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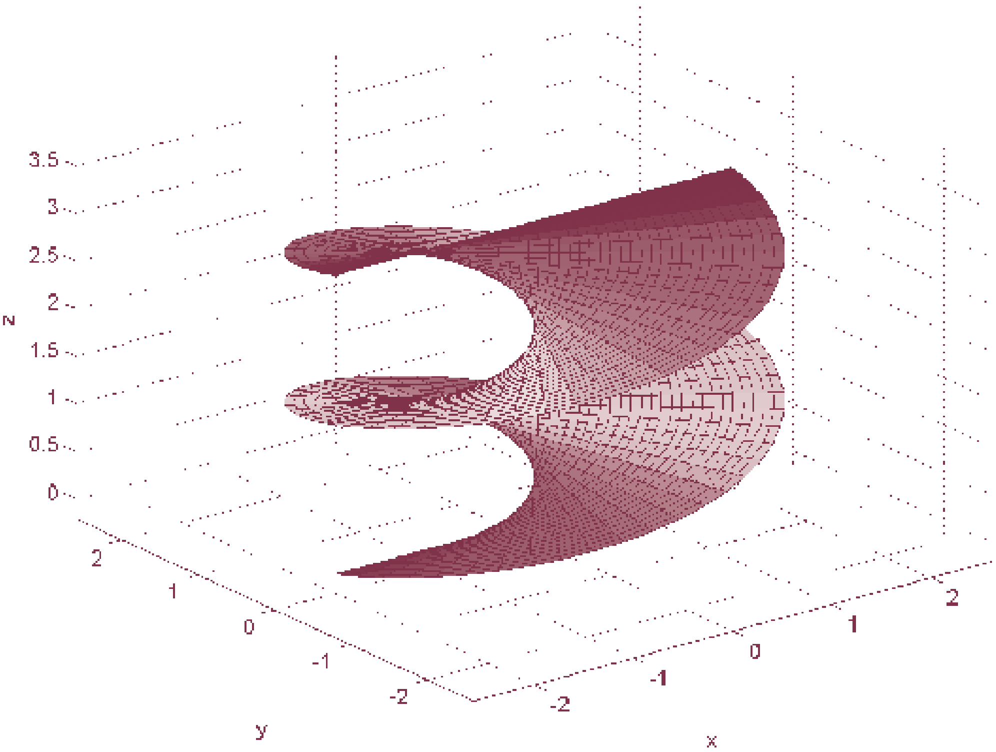

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INDEX OF THE ELASTICITY OPERATOR WITH CONTACT WITHOUT FRICTION BOUNDARY CONDITIONS

B. BENABDERRAHMANE, B. NOUIRI, AND Y. BOUKHATEM

Abstract. In this paper, one considers a contact without friction problem for the elasticity system, using the results given by P. Grisvard and B. Benabderrahmane respectively in ([1]: Far East J.Appl. Maths., Vol.24, No.3, p.373-380, (2006) and [2]: C.R. Acad. Sci. Paris, Ser.I Math. 304(3) (1987), 71-73), one proves that the Laplace operator is injective and with closed image of codimension N in $H^s(\Omega)^2$, and consequently Δ have an index which is equal to $-N$, where N denotes the number of the singular solutions of the considered problem. Using the above results one proves that the elasticity operator, denoted by $\mathcal{L}$ has an index which is equal to $-2N$, by basing on the *Fredholm* alternative. This enables us to deduce the explicitly singular solutions and to describe the singular behavior of the solutions in the polygon.

1. **Problem statement**

The aim of this statement is to deduce the index results for the contact without friction problem which is governed by the *Lamé* system in a polygon. Consequently, it can be given the explicitly singular solutions and to describe the singular behavior of the solutions in the polygon. Let $f \in L^2(\Omega)$, consider the following problem

$$(P): \begin{cases} \mathcal{L}u = f \quad \text{in } \Omega \\ \begin{cases} u.\eta^j = 0 \\ \left(\Sigma(u).\eta^j\right).\tau^j = 0 \end{cases} \quad \text{on } \Gamma_j, j = 1, ..., J, \end{cases}$$

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where Ω is homogeneous, elastic and isotropic medium occupying a bounded domain in $\mathbb{R}^2$, limited by straight polygonal boundary $\Gamma = \bigcup_{j=1}^{J} \overline{\Gamma}_j$, $\Gamma_i \cap \Gamma_j = \varnothing, \forall i \neq j$, $\Gamma_j =]S_j, S_{j+1}[$, where S_j are the different corners of Ω. $\eta^j = \left(\eta_1^j, \eta_2^j\right)$, $\tau^j = \left(\tau_1^j, \tau_2^j\right)$ designate the outward unit normal vector, and the tangential unit vector in Γ_j respectively. $\omega_j, (0 < \omega \leq 2\pi)$ represents the opening of the angle that makes Γ_j and Γ_{j+1} toward the interior of Ω.

$\mathcal{L}$ is the *Lamé* operator defined by:

$$\mathcal{L} : \lambda\Delta + (\lambda + \mu)\nabla div;$$

where Δ, ∇ and div represent respectively the *Laplace*, *Gradient* and *Divergence* . u, f is the displacement vector, and external forces density respectively. $\Sigma(u) = (\sigma_{ij}(u))_{ij}$ is the stress tensor given by *Hook*'s law using *Lamé* coefficients λ and μ which are strictly positive and such that $(\lambda + \mu) > 0$,

$$\sigma_{ij}(u) = 2\mu\varepsilon_{ij}(u) + \lambda tr(\varepsilon(u))\delta_{ij},$$

where δ_{ij} is a *Kronecker* symbol and $\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the linearized tensor of linear elasticity.

The particular case $\lambda + \mu \to 0$, reduces the problem to

$$(E) : \begin{cases} \Delta u = \frac{f}{\lambda}(= f) \text{ in } \Omega \\ \begin{cases} u.\eta^j = 0 \\ \left(\Sigma(u).\eta^j\right).\tau^j = 0 \end{cases} \text{ on } \Gamma_j, j = 1, ..., J, \end{cases}$$

with $\sigma_{ij}(u) = \mu(\partial_i u_j + \partial_j u_i) - (\partial_1 u_2 + \partial_2 u_1)\delta_{ij}$, where $\partial_j u$ is used as the partial derivative of u with respect to x_j.

Generally, the problem (E) hasn't sufficiently regular solution, hence we try to impose conditions on f in order to obtain desired solutions, i.e., we search for necessary and sufficient conditions on f allowing variational solution included in the space V such as

$$V = \left\{u \in H^1(\Omega)^2;\ u.\eta = 0 \text{ on } \Gamma\right\}$$

is in $H^{s+2}(\Omega)^2 \cap V$ $(s \geq 0)$, where H^{s+2} denotes $(s + 2)$ order *Sobolev* space.

The resolution of this problem is based on the following inequality

$$\|u\|_{s+2} \leq C_s \|u\|_s\,,\ u \in H^{s+2}(\Omega)^2 \cap V. \tag{1.1}$$

This inequality is not always true, for example the case when Ω is a polygon. However, we may prove this a priori inequality is verified all the same, when imposing a supplementary condition: $D_x u + D_y u \in H_0^s(\Omega)$ i.e.

$$u \in W_s(\Omega) = \left\{u \in H^{s+2}(\Omega)^2 \cap V;\ \ (\Sigma(u).\eta)\,.\tau = 0 \text{ on } \Gamma,\ D_x u + D_y u \in H_0^s(\Omega)\right\}.$$

By explicit calculations (see [1]), studying the boundary conditions considered, we prove the following Lemma.

Lemma 1. *The problem (P) amount to the two problems of oblique derivatives boundary conditions without coupling:*

$$(E_k) : \begin{cases} \Delta u_k = f_k \ \ on \ \ \Omega \\ \alpha_j D_x u_k + \beta_j D_y u_k = 0 \ \ in\ \Gamma_j,\ \ j = 1, ..., J \\ \alpha_j^2 + \beta_j^2 \neq 0. \end{cases} ,\ \ k = 1, 2$$

2. A priori inequality

This section is dedicated to demonstration of a priory inequality (1.1). To simplify the study, in all of this section we will write u instead u_k and (E) instead (E_k), because the $(E_k), k = 1, 2$ are two similar one-dimensional problems.

The inequality (1.1) follows from the following simple inequality

$$\|u\|_2 \leq C_0 \left(\|\Delta u\|_0 + \|u\|_1\right),\ \forall u \in K_{\alpha,\beta}(\Omega), \tag{1.2}$$

with

$$K_{\alpha,\beta}(\Omega) = \left\{\ u \in H^2(\Omega); \alpha_j D_x u + \beta_j D_y u = 0 \text{ on } \Gamma_j,\ \alpha_j^2 + \beta_j^2 \neq 0,\ j = 1, ..., J\right\}.$$

Remark 1. *It is known (in [1]) that there is a constant C_0 such as the inequality (1.2) is verified for all $u \in K_{\alpha,\beta}(\Omega)$.*

Proposition 1. *There is a constant C such that the inequality* (1.1) *takes place for all* $u \in W_s(\Omega)$.

Proof. The essential idea of the demonstration is to search the boundary conditions verified by :

$$v = D_x^n D_y^m u, \text{ with } n + m \leq s.$$

For this, we parameterize the segments Γ_j using the following applications:

$$[0,1] \longrightarrow \mathbb{R}^2$$

$$\lambda \mapsto \left(t_{1j}\lambda + t_{1j}^{'}, t_{2j}\lambda + t_{2j}^{'}\right).$$

We have the condition $\alpha_j D_x u + \beta_j D_y u = 0$ on Γ_j, $j = 1,..,J$, therefore

$$\alpha_j D_x u\left(t_{1j}\lambda + t_{1j}^{'}, t_{2j}\lambda + t_{2j}^{'}\right) + \beta_j D_y u\left(t_{1j}\lambda + t_{1j}^{'}, t_{2j}\lambda + t_{2j}^{'}\right) = 0,$$

$\lambda \in [0,1]$, $j = 1,...,J$, hence by derivation, we obtain

$$\alpha_j \sum_{k=0}^{s} C_s^k t_{1j}^k t_{2j}^{s-k} D_x^{k+1} D_y^{s-k} u + \beta_j \sum_{k=0}^{s} C_s^k t_{1j}^k t_{2j}^{s-k} D_x^k D_y^{s+1-k} u = 0, \tag{1.3}$$

on the other hand we have $D_x u + D_y u \in H_0^s(\Omega)^2$, therefore

$$D_x^p D_y^q (D_x u + D_y u) = 0, \text{ on } \Gamma_j, \ j = 1,...,J, \text{ for } p + q = s (s \geq 1)$$

and we obtain consequently:

$$D_x^{p+1} D_y^q u = -D_x^p D_y^{q+1} u \text{ on } \Gamma_j \tag{1.4}$$

from which we deduce that:

$$\left\{\begin{array}{l} D_x^k D_y^{s+1-k} u = (-1)^k D_y^{s+1} u, \ k = 0,1,..... \\ D_x^{k+1} D_y^{s-k} u = (-1)^k D_x D_y^s u, \ k = 0,1,..... \end{array}\right. \text{ on } \Gamma_j, \ j = 1,...,J$$

from which, we can rewrite the equation (1.3) as follow:

$$\alpha_j \sum_{k=0}^{s} C_s^k t_{1j}^k t_{2j}^{s-k} (-1)^k D_x D_y^s u + \beta_j \sum_{k=0}^{s} C_s^k t_{1j}^k t_{2j}^{s-k} (-1)^k D_y D_y^s u = 0. \tag{1.5}$$

Thus we obtain the condition of oblique derivative verified by $v = D_y^s u$. This condition is

$$\alpha_j^{'} D_x v + \beta_j^{'} D_y v = 0 \text{ on } \Gamma_j, \quad j = 1, ..., J \tag{1.6}$$

with

$$\begin{cases} \alpha_j^{'} = \sum\limits_{k=0}^{s} (-1)^{\alpha} \alpha_j C_s^k t_{1j}^k t_{2j}^{s-k} \\ \beta_j^{'} = \sum\limits_{k=0}^{s} (-1)^{\alpha} \beta_j C_s^k t_{1j}^k t_{2j}^{s-k} \end{cases} \tag{1.7}$$

According to (1.4) the condition is also verified by $D_x^1 D_y^{s-1} u, D_x^2 D_y^{s-2} u, ...,$ we can apply the previous remark, if $\left(\alpha_j^{'}\right)^2 + \left(\beta_j^{'}\right)^2 \neq 0$ is verified. For this we note that

$$\left(\alpha_j^{'}\right)^2 + \left(\beta_j^{'}\right)^2 = \left[\sum_{k=0}^{s} (-1)^{\alpha} C_s^k t_{1j}^k t_{2j}^{s-k}\right]^2 \left(\ \alpha_j^2 + \beta_j^2\right)$$

$$= \left[(t_{1j} + t_{2j})^s\right]^2 \left(\ \alpha_j^2 + \beta_j^2\right) \neq 0.$$

Since the two numbers t_{1j}, t_{2j} can't vanish simultaneously (because they are the coefficients of parameterization of Γ_j). Then we have proved that for $n + m \leq s$ $(s \geq 1)$, there exist α and β such as $D_x^n D_y^m u \in K_{\alpha,\beta}$ for all $u \in W_s(\Omega)$. In the case $s = 0$, the condition of oblique derivative (1.2) is verified by u. Then

$$\sum_{n+m=0}^{s} \left\|D_x^n D_y^m u\right\|_2 \leq C_0 \left(\sum_{n+m=0}^{s} \left\|D_x^n D_y^m \Delta u\right\|_0 + \sum_{n+m=0}^{s} \left\|D_x^n D_y^m u\right\|_1\right)$$

from which we deduce the inequality $\|u\|_{s+2} \leq C_S \left(\|\Delta u\|_s + \|u\|_1\right)$ and we obtain the inequality (1.1), using the well know inequality $\|u\|_1 \leq C \|\Delta u\|_0$ in V. □

3. **Fredholm alternative**

Let $R_s(\Omega)$ be the subspace of $H^s(\Omega)^2$ defined by

$$R_s(\Omega) = \{f = \Delta u; u \in W_s(\Omega)\}.$$

Remark 2. *Using the inequality* (1.1)*, there can be seen that* $R_s(\Omega)$ *is a closed subspace of* $H^s(\Omega)^2$. *Let* $N_s(\Omega)$ *be the orthogonal of* $R_s(\Omega)$ *in* $H^{-s}(\Omega)^2$, *i.e.*

$$N_s(\Omega) = \left\{v \in H^{-s}(\Omega)^2,\ (v, f) = 0 \textit{ for all } f \in R_s(\Omega)\right\},$$

where (v, f) *represents the duality pairing between* $H^s(\Omega)^2$ *and* $H^{-s}(\Omega)^2$.

Thanks to a generalization of *Green* formula, it can be proved the following Lemma:

Lemma 2. *The orthogonal of* $R_s(\Omega)$ *in* $H^{-s}(\Omega)$ *is the vector subspace,* $N_s(\Omega)$, *of* $H^{-s}(\Omega)$ *defined by :*

$$N_s(\Omega) = \left\{ v \in H^{-s}(\Omega); \Delta v = 0 \text{ in } \Omega ; \gamma_s \left(v.\eta^j, \left(\Sigma(v).\eta^j \right).\tau^j \right) = \mathbf{0} \right\},$$

where γ_s *is a generalized operator trace defined by duality.*

According to a generalization of *Green* formula, we will see that the orthogonal of R_s in H^{-s} is

$$N_s(\Omega) = \left\{ v \in H^{-s}; \Delta v = 0, \text{ in } \Omega; \ \gamma_s \left(u.\eta, (\Sigma(u).\eta).\tau \right) = 0 \right\}.$$

We will have the necessary and sufficiencies conditions on $f \in H^s(\Omega)^2$, in order to allow for a variational solution to be in $H^{s+2} \cap V$. This condition is expressed as follows:

$$(f, v) = 0, \text{ for all } v \in N_s(\Omega).$$

3.1. **Laplace operator Index.** In the case, when Ω doesn't have any angle ω of the form

$$\frac{\ell\pi}{k+2}; \ \ell, k \in \mathbb{N}, \ \ell \neq (k+2), \ k = 1, ..., s$$

the dimension of $N_s(\Omega)$ is exactly equal to N, where

$$N = \left\{ k \in \mathbb{N}; 1 \leq k \leq \left(\frac{\omega}{\pi} \right) s \right\}.$$

The ω are well specified at the end of the last section.

Using the techniques of Grisvard [2], it is shown the following result:

Lemma 3. *Suppose* Ω *is a simply connected,* $\Delta : W_s(\Omega) \to H^s(\Omega)^2$ *is an operator with index. More precisely, thanks to inequality* (1.1), *the Laplace operator* Δ *is injective, has a closed image of codimension equal to* $N < +\infty$ *in* $H^s(\Omega)^2$. *Consequently*

$$Ind(\Delta) = \dim Ker(\Delta) - co\dim(\Delta) = 0 - N = -N.$$

3.2. **Calculation of the operator $\mathcal{L}$ index.** Now come back to the problem (P), as defined above. In the following, and for the problem (P), essentially we are interested by the demonstration of the following inequality (1.8) :

$$\|u\|_{H^2(\Omega)^2} \leq C \|u\|_{L^2(\Omega)^2}, \tag{1.8}$$

where C is an independent constant of Lamé coefficients.

Lemma 4. *We have*

$$\left(D_x^2 u, D_y^2 u\right) = \|D_x D_y u\|^2_{L^2(\Omega)^2}, \forall u \in H^2(\Omega)^2 \cap V. \tag{1.9}$$

The proof of this Lemma is made by a party integration, using the density of $H^3(\Omega)^2 \cap V$ in $H^2(\Omega)^2 \cap V$.

Thus, we recover the restriction on the coefficients of Lamé $|\lambda| < \sqrt{3}|\mu|$ (see [2]), which is necessary so that the inequality (1.8) is verified. Thanks to inequality (1.8), the operator of Lamé is injective, has a closed image of $H^2(\Omega)^2 \cap V$ in $L^2(\Omega)^2$. Therefore, $\mathcal{L}$ is semi-*Fredholm* operator. As the operator $\mathcal{L}$ depends continuously of λ, its index (see [4]) is independent of λ. In the particular case where $\lambda = -\mu$ and according to the Lemma 1, the problem (P) amounts to problems $(E_k), k = 1, 2$, where $Ind(E_k) = N, k = 1, 2$, and consequently the index of the operator $\mathcal{L}$ is equal to $-2N$.

4. Singular solutions

Thanks to the index of the operator $\mathcal{L}$ that there exist $2N$ linearly independent functions S_j and $S'_j \in V$, such as

$$S_j, S'_j \notin H^2(\Omega)^2 \text{ and } \mathcal{L}S_j, \mathcal{L}S'_j \in L^2(\Omega)^2$$

and as $\mathcal{L}$ is an isomorphism of

$$Sp\left(H^2(\Omega)^2, S_j, S'_j\right) \cap V \text{ on } L^2(\Omega)^2, j = 1, ..., J,$$

where the symbol Sp designates the vector space generated by the elements continued in the bracket that follow.

We can calculate these functions explicitly, by searching S such as

$$S(r,\theta) = r^{\alpha}\Psi_{\alpha}(\theta),$$

solution of $\mathcal{L}S = 0$ in the sector

$$\Sigma = \{\theta; 0 < \theta < \omega\},$$

where

$$\Psi_{\alpha}(\theta) = (W_1(\theta)\cos\theta - W_2(\theta)\sin\theta, W_1(\theta)\sin\theta + W_2(\theta)\cos\theta)^t,$$

with

$$\begin{cases} W_1'(0) = W_2(0) = 0 \\ W_1'(\omega) = W_2(\omega) = 0. \end{cases}$$

Then, we find that the number α must be a solution of the following transcendent equation

$$\sin^2\alpha\omega = \sin\omega \tag{1.10}$$

and such that

$$\Psi_{\alpha}(\theta) = \begin{cases} ((\rho_0+\rho_1)\cos(\alpha-2)\theta - (\rho_1-\rho_0)\cos\alpha\theta)\sin(\alpha+1)\omega + \\ +2\rho_1\sin(\alpha-1)\omega\cos\alpha\theta \\ (-(\rho_0+\rho_1)\sin(\alpha-2)\theta - (\rho_1-\rho_0)\sin\alpha\theta)\sin(\alpha+1)\omega + \\ +2\rho_1\sin(\alpha-1)\omega\sin\alpha\theta \end{cases}$$

with $\rho_0 = \nu_0(\alpha-1)-2, \rho_1 = \nu_0(\alpha+1)+2$, $\nu_0 = \frac{1}{1-2\nu}$, where ν is the *Poisson* coefficient.

It is well clear that the solutions of the transcendent equation (1.10) are real and explicitly given by

$$\alpha_\ell = \frac{\ell\pi}{\omega} \pm 1, \ell \in \mathbb{Z}^*.$$

Besides, if $\omega \neq \frac{k\pi}{\omega}, k \in \mathbb{Z}^*$ then the solutions are simple, else they are double.

In conclusion the transcendent equation (1.10) possesses one simple solution α in $]0,1[$, when $\omega \in \left]\frac{\pi}{2},\pi\right[\cup \left]\frac{3\pi}{2},2\pi\right[$ and it has only one solution double $\alpha' = \frac{1}{3}$, when $\omega = \frac{3\pi}{2}$. This will permit the demonstration of the following theorem which is described the singular behavior of the solution of the problem (P).

Theorem 1. *For $f \in L^2(\Omega)^2$, if $u \in V$ is a variational solution of the problem (P), then there are constants C_α and C'_α such as*

$$u - C_\alpha r^\alpha \Psi_\alpha(\theta) - C'_\alpha \left(\log r \Psi_\alpha(\theta) + \frac{\partial \Psi_\alpha(\theta)}{\partial \alpha} \right)_{\alpha = \frac{1}{3}} \in H^2(\Omega)^2 .$$

The first sum in this expression is extended to all real numbers α simple solution of the transcendent equation (1.10), whereas the second sum is extended to all real numbers α double solution of the equation (1.10).

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RADIATION AND VARIABLE VISCOSITY EFFECTS IN FORCED CONVECTION FROM A HORIZONTAL PLATE EMBEDDED IN A POROUS MEDIUM

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Abstract. Radiation and temperature dependent viscosity effects on forced convection boundary layer flow over a horizontal plate embedded in a fluid-saturated porous media is studied in this paper. Darcy's law model, Rosseland model for radiation and an inverse proportional law for temperature dependent viscosity have been considered. The transformed ordinary differential equations are solved numerically, and a very good agreement between the present results and those reported for particular situations were found.

1. **Introduction**

Many technological applications in geophysics and conservation energy systems, thermal insulations, cooling, water waste disposal, petroleum industry involve mathematical models related to flows in fluid-saturated porous media. Recent monographs by Ingham and Pop [1,2,3], Pop and Ingham [4], Bejan et al.[5] and Vafai [6] give an excellent summary of the work on the subject.

It is well known that viscosity of many fluids depends strongly by temperature and this change influence also the flow. Water's viscosity decreases by about 240 percent when temperatures varies form 10^oC ($\mu = 0.0131$ g/cm.s) to 50^oC ($\mu = 0.00548$ g/cm.s) (see Ling and Dybs [7]), where μ is the dynamic viscosity of water. Thus, one can make significant errors when such viscosity variations are not considered.

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When technological processes take place at high temperatures (metal and glass cooling) thermal radiation effects start to play an important role and cannot be neglected (see Modest [8]). Previous works in this area were done by Ling and Dybs [7] who examined the effect of variable viscosity on forced convection past a horizontal flat plate embedded in a fluid-saturated porous medium. Postelnicu et al. [9] considered also in addition to the variable viscosity in this problem the internal heat generation effects. For viscous fluids, Kafoussias and Williams [10] and Kafoussias et al. [11] studied the combined free and forced convection on an isothermal vertical flat plate with temperature dependent viscosity while, Soundalgekar et al. [12] and Ali [13] considered the same problem for moving surfaces. The combined effects radiation and variable viscosity were also considered by Elbashbesy and Dimian [14] on the flow over a wedge.

2. **Basic Equations**

Consider the steady forced convection flow adjacent to a heated horizontal flat plate, which is embedded in an opaque fluid-saturated porous medium of ambient temperature T_∞ and velocity U_∞ as shown in Figure 1. It is assumed that the temperature of the plate is constant T_w ($T_w > T_\infty$) and there is a radiation heat transfer effect modeled by the Rosseland approximation. Following Ling and Dybs [7] we consider the temperature dependent dynamic viscosity, μ, given by:

$$\frac{1}{\mu} = \frac{1}{\mu_\infty}[1 + \gamma(T - T_\infty)] \tag{1}$$

where μ_∞ is the dynamic viscosity of the ambient fluid and γ is a constant.

Under the boundary-layer and Boussinesq approximations the governing boundary layer equations can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

$$\frac{\partial}{\partial y}(\mu u) = 0 \tag{3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho_\infty c_p}\frac{\partial q^r}{\partial y} \tag{4}$$

where x and y are the Cartesian co-ordinate along the plate and normal to it, respectively, u and v are the velocity components along x and y -axes, T is the temperature, α is the effective thermal diffusivity of the porous medium ρ_∞ is the ambient density and c_p is the specific heat at constant pressure. We assume that the radiation heat flux, q^r, is given by, see Modest[8],

$$q^r = -\left(\frac{4\sigma}{3\chi}\right)\frac{\partial T^4}{\partial y} \tag{5}$$

where σ is the Stefan-Boltzman's constant and χ is the average absorption coefficient in Rosseland approximation. The boundary conditions of equations (2)-(4) are:

$$\begin{aligned} v = 0, T = T_w \qquad & \text{at} \qquad y = 0 \\ u = U_\infty, T \to T_\infty \qquad & \text{as} \qquad y \to \infty \end{aligned} \tag{6}$$

Equation (1) is written, for convenience, as

$$\frac{1}{\mu} = a(T - T_e) \tag{7}$$

where $a = \gamma/\mu_\infty$ is a constant with $a > 0$ for liquids and $a < 0$ for gases, see Soundalgekar et al.[12], and $T_e = T_\infty - (1/\gamma)$ is a reference temperature.

Further, introducing the stream function,ψ, defined as usual by $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, and using (7) in (2) equations (2)-(4) become:

$$\frac{T - T_e}{T_\infty - T_e} = \frac{1}{U_\infty}\frac{\partial\psi}{\partial y} \tag{8}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{1}{\rho_\infty c_p}\frac{16\sigma}{3\chi}\left(T^3\frac{\partial T}{\partial y}\right) \tag{9}$$

These partial differential equations can now be reduced to ordinary differential equations by introducing the following similarity variables

$$\psi = (\alpha U_\infty x)^{1/2} f(\eta), \qquad \theta(\eta) = \frac{T - T_w}{T_\infty - T_w}, \qquad \eta = (U_\infty x/\alpha)^{1/2}\frac{y}{x} \tag{10}$$

Using transformations (10) in equations (8) - (10) we get:

$$f' = \frac{\theta + \theta_e}{1 + \theta_e} \tag{11}$$

$$\left(\left(1 + \frac{4}{3}N\left(\theta_w + (1 - \theta_w)\theta\right)^3\right)\theta'\right)' + \frac{1}{2}f\theta' = 0 \tag{12}$$

$$f'(0) = 0, \qquad \theta(0) = 0, \qquad \theta(\infty) = 1 \tag{13}$$

where primes denote differentiation with respect to η and the parameter θ_e is the dimensionless reference temperature given by

$$\theta_e = \frac{T_w - T_e}{T_\infty - T_w} = \frac{1}{\gamma(T_\infty - T_w)} \tag{14}$$

In the energy equation (12) θ_w and N are the wall temperature and the radiation parameters defined as:

$$\theta_w = \frac{T_w}{T_\infty}, \qquad N = \frac{4\sigma T_\infty^3}{k\chi} \tag{15}$$

Using the energetic balance at the plate we deduce the convection heat transfer coefficient, h, defined as:

$$-k\left[\frac{\partial T}{\partial y}\right]_{y=0} + q^r = h(T_w - T_\infty) \tag{16}$$

and the local Nusselt number, Nu_x, which is given by:

$$\frac{Nu_x}{Pe_x^{1/2}} = \theta'(0)\left(1 + \frac{4}{3}N\theta_w^3\right) \tag{17}$$

where $Pe_x = U_\infty x/\alpha$ is the local Pclet number. We mention that for $N = 0$ (radiation is absent) equations (11) - (13) reduce to those obtained by Ling and Dybs [7].

3. **Results and Discussions**

Equations (11) and (12), subject to the boundary conditions (13) have been solved numerically using a 4th Runge-Kutta method coupled with a shooting technique for some values of the parameters θ_e, N and θ_w. In the particular case $N = 0$ (i.e. radiation is absent) the results for the dimensionless heat transfer at the plate, $-\theta'(0)$, were compared with those obtained by Ling and Dybs [7], see Table 1. It is seen that these results are in a very good agreement (see Table 1). The results obtained in the presence of radiation ($N \neq 0$) are shown in Table 2 for some values of the radiation parameter N and temperature parameter θ_w.

i) Influence of the parameter θ_e.

The dimensionless temperature and viscosity profiles $f(\eta)$ and $\theta(eta)$ are shown in Figures 2 and 3 for some values of the parameter θ_e when $N = 1$ and 10, and

$\theta_w = 1.5$. We can see from these figures that the thickness of the thermal boundary layer and viscous boundary layer increase with the decreasing of the parameter θ_e. It should also be noticed that for $\theta_e >> 1$, the predicted temperature profiles are close to those when viscosity is constant. We also notice that for the constant viscosity ($\theta_e \to \infty$) the value of $-\theta'(0) = 0.564$, agrees with the value reported by Bejan [15]. When $\theta_e \to \infty$ the velocity profiles $f'(\eta)$ are convergent to the constant profile $f'(\eta) = 1$ corresponding to the constant viscosity case (see Figure 3).

ii) Influence of the parameter N.

Figures 4 and 5 show that increasing of the radiation parameter N leads to an increasing of the thermal and viscous boundary layers. The influence of the radiation parameter N is higher for small values of θ_e (i.e. the radiation effects are more pregnant if the dependence of the viscosity with the temperature is stronger) .

iii) Influence of the parameter θ_w

Figures 6 and 7 present temperatures and velocity profiles for different values of the parameter θ_w. It is seen that the thermal and viscous boundary layer increase with the increasing of θ_w, the effect being more pregnant for small values of θ_e.

TABLE 1. Values of $-\theta'(0)$ for different values of θ_e and $N = 0$

θ_e	Ling and Dybs [7]	Present results
0	0.332	0.3320
0.05	0.347	0.3474
0.10	0.361	0.3606
0.25	0.392	0.3916
0.5	0.426	0.4260
1.00	0.465	0.4649
2.00	0.500	0.5004
5.00	0.533	0.5333
10.00	0.548	0.5476
∞	0.564	0.5641

TABLE 2. Values of $-\theta'(0)$ for different values of θ_e, N and θ_w

θ_e	N	$\theta_w = 1.1$	$\theta_w = 1.5$	$\theta_w = 2$
0	1	0.190643	0.114643	0.067368
	5	0.099127	0.052877	0.029892
	10	0.071675	0.037508	0.021103
0.1	1	0.207738	0.126601	0.075515
	5	0.108174	0.058657	0.033674
	10	0.078238	0.041640	0.023791
1	1	0.269628	0.168878	0.103644
	5	0.140838	0.078948	0.046644
	10	0.101924	0.056126	0.032998
10	1	0.318507	0.201712	0.125149
	5	0.166585	0.094628	0.056516
	10	0.120587	0.067311	0.040002
∞	1	0.328301	0.208257	0.129416
	5	0.171739	0.097749	0.058472
	10	0.124323	0.069537	0.041389

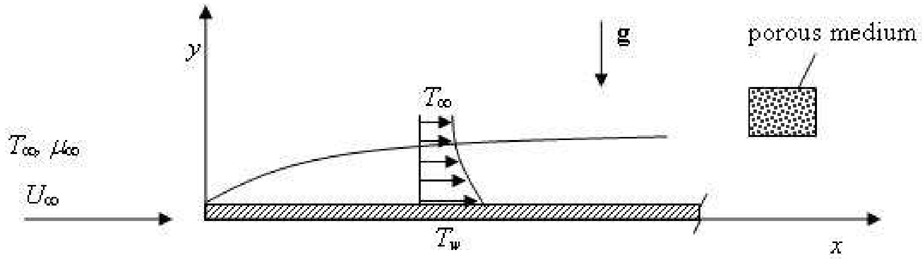

FIGURE 1. Physical model and co-ordinate system

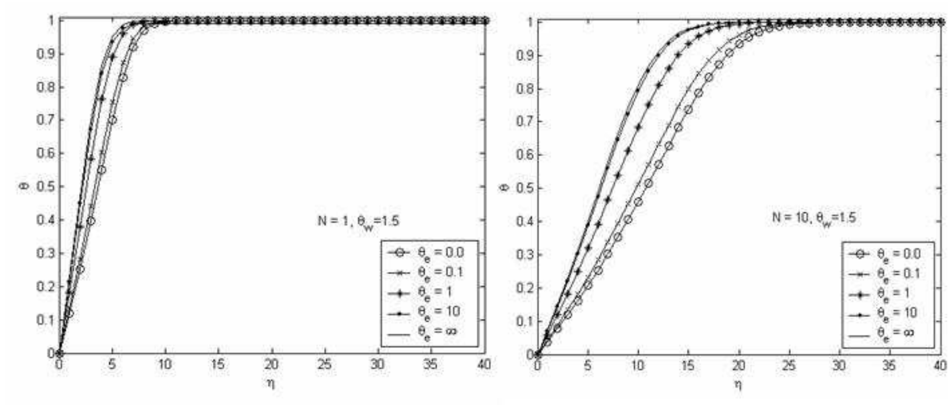

FIGURE 2. Variation of dimensionless temperature θ for different values of the parameter θ_e

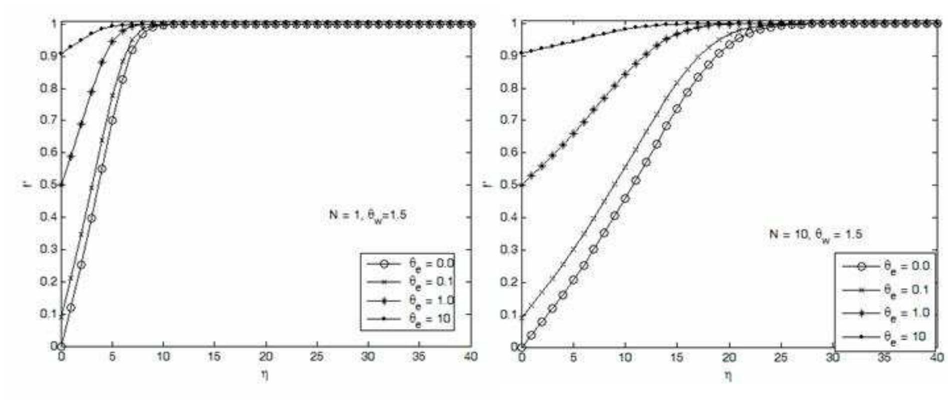

FIGURE 3. Variation of dimensionless velocity $f'(\eta)$ for different values of the parameter θ_e

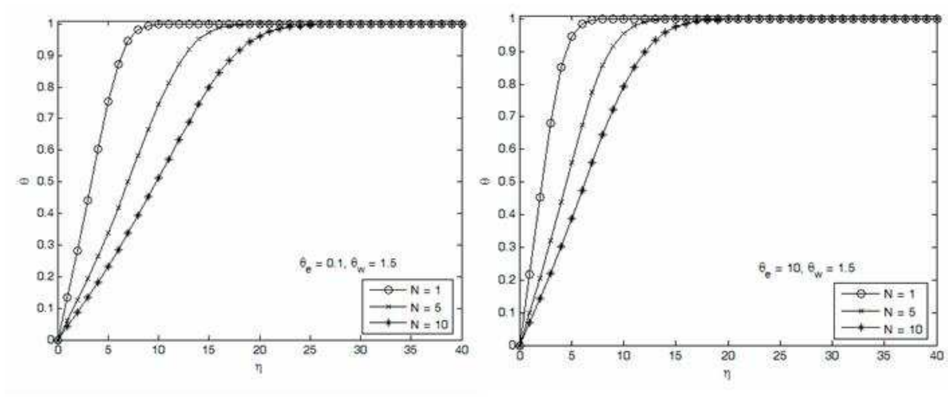

FIGURE 4. Variation of dimensionless temperature $\theta(\eta)$ for different values of the parameter N

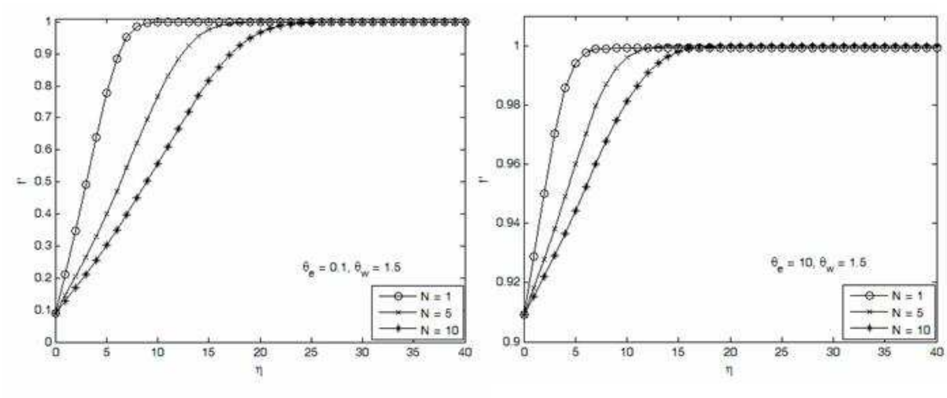

FIGURE 5. Variation of dimensionless velocity $f'(\eta)$ for different values of the parameter N

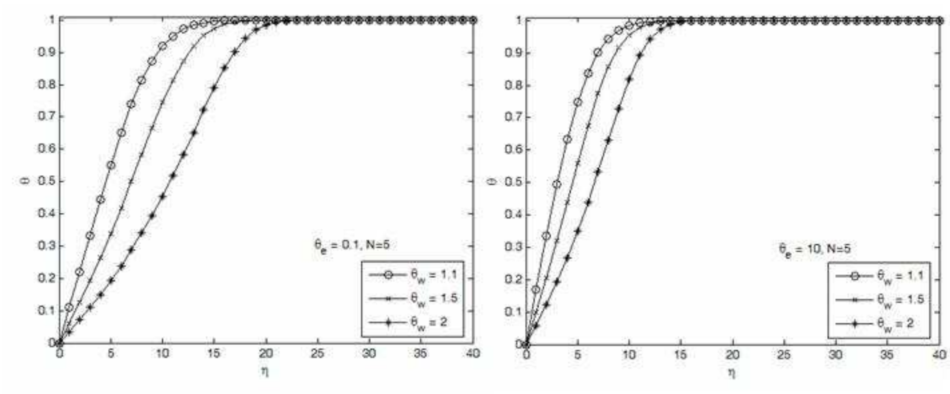

FIGURE 6. Variation of dimensionless temperature θ for different values of the parameter θ_w

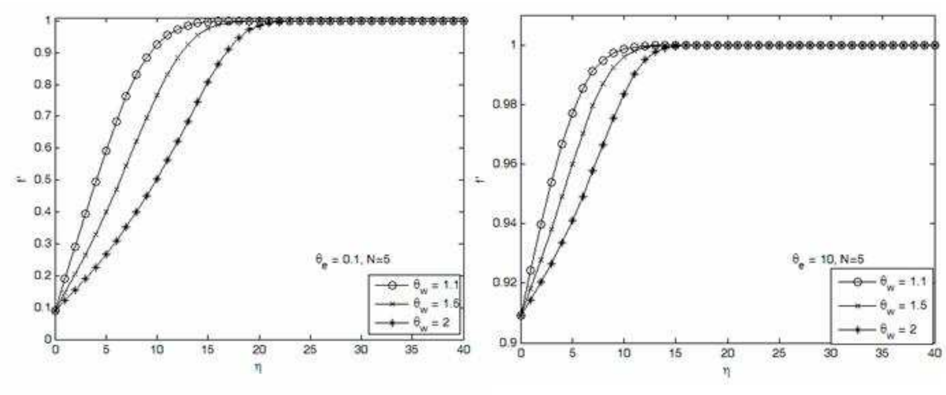

FIGURE 7. Variation of dimensionless velocity $f'(\eta)$ for different values of the parameter θ_w

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NUMERICAL METHOD FOR FREE SURFACE VISCOUS FLOWS

TIBERIU IOANA AND TITUS PETRILA

Abstract. In this paper we present a new algorithm for studying the flow of viscous fluids with a free surface. This algorithm is based on an optimization solution strategy. Numerical results are presented in the case of a particular fluid flow problem.

1. Mathematical model and solution strategy

A viscous incompressible fluid of dynamic viscosity η, pressure p, density ρ and velocity $\mathbf{u}$ flows over a solid boundary Γ . The fluid is up bounded by a free surface S, the fluid domain being denoted by D. It is assumed that the flow is steady and the exterior force is represented by gravity.

For this problem we write the system of equations

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\varphi-Re^{-1}\nabla\cdot\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right) &= 0, && \mathbf{x}\in D && (1)\\
\nabla\cdot\mathbf{u} &= 0, && \mathbf{x}\in D && (2)\\
\varphi-Fr^{-2}y &= 0, && \mathbf{x}\in S && (3)\\
Re^{-1}\mathbf{t}\cdot\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right)\mathbf{n} &= 0, && \mathbf{x}\in S && (4)\\
\mathbf{u}\cdot\mathbf{n} &= 0, && \mathbf{x}\in S && (5)\\
\mathbf{u} &= 0, && \mathbf{x}\in\Gamma && (6)\\
\mathbf{u} &= \mathbf{u}_d, && x\to\infty, && (7)
\end{aligned}
$$

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where φ is the hydrodynamic component of the fluid pressure, $\varphi(x,y) = p(x,y) + Fr^{-2}y$. Here we denote by Re and Fr the Reynolds and Froude numbers respectively, while $\mathbf{t}$ and $\mathbf{n}$ are the unit tanget and outward normal vectors respectively.

For solving this problem we use a solution strategy based on an optimization approach. Let's denote by S_0 the initial position of the free boundary. We assume that the new position of the free boundary S_ϵ, is related with its original position by

$$(x_0, y_0) \longrightarrow (x_\epsilon, y_\epsilon) = (x_0, y_0) - \epsilon \mathbf{n}.$$

(such a mapping is used for instance in [7])

Now an algorithm is constructed to obtain the shape of the free surface and the velocity field. Precisely, we have to chose first an initial position of the unknown free boundary and this position will be updated with $-\gamma \mathrm{grad}_{\mathbf{n}} J^*$, where

$$\begin{aligned} J^*(S) &= \int_S p^2 dS + \int_D \mathbf{v} \cdot \left((\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla\varphi - Re^{-1} \nabla \cdot \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \right) dD \\ &+ \int_D w \, (\nabla \cdot \mathbf{u}) \, dD. \end{aligned} \tag{8}$$

Then the procedure restart, with that new free boundary and so on. This procedure will stop when $\mathrm{grad}_{\mathbf{n}} J^* < \epsilon$. Folowing [2], [3] we get

$$\begin{aligned} \mathrm{grad}_{\mathbf{n}} J^* &= \int_S \{ (1 - \varphi + Fr^{-2} y) \mathbf{n} \cdot \nabla\varphi - Fr^{-2} \mathbf{n} \cdot \mathbf{j} \\ &\quad - \mathbf{v} \cdot Re^{-1} \left((\nabla (\mathbf{n} \cdot \nabla \mathbf{u})) + (\nabla (\mathbf{n} \cdot \nabla \mathbf{u}))^T \right) \mathbf{n} \} dS, \end{aligned} \tag{9}$$

where w and $\mathbf{v}$ are Lagrange multipilers, $\mathbf{v}$ is got from the boundary value problem (10) - (15), i.e.,

$$\begin{aligned}
\mathbf{u}\cdot\nabla\mathbf{v}+(\nabla\mathbf{u})\,\mathbf{v}+\nabla w+\nabla\cdot Re^{-1}\left(\nabla\mathbf{v}+(\nabla\mathbf{v})^T\right) &= 0, \quad \mathbf{x}\in D && (10)\\
\nabla\cdot\mathbf{v} &= 0, \quad \mathbf{x}\in D && (11)\\
\mathbf{v}\cdot\mathbf{n} &= -p, \ \mathbf{x}\in S && (12)\\
Re^{-1}\left(\nabla\mathbf{v}+(\nabla\mathbf{v})^T\right)\mathbf{n} &= 0, \quad \mathbf{x}\in S && (13)\\
\mathbf{v} &= 0, \quad \mathbf{x}\in\Gamma && (14)\\
\mathbf{v} &= 0, \quad |x|\to\infty. && (15)
\end{aligned}$$

Obviously, this is a linear problem in $\mathbf{v}$ and w, while $\mathbf{u}$ is the solution of the boundary value problem (1), (2), (3), (5), (6), (7).

2. **Numerical results**

Let's apply the optimization algorithm for a fluid configuration D considered in the sequel. Precisely let's consider the fluid flow of parameters $\rho = 1kg/m^3$, $\eta = 10Pa\cdot s$, $Fr = 0.7$, $u1_0 = 7m/s$, $u2_0 = 0m/s$ where $\mathbf{u_0} = (u1_0, u2_0)$ is the inflow velocity. For the output flow we have used some appropriate Neumann boundary conditions. Let's define the initial shape of the free boundary by a straight line (1).

To solve the respective boundary value problem and to update successively the free surface we have used the software packages Comsol and Matlab.

Using a step size $\gamma = 10^-2$ we get the new shape of the free surface (2), the fluid flow domain and the velocity field (3), the velocity surface (4), the stream lines (5), the velocity vector field (6).

We remark that the optimization algorithm proposed in this paper could be extended for three dimensional flows and this will be the target of a next paper.

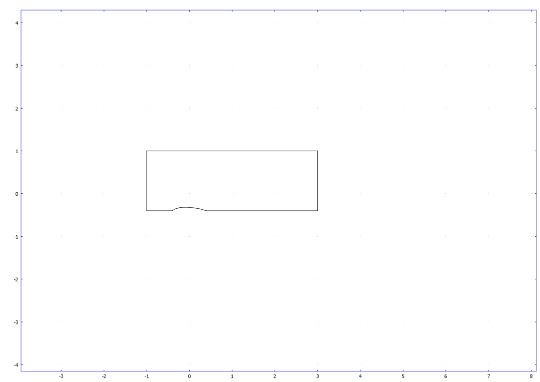

FIGURE 1. Initial fluid flow domain

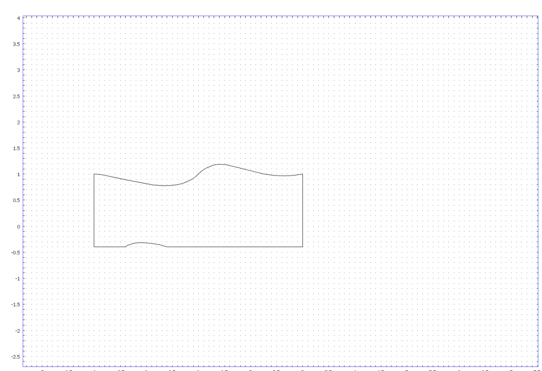

FIGURE 2. Fluid flow domain

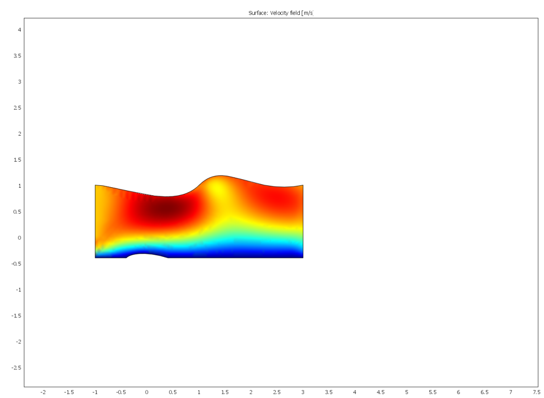

FIGURE 3. Fluid flow domain and the velocity filed

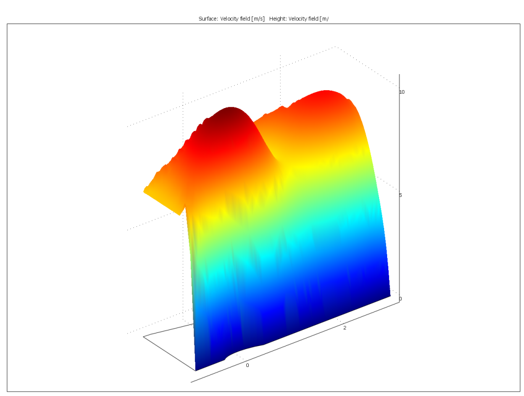

FIGURE 4. Velocity surface

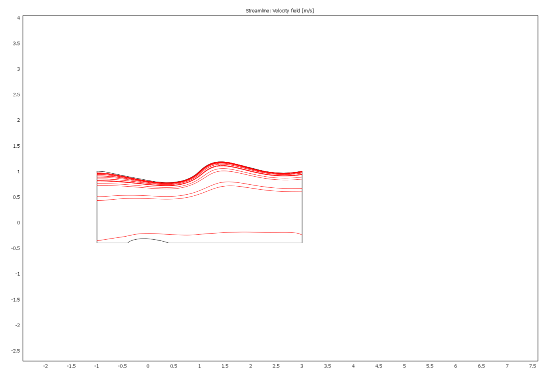

Figure 5. Stream lines

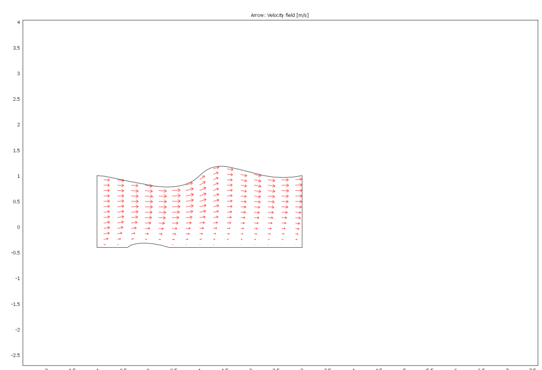

Figure 6. Velocity vector filed

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MULTIPLE SOLUTIONS FOR A DOUBLE EIGENVALUE ELLIPTIC PROBLEM IN DOUBLE WEIGHTED SOBOLEV SPACES

ILDIKÓ ILONA MEZEI

Abstract. In this paper we study a semilinear double eigenvalue problem with nonlinear boundary conditions in an unbounded domain $\Omega \in \mathbb{R}^N$. To obtain existence and multiplicity results for this problem we use the Mountain Pass Theorem applied to double weighted Sobolev spaces and a recent result proved by G. Bonanno (Nonlinear Analysis, **54**(2003), 651-665) concerning critical points. This result completes some recent results obtained in this direction.

1. Main result

Let $\Omega \subset \mathbb{R}^N,\ (N \geq 3)$ be an unbounded domain with smooth boundary Γ. For a positive measurable function u and a positive measurable function w defined in Ω, we define the weighted p-norm $(1 \leq p < \infty)$

$$||u||_{p,\Omega,w} = \left(\int_\Omega |u(x)|^p w(x) dx \right)^{\frac{1}{p}}$$

and denote by $L^q(\Omega; w)$ the space of all measurable functions u such that $||u||_{q,\Omega,w}$ is finite. The double weighted Sobolev space

$$W^{1,p}(\Omega; v_0, v_1)$$

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is defined as the space of all functions $u \in L^p(\Omega; v_0)$ such that all derivatives $\frac{\partial u}{\partial x_i}$ belong to $L^p(\Omega; v_1)$. The corresponding norm is defined by

$$||u||_{p,\Omega,v_0,v_1} = \left(\int_\Omega |\nabla u(x)|^p v_1(x) + |u(x)|^p v_0 dx \right)^{\frac{1}{p}}.$$

The Muckenhoupt class A_p is defined as the set of all positive functions v in $\mathbb{R}^N$, which satisfy

$$\frac{1}{|Q|} \left(\int_\Omega v \ dx \right)^{\frac{1}{p}} \left(\int_\Omega v^{-\frac{1}{p-1}} \ dx \right)^{\frac{p-1}{p}} \leq \bar{C}, \text{ if } 1 < p < \infty$$

$$\frac{1}{|Q|} \int_\Omega v \ dx \leq \bar{C} \text{ ess} \inf_{x \in Q} v(x), \text{ if } p = 1,$$

for all cubes $Q \in \mathbb{R}^N$ and some $\bar{C} > 0$.

In this paper we always assume that the weight functions $v_0, \ v_1, \ w$ are defined in Ω, belong to A_p and are choosen such that the embeddings

$$W^{1,2}(\Omega; v_0, v_1) \hookrightarrow L^p(\Omega; w) \tag{1}$$

and the trace

$$W^{1,2}(\Omega; v_0, v_1) \hookrightarrow L^q(\Gamma; w) \tag{2}$$

are compact for $2 < p < 2N/(N-2)$, $2 < q < 2(N-1)/(N-2)$ and continuous for $2 \leq p \leq 2N/(N-2)$, $2 \leq q \leq 2(N-1)/(N-2)$ respectively. Such weight functions there exist, see for example [4], [5]. The best embedding constants are denoted by $C_{p,\Omega}$ and $C_{q,\Gamma}$, i.e. we have the inequalities

$$||u||_{p,\Omega,w} \leq C_{p,\Omega} ||u||_{v_0,v_1}, \quad \text{for all } u \in W^{1,2}(\Omega; v_0, v_1) \tag{3}$$

$$||u||_{q,\Gamma,w} \leq C_{q,\Gamma} ||u||_{v_0,v_1}, \quad \text{for all } u \in W^{1,2}(\Omega; v_0, v_1) \tag{4}$$

where we used the abbreviation $||u||_{v_0,v_1} = ||u||_{2,\Omega,v_0,v_1}$.

For $\lambda > 0$ and $\mu \in \mathbb{R}$ we consider the following semilinear elliptic double eigenvalue problem

$$(P_{\lambda,\mu}) \qquad \begin{cases} Au \equiv -\Delta u + b(x)u = \lambda f(x,u) \text{ in } \Omega \\ \partial_n u = \lambda \mu g(x,u) \text{ on } \Gamma \end{cases},$$

where b is a positive measurable function, n denotes the unit outward normal on Γ and ∂_n is the outer normal derivative on Γ.

We define a bilinear form associated with A by

$$\langle u,v\rangle_A=\int_\Omega(\nabla u\nabla v+b(x)uv)dx.$$

A weak solution of the problem $(P_{\lambda,\mu})$ is a function $u\in W^{1,2}(\Omega;v_0,v_1)$, such that for every $v\in W^{1,2}(\Omega;v_0,v_1)$ we have

$$\langle u,v\rangle_A-\lambda\int_\Omega f(x,u(x))v(x)dx-\lambda\mu\int_\Gamma g(x,u(x))v(x)d\Gamma=0.$$

Furthermore we consider the following assumptions:

(A) we assume that A defines a continuous bilinear form $\langle\cdot,\cdot\rangle_A$ on $W^{1,2}(\Omega;v_0,v_1)$ and satisfies the ellipticity condition

$$\langle u,u\rangle_A\geq 2K||u||^2_{v_0,v_1}\text{ for every } u\in W^{1,2}(\Omega;v_0,v_1), \tag{5}$$

with some positive constant $K>0$;

(F1) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function with $f(\cdot,0)=0$ and

$$|f(x,s)|\leq f_0(x)+f_1(x)|s|^{p-1}\text{ for } x\in\Omega,\ s\in\mathbb{R},$$

where $2<p<\dfrac{2N}{N-2}$, and f_0,f_1 are positive measurable functions satisfying $f_0\in L^{\frac{p}{p-1}}(\Omega;w^{\frac{1}{1-p}})$, $f_0(x)\leq C_fw(x)$ and $f_1(x)\leq C_fw(x)$ for a.e. $x\in\Omega$, with an appropiate constant C_f;

(F2) $\lim\limits_{s\to 0}\dfrac{f(x,s)}{f_0(x)|s|}=0$, uniformly in $x\in\Omega$;

(F3) $\lim\limits_{s\to\infty}\dfrac{F(x,s)}{f_0(x)|s|^2}=0$, uniformly in $x\in\Omega$,
$\max\limits_{|s|\leq M}F(\cdot,s)\in L^1(\Omega)$, for all $M>0$, where

$$F(x,u)=\int_0^u f(x,s)ds;$$

(F4) there exist $x_0\in\Omega,\ s_0\in\mathbb{R}$ and $R_0>0$ such that $\min\limits_{|x-x_0|<R}F(x,s_0)>0$.

(G1) Let $g:\Gamma\times\mathbb{R}\to\mathbb{R}$ be a a Carathéodory function with $g(\cdot,0)=0$ and

$$|g(x,s)|\leq g_0(x)+g_1(x)|s|^{q-1},\text{ for } x\in\Gamma,\ s\in\mathbb{R}$$

where $2 < q < \dfrac{2(N-1)}{N-2}$, and g_0, g_1 are positive measurable functions satisfying $g_0 \in L^{\frac{q}{q-1}}(\Gamma; w^{\frac{1}{1-q}})$, $g_0(x) \leq C_g w(x)$ and $g_1(x) \leq C_g w(x)$, a.e. $x \in \Gamma$, with an appropiate constant C_g;

(G2) $\lim\limits_{s\to 0} \dfrac{g(x,s)}{g_0(x)|s|} = 0$, uniformly in $x \in \Gamma$;

(G3) $\lim\limits_{s\to +\infty} \dfrac{G(x,s)}{g_0(x)|s|^2} = 0$, uniformly for $x \in \Gamma$, $\max\limits_{|s|\leq M} G(\cdot, s) \in L^1(\Gamma)$, for every $M > 0$, where $G(x,s) = \int_0^u g(x,s)ds$.

Next, we introduce the functionals $J_F, J_G, J_\mu : W^{1,2}(\Omega; v_0, v_1) \to \mathbb{R}$, defined by

$$J_F(u) = \int_\Omega F(x, u(x))dx, \qquad J_G(u) = \int_\Gamma G(x, u(x))d\Gamma,$$

$$J_\mu(u) = J_F(u) + \mu J_G(u)$$

and the energy functional $\mathcal{E}_{\lambda,\mu}(u) : W^{1,2}(\Omega; v_0, v_1) \to \mathbb{R}$ associated to $(P_{\lambda,\mu})$, defined by

$$\mathcal{E}_{\lambda,\mu}(u) = \frac{1}{2}\langle u, u\rangle_A - \lambda J_\mu(u).$$

The main result of this paper is the following

Theorem 1.1. *We suppose that the assumption (A) is satisfied and the functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfy the conditions $(F1) - (F4)$ and $(G1) - (G3)$ respectively.*

(a) *Then there exists $\lambda_0 > 0$ such that to every $\lambda \in]\lambda_0, +\infty[$ it corresponds a nonempty open interval $I_\lambda \subset \mathbb{R}$ such that for every $\mu \in I_\lambda$ the problem $(P_{\lambda,\mu})$ has at least two distinct, nontrivial weak solutions $u_{\lambda,\mu}$ and $v_{\lambda,\mu}$, with the property*

$$\mathcal{E}_{\lambda,\mu}(u_{\lambda,\mu}) < 0 < \mathcal{E}_{\lambda,\mu}(v_{\lambda,\mu}).$$

(b) *Then there exists $\mu_0 > 0$ such that to every $\mu \in [-\mu_0, \mu_0]$ it corresponds a nonempty open interval $\Gamma_\mu \in]0, +\infty[$ and a number $\sigma_\mu > 0$ for which*

$(P_{\lambda,\mu})$ *has at least two distinct, nontrivial weak solutions:* $u^1_{\lambda,\mu}$ *and* $u^2_{\lambda,\mu}$, *with the property*

$$max\{||u^1_{\lambda,\mu}||_{v_0,v_1}, ||u^2_{\lambda,\mu}||_{v_0,v_1}\} \leq \sigma_\mu,$$

whenever $\lambda \in \Gamma_\mu$.

2. **Preliminaries**

In this section we denote by p' and q' the conjugates of p respective q, i.e. $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$.

The following result deals with the Nemytskii operator of a Carathéodory function $h : \Omega \times \mathbb{R} \to \mathbb{R}$, which is the function defined by $N_h(u) = h(x, u(x))$. Then we have the following result.

Lemma 2.1. *Assume that the conditions* (F1), (G1) *are satisfied. Then the Nemytskii operators* $N_f : L^p(\Omega; w) \to L^{\frac{p}{p-1}}(\Omega; w^{\frac{1}{1-p}})$, $N_F : L^p(\Omega; w) \to L^1(\Omega)$, $N_g : L^q(\Gamma; w) \to L^{\frac{q}{q-1}}(\Gamma; w^{\frac{1}{1-q}})$ *and* $N_F : L^q(\Gamma; w) \to L^1(\Gamma)$ *are bounded and continuous.*

Proof. We will use the following result: for all $s \in (0, \infty)$ there is a constant $C_s > 0$ such that

$$(x + y)^s \leq C_s(x^s + y^s), \quad \text{for any } x, y \in (0, \infty). \tag{6}$$

To prove that N_f is bounded, we choose an arbitrary set $A \subseteq L^p(\Omega; w)$ and prove that $N_f(A)$ is bounded. For this, let $u \in A$ be an arbitrary element and we claim that $N_f(u)$ is bounded in $L^{\frac{p}{p-1}}(\Omega; w^{\frac{1}{1-p}})$. Using the (F1) condition, the (6), the Hölder's inequalities, we have

$$||N_f(u)||^{\frac{1}{p'}}_{\frac{p}{p-1},\Omega,w^{\frac{1}{1-p}}} = \int_\Omega |f(x, u(x))|^{p'} w(x)^{\frac{1}{1-p}} dx \leq$$

$$\leq \int_\Omega \left(f_0(x) + f_1(x)|u(x)|^{p-1}\right)^{p'} w(x)^{\frac{1}{1-p}} dx \leq$$

$$\leq C_{p'} \left(\int_\Omega f_0(x)^{p'} w(x)^{\frac{1}{1-p}} dx + \int_\Omega f_1(x)^{p'} |u(x)|^{(p-1)p'} w(x)^{\frac{1}{1-p}} dx \right) \leq$$

$$\leq C_{p'} \left(C + \int_\Omega C_f^{p'} w(x)^{p'} w(x)^{\frac{1}{1-p}} |u(x)|^p dx \right) =$$

$$= C_{p'}C + C_{p'}C_f^{p'} \int_\Omega |u(x)|^p w(x)dx = C_{p'}C + C_{p'}C_f^{p'} ||u||^p_{p,\Omega,w},$$

where in the last inequality we used that $f_0 \in L^{\frac{p}{p-1}}(\Omega; w^{\frac{1}{1-p}})$, so there exists $C > 0$ such that $\int_\Omega f_0(x)^{\frac{p}{p-1}} w(x)^{\frac{1}{1-p}} dx \leq C$. Since $u \in A \subseteq L^p(\Omega; w)$, we have that $||u||^p_{p,\Omega,w}$ is finite, therefore N_f is bounded. Then the continuity follows from standard properties of the Nemytskii operators.

In the same way we obtain for $u \in L^p(\Omega; w)$

$$\int_\Omega |F(x, u(x))|dx \leq \int_\Omega (f_0(x)|u(x)| + f_1(x)|u(x)|^p)\, dx =$$

$$= \int_\Omega f_0(x)w(x)^{-\frac{1}{p}} |u(x)|w(x)^{\frac{1}{p}} dx + \int_\Omega f_1(x)|u(x)|^p dx \leq$$

$$\leq \left(\int_\Omega f_0(x)^{p'} w(x)^{\frac{1}{1-p}} dx \right)^{\frac{1}{p'}} \left(\int_\Omega |u(x)|^p w(x) dx \right)^{\frac{1}{p}} + C_f \int_\Omega |u(x)|^p w(x)dx \leq$$

$$\leq C^{\frac{1}{p'}} ||u||_{p,\Omega,w} + C_f ||u||^p_{p,\Omega,w},$$

therefore N_F is bounded. For the operators N_g and N_G the arguments are identical, therefore we omit the details here. □

Lemma 2.2. [5] *The energy functional $\mathcal{E}_{\lambda,\mu}$ is Fréchet differentiable in $W^{1,2}(\Omega; v_0, v_1)$ and its derivative is given by*

$$\langle \mathcal{E}'_{\lambda,\mu}(u), v\rangle = \langle u, v\rangle_A - \lambda \int_\Omega f(x, u(x))v(x)dx - \lambda\mu \int_\Gamma g(x, u(x))v(x)d\Gamma. \qquad (7)$$

for every $v \in W^{1,2}(\Omega; v_0, v_1)$.

Remark 2.1. Due to this result, one can see, that the critical points of $\mathcal{E}_{\lambda,\mu}$ are exactly the weak solutions of $(P_{\lambda,\mu})$.

Lemma 2.3. *Suppose that the conditions* (F2), (F3), (G2) *and* (G3) *are satisfied. Then, for every $\lambda > 0$ and $\mu \in \mathbb{R}$ the functional $\mathcal{E}_{\lambda,\mu}$ is coercive and bounded from below on $W^{1,2}(\Omega; v_0, v_1)$.*

Proof. Let us fix $\lambda > 0$ and $\mu \in \mathbb{R}$ arbitrarily and $a,\ b > 0$ such that

$$\lambda a C_f C^2_{2,\Omega} + \lambda|\mu| b C_g C^2_{2,\Gamma} < K.$$

By the conditions (F2),(F3) and (G2),(G3) there exist the positive functions $k_a \in L^1(\Omega; w)$ and $k_b \in L^1(\Gamma; w)$ such that

$$|F(x,s)| \leq af_0(x)|s|^2 + k_a(x)w(x), \quad \forall (x,s) \in \Omega \times \mathbb{R}$$

$$|G(x,s)| \leq bg_0(x)|s|^2 + k_b(x)w(x), \quad \forall (x,s) \in \Omega \times \mathbb{R}.$$

Thus, for every $u \in W^{1,2}(\Omega; v_0, v_1)$ we obtain

$$\begin{aligned}
\mathcal{E}_{\lambda,\mu}(u) &= \frac{1}{2}\langle u,u\rangle_A - \lambda \int_\Omega F(x,u(x))dx - \lambda\mu \int_\Gamma G(x,u(x)dx) \geq \\
&\geq K||u||^2_{v_0,v_1} - \lambda \int_\Omega af_0(x)|u(x)|^2 dx - \lambda \int_\Omega k_a(x)w(x)dx - \\
&\quad -\lambda|\mu| \int_\Gamma bg_0(x)|u(x)|^2 d\Gamma - \lambda|\mu| \int_\Gamma k_b(x)w(x)d\Gamma \geq \\
&\geq K||u||^2_{v_0,v_1} - \lambda aC_f||u||^2_{2,\Omega,w} - \lambda||k_a||_{1,\Omega,w} - \\
&\quad -\lambda|\mu|bC_g||u||^2_{2,\Gamma,w} - \lambda|\mu|||k_b||_{1,\Gamma,w} \geq \\
&\geq \left(K - \lambda aC_f C^2_{2,\Omega} - \lambda|\mu|bC_g C^2_{2,\Gamma}\right)||u||^2_{v_0,v_1} - \\
&\quad -\lambda||k_a||_{1,\Omega,w} - \lambda|\mu|||k_b||_{1,\Gamma,w}.
\end{aligned}$$

Since $k_a \in L^1(\Omega; w)$, $k_b \in L^1(\Gamma; w)$, we have that $||k_a||_{1,\Omega,w}$, $||k_b||_{1,\Gamma,w}$ are finite. Therefore $\mathcal{E}_{\lambda,\mu}$ is bounded from below on $W^{1,2}(\Omega; v_0, v_1)$ and $\mathcal{E}_{\lambda,\mu}(u) \to \infty$, whenever $||u||_{v_0,v_1} \to \infty$. Hence $\mathcal{E}_{\lambda,\mu}$ is coercive. □

Lemma 2.4. $\mathcal{E}_{\lambda,\mu} : W^{1,2}(\Omega; v_0, v_1) \to \mathbb{R}$ *satisfies the Palais-Smale condition on* $W^{1,2}(\Omega; v_0, v_1)$*, for every* $\lambda > 0$ *and* $\mu \in \mathbb{R}$.

Proof. Let $\{u_n\} \subset W^{1,2}(\Omega; v_0, v_1)$ be an arbitrary Palais-Smale sequence for $\mathcal{E}_{\lambda,\mu}$, i.e.

(a) $\{\mathcal{E}_{\lambda,\mu}(u_n)\}$ is bounded;

(b) $\mathcal{E}'_{\lambda,\mu}(u_n) \to 0$.

We have to prove that $\{u_n\}$ contains a strongly convergent subsequence. Since $\mathcal{E}_{\lambda,\mu}$ is coercive, we have that $\{u_n\}$ is bounded. $W^{1,2}(\Omega; v_0, v_1)$ is a reflexive Banach space, so taking a subsequence if necessary (denoted in the same way), we get an element $u \in W^{1,2}(\Omega; v_0, v_1)$ such that $u_n \to u$ weakly in $W^{1,2}(\Omega; v_0, v_1)$. Because the embeddings (1) and (2) are compact for $2 < p < 2N/(N-2)$, $2 < q < 2(N-1)/(N-2)$, we have that $u_n \to u$ strongly in $L^p(\Omega; w)$ and $L^q(\Gamma; w)$.

From the condition (b) we have that $\left|\langle\mathcal{E}'_{\lambda,\mu}(u_n), \frac{u_n}{||u_n||_{v_0,v_1}}\rangle\right| \le \varepsilon$, for every $\varepsilon > 0$ and large $n \in \mathbb{N}$. Then

$$-\langle u_n, u_n\rangle_A + \lambda\int_\Omega f(x,u_n(x))u_n(x)dx + \lambda\mu\int_\Gamma g(x,u_n(x))u_n(x)d\Gamma \le \varepsilon||u_n||_{v_0,v_1}.$$

Then we have

$$2K||u_n-u||^2_{v_0,v_1} \le \langle u_n-u, u_n-u\rangle_A \le |\langle u_n, u_n-u\rangle_A| + |\langle u, u_n-u\rangle_A| \le$$

$$\le 2\varepsilon||u_n-u||_{v_0,v_1}+$$

$$+\lambda\left|\int_\Omega f(x,u_n(x))(u_n(x)-u(x))dx\right| + \lambda\left|\int_\Omega f(x,u(x))(u_n(x)-u(x))dx\right| +$$

$$+\lambda|\mu|\left|\int_\Gamma g(x,u_n(x))(u_n(x)-u(x))d\Gamma\right| + \lambda|\mu|\left|\int_\Gamma g(x,u(x))(u_n(x)-u(x))d\Gamma\right|.$$

Using the Hölder's inequality we get

$$\left|\int_\Omega f(x,u_n(x))(u_n(x)-u(x))dx\right| \le$$

$$\le \int_\Omega \left|f(x,u_n(x))w(x)^{-\frac{1}{p}}\right| \left|(u_n(x)-u(x))w(x)^{\frac{1}{p}}\right| dx \le$$

$$\le \left(\int_\Omega |f(x,u_n(x))|^{p'} w(x)^{-\frac{p'}{p}}dx\right)^{\frac{1}{p'}} \left(\int_\Omega |u_n(x)-u(x)|^p w(x)dx\right)^{\frac{1}{p}} =$$

$$= \left(\int_\Omega |f(x,u_n(x))|^{p'} w(x)^{\frac{1}{1-p}}dx\right)^{\frac{1}{p'}} ||u_n-u||_{p,\Omega,w}$$

and arguing in the same way for g, we obtain

$$\left|\int_\Gamma g(x,u_n(x))(u_n(x)-u(x))dx\right| \le \left(\int_\Gamma |g(x,u_n(x))|^{q'} w(x)^{\frac{1}{1-q}}d\Gamma\right)^{\frac{1}{q'}} ||u_n-u||_{q,\Gamma,w}.$$

Since $\varepsilon > 0$ is arbitrary, $||u_n-u||_{p,\Omega,w}$ and $||u_n-u||_{q,\Gamma,w}$ tend to zero and $\int_\Omega |f(x,u_n(x))|^{p'} w(x)^{\frac{1}{1-p}}dx$, $\int_\Gamma |g(x,u_n(x))|^{q'} w(x)^{\frac{1}{1-q}}d\Gamma$ are bounded (by Lemma 2.1, using that $\{u_n\}$ is bounded), it follows that $||u_n-u||_{v_0,v_1}$ tends to zero. □

Lemma 2.5. [3, Lemma 3.2] *Assume that (F4) is satisfied. Then there exist an* $u_0 \in W^{1,2}(\Omega; v_0, v_1)$ *such that* $J_F(u_0) > 0$.

Let us define $m = \int_\Gamma |G(x,u_0(x))|d\Gamma$, $\lambda_0 = \dfrac{\frac{1}{2}\langle u_0, u_0\rangle_A}{J_F(u_0)} > 0$ and $\mu^*_\lambda = \frac{1}{\lambda(1+m)}$. $(\lambda-\lambda_0)J_F(u_0) > 0$.

Lemma 2.6. *For $\lambda > \lambda_0$ and $|\mu| \in]0, \mu_\lambda^*]$ we have*

$$\inf_{u \in W^{1,2}(\Omega; v_0, v_1)} \mathcal{E}_{\lambda,\mu}(u) < 0.$$

Proof. It is sufficient to prove, that for $\lambda > \lambda_0$ and $|\mu| \in]0, \mu_\lambda^*]$ we have $\mathcal{E}_{\lambda,\mu}(u_0) < 0$. Indeed,

$$\begin{aligned}
\mathcal{E}_{\lambda,\mu}(u_0) &= \frac{1}{2}\langle u_0, u_0\rangle_A - \lambda J_F(u_0) - \lambda\mu J_G(u_0) \leq \\
&\leq \lambda_0 J_F(u_0) - \lambda J_F(u_0) + \lambda|\mu|m = \\
&= (\lambda_0 - \lambda)J_F(u_0) + \lambda|\mu|m = \\
&= (\lambda_0 - \lambda)\frac{\lambda(1+m)\mu_\lambda^*}{\lambda - \lambda_0} + \lambda|\mu|m = \\
&= -(1+m)\lambda\mu_\lambda^* + \lambda|\mu|m = \\
&= -\lambda\mu_\lambda^* - m\lambda(\mu_\lambda^* - |\mu|) < 0.
\end{aligned}$$

for all $\lambda > \lambda_0$ and $|\mu| \in]0, \mu_\lambda^*]$. □

Lemma 2.7. *For every $\lambda > \lambda_0$ and $\mu \in]0, \mu_\lambda^*]$, the functional $\mathcal{E}_{\lambda,\mu}$ satisfies the Mountain Pass geometry.*

Proof. From the assumptions (F1), (F2), (G1) and (G2) results the existence of $\hat{c}_1(\varepsilon)$, $\hat{c}_2(\varepsilon) > 0$ such that, for every $\hat{\varepsilon} > 0$ we have

$$|f(x,s)| \leq \hat{\varepsilon} f_0(x)|s| + \hat{c}_1(\varepsilon) f_1(x)|s|^{p-1}, \text{ for } 2 < p < \frac{2N}{N-2}, \tag{8}$$

$$|g(x,s)| \leq \hat{\varepsilon} g_0(x)|s| + \hat{c}_2(\varepsilon) g_1(x)|s|^{q-1}, \text{ for } 2 < q < \frac{2(N-1)}{N-2}. \tag{9}$$

Then integrating with respect to the second variable, from 0 to $u(x)$, we get the existence of $c_1(\varepsilon)$, $c_2(\varepsilon) > 0$ such that, for every $\varepsilon > 0$ we have

$$|F(x,u(x))| \leq \varepsilon f_0(x)|u(x)|^2 + c_1(\varepsilon) f_1(x)|u(x)|^p, \text{ for } 2 < p < \frac{2N}{N-2}, \tag{10}$$

$$|G(x,u(x))| \leq \varepsilon g_0(x)|u(x)|^2 + c_2(\varepsilon) g_1(x)|u(x)|^q, \text{ for } 2 < q < \frac{2(N-1)}{N-2}. \tag{11}$$

Fix $\lambda > \lambda_0$ and $\mu \in]0, \mu_\lambda^*[$, then using the (10)and (11) inequalities for every $u \in W^{1,2}(\Omega; v_0, v_1)$ we have

$$\begin{aligned}
\mathcal{E}_{\lambda,\mu}(u) &= \frac{1}{2}\langle u, u\rangle_A - \lambda J_\mu(u) \geq \\
&\geq K||u||^2_{v_0,v_1} - \lambda \int_\Omega |F(x, u(x))|dx - \lambda|\mu| \int_\Gamma |G(x, u(x))|d\Gamma \geq \\
&= K||u||^2_{v_0,v_1} - \lambda\varepsilon C_f ||u||^2_{2,\Omega,w} - \lambda c_1(\varepsilon) C_f ||u||^p_{p,\Omega,w} - \\
&\quad -\lambda|\mu|\varepsilon C_g ||u||^2_{2,\Gamma,w} - \lambda|\mu| c_2(\varepsilon) C_g ||u||^q_{q,\Omega,w} \geq \\
&\geq \left(K - \lambda\varepsilon C_f C^2_{2,\Omega} - \lambda|\mu|\varepsilon C_g C^2_{2,\Gamma}\right) ||u||^2_{v_0,v_1} - \\
&- \lambda c_1(\varepsilon) C_f C^p_{p,\Omega} ||u||^p_{v_0,v_1} - \lambda|\mu| c_2(\varepsilon) C_g C^q_{q,\Gamma} ||u||^q_{v_0,v_1}.
\end{aligned}$$

Using the notations $A = \left(K - \lambda\varepsilon C_f C^2_{2,\Omega} - \lambda|\mu|\varepsilon C_g C^2_{2,\Gamma}\right)$, $B = \lambda c_1(\varepsilon) C_f C^p_{p,\Omega}$, $C = \lambda|\mu| c_2(\varepsilon) C_g C^q_{q,\Gamma}$, we get

$$\mathcal{E}_{\lambda,\mu}(u) \geq (A - B||u||^{p-2}_{v_0,v_1} - C||u||^{q-2}_{v_0,v_1})||u||^2_{v_0,v_1}.$$

We choose $\varepsilon \in \left]0, \frac{K}{2} \frac{1}{\lambda(C_f C^2_{2,\Omega} + |\mu| C_g C^2_{2,\Gamma})}\right[$, so $A > 0$. Now, let $l : \mathbb{R}_+ \to \mathbb{R}$ be the function defined by $l(t) = A - Bt^{p-2} - Ct^{q-2}$. We can see, that $l(0) = A > 0$, so because l is continuous, there exists an $\varepsilon^* > 0$ such that $l(t) > 0$, for every $t \in]0, \varepsilon^*[$. Then for every $u \in W^{1,2}(\Omega; v_0, v_1)$, with $||u||_{v_0,v_1} = \varepsilon^{**} < \min\{\varepsilon^*, ||u_0||_{v_0,v_1}\}$, we have $\mathcal{E}_{\lambda,\mu}(u) \geq \eta(\lambda, \mu, \varepsilon^*) > 0$. From Lemma 2.6 we have $\mathcal{E}_{\lambda,\mu}(u_0) < 0$.

Therefore the functional $\mathcal{E}_{\lambda,\mu}$ satisfies the Mountain Pass geometry, meaning that $\mathcal{E}_{\lambda,\mu}$ satisfies the conditions of the Mountain Pass Theorem (see Theorem 3.1). □

Lemma 2.8. *For every $\mu \in \mathbb{R}_+$, we have*

$$\lim_{\rho \to 0} \frac{\sup\{J_\mu(u) : \frac{1}{2}\langle u, u\rangle_A < \rho\}}{\rho} = 0.$$

Proof. Fix arbitrarily $\varepsilon > 0$ and $p \in \left]2, \dfrac{2N}{N-p}\right[$, $q \in \left]2, \dfrac{2(N-1)}{N-2}\right[$, then from (10) and (11) and the ellipticity condition (A), it follows that

$$\begin{aligned} J_\mu(u) &= J_F(u) + \mu J_G(u) \leq \\ &\leq \varepsilon \left(C_f C_{2,\Omega}^2 + |\mu| C_g C_{2,\Gamma}^2 \right) ||u||_{v_0,v_1}^2 + c_1(\varepsilon) C_f C_{p,\Omega}^p ||u||_{v_0,v_1}^p + \\ &+ |\mu| c_2(\varepsilon) C_g C_{q,\Gamma}^q ||u||_{v_0,v_1}^q \leq \\ &\leq \varepsilon \left(C_f C_{2,\Omega}^2 + |\mu| C_g C_{2,\Gamma}^2 \right) \frac{\langle u,u\rangle_A}{2K} + c_1(\varepsilon) C_f C_{p,\Omega}^p \left(\frac{\langle u,u\rangle_A}{2K} \right)^{\frac{p}{2}} + \\ &+ |\mu| c_2(\varepsilon) C_g C_{q,\Gamma}^q \left(\frac{\langle u,u\rangle_A}{2K} \right)^{\frac{q}{2}}. \end{aligned}$$

Therefore, we have

$$\sup\{J_\mu(u) : \frac{1}{2}\langle u,u\rangle_A < \rho\} \leq$$

$$\leq \varepsilon \frac{\left(C_f C_{2,\Omega}^2 + |\mu| C_g C_{2,\Gamma}^2 \right)}{K} \rho + \frac{c_1(\varepsilon) C_f C_{p,\Omega}^p}{K^{\frac{p}{2}}} \rho^{\frac{p}{2}} + |\mu| \frac{c_2(\varepsilon) C_g C_{q,\Gamma}^q}{K^{\frac{q}{2}}} \rho^{\frac{q}{2}}.$$

Since $p > 2,\ q > 2$, dividing this last inequality with ρ and taking the limit whenever $\rho \to 0$, we have the required equality.

Lemma 2.9. *We assume that the conditions* (F1)-(F3) *and* (G1)-(G3) *are satisfied. Then the functional $J_\mu = J_F + \mu J_G$ is sequentially weakly continuous.*

Proof. We argue by contradiction. Let u_n be a sequence from $W^{1,2}(\Omega; v_0, v_1)$ weakly convergent to some $u \in W^{1,2}(\Omega; v_0, v_1)$ and $d > 0$ such that

$$|J_\mu(u_n) - J_\mu(u)| \geq d, \qquad \text{for all } n \in \mathbb{N}.$$

At the same time we have

$$\begin{aligned} |J_\mu(u_n) - J_\mu(u)| &\leq \int_\Omega |F(x, u_n(x)) - F(x, u(x))| dx + \\ &+ |\mu| \int_\Gamma |G(x, u_n(x)) - G(x, u(x))| d\Gamma. \end{aligned}$$

In the sequel, we will estimate the previous two integrals. For this end, first we use the Mean Value Theorem for the function F on the interval $(u_n(x), u(x))$, then we make use of the (3), (8) and the Hölder inequalities. So, there exists a $\theta \in]0,1[$ such that

$$\begin{aligned}
& \int_\Omega |F(x,u_n(x)) - F(x,u(x))|dx = \\
= \quad & \int_\Omega |f(x,(1-\theta)u_n(x)+\theta u(x))||u_n(x)-u(x)|dx \leq \\
\leq \quad & \hat{\varepsilon}\int_\Omega f_0(x)|(1-\theta)u_n(x)+\theta u(x)||u_n(x)-u(x)|dx + \\
+ \quad & \hat{c}_1(\varepsilon)\int_\Omega f_1(x)|(1-\theta)u_n(x)+\theta u(x)|^{p-1}|u_n(x)-u(x)|dx \leq \\
\leq \quad & \hat{\varepsilon}\int_\Omega f_0(x)(|u_n(x)|+|u(x)|)|u_n(x)-u(x)|dx + \\
& +\hat{c}_1(\varepsilon)\int_\Omega f_1(x)(|u_n(x)|^{p-1}+|u(x)|^{p-1})|u_n(x)-u(x)|dx \leq \\
\leq \quad & \hat{\varepsilon}C_f\int_\Omega |u_n(x)-u(x)|w(x)^{\frac{1}{2}}w(x)^{\frac{1}{2}}(|u_n(x)|+|u(x)|)dx + \\
& +\hat{c}_1(\varepsilon)C_f\int_\Omega |u_n(x)-u(x)|w(x)^{\frac{1}{p}}w(x)^{\frac{1}{p'}}\left(|u_n(x)|^{p-1}+|u(x)|^{p-1}\right)dx \leq \\
\leq \quad & \hat{\varepsilon}C_f\left(\int_\Omega |u_n(x)-u(x)|^2w(x)dx\right)^{\frac{1}{2}} \cdot \\
& \cdot\left[\left(\int_\Omega |u_n(x)|^2w(x)dx\right)^{\frac{1}{2}}+\left(\int_\Omega |u(x)|^2w(x)dx\right)^{\frac{1}{2}}\right] + \\
& +\hat{c}_1(\varepsilon)C_f\left(\int_\Omega |u_n(x)-u(x)|^pw(x)dx\right)^{\frac{1}{p}} \cdot \\
& \cdot\left[\left(\int_\Omega |u_n(x)|^{(p-1)p'}w(x)dx\right)^{\frac{1}{p'}}+\left(\int_\Omega |u(x)|^{(p-1)p'}w(x)dx\right)^{\frac{1}{p'}}\right] \leq \\
\leq \quad & \hat{\varepsilon}C_f||u_n-u||_{2,\Omega,w}(||u_n||_{2,\Omega,w}+||u_n||_{2,\Omega,w}) + \\
& +\hat{c}_1(\varepsilon)C_f||u_n-u||_{p,\Omega,w}\left(||u_n||_{p,\Omega,w}^{\frac{p}{p'}}+||u||_{p,\Omega,w}^{\frac{p}{p'}}\right) \leq \\
\leq \quad & \hat{\varepsilon}C_fC_{2,\Omega}^2||u_n-u||_{v_0,v_1}(||u_n||_{v_0,v_1}+||u||_{v_0,v_1}) + \\
& +\hat{c}_1(\varepsilon)C_fC_{p,\Omega}^{p-1}||u_n-u||_{p,\Omega,w}\left(||u_n||_{v_0,v_1}^{p-1}+||u||_{v_0,v_1}^{p-1}\right).
\end{aligned}$$

Since u_n is weakly convergent to $u \in W^{1,2}(\Omega; v_0, v_1)$, we can assume without loss of generality that there exist a constant $M > 0$ such that

$$||u_n||_{v_0,v_1} \leq M \text{ and } ||u_n - u||_{v_0,v_1} \leq M, \text{ for all } n \in \mathbb{N}.$$

Then we have

$$|F(x, u_n(x)) - F(x, u(x))| \leq 2\hat{\varepsilon} C_f C_{2,\Omega}^2 M^2 + 2\hat{c}_1(\varepsilon) C_f C_{p,\Omega}^{p-1} M^{p-1} ||u_n - u||_{p,\Omega,w}.$$

Arguing as above for the function G, we obtain

$$|G(x, u_n(x)) - G(x, u(x))| \leq 2\hat{\varepsilon} C_g C_{2,\Gamma}^2 M^2 + 2\hat{c}_2(\varepsilon) C_g C_{q,\Gamma}^{q-1} M^{q-1} ||u_n - u||_{q,\Gamma,w}.$$

Therefore

$$d \leq |J_\mu(u_n) - J_\mu(u)| \leq 2\hat{\varepsilon} M^2 (C_f C_{2,\Omega}^2 + C_g C_{2,\Gamma}^2) +$$
$$+ 2\hat{c}_1(\varepsilon) C_f C_{p,\Omega}^{p-1} M^{p-1} ||u_n - u||_{p,\Omega,w} + 2\hat{c}_2(\varepsilon) C_g C_{q,\Gamma}^{q-1} M^{q-1} ||u_n - u||_{q,\Gamma,w}.$$

Because the embeddings (1) and (2) are compact for $2 < p < 2N/(N-2)$, $2 < q < 2(N-1)/(N-2)$, it follows that $||u_n - u||_{p,\Omega,w} \to 0$ and $||u_n - u||_{q,\Gamma,w} \to 0$. Therefore, if $\hat{\varepsilon} > 0$ is sufficiently small and $n \in \mathbb{N}$ is large enough, we have

$$d \leq |J_\mu(u_n) - J_\mu(u)| < d,$$

which is a contradiction.

3. **Proof of Theorem 1.1**

For the reader's convenience we recall here the Mountain Pass Theorem used in the proof of Theorem 1.1 (a).

Theorem 3.1. [6, Theorem 2.2] *Let E be a Banach space and $I \in C^1(E, \mathbb{R})$ a functional, satisfying the Palais-Smale condition. Suppose $I(0) = 0$ and*

(*a*) *there are constants $\alpha > 0$ and $\rho > 0$ such that $I(u) \geq \alpha$, for every $||u|| = \rho$;*

(*b*) *there is an $e \in E$ with $||e|| > \rho$ and $I(e) \leq 0$.*

Then the number

$$c = \inf_{g \in \Gamma} \max_{v \in g([0,1])} I(v),$$

where

$$\Gamma = \{g \in C([0,1], E) : g(0) = 0, g(1) = e\},$$

is a critical value of I, with $c \geq \alpha$.

The main tool in the proof of Theorem 1.1 (b) is the following refinement of a B. Ricceri-type critical point theorem ([7], [8]) established by G. Bonanno in [1].

Theorem 3.2. *Let X be a separable and reflexive real Banach space and let $\Phi, J : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = J(x_0) = 0$ and $\Phi(x) \geq 0$ for every $x \in X$, and there exists $x_1 \in X$, $\rho > 0$ such that*

(i) $\rho < \Phi(x_1)$ and $\sup_{\Phi(x)<\rho} J(x) < \rho \frac{J(x_1)}{\Phi(x_1)}$. Further put

$$\bar{a} = \frac{\zeta \rho}{\rho \frac{J(x_1)}{\Phi(x_1)} - \sup_{\Phi(x)<\rho} J(x)},$$

with $\zeta > 1$, assume that the functional $\Phi - \lambda J$ is sequentially weakly lower semicontinuous, satisfies the Palais-Smale condition and

(ii) $\lim_{||x||\to+\infty} [\Phi(x) - \lambda J(x)] = +\infty$, for every $\lambda \in [0, \bar{a}]$.

Then there is an open interval $\Lambda \subset [0, \bar{a}]$ and a number $\sigma > 0$ such that for each $\lambda \in \Lambda$, the equation $\Phi'(x) - \lambda J'(x) = 0$ admits at least three distinct solutions in X, having norm less than σ.

Proof of Theorem 1.1 (a). Fix $\lambda > \lambda_0$ and $\mu \in]0, \mu_\lambda^*[= I_\lambda$. From the Lemma 2.3 and Lemma 2.4 we have that the functional $\mathcal{E}_{\lambda,\mu}$ is bounded from below and satisfies the (PS)-condition. Then $\mathcal{E}_{\lambda,\mu}$ achieves its infimum, i.e. there exists an element $u_{\lambda,\mu} \in W^{1,2}(\Omega; v_0, v_1)$ such that $\mathcal{E}_{\lambda,\mu}(u_{\lambda,\mu}) = \inf_{v \in W^{1,2}(\Omega;v_0,v_1)} \mathcal{E}_{\lambda,\mu}(v)$ (see[6, Theorem 2.7]). So $\mathcal{E}'_{\lambda,\mu}(u_{\lambda,\mu}) = 0$ and by Lemma 2.6, we have $\mathcal{E}_{\lambda,\mu}(u_{\lambda,\mu}) < 0$.

On the other hand, there exists an element $v_{\lambda,\mu} \in W^{1,2}(\Omega; v_0, v_1)$ such that $\mathcal{E}'_{\lambda,\mu}(v_{\lambda,\mu}) = 0$ and $\mathcal{E}_{\lambda,\mu}(v_{\lambda,\mu}) \geq \eta(\lambda, \mu, \varepsilon^*) > 0$ (by Lemma 2.7 and Theorem 3.1), which completes the proof. □

Proof of Theorem 1.1 (b). Let $u_0 \in W^{1,2}(\Omega; v_0, v_1)$ be the function from Lemma 2.5 and fix

$$\mu_0 = \frac{J_F(u_0)}{1 + |J_G(u_0)|}.$$

Then for every $\mu \in [-\mu_0, \mu_0]$ we have

$$J_\mu(u_0) = J_F(u_0) + \mu J_G(u_0) \geq \frac{J_F(u_0)}{1 + |J_G(u_0)|} > 0.$$

Now, we apply the Theorem 3.2 of Bonanno, by choosing $X = W^{1,2}(\Omega; v_0, v_1)$, $\Phi(u) = \dfrac{1}{2}\langle u, u\rangle_A$ and $J = J_\mu$, for $\mu \in [-\mu_0, \mu_0]$.

Taking account the lema 2.8 and the inequalities $J_\mu(u_0) > 0$, $\Phi(u_0) > 0$, we can choose for every $\mu \in [-\mu_0, \mu_0]$ a $\rho_\mu > 0$ so small that

$$\rho_\mu < \frac{1}{2}\langle u_0, u_0\rangle_A = \Phi(u_0) \tag{12}$$

$$\frac{sup\{J_\mu(u) : \frac{1}{2}\langle u, u\rangle_A < \rho_\mu\}}{\rho_\mu} < \frac{J_\mu(u_0)}{\Phi(u_0)} \tag{13}$$

Now, choosing $x_1 = u_0$, $x_0 = 0$, $\zeta = 1 + \rho_\mu$ and

$$a = \bar{a}_\mu = \frac{1 + \rho_\mu}{\frac{J_\mu(u_0)}{\Phi(u_0)} - \frac{sup\{J_\mu(u): \frac{1}{2}\langle u,u\rangle_A < \rho_\mu\}}{\rho_\mu}},$$

all the assumptions of the Theorem 3.2 are verified. Then, there is an open interval $\Lambda_\mu \subset [0, \bar{a}_\mu]$ and a number $\sigma_\mu > 0$ such that for any $\lambda \in \Lambda_\mu$, the functional $\mathcal{E}_{\lambda,\mu} = \Phi - \lambda J_\mu$ admits at least three distinct critical points: $u^i_{\lambda,\mu} \in W^{1,2}(\Omega; v_0, v_1)$, $(i \in \{1, 2, 3\})$, having norms less than σ_μ.

We can see, that $u = 0$ is a solution of the problem $(P_{\lambda,\mu})$. So if we are looking for nontrivial solutions, we can affirm that $(P_{\lambda,\mu})$ has at least two distinct, nontrivial solutions in $W^{1,2}(\Omega; v_0, v_1)$, having norms less than σ_μ, concluding the proof of the Theorem 1.1.

Remark. As an example, we consider the weight functions (see [5])

$$v_0(x) = w(x) = \begin{cases} ||x||^{-2}, \text{ if } x \in \mathbb{R}^N \setminus B_1 \\ 1, \text{ if } x \in B_1 \end{cases},$$

$$v_1(x) = 1, \forall x \in \mathbb{R}^N,$$

where $B_1 = \{x \in \mathbb{R}^N : ||x|| \leq 1\}$. For these functions the embeddings $W^{1,2}(\Omega; v_0, 1) \hookrightarrow L^p(\Omega; w)$ and $W^{1,2}(\Omega; v_0, 1) \hookrightarrow L^q(\Gamma; w)$ are compact, if $2 < p < 2N/(N-2)$, $2 < q < 2(N-1)/(N-2)$. Assuming that f and g satisfy the conditions

(F1)-(F4), (G1)-(G3) respectively and A defines a bilinear form with (A), we can apply the Theorem 1.1.

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APPROXIMATION PROCEDURES IN CONNECTION WITH A PROBLEM OF STURM-LIOUVILLE TYPE

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Abstract. Some results concerning the superdense unbounded divergence or the convergence of a family of interpolating operators and point-interpolatory functionals, associated to a problem of Sturm-Liouville type, are established.

1. **Introduction**

Let consider the Sturm-Liouville problem

$$\begin{cases} u''(x) + [\lambda^2 - B(x)]u(x) = 0;\ 0 \le x \le \pi \\ u'(0) = au(0);\ u'(\pi) = Au(\pi), \end{cases} \tag{1.1}$$

where $B(x)$ is a continuous function with bounded variation and $a, A \in \overline{\mathbb{R}}$, with the convention $u(0) = 0$, if $|a| = \infty$ and $u(\pi) = 0$, if $|A| = \infty$, [4].

It is known that there exists an orthonormal system of eigenfunctions $u_n \in C^2[0,\pi]$, $n \ge 1$, with respect to the problem (1.1), [5]. Moreover, each eigenfunction u_n has n distinct roots x_n^k, $1 \le k \le n$, in the interval $(0,\pi)$, [4], i.e. $0 < x_n^1 < x_n^2 < x_n^3 < \cdots < x_n^n < \pi$; in this paper, we shall put $x_n^0 = 0$, $x_n^{n+1} = \pi$.

Introduce the natural numbers m_0, m_n as

$$m_0 = \begin{cases} 1, & \text{if} \quad a \in \mathbb{R} \\ 0, & \text{if} \quad |a| = \infty \end{cases} \quad ; \quad m_n = \begin{cases} n, & \text{if} \quad A \in \mathbb{R} \\ n+1, & \text{if} \quad |A| = \infty \end{cases}$$

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and define, analogously to Lagrange interpolation, the *"fundamental interpolating" functions* $s_n^k \in C[0,\pi]$, $m_0 \le k \le m_n$, by

$$s_n^k(x) = \begin{cases} \dfrac{u_n(x)}{(x - x_n^k)u_n'(x_n^k)}, & \text{if} \quad x \neq x_n^k \\ 1, & \text{if} \quad x = x_n^k \end{cases}$$

The linear operators

$$\begin{cases} S_n : C[0,\pi] \to C[0,\pi];\ f \mapsto S_n f;\ n \ge 1 \\ (S_n f)(x) = \displaystyle\sum_{k=m_0}^{m_n} f(x_n^k) s_n^k(x) \end{cases} \tag{1.2}$$

are said to be the *"interpolating" operators* associated to the *Sturm-Liouville node matrix*

$$\mathcal{M}_{SL} = \{x_n^k :\ n \ge 1;\ m_0 \le k \le m_n\}.$$

Further, if x_0 is a given point of $[0,\pi]$, the linear functionals

$$S_n^0 : C[0,\pi] \to \mathbb{R}, \quad S_n^0 f = (S_n f)(x_0), \quad n \ge 1 \tag{1.3}$$

are named the *"interpolatory" functionals* associated to x_0 or *"point-interpolatory" functionals* at x_0.

Now, denote by $\omega(f;\cdot)$ the modulus of continuity of a function $f \in C[0,\pi]$ and let $\|\cdot\|$ be the uniform norm of f. G.I. Natanson, [3] and L.I. Tuchinskiĭ, [4] established the following estimation concerning the "interpolating" operators S_n:

The relation

$$|f(x) - (S_n f)(x)| = \left[\omega\left(f;\frac{1}{n}\right) + \frac{1}{n}\|f\|\right] O(\ln n), \quad f \in C[0,\pi] \tag{1.4}$$

holds uniformly on each interval $[a, \pi - a]$*, for any given* $a \in \left(0, \dfrac{\pi}{2}\right)$.

Based on this result, they proved the following convergence theorem:

If $f \in C[0,\pi]$ *satisfies a Dini-Lipschitz condition*

$$\lim_{\delta \searrow 0} \omega(f;\delta) \ln \delta = 0, \tag{1.5}$$

then the sequence $(S_n f)_{n \ge 1}$ *is uniformly convergent to* f *on each segment* $[a, \pi - a]$, $0 < a < \dfrac{\pi}{2}$.

The main aim of this paper is to establish the superdense unbounded divergence of the family of interpolating operators $\{S_n:\ n\geq 1\}$. (Recall that a subset of a topological space T is said to be *superdense* in T if it is residual, i.e. its complement is of first Baire category, uncountable and dense in T). To this end, we shall define, in the next section, the functions and the constants of Lebesgue with respect to the node matrix $\mathcal{M}_{SL}$.

2. The functions and the constants of Lebesgue associated to the Sturm-Liouville node matrix

The functions $L_n:[0,\pi]\to\mathbb{R}$, $L_n(x)=\sum_{l=m_0}^{m_n}|s_n^k(x)|$, $0\leq x\leq\pi$ and the positive numbers $\Lambda_n=\|L_n\|$, $n\geq 1$, are said to be the *Lebesgue functions*, respectively the *Lebesgue constants* associated to the node matrix $\mathcal{M}_{SL}$.

Standard arguments show that S_n, $n\geq 1$, are linear continuous operators and

$$\|S_n\|=\Lambda_n,\quad n\geq 1. \tag{2.1}$$

Indeed, using (1.2) we obtain:

$$|(S_nf)(x)|\leq L_n(x)\|f\|\leq\Lambda_n\|f\|,\ \text{for all } x\in[0,\pi],$$

so

$$\|S_nf\|\leq\Lambda_n\|f\|,\ \text{i.e. } \|S_n\|\leq\Lambda_n.$$

Conversely, for an arbitrary $t\in[0,\pi]$ let us define the function f_t by:

$$f_t(x)=\begin{cases}\operatorname{sign} s_n^k(t), & \text{if } x\in\{x_n^k:\ m_0\leq k\leq m_n\}\\ \text{linear}, & \text{otherwise;}\end{cases}$$

we have $f_t\in C[0,\pi]$, $\|f_t\|=1$ and

$$\|S_n\|=\sup\{\|S_nf\|:\ f\in C[0,\pi],\ \|f\|\leq 1\}\geq\|S_nf_t\|$$

$$\geq|(S_nf_t)(t)|=\left|\sum_{k=m_0}^{m_n}f_t(x_n^k)s_n^k(t)\right|=L_n(t),\ \forall\, t\in[0,\pi],$$

which leads to $\|S_n\|\geq\Lambda_n$.

Similarly, concerning the functionals S_n^0 of (1.3), we get:

$$\|S_n^0\| = L_n(x_0) \tag{2.2}$$

for every $x_0 \in [0, \pi]$.

In what follows, we shall use the following estimation regarding the Lebesgue functions L_n, [4]:

$$L_n(x_0) = 1 + \frac{1}{\sqrt{2\pi}}|u_n(x_0)|[\ln n + \ln(n \sin x_0 + 1) + O(1)], \tag{2.3}$$

where $x_0 \in [0, \pi]$ and

$$u_n(x_0) = \sqrt{\frac{2}{\pi}} \cos(\alpha_n x_0 + \varepsilon\pi) + O\left(\frac{1}{n}\right) \tag{2.4}$$

where

$$\varepsilon = \varepsilon(a) = \begin{cases} 0, & \text{if} \quad a \in \mathbb{R} \\ 1/2, & \text{if} \quad |a| = \infty \end{cases} \tag{2.5}$$

$$\alpha_n = \alpha_n(a, A) = \begin{cases} n, & \text{if} \quad a \in \mathbb{R},\ A \in \mathbb{R} \\ n + 1/2, & \text{if} \quad a \in \mathbb{R},\ |A| = \infty \text{ or } |a| = \infty,\ A \in \mathbb{R} \\ n + 1, & \text{if} \quad |a| = |A| = \infty \end{cases} \tag{2.6}$$

for every $x_0 \in [0, \pi]$.

3. Superdense unbounded divergence of the family of "interpolating" operators

The main result of this paper is the following

Theorem 3.1. *The set of unbounded divergence of the family of "interpolating" operators $\{S_n : \ n \geq 1\}$, i.e.*

$$\left\{ f \in C[0, \pi] : \ \limsup_{n \to \infty} \|S_n f\| = \infty \right\},$$

is superdense in the Banach space $(C[0, \pi], \|\cdot\|)$.

Proof. In what follows M_k, $k \geq 1$, will be positive constants which do not depend on n. We deduce from (2.3) and (2.4):

$$|u_n(x_0)| \leq M_1; \quad |L_n(x_0)| \leq M_2 \ln n,\ \forall\ x_0 \in [0, \pi],$$

so that, according to (2.1), we get:

$$S_n = \Lambda_n \le M_2 \ln n, \text{ for sufficiently large } n. \tag{3.1}$$

Let us establish the converse of (3.1). According to (2.5) and (2.6), there are four possibilities.

1°. If $a \in \mathbb{R}$ and $A \in \mathbb{R}$, then $\alpha_n = n$ and $\varepsilon = 0$, so we deduce from (2.3) and (2.4):

$$L_n(0) = 1 + \left(\frac{1}{\pi} + O\left(\frac{1}{n}\right)\right)(\ln n + O(1))$$

and $\|S_n\| = \Lambda_n \ge L_n(0) \ge M_3 \ln n$.

2°. If $a \in \mathbb{R}$ and $|A| = \infty$, then $\varepsilon = 0$ and $\alpha_n = n + 1/2$, so (2.3), (2.4) give:

$$L_n\left(\frac{\pi}{2}\right) = 1 + \left(\frac{1}{\pi\sqrt{2}} + O\left(\frac{1}{n}\right)\right)(\ln(n^2+n) + O(1)),$$

therefore

$$\|S_n\| = \Lambda_n \ge L_n\left(\frac{\pi}{2}\right) \ge M_4 \ln n.$$

3°. If $|a| = \infty$ and $A \in \mathbb{R}$, then $\varepsilon = 1/2$ and $\alpha_n = n + \dfrac{1}{2}$, so:

$$\|S_n\| = \Lambda_n \ge L_n(\pi) \ge M_5 \ln n.$$

4°. If $|a| = |A| = \infty$, then $\varepsilon = 1/2$ and $\alpha_n = n+1$, so we get from (2.3) and (2.4):

$$L_{2n}\left(\frac{\pi}{2}\right) = 1 + \frac{1}{\sqrt{2\pi}}\left|\sqrt{\frac{2}{\pi}}(-1)^{n+1} + O\left(\frac{1}{n}\right)\right|(\ln(4n^2+2n) + O(1)),$$

so:

$$\|S_{2n}\| \ge L_{2n}(\pi) \ge M_6 \ln n \tag{3.2}$$

$$L_{2n+1}\left(\frac{\pi}{4}\right) = 1 + \frac{1}{\sqrt{2\pi}}\left|\cos\left(\frac{n\pi}{2} + \frac{3\pi}{4}\right) + O\left(\frac{1}{n}\right)\right|(2\ln n + O(1)),$$

so:

$$\|S_{2n+1}\| \ge L_{2n+1}\left(\frac{\pi}{4}\right) \ge M_7 \ln n. \tag{3.3}$$

Now, (3.2) and (3.3) give:

$$\|S_n\| = \Lambda_n \ge M_8 \ln n.$$

It follows from 1°, 2°, 3° and 4°:

$$\|S_n\| = \Lambda_n \geq M_9 \ln n, \text{ for sufficiently large } n. \tag{3.4}$$

The relations (3.1) and (3.4) lead to the estimation:

$$\|S_n\| = \Lambda_n \sim \ln n, \tag{3.5}$$

i.e.

$$M_9 \ln n \leq \|S_n\| = \Lambda_n \leq M_2 \ln n, \text{ for sufficiently large } n.$$

To prove the conclusion of this theorem, we shall apply the following principle of condensation of the singularities, [1]:

If X is a Banach space, Y is a normed space and $(A_n)_{n\geq 1}$ is a sequence of continuous linear operators from X into Y so that the set of norms $\{\|A_n\| : n \geq 1\}$ is unbounded, then the set of singularities of the family $\{A_n : n \geq 1\}$, i.e.

$$\left\{x \in X : \limsup_{n\to\infty} \|A_n x\| = \infty\right\},$$

is superdense in X.

Now, choose $X = Y = (C[0,\pi], \|\cdot\|)$ and take into account the estimation (3.5), which completes the proof.

4. On the convergence of the point-interpolatory functionals

Let consider the point-interpolatory functionals S_n^0, $n \geq 1$, given by (1.3) and suppose $x_0 \in (0, \pi)$.

According to (2.2), we have $\|S_n^0\| = L_n(x_0)$; moreover, if $\dfrac{x_0}{\pi} \in \mathbb{Q}$, then the set of values of Lebesgue functions at x_0 is unbounded, [2], so that the following result, similar to that of Theorem 3.1, holds:

If $\dfrac{x_0}{\pi} \in \mathbb{Q} \cap (0,1)$, then the set of unbounded divergence of the family of point-interpolatory functionals $\{S_n^0 : n \geq 1\}$, i.e.

$$\left\{f \in C[0,\pi] : \limsup_{n\to\infty} |S_n^0 f| = \infty\right\},$$

is superdense in the Banach space $(C[0,\pi], \|\cdot\|)$.

On the other hand, let $a \in \left(0, \dfrac{\pi}{2}\right)$ and $x_0 \in [a, \pi - a]$. It follows from (1.4):

$$|S_n^0 f - f(x_0)| \leq M_{10} \left[\omega\left(f; \frac{1}{n}\right) + \frac{1}{n}\|f\|\right] \ln n, \ \forall \ f \in C[0, \pi],$$

which leads to the following statement:

Theorem 4.1. *If $x_0 \in (0, \pi)$, then the family of point-interpolatory functionals $\{S_n^0 : \ n \geq 1\}$ is convergent on the set $DLC[0, \pi]$ of all functions $f \in C[0, \pi]$ satisfying the Dini-Lipschitz condition (1.5), namely*

$$\lim_{n \to \infty} S_n^0 f = f(x_0), \ \forall \ f \in DLC[0, \pi].$$

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SOME NOTES ON (σ,τ)-AMENABILITY OF BANACH ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a Banach algebra and σ,τ be continuous homomorphisms on $\mathcal{A}$. Suppose that $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. A linear mapping $d:\mathcal{A}\longrightarrow\mathcal{X}$ is a (σ,τ)-derivation if

$$d(ab)=d(a)\sigma(b)+\tau(a)d(b)\qquad(a,b\in\mathcal{A}),$$

and is a (σ,τ)-inner derivation if there exists $x\in\mathcal{X}$ such that

$$d(a)=x\sigma(a)-\tau(a)x\qquad(a\in\mathcal{A}).$$

The Banach algebra $\mathcal{A}$ is called (σ,τ)-amenable if every (σ,τ)-derivation is (σ,τ)-inner. In this paper, we investigate the relation between amenability and (σ,τ)-amenability of Banach algebras and also hereditary properties of (σ,τ)-amenability. We give the notion σ-virtual diagonal and σ-approximate diagonal and apply them in study of σ-amenability.

1. **Introduction and preliminaries**

The notion of amenable Banach algebra was introduced by B.E. Johnson in his monograph [4]. This class of Banach algebras arises naturally out of the cohomology theory for Banach algebras, the algebraic version of which was developed by Hochschild [3]. For a comprehensive account on amenability the reader is referred to the books [2, 10, 11].

Throughout the paper, $\mathcal{A}$ is a Banach algebra and $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule. We denote by $\mathcal{A}\mathcal{X}$ and $\mathcal{X}\mathcal{A}$ the closed linear span of $\{ax:a\in\mathcal{A},x\in\mathcal{X}\}$ and $\{xa:a\in\mathcal{A},x\in\mathcal{X}\}$, respectively. A Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is pseudo-unital if $\mathcal{A}\mathcal{X}\mathcal{A}=\mathcal{X}$,

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where $\mathcal{A}\mathcal{X}\mathcal{A}$ is the closed linear span of $\{axb : a, b \in \mathcal{A}, x \in \mathcal{X}\}$. The space $\mathcal{X}^*$ is a Banach $\mathcal{A}$-bimodule via the following module actions:

$$\begin{aligned}(a \cdot f)(x) &= f(xa),\\ (f \cdot a)(x) &= f(ax),\end{aligned}$$

$a \in \mathcal{A}$, $x \in \mathcal{X}$, $f \in \mathcal{X}^*$.

We also denote *weak**-limits with $w^* - \lim$. For a closed subspace $\mathcal{M}$ of $\mathcal{X}$, we denote by $\mathcal{M}^\perp$ the set $\{f \in \mathcal{X}^* : f|_{\mathcal{M}} = 0\}$.

Let σ, τ be continuous homomorphisms from $\mathcal{A}$ to $\mathcal{A}$. A linear mapping $d : \mathcal{A} \longrightarrow \mathcal{X}$ is a (σ, τ)-*derivation* if

$$d(ab) = d(a)\sigma(b) + \tau(a)d(b),$$

for all $a, b \in \mathcal{A}$. A linear map $d : \mathcal{A} \longrightarrow \mathcal{X}$ is a (σ, τ)-*inner derivation* if there exists $x \in \mathcal{X}$ such that $d(a) = x\sigma(a) - \tau(a)x$ for all $a \in \mathcal{A}$.

A wide range of examples are as follows (see [5, 6]):

(i) Every ordinary derivation of an algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{X}$ is an $id_{\mathcal{A}}$-derivation, where $id_{\mathcal{A}}$ is the identity map on the algebra $\mathcal{A}$.

(ii) Every endomorphism α on $\mathcal{A}$ is an $\frac{\alpha}{2}$-derivation.

(iii) Given a character θ on $\mathcal{A}$, a θ-derivation is nothing than a point derivation $d : \mathcal{A} \to \mathbb{C}$ at the character θ.

We use notations $Z^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ for the space of all continuous (σ, τ)-derivations $d : \mathcal{A} \longrightarrow \mathcal{X}$, $B^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ for those which are inner (σ, τ)-derivations, and $H^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ for the quotient space $Z^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})/B^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ which we call the first (σ, τ)-cohomology group of $\mathcal{X}$.

A Banach algebra $\mathcal{A}$ is said to be (σ, τ)-*amenable*(resp. (σ, τ)-*contractible*) if $H^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}^*) = 0$ (resp. $H^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}) = 0$) for all $\mathcal{A}$-bimodules $\mathcal{X}$. See [8, 7].

If $\sigma = \tau$ we simply use the terminologies σ-derivation, σ-amenability, etc. For definitions and elementary properties of Banach algebras we refer the reader to [1, 2, 9].

Modifying some known definition and techniques in the theory of amenability of Banach algebras and using some ideas and terminology of [11], we investigate the relation between amenability and (σ,τ)-amenability of Banach algebras and also hereditary properties of (σ,τ)-amenability. We give σ-virtual diagonal and σ-approximate diagonal and apply them in study of σ-amenability.

2. General properties of (σ,τ)-amenability

In this section we study general properties of (σ,τ)-amenable Banach algebras. Our first result reads as follows.

Proposition 2.1. *Let σ,τ be two continuous homomorphisms on Banach algebra $\mathcal{A}$. If $\mathcal{A}$ is (σ,τ)-amenable then it is $(\lambda\circ\sigma,\mu\circ\tau)$-amenable too, for any continuous homomorphisms λ,μ on $\mathcal{A}$.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and $D:\mathcal{A}\longrightarrow\mathcal{X}^*$ be a continuous $(\lambda\circ\sigma,\mu\circ\tau)$-derivation. We define another $\mathcal{A}$-module product on $\mathcal{X}$ by

$$\begin{aligned} a\Box x &= \lambda(a)x \\ x\Box a &= x\mu(a) \end{aligned}$$

for all $a\in\mathcal{A}$ and $x\in\mathcal{X}$. Then $\mathcal{X}$ with this product is a Banach $\mathcal{A}$-bimodule, and $D(ab)=D(a)(\lambda\circ\sigma)(b)+(\mu\circ\tau)(a)D(b)=D(a)\Box\sigma(b)+\tau(a)\Box D(b)$. Therefore D is a (σ,τ)-derivation, and so, by (σ,τ)-amenability of $\mathcal{A}$, there exists $f\in\mathcal{X}^*$ such that $D(a)=f\Box\sigma(a)-\tau(a)\Box f=f(\lambda\circ\sigma)(a)-(\mu\circ\tau)(a)f$. □

Corollary 2.2. *If $\mathcal{A}$ is amenable, then $\mathcal{A}$ is (σ,τ)-amenable for every two homomorphisms σ and τ.*

The following proposition provides a converse for Proposition 2.1 in a special case.

Proposition 2.3. *Let $\sigma:\mathcal{A}\longrightarrow\mathcal{A}$ be an epimorphism. If $\mathcal{A}$ is σ-amenable, then $\mathcal{A}$ is amenable.*

Proof. Suppose $\mathcal{X}$ is an $\mathcal{A}$-bimodule and $d : \mathcal{A} \longrightarrow \mathcal{X}^*$ is a derivation. Then $D = d \circ \sigma$ is a σ-derivation, since

$$\begin{aligned} D(ab) &= d \circ \sigma(ab) \\ &= d(\sigma(a)\sigma(b)) \\ &= d(\sigma(a))\sigma(b) + \sigma(a)d(\sigma(b)) \\ &= D(a)\sigma(b) + \sigma(a)D(b). \end{aligned}$$

By σ-amenability of $\mathcal{A}$, there exists $f \in \mathcal{X}^*$ such that $D(a) = f \cdot \sigma(a) - \sigma(a) \cdot f$ for each $a \in \mathcal{A}$. Let $b \in \mathcal{A}$. Then there exists $a \in \mathcal{A}$ such that $\sigma(a) = b$ and so $d(b) = d \circ \sigma(a) = D(a) = f \cdot \sigma(a) - \sigma(a) \cdot f = f \cdot b - b \cdot f$. Therefore d is inner. □

The proof of next result is clear and we omit it.

Proposition 2.4. *Let $\mathcal{A}$ be a Banach algebra with a left identity e. Let σ be a homomorphism on $\mathcal{A}$ and $\tau = 0$. Then $\mathcal{A}$ is (σ, τ)-contractible.*

3. **Hereditary properties of (σ, τ)-amenability**

This section is devoted to study hereditary properties of (σ, τ)-amenable Banach algebras. The results of this section are extensions of the known theorems in the classical setting; cf. [11, Subsection 2.3]. Suppose that $\tau, \sigma : \mathcal{A} \longrightarrow \mathcal{A}$ are two endomorphisms, and $\mathcal{I}$ is a closed ideal of $\mathcal{A}$ such that $\sigma(\mathcal{I}) \subseteq \mathcal{I}, \tau(\mathcal{I}) \subseteq \mathcal{I}$. Then one can define the map $\widehat{\tau}, \widehat{\sigma} : \frac{\mathcal{A}}{\mathcal{I}} \longrightarrow \frac{\mathcal{A}}{\mathcal{I}}$ by $\widehat{\sigma}(a + \mathcal{I}) = \sigma(a) + \mathcal{I}, \widehat{\tau}(a + \mathcal{I}) = \tau(a) + \mathcal{I}$.

Let $\mathcal{M}$ be a closed subspace of $\mathcal{X}$. We identify $\mathcal{M}^\perp$ with $(\frac{\mathcal{X}}{\mathcal{M}})^*$ via $f \mapsto \widetilde{f}$, $\widetilde{f}(x + \mathcal{M}) = f(x)$.

Proposition 3.1. *Let $\mathcal{I}, \sigma, \tau$ be as above. If $\mathcal{A}$ is (σ, τ)-amenable then $\frac{\mathcal{A}}{\mathcal{I}}$ is $(\widehat{\sigma}, \widehat{\tau})$-amenable.*

Proof. Let $\mathcal{X}$ be a $\frac{\mathcal{A}}{\mathcal{I}}$-bimodule, and $d : \frac{\mathcal{A}}{\mathcal{I}} \longrightarrow \mathcal{X}^*$ is a $(\widehat{\sigma}, \widehat{\tau})$-derivation. $\mathcal{X}$ is a $\mathcal{A}$-bimodule via, $a \cdot x = (a + \mathcal{I})x, x \cdot a = x(a + \mathcal{I})$. Define $D = d \circ \pi : \mathcal{A} \longrightarrow \mathcal{X}^*$, where $\pi : \mathcal{A} \longrightarrow \frac{\mathcal{A}}{\mathcal{I}}$ is the natural homomorphism. Clearly D is a (σ, τ)-derivation on $\mathcal{A}$. Since $\mathcal{A}$ is (σ, τ)-amenable, D is (σ, τ)-inner. Hence there exists $f \in \mathcal{X}^*$ such that $D = d_f$. So that, $D(a) = d \circ \pi(a) = \sigma(a)\cdot f - f\cdot\tau(a)$ for all $a \in \mathcal{A}$. Hence $d(a+\mathcal{I}) = (\sigma(a)+\mathcal{I})f - f(\tau(a)+\mathcal{I}) = \widehat{\sigma}(a+\mathcal{I})f - f\widehat{\tau}(a+\mathcal{I})$ for all $a \in \mathcal{A}$. $\square$

Proposition 3.2. *Let $\mathcal{I},\sigma,\tau$ be as above and let σ,τ be idempotent homomorphisms. If $\mathcal{I}$ is (σ,τ)-amenable and $\frac{\mathcal{A}}{\mathcal{I}}$ is $(\widehat{\sigma},\widehat{\tau})$-amenable, then $\mathcal{A}$ is (σ,τ)-amenable.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and $D' : \mathcal{A} \longrightarrow \mathcal{X}^*$ is an arbitrary (σ,τ)-derivation. Then $D'|_{\mathcal{I}} : \mathcal{I} \longrightarrow \mathcal{X}^*$is a (σ,τ)-derivation. Then there exists $f \in \mathcal{X}^*$ such that $D'|_{\mathcal{I}} = d_f$, and so $(D'-d_f)(\mathcal{I}) = 0$. Define $D := D' - d_f$. Then $D(\mathcal{I}) = 0$. We have $0 = D(ab)(x) = D(a)(\sigma(b)(x)) + \tau(a)D(b)(x) = D(a)(\sigma(b)(x))$ for all $a \in \mathcal{A}, b \in \mathcal{I}, x \in \mathcal{X}$. Also $0 = D(ba)(x) = D(b)(\sigma(a)x) + \tau(b)D(a)(x)$. Hence $\tau(b)D(a)(x) = 0$ and so $D(a)(x\tau(b)) = 0$ for all $a \in \mathcal{A}, b \in \mathcal{I}, x \in \mathcal{X}$. Let $\mathcal{X}_{\mathcal{I}}$ be the closed submodule generated by $\sigma(\mathcal{I})\mathcal{X} \cup \mathcal{X}\tau(\mathcal{I})$. Then $D(\mathcal{A}) \subseteq \mathcal{X}_I^{\perp} = (\frac{\mathcal{X}}{\mathcal{X}_{\mathcal{I}}})^*$. But $\frac{\mathcal{X}}{\mathcal{X}_{\mathcal{I}}}$ is a Banach $\frac{\mathcal{A}}{\mathcal{I}}$-bimodule via

$$\begin{aligned}(a+\mathcal{I})(x+\mathcal{X}_{\mathcal{I}}) &= \sigma(a)x + \mathcal{X}_{\mathcal{I}}\\ (x+\mathcal{X}_{\mathcal{I}})(a+\mathcal{I}) &= x\tau(a) + \mathcal{X}_{\mathcal{I}}\end{aligned}$$

for all $a \in \mathcal{A}, x \in \mathcal{X}$. Since $D(\mathcal{I}) = 0$, we can define $\widetilde{D} : \frac{\mathcal{A}}{\mathcal{I}} \longrightarrow \mathcal{X}_{\mathcal{I}}^{\perp}$ by $\widetilde{D}(a+\mathcal{I}) = \widetilde{D(a)}$, $\widetilde{D(a)}(x+\mathcal{X}_{\mathcal{I}}) = D(a)(x)$ $(a \in \mathcal{A}, x \in \mathcal{X})$ which is a $(\widehat{\sigma},\widehat{\tau})$-derivation, since

$$\begin{aligned}\widetilde{D}(ab+\mathcal{I})(x+\mathcal{X}_{\mathcal{I}}) &= \widetilde{D(ab)}(x+\mathcal{X}_{\mathcal{I}})\\ &= D(ab)(x)\\ &= D(a)\sigma(b)(x) + \tau(a)D(b)(x)\\ &= D(a)(\sigma(b)x) + D(b)(x\tau(b))\\ &= \widetilde{D}(a+\mathcal{I})(\sigma(b)x+\mathcal{X}_{\mathcal{I}}) + \widetilde{D}(b+\mathcal{I})(x\tau(a)+\mathcal{X}_{\mathcal{I}})\\ &= \widetilde{D}(a+\mathcal{I})(\sigma(b)+\mathcal{I})(x+\mathcal{X}_{\mathcal{I}}) + \widetilde{D}(b+\mathcal{I})(x+\mathcal{X}_{\mathcal{I}})(\tau(a)+\mathcal{I})\\ &= [\widetilde{D}(a+\mathcal{I})\widehat{\sigma}(b+\mathcal{I}) + \widehat{\tau}(a+\mathcal{I})\widetilde{D}(b+\mathcal{I})](x+\mathcal{X}_{\mathcal{I}}).\end{aligned}$$

Hence $\widetilde{D}$ is $(\widehat{\sigma},\widehat{\tau})$-inner, so there exists $\tilde{g} \in (\frac{\mathcal{X}}{\mathcal{X}_{\mathcal{I}}})^*$ such that $\widetilde{D} = d_{\tilde{g}}$. Therefore $D' = d_{f+g}$. $\square$

Proposition 3.3. *Let $\mathcal{A}, \mathcal{B}$ be Banach algebras and σ, τ be continuous endomorphisms of $\mathcal{A}$ and $\mathcal{B}$, respectively. If there is a continuous homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ such that $\varphi(\mathcal{A})$ is a dense subalgebra of $\mathcal{B}$ and $\tau\varphi = \varphi\sigma$, then σ-amenability of $\mathcal{A}$ implies τ-amenability of $\mathcal{B}$.*

Proof. Let $\mathcal{X}$ be a Banach B-module. Then $\mathcal{X}$ can be considered as a Banach $\mathcal{A}$-module, via $a \circ x = \varphi(a)x, x \circ a = x\varphi(a)$. Let $d : \mathcal{B} \longrightarrow \mathcal{X}^*$ be a τ-derivation, then $D = d \circ \varphi : \mathcal{A} \longrightarrow \mathcal{X}^*$ is a σ-derivation. It follows from σ-amenability of $\mathcal{A}$ that there exist $f \in \mathcal{X}^*$ such that $D = d_f$. Therefore

$$\begin{aligned} d(\varphi(a)) = D(a) &= \sigma(a) \circ f - f \circ \sigma(a) \\ &= \varphi(\sigma(a))f - f\varphi(\sigma(a)) \\ &= \tau(\varphi(a))f - f\tau(\varphi(a)) \end{aligned}$$

Hence $d(c) = \tau(c)f - f\tau(c)(c \in B)$. □

4. **Approximate identity and σ-amenability**

We start our work with following extension of [11, Proposition 2.2.1] to show the existence of a bounded approximate identity.

Proposition 4.1. *If σ, τ are two idempotent endomorphisms on $\mathcal{A}$ such that $\sigma(\mathcal{A})$ and $\tau(\mathcal{A})$ are dense subalgebras of $\mathcal{A}$ and $\mathcal{A}$ is (σ, τ)-amenable Banach algebra,then it has a bounded approximate identity.*

Proof. Suppose that $\mathcal{A}$ is (σ, τ)-amenable. Note that $\mathcal{X} = \mathcal{A}$ is a Banach $\mathcal{A}$-bimodule under the actions $a \bullet x = \tau(a)x$ and $x \circ a = 0$. Then $\mathcal{A}^{**}$, as a Banach $\mathcal{A}$-bimodule, has the property $\mathcal{A}^{**} \cdot \mathcal{A} = \{0\}$. The linear map $d : \mathcal{A} \longrightarrow \mathcal{A}^{**}$ defined by $d(a) = \widehat{\tau(a)}$ is a bounded (σ, τ)-derivation, where$\widehat{\ }$denotes the Gelfand transform. In fact,

$$\begin{aligned} d(ab)(f) &= \widehat{\tau(ab)}(f) = f(\tau(ab)) \\ &= f(\tau(a)\tau(b)) = f(\tau^2(a)\tau(b)) \\ &= (\tau(a) \bullet f)(\tau(b)) = \widehat{\tau(b)}(\tau(a) \bullet f) \\ &= \big(\tau(a) \bullet \widehat{\tau(b)}\big)(f) = \big(\tau(a) \bullet d(b)\big)f). \end{aligned}$$

Therefore $d(ab) = \tau(a) \bullet d(b) = d(a) \circ \sigma(b) + \tau(a) \bullet d(b)$. Also $\|d(a)\| = \|\widehat{\tau(a)}\| = \|\tau(a)\| \leq \|\tau\|\|a\|$. By (σ,τ)-amenability of $\mathcal{A}$ there exists $E \in \mathcal{A}^{**}$ such that $d(a) = \widehat{\tau(a)} = -E \circ \sigma(a) + \tau(a) \bullet E$ for all $a \in \mathcal{A}$. Therefore $\widehat{\tau(a)} = \tau(a) \bullet E$. By the Goldstine theorem, let $\{e_\alpha\}$ be a bounded net in $\mathcal{A}$ such that $w^* - \lim \widehat{e_\alpha} = E$. Then $w^* - \lim \tau(a) \bullet \widehat{e_\alpha} = \tau(a) \bullet E = \widehat{\tau(a)}$. Hence

$$\begin{aligned} f(\tau(a)) &= \widehat{\tau(a)}(f) = \lim_\alpha (\tau(a) \bullet \widehat{e_\alpha})(f) \\ &= \lim_\alpha \widehat{e_\alpha}(\tau(a) \bullet f) = \lim_\alpha (\tau(a) \bullet f)(e_\alpha) \\ &= \lim_\alpha f(\tau(a) \bullet e_\alpha) = \lim_\alpha f(\tau(a) e_\alpha). \end{aligned}$$

It follows from $\overline{\tau(\mathcal{A})} = \mathcal{A}$, boundedness of f and boundedness of $\{e_\alpha\}$ that $f(ae_\alpha) \longrightarrow f(a)$ for all $a \in \mathcal{A}, f \in \mathcal{A}^*$. Hence $\{e_\alpha\}$ is a weakly right approximate identity for $\mathcal{A}$. It induces a right approximate identity for $\mathcal{A}$ say $\{d_\alpha\}$. With similar argument we can find a left approximate identity $\{c_\beta\}$. Thus by [1, Proposition 11.1.5] $d_\alpha \Box c_\beta = d_\alpha + c_\beta - d_\alpha c_\beta$ give us an approximate identity for $\mathcal{A}$. $\Box$

Let $\mathcal{A}$ be a Banach algebra. Recall that $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule under the module actions $a(b \otimes c) = ab \otimes c$ and $(b \otimes c)a = b \otimes ca$. Moreover, let $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow A$ be the canonical linear mapping defined by $\pi(a \otimes b) = ab$. We denote the first and the second conjugates of π by $\pi^* : \mathcal{A}^* \longrightarrow (\mathcal{A}\widehat{\otimes}\mathcal{A})^*$ and $\pi^{**} : (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**} \longrightarrow \mathcal{A}^{**}$, respectively.

A net $\{m_\alpha\} \subseteq \mathcal{A}\widehat{\otimes}\mathcal{A}$ is said to be an *σ-approximate diagonal* for $\mathcal{A}$ if $\lim_\alpha m_\alpha \sigma(a) - \sigma(a) m_\alpha = 0$ and $\lim_\alpha \pi(m_\alpha).\sigma(a) = \sigma(a)$ for all $a \in \mathcal{A}$.

An element $M \in (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ is said to be a *σ-virtual diagonal* for $\mathcal{A}$, if $\sigma(a).M = M.\sigma(a)$, $\pi^{**}(M).\sigma(a) = \sigma(a)$ for all $a \in \mathcal{A}$.

The following theorem is an extension of [11, Theorem 2.2.4]

Theorem 4.2. *Let σ be a continuous idempotent homomorphism on a Banach algebra $\mathcal{A}$ with a bounded approximate identity $\{e_\alpha\}$. If $\mathcal{A}$ is σ-amenable then it has a σ-virtual diagonal.*

Proof. Let $\mathcal{A}$ be σ-amenable. The bounded net $\{e_\alpha \otimes e_\alpha\}$ has a w^*-cluster point in $(\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$; say E. We can assume that $w^* - \lim e_\alpha \otimes e_\alpha = E$. Consider the inner

σ-derivation $d_E : \mathcal{A} \longrightarrow (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ defined by $d_E(a) = E.\sigma(a) - \sigma(a).E$. We have

$$\begin{aligned}\pi^{**}(d_E(a)) &= \pi^{**}(E.\sigma(a) - \sigma(a).E)\\ &= w^* - \lim \pi(e_\alpha \otimes e_\alpha\sigma(a) - \sigma(a)e_\alpha \otimes e_\alpha)\\ &= \lim_\alpha(e_\alpha^2\sigma(a) - \sigma(a)e_\alpha^2)\\ &= \sigma(a) - \sigma(a)\\ &= 0.\end{aligned}$$

Therefore $d_E(\mathcal{A}) \subseteq ker(\pi^{**})$. It is Known that $ker(\pi^{**}) = (ker\pi)^{**}$ [11, Page 45]. Thus $d_E \in Z_{(\sigma,\sigma)}(\mathcal{A}, (ker\pi)^{**})$. By σ-amenability of $\mathcal{A}$, there exists $N \in (ker\pi)^{**}$ such that $d_E = d_N$. Put $M = E - N$. Then $M \in (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ and for all $a \in \mathcal{A}$ we have $\sigma(a) \cdot M - M \cdot \sigma(a) = d_M(a) = d_E(a) - d_N(a) = 0$ and

$$\begin{aligned}\pi^{**}(M) \cdot \sigma(a) &= \pi^{**}(E) \cdot \sigma(a) - \pi^{**}(N) \cdot \sigma(a)\\ &= \pi^{**}(E) \cdot \sigma(a)\\ &= w^* - \lim_\alpha(\pi(e_\alpha \otimes e_\alpha)\sigma(a))\\ &= \lim_\alpha e_\alpha^2\sigma(a)\\ &= \sigma(a).\end{aligned}$$

□

The following result is a generalization of [1, Lemma 8].

Proposition 4.3. *Let σ be a continuous idempotent homomorphism on a Banach algebra $\mathcal{A}$ with a bounded approximate identity $\{e_\alpha\}$. If $\mathcal{A}$ has a σ-virtual diagonal then it has a σ-approximate diagonal.*

Proof. Let M be a σ-virtual diagonal for $\mathcal{A}$. Since $M \in (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ there exists a bounded net $\{p_\alpha\}$ in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $w^* - \lim_\alpha p_\alpha = M$. Then $w^* - \lim_\alpha(\sigma(a).p_\alpha - p_\alpha.\sigma(a)) = \sigma(a)M - M\sigma(a) = 0$ and so $w^* - \lim_\alpha(\sigma(a).p_\alpha - p_\alpha.\sigma(a)) = 0$. Moreover, $w^* - \lim_\alpha \pi(p_\alpha).\sigma(a) = \pi^{**}(M).\sigma(a)$ and so $w^* - \lim_\alpha \pi(p_\alpha)\sigma(a) = \sigma(a)$. By passing to a convex combination, we conclude the existence of a net $\{m_\alpha\}$ in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\sigma(a)\cdot m_\alpha - m_\alpha\cdot\sigma(a)\longrightarrow 0$ and $\sigma(a)\cdot\pi(m_\alpha)\longrightarrow\sigma(a)$. □

Proposition 4.4. *Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and $\mathcal{A}$ has a bounded approximate identity. If $\mathcal{A}\mathcal{X}=\{0\}$ (or $\mathcal{X}\mathcal{A}=\{0\}$) then any (σ,τ)-derivation $d:\mathcal{A}\longrightarrow\mathcal{X}^*$ is inner.*

Proof. Let $\{e_\alpha\}$ be a bounded approximate identity for $\mathcal{A}$. $\mathcal{A}\mathcal{X}=\{0\}$ implies that $\mathcal{X}^*\mathcal{A}=\{0\}$. Without lose of generality we can assume that $w^*-\lim d(e_\alpha)=-f$ for some $f\in\mathcal{X}^*$. Then

$$\begin{aligned} d(a) &= w^*-\lim d(ae_\alpha)\\ &= w^*-\lim(d(a)\sigma(e_\alpha)+\tau(a)d(e_\alpha))\\ &= -\tau(a)f\\ &= f\sigma(a)-\tau(a)f\\ &= d_f(a). \end{aligned}$$

□

The next proposition is an extension of [10, Proposition 0.3]

Proposition 4.5. *Let σ,τ be two homomorphisms on a Banach algebra $\mathcal{A}$ having a bounded approximate identity. Then $H^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}^*)=H^1_{(\sigma,\tau)}(\mathcal{A},(\mathcal{A}\mathcal{X}\mathcal{A})^*)$ for each Banach $\mathcal{A}$-bimodule $\mathcal{X}$.*

The following proposition is an extension of [11, Theorem 2.2.4].

Proposition 4.6. *Let σ be a continuous idempotent epimorphism on a Banach algebra $\mathcal{A}$ which has a σ-approximate diagonal. Then $\mathcal{A}$ is σ-amenable.*

Proof. Let $\{m_\alpha\}$ be a σ-approximate diagonal for $\mathcal{A}$. For each α there are two sequences $\{a_n^\alpha\}$, $\{b_n^\alpha\}$ such that $m_\alpha=\sum_{n=1}^\infty a_n^\alpha\otimes b_n^\alpha$ and $\sum_{n=1}^\infty ||a_n^\alpha||||b_n^\alpha||<\infty$. Let $\mathcal{X}$ be a pseudo-unital $\mathcal{A}$-bimodule and $D\in Z_{(\sigma,\sigma)}(\mathcal{A},\mathcal{X}^*)$. The bounded net $\{\sum_{n=1}^\infty\sigma(a_n^\alpha)D(b_n^\alpha)\}$ in $\mathcal{X}^*$ has a w^*-cluster point, say $\phi\in\mathcal{X}^*$. Then by passing to a subnet, if necessary, we have

$$\begin{aligned}
\sigma(a)\phi &= \sigma(a)\lim_{\alpha}\sum_{n=1}^{\infty}\sigma(a_n^{\alpha})D(b_n^{\alpha}) \\
&= \lim_{\alpha}\sum_{n=1}^{\infty}\sigma(a)\sigma(a_n^{\alpha})D(b_n^{\alpha}) \\
&= \lim_{\alpha}F(\sigma(a)m_{\alpha}) \\
&= \lim_{\alpha}F(m_{\alpha}\sigma(a)) \\
&= \lim_{\alpha}\sum_{n=1}^{\infty}\sigma(a_n^{\alpha})D(b_n^{\alpha}\sigma(a)) \\
&= \lim_{\alpha}\sum_{n=1}^{\infty}\sigma(a_n^{\alpha})\Big(\sigma(b_n^{\alpha})D(\sigma(a)) + D(b_n^{\alpha})\sigma(a)\Big) \\
&= \lim_{\alpha}\Big(\sum_{n=1}^{\infty}\sigma(a_n^{\alpha})\sigma(b_n^{\alpha})D(\sigma(a)) + \sum_{n=1}^{\infty}\sigma(a_n^{\alpha})D(b_n^{\alpha})\sigma(a)\Big) \\
&= \lim_{\alpha}\sigma(\pi(m_{\alpha}))D(\sigma(a)) + \phi\sigma(a) \qquad (a \in \mathcal{A}),
\end{aligned}$$

where $F : \mathcal{A}\otimes\mathcal{A} \to \mathcal{X}^*$ is defined by $F(a\otimes b) = \sigma(a)D(b)$. Therefore for each $b \in \mathcal{A}$ is implied that $\sigma(b)(\sigma(a)\phi - \phi\sigma(a) - D(\sigma(a))) = 0$. Since σ is an epimorphism and $\lim_{\alpha}\pi(m_{\alpha})b = b$ for all $b \in \mathcal{A}$, we conclude that $D(a) = \sigma(a)\phi - \phi\sigma(a)$. □

5. **An example**

We use a Banach algebra introduced by Yong Zhang [12] to introduce a Banach algebra that is (σ,τ)-weak amenable for all homomorphisms σ,τ but for some homomorphisms σ and τ it is not (σ,τ)-amenable.

It is easy to see that ℓ^1 is a Banach algebra equipped with the following product

$$a\cdot b = a(1)b \qquad (a, b \in \ell^1).$$

and ℓ^1 has a left identity e_1 defined by

$$e_1(n) = \begin{cases} 1 & if \quad n = 1 \\ 0 & if \quad n \neq 1. \end{cases}$$

The dual space $(\ell^1)^* = \ell^\infty$ is a ℓ^1-bimodule via the ordinary actions as follows

$$a \cdot f = f(a)e_1, \quad f \cdot a = a(1)f \qquad (a \in \ell^1, f \in \ell^\infty)$$

where e_1 is regarded as an element of ℓ^∞.

Next let $\sigma : \ell^1 \longrightarrow \ell^1$ be a bounded homomorphism. We have $a(1)\sigma(b) = \sigma(a \cdot b) = \sigma(a) \cdot \sigma(b) = \sigma(a)(1)\sigma(b)$ and so $(a(1) - \sigma(a)(1))\sigma(b) = 0$ for all $a, b \in \mathbb{N}$. Since $\sigma \neq 0$, we have

$$\sigma(a)(1) = a(1) \quad (a \in \ell^1). \tag{5.1}$$

Now let σ, τ be homomorphisms and let $D : \ell^1 \longrightarrow \ell^\infty$ be a bounded (σ, τ)-derivation. Then for all $a, b \in \ell^1$ we have

$$D(a \cdot b) = D(a) \cdot \sigma(b) + \tau(a) \cdot D(b) \tag{5.2}$$

$$a(1)D(b) = \sigma(b)(1)D(a) + \tau(a) \cdot D(b). \tag{5.3}$$

By taking $b = a$ in (5.2) and using (5.1), we get $\tau(a) \cdot D(a) = 0$ for all $a \in \ell^1$. Therefore we have $\tau(a + b) \cdot D(a + b) = 0$ and so $\tau(a) \cdot D(b) = -\tau(b) \cdot D(a)$ for all $a, b \in \ell^1$.

Then

$$\begin{aligned} D(a) = D(e_1 \cdot a) &= D(e_1) \cdot \sigma(a) + \tau(e_1) \cdot D(a) \\ &= D(e_1) \cdot \sigma(a) - \tau(a) \cdot D(e_1) \end{aligned}$$

for all $a \in \ell^1$. Therefore D is (σ, τ)-inner. Thus ℓ^1 is (σ, τ)-weakly amenable for all homomorphisms σ and τ on ℓ^1.

Remark 5.1. The Banach algebra ℓ^1 is not amenable since it clearly has no bounded right approximate identity. Then, by Corollary 2.2, there exist homomorphisms σ_1 and τ_1 that ℓ^1 is not (σ_1, τ_1)-amenable. By Proposition 2.4, it is however $(\sigma, 0)$-amenable for all homomorphisms σ on ℓ^1.

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ON SOME CLASSES OF ANALYTIC FUNCTIONS DEFINED BY A MULTIPLIER TRANSFORMATION

VERONICA OANA NECHITA

Abstract. We introduce two new classes of analytic functions defined by applying a multiplier transformation to functions $f \in \mathcal{A}(p)$ and study some containment properties of this classes.

1. **Preliminaries**

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$. For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + ...\}.$$

For $p \in \mathbb{N}^*$, we consider $\mathcal{A}(p)$ to be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,$$

which are analytic in the unit disk U.

We denote by $\mathcal{Q}$ the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Since we use the terms of subordination and superordination, we review here those definitions. Let $f, F \in \mathcal{H}$. The function f is said to be *subordinate* to F, or F is said to be *superordinate* to f, if there exists a function w analytic in U, with

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$w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case we write $f \prec F$ or $f(z) \prec F(z)$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

The functions considered in this paper and conditions on them are defined uniformly in the unit disk U, so we shall omit the requirement "$z \in U$".

For $c > -p$, $\delta \in \mathbb{R}$ and for a given function $f \in \mathcal{A}(p)$, we consider the multiplier transformation of functions $f \in \mathcal{A}(p)$, introduced in [5] by

$$K_p^\delta f(z) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{c+p}{c+n} \right)^\delta a_n z^n .$$

For $\delta \geq 0$ we find that K_p^δ is the Komatu linear operator, defined in [2] by

$$K_p^\delta f(z) = \frac{(c+p)^\delta}{\Gamma(\delta)} \frac{1}{z^c} \int_0^z t^{c-1} \left(\log \frac{z}{t} \right)^{\delta-1} f(t)\, dt.$$

We introduce and study some properties of the following classes of functions.

Definition 1.1. *Let ϕ be an analytic functions in the unit disk, with $\phi(0) = 1$ and $\lambda \geq 0$. A function $f \in \mathcal{A}(p)$ is said to be in the class $\Omega_p^\delta(\phi, \lambda)$ if it satisfies the following subordination:*

$$\frac{\lambda}{p} \frac{K_p^{\delta-1} f(z)}{z^p} + \frac{p-\lambda}{p} \frac{K_p^\delta f(z)}{z^p} \prec \phi(z),$$

and is said to be in the class $\overline{\Omega}_p^\delta(\phi, \lambda)$ if it satisfies the superordination

$$\phi(z) \prec \frac{\lambda}{p} \frac{K_p^{\delta-1} f(z)}{z^p} + \frac{p-\lambda}{p} \frac{K_p^\delta f(z)}{z^p}.$$

In our investigation we shall need the following results.

Theorem 1.2 ([1]). *Let h be a convex function in U, with $h(0) = a$, $\gamma \neq 0$ and* $\mathrm{Re}\gamma \geq 0$. *If $p \in \mathcal{H}[a, 1]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{z^{\gamma}} \int_0^z h(t)\, t^{\gamma-1} dt.$$

The function q is convex and is the best dominant.

Theorem 1.3 ([3])**.** *Let h be a convex function in U, with $h(0) = a$, $\gamma \neq 0$ and* $\mathrm{Re}\gamma \geq 0$. *If $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$, $p(z) + \frac{zp'(z)}{\gamma}$ is univalent in U and*

$$h(z) \prec p(z) + \frac{zp'(z)}{\gamma},$$

then

$$q(z) \prec p(z),$$

where

$$q(z) = \frac{\gamma}{z^{\gamma}} \int_0^z h(t)\, t^{\gamma-1} dt.$$

The function q is convex and is the best subordinant.

2. **Main results**

Theorem 2.1. *Let ϕ be a convex function in the unit disk, with $\phi(0) = 1$ and $\lambda > 0$. If $f \in \Omega_p^{\delta}(\phi, \lambda)$, then there exists a convex function q, such that $q(z) \prec \phi(z)$ and $f \in \Omega_p^{\delta}(q, 0)$.*

Proof. We set

$$p(z) = \frac{K_p^{\delta} f(z)}{z^p} = 1 + \sum_{n=1}^{\infty} \left(\frac{c+p}{c+p+n} \right)^{\delta} a_{p+n} z^n$$

and observe that $p \in \mathcal{H}[1,1]$.

A short calculation leads us to

$$\frac{z\left\{K_p^{\delta} f(z)\right\}'}{pz^p} = p(z) + \frac{zp'(z)}{p}$$

and, by using the identity

$$z\left\{K_p^{\delta} f(z)\right\}' = (c+p)\, K_p^{\delta-1} f(z) - cK_p^{\delta} f(z),$$

we get

$$\frac{K_p^{\delta-1} f(z)}{z^p} = p(z) + \frac{zp'(z)}{c+p}.$$

Therefore, since $f \in \Omega_p^\delta(\phi,\lambda)$, we can conclude that

$$p(z) + \frac{\lambda}{p(c+p)} z p'(z) \prec \phi(z).$$

By Theorem 1.2 for $\gamma = \dfrac{p(c+p)}{\lambda}$ it follows now that

$$\frac{K_p^\delta f(z)}{z^p} \prec q(z) \prec \phi(z),$$

where

$$q(z) = \frac{p(c+p)}{\lambda} z^{-p(c+p)/\lambda} \int_0^z \phi(t)\, t^{p(c+p)/\lambda - 1} dt$$

is convex and the best dominant.

Thus, $f \in \Omega_p^\delta(q,0)$ and $f \in \Omega_p^\delta(\widetilde{q},0)$ for all convex functions $\widetilde{q}$ that satisfy $q \prec \widetilde{q}$. □

For suitable choises of the function ϕ, we can obtain some corollaries. Let us first consider the function $\phi(z) = \dfrac{1+Az}{1+Bz}$, for $-1 \le B < A \le 1$. The class $\Omega_p^\delta(\phi,\lambda)$ becomes in this case the class $\Omega_p^\delta(A,B,\lambda)$ from [4].

Corollary 2.2 ([4])**.** *Let $\lambda > 0$ and $f \in \Omega_p^\delta(A,B,\lambda)$. Then $f \in \Omega_p^\delta(A,B,0)$.*

We take now ϕ to be the function given by $\phi(z) = \dfrac{1+\beta z}{1-\alpha\beta z}$, with $0 < \alpha \le 1$ and $0 < \beta < 1$. In this case let us denote the class $\Omega_p^\delta(\phi,\lambda)$ by $\Omega_p^\delta(\alpha,\beta,\lambda)$.

Corollary 2.3. *Let $\lambda > 0$ and $f \in \Omega_p^\delta(\alpha,\beta,\lambda)$. Then $f \in \Omega_p^\delta(\alpha,\beta,0)$.*

Theorem 2.4. *Let ϕ be a convex function in the unit disk, with $\phi(0) = 1$ and $\lambda > 0$. If $f \in \overline{\Omega}_p^\delta(\phi,\lambda)$, $\dfrac{K_p^\delta f(z)}{z^p} \in \mathcal{H}[a,1] \cap \mathcal{Q}$ and $\dfrac{\lambda}{p}\dfrac{K_p^{\delta-1} f(z)}{z^p} + \dfrac{p-\lambda}{p}\dfrac{K_p^\delta f(z)}{z^p}$ is univalent in U, then there exists a convex function q such that $f \in \overline{\Omega}_p^\delta(q,0)$.*

Proof. We set

$$p(z) = \frac{K_p^\delta f(z)}{z^p} = 1 + \sum_{n=1}^{\infty} \left(\frac{c+p}{c+p+n}\right)^\delta a_{p+n} z^n$$

and observe that $p \in \mathcal{H}[1,1] \cap \mathcal{Q}$.

After a short calculation and considering that $f \in \overline{\Omega}_p^\delta(\phi,\lambda)$, we can conclude that

$$\phi(z) \prec p(z) + \frac{\lambda}{p(c+p)} z p'(z)$$

and $p(z) + \dfrac{\lambda}{p(c+p)} zp'(z)$ is univalent in U. We can apply now Theorem 1.3 for $\gamma = \dfrac{p(c+p)}{\lambda}$ and it follows that

$$q(z) \prec \frac{K_p^{\delta} f(z)}{z^p},$$

where

$$q(z) = \frac{p(c+p)}{\lambda} z^{-p(c+p)/\lambda} \int_0^z \phi(t)\, t^{p(c+p)/\lambda - 1} dt$$

is convex and the best subordinant.

Thus, $f \in \overline{\Omega}_p^{\delta}(q, 0)$ and $f \in \overline{\Omega}_p^{\delta}(\widetilde{q}, 0)$, for all convex functions $\widetilde{q}$ that satisfy $\widetilde{q} \prec q$. □

If we combine the results of Theorem 2.1 and Theorem 2.4, we obtain the following differential "sandwich theorem".

Corollary 2.5. *Let* ϕ_1, ϕ_2 *be convex functions in the unit disk, with* $\phi_1(0) = \phi_2(0) = 1$ *and* $\lambda > 0$. *If* $f \in \Omega_p^{\delta}(\phi_1, \lambda) \cap \overline{\Omega}_p^{\delta}(\phi_2, \lambda)$, $\dfrac{K_p^{\delta} f(z)}{z^p} \in \mathcal{H}[a, 1] \cap \mathcal{Q}$ *and* $\dfrac{\lambda}{p} \dfrac{K_p^{\delta-1} f(z)}{z^p} + \dfrac{p-\lambda}{p} \dfrac{K_p^{\delta} f(z)}{z^p}$ *is univalent in* U, *then*

$$f \in \Omega_p^{\delta}(q_1, 0) \cap \overline{\Omega}_p^{\delta}(q_2, 0)$$

where

$$q_1(z) = \frac{p(c+p)}{\lambda} z^{-p(c+p)/\lambda} \int_0^z \phi_1(t)\, t^{p(c+p)/\lambda - 1} dt$$

and

$$q_2(z) = \frac{p(c+p)}{\lambda} z^{-p(c+p)/\lambda} \int_0^z \phi_2(t)\, t^{p(c+p)/\lambda - 1} dt.$$

The functions q_1 *and* q_2 *are convex.*

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FILTERING FOR STOCHASTIC VOLATILITY FROM POINT PROCESS OBSERVATION

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Abstract. In this note we consider the filtering problem for financial volatility that is an Ornstein-Uhlenbeck process from point process observation. This problem is investigated for a Markov-Feller process of which the Ornstein-Uhlenbeck process is a particular case.

Introduction

Stochastic volatility is one of main objective to study of financial mathematics. It reflects qualitively random effects on change of financial derivatives, interest rates and other financial product prices.

Many results have been received recently for volatility estimation by filtering approach. Rüdiger Frey and W. J. Runggaldier [3] studied the case of high frequency data. Frederi G. Viens [10] considered the problem of portfolio optimization under partially observed stochastic volatility. Wolfgang J. Runggaldier [7] used filtering methods to specify coefficients of financial market models.

A filtering approach was introduced by J. Cvitanic, R. Liptser and B. Rozovskii [2] to tracking volatility from prices observed at random times. A filtering problem for Ornstein-Uhlenbeck signal from discrete noises was investigated by Y.Zeng and L.C.Scott [11] and applied to the micro-movement of stock prices. Also a practical method of filtering for stochastic volatility models was given by J. R. Stroud, N. G. Polson and P. Müller [8].

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These authors introduced also a sequential parameter estimation in stochastic volatility models with jumps [4]. And other contributions were given recently by A. Bhatt, B. Rajput and Jie Xiong, R. Elliott, R. Mikulecivius and B. Rozovskii, etc.

Filtered multi-factor models are studied by E. Platen and W. J. Runggaldier [6] by a so-called benchmark approach to filtering.

1. Filtering from point process observation

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which all processes are defined and adapted to a filtration $(\mathcal{F}_t, \quad t \geq 0)$ that is supposed to satisfy the "usual conditions".

For the sake of simplicity, all stochastic processes considered here are supposed to be 1-dimensional real processes.

We consider a filtering problem where the signal processes is a semimartingale

$$X_t = X_0 + \int_0^t H_s ds + Z_t, \tag{1}$$

where Z_t is a square integrable $\mathcal{F}_t$- martingale, H_t is bounded $\mathcal{F}_t$-progressive process and $E[\sup_{s \leq t} |X_s|] < \infty$ for every $t \geq 0$, X_0 is a random variable such that $E|X_0|^2 < \infty$; the observation is given by a point process $\mathcal{F}_t$- semimartingale of the form

$$Y_t = \int_0^t h_s ds + M_t, \tag{2}$$

where M_t is a square integrable $\mathcal{F}_t$-martingale with mean 0, $M_0 = 0$ such that the future σ- field $\sigma(M_u - M_t; u \geq t)$ is independent of the past one $\sigma(Y_u, h_u; u \leq t)$, $h_t = h(X_t)$ is a positive bounded $\mathcal{F}_t$- progressive process such that $E \int_0^t h_s^2 ds < \infty$ for every t.

Denote by $\mathcal{F}_t^Y$ the σ-algebra generated by all random variables $Y_s, s \leq t$. Thus $\mathcal{F}_t^Y$ records all information about the observation up to the time t.

Suppose that the process $u_s = \dfrac{d}{ds} < Z, M >_s$ is $\mathcal{F}_s$- predictable $(s \leq t)$ where $<, >$ stands for the quadratic variation of Z_t and M_t. Denote also by $\hat{u}_s$ the $\mathcal{F}_t^Y$- predictable projection of u_s. By assumptions imposed on Z and M we see that $< Z, M >= 0$, so $u_s = 0$.

The filter of (X_t) based on information given by (Y_t) is defined as the conditional expectation

$$\pi(X_t) := E(X_t|\mathcal{F}_t^Y), \tag{3}$$

or more general

$$\pi_t(f) := E[f(X_t)|\mathcal{F}_t^Y], \tag{4}$$

where f is a bounded continuous function $f \in C_b(\mathbb{R})$.

Denote by $\pi(h_t)$ the filtering process corresponding to the process h_t in (2).

Let m_t be the process defined by

$$m_t = Y_t - \int_0^t \pi(h_s)ds. \tag{5}$$

The process m_t is called the innovation from the observation process Y_t.

Lemma 1.1. *m_t is a point process $\mathcal{F}_t^Y$-martingale and for any t, the future σ-field $\sigma(m_t - m_s\ ;\ t \geq s)$ is independent of $\mathcal{F}_s^Y$.*

Proof. We have by definitions (2) and (5):

$$\begin{aligned} m_t - m_s &= Y_t - Y_s - \int_s^t \pi(h_u)du \\ &= M_t - M_s + \int_s^t [h_u - \pi(h_u)]du. \end{aligned} \tag{6}$$

It follows from the assumption on M_t that

$$E[(M_t - M_s)|\mathcal{F}_s^Y] = 0. \tag{7}$$

On the other hand, since for $u \geq s$

$$E(h_u|\mathcal{F}_s^Y) = E[E(h_u|\mathcal{F}_u^Y)|\mathcal{F}_s^Y] = E[\pi(h_u)|\mathcal{F}_s^Y],$$

or

$$E[\int_s^t [h_u - \pi(h_u)]du|\mathcal{F}_s^Y] = 0, \tag{8}$$

and then

$$E[m_t - m_s|\mathcal{F}_s^Y] = 0\ ,\ t \geq s. \tag{9}$$

Now for any s, t such that $0 \leq s \leq t$ we consider two families $\mathcal{C}_t$ and $\mathcal{D}_t$ of sets of random variables defined as follows:

$\mathcal{C}_{s,t} = \{\text{sets } C_a\,, s \le a \le t\}$ where $C_a = \{m_t - m_\alpha\,; a \le \alpha \le t\}$

$\mathcal{D}_s = \{\text{sets } D_b\,, 0 \le b \le t\}$ where $D_b = \{Y_\beta\,; b \le \beta \le s\}$.

It is easy to check that $\mathcal{C}_{s,t}$ and $\mathcal{D}_s$ are π-systems, i.e. they are closed with respect to finite intersection. Also they are independent each of other by (9). It follows that (refer to [5]) the σ-algebra $\sigma(\mathcal{C}_{s,t}) = \sigma(m_t - m_s, s \le t)$ generated by $\mathcal{C}_{s,t}$ is independent of σ-algebra $\sigma(\mathcal{D}_s) = \mathcal{F}_s^Y$ generated by $\mathcal{D}_s$. The second assertion of Lemma 1.1 as thus established.

We state here an important result by P. Bremaud (refer to [1]) on an integral representation for $\mathcal{F}_t^Y$-martingale:

Lemma 1.2. *Let R_t be a $\mathcal{F}_t^Y$-martingale. Then there exists a $\mathcal{F}_t^Y$-predictable process K_t such that for all $t \ge 0$,*

$$\int_0^t K_s \pi(h_s) ds < \infty \ P.a.s, \tag{10}$$

and such that R_t has the following representation:

$$R_t = R_0 + \int_0^t K_s dm_s. \tag{11}$$

Remark. Since the innovation process m_t is a $\mathcal{F}_t^Y$- martingale so it can represented by

$$m_t = m_0 + \int_0^t K_s dm_s, \tag{12}$$

where K_t is some $\mathcal{F}_t^Y$- predictable process satisfying (10). It is known from [9] that K_t is of the form

$$K_t = \pi(h_t)^{-1}[\pi(X_{t^-} h_t) - \pi(X_{t^-})\pi(h_t) + \hat{u}_t],$$

and since $\hat{u}_t = 0$ we have

Theorem 1.1. *The filtering equation for the filtering problem (1)- (2) is given by:*

$$\pi(X_t) = \pi(X_0) + \int_0^t \pi(H_s) ds + \int_0^t \pi^{-1}(h_s)[\pi(X_{s^-} h_s) - \pi(X_{s^-})\pi(h_s)] dm_s. \tag{13}$$

provided $\pi(h_t) \neq 0$ a.s., and H_t and h_t are processes indicated in (1) and (2).

Remark. If the observation is given by a standard Poisson process Y_t then the filtering equation takes the following form

$$\pi(X_t) = \pi(X_0) + \int_0^t \pi(H_s)ds + \int_0^t \pi^{-1}(h_s)X_{s^-}[\pi(h_s) - 1]dm_s, \tag{14}$$

where $m_t = Y_t - t$.

Quasi-filtering

There is some inconvenience in the application of (13) because the appearance of the factor$[\pi(h_s)]^{-1}$. To avoid this difficulty we introduce the unnormalized conditional filtering or quasi-filtering in the other term.

As we know in the method of reference probability, the probability P actually governing the statistics of the observation Y_t is obtained from a probability Q by an absolutely continuous change $Q \to P$. We assume that Q is the reference probability such that Y is a $(Q, \mathcal{F}_t)$- Poisson process of intensity 1, where $\mathcal{F}_t = \mathcal{F}_t^Y \vee \mathcal{F}_\infty^X$.

Denoting for every $t \geq 0$ by P_t and Q_t the restrictions of P and Q respectively to $(\Omega, \mathcal{F}_t)$ we have $P_t << Q_t$. It is known that the corresponding Radon-Nikodym derivative is the unique solution of the Doleans-Dade equation:

$$L_t = 1 + \int_0^t L_{s-}(h_s - 1)dM_s, \tag{15}$$

where h_t and M_t are given in (2).

The explicit solution of (15) is

$$L_t = \frac{dP_t}{dQ_t} = \sqcap_{0 \leq s \leq t} h_s \Delta Y_s \exp \int_0^t (1 - h_s)ds. \tag{16}$$

Let Z_t be a real valued and bounded process adapted to $\mathcal{F}_t$, then for every history $\mathcal{G}_t$ such that $\mathcal{G}_t \subseteq \mathcal{F}_t,\ t \geq 0$ we have the Bayes formula

$$E_P(Z_t|\mathcal{G}_t) = \frac{E_Q(Z_t L_t|\mathcal{G}_t)}{E_Q(L_t|\mathcal{G}_t)}, \tag{17}$$

where $E_P(.|\mathcal{G}_t)$ and $E_Q(.|\mathcal{G}_t)$ are conditional expectations under probabilities P and Q respectively.

Definition. The process $\sigma(X_t)$ defined by

$$\sigma(X_t) = E_Q(L_t X_t | \mathcal{F}_t) \tag{18}$$

is call the optimal quasi-filter (or quasi-filter) of X_t based on data $\mathcal{F}_t$. It is in fact an unnormalized filter of X_t.

Remarks.

(i) If under the probability Q, Y_t is a standard Poisson process (i.e of intensity 1) and the process $\mu_t \equiv Y_t - t$ is then a $(\mathcal{F}_t, Q)$-martingale.

(ii) We have by consequence of the definition

$$\pi(X_t) = \frac{\sigma(X_t)}{\sigma(1_t)}, \tag{19}$$

where 1 stands for function identified to 1 for every t: $1(t) \equiv 1$.

Replacing $\pi(.)$ by its expression given by (19) we can rewrite the filtering equation (14) as an equation for quasi-filtering $\sigma(.)$:

Theorem 1.2. *The assumptions are those prevailing in Theorem 1.1. Moreover, assume that Z_t and M_t have no common jumps. Then the quasi-filter $\sigma(X_t)$ satisfies the following equation*

$$\sigma(X_t) = \sigma(X_0) + \int_0^t \sigma(H_s)ds + \int_0^t [\sigma(X_{s^-} h_s) - \sigma(X_{s^-})]ds_s, \tag{20}$$

where

$$n_t = Y_t - t\,. \tag{21}$$

Proof. Suppose we have (13) already:

$$\pi(X_t) = \pi(X_0) + \textstyle\int_0^t H(X_s)ds + \int_0^t \pi^{-1}(h_s)\gamma_s dm_s \qquad (1.13)’$$

where $\gamma_s = \pi(X_{s^-} h_s) - \pi(X_{s^-})\pi(h_s)$ and $m_s = Y_s - \int_0^t \pi(h_s)ds$.

By definition $\sigma(X_t) = \pi(L_t)\pi(X_t)$. Applying a formula of integration by part we get

$$\pi(L_t)\pi(X_t) = \pi(X_0) + \int_0^t \pi(X_s)\pi(H_s)ds + \int_0^t \pi(L_{s^-})\gamma_s dm_s$$
$$+ \int_0^t \pi(X_{s^-})\pi(L_{s^-})[\pi(h_s) - 1]dn_s + [\pi(L), \pi(X)]_t \tag{22}$$

where $n_t = Y_t - t$ and $[.,.]$ stands for the quadratic variation.

Because $\pi(X_0) = \sigma(X_0)$ and there are at most countably many points where $\pi(L_{t^-}) \neq \pi(L_t)$ so

$$\int_0^t \pi(L_{s^-})\pi(H_s)ds = \int_0^t \pi(L_s)\pi(H_s)ds = \int_0^t \sigma(H_s)ds.$$

On the other hand we have

$$[\pi(L), \pi(X)]_t = \sum_{0\le s\le t} \Delta\pi(L_s)\Delta\pi(X_s) = \int_0^t \gamma_s \pi(h_{s^-})[\pi(h_s) - 1]dY_s. \tag{23}$$

Then

$$\begin{aligned}\pi(L_t)\pi(X_t) = \sigma(X_t) = \sigma(X_0) + \int_0^t \sigma(H_s)ds+ \\ + \int_0^t \pi(L_{s^-})\big[\pi(X_{s^-}h_s) - \pi(X_s)\pi(h_s)\big]dn_s \\ + \int_0^t \pi(L_{s^-})\pi(X_{s^-})\big[\pi(h_s) - 1\big]dn_s \\ = \sigma(X_0) + \int_0^t \sigma(H_s)ds + \int_0^t \big[\sigma(X_{s^-}h_s) - \sigma(X_{s^-})\big]dn_s\,. \end{aligned} \tag{24}$$

The proof of Theorem 1.2 is thus completed.

□

2. **Filtering for a Fellerian system**

Suppose that X_t is a Markov process taking values in a compact separable Hausdorff space S and that the semigroup $(P_t\ ,\ t \ge 0)$ associated with the transition probability $P_t(x, E)$ with $x \in S$ and $E \subset S$ is a Feller semigroup, that is

$$P_t f(x) = \int_0^t P_t(x, dy)f(y), \tag{1}$$

maps $C(S)$ into itself for all $t \ge 0$ satisfies

$$\lim_{t\downarrow 0} P_t f(x) = f(x), \tag{2}$$

uniformly in S for all $f \in C(S)$, where $C(S)$ is the space of all real continuous function over S. Assume that the observation Y_t is a Poisson process of intensity $h_t = h(X_t) \in C(S)$.

As before the filter π_t is defined as:

$$\pi_t(f) = \pi(f(X_t)) := E[f(X_t)|\mathcal{F}_t^Y]. \tag{3}$$

Also we have

$$\sigma_t(f) := \sigma(f(X_t)) = E_Q[L_t f(X_t)|\mathcal{F}_t^Y], \tag{4}$$

where the probability Q and the likelihood ratio are defined as in subsection I.2.

Denote by m_t the innovation process of Y_t:

$$m_t := Y_t - \int_0^t \pi_s(h)ds = Y_t - \int_0^t \frac{\sigma_s(h)}{\sigma_s(1)}ds. \tag{5}$$

The following results are given in [9]:

Theorem 2.1 (Filtering equation for Feller process with point process observation)**.** *If $\mathcal{A}$ is infinitesimal generator of the semigroup P_t of the signal process, then the optimal filter $\pi_t(f) = \pi(f(X_t))$ satisfies the following two equations provided $\pi_s(h) \neq 0$ a.s.*

a)

$$\begin{aligned}\pi_t(f) &= \pi_0(f) + \int_0^t \pi_s(Af)ds + \\ &+ \int_0^t \pi_s^{-1}(h)[\pi_{s^-}(fh) - \pi_{s^-}(f)\pi_s(h)]dm_s \ , \ f \in C_b(S),\end{aligned} \tag{6}$$

b)

$$\begin{aligned}\pi_t(f) &= \pi_0(P_t f) + \int_0^t \pi_s^{-1}(h)[\pi_{s^-}(hP_{t-s}f) \\ &\quad -\pi_{s^-}(P_{t-s}f)\pi_s(h)]dm_s \ , f \in C_b(S).\end{aligned} \tag{7}$$

Theorem 2.2 (Quasi-filtering equation for Feller process with point process observation.)**.** *The quasi-filter σ_t satisfies the following two equations:*

a)

$$\sigma_t(f) = \sigma_0(f) + \int_0^t \sigma_s(Af)ds + \int_0^t [\sigma_{s-}(hf) - \sigma_{s-}(f)]dm_s \ , \ f \in C_b(S), \tag{8}$$

b)

$$\sigma_t(f) = \sigma_0(P_t f) + \int_0^t [\sigma_{s-}(hP_{t-s}f) - \sigma_{s-}(P_{t-s}f)]dm_s \ f \in C_b(S). \tag{9}$$

3. **Ornstein-Uhlenbeck process and financial filtering**

We recall in this Section some facts on Ornstein- Uhlenbeck and show how to use it to our filtering problems. This process is of importance in studies in finance. It has various 'good properties' to describe many elements in financial models as that of interest rate (Vasicek, Ho-Lee, Hull-White, etc.) or stochastic volatility of asset pricing.

Let $X = (X_t, t \geq 0)$ be a stochastic process with initial value X_0 which is standard normal distributed: $X_0 \in \mathcal{N}(0,1)$.

Definition. If (X_t) is a Gaussian process with

a) mean $EX_t = 0$, $\forall t \geq 0$

b) covariance function

$$R(s,t) = E(X_s X_t) = \gamma \exp(-\alpha|t-s|) \ , s,t \geq 0; \quad \alpha, \gamma \in \mathbb{R}^+, \tag{1}$$

then X_t is called an Ornstein-Ulhenbeck.

It follows from this definition that (X_t) is a stationary process in wide-sense. It is also a stationary process in strict sense since its density of the transition probability is given by

$$p(s,x;t,y) = \frac{1}{\sqrt{\gamma\pi(1-e^{-2\alpha(t-s)})}} \exp\left\{ -\frac{(y - xe^{-2\alpha(t-s)})^2}{\gamma(1-2e^{-2\alpha(t-s)})} \right\}, \tag{2}$$

that depends only on $(t-s)$, where γ is some positive constant.

3.1. **Stochastic Langevin equation.**

An Ornstein-Uhlenbeck process (X_t) can be defined also as the unique solution of the form

$$dX_t = -\alpha X_t dt + \gamma dW_t \ , X_0 \sim \mathcal{N}(0,1), \tag{3}$$

where $\alpha > 0$ and γ are constants.

The explicit form of this solution is

$$X_t = X_0 e^{-\alpha t} + \gamma \int_0^t e^{-\alpha(t-s)} dW_s,$$

and its expectation, variance and covariance are given by

$$EX_t = e^{-\alpha t} \ ,$$

$$V_t := Var(X_t) = \frac{\gamma^2}{2\alpha} \ ,$$

$$R(s,t) = \frac{\gamma^2}{2\alpha} e^{-\alpha|t-s|},$$

where $\dfrac{\gamma^2}{2\alpha}$ is denoted by β in (1)

3.2. **Ornstein - Uhlenbeck process as a Feller process.** Consider a standard Gaussian measure on R

$$\mu(dx) = \frac{1}{\sqrt{2\pi}} \exp\left(- \frac{x^2}{2} \right) dx.$$

It is known that an Orntein - Uhlenbeck process (X_t) is a Markov process and its semigroup is defined by a family $(P_t \ , t \geq 0)$ of operations on bounded Borelian functions f:

$$(P_t f)(x) = \int_R f(e^{-\alpha t} x + \frac{\gamma^2}{2\alpha} \sqrt{1 - e^{-2\alpha t}} y) \mu(dy). \tag{4}$$

It is obvious that

$$\lim_{t \downarrow 0} (P_t f)(x) = f(x), \tag{5}$$

then X_t is really a Feller process and the family $(P_t, t \geq 0)$ is called an Ornstein-Uhlenbeck semigroup.

3.3. **Filtering for Ornstein-Uhlenbeck process from point process observation.** We will apply the results of Section II to the following filtering problem:

• Signal process: An Ornstein-Ulhenbeck process X_t that is the solution of the equation (3).

• Observation process: A point process N_t of intensity $\lambda_t > 0$.

So the signal and observation processes (X_t, N_t) can be expressed in the form

$$dX_t = -\alpha X_t dt + \gamma dW_t \ , X_0 \sim \mathcal{N}(0,1), \tag{6}$$

$$dN_t = \lambda_t dt + M_t, \tag{7}$$

where $\alpha, \gamma > 0$, λ_t is a $\mathcal{F}_t$-adapted process, M_t is a point process martingale independent of W_t.

Denote by $\mathcal{F}_t^N$ the σ-algebra of observation that is generated by $(N_s, s \leq t)$

The filter of (X_t) based on data given by $(\mathcal{F}_t^N)$ is denoted now by $\hat{X}_t$:

$$\hat{X}_t = \pi_t(X) = E(X_t|\mathcal{F}_t^Y)$$

and also $\pi_t(f) = f(\hat{X}_t) = E(f(X_t)|\mathcal{F}_t^Y)$, $f \in C_b(R)$.

The innovation process m_t is given by

$$m_t = Y_t - \int_0^t \hat{\lambda}_t ds, \tag{8}$$

and $dm_t = dY_t - \hat{\lambda}_t dt$.

Since the semigroup $(P_t \ , t \geq 0)$ for X_t is defined by (4), the infinitesimal operator A_t is given by

$$A_t f = \lim_{t \to 0} \frac{1}{t}(P_t f - f) = -\alpha x f'(x) + \frac{1}{2\alpha}\gamma^2 f''(x). \tag{9}$$

On the other hand, $P_t f$ can be expressed under the form:

$$(P_t f)(x) = E[f(e^{-\alpha t}x + \frac{\gamma^2}{2\alpha}\sqrt{1 - e^{-2\alpha t}}Y)], \tag{10}$$

where Y is a standard Gaussian variable, $Y \sim \mathcal{N}(0,1)$.

Then from Theorem 2.1 we can get:

Theorem 3.1. *a)*

$$\begin{aligned} \pi_t(f) \quad &= \quad \pi_0(f) + \int_0^t \pi_s[-\alpha X f'(X) + \frac{\gamma^2}{2\alpha} f''(X)] ds \\ &\quad + \int_0^t \pi_s^{-1}(\lambda)[\pi_{s^-}(\lambda f) - \pi_{s-}(f)\pi_s(\lambda)](dY_s - \pi_s(\lambda)ds), \end{aligned} \tag{11}$$

b)

$$\pi_t(f) = \pi_0(P_t f) + \int_0^t \pi_s^{-1}(\lambda)[\pi_{s^-}(\lambda P_{t-s} f) - \pi_{s-}(P_{t-s} f)\pi_s(\lambda)][dY_s - \pi_s(\lambda)ds], \tag{12}$$

where P_t is given by (10).

Theorem 3.2. *The quasi-filter $\sigma_t(f)$ for the filtering (6)- (7) is given by one of following two equations:*

a)

$$\begin{aligned}\sigma_t(f) &= \sigma_0(f) + \int_0^t \sigma_s[-\alpha X f'(X) + \frac{\gamma^2}{2\alpha} f''(X)]ds \\ &+ \int_0^t [\sigma_{s-}(\lambda f) - \sigma_{s-}(f)][dY_s - \pi_s(\lambda)ds], \end{aligned} \quad (13)$$

b) $\sigma_t(f) = \sigma_0(P_t f) + \int_0^t [\sigma_{s-}(\lambda P_{t-s} f) - \sigma_{s-}(P_{t-s} f)][dY_s - \pi_s(\lambda)ds].$

Remarks.

(i) The above results can be applied also to term structure models for interest rates, where the rate is expressed as an Orstein-Ulhenbeck process and the observation is given by a point process of the form

$$N_t = \int_0^t h(S_s)ds + M_t \ , \quad 0 \le t \le T,$$

where S_t is the a process observed stock prices the models for Vasicek, Ho-Lee, Hull-White,..., can be included in this context.

(ii) The assumption that the volatility of asset pricing is of form of an Ornstein-Ulhenbeck process is quite frequently met in various financial models. So, the above results can give another approach to estimate this volatility.

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APPROXIMATION OF SOLUTION OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH CONDITIONS INSIDE THE INTERVAL $(0, 1)$ USING CUBIC B-SPLINE FUNCTIONS

DANIEL POP

Abstract. In this paper, we present a numerical algorithm, based on cubic B-splines collocation method for solving the second order differential equations with condition inside the (0,1) interval. The scheme is shown to be accurate and only a few terms are required to obtain approximate solution.

1. **Introduction**

The purpose of this paper is to approximate the solution of the following problem:

$$\begin{cases} Ly(x) = r(x),\ \ 0 \le x \le 1, \\ y(a) = A,\ \ y(b) = B \\ 0 < a < b < 1,\ \ a, b, A, B \in \mathbb{R}, \end{cases} \tag{1.1}$$

where:

$$Ly(x) := -\frac{d}{dx}(\frac{dy}{dx}) + q(x) \cdot y(x),\ \ 0 \le x \le 1 \tag{1.2}$$

and $q(x), r(x) \in C(0, 1),\ \ q(x) < 0,\ \ y(x) \in C^2(0, 1)$, using cubic B-splines collocation method.

In fall-back we assume that the functions $q(x)$, $r(x)$ are such that the problem (1.1) has a unique solution.

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2. **Preliminaries**

To describe the basic method in this and later sections we choose a uniform mesh of the given interval $(0,1)$

$$\Delta \coloneqq 0 = x_0 < x_1 < \ldots < x_{n+1} = 1,\ x_j = j \cdot h,\ j = 0, 1, \ldots, n+1,\ h = \tfrac{1}{n+1}. \qquad (2.1)$$

We introduce the domain of definition of the operator L, defined by (1.2), as

$$D_B(L) := \{u \in C^2(0,1) |\ Lu \in C^2(0,1) \text{ and } u(a) = A,\ u(b) = B\},$$

and suppose that $D_B(L)$ is dense in $C^2(0,1)$ such that:

$$L : D_B(L) \subseteq C^2(0,1) \to C^2(0,1).$$

We also define

$$U := \{u \in D_B(L) : u|_{[x_i, x_{i+1}]} \in \Pi_3, u' \in C[x_j, x_{j+1}]\},$$

where Π_3 is the set of polynomials of degree at most three. We use the following notations: $q_j := q(x_j)$, $r_j := r(x_j)$.

Obviously, dim $U = n + 2$. Because $U \subset C^2(0,1)$ (see [3]) there exists in U a basis consisting of cubic B-splines functions:

$$\{s_0, s_1, ... s_{n+1}\}$$

and moreover (see [4, page 555], [5, page 69]):

$$s_i(x) = S(\frac{x}{h} - i),\ i = 0, 1, 2, ..., n+1 \qquad (2.2)$$

$$s_i''(x) = \frac{1}{h^2} \cdot S''(\frac{x}{h} - i), i = 0, 1, ..., n+1 \qquad (2.3)$$

where:

$$S(x) = \frac{1}{4!}\begin{cases} 0 & \text{if } x \in \mathbb{R}\backslash[-2,2] \\ (2-x)^3 - 4(1-x)^3 - 6x^3 + 4(1+x)^3 & \text{if } x \in [-2,-1] \\ (2-x)^3 - 4(1-x)^3 - 6x^3 & \text{if } x \in [-1,0] \\ (2-x)^3 - 4(1-x)^3 & \text{if } x \in [0,1] \\ (2-x)^3 & \text{if } x \in [1,2] \end{cases}$$

The graph of S is shown in Figure 1.

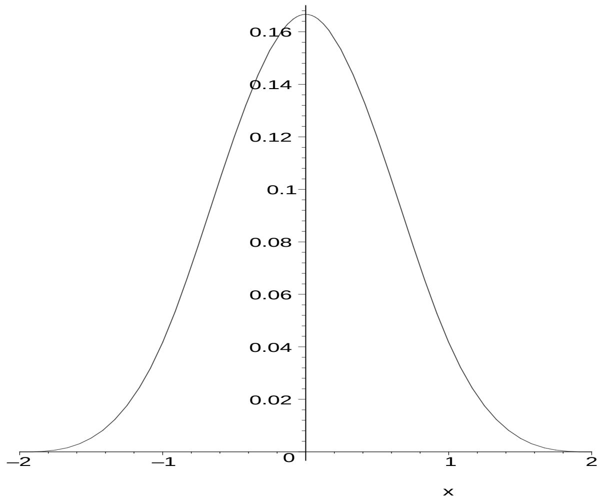


FIGURE 1. The graph of basic function S.

We observe that:

$$S(0) = \frac{1}{6};\ S(1) = S(-1) = \frac{1}{24};\ S(1) = S(-1) = \frac{1}{4}S(0) \tag{2.4}$$

$$S^{''}(x) = \frac{1}{4}\begin{cases} 0 & \text{if } x \in \mathbb{R} - (-2,2) \\ x+2 & \text{if } -2 \leq x \leq -1 \\ -3x-2 & \text{if } -1 < x \leq 0 \\ 3x-2 & \text{if } 0 < x \leq 1 \\ 2-x & \text{if } 1 < x \leq 2 \end{cases}$$

The graph of S'' is shown in Figure 2.

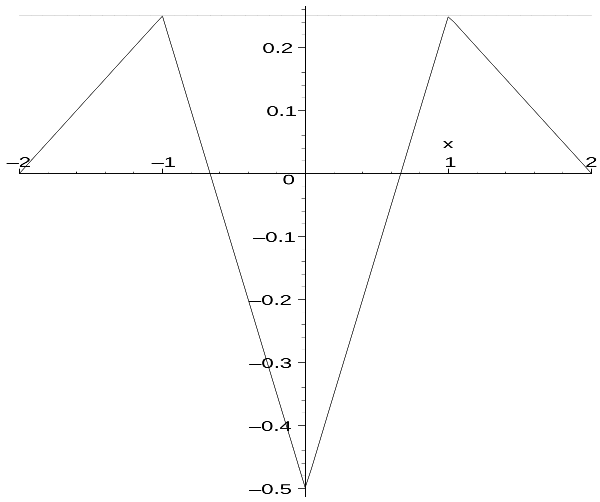


FIGURE 2. The graph of S''.

We also see:

$$S''(0) = -\frac{1}{2};\ S''(-1) = S''(1) = \frac{1}{4};\ S''(1) = S''(-1) = -\frac{1}{2} \cdot S''(0) \tag{2.5}$$

Using Orthogonal Spline Collocation Methods (see [2, pp. 2-5]) an approximate solution $s_y(x) \in U$ has the form:

$$s_y(x) = \sum_{i=0}^{n+1} c_i \cdot s_i(x),\ c_i \in \mathbb{R}, i = 0, 1, ..., n+1.$$

Moreover, from (2.2):

$$s_y(x) = \sum_{i=0}^{n+1} c_i \cdot S(\frac{x}{h} - i),\ c_i \in \mathbb{R}, i = 0, 1, ..., n+1.$$

3. Main Result

Lemma 3.1. *For any $i = 0, 1, 2, ..., n, n+1$, $j = 1, 2, ...n$ the following relations hold:*

$$s_i(x_j) = \begin{cases} \frac{1}{24},\ \text{if } i = j-1, i = j+1 \\ \frac{1}{6},\ \text{if } i = j \\ 0,\ \text{otherwise} \end{cases} \tag{3.1}$$

$$s_i''(x_j) = -\frac{1}{2 \cdot h^2} \begin{cases} -\frac{1}{2}, \ \textit{if } i = j-1, i = j+1 \\ 1, \ \textit{if } i = j \\ 0, \ \textit{otherwise.} \end{cases} \tag{3.2}$$

Proof. Because $s_i(x_j) \neq 0$, only for $i = j-1, j, j+1$ (see Figure 3), using (2.2) we obtain:

if $i = j-1 \ => s_i(x_j) = S(j-i) = S(1) = \frac{1}{24}$

if $i = j => s_i(x_j) = S(j-i) = S(0) = \frac{1}{6}$

if $i = j+1 => s_i(x_j) = S(j-i) = S(-1) = \frac{1}{24}$.

Also $s_i''(x_j) \neq 0$, only for $i = j-1, j, j+1$ using (2.3) we obtain:

if $i = j-1 => s_i''(x_j) = \frac{1}{h^2} \cdot S''(j-i) = \frac{1}{h^2} \cdot S''(1) = \frac{1}{4 \cdot h^2}$

if $i = j \ => s_i''(x_j) = \frac{1}{h^2} \cdot S''(j-i) = \frac{1}{h^2} \cdot S''(0) = -\frac{1}{2 \cdot h^2}$

if $i = j+1 => s_i''(x_j) = \frac{1}{h^2} \cdot S''(j-i) = \frac{1}{h^2} \cdot S''(-1) = \frac{1}{4 \cdot h^2}$ □

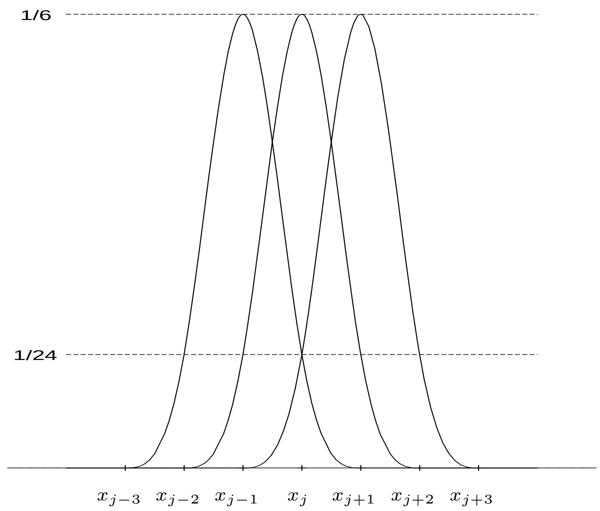


FIGURE 3. Non-zero B-splines on interval $[x_{j-3}, x_{j+3}]$.

Lemma 3.2. *For any $x_0 \in [x_j, x_{j+1}]$, $j = 1, 2, ..., n-1$ the following inequalities hold:*

$$0 < \lim_{h->0} s_{j+2}(x_0) = \lim_{h->0} s_{j-1}(x_0) < \frac{1}{24}, \tag{3.3}$$

$$\frac{1}{24} < \lim_{h->0} s_{j+1}(x_0) = \lim_{h->0} s_j(x_0) < \frac{1}{8}. \tag{3.4}$$

Proof. Let $\lambda(h) := s_{j+2}(x_0)$; $\eta(h) := s_{j+1}(x_0)$; $\rho(h) := s_j(x_0)$; $\varphi(h) := s_{j-1}(x_0)$. In ([5, page 72]) one shows that:

$$s_{j+2}(x) = S(\frac{x}{h} - j - 2) =$$

$$\frac{1}{24} \cdot \begin{cases} [(j+4)h - x]^3 - 4[(j+3)h - x]^3 + 6[(j+2)h - x]^3 - 4[(j+1)h - x]^3 \\ \quad \text{if } \; h \le x \le (j+1)h \\ [(j+4)h - x]^3 - 4[(j+3)h - x]^3 + 6[(j+2)h - x]^3 \\ \quad \text{if } (j+1)h \le x \le (j+2)h \\ [(j+4)h - x]^3 - 4[(j+3)h - x]^3; \\ \quad \text{if } (j \; +2)h \le x \le (j+3)h \\ [(j+4)h - x]^3 \\ \quad \text{if } (j+3)h \le x \le (j+4)h \\ 0, \text{ otherwise.} \end{cases}$$

Because $x_0 \in [x_j, x_{j+1}]$ and $x_j = jh$ it follows $x_0 \in [jh, (j+1)h]$. Then

$$\begin{aligned} 0 &< \lim_{h->0} \lambda(h) = \\ &= \frac{1}{24} \lim_{h->0} \{[(j+4)h - x_0]^3 - 4[(j+3)h - x_0]^3 + \\ &\quad + 6[(j+2)h - x_0]^3 - 4[(j+1)h - x_0]^3\} = \frac{x_0^3}{24}. \end{aligned}$$

Since $0 < x_0 < 1$, we obtain the following relations on λ, φ, η, ρ:

$$0 < \lim_{h->0} \lambda(h) < \frac{1}{24},$$

$$0 < \lim_{h->0} \varphi(h) = \frac{1}{24} \lim_{h->0} \{[(j+1)h - x_0]^3 - 4[jh - x_0]^3 +$$

$$6[(j-1)h - x_0]^3\} = \frac{x_0^3}{24},$$

$$0 < \lim_{h->0} \varphi(h) < \frac{1}{24},$$

$$0 < \lim_{h->0} \eta(h) = \frac{1}{24} \lim_{h->0} \{[(j+3)h - x_0]^3 - 4 \cdot [j \cdot h - x_0]^3\} = \frac{x_0^3}{8},$$

$$\frac{1}{24} < \lim_{h->0} \eta(h) < \frac{1}{8},$$

and

$$0 < \lim_{h->0} \rho(h) = \frac{1}{24} \lim_{h->0} \{[(j+2)h - x_0]^3 - 4 \cdot [(j+1) \cdot h - x_0]^3\} = \frac{x_0^3}{8}.$$

Finally, $0 < x_0 < 1$, implies

$$\frac{1}{24} < \lim_{h->0} \rho(h) < \frac{1}{8}.$$

□

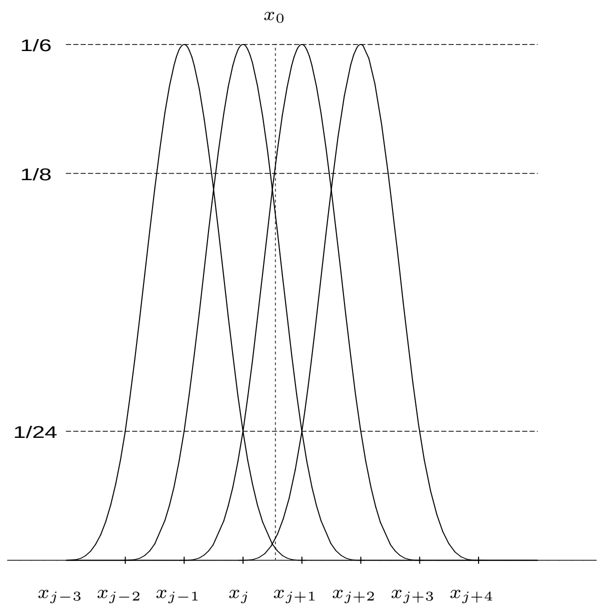


FIGURE 4. Behaviour of B-splines within the interval $[x_j, x_{j+1}]$.

Lemma 3.3. *For $h \to 0$, it holds*

$$\begin{aligned} 0 < \varphi(h) < \eta(h) < \frac{1}{8}, \\ 0 < \lambda(h) < \eta(h) < \frac{1}{8}, \\ 0 < \varphi(h) < \rho(h) < \frac{1}{8}, \\ 0 < \lambda(h) < \rho(h) < \frac{1}{8}, \end{aligned} \tag{3.5}$$

$$\frac{1}{12} < \varphi(h) + \eta(h) + \lambda(h) + \rho(h) < \frac{1}{3}. \tag{3.6}$$

Proof. The relations (3.5) and (3.6) are immediate consequences of the previous lemma. ☐

We show that

$$s_y(x) = \sum_{i=0}^{n+1} c_i \cdot S(\frac{x}{h} - i) \tag{3.7}$$

can be determinate to be an approximate solution of problem (1.1). We impose the conditions:

$$Ls_y(x) = r(x),\ 0 < x < 1 \tag{3.8}$$

$$s_y(a) = A, s_y(b) = B,\ 0 < a < b < 1. \tag{3.9}$$

Theorem 3.4. *If the problem (1.1) has a unique solution, then there exists and it is unique a function $s_y(x)$ which verifies (3.9) and (3.8) on mesh points*

$$x_j = jh;\ \ j = 1, 2, 3, ..., n.$$

Proof. We suppose that $a \in [x_j, x_{j+1}]$. Because $s_i(x) \neq 0$, only for $x_{j-2} < x < x_{j+2}$ then

$$S(\frac{a}{h} - i) \neq 0,\ \text{only for } i = j-1, j, j+1, j+2,$$

and

$$A = c_{j-1} \cdot S(\frac{a}{h} - j + 1) + c_j \cdot S(\frac{a}{h} - j) + c_{j+1} \cdot S(\frac{a}{h} - j - 1) + c_{j+2} \cdot S(\frac{a}{h} - j - 2).$$

Let $\alpha := S(\frac{a}{h} - j + 1); \beta := S(\frac{a}{h} - j); \gamma := S(\frac{a}{h} - j - 1); \delta := S(\frac{a}{h} - j - 2)$ then:

$$c_{j-1} \cdot \alpha + c_j \cdot \beta + c_{j+1} \cdot \gamma + c_{j+2} \cdot \delta = A \tag{3.10}$$

Since $a < b$, then $b \in [x_{j+m}, x_{j+m+1}]$, $j = 1, 2, ..., n$, $m = 1, 2, ..., n-j$ and

$$S(\frac{b}{h} - i) \neq 0, \text{ only for } i = j+m-1, j+m, j+m+1, j+m+2.$$

Also,

$$\begin{aligned} B = s_y(b) = \\ &= c_{j+m-1} \cdot S(\frac{b}{h} - j - m + 1) + c_{j+m} \cdot S(\frac{b}{h} - j - m) + \\ &+ c_{j+m+1} \cdot S(\frac{b}{h} - j - m - 1) + c_{j+m+2} \cdot S(\frac{b}{h} - j - m - 2). \end{aligned}$$

Let $\mu := S(\frac{b}{h} - j - m + 1); \varepsilon := S(\frac{b}{h} - j - m); \tau := S(\frac{b}{h} - j - m - 1); \xi := S(\frac{b}{h} - j - m - 2)$ then:

$$c_{j+m-1} \cdot \mu + c_{j+m} \cdot \varepsilon + c_{j+m+1} \cdot \tau + c_{j+m+2} \cdot \xi = B. \tag{3.11}$$

We impose the conditions:

$$Ls_y(x_j) = r_j; \;\; j = 1, 2, ..., n.$$

Using (2.2) and (2.3) we have

$$Ls_y(x_j) = -\sum_{i=0}^{n+1} c_i \cdot [\frac{1}{h^2} \cdot S''(j-i) - q_j \cdot S(j-i)] = r_j; \; j = 1, 2, ..., n.$$

Because $s_i(x_j) \neq 0$ and $s_i''(x_j) \neq 0$ only for $i = j-1, j, j+1$, using Lemma 3.1 we obtain:

$$\begin{aligned} Ls_y(x_j) = -\frac{1}{h^2} S''(0)[-\frac{1}{2}c_{j-1} + c_j - \frac{1}{2}c_{j+1}] + \\ q_j S(0)[\frac{1}{4}c_{j-1} + c_j + \frac{1}{4}c_{j+1}] = r_j; \; j = 1, 2, ..., n. \end{aligned}$$

Relations (2.5) and (2.4) yield

$$\begin{aligned} Ls_y(x_j) &= \frac{1}{4}c_{j-1}(\frac{q_j}{6} - \frac{1}{h^2}) + \frac{1}{2}c_j(\frac{q_j}{3} + \frac{1}{h^2}) + c_{j+1}(\frac{q_j}{6} - \frac{1}{h^2}) \\ &= r_j, \quad j = 1, 2, ..., n. \end{aligned} \tag{3.12}$$

Because $q(x) < 0$, for any $0 < x < 1$, then

$$\frac{1}{4} \cdot (\frac{q_j}{6} - \frac{1}{h^2}) < 0, \; j = 0, 1, ..., n+1$$

we may divide the relation(3.12) by $\frac{1}{4} \cdot (\frac{q_j}{6} - \frac{1}{h^2})$ and let:

$$p_j := \frac{24 \cdot h^2}{h^2 \cdot q_j - 6} \cdot r_j; \; t_j := \frac{4(h^2 \cdot q_j + 3)}{h^2 \cdot q_j - 6};$$

then the relation(3.12) has the form:

$$c_{j-1} + t_j \cdot c_j + c_{j+1} = p_j. \tag{3.13}$$

We also observe that:

$$\lim_{h->0} t_j = -2. \tag{3.14}$$

The relations (3.10), (3.11), and (3.13) form a linear system of $(n+2)$ equations with $(n+2)$ unknowns $c_0, c_1, ..., c_{n+1.}$:

$$\begin{cases} c_0 \;\; + t_1 c_1 + c_2 & = p_0 \\ c_1 + t_2 c_2 + c_3 & = p_1 \\ \vdots & \\ c_{j-1} + t_j c_j + c_{j+1} & = p_j \\ \alpha c_{j-1} + \beta c_j + \gamma c_{j+1} + \delta c_{j+2} & = A \\ c_j + t_{j+1} c_{j+1} + c_{j+2} & = p_{j+1} \\ \vdots & \\ c_{j+m-1} + t_{j+m} c_{j+m} + c_{j+m+1} & = p_{j+m} \\ \mu c_{j+m-1} + \varepsilon c_{j+m} + \tau c_{j+m+1} + \xi c_{j+m+2} & = B \\ c_{j+m} + t_{j+m+1} c_{j+m+1} + c_{j+m+2} & = p_{j+m+1} \\ \vdots & \vdots \\ c_{n-1} + t_n c_n + c_{n+1} & = p_n \end{cases} \tag{3.15}$$

The system matrix is a band matrix with at most four nonzero diagonals. We note this matrix with C and his determinant with $\det C$. If we develop $\det C$ after the columns $0, 1, 2, ..., j-2, j+m+3, ..., n, n+1$ we obtain:

$$\det C = \begin{vmatrix} 1 & t_j & 1 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ \alpha & \beta & \gamma & \delta & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 1 & t_{j+1} & 1 & 0 & ... & 0 & 0 & 0 & 0 \\ .. & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & 1 & t_{j+m} & 1 & 0 \\ ... & ... & ... & ... & ... & ... & \mu & \varepsilon & \tau & \xi \\ ... & ... & ... & ... & ... & ... & 0 & 1 & t_{j+m+1} & 1 \end{vmatrix} \qquad (3.16)$$

If in Lemma 3.2 we set $x_0 := a$; $x_0 := b$ then applying Lemma 3.3, it follows α, ξ, δ, μ are nonzero. In $\det C$ from (3.16) using the properties of determinants, we obtain the following determinant:

$$\det C = \begin{vmatrix} c & d & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & t_{j+1} & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & t_{j+2} & 1 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 1 & t_{j+m-1} & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 & 1 & t_{j+m} & 1 \\ \dots & \dots & \dots & \dots & \dots & 0 & 0 & e & f \end{vmatrix}, \qquad (3.17)$$

where

$$\begin{aligned} c &:= \beta - \alpha t_j - \delta; \\ d &:= \gamma - \alpha - \delta t_{j+1}; \\ e &:= \varepsilon - \xi - \mu t_{j+m}; \\ f &:= \tau - \xi t_{j+m+1} - \mu. \end{aligned}$$

From (3.14) and Lemma 3.2 we have:

$$\lim_{h->0} c = \lim_{h->0} (\gamma - \alpha - \delta t_{j+1}) = \frac{a^3}{6}, \forall a \in (0,1),$$

$$\lim_{h->0} d = \lim_{h->0} (\beta - \alpha t_j - \delta) = \frac{a^3}{6}, \forall a \in (0,1),$$

$$\lim_{h->0} e = \lim_{h->0} (\varepsilon - \xi - \mu t_{j+m}) = \frac{b^3}{6}; \forall b \in (0,1),$$

$$\lim_{h->0} f = \lim_{h->0} (\tau - \xi t_{j+m+1} - \mu) = \frac{b^3}{6}; \forall b \in (0,1).$$

Let $x := \frac{\beta - \alpha t_j - \delta}{\gamma - \alpha - \delta t_{j+1}}$, $z := \frac{\tau - \xi t_{j+m+1} - \mu}{\varepsilon - \xi - \mu t_{j+m}}$, then from (3.17),

$$\det C = ce \begin{vmatrix} x & 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 1 & t_{j+1} & 1 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 1 & t_{j+2} & 1 & 0 & ... & 0 & 0 & 0 & 0 \\ .. & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & 1 & t_{j+m-1} & 1 & 0 \\ ... & ... & ... & ... & ... & ... & 0 & 1 & t_{j+m} & 1 \\ ... & ... & ... & ... & ... & ... & 0 & 0 & 1 & z \end{vmatrix}$$

But, $\lim_{h->0} x(h) = \lim_{h->0} z(h) = 1$ and from (3.14), we have

$$\lim_{h->0} |t_j(h)| = 2.$$

Let:

$$D := \begin{bmatrix} x & 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 1 & t_{j+1} & 1 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 1 & t_{j+2} & 1 & 0 & ... & 0 & 0 & 0 & 0 \\ .. & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & 1 & t_{j+m-1} & 1 & 0 \\ ... & ... & ... & ... & ... & ... & 0 & 1 & t_{j+m} & 1 \\ ... & ... & ... & ... & ... & ... & 0 & 0 & 1 & z \end{bmatrix}$$

The matrix D is symmetrical, and for $h \to 0$, is diagonal dominant, therefore is nonsingular. Then $\det D \neq 0, \det C \neq 0$ and the system (3.15) has a unique solution.

□

4. **Numerical Results**

We shall approximate the solution of following boundary value problem:

$$-Z''(t) - 243Z(t) = t; 0 \leq t \leq 1 \tag{4.1}$$

$$Z(0) = Z(1) = 0$$

with conditions:

$$Z\left(\frac{\pi}{6}\right) = \frac{\sin\left(\frac{3\sqrt{3}}{2}\pi\right) - \frac{1}{6}\pi\sin\left(9\sqrt{3}\right)}{243\sin\left(9\sqrt{3}\right)}$$

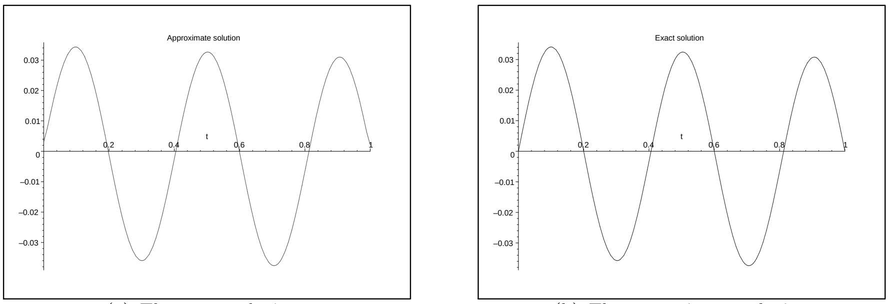
$$Z\left(\frac{\pi}{4}\right) = \frac{\sin\left(\frac{9\sqrt{3}}{4}\pi\right) - \frac{1}{4}\pi\sin\left(9\sqrt{3}\right)}{243\sin\left(9\sqrt{3}\right)}.$$

The problem (4.1) has a unique solution:

$$Z(t) = \frac{\sin\left(9\sqrt{3}t\right) - t\sin\left(9\sqrt{3}\right)}{243\sin\left(9\sqrt{3}\right)}.$$

We used Maple 8 to solve the problem exactly and to approximate the solution. The mesh considered was uniform, with $h = \frac{1}{52}$.

Figure 5(a) illustrates the graph of the exact solution. Figure 5(b) shows the graph of the approximate solution. Both solution are represented on the same graph in Figure 6. The graph of error in a semilogarithmic scale is given in Figure 7.



(a) The exact solution (b) The approximate solution

FIGURE 5. Exact (left) and approximate solution

Table 1 gives the coefficients of approximation.

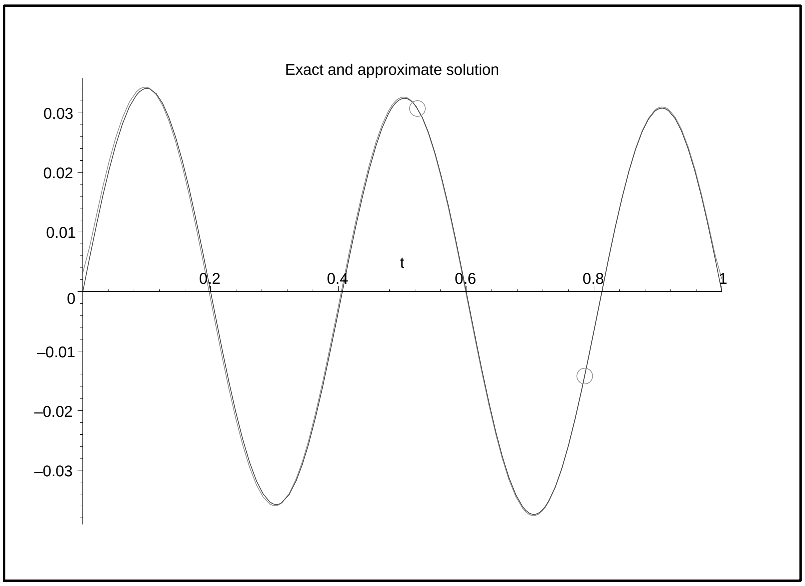


FIGURE 6. The exact and approximate solution on the same graph

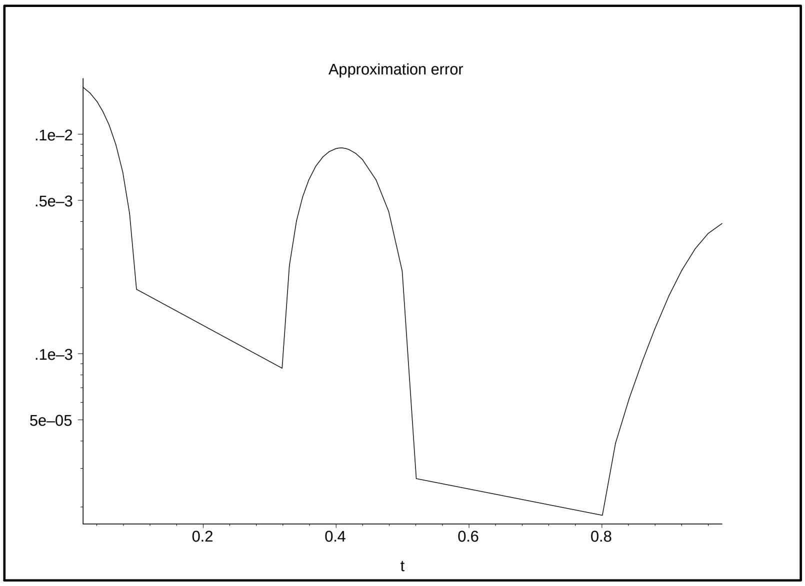


FIGURE 7. Error, plotted in semilogarithmic scale

0.00001264126797	0.01478815919	0.02952783767	0.04419585544
0.05875642949	0.07317381026	0.08741229213	0.1014362548
0.1152101532	0.1286985421	0.1418660874	0.1546775626
0.1670978974	0.1790921693	0.1906256025	0.2016636227
0.2121718270	0.2221160295	0.2314622429	0.2314622429
0.2401767329	0.2482259974	0.2555767791	0.2621961090
0.2680512859	0.2731099106	0.2773398850	0.2807094234
0.2831870848	0.2847417600	0.2853427074	0.2849595322
0.2835622375	0.2811212005	0.2776072155	0.2729914830
0.2672456113	0.2603416764	0.2522521606	0.2429500325
0.2324087120	0.2206021009	0.2075045846	0.1930910568
0.1773368827	0.1602179832	0.1417107936	0.1217922539
0.1004398914	0.07763175011	0.05334645858	0.02756317540
0.0002616852659			

TABLE 1. Coefficients of approximation.

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NUMERICAL GENERATION OF SYMMETRIC α-STABLE RANDOM VARIABLES

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Abstract. The paper discusses two extensions to higher order of the fast, accurate algorithm due to Mantegna [9] for the numerical generation of symmetric α-stable random variables. These extensions result in improved computing time over the most usual range of the index of stability, $\alpha > 1$, for which expectations exist.

1. **Introduction**

Lévy processes are a class of stochastic processes which enjoy a rich mathematical structure and are increasingly used in applications ranging from finance [3] to the study of non-Fickian diffusion in physical systems [8]. Since exact solutions to stochastic differential equations (SDEs) driven by Lévy noise are not usually available, the numerical approximation of such SDEs is often needed. When the path of the Lévy process has to be constructed explicitly, i.e. in the case of strong approximation, the numerical generation of a large number of random variables with the corresponding Lévy distribution is necessary. Even more so, in numerical approximations of some integro-differential nonlinear partial equations of evolution based on the interacting particles approximation [14], the position of each particle is governed by a SDE driven by Lévy noise, hence a system of SDEs of size equal to the number of particles must be integrated numerically. In such a case, the use of a fast and accurate algorithm for the generation of these random variables (which represent discrete approximations to the time increments of the stochastic process) is crucial if reliable numerical results are to be obtained in a convenient time frame.

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A first numerical algorithm for the generation of random variables with a general Lévy distribution, including those with skewness, has been presented by Chambers, Mallows and Stuck [4]. More recently, Mantegna [9] devised a different numerical method for the class of symmetric α-stable Lévy distributions based on the asymptotic expansion of the integral expression of their probability density function. Mantegna's algorithm makes use of the generalized version of the central limit theorem together with a nonlinear transformation to achieve an accurate approximation of the probability density function. However, the use of the generalized central limit theorem by this latter algorithm implies summation of several independent realizations of a random variable with a probability distribution close to the targeted distribution. Although the number of these samples is reduced by a nonlinear transformation, the generation of several independent samples reduces the efficiency of the algorithm.

In this paper, we propose a new algorithm for the numerical generation of α-stable random variables. It is based, as Mantegna's algorithm [9], on the asymptotic expansion of the probability density function, but to the next higher order. The use of the higher-order term introduces some complications in the evaluation of the associated probabilities, which could not be surmounted analytically so that numerical approximations were needed. The paper is organized as follows. The next two sections briefly recall the basic properties of symmetrical Lévy α-stable random variables that we need and the algorithm due to Mantegna. Section 4 develops our proposed algorithm, while the last section presents pertinent numerical results comparing the algorithms. A brief conclusion section ends the paper.

2. Symmetric $\alpha-$stable distributions

We recall that a univariate random variable Z has a (strictly) stable distribution if for any $a, b > 0$ there exists $c > 0$ such that $aZ_1 + bZ_2 \stackrel{d}{=} cZ$, where Z_1 and Z_2 are independent copies of Z, and $\stackrel{d}{=}$ denotes equality in distribution. For given α, the distribution of Z is called symmetric α-stable if it equals the distribution of $-Z$, and in this case its probability density function (PDF) can be expressed as the improper integral

$$f_Z^{\alpha,\gamma}(z) = \frac{1}{\pi}\int_0^\infty \exp(-\gamma q^\alpha)\cos(qz)\,dq, \quad 0 < \alpha \le 2 \tag{1}$$

The parameter α is known as the index of stability, or characteristic exponent, of the distribution, while γ is a scale factor ($\gamma > 0$). For $\alpha = 2$, $\alpha = 1$ and $\alpha = 1/2$, the Gauss, Cauchy and Lévy distributions are obtained, respectively.

Humbert [5] discusses the problem of representing the derivative of e^{-q^α} as a Laplace integral. His result leads to the following expression for $f_Z^{\alpha,\gamma}(z)$:

$$f_Z^{\alpha,\gamma}(z) = -\frac{1}{\pi}\sum_{k=1}^{N}\frac{(-1)^k}{k!}\frac{\Gamma(\alpha k+1)}{z^{\alpha k+1}}\sin\left(\frac{k\pi\alpha}{2}\right) + R(z) \tag{2}$$

where $\Gamma(z)$ is the gamma function and $R(z) = O\left(z^{-\alpha(N+1)-1}\right)$. From (2), one can obtain the two-term asymptotic approximation of a symmetric stable PDF for large z as a function of the parameter α,

$$f_Z^{\alpha,\gamma}(z) \approx \frac{\Gamma(1+\alpha)\sin(\pi\alpha/2)}{\pi z^{1+\alpha}} - \frac{\Gamma(1+2\alpha)\sin(\pi\alpha)}{\pi z^{1+2\alpha}} \tag{3}$$

For more details about stable distributions we refer to [6, 13].

3. Computer generation of symmetric α-stable random variables

While the work of Chambers *et al.* [4] describes a method for generation of α-stable random variables with general distributions that may include skewness, a different approach valid for the symmetric α-stable case was taken by Mantegna [9]. The latter results in an algorithm allowing the generation of a random variable Z whose probability density is *a*rbitrarily close to the PDF (1) for $0.3 \le \alpha \le 1.99$. The main idea stems from the generalized central limit theorem: the sum of independent random variables having the same symmetric α-stable distribution will eventually converge to a random variable characterized by the same law. Given α, consider the random variable

$$V = \frac{X}{|Y|^{1/\alpha}}, \tag{4}$$

where X and Y are two normal random variables with standard deviation σ_x and σ_y, respectively. One can then choose these values such that the probability density function of V, $f_V(v)$ matches the exact PDF $f_Z^{\alpha,1}(z)$ in the origin and for large values of z. To obtain better results, one can then generate a number of independent copies of V, say $V_1, V_2, \ldots, V_n$, and use the central limit theorem to construct

$$\tilde{Z} = \frac{1}{n^{1/\alpha}} \sum_{k=1}^{n} V_k. \tag{5}$$

The random variable $\tilde{Z}$ may be expected to converge to a symmetric α-stable random variable. Because the convergence is quite slow, i.e. one needs a relatively large n in equation (5), Mantegna introduced a nonlinear transformation which gives an exponential tilt to the distribution of the random variable V by defining a new random variable,

$$W = \{[K(\alpha) - 1][\exp(-|V|/C(\alpha))] + 1\} V, \tag{6}$$

with parameters $K(\alpha)$ and $C(\alpha)$ determined by requiring

$$P(W = 0) = f_Z^{\alpha,1}(0) \tag{7}$$

and respectively

$$P[W = W(C(\alpha))] = f_Z^{\alpha,1}[W(C(\alpha))]. \tag{8}$$

A fast convergence toward a stable random variable is then obtained by constructing

$$\tilde{Z} = \frac{1}{n^{1/\alpha}} \sum_{k=1}^{n} W_k, \tag{9}$$

instead of (5). Note that the cost of Mantegna's algorithm depends on the number n of samples of the random variable W used in equation (9). Larger values of n make the algorithm more accurate at the price of generating many copies of W, each of which requires two samples from a normal distribution.

4. **High-Order Algorithm Using Independent Samples**

In the following, we propose a new algorithm for the numerical generation of a symmetric α-stable random variable which has the same starting point as Mantegna's algorithm [9], but is much faster for comparable accuracy. Note that one can set $\gamma = 1$ for simplicity, since rescaling of the generated random variable is straightforward.

First, consider equation (2) with $N = 2$ and $\gamma = 1$ and let us compute

$$V_1 = \frac{X_1}{|Y_1|^{1/\alpha}} \text{ and } V_2 = \frac{X_2}{|Y_2|^{1/2\alpha}}, \tag{10}$$

where X_1, X_2, Y_1, Y_2 are four independent normal random variables with standard deviation $\sigma_{x_1}, \sigma_{x_2}, \sigma_{y_1}, \sigma_{y_2}$ respectively. Using the method of transformations for the bivariate case, see e.g. [12], the probability densities of the continuous variables V_1 and V_2 can be found to be

$$f_{V_1}(v_1) = \frac{1}{\pi\sigma_{x_1}\sigma_{y_1}} \int_0^\infty y^{1/\alpha} \exp\left[-\frac{y^2}{2\sigma_{y_1}^2} - \frac{v_1^2 y^{2/\alpha}}{2\sigma_{x_1}^2}\right] dy,$$
$$f_{V_2}(v_2) = \frac{1}{\pi\sigma_{x_2}\sigma_{y_2}} \int_0^\infty y^{1/2\alpha} \exp\left[-\frac{y^2}{2\sigma_{y_2}^2} - \frac{v_2^2 y^{1/\alpha}}{2\sigma_{x_2}^2}\right] dy, \tag{11}$$

For large arguments, the above probability densities are very well described by the asymptotic approximation

$$f_{V_1}(v_1 \gg 0) \approx \frac{\alpha 2^{(\alpha-1)/2}\sigma_{x_1}^{\alpha}\Gamma((\alpha+1)/2)}{\pi\sigma_{y_1} v_1^{\alpha+1}},$$
$$f_{V_2}(v_2 \gg 0) \approx \frac{\alpha 2^{(2\alpha+1)/2}\sigma_{x_2}^{2\alpha}\Gamma((2\alpha+1)/2)}{\pi\sigma_{y_2} v_2^{2\alpha+1}} \tag{12}$$

and in the origin

$$f_{V_1}(0) = \frac{2^{(1-\alpha)/2\alpha}\sigma_{y_1}^{1/\alpha}\Gamma((\alpha+1)/2\alpha)}{\pi\sigma_{x_1}},$$
$$f_{V_2}(0) = \frac{2^{(1-2\alpha)/4\alpha}\sigma_{y_2}^{1/2\alpha}\Gamma((2\alpha+1)/4\alpha)}{\pi\sigma_{x_2}}. \tag{13}$$

The second step in our algorithm is to compute another random variable V given by

$$V = V_1 + V_2. \tag{14}$$

The density of the sum of two independent continuous random variables is the convolution of their individual densities. Considering, without loss of generality, the particular case where $\sigma_{y_1} = \sigma_{y_2} = 1$, it follows then [12] that the probability density of the random variable V is given by

$$f_V(v) \approx \int_{-\infty}^{\infty} \left[\int_0^\infty \frac{1}{\pi\sigma_{x_1}} s^{\frac{1}{\alpha}} \exp\left(-\frac{s^2}{2} - \frac{t^2}{2\sigma_{x_1}} s^{\frac{2}{\alpha}}\right) ds\right] \cdot$$
$$\cdot \left[\int_0^\infty \frac{1}{\pi\sigma_{x_2}} s^{\frac{1}{2\alpha}} \exp\left(-\frac{s^2}{2} - \frac{(v-t)^2}{2\sigma_{x_2}} s^{\frac{1}{\alpha}}\right) ds\right] dt \tag{15}$$

hence its value at the origin is

$$f_V(0) = \int_{-\infty}^{\infty} \left[\int_0^{\infty} \frac{1}{\pi\sigma_{x_1}} s^{\frac{1}{\alpha}} \exp\left(-\frac{s^2}{2} - \frac{t^2}{2\sigma_{x_1}} s^{\frac{2}{\alpha}}\right) ds \right] \cdot \left[\int_0^{\infty} \frac{1}{\pi\sigma_{x_2}} s^{\frac{1}{2\alpha}} \exp\left(-\frac{s^2}{2} - \frac{t^2}{2\sigma_{x_2}} s^{\frac{1}{\alpha}}\right) ds \right] dt \tag{16}$$

We now obtain values for σ_{x_1} and σ_{x_2} such that the following conditions are satisfied simultaneously for a given value of α:

- The approximate PDF matches the exact one in the origin,

$$f_Z^{\alpha,1}(0) = f_V(0) \tag{17}$$

- The least-squares error in the approximate PDF is minimized over a bounded interval $[-L, L]$:

$$F(\sigma_{x_1}, \sigma_{x_2}) = \int_{-L}^{L} [f_Z^{\alpha,1}(z) - f_V(z)]^2 dz = \min. \tag{18}$$

From these conditions one obtains a system of equations that can be solved numerically for the values of $\sigma_{x_1}, \sigma_{x_2}$ once α and a value for L are specified.

5. **High-Order Algorithm Using Dependency**

Another approach which is less computationally expensive but involves some tedious, albeit straightforward algebraic manipulation, is to reduce the number of independent normal variables generated in the high-order algorithm. This can be done as follows. Note that in (10) four independent normal random variables are used, although there are only two free unknowns. To further reduce the computational cost, let (10) hold for $X_1 = X_2 = X$ and $Y_1 = Y_2 = Y$, where X, Y are two independent normal random variables with standard deviation σ_x, σ_y respectively. Hence (10) is equivalent to

$$V_1 = \frac{X}{|Y|^{1/\alpha}} \text{ and } V_2 = \frac{X}{|Y|^{1/2\alpha}}. \tag{19}$$

With this choice, the random variables V_1 and V_2 are dependent, and the joint density of V_1 and V_2 becomes more difficult to evaluate. The method of transformations [12] for the bivariate case will be used again to compute the probability density function

$f_{V_1,V_2}(v_1,v_2)$ of (V_1,V_2).

Let $g(x,y)=\left(\frac{x}{|y|^{1/\alpha}},\frac{x}{|y|^{1/2\alpha}}\right)=(v_1,v_2)$. Note that

$$g^{-1}(v_1,v_2)=\left(\frac{v_2^2}{v_1},\left(\frac{v_2}{v_1}\right)^{2\alpha}\right)=\left(\frac{x^2}{|y|^{1/\alpha}}\cdot\frac{|y|^{1/\alpha}}{x},\frac{x^{2\alpha}}{|y|}\cdot\frac{|y|^2}{x^{2\alpha}}\right)=(x,|y|).$$

Therefore, $g^{-1}(v_1,v_2)=\left(\frac{v_2^2}{v_1},\left(\frac{v_2}{v_1}\right)^{2\alpha}\right)$. The absolute value of the determinant of the Jacobian $J_{g^{-1}}$ is given by

$$|J_{g^{-1}}(v_1,v_2)| = abs\begin{vmatrix}\frac{\partial x}{\partial v_1} & \frac{\partial x}{\partial v_2}\\ \frac{\partial y}{\partial v_1} & \frac{\partial y}{\partial v_2}\end{vmatrix} = abs\begin{vmatrix}-\frac{v_2^2}{v_1^2} & 2\frac{v_2}{v_1}\\ v_2^{2\alpha}(-2\alpha v_1^{-2\alpha-1}) & 2\alpha v_2^{2\alpha-1}v_1^{-2\alpha}\end{vmatrix}$$

$$= |(-2\alpha v_2^{2\alpha+1}v_1^{-2\alpha-2}+4\alpha v_2^{2\alpha+1}v_1^{-2\alpha-2}| = 2\alpha|(v_2^{2\alpha+1}v_1^{-2\alpha-2})|.$$

Hence, the probability density function $f_{V_1,V_2}(v_1,v_2)$ of (V_1,V_2) is given by

$$f_{V_1,V_2}(v_1,v_2)=2\alpha|v_2^{2\alpha+1}v_1^{-2\alpha-2}|[f_X(\frac{v_2^2}{v_1})f_Y((\frac{v_2}{v_1})^{2\alpha})+f_X(\frac{v_2^2}{v_1})f_Y(-(\frac{v_2}{v_1})^{2\alpha})].$$

Next, let $\{V\}$ to be another random variable given by

$$V=V_1+V_2. \tag{20}$$

The probability density function of the random variable V is given by (see [12])

$$\begin{aligned} f_V(v) &= \int_{-\infty}^{\infty} f_{V_1,V_2}(w,v-w)dw \\ &= 4\alpha\frac{1}{2\pi\sigma_x\sigma_y}\int_{-\infty}^{\infty}|(v-w)^{2\alpha+1}w^{-2\alpha-2}|\exp\left(-\frac{1}{2\sigma_x^2}\frac{(v-w)^4}{w^2}-\frac{1}{2\sigma_y^2}\frac{(v-w)^{4\alpha}}{w^{4\alpha}}\right)dw. \end{aligned} \tag{21}$$

In order to obtain values for σ_x and σ_y for a given value of α, one can again impose conditions similar to those stated in equations (17) and (18). Lastly, let us note that this use of dependent variables reduces the cost of the algorithm in the previous section by a factor of two.

6. **Numerical tests**

Table 1 gives a sample set of values obtained for the two parameters σ_{x_1} and σ_{x_2} (independent case) as a function of α, with the choice $L=10$.

α	$\sigma_{x_1}(\alpha)$	$\sigma_{x_2}(\alpha)$
0.7	0.880	0.002
0.8	0.930	0.001
0.9	0.971	0.000
1	1	0.027
1.1	0.951	0.215
1.2	0.855	0.410
1.3	0.800	0.523
1.4	0.729	0.645
1.5	0.610	0.800
1.6	0.396	1.008
1.7	0.280	1.100
1.8	0.110	1.200
1.9	0.001	1.231

TABLE 1. Values obtained for the parameters σ_{x_1} and σ_{x_2} as a function of α.

Probability density functions obtained numerically by the proposed algorithms, as well as by the algorithm due to Mantegna for both $n = 1$ and $n = 10$ in equation (9) are compared with the exact density in figures 1 and 2 for $\alpha = 1.7$ and $\alpha = 1.3$, respectively. For completeness, we also include results obtained with the corrected version of the algorithm due to Chambers *et al.* [4, 13]. In these figures the dashed lines are the result of the simulation (histograms based on 10^6 samples), while the continuous line is the exact Lévy stable distribution, computed from the integral form evaluated with 20 decimal digits in the symbolic computation package **Maple**, see `http://www.maplesoft.com`. The L_2 error in the numerically generated probability distributions as a function of α, again based on 10^6 samples, is given in figure 3, with the actual CPU time needed shown in figure 4. As can be seen, under this measure, our algorithm offers an accuracy comparable to Mantegna's method with $n = 10$ for a much smaller computational cost.

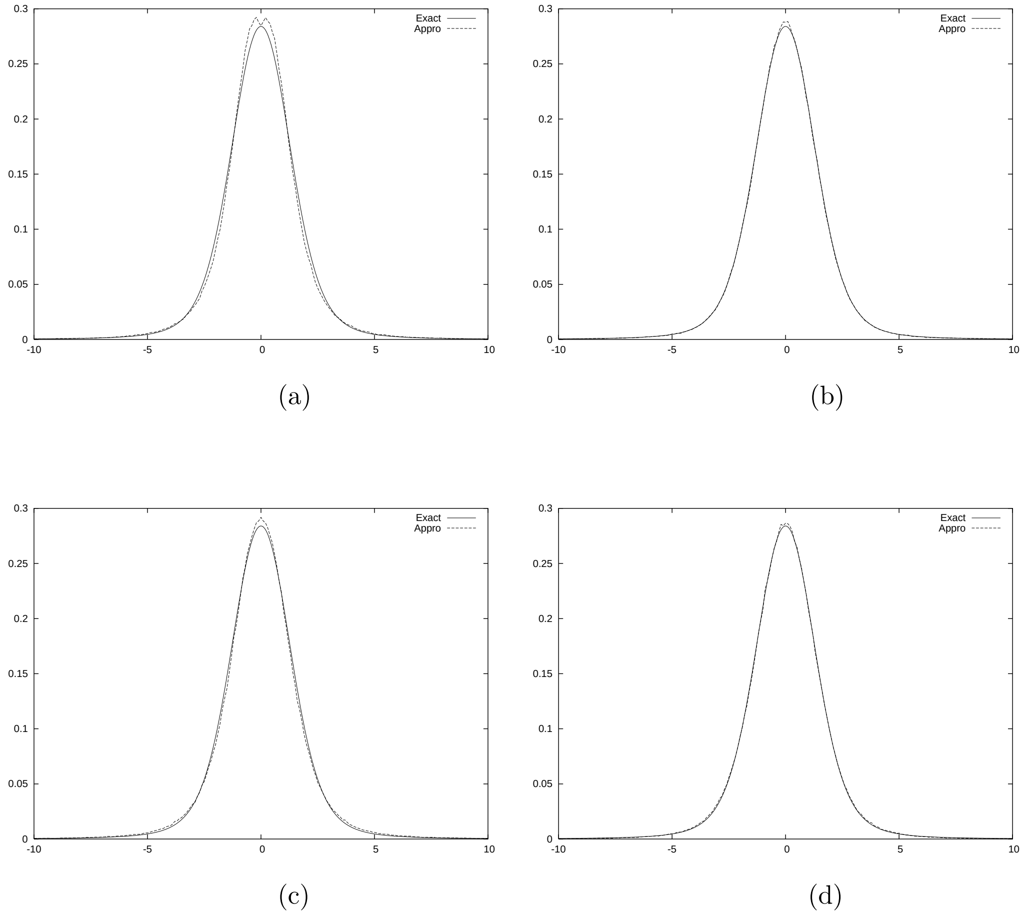

FIGURE 1. Exact and approximate Lévy density with $N = 1,000,000$ samples for $\alpha = 1.7$.
(a) Mantegna ($n = 1$)
(b) Mantegna ($n = 10$)
(c) Chambers *et al.* (d) Present

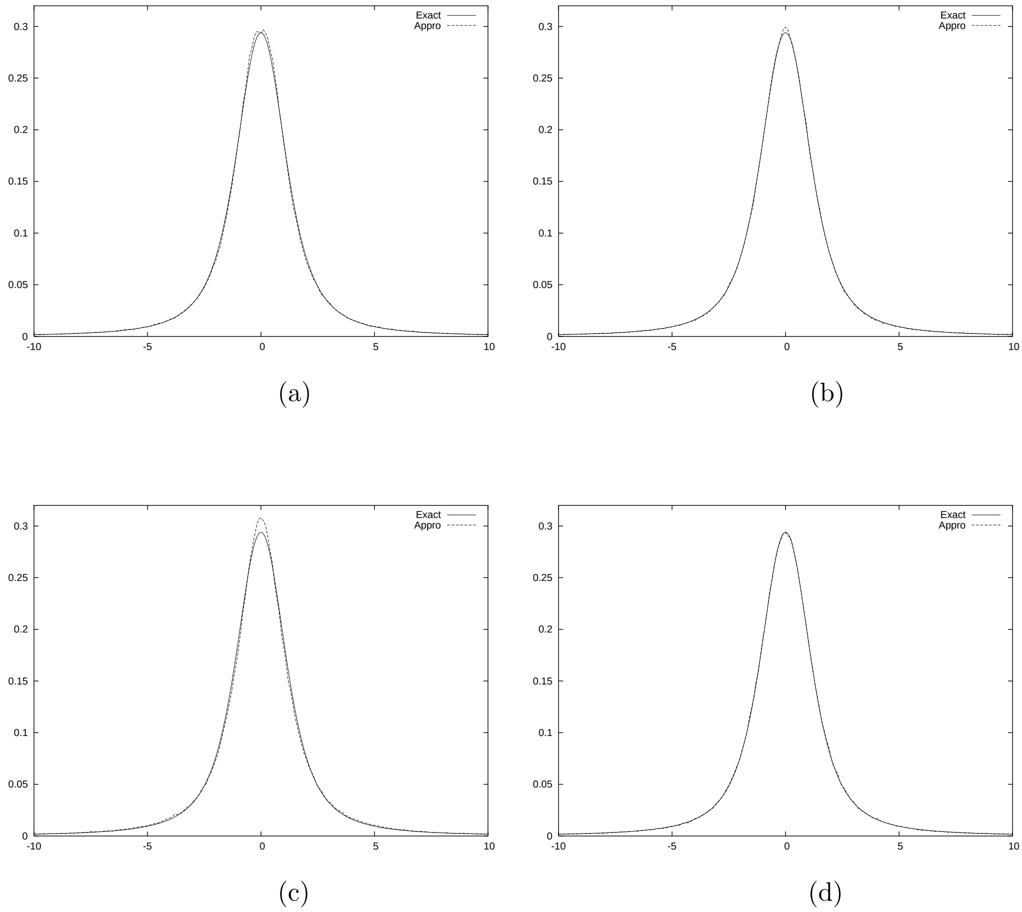

FIGURE 2. Exact and approximate Lévy density with $N = 1,000,000$ samples for $\alpha = 1.3$.
(a) Mantegna ($n = 1$)
(b) Mantegna ($n = 10$)
(c) Chambers *et al.*
(d) Present

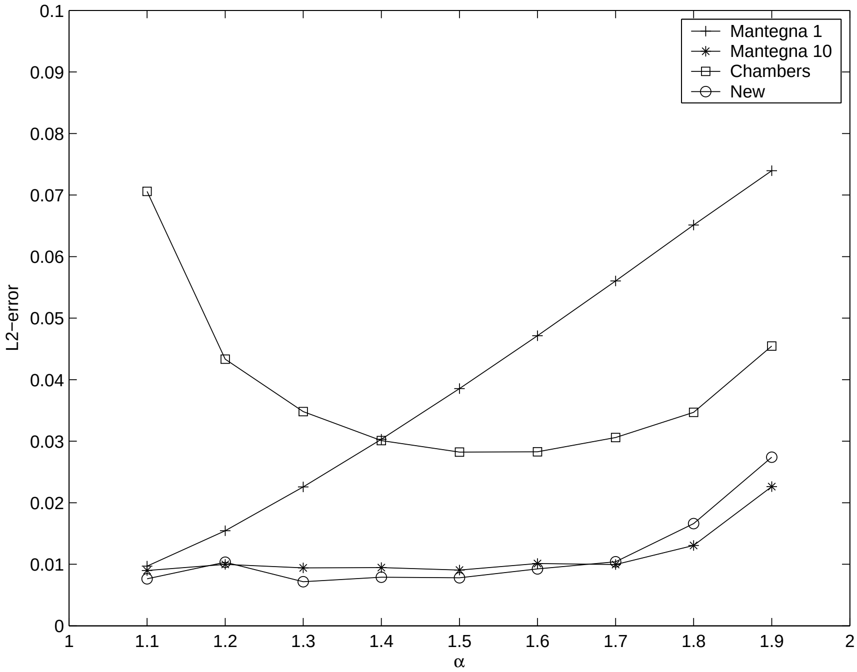

FIGURE 3. L_2 error as a function of α for Mantegna ($n = 1$ and $n = 10$), Chambers, and the proposed algorithm.

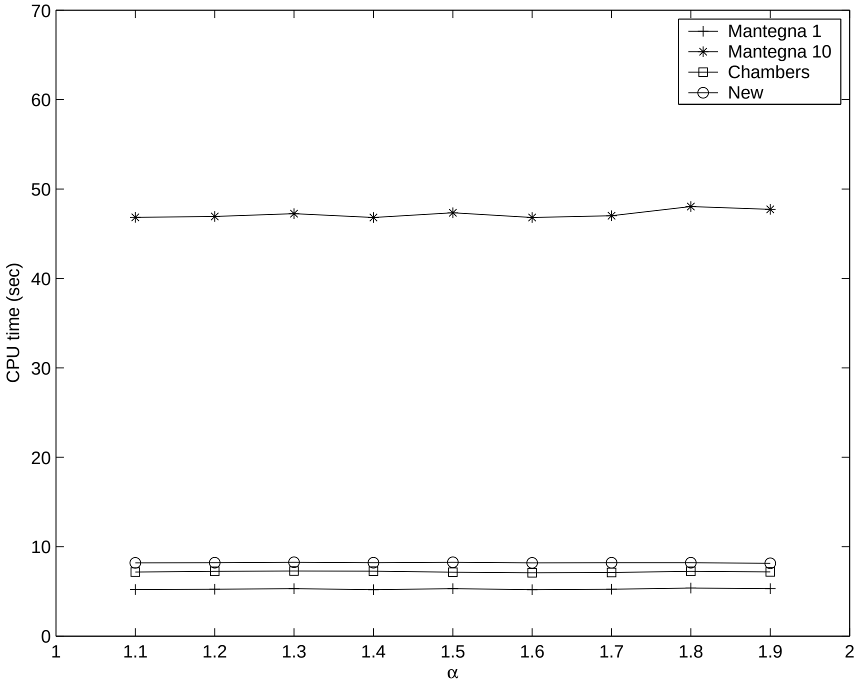

FIGURE 4. CPU time required for 10^6 samples using Mantegna's method ($n = 1$ and $n = 10$), Chambers, and the proposed algorithm.

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COMPARATIVE STUDY BETWEEN LEMKE'S METHOD AND THE INTERIOR POINT METHOD FOR THE MONOTONE LINEAR COMPLEMENTARY PROBLEM

ADNAN YASSINE

Abstract. In this paper, we present two different methods in order to solve a monotone linear complementarity problem (LCP): a simplicial method (Lemke's method) related to the Jordan's pivot and the interior point method based on the central path. We demonstrate that a quadratic convex program (QCP) can be written as a (LCP) form and, thus, be solved by means of one of these two methods. We provide numerical simulations as well as experimental and comparative results regarding these two methods.

1. Introduction

The introduction of the polynomial-time interior point algorithm in linear programs by Karmarkar in 1984, has led many authors to generalise this algorithm in order to solve non-linear optimization problems. Successive works were devoted to solving the (LCP) by means of interior point methods (Kojima, Mizuno and Yoshise [5,6], Gonzaga [4], Bonnans, Gilbert and Lemarechal [2],...). Apparently, these authors were unaware of a long-time existing tool able to solve the problem (LCP): the Lemke's method, which is based on the principle of the simplex method introduced by Dantzig in 1951. This method converges with a finite number of iterations when the problem admits a solution. In the literature, we know that the interior point methods are very fast and more effective than the methods based on the pivot and especially if

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the problem is of big dimension. This result is completely true if we know the starting point x^0, but in the opposite case (that is if we do not know the initial point x^0), the determination of x^0 by the interior point methods represents an inconvenience for these methods and makes them slow with regard to other algorithms. As it is well-known, though the interior point methods are robust and rapid, their major disadvantage is the determination of the initial point. Nevertheless, when a starting point is given, these methods prove to be the best, with a very fast convergence.

In this paper, we show that, in the particular cases of unknown starting points, their determination delays significantly the interior point methods and sometimes turns them slower than other classical approaches, when solving a convex quadratic problem. Our study also rivals that the evaluation of the starting point with Kojima's approach is expensive, and has been found to slower than Lemke's swivelling method, which is a simplicial method as pointed out in the literature (e.g. [1], [7]). We still underline that the interior point method is the best, faster than Lemk's method when the starting point is known.

This paper presents the two methods and well as comparative numerical simulations in order to show the importance (from theoretical and practical points of view) of the Lemke's method stability, efficiency and the longevity regarding interior points algorithms.

In the present paper, we are concerned by solving two important problems of non-linear optimisation:

1. The monotone linear complementarity problem (LCP)

   (LCP) consists in finding two vectors $(x, z) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$z = Mx + q \tag{1}$$

$$x \geqslant 0, z \geqslant 0 \tag{2}$$

$$< x, z >= 0 \tag{3}$$

   where $M \in \mathbb{R}^{n \times n}$ and $< x, z >$ denotes the scalar product of two vectors x and z.

2. The Quadratic Convex Program (QCP)

$$(QCP) : Min \left\{ f(x) = \frac{1}{2} < Cx, x > + < d, x >: Ax \leqslant b \right\}$$

where $C \in \mathbb{R}^{n\times n}$ is symmetric, positive semidefinite matrix, $d \in \mathbb{R}^n, A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

In Section 2, we present Lemke's method, a simplicial method known in the literature for solving (LCP) (Bazaraa, Sherali and Shetty [1], Yassine [7]). We provide the corresponding algorithm and his convergence theorem. The interior point algorithm based on the central path and its convergence properties for solving (LCP) are provided in Section 3. In Section 4, we give the transforming techniques for a quadratic convex program into (LCP) using the optimality conditions of Kuhn Tucker. Section 5 is dedicated to high-dimension numerical simulations and comparisons between numerical predictions of the two methods.

2. Lemke's method

2.1. **Preliminaries.** Let x_i (resp. z_i) be the component number i of vector x (resp. z). The component x_i (resp. z_i) is said basic variable if $x_i \geqslant 0$ (resp. $z_i \geqslant 0$). If x_i (resp z_i) is out of base (non-basic variable), then inevitably $x_i = 0$ (resp. $z_i = 0$).

Definition 2.1. *A solution (x, z) of (LCP) is said feasible-complementarity solution, if it verifies the two following conditions:*

- *(x, z) is a feasible solution of (1) and (2)*
- *one and only one component of (x_i, z_i) is a basic variable for $i = 1, ..., n$.*

We notice that if $q \geqslant 0$, then $(x, z) = (0, q)$ is a solution of (LCP). On the opposite, ($\exists i \in \{1, ..., n\}$ such that $q_i < 0$), we introduce the column vector e the components of which are equal to 1, and an artificial variable z_0 initialized as:

$$z_0 = max\{-q_i : 1 \leqslant i \leqslant n\}.$$

We consider the new system defined by: Find $(x, z, z_0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ such that

$$(P') : \begin{cases} z - Mx - ez_0 = 0 & (4) \\ x \geqslant 0, z \geqslant 0, z_0 \geqslant 0 & (5) \\ < x, z >= 0. & (6) \end{cases}$$

We notice that $x = 0$ and $z = q + ez_0$ is a feasible-complementarity solution of (P').

Definition 2.2. *(x, z, z_0) is said feasible-almost-complementarity of (P') if it verifies the three following conditions:*

- *(x, z, z_0) is a feasible solution of (4) and (5)*
- *$\exists s \in \{1, ..., n\}$ such that x_s and z_s are out of basis ($x_s = z_s = 0$)*
- *z_0 is a basic variable, and $\forall i \neq s, x_i$ or z_i is a basic variable.*

Remark 2.1. *In entering x_s or z_s in the base, we obtain an adjacent feasible-almost-complementarity solution. Then, each feasible-almost-complementarity solution admits two adjacent solutions, one when entering as x_s in the basis and the other when entering as z_s.*

2.2. **Lemke's algorithm (ALGI). Initialisation stage.** If $q \geqslant 0$, we stop: $(x, z) = (0, q)$ is a solution of (LCP). Else, we introduce the artificial variable z_0, we represent the problem (P') through a table and then, we choose

$$q_s = min\{q_i : 1 \leqslant i \leqslant n\}.$$

We update the table by pivoting the line s and the column of z_0, z_s leaves the base and z_0 enters it, then z_0 and z_i (for i=1,...,n and $i \neq s$) are positive (basic variables). Let us put $y_s = x_s$ and go to the main stage.

Main stage: This stage is divided into three phases:

Phase 1: Let d^s the column which corresponds to the variable y_s in the current table. If $d^s = 0$, we stop: (LCP) admits no solution. Else, we determine an index r such that:

$$\frac{q_r^*}{d_r^s} = Min\left\{\frac{q_i^*}{d_i^s} : d_i^s > 0 \ \forall i = 1, ..., n\right\}$$

(the vector q^* designates the second member column).

If the basic variable of the line r is z_0, then go to Phase 3, else go to Phase 2.

Phase 2: The basic variable of the line r is, either x_k, or z_k for some $k \neq s$.

The variable y_s enters the base and the table will define itself through the pivot of the line r and the column of y_s. If the variable, which has left the base, is z_k (resp. x_k), we put $y_s = x_k$ (resp. z_k) and return to Phase 1.

Phase 3: We pivot between the column of y_s and the line of z_0. Then, z_0 leaves the base and y_s enters it. We obtain a solution of (LCP).

2.3. **Convergence of Lemke's method.** Let $M \in \mathbb{R}^{n\times n}$ be a $n \times n$ symmetric matrix and $x \in \mathbb{R}^n$ be a n-dimensional real vector.

Definition 2.3. 1. *M is said copositive if and only if* $\forall x \geqslant 0$, $x^tMx \geqslant 0$

2. *M is said strictly copositive if and only if* $\forall x \geqslant 0$, $x \neq 0 \Longrightarrow x^tMx > 0$
3. *M is said copositive plus if and only if it verifies the two following conditions:*
   (i): *M is copositive*
   (ii): $x \geqslant 0$ *and* $x^tMx = 0 \Longrightarrow (M + M^t)x = 0.$

If M is symmetric, the property (ii) becomes:

$$(ii)\ x \geqslant 0 \text{ and } x^tMx = 0 \Longrightarrow Mx = 0.$$

Theorem 2.1. *([1]) We suppose that each feasible-almost-complementarity solution of* (P')*, is non-degenerated (each basic variable is strictly positive) and that the matrix M is copositive plus, then the algorithm (ALG1) stops after a finite number of iterations. If the system defined by* (1) *and* (2) *is consistent, then the algorithm* $(ALGI)$ *stops with an optimal solution of* (LCP)*, else, we notice that the problem* (LCP) *admits no solution.*

Corollary 2.1. *If the matrix M admits positive elements and if the diagonal elements are strictly positive, then the algorithm stops with a feasible complementary basic solution.*

3. Interior point algorithm

3.1. Introduction.
We consider the monotone linear complementarity problem as a standard given by (1), (2) and (3). The set of all the feasible solutions is defined by:

$$S = \{(x, z) \in \mathbb{R}^{2n} \ : \ z = Mx + q, \, x \geqslant 0, \, z \geqslant 0\}$$

and its relative interior

$$S_{int} = \{(x, z) \in S \ : \ x > 0, z > 0\}.$$

Then, we suppose that the two following hypotheses are satisfied:

(H_1): M is positive semidefinite

(H_2): $S_{int} \neq \emptyset$.

The size of problem (LCP) is defined by ([5]):

$$L = E\left(\sum_{i=1}^{n}\sum_{j=1}^{n+1} log(|\mathbb{M}_{ij}| + 1) + log(n^2)\right) + 1$$

where $\mathbb{M} = [M \quad q]$ and $E(u)$ is the largest integer, not greater than $u \in \mathbb{R}_+$.

Let $\quad H : \mathbb{R}_+ \times \mathbb{R}_+^{2n} \rightarrow \mathbb{R}^n \times \mathbb{R}^n$

$$(\mu, x, z) \quad \rightarrow \quad H(\mu, x, z) = (xz - \mu e, z - Mx - q)$$

for every $\mu > 0$ and $(x, z) \in \mathbb{R}_+^{2n}$, we consider the following system of equations:

$$H(\mu, x, z) = 0. \tag{7}$$

It is obvious that (x, z) is a solution of (LCP) if and only if it is a solution of the system (7) for $\mu = 0$. The Newton direction at (x, z) is defined as a solution (d_x, d_z) of the system of linear equations:

$$\begin{cases} Zd_x + Xd_z = -xz + \mu e \\ \qquad\qquad d_z = Md_x \end{cases} \tag{8}$$

where $X = diag(x_1, x_2, ..., x_n)$ and $Z = diag(z_1, z_2, ..., z_n)$.

By a simple calculation, we obtain:

$$\begin{cases} (M + X^{-1}Z)d_x = Ze + \mu X^{-1}e \\ d_z = Md_x. \end{cases} \tag{9}$$

Then the new point $(\overline{x}, \overline{z})$ will be given by:

$$(\overline{x}, \overline{z}) = (x + d_x, z + d_z). \tag{10}$$

We can easily verify that:

$$\overline{z} = M\overline{x} + q \text{ for any } (x, z) \in S \text{ and any } \mu > 0. \tag{11}$$

3.2. **Centralisation measures.** Note that S_{cen} is the central trajectory of (LCP):

$$\begin{aligned} S_{cen} &= \{(x, z) \in \mathbb{R}^{2n}_{+} : H(\mu, x, z) = 0 \; for \; \mu > 0\} \\ &= \{(x, z) \in S_{int} : xz = \mu e \; for \; \mu > 0\}. \end{aligned}$$

Proposition 3.1. *([5]) If $S_{int} \neq \emptyset$, the system* (7) *admits a unique solution called associated center to μ, for every $\mu > 0$.*

The algorithms of central trajectory generate a sequence of points (x^{μ}, z^{μ}) verifying the following system:

$$\begin{cases} z = Mx + q \\ xz = \mu e \\ x > 0 \text{ and } z > 0. \end{cases} \tag{12}$$

In tending μ to 0, (x^{μ}, z^{μ}) tends to a solution (x^*, z^*) of (LCP) which is located at the extremity of the central trajectory.

To control the non-linearity of xz, successive points are imposed to stay in the central trajectory neighbourhood. To evaluate deviation $(x, z) \in S_{int}$ of each point for the central trajectory, we define a centralisation measure:

$$\begin{aligned} \delta(x, z) &= Min\{\|H(\mu, x, z)\| : \mu \geq 0\} = Min\{\|xz - \mu e\| : \mu \geq 0\} \\ &= Min\left\{xz - (\frac{x^t z}{n})e : \mu \geq 0\right\}. \end{aligned}$$

For every point $(x,z) \in S_{int}$, we get: $\quad (x,z) \in S_{int} \Longleftrightarrow \delta(x,z) = 0$.

Definition 3.1. *Let $\alpha > 0$. We call α-center neighbourhood, the set*

$$S_\alpha = \{(x,z) \in S_{int} : \delta(x,z) \leq \frac{x^t z}{n}\alpha\}.$$

Theorem 3.1. *([5]) Let $0 < \alpha < 0.1$ and $\delta = \frac{\alpha}{1-\alpha}$. We assume that $(x,z) \in S_\alpha$ and $\mu = (1 - \frac{\delta}{\sqrt{(n)}})\frac{x^t z}{n}$ then $(\overline{x}, \overline{z})$, given by (10), verifies:*

$$(\overline{x}, \overline{z}) \in S_\alpha \tag{13}$$

$$(\overline{x}^t \overline{z}) \leq \left(1 - \frac{\delta}{6\sqrt{(n)}}\right) x^t z. \tag{14}$$

3.3. **Interior point algorithm (ALG2).**

Initialisation: (see Appendix 1): Let $0 < \alpha < 0.1$ and $\delta = \frac{\alpha}{1-\alpha}$.

We suppose that the initial point $(x^1, z^1) \in S_{int}$ are known, such that

$$\delta(x^1, z^1) \leq \frac{(x^1)^t z^1}{n}\alpha \textit{ and } (x^1)^t z^1 \leq 2^{0(L)},\ k = 1.$$

Stage 1: If $(x^k)^t z^k \leq 2^{-2L}$, we stop: $(x^*, z^*) = (x^k, z^k)$ is a solution of (LCP)

Stage 2: $\mu = (1 - \frac{\delta}{\sqrt{(n)}})\frac{(x^k)^t z^k}{n}$ and $(x,z) = (x^k, z^k)$.

Stage 3: We determine the Newton's direction (d_x, d_z) defined by (9) and $(x^{k+1}, z^{k+1}) = (\overline{x}, \overline{z})$ defined by (10).$k \longleftarrow k+1$ and return to Stage 1.

Theorem 3.2. *([5,6]) The algorithm $(ALG2)$ generates a sequence (x^k, z^k) verifying: $(x^k, z^k) \in S_\alpha$ and $(x^{k+1}, z^{k+1}) \leq (1 - \frac{\delta}{6\sqrt{(n)}})(x^k)^t z^k$ for $k = 1, ...,$ The sequence (x^k, z^k) converges to (x^*, z^*) solution of (LCP) after, at the most, $O(n^{0.5}L)$ iterations.*

4. **Transformation of a convex quadratic program into a monotone linear complementarity problem**

In this paragraph, we firstly present the transformation of convex quadratic program (based on the optimality conditions of Kuhn Tucker) into a complementarity linear problem. We distinguish the two following cases, may there be or not positivity constraints of the variable x components. We also provide the conditions ensuring the convergence of Lemke's algorithm for the solving of these monotone linear complementarity problems.

4.1. **Transformation with positivity constraints.** Let us consider the following problem:

$$(QCP_1)\ :\ Min\{f(x) = \frac{1}{2} < x, Cx > + < d, x > \quad : \quad Ax \leq b,\, x \geq 0\}$$

where $C \in \mathbb{R}^{n\times n}$ is symmetric, positive definite; $A \in \mathbb{R}^{m\times n}$; $x, d \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

The conditions of Kuhn Tucker's related to the problem (QCP_1) are written as follows:

x is a solution of (QCP_1) if and only if there exists $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ such that

$$(*) : \begin{cases} Cx + d + A^t u - v = 0 \\ < u, b - Ax >=< v, x >= 0 \\ Ax \leq b, x \geq 0, u \geq 0, v \geq 0. \end{cases}$$

Let

$$q = \begin{bmatrix} b \\ d \end{bmatrix} \in \mathbb{R}^{n+m},\ z = \begin{bmatrix} u \\ x \end{bmatrix} \in \mathbb{R}^{n+m} \text{ and } M = \begin{bmatrix} 0 & -A \\ A^t & C \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}.$$

It is readily verified that the quadratic program solving (QCP_1) is equivalent for solving the following linear complementarity problem:

$$(LCP_1)\ :\ Find\, z \in \mathbb{R}^{(n+m)}\ such\ that\ :\ z \geq 0, Mz + q \geq 0\ and\ z^t(Mz + q) = 0.$$

Theorem 4.1. *If C is symmetric, positive definite, then M is copositive plus and Lemke's algorithm converges to a solution of (LCP).*

Proof. Let $z = \begin{bmatrix} x \\ y \end{bmatrix} \geq 0$ where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, then $z^t = [x^t \quad y^t] \geq 0$.

$$z^t M z = [x^t \quad y^t]. \begin{bmatrix} 0 & -A \\ A^t & C \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix} = y^t C y.$$

$z \geq 0 \Longrightarrow y \geq 0 \Longrightarrow y^t.C.y \geq 0$ (by assumption, C is symmetric, positive definite) $\Longrightarrow z^t.M.z \geq 0 \Longrightarrow M$ is copositive.

We have, moreover, $M + M^t = \begin{bmatrix} 0 & 0 \\ 0 & 2C \end{bmatrix} \Longrightarrow (M + M^t)z = 2Cy$, then, $z^t.M.z = 0 \Longrightarrow y^t.C.y = 0 \Longrightarrow C.y = 0 \Longrightarrow (M + M^t).z = 0 \Longrightarrow M$ is copositive plus.

According to Theorem 2.1, $ALG1$ converges to a solution of (LCP). □

4.2. **No-constraint transformation of positivity.** Considering the following problem:

$$(QCP_2) \ : \ Min\{f(x) = \frac{1}{2} < x, Cx > + < d, x > : \ Ax \leq b$$

where $C \in \mathbb{R}^{n\times n}$ symmetric, positive definite; $A \in \mathbb{R}^{m\times n}$; $x, d \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

Kuhn Tucker's optimality conditions related to the problem (QCP_2) are written:

x is a solution of (QCP_2) if and only if there exists $u \in \mathbb{R}^m$ such that

$$(**) : \begin{cases} Cx + d + A^t u = 0 \\ < u, b - Ax >= 0 \\ Ax \leq b, u \geq 0 \end{cases}$$

which are equivalent in solving the following LCP:

$$(LCP_2) \ : \ Find\, u \in \mathbb{R}^m \; such\, that \ : u \geq 0, Mu + q \geq 0 \; and \; < u, Mu + q >= 0$$

where $M = AC^{-1}A^t \quad and \quad q = AC^{-1}d + b$.

Remark 4.1.
- *u^* is a solution of the problem LCP_2 if and only if $x^* = (-C^{-1}A^t u^* - C^{-1}d)$ is a solution of the problem (QCP_2).*
- *Given that $M = M^t = AC^{-1}A^t$ is positive definite, Lemke's algorithm leads to a solution of (LCP_2) (then of (QCP_2)) or concludes on the vacuity of the solution set of (LCP_2) (consequently that of (QCP_2)).*
- *In case 4.2., the transformation requires that C is positive definite. Moreover, applying Lemke's algorithm needs to calculate C^{-1}. These are the drawbacks when solving (QCP_2) through Lemke's algorithm.*

- *In case* 4.1.*, such drawbacks do not exist. On the opposite, we have to work in* $\mathbb{R}^{n+m}$ *(instead of* $\mathbb{R}^m$ *in* 4.2.*). Lemke's algorithm would be more expensive when* n *is quite big.*

5. **Numerical experiments**

In this section, we present the comparative numerical results between the two methods for the (LCP) problem. The numerical simulations are applied to quadratic problems. In our numerical applications, the matrix C is always definite positive (chosen in a random way) to ensure the convergence of Lemke's method. In the Table I, the first column represents the problem dimension ($M \in \mathbb{R}^{n\times n}$ *and* $q \in \mathbb{R}^n$), the second provides (resp. the third) the CPU calculation time in seconds for the Lemke's method to be carried out (resp. interior point algorithm).

N	Lemke	Interior Point
10	5	23
20	8	34
50	14	56
100	32	148
200	74	289
400	126	518
500	159	665
1000	334	875

Table I. Calculation time of two methods (in seconds).

According to the numerical results, the following remarks can be make:

- Lemke's method efficiency, stability and robustness compared with the interior point method, should be underlined.
- The interior point method becomes low for too small values of α (see Section 3.3.) (lower than 0.01) or too big (upper than 0.08).
- Interior point method difficulty lies in the determination of the initial point x^0 (initial stage).

- In some cases, and for a fixed value of α, the interior point method diverges leading to change the α value in order to obtain convergence on an optimal solution. This numerical instability does not exist in Lemke's method.

N	Interior Point method by knowing the initial point x^0
10	1
20	2
50	4
100	9
200	16
400	31
500	42
1000	78

Table II. Calculation time (in second) of the Interior Point Method by Knowing x^0.

The result of the table ($Table\,II.$) show clearly that if we know x^0 then the interior point method is much faster and more effective than the Lemkes' method.

Appendix 1

Initialisation. Let

$$M' = \begin{bmatrix} 0 & -e^t \\ e & M \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)} \text{ and } q' = [q_0 \quad q] \in \mathbb{R}^{(n+1)}$$

where

$$q_0 = \frac{2^{L*}.(n+1)}{n^2}, \quad L* = \sum_{i=1}^{n}\sum_{j=1}^{n+1} log(|\mathbb{M}_{ij}|+1) + log(n^2)) \text{ and } \mathbb{M} = [M \quad q].$$

We consider the following LCP:

$$(LCP'): \begin{cases} z' = M'x' + q' \\ x'z' = 0 \\ (x', z') \geq 0 \end{cases}$$

where $(x', z') = (x_0, x, z_0, z) \in \mathbb{R}_+^{2(n+1)}$.

Assumptions (H_1) and (H_2), its size is:

$$L' = E(\sum_{i=1}^{n+1}\sum_{j=1}^{n+2} log(|\widetilde{M}_{ij}|+1) + log((n+1)^2)) + 1$$

where $\widetilde{M} = [M' \quad q']$. Putting

$$x_0^1 = 2^{2L*}, x^1 = (\frac{2L*}{n^2}).e, z_0^1 = x_0^1.e + Mx^1 + q = 2^{2L*}.e + (\frac{2L*}{n^2}).Me + q$$

$$x_1' = (x_0^1, x^1) \; and \; z_1' = (z_0^1, z^1)$$

We denote by $S', S'_{int}, S'_{cen}, S'_{\alpha}$, the solutions set of (LCP'), its relative interior, its central trajectory and its α-center neighbourhood, respectively.

Lemme 5.1. *([5])*

1.

$$0 < (\frac{15}{16}.2^{2L*}.e) \leq 2^{2L*}.(1-\frac{1}{n^4}).e \leq z^1 \leq 2^{2L*}.(1+\frac{1}{n^4})e \leq (\frac{17}{16}.2^{2L*}).e$$

2.

$$(x'^1, z'^1) \in S'_{int}.$$

Lemme 5.2. *([5])*

1.

$$(x'^1)^t z'^1 \leq 2^{2L} \leq 2^{2L'}$$

2.

$$(x'^1, z'^1) \in S'_{0.1}.$$

Theorem 5.1. *([5]) Suppose that the (LCP) has a solution. Then $\overline{x}_0 = 0$ for any solution $(\overline{x}_0, \overline{x}, \overline{z}_0, \overline{z})$ of the (LCP').*

According to Theorem 5.1, (x'^1, z'^1) can be useful as an initial point to the algorithm $(ALG2)$. We calculate the solution $(\overline{x}_0, \overline{x}, \overline{z}_0, \overline{z})$ of (LCP') such that $\overline{x}^t\overline{z} < 2^{-2L'}$.

If $\overline{x}_0 = 0$ then $(\overline{x}, \overline{z})$ is a solution of (LCP), else the above-mentioned theorem ensures that (LCP) admits no solution.

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BOOK REVIEWS

C. E. Silva, *Invitation to ergodic theory*, Student Mathematical Library, Vol. 42, American Mathematical Society, Providence, Rhode Island 2008, ix+262 pp, ISBN: 978-0-8218-4420-5

The ergodic theory, started in 1931 by John von Neumann and G. D. Birkhoff, has its origins in the statistical physics of Boltzmann. The present book is intended to be an introduction to ergodic theory and covers topics as recurrence, ergodicity, the ergodic theorems and mixing. In order to make the book as self-contained as possible, measure theory is developed as needed in Chapters 2, *Lebesgue measure*, and 4, *The Lebesgue integral*, including an introduction to measure spaces, Carathéodori extension theorem, Lebesgue dominated convergence theorem and the Lebesgue spaces L^p.

The study of ergodic theory begins in Chapter 3, *Recurrence and ergodicity,* with the classical example of baker's transformation, doubling maps, measure-preserving transformations, ergodic transformations.

Chapter 5, *The ergodic theorem,* is devoted to the proof of Birkhoff's ergodic theorem (in fact, two proofs of this important result are given) and of the maximal ergodic theorem in L^1 and in L^p.

Chapter 6, *Mixing notions,* is concerned with the important notion of mixing, meaning that $\lim_{n\to\infty}\mu(T^{-n}(A)\cap B)=\mu(A)\mu(B)$, for all $A,B\in\mathcal{S}$, where $(X,\mathcal{S},\mu)$ is a probability measure space and $T:X\to X$ is a measure-preserving transformation. The notion of weak-mixing, meaning that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}|\mu(T^{-i}(A)\cap B)-\mu(A)\mu(B)|=0,$$

is also studied.

A word must be said about the numerous examples from physics, biology, and mathematics included in the book. I do mention the nice treatment of Weyl's result on the equidistribution of numbers, with references to some results of Furstenberg on the Szemeredi theorem and the recent solution by Green and Tao of the 300-year old problem on the existence of arithmetic progressions of arbitrary length in the primes.

There are also problems and exercises completing the main text and some open questions, suggesting possible topics for further research by the reader, are included.

The book is well written and can be used for an introductory course in measure theory or in ergodic theory, or for self-study.

S. Cobzaş

Massimiliano Berti, *Nonlinear oscillations of Hamiltonian PDEs*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 74, Series Editor: Haim Brezis, Birkhäuser Verlag, Basel-Boston-Berlin, 2007, x+235 pp; ISBN- 13: 978-0-8176-4680-6, e-ISBN-13: 978-0-8176-4681-3.

In the study of complex dynamical systems, the simplest invariant manifolds are the equilibria and, next, the periodic orbits. The relevance of periodic solutions for understanding the dynamics of a finite-dimensional Hamiltonian system was highlighted at the end of the 19th century by H. Poincaré in his famous treatise on Celestial Mechanics. In spite of the fact that the set of periodic orbits has measure zero, their study is important due to the possibility, conjectured by Poincaré, of approximating arbitrarily well in long time any solution of a Hamiltonian equation by periodic solutions (with very long periods). This conjecture motivated the systematic study of periodic orbits, initiated by Poincaré and continued by the work of Lyapunov, Birkhoff, Moser, Weinstein, and others. Great progress in the understanding the complex orbit structure of Hamiltonian systems was made by Kolmogorov (1954), Arnold (1963) and Moser (1962), leading to the so called KAM theory and to small divisor theory as well.

The present book is concerned with bifurcation results of nonlinear oscillations of Hamiltonian PDEs of the form

$$u_{tt} - u_{xx} + a_1(x)u = a_2(x)u^2 + a_3(x)u^3 + \dots \tag{1}$$

Previous results about this equations referred to the "nonresonant" PDEs, that is equation (1) with non identically null term $a_1(x)$. The term $a_1(x)$ allows to verify suitable nonresonance conditions on the linear eigenfrequencies of the small oscillations, and, further, the bifurcation equation is finite-dimensional.

The main concern of the present book is to present recent bifurcation results for the completely nonresonant wave equation (1) with $a_1(x) \equiv 0$. In this case infinite-dimensional bifurcation phenomena appear jointly with small-divisor difficulties.

A good idea on the content is given by the headings of the chapters: **1.** Finite dimension; **2.** Infinite dimension; **3.** A tutorial in Nash-Moser theory; **4.** Application to the nonlinear wave equation; **5.** Forced vibrations. There are also four appendices: A. Hamiltonian PDEs; B. Critical point theory; C. Free vibrations of nonlinear wave equations: A global result; D. Approximation of irrationals by rationals; E. The Banach algebra property of $X_{\sigma,s}$.

The book is a good introduction to this fascinating and rapidly growing field of investigation, closely related to fundamental problems in mechanics and physics. It can be warmly recommended to graduate students and researchers desiring to work in nonlinear Hamiltonian PDEs or in related domains (variational techniques, critical point theory, small divisors).

Radu Precup

Patrizia Pucci and James Serrin, *The maximum principle*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 73, Series Editor: Haim Brezis, Birkhäuser Verlag, Basel-Boston-Berlin, 2007, x+235 pp; ISBN: 978-3-7643-8144-8, e-ISBN: 978-3-7643-8145-5.

The maximum principle gives information about solutions of differential equations and inequalities without their explicit knowledge, being valuable tools not only for mathematicians but also for physicists, chemists, engineers, economists.

The maximum principle for elliptic partial differential equations has its origins in the maximum principle for harmonic functions proved by Gauss in 1839 on the basis of the mean value theorem. Extensions to elliptic equations and inequalities were done only at the beginning of the 20th century by Bernstein (1904), Picard (1905) and Lichtenstein (1912, 1924), with difficult proofs involving hard analysis tools as well as regularity conditions for the coefficients in the highest order term. It was Eberhard Hopf in 1927 who realized that the maximum principle can be obtained on an elementary basis. The comparison technique he invented for this purpose generated important applications in many directions. The remarkable simple proof given by Hopf to the maximum principle is given as an Appendix to Chapter 2 of the book.

The aim of the present monograph is to give a clear and thorough presentation of various maximum principles for second-order elliptic equations from their beginning in linear theory to recent work on nonlinear equations.

The maximum principles are exposed in 6 chapters of the book: **2.** *Tangency and comparison theorems for elliptic inequalities*; **3.** *Maximum principles for divergence structure elliptic differential inequalities*; **4.** *Boundary value problems for nonlinear ordinary differential equations*; **5.** *The Strong Maximum Principle and the Compact Support Principle*; **6.** *Non-homogeneous divergence structure inequalities*; **7.** *The Harnack inequality.* The first chapter, *Introduction and preliminaries*, beside some preliminary material and notation, contains the enounce of the Strong Maximum Principle and of the Compact Support Principle whose proofs are postponed to Chapter 5.

The book is clearly written, with proofs given in detail, which, although difficult, by the direct approach proposed by the authors are available to students with a basic knowledge in real analysis (including Sobolev spaces), but avoiding more advanced topics as linear operator theory, monotone operator theory, Orlicz-Sobolev spaces, or viscosity solutions, used in other treatments of the subject.

The book can be used as a good introduction to recent results in this important area of research, with the possibility for the reader to attack open problems waiting for solution.

Radu Precup