boldsymbol{\sigma}_\nu) - \gamma_\nu\beta^2 R_\nu(\boldsymbol{u}_\nu)|), \\ |\boldsymbol{\sigma}_\tau + \gamma_\tau\beta^2 R_\tau(\boldsymbol{u}_\tau)| < \mu p(|R(\boldsymbol{\sigma}_\nu) - \gamma_\nu\beta^2 R_\nu(\boldsymbol{u}_\nu)|) \Rightarrow \boldsymbol{u}_\tau = 0, \\ |\boldsymbol{\sigma}_\tau + \gamma_\tau\beta^2 R_\tau(\boldsymbol{u}_\tau) = \mu p(|R(\boldsymbol{\sigma}_\nu) - \gamma_\nu\beta^2 R_\nu(\boldsymbol{u}_\nu)|) \Rightarrow \exists\lambda \geq 0, \\ such\ that\ \boldsymbol{\sigma}_\tau + \gamma_\tau\beta^2 R_\tau(\boldsymbol{u}_\tau) = -\lambda\boldsymbol{u}_\tau, \end{cases} \quad on\ \Gamma_3 \times (0,T), \tag{2.8}$$

$$\dot{\beta} = -(\beta(\gamma_\nu R_\nu(\boldsymbol{u}_\nu)^2 + \gamma_\tau\|R_\tau(\boldsymbol{u}_\tau)\|^2) - \epsilon_a) \quad on\ \Gamma_3 \times (0,T), \tag{2.9}$$

$$\varphi = 0 \quad on\ \Gamma_a \times (0,T), \tag{2.10}$$

$$D \cdot \nu = q_2 \quad on\ \Gamma_b \times (0,T), \tag{2.11}$$

$$D \cdot \nu = 0 \quad on\ \Gamma_3 \times (0,T), \tag{2.12}$$

$$\beta(0) = \beta_0 \quad on\ \Gamma_3. \tag{2.13}$$

We now provide some comments on equations and conditions (2.1)-(2.13).

Equations (2.1) and (2.2) represent the electro-elastic constitutive law in which $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain tensor, $\boldsymbol{E}(\varphi) = -\nabla\,\varphi$ is the electric field, where φ is the electric potential, $\mathcal{F}$ is a given nonlinear function, $\mathcal{E}$ represents the piezoelectric operator, $\mathcal{E}^*$ is its transposed, $\mathcal{B}$ denotes the electric permittivity operator, and $\mathbf{D} = (D_1, \ldots, D_d)$ is the electric displacement vector. Details on the constitutive equations of the form (2.1) and (2.2) can be find, for instance, in [3] and in [4]. Next, equations (2.3) and (2.4) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which "Div" and "div" denote the divergence operator for tensor and vector valued functions, respectively. Equations (2.5) and (2.6) represent the displacement and traction boundary conditions Conditions (2.10) and (2.11) represent the electric boundary conditions.

Conditions (2.7) represent he Signorini's contact condition with adhesion where $\boldsymbol{u}_\nu$ is the normal displacement $\boldsymbol{\sigma}_\nu$ represents the normal stress, γ_ν denote a given adhesion coefficient and R_ν is the truncation operator define by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \leq s \leq 0, \\ 0 & \text{if } s > 0, \end{cases}$$

where $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of operator R_ν, together with the operator R_τ defined below , is motivated by the mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows. Thus, by choosing L very large, we can assume that $R_\nu(\boldsymbol{u}_\nu) = \boldsymbol{u}_\nu$ and, therefore, from (2.7) we recover the contact conditions

$$\boldsymbol{u}_\nu \leq 0,\ \boldsymbol{\sigma}_\nu - \gamma_\nu\beta^2\boldsymbol{u}_\nu \leq 0,\ \boldsymbol{u}_\nu(\boldsymbol{\sigma}_\nu - \gamma_\nu\beta^2\boldsymbol{u}_\nu) = 0 \text{ on } \Gamma_3 \times (0,T),$$

It follows from (2.7) that there is no penetration between the body and the foundation, since $\boldsymbol{u}_\nu \leq 0$ during the process.

Conditions (2.8) are a non local Coulomb's friction law conditions coupled with adhesion, where $\boldsymbol{u}_\tau$ and $\boldsymbol{\sigma}_\tau$ denote tangential components of vector $\boldsymbol{u}$ and tensor

$\boldsymbol{\sigma}$ respectively. R_τ is the truncation operator given by

$$R_\tau(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } \|\boldsymbol{v}\| \leq L, \\ L\dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|} & \text{if } \|\boldsymbol{v}\| > L. \end{cases}$$

This condition shows that the magnitude of the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L.

R will represent a normal regularization operator that is , linear and continues operator $R: H^{-\frac{1}{2}}(\Gamma) \to L^2(\Gamma)$. We shall need it to regularize the normal trace of the stress witch is too rough on Γ. p is a non-negative function, the so-called friction bound, $\mu \geq 0$ is the coefficient of friction. The friction law was used in some studies with $p\,(r) = r_+$ where $r_+ = \max\{0, r\}$. Recently, from thermodynamic considerations, a new version of *Coulomb's* law is proposed; its consists to take

$$p(r) = r(1 - \alpha r)_+, \tag{2.14}$$

where α is a small positive coefficient related to the hardness and the wear of the contact surface.

Also, note that when the bonding field vanishes, then the contact conditions (2.7) and (2.8) become the classic Signorini's contact with a non local Coulomb's friction law conditions were used in ([11]), that is

$$\boldsymbol{u}_\nu \leq 0,\ \boldsymbol{\sigma}_\nu \leq 0,\ \boldsymbol{u}_\nu \boldsymbol{\sigma}_\nu = 0 \text{ on } \Gamma_3 \times (0, T),$$

$$\begin{cases} |\boldsymbol{\sigma}_\tau| \leq \mu p(|R(\boldsymbol{\sigma}_\nu)|), \\ |\boldsymbol{\sigma}_\tau| < \mu p(|R(\boldsymbol{\sigma}_\nu)|) \Rightarrow \boldsymbol{u}_\tau = 0, & \text{on } \Gamma_3 \times (0, T), \\ |\boldsymbol{\sigma}_\tau| = \mu p(|R(\boldsymbol{\sigma}_\nu)|) \Rightarrow \exists \lambda \geq 0, \text{ such that } \boldsymbol{\sigma}_\tau = -\lambda \boldsymbol{u}_\tau. \end{cases}$$

The evolution of the bonding field is governed by the differential equation (2.9) with given positive parameters γ_ν, γ_τ and ϵ_a, where $r_+ = \max\{0, r\}$. Here and below in this paper, a dot above a function represents the derivative with respect to the time variable. We note that the adhesive process is irreversible and, indeed, once debonding occurs bonding cannot be reestablished, since $\dot{\beta} \leq 0$. Finally, (2.13) is the initial condition in which β_0 is a given bonding field.

3. **Variational formulation and preliminaries**

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end we need to introduce some notation and preliminary material.

Here and below $\mathbb{S}^d$ represents the space of second order symmetric tensors on $\mathbb{R}^d$. We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}_i \boldsymbol{v}_i \ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} \cdot \tau = \boldsymbol{\sigma}_{ij} \tau_{ij} \ , \qquad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma}, \tau \in \mathbb{S}^d.$$

Here and everywhere in this paper, i, j, k, l run from 1 to d, summation over repeated indices is applied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $\boldsymbol{u}_{i,j} = \frac{\partial \boldsymbol{u}_i}{\partial x_j}$.

Everywhere below, we use the classical notation for L^p and Sobolev spaces associated to Ω and Γ. Moreover, we use the notation $L^2(\Omega)^d$, $H^1(\Omega)^d$ and $\mathcal{H}$ and $\mathcal{H}_1$ for the following spaces :

$$L^2(\Omega)^d = \{\ \boldsymbol{v} = (\boldsymbol{v}_i) \mid \boldsymbol{v}_i \in L^2(\Omega)\ \}, \quad H^1(\Omega)^d = \{\ \boldsymbol{v} = (\boldsymbol{v}_i) \mid \boldsymbol{v}_i \in H^1(\Omega)\ \},$$

$$\mathcal{H} = \{\ \tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\ \}, \quad \mathcal{H}_1 = \{\ \tau \in \mathcal{H} \mid \tau_{ij,j} \in L^2(\Omega)\ \}.$$

The spaces $L^2(\Omega)^d$, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)^d} = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, dx, \quad (\boldsymbol{u}, \boldsymbol{v})_{H^1(\Omega)^d} = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, dx + \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v}\, dx,$$

$$(\boldsymbol{\sigma}, \tau)_{\mathcal{H}} = \int_\Omega \boldsymbol{\sigma} \cdot \tau\, dx, \quad (\boldsymbol{\sigma}, \tau)_{\mathcal{H}_1} = \int_\Omega \boldsymbol{\sigma} \cdot \tau\, dx + \int_\Omega \mathrm{Div}\, \boldsymbol{\sigma} \cdot \mathrm{Div}\, \tau\, dx,$$

and the associated norms $\|\cdot\|_{L^2(\Omega)^d}$, $\|\cdot\|_{H^1(\Omega)^d}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Here and below we use the notation

$$\nabla \boldsymbol{v} = (\boldsymbol{v}_{i,j}), \quad \boldsymbol{\varepsilon}(\boldsymbol{v}) = (\boldsymbol{\varepsilon}_{ij}(\boldsymbol{v})), \quad \boldsymbol{\varepsilon}_{ij}(\boldsymbol{v}) = \frac{1}{2}(\boldsymbol{v}_{i,j} + \boldsymbol{v}_{j,i}) \quad \forall\, \boldsymbol{v} \in H^1(\Omega)^d,$$

$$\mathrm{Div}\,\tau = (\tau_{ij,j}) \quad \forall\, \tau \in \mathcal{H}_1.$$

For every element $\boldsymbol{v} \in H^1(\Omega)^d$ we also write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by $\boldsymbol{v}_\nu$ and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ given by $\boldsymbol{v}_\nu = \boldsymbol{v} \cdot \nu$, $\boldsymbol{v}_\tau = \boldsymbol{v} - \boldsymbol{v}_\nu \nu$.

Let now consider the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \{\ \boldsymbol{v} \in H^1(\Omega)^d \mid \boldsymbol{v} = 0 \text{ on } \Gamma_1\ \}.$$

Since $meas\,(\Gamma_1) > 0$, the following Korn's inequality holds:

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \geq c_K\, \|\boldsymbol{v}\|_{H^1(\Omega)^d} \quad \forall\, \boldsymbol{v} \in V, \tag{3.1}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. Over the space V we consider the inner product given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \tag{3.2}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (3.1) that $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent norms on V and, therefore, $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, (3.1) and (3.2), there exists a constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leq c_0 \|\boldsymbol{v}\|_V \qquad \forall \boldsymbol{v} \in V. \tag{3.3}$$

We also introduce the following spaces.

$$W = \{\ \psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_a\ \}, \quad \mathcal{W}_1 = \{\ D = (D_i) \mid D_i \in L^2(\Omega),\ D_{i,i} \in L^2(\Omega)\ \}.$$

Since $meas\,(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\psi\|_{L^2(\Omega)^d} \geq c_F\, \|\psi\|_{H^1(\Omega)} \quad \forall\, \psi \in W, \tag{3.4}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a and $\nabla\psi = (\psi_{,i})$. Over the space W, we consider the inner product given by

$$(\varphi, \psi)_W = \int_\Omega \nabla\varphi \cdot \nabla\psi \, dx$$

and let $\|\cdot\|_W$ be the associated norm. It follows from (3.4) that $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_W$ are equivalent norms on W and therefore $(W, \|\cdot\|_W)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant c_0, depending only on Ω, Γ_a and Γ_C, such that

$$\|\zeta\|_{L^2(\Gamma_C)} \le \tilde{c}_0 \|\zeta\|_W \quad \forall\, \zeta \in W. \tag{3.5}$$

The space $\mathcal{W}_1$ is real Hilbert space with the inner product

$$(D, \boldsymbol{E})_{\mathcal{W}_1} = \int_\Omega D \cdot \boldsymbol{E}\, dx + \int_\Omega \operatorname{div} D \cdot \operatorname{div} \boldsymbol{E}\, dx,$$

where $\operatorname{div} = (D_{i,i})$, and the associated norm $\|\cdot\|_{\mathcal{W}_1}$.

For every real Hilbert space X we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $1 \le p \le \infty$, $k \ge 1$ and we also introduce the set

$$\mathcal{Q} = \{\, \theta \in L^\infty(0,T;L^2(\Gamma_3)) \mid 0 \le \theta(t) \le 1 \;\; \forall\, t \in [0,T], \text{ a.e. on } \Gamma_3 \,\}.$$

Finally, if X_1 and X_2 are two Hilbert spaces endowed with the inner products $(\cdot,\cdot)_{X_1}$ and $(\cdot,\cdot)_{X_2}$ and the associated norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$, respectively, we denote by $X_1 \times X_2$ the product space together with the canonical inner product $(\cdot,\cdot)_{X_1\times X_2}$ and the associated norm $\|\cdot\|_{X_1\times X_2}$.

In the study of the problem $\mathcal{P}$, we consider the following assumptions on the problem data.

The elasticity operator $\mathcal{F}$, the piezoelectric operator $\mathcal{E}$ and the electric permittivity operator $\mathcal{B}$ satisfy

$$\begin{cases} \text{(a) } \mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d, \\ \text{(b) there exists } L_{\mathcal{F}} > 0 \text{ such that} \\ \quad \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{a.e. } \boldsymbol{x} \in \Omega, \\ \text{(c) there exists } m > 0 \text{ such that} \\ \quad (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2), \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{a.e. } \boldsymbol{x} \in \Omega, \\ \text{(d) the mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable in } \Omega, \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \\ \text{(e) the mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, 0) \in \mathcal{H} \end{cases} \tag{3.6}$$

$$\begin{cases} \text{(a) } \mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d. \\ \text{(b) } \mathcal{E}(\boldsymbol{x}, \boldsymbol{\tau}) = (e_{ijk}(\boldsymbol{x}) \tau_{jk}) \quad \forall \boldsymbol{\tau} = (\boldsymbol{\tau}_{ij}) \in \mathbb{S}^d, \quad \text{a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) } e_{ijk} = e_{ikj} \in L^\infty(\Omega). \end{cases} \tag{3.7}$$

$$\begin{cases} \text{(a) } \mathcal{B} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d. \\ \text{(b) } \mathcal{B}(\boldsymbol{x}, \boldsymbol{E}) = (b_{ij}(\boldsymbol{x}) E_j) \quad \forall \boldsymbol{E} = (E_i) \in \mathbb{R}^d, \quad \text{a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) } b_{ij} = b_{ji} \in L^\infty(\Omega). \\ \text{(d) There exists } m_{\mathcal{B}} > 0 \text{ such that } b_{ij}(\boldsymbol{x}) E_i E_j \geq m_{\mathcal{B}} \|\boldsymbol{E}\|^2 \\ \qquad \forall \boldsymbol{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \end{cases} \tag{3.8}$$

From the assumptions (3.7) and (3.8), we deduce that the piezoelectric operator $\mathcal{E}$ and the electric permittivity operator $\mathcal{B}$ are linear, have measurable bounded components denoted e_{ijk} and b_{ij}, respectively, and moreover, $\mathcal{B}$ is symmetric and positive definite.

Recall also that the transposed operator $\mathcal{E}^*$ is given by $\mathcal{E}^* = (e^*_{ijk})$ where $e^*_{ijk} = e_{kij}$, and the following equality holds :

$$\mathcal{E}\boldsymbol{\sigma} \cdot \boldsymbol{v} = \boldsymbol{\sigma} \cdot \mathcal{E}^* \boldsymbol{v} \qquad \forall \boldsymbol{\sigma} \in \mathbb{S}^d, \ \boldsymbol{v} \in \mathbb{R}^d. \tag{3.9}$$

The friction function satisfies :

$$\left\{\begin{array}{l} \quad p:\,\Gamma_3\times\mathbb{R}\rightarrow\mathbb{R}_+ \text{ verifies} \\ (a)\ \text{there exists } M>0 \text{ such that :} \\ \quad |p\,(x,r_1)-p\,(x,r_2)|\leq M\,|r_{1-}r_2| \\ \quad \text{for every } r_1,r_2\in\mathbb{R}, \quad \text{a.e. } x\in\Gamma_3; \\ (b)\ x\mapsto p\,(x,r)\ \text{ is measurable on } \Gamma_3,\ \text{for every } r\in\mathbb{R}; \\ (c)\ p\,(x,0)=0, \quad \text{a.e. } x\in\Gamma_3. \end{array}\right. \tag{3.10}$$

We note that (3.10) is satisfied in the case of function p given by (2.14).

We also suppose that the body forces and surface tractions have the regularity

$$f_0\in W^{1,\infty}(0,T;L^2(\Omega)^d), \quad f_2\in W^{1,\infty}(0,T;L^2(\Gamma_2)^d), \tag{3.11}$$

and the densities of electric charges satisfy

$$q_0\in W^{1,\infty}(0,T;L^2(\Omega)), \quad q_2\in W^{1,\infty}(0,T;L^2(\Gamma_b)), \tag{3.12}$$

Note that we need to impose assumption (3.12) for physical reasons; indeed, the foundation is supposed to be insulator and therefore the electric boundary conditions on Γ_3 do not have to change in function of the status of the contact, are the same on the contact and on the separation zone, and are included in the boundary condition (2.11).

We define the function $f:[0,T]\rightarrow V$ and $q:[0,T]\rightarrow W$ by

$$(f(t),\boldsymbol{v})_V=\int_\Omega f_0(t)\cdot\boldsymbol{v}\,dx+\int_{\Gamma_2} f_2(t)\cdot\boldsymbol{v}\,da, \tag{3.13}$$

$$(q(t),\psi)_W=\int_\Omega q_0(t)\psi\,dx-\int_{\Gamma_b} q_2(t)\psi\,da,$$

for all $\boldsymbol{u},\boldsymbol{v}\in V$, $\psi\in W$ and $t\in[0,T]$, and note that conditions (3.11) and (3.12) imply that

$$f\in W^{1,\infty}(0,T;V), \qquad q\in W^{1,\infty}(0,T;W). \tag{3.14}$$

The adhesion coefficients γ_ν, γ_τ and the limit bound ϵ_a satisfy the conditions

$$\gamma_\nu,\ \gamma_\tau\in L^\infty(\Gamma_3), \quad \epsilon_a\in L^2(\Gamma_3), \quad \gamma_\nu,\,\gamma_\tau,\,\epsilon_a\geq 0 \quad \text{a.e. on } \Gamma_3 \tag{3.15}$$

while the friction coefficient μ is such that

$$\mu \in L^\infty(\Gamma_3), \quad \mu(x) \geq 0 \quad \text{a.e. on } \Gamma_3 \tag{3.16}$$

and finally, the initial condition β_0 satisfies

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1 \;\; \text{a.e. on } \Gamma_3. \tag{3.17}$$

We denote by U_{ad} the convex subset of admissible displacements fields given by

$$U_{ad} = \{\boldsymbol{v} \in H_1 \mid \;\; \boldsymbol{v} = 0 \text{ on } \Gamma_1\,,\;\; \boldsymbol{v}_\nu \leq 0 \text{ on } \Gamma_3\}\,. \tag{3.18}$$

We define the adhesion functional $j_{ad} : L^2(\Gamma_3) \times V \times V \to \mathbb{R}$ by

$$j_{ad}(\beta, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \big(-\gamma_\nu \beta^2 R_\nu(\boldsymbol{u}_\nu)\boldsymbol{v}_\nu + \gamma_\tau \beta^2 R_\tau(\boldsymbol{u}_\tau)\cdot \boldsymbol{v}_\tau \big)\, da, \tag{3.19}$$

and the friction functional $j_{fr} : L^2(\Gamma_3) \times \mathcal{H}_1 \times V \times V \to \mathbb{R}$ by

$$j_{fr}(\beta, \boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \mu p(|R(\boldsymbol{\sigma}_\nu) - \gamma_\nu \beta^2 R_\nu(\boldsymbol{u}_\nu)|)\cdot |\boldsymbol{v}_\tau|\, da. \tag{3.20}$$

By a standard procedure based on Green's formula we can derive the following variational formulation of the contact problem (2.1)-(2.13).

Problem 3.1 ($\mathcal{P}^V$)**.** *Find a displacement field $\boldsymbol{u} : [0,T] \longrightarrow V$, an electric potential field $\varphi : [0,T] \to W$ and a bonding field $\beta : [0,T] \to L^2(\Gamma_3)$ such that*

$$\begin{aligned} &\boldsymbol{u}(t) \in U_{ad} \quad (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}(t)))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi(t), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}(t))_{\mathcal{H}} + \\ &+ j_{ad}(\beta(t), \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) + j_{fr}(\beta(t), \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{E}^*\nabla\varphi(t), \boldsymbol{u}(t), \boldsymbol{v}) - \\ &- j_{fr}(\beta(t), \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{E}^*\nabla\varphi(t), \boldsymbol{u}(t), \boldsymbol{u}(t)) \geq (f(t), y - \boldsymbol{u}(t))_V \quad \forall \boldsymbol{v} \in U_{ad},\; t \in [0,T], \end{aligned} \tag{3.21}$$

$$(\mathcal{B}\nabla\varphi(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \nabla\psi)_{L^2(\Omega)^d} = (q(t), \psi)_W, \quad \forall \psi \in W, \forall t \in [0,T], \tag{3.22}$$

$$\dot{\beta}(t) = -(\beta(t)\,(\gamma_\nu R_\nu(\boldsymbol{u}_\nu(t))^2 + \gamma_\tau \|R_\tau(\boldsymbol{u}_\tau(t))\|^2) - \epsilon_a)_+ \;\; \textit{a.e. } t \in (0,T), \tag{3.23}$$

$$\beta(0) = \beta_0. \tag{3.24}$$

In the rest of this section, we derive some inequalities involving the functionals j_{ad}, and j_{fr} which will be used in the following sections. Below in this section β, β_1, β_2 denote elements of $L^2(\Gamma_3)$ such that $0 \leq \beta,\ \beta_1,\ \beta_2 \leq 1$ a.e. on Γ_3, $\boldsymbol{u}_1$, $\boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2$, $\boldsymbol{u}$ and $\boldsymbol{v}$ represent elements of V; $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ denote elements of $\mathcal{H}_1$ and c is a generic positive constants which may depend on Ω, Γ_1, Γ_3, p, γ_ν, γ_τ and L, whose value may change from place to place. For the sake of simplicity, we suppress in what follows the explicit dependence on various functions on $x \in \Omega \cup \Gamma_3$.

First, we remark that the j_{ad} is linear with respect to the last argument and therefore

$$j_{ad}(\beta, \boldsymbol{u}, -\boldsymbol{v}) = -j_{ad}(\beta, \boldsymbol{u}, \boldsymbol{v}). \tag{3.25}$$

Next, using (3.19) and the inequalities $|R_\nu(\boldsymbol{u}_{1\nu})| \leq L,\ \|R_\tau(\boldsymbol{u}_\tau)\| \leq L,$

$|\beta_1| \leq 1,\ |\beta_2| \leq 1$, for the previous inequality, we deduce that

$$j_{ad}(\beta_1, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{ad}(\beta_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leq c \int_{\Gamma_3} |\beta_1 - \beta_2| \, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\| \, da,$$

then, we combine this inequality with (3.3), to obtain

$$j_{ad}(\beta_1, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{ad}(\beta_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leq c \, \|\beta_1 - \beta_2\|_{L^2(\Gamma_3)} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V. \tag{3.26}$$

Next, we choose $\beta_1 = \beta_2 = \beta$ in (3.26) to find

$$j_{ad}(\beta, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{ad}(\beta, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leq 0. \tag{3.27}$$

Similar manipulations, based on the Lipschitz continuity of operators R_ν, R_τ show that

$$|j_{ad}(\beta, \boldsymbol{u}_1, \boldsymbol{v}) - j_{ad}(\beta, \boldsymbol{u}_2, \boldsymbol{v})| \leq c \, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}\|_V. \tag{3.28}$$

Also, we take $\boldsymbol{u}_1 = \boldsymbol{v}$ and $\boldsymbol{u}_2 = 0$ in (3.27), then we use the equalities $R_\nu(0) = 0$, $R_\tau(0) = 0$ and (3.26) to obtain

$$j_{ad}(\beta, \boldsymbol{v}, \boldsymbol{v}) \geq 0. \tag{3.29}$$

Next, we use (3.20), (3.10)(a), keeping in mind (3.3), propriety of a normal regularization operator and the inequalities $|R_\nu(\boldsymbol{u}_\nu)| \leq L,\ |\beta_1| \leq 1,\ |\beta_2| \leq 1$ and the regularity of the operator R we obtain

$$j_{fr}(\beta_1,\boldsymbol{\sigma}_1,\boldsymbol{u}_1,\boldsymbol{v}_2)-j_{fr}(\beta_1,\boldsymbol{\sigma}_1,\boldsymbol{u}_1,\boldsymbol{v}_1)+j_{fr}(\beta_2,\boldsymbol{\sigma}_2,\boldsymbol{u}_2,\boldsymbol{v}_1)-j_{fr}(\beta_2,\boldsymbol{\sigma}_2,\boldsymbol{u}_2,\boldsymbol{v}_2)\leq$$
$$\leq c_0^2M\|\mu\|_{L^\infty(\Gamma_3)}(||\boldsymbol{u}_2-\boldsymbol{u}_1\|_V+c(\|\beta_2-\beta_1\|_{L^2(\Gamma_3)}+\|\boldsymbol{\sigma}_{2-}\boldsymbol{\sigma}_1||_{\mathcal{H}_1}))||\boldsymbol{v}_2-\boldsymbol{v}_1\|_V. \tag{3.30}$$

now, by using (3.10)(a) and (3.16), it follows that the integral in (3.20) is well defined. Moreover, we have

$$j_{fr}(\beta,\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{v})\leq c_0^2M\|\mu\|_{L^\infty(\Gamma_3)}\|(||\boldsymbol{u}\|_V\|+c(\|\boldsymbol{\sigma}||_{\mathcal{H}_1}+\|\beta\|_{L^2(\Gamma_3)}))\boldsymbol{v}\|_V. \tag{3.31}$$

The inequalities (3.26)-(3.31) combined with equalities (3.25) will be used in various places in the rest of the paper.

4. **Existence and uniqueness result**

Our main result which states the unique solvability of Problem $\mathcal{P}^V$, is the following.

Theorem 4.1. *Assume that* (3.6)-(3.8), (3.10) *and* (3.15)-(3.17) *hold. Then, there exists* $\mu_0>0$ *depending only on* $\Omega,\Gamma_1,\Gamma_3,F,$ B *and* p *such that, if* $\|\mu\|_{L^\infty(\Gamma_3)}<\mu_0$, *then Problem* $\mathcal{P}^V$ *has a unique solution* $(\boldsymbol{u},\varphi,\beta)$*. Moreover, the solution satisfies*

$$\boldsymbol{u}\in W^{1,\infty}(0,T;V), \tag{4.1}$$

$$\varphi\in W^{1,\infty}(0,T;W). \tag{4.2}$$

$$\beta\in W^{1,\infty}(0,T;L^2(\Gamma_3))\cap\mathcal{Q}. \tag{4.3}$$

A "quintuple" of functions $(\boldsymbol{u}$, $\boldsymbol{\sigma}$, φ, D, $\beta)$ which satisfy (2.1), (2.2), (3.21)-(3.24) is called a *weak solution* of the contact problem $\mathcal{P}$. We conclude by Theorem 4.1 that, under the stated assumptions, Problem $\mathcal{P}$ has a unique weak solution. To precise the regularity of the weak solution we note that the constitutive relations (2.1) and (2.2), the assumptions (3.6), (3.8) and the regularities (4.1), (4.2) show that $\boldsymbol{\sigma}\in W^{1,\infty}([0,T];\mathcal{H})$, $\boldsymbol{D}\in W^{1,\infty}([0,T];L^2(\Omega)^d)$; moreover, (3.21), (3.22) combined with the definitions of $\boldsymbol{f}$ q and functionals j_{ad} and j_{fr} yield

$$\operatorname{Div}\boldsymbol{\sigma}(t)+\boldsymbol{f}_0(t)=\boldsymbol{0},\quad \operatorname{div}\boldsymbol{D}(t)=q_0(t)\quad\forall\, t\in[0,T].$$

It follows now from the regularities (3.11), (3.9) that $\mathrm{Div}\,\boldsymbol{\sigma} \in W^{1,\infty}(0,T;L^2(\Omega)^d)$ and $\mathrm{div}\,\boldsymbol{D} \in W^{1,\infty}(0,T;L^2(\Omega))$, which shows that

$$\boldsymbol{\sigma} \in W^{1,\infty}(0,T;\mathcal{H}_1), \tag{4.4}$$

$$\boldsymbol{D} \in W^{1,\infty}(0,T;\mathcal{W}_1). \tag{4.5}$$

We conclude that the weak solution $(\boldsymbol{u},\boldsymbol{\sigma},\varphi,\boldsymbol{D},\beta)$ of the piezoelectric contact problem $\mathcal{P}$ has the regularity (4.1), (4.2), (4.3), (4.4) and (4.5).

The proof of Theorem 4.1 is carried out in several steps and is based on the following abstract result for variational inequalities.

Let X be a real Hilbert space with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$, and consider the problem of finding $u \in X$ such that

$$(Au, v-u)_X + j(u,v) - j(u,u) \geqslant (f, v-u) \quad \forall v \in X. \tag{4.6}$$

To study problem (4.6) we need the following assumptions: The operator $A : X \to X$ is strongly monotone and Lipschitz continuous, i.e.,

$$\begin{cases} \text{(a) There exists } m_A > 0 \text{ such that} \\ \quad (Au_1 - Au_2, u_1 - u_2)_X \geq m_A \|u_1 - u_2\|_X^2 \quad \forall\, u_1, u_2 \in X. \\ \text{(b) There exists } L_A > 0 \text{ such that} \\ \quad \|Au_1 - Au_2\|_X \leq L_A \|u_1 - u_2\|_X \quad \forall\, u_1, u_2 \in X. \end{cases} \tag{4.7}$$

The functional $j : X \times X \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a) } j(u,\cdot) \text{ is convex and l.s.c. on } X \text{ for all } u \in X. \\ \text{(b) There exists } m > 0 \text{ such that} \\ \quad j(u_1,v_2) - j(u_1,v_1) + j(u_2,v_1) - j(u_2,v_2) \\ \quad\quad \leq m\,\|u_1 - u_2\|_X\,\|v_1 - v_2\|_X \quad \forall\, u_1,u_2,v_1,v_2 \in X. \end{cases} \tag{4.8}$$

Finally, we assume that

$$f \in X \tag{4.9}$$

The following existence, uniqueness was proved in [28].

Theorem 4.2. *Assume that (4.7), (4.8) and (4.9) hold. Then, if $m < m_A$, for all $f \in X$, there exists a unique solution $u \in Y$ of Problem 4.6.*

We return now to proof of theorem 4.1. To this end, we assume in the following that (3.6)-(3.8), (3.10)-(3.12) and (3.15)-(3.17) hold; below, c is a generic positive constants which may depend on Ω, Γ_1, Γ_3, $\mathcal{F}$, p, γ_ν, γ_τ and L, whose value may change from place to place. For the sake of simplicity, we suppress in what follows the explicit dependence on various functions on $\boldsymbol{x} \in \Omega \cup \Gamma_3$.

Let $\mathcal{L}$ denotes the closed set of the space $C([0,T]; L^2(\Gamma_3))$ defined by

$$\mathcal{L} = \left\{ \beta \in C([0,T]; L^2(\Gamma_3)) \cap \mathcal{Q} \mid \beta(0) = \beta_0 \right\} \tag{4.10}$$

and let $\beta \in \mathcal{L}$ and $g \in W^{1,\infty}(0,T;\mathcal{H}_1)$ are given. In the first step, we consider the following variational problem.

Problem 4.3 $(\mathcal{P}_{\beta g})$**.** *Find a displacement field* $\boldsymbol{u} : [0,T] \to V$*, an electric potential field* $\varphi : [0,T] \to W$ *such that*

$$\boldsymbol{u}_{\beta g}(t) \in U_{ad}, \quad (\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{\beta g}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{u}_{\beta g}(t))_{\mathcal{H}} + (\mathcal{E}^* \nabla \varphi_{\beta g}(t), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}_{\beta g}(t)))_{\mathcal{H}} +$$
$$+ j_{ad}(\beta(t), \boldsymbol{u}_{\beta g}(t), \boldsymbol{v} - \boldsymbol{u}_{\beta g}(t)) + j_{fr}(\beta(t), g(t), \boldsymbol{u}_{\beta g}(t), \boldsymbol{v}) -$$
$$- j_{fr}(\beta(t), g(t), \boldsymbol{u}_{\beta g}(t), \boldsymbol{u}_{\beta g}(t)) \geq (f(t), \boldsymbol{v} - \boldsymbol{u}_{\beta g}(t))_V \quad \forall \boldsymbol{v} \in U_{ad}, \tag{4.11}$$

$$(\mathcal{B} \nabla \varphi_{\beta g}(t), \nabla \psi)_{L^2(\Omega)^d} - (\mathcal{E} \boldsymbol{\varepsilon}(\boldsymbol{u}_{\beta g}(t)), \nabla \psi)_{L^2(\Omega)^d} = (q(t), \psi)_W \quad \forall\, \psi \in W. \tag{4.12}$$

In order to solve Problem $\mathcal{P}_{\beta g}$ we consider the product space $X = V \times W$ endowed with the inner product

$$(x, y)_X = (\boldsymbol{u}, \boldsymbol{v})_V + (\varphi, \psi)_W \qquad \forall x = (\boldsymbol{u}, \varphi),\ y = (\boldsymbol{v}, \psi) \in X \tag{4.13}$$

and the associated norm $||.||_X$. We also introduce the set $K \subset X$ and the function $A_{\beta g} : [0,T] \times X \to X$, $\mathbf{f}\ [0,T] \to X$, defined by

$$K = U_{ad} \times W, \tag{4.14}$$

$$(A_{\beta g}(t)x, y)_X = (\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\mathcal{B}\nabla\varphi(t), \nabla\psi)_{L^2(\Omega)^d} + \tag{4.15}$$

$$(\mathcal{E}^*\nabla\varphi(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \nabla\psi)_{L^2(\Omega)^d} \tag{4.16}$$

$$+j_{ad}(\beta(t), \boldsymbol{u}(t), \boldsymbol{v}) \quad \forall x = (\boldsymbol{u}, \varphi)_V,\ y = (\boldsymbol{v}, \psi)_W \in X, \quad t \in [0, T],$$

$$j_{\beta g}(x, y) = j_{fr}(\beta(t), g(t), \boldsymbol{u}(t), \boldsymbol{u}(t)) \quad \forall x = (\boldsymbol{u}, \varphi),\ y = (\boldsymbol{v}, \psi) \in X \tag{4.17}$$

$$\mathbf{f}{=}(f(t), q(t)) \ \ \forall t \in [0, T]. \tag{4.18}$$

We start with the following equivalence result.

Lemma 4.4. *The couple* $(x_{\beta g}, \varphi_{\beta g})$: $[0, T] \to V \times W$ *is a solution to Problem* $\mathcal{P}_{\beta g}$ *if and only if* $x_{\beta g}$: $[0, T] \to X$ *satisfies*

$$x_{\beta g} \in K, \quad (A_{\beta g}(t)x_\beta(t), y - x_{\beta g}(t))_X + j_{\beta g}(x_{\beta g}(t), y(t)) - \tag{4.19}$$

$$-j_{\beta g}(x_{\beta g}(t), x_{\beta g}(t)) \geqslant (\mathbf{f(t)}, y - x_{\beta g}(t))_X \quad \forall y \in K, \ \textit{for all}\ t \in [0, T].$$

Proof. Let $x_{\beta g} = (\boldsymbol{u}_{\beta g}, \varphi_{\beta g})$: $[0, T] \to V \times W$ be a solution to Problem $\mathcal{P}_{\beta g}$. Let $y = (\boldsymbol{v}, \psi) \in K$ and let $t \in [0, T]$. We use the test function $\psi - \varphi_{\beta g}(t)$ in (4.12), add the corresponding inequality to (4.11), and use (4.13)-(4.18) to obtain (4.19). Conversely, assume that $x_{\beta g} = (\boldsymbol{u}_{\beta g}, \varphi_{\beta g})$: $[0, T] \to X$ satisfies (4.19) and let $t \in [0, T]$. For any $\boldsymbol{v} \in U_{ad}$, we take $y = (\boldsymbol{v}, \varphi_{\beta g}(t))$ in (4.19) to obtain (4.11). Then, for any $\psi \in W$, we take successively $y = (\boldsymbol{u}_{\beta g}, \varphi_{\beta g}(t) + \psi)$ and $y = (\boldsymbol{u}_{\beta g}, \varphi_{\beta g}(t) - \psi)$ in (4.19) to obtain (4.12). □

We use now Lemma 4.4 to obtain the following existence and uniqueness result.

Lemma 4.5. *There exists* $\mu_0 > 0$ *depending only on* $\Omega, \Gamma_1, \Gamma_3, \mathcal{F}$, $\mathcal{B}$ *and* p *such that, if* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, *Problem* $\mathcal{P}_{\beta g}$ *has a unique solution* $(\boldsymbol{u}_{\beta g}, \varphi_{\beta g}) \in C([0, T]; V \times W)$.

Proof. We apply Theorem 4.2 where $X = V \times W$ and $Y = K = U_{ad} \times W$. Let $t \in [0, T]$.We use (3.6)-(3.9), (3.28), and (3.29) to see that $A_{\beta g}(t)$ is a strongly monotone Lipschitz continuous operator on X and it satisfies

$$(A_{\beta g}(t)x_1(t) - A_{\beta g}(t)x_2(t), x_1(t) - x_2(t))_X \geqslant \min(m_{\mathcal{F}}, m_{\mathcal{B}})||x_1(t) - x_2(t)||_X. \tag{4.20}$$

Using (3.20), we can easily check that $j_{\beta g}(x,.)$ is a continuous seminorm on X and moreover, it satisfies (3.30) and(3.31) which shows that the functional $j_{\beta g}$ satisfies condition (4.8) on $.X$. By (3.14) and (4.18) it is easy to see that the function $\mathbf{f}$ defined by (4.18) satisfies $\mathbf{f}(t) \in X$.

Let

$$\mu_0 = \frac{\min(m_{\mathcal{F}}, m_{\mathcal{B}})}{c_0^2 M},$$

where μ, $m_{\mathcal{F}}$, $m_{\mathcal{B}}$, c_0 and M are given in (2.8), (3.6), (3.8), (3.3) and (3.10), respectively. We note that μ_0 depends on $\Omega, \Gamma_1, \Gamma_3, \mathcal{F}$, $\mathcal{B}$ *and* p. Assume that $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, then

$$c_0^2 M \|\mu\|_{L^\infty(\Gamma_3)} < \min(m_{\mathcal{F}}, m_{\mathcal{B}}), \tag{4.21}$$

and note that this smallness assumption involves only the geometry and the electrical part, and does not depend on the mechanical data of problem.

Using (3.30), (3.31), 4.20, the existence and uniqueness part in Lemma 4.5 is now a consequence of Lemma 4.4 and theorem 4.2.

For $t_1, t_2 \in [0,T]$, an argument based on (3.6), (3.28) and (3.30) shows that

$$\begin{aligned} ||x_{\beta g}(t_2) - x_{\beta g}(t_1)||_X \le \frac{c}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2 \|\mu\|_{L^\infty(\Gamma_3)} M} (||\beta(t_2) - \beta(t_1)||_{L^2(\Gamma_3)} + \\ + ||g(t_2) - g(t_1)||_{\mathcal{H}_1} + ||\mathbf{f}(t_2) - \mathbf{f}(t_1)||_X). \end{aligned} \tag{4.22}$$

The last inequality implies that

$$\begin{aligned} ||u(t_2) - u(t_1)||_V \quad \le \quad & \frac{c}{m_{\mathcal{F}} - c_0^2 \|\mu\|_{L^\infty(\Gamma_3)} M} (||\beta(t_2) - \beta(t_1)||_{L^2(\Gamma_3)} + \\ & + ||g(t_2) - g(t_1)||_{\mathcal{H}_1} + ||\mathbf{f}(t_2) - \mathbf{f}(t_1)||_X). \end{aligned} \tag{4.23}$$

Keeping in mind that $\mathbf{f} \in W^{1,\infty}(0,T;X)$ and recall that $\beta \in C([0,T];X)$, $g \in W^{1,\infty}(0,T;\mathcal{H}_1)$, it follows now from (4.22) that the mapping $t \to x_{\beta g} = (\boldsymbol{u}_{\beta g}, \varphi_{\beta g}) : [0,T] \to X$ is continuous. □

We assume in what follows that $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$ and therefore (4.21) is valid. In the next step, we use the displacement field $\boldsymbol{u}_{\beta g}$ obtained in Lemma 4.5, denote by $\boldsymbol{u}_{\beta g \nu}$, $\boldsymbol{u}_{\beta g \tau}$ its normal and tangential components, and we consider the following initial value problem.

Problem 4.6 ($\mathcal{P}^{\theta}_{\beta g}$)**.** *Find a bonding field* $\theta_{\beta g}$: $[0,T] \to L^2(\Gamma_3)$ *such that*

$$\dot{\theta}_{\beta g}(t) = -\Big(\theta_{\beta g}(t)\big(\gamma_\nu R_\nu(\boldsymbol{u}_{\beta g\nu}(t))^2 + \gamma_\tau \|R_\tau(\boldsymbol{u}_{\beta g\tau}(t))\|^2\big) - \epsilon_a\Big)_+ \quad \text{a.e. } t \in (0,T), \tag{4.24}$$

$$\theta_{\beta g}(0) = \beta_0. \tag{4.25}$$

We obtain the following result.

Lemma 4.7. *There exists a unique solution to Problem* $\mathcal{P}^{\theta}_{\beta g}$ *and it satisfies* $\theta_{\beta g} \in W^{1,\infty}(0,T,L^2(\Gamma_3)) \cap \mathcal{Q}$

Proof. Consider the mapping $F_{\beta g} : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_{\beta g}(t,\theta) = -(\theta(t)(\gamma_\nu R_\nu((\boldsymbol{u}_{\beta g})_\gamma(t))^2 + \gamma_\tau \|R_\tau((\boldsymbol{u}_{\beta g})_\tau(t))\|^2) - \epsilon_a)_+, \tag{4.26}$$

for all $t \in [0,T]$ and $\theta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operators R_ν and R_τ that F_β is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, for any $\theta \in L^2(\Gamma_3)$, the mapping $t \mapsto F_{\beta g}(t,\theta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Using now a version of Cauchy-Lipschitz theorem, we obtain the existence of a unique function $\theta_{\beta g} \in W^{1,\infty}(0,T,L^2(\Gamma_3))$ which solves (4.24), 4.25). We note that the restriction $0 \le \beta \le 1$ is implicitly included in the variational problem $\mathcal{P}_{\mathbf{V}}$. Indeed, (3.23) and (3.24) guarantee that $\beta(t) \le \beta_0$ and, therefore, assumption (3.17) shows that $\beta(t) \le 1$ for $t \ge 0$, a.e. on Γ_3. On the other hand, if $\beta(t_0) = 0$ at $t = t_0$, then it follows from (3.23) and (3.24) that $\dot{\beta}(t) = 0$ for all $t \ge t_0$ and therefore, $\beta(t) = 0$ for all $t \ge 0$, a.e. on Γ_3. We conclude that $0 \le \beta(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set $\mathcal{Q}$, we find that $\theta_{\beta g} \in \mathcal{Q}$, which concludes the proof of Lemma. □

It follows from Lemma 4.7 that for all $\beta \in \mathcal{L}$ and $g \in W^{1,\infty}(0,T,\mathcal{H}_1)$ the solution $\theta_{\beta g}$ of Problem $\mathcal{P}^{\theta}_{\beta g}$ belongs to $\mathcal{L} \times W^{1,\infty}(0,T,L^2(\Gamma_3))$, see (4.10).

We denote now by $\sigma_{\beta g}$ the tensor given by

$$\sigma_{\beta g} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_{\beta g}) + \mathcal{E}^*\nabla(\varphi_{\beta g}). \tag{4.27}$$

From see (3.6), (3.6) and Lemma 4.5, its follows that $\sigma_{\beta g} \in C(0,T,\mathcal{H}_1)$. Therefore, we may consider the operator $\Lambda : \mathcal{L} \times C(0,T,L^2(\Gamma_3) \times \mathcal{H}_1) \to \mathcal{L} \times C(0,T,L^2(\Gamma_3) \times \mathcal{H}_1)$

given by

$$\Lambda(\beta, g) = (\theta_{\beta g}, \sigma_{\beta g}). \tag{4.28}$$

The third step consists in the following result.

Lemma 4.8. *There exists a unique element* $(\beta^*, g^*) \in \mathcal{L} \times C(0, T, L^2(\Gamma_3) \times \mathcal{H}_1)$ *such that* $\Lambda(\beta^*, g^*) = (\beta^*, g^*)$.

Proof. Suppose that (β_i, g_i) are two couples of functions in $\mathcal{L} \times W^{1,\infty}(0, T, L^2(\Gamma_3) \times \mathcal{H}_1)$ and denote by $(\boldsymbol{u}_i, \varphi_i)$, θ_i the functions obtained in Lemmas 4.5 and 4.7, respectively, for (β, g)=(β_i, g_i), $i = 1, 2$. Let $t \in [0, T]$. We use arguments similar to those used in the proof of (4.22) to deduce that

$$||x_{\beta_1 g_1}(t_2) - x_{\beta_2 g_2}(t_1)||_X \leq$$
$$\leq \frac{c}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2\|\mu\|_{L^\infty(\Gamma_3)}M}(||\beta_2(t) - \beta_1(t)||_{L^2(\Gamma_3)} + ||g_2(t) - g_1(t)||_{\mathcal{H}_1}), \tag{4.29}$$

which implies

$$||\boldsymbol{u}_2(t) - \boldsymbol{u}_1(t))||_V \leq \frac{c}{m_{\mathcal{F}} - c_0^2\|\mu\|_{L^\infty(\Gamma_3)}M}(||\beta_2(t) - \beta_1(t)||_{L^2(\Gamma_3)} + ||g_2(t) - g_1(t)||_{\mathcal{H}_1}). \tag{4.30}$$

On the other hand, it follows from (4.24) and (4.25) that

$$\theta_i(t) = \beta_0 - \int_0^t (\theta_i(s)(\gamma_\nu R_\nu(\boldsymbol{u}_{i\nu}(s))^2 + \gamma_\tau\|R_\tau(\boldsymbol{u}_{i\tau}(s))\|^2) - \epsilon_a)_+ ds \tag{4.31}$$

and then

$$||\theta_2(t) - \theta_1(t)||_{L^2(\Gamma_3)} \leq c(\int_0^t ||\theta_2(s)R_\nu(\boldsymbol{u}_{2\nu}(s))^2 - \theta_1(s)R_\nu(\boldsymbol{u}_{1\nu}(s))^2||_{L^2(\Gamma_3)}ds +$$
$$+ \int_0^t ||\theta_2(s)||R_\tau(\boldsymbol{u}_{2\tau}(s))^{||2} - \theta_1(s)||R_\tau(\boldsymbol{u}_{1\tau}(s))||^2||_{L^2(\Gamma_3)}ds). \tag{4.32}$$

Using the definition of R_ν and R_τ and writing , $\theta_1 = \theta_1 - \theta_2 + \theta_2$ we get

$$||\theta_2(t) - \theta_1(t)||_{L^2(\Gamma_3)} \leq c(\int_0^t ||\theta_2(s) - \theta_1(s)||_{L^2(\Gamma_3)}ds + \int_0^t ||\boldsymbol{u}_2(s)) - \boldsymbol{u}_1(s))||_{L^2(\Gamma_3)}ds). \tag{4.33}$$

By Gronwall's inequality, it follows that

$$||\theta_2(t) - \theta_1(t)||_{L^2(\Gamma_3)} \leq c\int_0^t ||\boldsymbol{u}_2(s)) - \boldsymbol{u}_1(s))||_{L^2(\Gamma_3)}ds \tag{4.34}$$

and, using (3.3), we obtain

$$||\theta_2(t)-\theta_1(t)||_{L^2(\Gamma_3)} \leq c\int_0^t ||\boldsymbol{u}_2(s))-\boldsymbol{u}_1(s))||_V ds. \tag{4.35}$$

We now combine (4.30) and (4.35) to see that

$$||\theta_2(t)-\theta_1(t)||_{L^2(\Gamma_3)} \leq$$

$$\leq \frac{c}{\min(m_{\mathcal{F}},m_{\mathcal{B}})-c_0^2\|\mu\|_{L^\infty(\Gamma_3)}M}\int_0^t (||\beta_2(t)-\beta_1(t)||_{L^2(\Gamma_3)}+||g_2(t)-g_1(t)||_{\mathcal{H}_1})ds. \tag{4.36}$$

Using now (3.6), (3.7) and (4.27) (4.29) it is easy to see that

$$\|\sigma_{\beta_1 g_1}(t)-\sigma_{\beta_2 g_2}(t)||\leq \frac{c}{\min(m_{\mathcal{F}},m_{\mathcal{B}})-c_0^2\|\mu\|_{L^\infty(\Gamma_3)}M}(||\beta_2(t)-\beta_1(t)\|+\|g_2(t)-g_1(t)\|). \tag{4.37}$$

From (4.28), (4.30) and the last inequality, it results that

$$||\Lambda(\beta_1,g_1)(t)-\Lambda(\beta_2,g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1} \leq N||(\beta_1,g_1)(t)-(\beta_2,g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1}+$$

$$+c\int_0^t ||(\beta_1,g_1)(t)-(\beta_2,g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1}ds. \tag{4.38}$$

such that :

$$N=\frac{c}{\min(m_{\mathcal{F}},m_{\mathcal{B}})-c_0^2\|\mu\|_{L^\infty(\Gamma_3)}M}. \tag{4.39}$$

Using the following notations

$$\begin{aligned} I_0(t) &= ||(\beta_1,g_1)(t)-(\beta_2,g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1}, \qquad (4.40)\\ I_1(t) &= \int_0^t ||(\beta_1,g_1)(t)-(\beta_2,g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1}ds,\\ I_k(t) &= \int_0^t\int_0^{s_{k-1}}...\int_0^{s_1} ||(\beta_1,g_1)(t)-(\beta_2,g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1}drds_1...ds_{k-1},\ \forall k\geqslant 2, \end{aligned}$$

and denoting now by Λ^p the powers of operator Λ, (4.38) and (4.40) imply by recurrence that

$$\|\Lambda^p(\beta_1,g_1)(t)-\Lambda^p(\beta_2,g_2)(t)\| \leq (\sum_{k=0}^{p} C_p^k\frac{N^{p-k}M^pT^p}{p!}))\,\|(\beta_1,g_1)(t)-(\beta_2,g_2)(t)\|$$

$$\leq \frac{(Np+MT)^p}{p!}\,\|(\beta_1,g_1)(t)-(\beta_2,g_2)(t)\|_{L^2(\Gamma_3)\times\mathcal{H}_1}. \tag{4.41}$$

Using the Stirling's formula, we obtain under the condition $N \leq \dfrac{1}{e}$ that

$$\lim_{p\to\infty} \frac{(Np + MT)^p}{p!} = 0,$$

which shows that for p sufficiently large $\Lambda^p : \mathcal{L} \times C(0, T, \mathcal{H}_1) \to \mathcal{L} \times C(0, T, \mathcal{H}_1)$ is a contraction. Then, we conclude by using the Banach fixed point theorem that Λ has a unique fixed point $(\beta^*, g^*) \in \mathcal{L} \times C(0, T, \mathcal{H}_1)$ such that $\Lambda(\beta^*, g^*) = (\beta^*, g^*)$. Hence, from (4.28) it results for all $t \in [0, T]$,

$$(\beta^*, g^*)(t) = (\theta_{\beta^* g^*}(t), \sigma_{\beta^* g^*}(t)). \tag{4.42}$$

□

Now, we have all the ingredients to provide the proof of Theorem 4.1.

Proof of Theorem 4.1. Existence. Let $(\beta^*, g^*) \in \mathcal{L} \times C(0, T, \mathcal{H}_1)$ be the fixed point of Λ and let $(\boldsymbol{u}^*, \varphi^*)$ be the solution of Problem $\mathcal{P}_{\beta g}$V for $(\beta, g) = (\beta^*, g^*)$, that is, $\boldsymbol{u}^* = \boldsymbol{u}_{\beta^* g^*}$ and $\varphi^* = \varphi_{\beta^* g^*}$. Since $\theta_{\beta^* g^*} = \beta^*$, we conclude by (4.11)), (4.12), (4.24) and (4.25) that $(\boldsymbol{u}^*, \varphi^*, \beta^*)$ is a solution of Problem $\mathcal{P}^V$ and, moreover, β^* satisfies the regularity (4.3). Also, since $\beta^* = \theta_{\beta^*} \in W^{1,\infty}(0, T; L^2(\Gamma_3))$, $\sigma_{\beta^* g^*} \in W^{1,\infty}(0, T, \mathcal{H}_1)$ and $\mathbf{f} \in W^{1,\infty}(0, T; X)$, inequality (4.22) implies that the function $x^* = (\boldsymbol{u}^*, \varphi^*) : [0, T] \to X$ is Lipschitz continuous; therefore, x^* belongs to $W^{1,\infty}(0, T; X)$, which shows that the functions x^* and φ^* have the regularity expressed in (4.1), (4.2).

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator Λ defined by (4.28). Indeed, let $(\boldsymbol{u}, \varphi, \beta)$ be a another solution of Problem $\mathcal{P}^V$ which satisfies (4.1)-(4.3).
We denote by $(\beta, g) \in W^{1,\infty}(0, T, L^2(\Gamma_3) \times \mathcal{H}_1)$ the couple of function defined by

$$\dot{\beta}(t) = -(\beta(t)\,(\gamma_\nu R_\nu(\boldsymbol{u}_\nu(t))^2 + \gamma_\tau \|R_\tau(\boldsymbol{u}_\tau(t))\|^2) - \epsilon_a)_+ \text{ a.e. } t \in (0, T), \tag{4.43}$$

$$\beta(0) = \beta_0, \tag{4.44}$$

$$g(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{E}^*\nabla(\varphi(t)). \tag{4.45}$$

It follows from (4.11), (4.12) that $(\boldsymbol{u}, \varphi)$ is a solution to Problem $\mathcal{P}_{\beta g}$ and, since by Lemma 4.5 this problem has a unique solution denoted by $(\boldsymbol{u}_{\beta g}, \varphi_{\beta g})$, we obtain

$$\boldsymbol{u} = \boldsymbol{u}_{\beta g}, \tag{4.46}$$

$$\varphi = \varphi_{\beta g}. \tag{4.47}$$

Then, we replace $\boldsymbol{u} = \boldsymbol{u}_{\beta g}$ in (3.23) and use the initial condition (3.24) to see that β is a solution to Problem $\mathcal{P}^{\theta}_{\beta g}$. Since by Lemma 4.7 this last problem has a unique solution denoted by $\theta_{\beta g}$, we find

$$\beta = \theta_{\beta g}. \tag{4.48}$$

We use now (4.28), (4.48) and Lemma 4.8, it follows that

$$\beta = \beta^*. \tag{4.49}$$

On a other hand, it follows from (4.46), (4.45), (4.46), (4.46) and Lemma 4.8 that

$$g = g^* \tag{4.50}$$

The uniqueness part of the theorem is now a consequence of (4.46), (4.47), (4.49) and the last inequality. □

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INTEGRAL PROPERTIES OF SOME FAMILIES OF MULTIVALENT FUNCTIONS WITH COMPLEX ORDER

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Abstract. In the present paper, we study integral properties of two families of p-valently analytic functions of complex order defined of the derivative operator of order m. The obtained results improve known results.

1. **Introduction**

Let $\mathcal{A}_p(n)$ denote the class of functions of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0; n, p \in \mathbb{N} := \{1, 2, 3, \cdots\}), \tag{1}$$

which are analytic and p-valent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Upon differentiating both sides (1) m-times with respect to z, we have

$$f^{(m)}(z) = \frac{p!}{(p-m)!} z^{p-m} - \sum_{k=n+p}^{\infty} \frac{k!}{(k-m)!} a_k z^{k-m} \tag{2}$$

where $n, p \in \mathbb{N}; m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; p > m$.

Making use of the function $f^{(m)}(z)$ given by (2), Srivastava and Orhan [1]introduced the subclasses $\mathcal{R}^p_{n,m}(\lambda, b)$ and $\mathcal{L}^p_{n,m}(\lambda, b)$ of the p-valently analytic function class $\mathcal{A}_p(n)$, which consist of functions $f(z)$ satisfying the following inequality, respectively:

$$\left| \frac{1}{b} \left(\frac{z f^{(1+m)}(z) + \lambda z^2 f^{(2+m)}(z)}{\lambda z f^{(1+m)}(z) + (1-\lambda) f^{(m)}(z)} - (p-m) \right) \right| < 1 \tag{3}$$

and

$$\left| \frac{1}{b} \left(f^{(1+m)}(z) + \lambda z f^{(2+m)}(z) - (p-m) \right) \right| < p - m, \tag{4}$$

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where $z \in \mathbb{U}; p \in \mathbb{N}; m \in \mathbb{N}_0; p > m; 0 \leq \lambda \leq 1; b \in \mathbb{C} \setminus \{0\}$.

Also, in [1], Srivastava and Orhan proved the following characterization theorems of these subclasses.

Theorem A. *Let $f(z) \in \mathcal{A}_p(n)$ be given by (1). Then $f(z) \in \mathcal{R}^p_{n,m}(\lambda, b)$ if and only if*

$$\sum_{k=n+p}^{\infty} \frac{(k+|b|-p)k![\lambda(k-m-1)+1]}{(k-m)!} a_k \leq \frac{|b|p![\lambda(p-m-1)+1]}{(p-m)!}. \tag{5}$$

Theorem B. *Let $f(z) \in \mathcal{A}_p(n)$ be given by (1). Then $f(z) \in \mathcal{L}^p_{n,m}(\lambda, b)$ if and only if*

$$\sum_{k=n+p}^{\infty} \binom{k}{m}(k-m)[\lambda(k-m-1)+1]a_k \leq (p-m)\left[\frac{|b|-1}{m!} + \binom{p}{m}[\lambda(p-m-1)+1]\right]. \tag{6}$$

Also, let $\mathcal{I}_c : \mathcal{A}_p(n) \to \mathcal{A}_p(n)$ be an *integral operator* defined by $g = \mathcal{I}_c(f)$, where $c \in (-p, \infty), f \in \mathcal{A}_p(n)$ and

$$g(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt. \tag{7}$$

We note that if $f \in \mathcal{A}_p(n)$ is a function of the form (1), then

$$g(z) = \mathcal{I}_c(f)(z) = z^p - \sum_{k=n+p}^{\infty} \frac{c+p}{c+k} a_k z^k. \tag{8}$$

The main object of the present work is to investigate the integral properties of p-valently functions belonging to the subclasses $\mathcal{R}^p_{n,m}(\lambda, b)$ and $\mathcal{L}^p_{n,m}(\lambda, b)$.

Our properties of the function classes $\mathcal{R}^p_{n,m}(\lambda, b)$ and $\mathcal{L}^p_{n,m}(\lambda, b)$ are motivated essentially by several earlier investigations including in [2].

2. Integral properties of the class $\mathcal{R}^p_{n,m}(\lambda, b)$

Theorem 1. *Let $p \in \mathbb{N}; m \in \mathbb{N}_0; p > m; b \in \mathbb{C} \setminus \{0\}$ and $c \in (-p, \infty)$. If $f \in \mathcal{R}^p_{n,m}(\lambda, b)$ and $g = \mathcal{I}_c(f)$, then $g \in \mathcal{R}^p_{n,m}(\lambda, \gamma)$ where*

$$|\gamma| = \frac{(c+p)|b|}{(c+p+n+|b|)} \tag{9}$$

and $|\gamma| < |b|$. The result is sharp.

Proof. From Theorem A and from (8), we have $g \in \mathcal{R}^p_{n,m}(\lambda,\gamma)$ if and only if

$$\sum_{k=n+p}^{\infty} \frac{(k+|\gamma|-p)k!(p-m)![\lambda(k-m-1)+1](c+p)}{(k-m)!|\gamma|p![\lambda(p-m-1)+1](c+k)} a_k \leq 1. \tag{10}$$

We note that for $k \geq n+p$ the inequalities

$$\frac{(k+|\gamma|-p)k!(p-m)![\lambda(k-m-1)+1](c+p)}{(k-m)!|\gamma|p![\lambda(p-m-1)+1](c+k)}$$
$$\leq \frac{(k+|b|-p)k!(p-m)![\lambda(k-m-1)+1]}{(k-m)!|b|p![\lambda(p-m-1)+1]} \tag{11}$$

imply (10), because $f \in \mathcal{R}^p_{n,m}(\lambda,b)$ and it is satisfies (5). This inequality is equivalent to

$$\frac{(k+|\gamma|-p)(c+p)}{|\gamma|(c+k)} \leq \frac{(k+|b|-p)}{|b|}$$

and we obtain

$$|\gamma| \geq \frac{(k-p)(c+p)|b|}{(k+|b|-p)(c+k)-|b|(c+p)} \ ; \ k \geq n+p \ ; \ \gamma = \gamma(p,k,c,b). \tag{12}$$

And now, we show that $|\gamma|$ is a decreasing function of k, $k \geq n+p$. Indeed, let

$$h(x) = \frac{(x-p)(c+p)|b|}{(x+|b|-p)(c+x)-|b|(c+p)} \quad ; x \in [n+p,\infty) \subset [n,\infty). \tag{13}$$

We have

$$h'(x) = \frac{-(x-p)^2(c+p)|b|}{((x+|b|-p)(c+x)-|b|(c+p))^2} < 0. \tag{14}$$

This implies

$$|\gamma(p,k,c,b)| \leq |\gamma| = |\gamma(p,n+p,c,b)| \ ; k \geq n+p. \tag{15}$$

The result is sharp, because

$$\mathcal{I}_c(f_b) = f_\gamma \tag{16}$$

where

$$f_b(z) = z^p - \frac{|b|p!(n+p-m)![\lambda(p-m-1)+1]}{(p-m)!(n+p)![n+|b|][\lambda(n+p-m-1)+1]} z^{n+p} \tag{17}$$

and

$$f_\gamma(z) = z^p - \frac{|\gamma|p!(n+p-m)![\lambda(p-m-1)+1]}{(p-m)!(n+p)![n+|\gamma|][\lambda(n+p-m-1)+1]} z^{n+p} \tag{18}$$

are extremal functions of $\mathcal{R}^p_{n,m}(\lambda, b)$ and $\mathcal{R}^p_{n,m}(\lambda, \gamma)$, respectively. Indeed we have

$$\mathcal{I}_c(f_b)(z) = z^p - \frac{|b|p!(n+p-m)![\lambda(p-m-1)+1](c+p)}{(p-m)!(n+p)![n+|b|][\lambda(n+p-m-1)+1](c+p+n)} z^{n+p} \tag{19}$$

Thus we deduce

$$\frac{|\gamma|}{n+|\gamma|} = \frac{|b|(c+p)}{[n+|b|](c+p+n)} \tag{20}$$

and this implies (16). From $\frac{|\gamma|}{n+|\gamma|} = \frac{|b|(c+p)}{[n+|b|](c+p+n)}$, we obtain $|\gamma| > 0$. Also, we have $|\gamma| < |b|$. Indeed,

$$|\gamma| - |b| = -\frac{[n+|b|]|b|}{|b| + (c+p+n)} < 0. \tag{21}$$

Remark 1. In Theorem 1, if we take $p = 1, m = 0, b = 1 - \alpha$ and $\gamma = 1 - \beta$, we obtain

$$\beta = 1 - \frac{(c+1)(1-\alpha)}{2 - \alpha + c + n} \tag{22}$$

which was proved by Salagean [2].

3. Integral properties of the class $\mathcal{L}^p_{n,m}(\lambda, b)$

Theorem 2. *Let $p \in \mathbb{N}; m \in \mathbb{N}_0; p > m; b \in \mathbb{C} \setminus \{0\}$ and $c \in (-p, \infty)$. If $f \in \mathcal{L}^p_{n,m}(\lambda, b)$ and $g = \mathcal{I}_c(f)$, then $g \in \mathcal{L}^p_{n,m}(\lambda, \beta)$ where*

$$|\beta| = \frac{(c+p)|b| + n(1 - \frac{p!}{(p-m)!}[\lambda(p-m-1)+1])}{c+p+n} \tag{23}$$

and $|\beta| < |b|$. The result is sharp.

Proof. Using similar arguments as given by Theorem 1, we can get the result.

Remark 2. In (7), for $p = n = 1$, we obtain the integral operator of Bernardi [3],

$$I_c : \mathcal{A}_1(1) \to \mathcal{A}_1(1)$$

defined by $h = \mathcal{I}_c(f)$, where $c > -1$, $f \in \mathcal{A}_1(1)$,

$$h(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt.$$

Also, for $p = c = n = 1$, we obtain the integral operator of Libera [4],

$$I_1 : \mathcal{A}_1(1) \to \mathcal{A}_1(1)$$

defined by $h_1 = \mathcal{I}_1(f)$, where $f \in \mathcal{A}_1(1)$,

$$h_1(z) = \frac{2}{z} \int_0^z f(t)dt.$$

Corollary 1. *Let $b \in \mathbb{C} - \{0\}$, $c \in (-1, \infty)$. If $f \in \mathcal{L}^1_{1,0}(\lambda, b)$ and $h = I_c(f)$ is the Bernardi operator, then $h \in \mathcal{L}^1_{1,0}(\lambda, \beta)$, where*

$$|\beta| = \frac{c+1}{c+2} |b| .$$

Proof. In Theorem 2, we consider $p = n = 1$ and $m = 0$.

Corollary 2. *Let $b \in \mathbb{C} - \{0\}$. If $f \in \mathcal{L}^1_{1,0}(\lambda, b)$ and $h_1 = I_1(f)$ is the Libera operator, then $h \in \mathcal{L}^1_{1,0}(\lambda, \beta)$, where*

$$|\beta| = \frac{2}{3} |b| .$$

Proof. In Theorem 2, we consider $p = n = 1$, $m = 0$ and $c = 1$.

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APPROXIMATION OF CONTINUOUS FUNCTIONS ON V.K. DZJADYK CURVES

SADULLA Z. JAFAROV

Abstract. In the given paper rational approximation is studied on closed curves of a complex plane for continuous functions in terms of the k-th modulus of continuity, $k \geq 1$. Here a rational function interpolates a continuous function at definite points.

1. **Introduction and main result**

Let Γ be an arbitrary restricted Jordan curve with two -component complements $\Omega = C\Gamma = \Omega_1 \bigcup \Omega_2, (0 \in \Omega_1, \infty \in \Omega_2)$. Let's consider the functions $w = \Phi_i(z)$, $(i = 1, 2)$, that conformally and univalently maps respectively Ω_i onto $\Omega_i^{'}$, $(\Omega_1^{'} = \{w : |w| < 1\}, \ \Omega_2^{'} = \{w : |w| > 1\})$, with norm $\Phi_1(0) = 0$, $\Phi_1^{'}(0) > 0$, $\Phi_2(\infty) = \infty$, $\lim\limits_{z\to\infty} \frac{1}{z}\Phi_2(z) > 0$. Let's extend each $\Phi_i(z), (i = 1, 2)$ continuously up to the bound $\Gamma = \partial\Omega_1 = \partial\Omega_2$ (generally speaking $\Phi_1(z) \neq \Phi_2(z)$ for $z \in \Gamma$). We preserve the notation $\Phi_i, (i = 1.2)$ for the extension.Let $z = \Psi_i(w)$ be the inverse mapping of $w = \Phi_i(z), (i = 1, 2)$.

For $A > 0$ and $B > 0$, we use the expression $A \preceq B$ (order inequality) if $A \leq CB$. The expression $A \asymp B$ means that $A \preceq B$ and $B \preceq A$ hold simultaneously.

By $C(\Gamma)$ we denote a class of functions continuous on Γ. For $\delta > 0$ and fixed $u_o \in (0, 1)$ we assume

$$U(z, \delta)) = \{\zeta : |\zeta - z| < \delta\}, d(\zeta, \Gamma) = \inf_{z\in\Gamma} |\zeta - z|,$$

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$$\Gamma_\delta = \bigcup_{z\in\Gamma} U\left(z,\delta\right) = \left\{\zeta : d\left(\zeta,\Gamma\right) < \delta\right\},$$

$$\Gamma_{1+u_0} = \left\{\zeta : \zeta \in \Omega_2; \left|\Phi_2(\zeta)\right| = 1+u_0\right\},$$

$$\Gamma_{1-u_0} = \left\{\zeta : \zeta \in \Omega_1; \left|\Phi_1(\zeta)\right| = 1-u_o\right\},$$

$$D^1_{u_o} = int\Gamma_{1+u_0}, D^2_{u_o} = ext\Gamma_{1-u_o},$$

$$D_{u_o} = D^1_{u_0}\bigcap D^2_{u_0},$$

$$\delta^*_n = \sup_{\zeta\in int\Gamma_{1+\frac{1}{n}}\cap\, ext\Gamma_{1-\frac{1}{n}}} d\left(\zeta,\Gamma\right), n = 1,2,\cdots,$$

where under $int\,\Gamma$ we understand a finite domain whose boundary coincides with Γ, under $ext\,\Gamma$ we understand a finite domain whose domain coincides with Γ. Let R_n, $n = 0,1,2,...$ be a set of all complex rational functions of power no higher than n. For $f \in C(\Gamma)$ we define

$$E_n(f,\Gamma) := \inf_{r_n\in R_n}\sup_{z\in\Gamma}|f(z) - r_n(z)| = \inf_{r_n\in R_n}\|f - r_n\|_\Gamma.$$

In connection with "simultaneous approximation and interpolation" the following claims are suggested in the paper $[14, p.310]$. Let $z_1, z_2, ..., z_p \in \Gamma$ be definite points and $f \in C(\Gamma)$. In this case for $\forall n \in N, n \geq p-1$, there exists a rational function $r_n \in R_n$, for which

$$\|f - r_n\|_\Gamma \leq cE_n(f,\Gamma),$$

$$r_n(z_j) = f(z_j),\ \ (j = 1,2,...,p),$$

where $c > 0$ is independent n and f. The appropriate rational function is written in the following form

$$r_n(z) = r^*_n(z) + \sum_{j=1}^{p}\frac{q(z)}{q'(z_j)(z-z_j)}(f(z_j) - r^*_n(z_j)),$$

where

$$q(z) = \prod_{j=1}^{p}(z - z_j)$$

and $r^*_n \in R_n$ satisfies the following condition

$$\|f - r^*_n\|_\Gamma = E_n(f,\Gamma).$$

Definition 1.[11] *Let* Γ *be a rectifiable Jordan curve. By* $\theta(z,\delta), z \in \Gamma, 0 < \delta < +\infty$ *we denote the length of a part of* Γ *getting into the* $U(z,\delta) = \zeta : |\zeta - z| < \delta$. *We attribute the curve* Γ *to the class* S *if it is fulfilled the condition*

$$\theta_\Gamma(\delta) \stackrel{df}{=} \sup_{z \in \Gamma} \theta(z,\delta) \asymp \delta.$$

Let's give the definition of a class of V.K. Dzjadyk curves B_k^* in a briefly and slightly modified form. (see [7, p.439-440]).

Definition 2. *We'll say that a rectifiable Jordan curve* Γ *belongs to the class* B_k^* *for some natural* k *if* $\Gamma \in S$ *and satisfies the following conditions*

(i) $|\tilde{z} - z| \asymp \rho_{1+\frac{1}{n}}(z)$,

where $\forall z \in \Gamma$, $\tilde{z} = \Psi_2((1+\frac{1}{n})\Phi_2(z))$, $\rho_{1+\frac{1}{n}}(z) = \inf\limits_{\zeta \in \Gamma_{1+\frac{1}{n}}} |\zeta - z|$,

(ii) $|\tilde{\zeta} - \zeta|^k \leq |\tilde{\zeta} - z|^{k-1} |\tilde{z} - z|, \forall z, \zeta \in \Gamma$.

We'll study the functions for which the k-th modulus of continuity ($k \in N$) have been defined. There are some different definitions of such continuity modulus (see [6], [8], [13], [15]). The most convenient for our aim is the definition given by E.M. Dynkin [8].

Definition 3. *The k-th local modulus of continuity we'll call the quantity*

$$\omega_{f,k,z,\Gamma}(\delta) = E_{k-1}(f, \Gamma \cap U(z,\delta)),$$

where $f \in C(\Gamma), k \in N, z \in \Gamma, \delta > 0$.

Definition 4. *The k-th global modulus of continuity we'll call the quantity*

$$\omega_{f,k,\Gamma}(\delta) = \sup_{z \in \Gamma} \omega_{f,k,z,\Gamma}(\delta).$$

In particular,

$$\omega_{f,k,\Gamma}(t\delta) \leq c_1 t^k \omega_{f,k,\Gamma}(\delta) \quad (t > 1, \delta > 0). \tag{1}$$

If $0 < \delta < 1$, there exists a constant c_2, that

$$\int\limits_0^\delta \omega_{f,k,\Gamma}(t) \frac{dt}{t} \leq c_2 \omega_{f,k,\Gamma}(\delta). \tag{2}$$

The following theorems is the main result of the report.

Theorem 1. *Let $\Gamma \in B_k^*,\ f \in C(\Gamma), k \in N$ and $z_1, z_2, ..., z_p \in \Gamma$ be distinct points. Then for each $n \in N, n \geq p+k$ there exists a rational function $r_n \in R_n$ for which the following conditions are fulfilled*

$$|f(z) - r_n(z)| \leq c_1 \omega_{f,k,\Gamma}(\delta_n^*) \quad (z \in \Gamma), \tag{3}$$

$$r_n(z_j) = f(z_j) \quad (j = 1, 2, ..., p), \tag{4}$$

where the constant c_1 is independent of n.

The rational functions play an important role in many areas of applied mathematics and mechanics. It is actually the approximation of continuous functions by rational functions or some other functions, which can be found easily. The Theorem 1 studies rational approximations (in terms of the k-th modulus of continuity, $k \geq 1$) for continuous functions defined on closed curves Γ in the complex plane, which simultaneously interpolate at given points of Γ. The similar results for the analytic functions in different continua were obtained in the papers [5], [15], [1].

2. **Subsidiary facts**

By obtaining the main result we use an approximation of Cauchy kernel $(s-z)^{-1}$ by rational functions of the form

$$K_n(\zeta, z) = \sum_{j=-n}^{n} a_j(\zeta) z^j.$$

To construct rational functions a rational kernel suggested by V.K. Dzjadyk (see [7, ch.9] or [3, ch.3]) is used.

Lemma 1. *Let Γ be an arbitrary Jordan curve, $0 < u_0 < 1$ be an arbitrary fixed number, $c = 2(1+u_0)e^{2\pi}$. Then for all natural $n = 1.2, ...$ and $\zeta \in D_{u_0} \backslash \Gamma_{\delta^*(\frac{c}{n})}$ there exists the function $\Pi_n(\zeta, z) = \sum\limits_{j=-n}^{n} a_j(\zeta) z^j$ with continuous with respect to ζ coefficients $a_j, j = \overline{-n, n}$, that for $z \in \Gamma$ and $p = 0, 1$ satisfies the inequalities*

$$\left|\frac{\partial^p}{\partial z^p}\left[\frac{1}{\zeta - z} - \Pi_n(\zeta, z)\right]\right| \preceq {\delta^*}^2\left(\frac{1}{n}\right)|\zeta - z|^{-p-3} \tag{5}$$

$$\left|\frac{\partial^p}{\partial z^p}\Pi_n(\zeta, z)\right| \preceq |\zeta - z|^{-p-1}. \tag{6}$$

The lemma is proved similarly to Corollary 4 of the paper [2]. It should be only noted that $\Pi_n(\zeta, z)$ is a polynomial kernel for $\zeta \in \left\{D^1_{u_o} \backslash \Gamma_{\delta^*\left(\frac{c}{n}\right)}\right\} \cap \Omega_2$. In the case $\zeta \in \left\{D^2_{u_o} \backslash \Gamma_{\delta^*\left(\frac{c}{n}\right)}\right\} \cap \Omega_1$ as it is shown in the paper [4] it gives us a rational function.

Lemma 2. *Let $\Gamma \in B^*_k, 0 < u_0 < 1$ be an arbitrary number. Then for any $n = 1, 2...$ and $\zeta \in D_{u_o}$ there exists a rational function $K_n(\zeta, z)$ with respect to the variable z with summable with respect to s coefficients for which for $z \in \Gamma$ and $p = 0, 1$ the inequalities*

$$\left|\frac{\partial^p}{\partial z^p}\left[\frac{1}{\zeta - z} - K_n(\zeta, z)\right]\right| \preceq \frac{1}{|\zeta - z|^{p+1}}\left[\frac{\delta^*_n}{|\zeta - z| + \delta^*_n}\right]^2, \tag{7}$$

$$\left|\frac{\partial^p}{\partial z^p} K_n(\zeta, z)\right| \preceq \left[|\zeta - z| + \delta^*_n\right]^{-p-1}. \tag{8}$$

are fulfilled.

Proof. Let n be sufficiently large. Assume $c = 2(1 + u_0)e^{2\pi}$. By compactness of $\overline{\Gamma_{\delta^*\left(\frac{c}{n}\right)}}$ we can distinguish a finite number of points $\zeta_1, \zeta_2, ..., \zeta_m \in \overline{\Gamma_{\delta^*\left(\frac{c}{n}\right)}}$, for which

$$\Gamma_{\delta^*\left(\frac{c}{n}\right)} \subset \bigcup_{k=1}^{m} U(\zeta_k, \delta^*_n).$$

Since $\Gamma \in B^*_k$, at each point $\zeta_k, k = \overline{1, m}$ we can construct the point $\zeta'_k \in D_{u_0} \backslash \Gamma_{\delta^*\left(\frac{c}{n}\right)}$ with the following condition

$$\left|\zeta_k - \zeta'_k\right| \preceq \delta^*_n. \tag{9}$$

We can easily see that

$$\left|\zeta'_k - z\right| \asymp |\zeta - z| + \delta^*_n, \ \ z \in \Gamma, \ \zeta \in U(\zeta_k, \delta^*_n). \tag{10}$$

By the identity

$$\frac{1}{\zeta - z} = \frac{1}{\zeta'_k - z} + \frac{\zeta'_k - \zeta}{\left(\zeta'_k - z\right)^2} + \left(\frac{\zeta'_k - \zeta}{\zeta'_k - z}\right)^2 \frac{1}{\zeta - z}$$

consider for $\zeta \in U(\zeta_k, \delta^*_n), k = \overline{1, m}$ the function

$$\lambda^{(k)}_n(\zeta, z) = \Pi_n\left(\zeta'_k, z\right) + \left(\zeta'_k - \zeta\right)\left(\Pi_{\left[\frac{n}{2}\right]}\left(\zeta'_k, z\right)\right)^2,$$

where $\Pi_n\left(\zeta_k^{'},z\right))$ is the function from Lemma 1. We construct the required function $K_n(\zeta,z)$ as follows:

1) If $\zeta\in U(\zeta_1,\delta_n^*)$ we assume

$$K_n(\zeta,z)=\lambda_n^{(1)}(\zeta,z)$$

2) If $\zeta\in U(\zeta_k,\delta_n^*)\setminus\bigcup_{j=1}^{k-1}U(\zeta_j,\delta_n^*)\,,k=\overline{2,m}$ we assume

$$K_n(\zeta,z)=\lambda_n^{(k)}(\zeta,z)\,.$$

3) If $\zeta\in D_{u_0}\setminus\left\{\bigcup_{j=1}^{m}U(\zeta_j,\delta_n^*)\right\}$, we take the appropriate function from Lemma 1. Now the affirmation of Lemma 2 follows from the above mentioned constructions, Lemma 1, estimates of the following easily verifiable relations

$$\frac{1}{\zeta-z}-\lambda_n^{(k)}(\zeta,z)=\left(\frac{\zeta_k^{'}-\zeta}{\zeta_k^{'}-z}\right)^2\frac{1}{\zeta-z}+\left(\frac{1}{\zeta_k^{'}-z}-\Pi_n\left(\zeta_k^{'},z\right)\right)+$$

$$+\left(\zeta_k^{'}-\zeta\right)\left[\frac{1}{\left(\zeta_k^{'}-z\right)^2}-\left(\Pi_{\left[\frac{n}{2}\right]}\left(\zeta_k^{'},z\right)\right)^2\right],$$

$$\left|\frac{\partial^p}{\partial z^p}\left[\frac{1}{\zeta-z}-\lambda_n^{(k)}(\zeta,z)\right]\right|\preceq\frac{1}{|\zeta-z|^{p+1}}\left(\frac{\delta_n^*}{|\zeta-z|+\delta_n^*}\right)^2,$$

$$\left|\frac{\partial^p}{\partial z^p}\lambda_n^{(k)}(\zeta,z)\right|\preceq\left(|\zeta-z|+\delta_n^*\right)^{-p-1},$$

where $\zeta\in U(\zeta_k,\delta_n^*)\,,\ \ z\in\Gamma,\ \ p=0,1.$

Let's give a result in slightly modified form cited in the papers [8], [9], [12], [13, p.13-15].

Lemma 3. *Let $\Gamma\in B_k^*$ and $F\in C(\Gamma)$. Then we can continue the function $F(z)$ on the complex plane $\mathbf{C}$ so that the following relations be fulfilled (we keep denotation $F(z)$): (i) for $z\in\mathbf{C}\backslash\Gamma$*

$$\left|\frac{\partial F(z)}{\partial\bar z}\right|\le c_1\frac{\omega_{F,k,z,\Gamma}\left(c_2 d(z,\Gamma)\right)}{d(z,\Gamma)},$$

where $c_1=c_1(k,diam\Gamma)$.

(ii) if $\zeta\in\Gamma,z\in\mathbf{C}$, $|z-\zeta|<\delta,\ 0<\delta<\frac{1}{2}diam\Gamma$, then

$$|F(z)-R_{F,k,\zeta,\Gamma,\delta}(z)|\le c_3\omega_{F,k,\zeta,\Gamma}(c_4\delta)\,,$$

where $R_{F,k,\zeta,\Gamma,\delta}(z) \in R_{k-1}$ *is such a rational function that*

$$\|F - R_{F,k,\zeta,\Gamma,\delta}\|_{\Gamma\cap D(\zeta,\delta)} = \omega_{F,k,\zeta,\Gamma}(\delta),$$

and $c_3 = c_3(k)$.

(iii) If F *satisfies the Lipschitz condition on* Γ, *i.e.*

$$|F(z) - F(\zeta)| \le c\,|z-\zeta|, z, \zeta \in \Gamma,$$

then the continued function for $z, \zeta \in \mathbf{C}$ *satisfies the same condition. Here, instead of* c *there will be the constant* $c_4 = c_4(c, diam\Gamma, k)$.

3. **The proof of the main result (Theorem 1)**

Let's fix a point $z_0 \in \Gamma$ and assume for $\zeta \in \Gamma$

$$F(\zeta) = \int\limits_{\gamma(z_0,\zeta)} f(\xi)\,d\xi,$$

where $\gamma(z_0,\zeta) \subset \Gamma$ is an arc connecting the points z_0 and ζ.

We extend the function $F(\zeta)$ continuously on the complex plane $\mathbf{C}$. Let z and $\zeta \in \Gamma, |\zeta - z| \le \delta$, the arc $\gamma(z,\zeta) \subset int\Gamma$ connects these points, $mes\gamma(z,\zeta) \le c\,|z-\zeta|, c = c(\Gamma) \ge 1$. We'll have

$$F(\zeta) = F(z) + \int\limits_{\gamma(z,\zeta)} f(\xi)\,d\xi = \nu_\delta(\zeta, z) + + \int\limits_{\gamma(z,\zeta)} (f(\xi) - R_{f,k,z,\Gamma,c\delta}(\xi))\,d\xi,$$

$$\omega_{F,k+1,z,\Gamma}(\delta) \le \|F - \nu_\delta(.,z)\|_{\Gamma\cap D(z,\delta)} \preceq \delta\omega(\delta),$$

where $\omega(\delta) := \omega_{f,k,\Gamma}(\delta)$.

Using Lemma 3 for $\zeta \in G := \overline{int\Gamma_{1+\frac{1}{2}}} \cap \overline{ext\Gamma_{\frac{1}{2}}}$ we have

$$\left|\frac{\partial F(\zeta)}{\partial\bar{\zeta}}\right| \preceq \omega(d(\zeta,\Gamma)). \tag{11}$$

Besides, for $z \in \Gamma, \zeta \in \mathbf{C}$, $|z - \zeta)| \le \delta < diam\Gamma$ we have

$$|F(\zeta) - \nu_\delta(\zeta, z)| \preceq \delta\omega(\delta). \tag{12}$$

Indeed, for $\zeta \in \Gamma \cap D(z,\delta)$ the following inequality is valid

$$|\nu_\delta(\zeta,z) - R_{F,k+1,z,\Gamma,\delta}(\zeta)| \le |F(\zeta) - \nu_\delta(\zeta,z)| + |F(\zeta) - R_{F,k+1,z,\Gamma,\delta}(\zeta)| \preceq \delta\omega(\delta).$$

By Bernstein-Walsh lemma (see [14, p.77]) we have

$$\|\nu_\delta(.,z) - R_{F,k+1,z,\Gamma,\delta}\|_{D(z,\delta)} \preceq \delta\omega(\delta) \tag{13}$$

We introduce a rational kernel $Q_{\frac{n}{2}}(\zeta,z) := K_{\left[\frac{n}{2}\right]}(\zeta,z)$, where $K_n(\zeta,z)$ is a rational kernel from Lemma 2. By virtue of (3) and (4) $\zeta \in int\Gamma_{1+\frac{1}{2}} \cap ext\Gamma_{\frac{1}{2}}, z \in \Gamma$ we have

$$\left|\frac{1}{\zeta - z} - Q_{\frac{n}{2}}(\zeta,z)\right| \preceq \frac{1}{|\zeta - z|}\left(\frac{\delta_n^*}{|\zeta - z| + \delta_n^*}\right)^2, \tag{14}$$

$$\left|Q_{\frac{n}{2}}(\zeta,z)\right| \preceq \frac{1}{[|\zeta - z| + \delta_n^*]} \tag{15}$$

For $z \in \Gamma$ we give the approximate rational function by the formula

$$R_n(z) = -\frac{1}{\pi}\int\limits_G \frac{\partial F(\zeta)}{\partial\bar{\zeta}} Q^2_{\frac{n}{2}}(\zeta,z)\,dm(\zeta),$$

where $dm(\zeta)$ means integration with respect to the two-dimensional Lebesgue measure (area).

Let $z \in \Gamma$ and assume $U_n := U(z,\delta_n^*), \gamma_n := \partial U_n$. By lemma (iii) of the Lemma 3 $F \in ACL$ in $\mathbf{C}$ (absolutely continuous on all horizontal and verticals in $\mathbf{C}$). Then we apply the Green formula (see [10]) and have

$$\begin{aligned} f(z) - R_n(z) &= \tfrac{1}{\pi}\int\limits_{G\setminus U_n} \tfrac{\partial F(\zeta)}{\partial\bar{\zeta}}\left(Q^2_{\frac{n}{2}}(\zeta,z) - \tfrac{1}{(\zeta-z)^2}\right)dm(\zeta) \\ &+\tfrac{1}{\pi}\int\limits_{U_n} \tfrac{\partial F(\zeta)}{\partial\bar{\zeta}} Q^2_{\frac{n}{2}}(\zeta,z)\,dm(\zeta) + \\ &+f(z) - \tfrac{1}{2\pi i}\int\limits_{\gamma_n} \tfrac{F(\zeta)}{(\zeta-z)^2}d\zeta = U_1(z) + U_2(z) + U_3(z). \end{aligned} \tag{16}$$

Now, we estimate each $U_i(z), i = 1,2$. In relation (16) passing to polar coordinates and using (1), (2), (11), (14) and (15) the first two integrals are estimated in the following way:

$$|U_1(z)| = \left|\frac{1}{\pi}\int\limits_{G\setminus U_n} \frac{\partial F(\zeta)}{\partial\bar{\zeta}}\left(Q^2_{\frac{n}{2}}(\zeta,z) - \frac{1}{(\zeta - z)^2}\right)dm(\zeta)\right| \leq$$

$$\leq \frac{1}{\pi}\int\limits_{G\setminus U_n} \left|\frac{\partial F(\zeta)}{\partial\bar{\zeta}}\right|\left|Q^2_{\frac{n}{2}}(\zeta,z) - \frac{1}{(\zeta - z)^2}\right|dm(\zeta) \preceq \tag{17}$$

$$\preceq \int\limits_{\delta_n^*}^{c} \omega(r) \frac{(\delta_n^*)^3}{r^4} dr \preceq \omega(\delta_n^*) \delta_n^* \int\limits_{\delta_n^*}^{c} \frac{dr}{r^2} \preceq \omega(\delta_n^*),$$

$$|U_2(z)| = \left| \frac{1}{\pi} \int\limits_{U_n} \frac{\partial F(\zeta)}{\partial \bar{\zeta}} Q_{\frac{n}{2}}^2(\zeta, z)\, dm(\zeta) \right| \leq$$

$$\leq \frac{1}{\pi} \int\limits_{U_n} \left| \frac{\partial F(\zeta)}{\partial \bar{\zeta}} \right| \left| Q_{\frac{n}{2}}^2(\zeta, z) \right| dm(\zeta) \preceq \int\limits_{o}^{\delta_n^*} \frac{\omega(r)}{r} dr \preceq \omega(\delta_n^*). \tag{18}$$

Now let's estimate $U_3(z)$. We have

$$|U_3(z)| = \left| f(z) - \frac{1}{2\pi i} \int\limits_{\gamma_n} \frac{F(\zeta)}{(\zeta - z)^2} d\zeta \right| \leq \left| f(z) - \left(\nu_{\delta_n^*}\right)'_\zeta (z, z) \right| +$$

$$+ \frac{1}{2\pi} \left| \int\limits_{\gamma_n} \frac{F(\zeta) - \nu_{\delta_n^*}(\zeta, z)}{(\zeta - z)^2} d\zeta \right|. \tag{19}$$

The estimate

$$\left| f(z) - \left(\nu_{\delta_n^*}\right)'_\zeta (z, z) \right| = \left| f(z) - R_{f,k,z,\Gamma,c\delta_n^*}(z) \right| \leq \omega(c\delta_n^*) \preceq \omega(\delta_n^*) \tag{20}$$

is true. By inequalities (19), (20), and (12) we have

$$|U_3(z)| \preceq \omega(\delta_n^*) \tag{21}$$

Comparing estimates (16), (17), (18) and (21) we have

$$|f(z) - R_n(z)| \preceq \omega(\delta_n^*),\, z \in \Gamma. \tag{22}$$

Now, let's construct rational function for which conditions (3) and (4) are fulfilled. Let $n > 2p$. Let's construct the following functions

$$V_{\frac{n}{2+1}}(\zeta, z) = 1 - (\zeta - z) Q_{\frac{n}{2}}(\zeta, z),\, \zeta, z \in \Gamma,$$

$$u_n(z) = \sum_{j=1}^{p} \frac{q(z)}{q'(z_j)(z - z_j)} \left(f(z_j) - t_n(z_j)\right) V_{\frac{n}{2+1}}(z_j, z).$$

By (14) and (22) we have

$$|u_n(z)| \preceq \sum_j{}' \omega(\delta_n^*) \left(\frac{\delta_n^*}{|z - z_j| + \delta_n^*} \right)^2,\, z \in \Gamma,$$

where $\sum\limits_{j}^{\prime}$ means the sum in all j with $z_j \in \Gamma$.

Let's construct the required rational function in the following form

$$r_n(z) = t_n(z) + u_n(z). \tag{23}$$

Obviously the rational function of the form (23) satisfies conditions (3) and (4).

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THE STABILITY OF THE EQUILIBRIUM STATES FOR SOME MECHANICAL SYSTEMS

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Abstract. In the first part of the paper some theoretical results (including the Lyapunov-Malkin theorem) are presented, followed in the second part by some of its applications in geometrical mechanics.

1. **Theoretical aspects**

To explain the stability concept, we need some basic notions and results from the theory of dynamical systems.

The lows of Dynamics are usually presented as equations of motion, which we will write as differential equations:

$$\dot{x} = f(x) \tag{1}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

is a variable describing the state of the system. The function

$$f : \mathbb{R}^n \to \mathbb{R}^n$$

is smooth of x, and

$$\dot{x} = \frac{dx}{dt}.$$

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The set of all allowed x forms the state space of (1). When the time advances, the system's state is changed.

Definition 1. *An point $x_e \in \mathbb{R}^n$ is called equilibrium point for the system (1) if:*

$$f(x_e) = 0.$$

Remark 1. It is clear that the constant function

$$x(t) = x_e$$

is a solution for (1) and from the existence and uniqueness theorem it results that does not exists other solution containing x_e. So the unique trajectory starting at x_e is x_e itself, i.e. x_e does not changes in time.

Definition 2. *Let x_e be an equilibrium state for (1). We will say that x_e is nonlinear stable (or Lyapunov stable) if for any neighborhood U of x_e there exists a neighborhood V of x_e, $V \subset U$ such that any solution $x(t)$, initially in V (i.e. $x(0) \in V$), never leaves U.*

Definition 3. *If V in Definition 2 can be chosen such that*

$$\lim_{t\to\infty} x(t) = x_e,$$

then x_e is called asymptotically stable.

Definition 4. *An equilibrium state x_e that is not stable is called unstable.*

Let us consider the following system of differential equations of order one:

$$\begin{cases} \dot{x} = Ax + X(x,y) \\ \dot{y} = By + Y(x,y) \end{cases} \tag{2}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, A and B are constant matrices such that all eigenvalues of A are of nonzero real parts and all eigenvalues of B are of zero real parts, and the functions X, Y satisfy the following conditions:

i) $X(0,0) = 0$,

ii) $dX(0,0) = 0$,

iii) $Y(0,0) = 0$,

iv) $dY(0,0) = 0$.

We will take now the particular case of (2) in which the matrix B is O_n. The equations (2) become:

$$\begin{cases} \dot{x} = Ax + X(x,y) \\ \dot{y} = Y(x,y). \end{cases} \tag{3}$$

Theorem 1. *(Lyapunov-Malkin) Under the above conditions, if all eigenvalues of A have negative real parts and $X(x,y)$ and $Y(x,y)$ vanish when $x = 0$, then the equilibrium state*

$$x = 0,\ y = 0$$

of the system (2) is nonlinear stable with respect to (x,y) and asymptotically stable with respect to X.

For the proof of this basic result see Zenkov, Bloch and Marsden [5].

2. **Two application of Lyapunov-Malkin theorem**

In this section we study the stability of the equilibrium points for some concrete mechanical systems.

Example 1. (3-dimensional Toda lattice with two controls)

The dynamics of the generalized 3-dimensional Toda lattice with two controls is described by the following system:

$$\begin{cases} \dot{q}_1 = 2p_1^2 \\ \dot{q}_2 = 2p_2^2 - 2p_1^2 \\ \dot{q}_3 = -2p_2^2 \\ \dot{p}_1 = p_1q_2 - p_1q_1 + u_1 \\ \dot{p}_2 = p_2q_3 - p_2q_2 + u_2. \end{cases} \tag{4}$$

In what follows we shall employ the controls:

$$\begin{cases} u_1 = \alpha p_1 \\ u_2 = \beta p_2 \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta < 0$. Then the system (4) takes the following form:

$$\begin{cases} \dot{q}_1 = 2p_1^2 \\ \dot{q}_2 = 2p_2^2 - 2p_1^2 \\ \dot{q}_3 = -2p_2^2 \\ \dot{p}_1 = p_1q_2 - p_1q_1 + \alpha p_1 \\ \dot{p}_2 = p_2q_3 - p_2q_2 + \beta p_2. \end{cases} \tag{5}$$

Now we can consider:

$$x = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \quad y = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$X(x,y) = \begin{bmatrix} p_1q_2 - p_1q_1 \\ p_2q_3 - p_2q_2 \end{bmatrix}$$

$$Y(x,y) = \begin{bmatrix} 2p_1^2 \\ 2p_2^2 - 2p_1^2 \\ -2p_2^2 \end{bmatrix}.$$

In those conditions the system (5) can be written in the following equivalent form:

$$\begin{cases} \dot{x} = Ax + X(x,y) \\ \dot{y} = Y(x,y) \end{cases}$$

where:

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}.$$

We will verify the conditions in the Lyapunov-Malkin theorem. We have successively:

i) $$X(0,0) = \begin{bmatrix} p_1q_2 - p_1q_1 \\ p_2q_3 - p_2q_2 \end{bmatrix}_{(0,0,0,0,0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$Y(0,0) = \begin{bmatrix} 2p_1^2 \\ 2p_2^2 - 2p_1^2 \\ -2p_2^2 \end{bmatrix}_{(0,0,0,0,0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

ii) $X(0,y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, for any $y \in \mathbb{R}^3$,

$Y(0,y) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, for any $y \in \mathbb{R}^3$.

iii)
$$\frac{DX}{Dx}\Big|_{(0,0)} = \begin{bmatrix} \frac{\partial(p_1q_2-p_1q_1)}{\partial p_1} & \frac{\partial(p_1q_2-p_1q_1)}{\partial p_2} \\ \frac{\partial(p_2q_3-p_2q_2)}{\partial p_1} & \frac{\partial(p_2q_3-p_2q_2)}{\partial p_2} \end{bmatrix}_{(0,0,0,0,0)}$$
$$= \begin{bmatrix} q_2-q_1 & 0 \\ 0 & q_3-q_2 \end{bmatrix}_{(0,0,0,0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\frac{DX}{Dy}\Big|_{(0,0)} = \begin{bmatrix} \frac{\partial(p_1q_2-p_1q_1)}{\partial q_1} & \frac{\partial(p_1q_2-p_1q_1)}{\partial q_2} & \frac{\partial(p_1q_2-p_1q_1)}{\partial q_3} \\ \frac{\partial(p_2q_3-p_2q_2)}{\partial q_1} & \frac{\partial(p_2q_3-p_2q_2)}{\partial q_2} & \frac{\partial(p_2q_3-p_2q_2)}{\partial q_3} \end{bmatrix}_{(0,0,0,0,0)}$$
$$= \begin{bmatrix} -p_1 & p_1 & 0 \\ 0 & -p_2 & p_2 \end{bmatrix}_{(0,0,0,0,0)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\frac{DY}{Dx}\Big|_{(0,0)} = \begin{bmatrix} \frac{\partial(2p_1^2)}{\partial p_1} & \frac{\partial(2p_1^2)}{\partial p_2} \\ \frac{\partial(2p_2^2-2p_1^2)}{\partial p_1} & \frac{\partial(2p_2^2-2p_1^2)}{\partial p_2} \\ \frac{\partial(-2p_2^2)}{\partial p_1} & \frac{\partial(-2p_2^2)}{\partial p_1} \end{bmatrix}_{(0,0,0,0,0)}$$
$$= \begin{bmatrix} 4p_1 & 0 \\ -4p_1 & 4p_2 \\ 0 & -4p_2 \end{bmatrix}_{(0,0,0,0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\frac{DY}{Dy}\Big|_{(0,0)} = \begin{bmatrix} \frac{\partial(2p_1^2)}{\partial q_1} & \frac{\partial(2p_1^2)}{\partial q_2} & \frac{\partial(2p_1^2)}{\partial q_3} \\ \frac{\partial(2p_2^2-2p_1^2)}{\partial q_1} & \frac{\partial(2p_2^2-2p_1^2)}{\partial q_2} & \frac{\partial(2p_2^2-2p_1^2)}{\partial q_3} \\ \frac{\partial(-2p_2^2)}{\partial q_1} & \frac{\partial(-2p_2^2)}{\partial q_1} & \frac{\partial(-2p_2^2)}{\partial q_3} \end{bmatrix}_{(0,0,0,0,0)}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

iv) The characteristic polynomial of matrix A is

$$P_A(x) = \det \begin{bmatrix} \alpha - x & 0 \\ 0 & \beta - x \end{bmatrix} = (\alpha - x)(\beta - x).$$

It has negative roots α and β. Hence the eigenvalues of the matrix A are $x_1 = \alpha < 0,\ x_2 = \beta < 0$.

We can conclude with:

Remark 2. The equilibrium state $(0,0,0,0,0)$ for the system (4) is nonlinear stable.

Example 2. (Maxwell-Bloch equations with two controls)

The Maxwell-Bloch equations with two controls on axes Ox_1 and Ox_2 are writing in the following form:

$$\begin{cases} \dot{x}_1 = x_2 + u_1 \\ \dot{x}_2 = x_1 x_3 + u_2 \\ \dot{x}_3 = -x_1 x_2. \end{cases} \tag{6}$$

The controls u_1 and u_2 will be written as:

$$u_1(x_1, x_2, x_3) = \alpha x_1$$
$$u_2(x_1, x_2, x_3) = \beta x_2,$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta < 0$. Then our dynamics takes the following form:

$$\begin{cases} \dot{x}_1 = x_2 + \alpha x_1 \\ \dot{x}_2 = x_1 x_3 + \beta x_2 \\ \dot{x}_3 = -x_1 x_2. \end{cases} \tag{7}$$

If we take:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = x_3,$$
$$X(x,y) = \begin{bmatrix} 0 \\ x_1 x_3 \end{bmatrix},$$
$$Y(x,y) = -x_1 x_2$$

then the system (7) can be written in the following equivalent form:

$$\begin{cases} \dot{x} = Ax + X(x,y) \\ \dot{y} = Y(x,y) \end{cases} \tag{8}$$

where:

$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \beta \end{bmatrix}.$$

Now we must verify the conditions from the Lyapunov-Malkin theorem for system (8). We have successively:

i) $X(0,0) = \begin{bmatrix} 0 \\ x_1 x_3 \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$Y(0,0) = [x_3]_{(0,0,0)} = 0.$

ii) $X(0,y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $(\forall)\ y \in R^3$,

$Y(0,y) = 0,\ \ (\forall)\ y \in R^3.$

iii) $\frac{DX}{Dx}\Big|_{(0,0)} = \begin{bmatrix} \frac{\partial 0}{\partial x_1} & \frac{\partial 0}{\partial x_2} \\ \frac{\partial (x_1 x_3)}{\partial x_1} & \frac{\partial (x_1 x_3)}{\partial x_2} \end{bmatrix}_{(0,0,0)}$

$= \begin{bmatrix} 0 & 0 \\ x_3 & 0 \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

$\frac{DX}{Dy}\Big|_{(0,0)} = \begin{bmatrix} \frac{\partial 0}{\partial x_3} \\ \frac{\partial (x_1 x_3)}{\partial x_3} \end{bmatrix}_{(0,0,0)}$

$= \begin{bmatrix} 0 \\ x_1 \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$\frac{DY}{Dx}\Big|_{(0,0)} = \begin{bmatrix} -x_2 & -x_1 \end{bmatrix}_{(0,0,0)}$

$= \begin{bmatrix} 0 & 0 \end{bmatrix}_{(0,0,0)}$,

$\frac{DY}{Dy}\Big|_{(0,0)} = [0]_{(0,0,0)} = [0].$

iv) Again the characteristic polynomial of A is

$$P_A(x) = \det \begin{bmatrix} \alpha - x & 1 \\ 0 & \beta - x \end{bmatrix} = (\alpha - x)(\beta - x),$$

and it has negative roots. So we have the eigenvalues of the matrix A, $x_1 = \alpha < 0,\ x_2 = \beta < 0$.

We can conclude, by Lyapunov-Malkin theorem, that:

The equilibrium state $(0,0,0)$ for the system (6) is nonlinear stable.

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ABOUT THE UNIVALENCE OF THE BESSEL FUNCTIONS

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Abstract. The authors of [1] and [3] deduced univalence criteria concerning Bessel functions. In [3] the author used the theory developed in [2] to obtain the desired result. In this paper we will extend a few results obtained in [3] employing elementary methods.

1. **Introduction**

Let

$$U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

be the disc with center z_0 and of the radius r, the particular case $U(0, 1)$ will be denoted by U. The Bessel function of the first kind is defined by

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}.$$

The series, which defines J_ν is everywhere convergent and the function defined by the series is generally not univalent in any disc $U(0, r)$. We will study the univalence of the following normalized form:

$$f_\nu(z) = 2^\nu \Gamma(1+\nu) z^{-\frac{\nu}{2}} J_\nu(z^{\frac{1}{2}}), \quad g_\nu(z) = z f_\nu(z). \tag{1}$$

2. **Preliminaries**

In order to prove our main result we need the following lemmas.

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Lemma 1 ([3], equality (6))**.** *The function f_ν satisfies the equality:*

$$f'_\nu(z) = -\frac{1}{2} f_{\nu+1}(z).$$

Lemma 2. *Let R be the function defined by the equality*

$$R(\theta) = \sum_{n=3}^{\infty} \frac{(-1)^n(\nu+1)^n \cos n\theta}{n!(\nu+1)...(\nu+n)},\ \ \theta \in \mathbb{R}, \nu \in (-1,\infty).$$

The following inequality holds

$$\left|R(\theta)\right| \leq \frac{(\nu+1)^2}{4(\nu+2)(\nu+3)},\ \ \theta \in \mathbb{R}.$$

Proof. Since

$$R(\theta) = \frac{\nu+1}{\nu+2} \sum_{n=3}^{\infty} \frac{(-1)^n(\nu+1)^{n-2} \cos n\theta}{n!(\nu+3)...(\nu+n)}$$

it follows that

$$\left|R(\theta)\right| \leq \frac{\nu+1}{\nu+2} \sum_{n=3}^{\infty} \left| \frac{(-1)^n(\nu+1)^{n-2} \cos n\theta}{n!(\nu+3)...(\nu+n)} \right| \leq$$

$$\frac{\nu+1}{\nu+2} \sum_{n=3}^{\infty} \frac{(\nu+1)^{n-2}}{n!(\nu+3)...(\nu+n)} \leq \frac{(\nu+1)^2}{(\nu+2)(\nu+3)} \sum_{n=3}^{\infty} \frac{1}{n!} \leq \frac{(\nu+1)^2}{4(\nu+2)(\nu+3)}.$$

□

Lemma 3. *If $z \in U$ then*

$$|g'_\nu(z) - \frac{g_\nu(z)}{z}| \leq \frac{2+\nu}{(1+\nu)(4\nu+7)}, \tag{2}$$

$$|f_\nu(z)| = |\frac{g_\nu(z)}{z}| \geq \frac{4\nu^2+10\nu+5}{(1+\nu)(4\nu+7)}, \tag{3}$$

$$\left|f'_\nu(z)\right| \leq \frac{\nu+2}{(\nu+1)(4\nu+7)}. \tag{4}$$

Proof. If $z \in U$ then the triangle inequality implies that:

$$\left|g'_\nu(z) - \frac{g_\nu(z)}{z}\right| = \left| \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n n!(\nu+1)...(\nu+n)} z^n \right| \leq$$

$$\sum_{n=1}^{\infty} \frac{n}{4^n n!(\nu+1)...(\nu+n)}.$$

Since

$$\sum_{n=1}^{\infty} \frac{n}{4^n n!(\nu+1)...(\nu+n)} \leq \frac{1}{4(\nu+1)} \sum_{n=0}^{\infty} \left(\frac{1}{4(\nu+2)} \right)^n = \frac{2+\nu}{(1+\nu)(4\nu+7)}$$

we obtain (2).

By using again the triangle inequality, we deduce that

$$|\frac{g_\nu(z)}{z}| \geq 1 - \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{4^n n!(\nu+1)...(\nu+n)} z^n \right| \geq 1 - \sum_{n=1}^{\infty} \frac{1}{4^n n!(\nu+1)...(\nu+n)}$$

and so the inequality

$$1 - \sum_{n=1}^{\infty} \frac{1}{4^n n!(\nu+1)...(\nu+n)} \geq 1 - \frac{1}{4(\nu+1)} \sum_{n=1}^{\infty} \frac{1}{[4(\nu+2)]^{n-1}} = \frac{4\nu^2+10\nu+5}{(1+\nu)(4\nu+7)}$$

leads to (3). Using similar ideas we obtain the following inequality chain

$$\left| f'_\nu(z) \right| \leq \sum_{n=1}^{\infty} \left| \frac{(-1)^n z^n}{4^n (n-1)!(1+\nu)...(n+\nu)} \right| \leq \sum_{n=1}^{\infty} \frac{1}{4^n (n-1)!(1+\nu)...(n+\nu)} \leq \frac{1}{4(1+\nu)} \sum_{n=0}^{\infty} \left(\frac{1}{4(2+\nu)} \right)^n = \frac{\nu+2}{(\nu+1)(4\nu+7)}.$$

This means that (4) also holds. □

3. **The main result**

Theorem 4. *If* $\nu > -1$ *then*

$$Ref_\nu(z) > 0, \ \textit{for all} \ \ z \in U(0, 4(1+\nu)).$$

Proof. The minimum principle for harmonic functions implies that

$$\text{Re} f_\nu(z) \geq \inf_{\theta \in \mathbb{R}} \text{Re} f_\nu(r_\nu e^{i\theta}), \ \text{for all} \ \ z \in U(0, 4(1+\nu))$$

where $r_\nu = 4(1+\nu)$. According to the definition of f_ν, we have

$$f_\nu(z) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{4^n n!(\nu+1)...(\nu+n)}$$

and

$$\text{Re}f_\nu(r_\nu e^{i\theta}) = 1 + \text{Re}\sum_{n=1}^{\infty} \frac{(-1)^n(\nu+1)^n e^{in\theta}}{n!(\nu+1)...(\nu+n)}$$

$$= 1 - \cos\theta + \frac{\nu+1}{2(\nu+2)}\cos 2\theta + \sum_{n=3}^{\infty} \frac{(-1)^n(\nu+1)^n \cos n\theta}{n!(\nu+1)...(\nu+n)}.$$

If we let

$$P(\theta) = 1 - \cos\theta + \frac{\nu+1}{2(\nu+2)}\cos 2\theta \quad \text{and} \quad R(\theta) = \sum_{n=3}^{\infty} \frac{(-1)^n(\nu+1)^n \cos n\theta}{n!(\nu+1)...(\nu+n)}$$

then

$$\text{Re}f_\nu(r_\nu e^{i\theta}) = P(\theta) + R(\theta). \tag{5}$$

A study of the behaviour of the function

$$P : \mathbb{R} \to \mathbb{R},\ P(\theta) = 1 - \cos\theta + \frac{\nu+1}{2(\nu+2)}\cos 2\theta$$

leads to the inequalities

$$P(\theta) \geq \frac{\nu+1}{2(\nu+2)},\ \theta \in \mathbb{R},\ \nu \in (-1, 0) \ \text{ and}$$

$$P(\theta) \geq \frac{\nu^2 + 4\nu + 2}{4(\nu+1)(\nu+3)},\quad \theta \in \mathbb{R},\ \nu \in (0, \infty). \tag{6}$$

From (5), Lemma 1 and (6) it follows that

$$\text{Re}f_\nu(r_\nu e^{i\theta}) \geq \min_{\theta\in\mathbb{R}} P(\theta) - \max_{\theta\in\mathbb{R}} R(\theta) \geq 0.$$

□

Now Lemma 1 and Theorem 1 imply the following result:

Theorem 5. *If $\nu > -2$ then $\text{Re}f'_\nu(z) < 0$ for $z \in U(0, 4(\nu+2))$ and hence f_ν is univalent in $U(0, 4(\nu+2))$.*

Remark 6. *Theorem 1 and Theorem 2 improves Lemma 1 and Theorem 1 from* [3].

Theorem 7. *If $\nu > \frac{-17+\sqrt{33}}{8}$ then the function f_ν is convex in U.*

Proof. We introduce the notation $p_1(z) = 1 + \frac{zf''_\nu(z)}{f'_\nu(z)}$. The function f_ν is convex if and only if

$$\text{Re}p_1(z) > 0,\ z \in U. \tag{7}$$

It is simple to prove that if

$$\big|p_1(z) - 1\big| < 1, \ \ z \in U \tag{8}$$

then results (7).

Lemma 1 leads to the equality

$$\big|p_1(z) - 1\big| = \left|\frac{zf'_{\nu+1}(z)}{f_{\nu+1}(z)}\right|.$$

In (3) and (4) replacing ν by $\nu + 1$, we deduce that if $z \in U$ then

$$\left|\frac{zf'_{\nu+1}(z)}{f_{\nu+1}(z)}\right| \leq \frac{\nu + 3}{4\nu^2 + 18\nu + 19}.$$

Now to prove (7) it is enough to show that $\frac{\nu+3}{4\nu^2+18\nu+19} < 1$, but this is immediately using the condition $\nu > \frac{-17+\sqrt{33}}{8}$. □

Theorem 8. *If $\nu > \frac{\sqrt{3}}{2} - 1$ then the function g_ν defined by (1) is starlike of order $\frac{1}{2}$ in U.*

Proof. Let p be the function defined by the equality $p_2(z) = \frac{2zg'_\nu(z)}{g_\nu(z)} - 1$. Since $\frac{g_\nu(z)}{z} \neq 0, \ z \in U$ the function p_2 is analytic in U and $p_2(0) = 1$. The assertion of Theorem 2 is equivalent to

$$\mathrm{Re} p_2(z) > 0, \ \ z \in U. \tag{9}$$

It is simple to prove that if

$$\big|p_2(z) - 1\big| < 1, \ \ z \in U \tag{10}$$

then results (9).

On the other hand inequalities (2) and (3) lead to

$$\big|p_2(z) - 1\big| = 2\left|\frac{g'_\nu(z) - \frac{g_\nu(z)}{z}}{\frac{g_\nu(z)}{z}}\right| < \frac{2(2+\nu)}{4\nu^2 + 10\nu + 5}, \ \ z \in U.$$

This means that if $\frac{2(2+\nu)}{4\nu^2+10\nu+5} < 1$ then (8) holds, but this inequality is a consequence of the condition $\nu > \frac{\sqrt{3}}{2} - 1$. □

Corollary 9. *If $\nu > \frac{\sqrt{3}}{2} - 1$ then the function h_ν defined by the equality $h_\nu(z) = z^{1-\nu} J_\nu(z)$ is starlike in U.*

The proof of this result is based on Theorem 3 and is similar to the proof of Corollary 2 in [3], hence we do not reproduce it here again.

Remark 10. *Theorem 3, Theorem 4 and Corollary 1 improves the results of Theorem 2, Theorem 3 and Corollary 2 in* [3].

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