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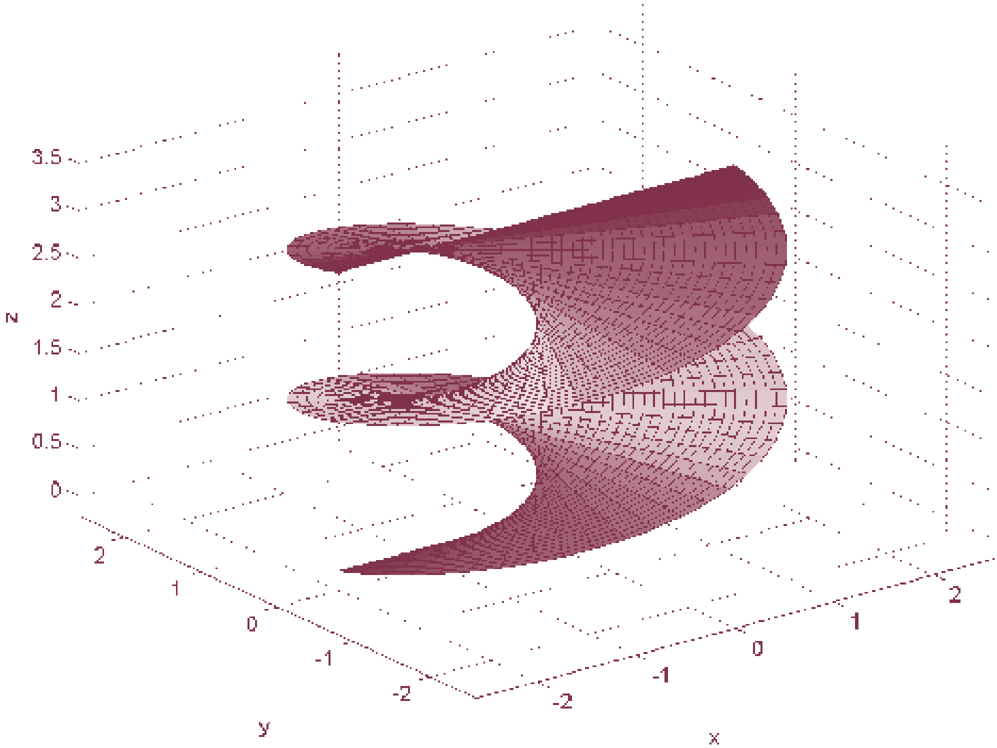

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CONTENTS

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Existence of solutions for fractional boundary value problems with Riesz space derivative and nonlocal conditions

Şuayip Toprakseven

Abstract. By using the fixed point theorems, we give sufficient conditions for the existence and uniqueness of solutions for the nonlocal fractional boundary value problem of nonlinear Riesz-Caputo differential equation. The boundedness assumption on the nonlinear term is replaced by growth conditions or by a continuous function. Finally, some examples are presented to illustrate the applications of the obtained results.

Mathematics Subject Classification (2010): 26A33, 26D10, 34A60.

Keywords: Fractional boundary value problem, Riesz-Caputo fractional derivative, existence and uniqueness, fixed point, nonlocal conditions.

1. Introduction

Fractional differential equations can be thought as an extension of the ordinary differential equation of real order. Fractional calculus is as old as differential calculus which goes back to Leibniz and Newton. In recent years, there has been an active movement in fractional differential equations which have been used for modelling real world phenomena in different fields. The reason is that they represent better these phenomena than ordinary differential equations. Geometric and physical interpretation of fractional differentiation and integration can be found in the paper [23]. Very recently, the existence of the solutions for fractional differential equations have attracted a good deal of attention and have been developed by many authors; see the books [17, 22, 19] and papers [1, 2, 3, 12, 11, 16, 33, 25, 26, 27, 30, 28, 29] and the references therein. A large number of studies on fractional differential equations has

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been presented for the existence and uniqueness of initial value problems. Multi-point and nonlocal boundary value problems for fractional differential equation are sparse and have received attention in the last decades [3, 20, 14, 18].

It should be pointed out that the most of the papers and monographs on fractional calculus have focused on the fractional differential equations involving Riemann-Lioville and Caputo derivatives in the literature. Both two fractional operators are one sided operator and thus, they hold either past or future memory effects. In contrast, the main feature of the Riesz fractional operator is that it is both left and right sided operator which holds both the history and future non-local memory effects. This property of the Riesz fractional operator is important in the mathematical modelling for physical processes on a finite domain because the present states depend both on the past and future memory effects. As an example, the Riesz fractional derivative has been used for the memory effects in both past and future concentrations in the anomalous diffusion problem [7, 24].

A variety of papers are devoted to numerical solutions of the fractional calculus, specifically in the anomalous diffusion that involves the Riesz derivative [13, 32, 24, 24, 7, 4, 21, 31]. Recently, there are papers on existence and positive solutions for the fractional boundary value problems of the Riesz-Caputo derivative [8, 15].

To the best of our knowledge, there does not exist a paper on the fractional boundary value problems (FBVP) of the Riesz-Caputo differential equations with nonlocal boundary conditions. In this paper, we investigate the existence and uniqueness of solutions for the following nonlocal boundary value problems of the Riesz-Caputo fractional differential equations

$$\begin{aligned} {}^{RC}_{\;0}D^{\nu}_{T}u(\eta) &= F(\eta, u(\eta)) \quad \nu \in (1,2], \quad 0 \le \eta \le T, \\ u(0) &= g(u), \quad u(T) = u_T, \end{aligned} \tag{1.1}$$

where ${}^{RC}_{\;0}D^{\nu}_{T}$ is the Riesz-Caputo derivative defined below and $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $g : C[0,T] \to \mathbb{R}$ is a continuous function and $u_T \in \mathbb{R}$.

Byszewski [6] first time investigated the existence and uniqueness of a solution of nonlocal Cauchy problems. It should be noted that some psychical processes can be better described by the nonlocal boundary conditions than the usual initial/boundary conditions [5]. For instance, the initial condition $g(u)$ can be taken as

$$g(u) = \sum_{k=1}^{n} a_k u(t_k)$$

where $a_k, k = 1, 2, \ldots, n$ constant and $0 < t_1 < t_2 < \cdots < t_n \le T$.

The remainder of paper is organized as follows. Section 2 introduces some preliminaries, definitions and lemmas which are useful in proving main results. Section 3 provides some sufficient conditions for the existence and the uniqueness of solutions of the problem (1.1) with nonlocal boundary conditions. We establish these results by using the contraction principle in the Banach space and Schaefer's fixed point theorem and Leray-Schauder fixed point theorem, respectively. Finally, some numerical examples are given to illustrate the applications of the main results.

2. Preliminaries

In this section, we give some useful definitions and lemmas that will be used in this paper.

Definition 2.1. [17] Let $\nu > 0$. The left-sided and right-sided Riemann-Liouville fractional integral of a function $f \in C[a,b]$ of order ν defined as, respectively

$$I_a^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-s)^{\nu-1} f(s)\, ds, \quad x \in [a,b].$$

$${}_bI^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_x^b (s-x)^{\nu-1} f(s)\, ds, \quad x \in [a,b].$$

Definition 2.2. (Riesz Fractional Integral) Let $\nu > 0$. The Riesz fractional integral of a function $f \in C[a,b]$ of order ν defined as

$${}_bI_a^\nu f(x) = \frac{1}{2\Gamma(\nu)} \int_a^b |x-s|^{\nu-1} f(s)\, ds, \quad x \in [a,b].$$

Note that the Riesz fractional integral operator can be written as

$${}_bI_a^\nu f(x) = \frac{1}{2}\Big(I_a^\nu f(x) +_b I^\nu f(x) \Big) \tag{2.1}$$

Definition 2.3. [17] Let $\nu \in (n, n+1], n \in \mathbb{N}$. The left-sided and right-sided Caputo fractional derivative of a function $f \in C^{n+1}[a,b]$ of order ν defined as, respectively

$${}_a^C D_x^\nu f(x) = \frac{1}{\Gamma(n+1-\nu)} \int_a^x (x-s)^{n-\nu} f^{(n+1)}\, ds = (I_a^{n+1-\nu} D^{n+1}) u(x).$$

$${}_x^C D_b^\nu f(x) = \frac{(-1)^{n+1}}{\Gamma(n+1-\nu)} \int_x^b (s-x)^{n-\nu} f^{(n+1)}\, ds = (-1)^{n+1} ({}_bI^{n+1-\nu} D^{n+1}) u(x).$$

where D is ordinary differential operator.

Definition 2.4. Let $\nu \in (n, n+1], n \in \mathbb{N}$. The Riesz-Caputo fractional derivative ${}_a^{RC}D^\nu$ of order ν of a function $f \in C^{n+1}[a,b]$ defined by

$$\begin{aligned}
{}_a^{RC}D_b^\nu f(x) &= \frac{1}{\Gamma(n+1-\nu)} \int_a^b |x-s|^{n-\nu} f^{(n+1)}(s)\, ds \\
&= \frac{1}{2}\Big({}_a^C D_x^\nu f(x) + (-1)^{n+1} {}_x^C D_b^\nu f(x) \Big) \\
&= \frac{1}{2}\Big((I_a^{n+1-\nu} D^{n+1}) u(x) + (-1)^{n+1} ({}_bI^{n+1-\nu} D^{n+1}) u(x) \Big).
\end{aligned}$$

In the case when $\nu \in (1,2]$ and $f(x) \in C^2(a,b)$ we then have

$${}_a^{RC}D_b^\nu f(x) = \frac{1}{2}\Big({}_a^C D_x^\nu f(x) + {}_x^C D_b^\nu f(x) \Big). \tag{2.2}$$

Lemma 2.5. [17] *Let $f \in C^n[a,b]$ and $\nu \in (n, n+1]$. Then we have the following relations*

$$I^{\nu}_{a}\,{}^{C}_{a}D^{\nu}_{x} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

$${}_{b}I^{\nu}_{x}\,{}^{C}_{x}D^{\nu}_{b} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{k!}(b-x)^k.$$

Lemma 2.6. [8] *Assume that $u \in C[0,T]$ satisfies*

$$|u(t)| \leq c_1 + c_2 \int_0^T |t-s|^{\nu-1}|u(s)|^{\beta}\, ds + c_3 \int_0^T (T-s)^{\nu-2}|u(s)|^{\beta}\, ds,$$

where $\nu \in (1,2], \beta \in (0,\sigma)$ for some $0 < \sigma < \nu - 1$ and $c_i, (i = 1,2,3)$ are positive constants. Then there is a positive constant C such that

$$|u(t)| \leq C.$$

Lemma 2.7. [10] *Let X be a Banach space and B be a closed and convex subset of X. If C is a open subset of B and $T : C \to C$ is a continuous and compact operator, then one of the following hols:*

1. *The operator has a fixed point in C,*
2. *There is a point $c \in \partial C$ with $0 < \mu < 1$ such that $c = \mu T(c)$.*

3. Existence results

Let $E = C[0,T]$ denote the Banach space with the norm defined as $\|u\| = \sup\{|u(t)| : t \in J = [0,T]\}$.

We say that $u \in C^2(J)$ with ${}^{RC}_{0}D^{\nu}_{T}u$ exists on J is a solution of the problem (1.1) if u solves the equation ${}^{RC}_{0}D^{\nu}_{T}u(t) = F(t, u(t))$ for each $t \in J$ and the conditions $u(0) = g(u)$ and $u(T) = u_T$ are fulfilled.

In order to prove the existence results for the problem (1.1), the following lemmas are useful.

Lemma 3.1. [8] *Assume that $h \in C[0,T]$ and $\nu \in (1,2]$. Then the following boundary value problem of Riesz-Caputo fractional differential equation*

$$\begin{cases} {}^{RC}_{0}D^{\nu}_{T}u(t) = h(t), & 0 \leq t \leq T, \\ u(0) = g(u), \quad u(T) = u_T, \end{cases}$$

has a unique solution $u(x)$ given by

$$\begin{aligned} u(t) = &\frac{1}{\Gamma(\nu)} \int_0^T |t-s|^{\nu-1} h(s)\, ds + \frac{T-2t}{T\Gamma(\nu)} \int_0^T (T-s)^{\nu-1} h(s)\, ds \\ &- \frac{T-t}{\Gamma(\nu-1)} \int_0^T (T-s)^{\nu-2} h(s)\, ds + (1 - \frac{t}{T})g(u) + \frac{t}{T}u_T. \end{aligned} \tag{3.1}$$

Proof. From the above definitions and Lemma 2.5 we have for $\nu \in (1, 2]$

$$ {}_bI_a^{\nu}\, {}^{RC}_{\ 0}D_T^{\nu} u(t) = u(t) - \frac{1}{2}(u(0) + u(T)) - \frac{t}{2}(u'(0) + u'(T)) + \frac{T}{2}u'(T). $$

This implies that

$$ u(t) = \frac{1}{2}(u(0)+u(T)) + \frac{t}{2}(u'(0)+u'(T)) - \frac{T}{2}u'(T) + \frac{1}{\Gamma(\nu)}\int_0^T |t-s|^{\nu-1}h(s)\,ds. \quad (3.2) $$

We compute the first derivative of u

$$ \begin{aligned} u'(t) = {} & \frac{1}{2}(u'(0) + u'(T)) + \frac{1}{\Gamma(\nu-1)}\int_0^t (t-s)^{\nu-2}h(s)\,ds \qquad (3.3) \\ & - \frac{1}{\Gamma(\nu-1)}\int_t^T (s-t)^{\nu-2}h(s)\,ds. \end{aligned} $$

The equation (3.2) can be rewritten as follows

$$ \begin{aligned} u(t) = {} & u(0) + \frac{T}{2}(u'(0) - u'(T)) + \frac{t}{2}(u'(0) + u'(T)) \qquad (3.4) \\ & + \frac{1}{\Gamma(\nu)}\int_0^T [(T-s)^{\nu-1} + |t-s|^{\nu-1}]h(s)\,ds. \end{aligned} $$

From the equation (3.3), we have

$$ u'(T) = u'(0) + \frac{2}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}h(s)\,ds. \quad (3.5) $$

We plug the equation above (3.5) into the equation (3.4) to obtain

$$ \begin{aligned} u(t) = {} & u(0) + tu'(0) - \frac{T-t}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}h(s)\,ds \qquad (3.6) \\ & + \frac{1}{\Gamma(\nu)}\int_0^T [(T-s)^{\nu-1} + |t-s|^{\nu-1}]h(s)\,ds. \end{aligned} $$

Applying the boundary conditions to the equation (3.6) yields the desired result (3.1). □

By making use of Lemma 3.1, we consider the operator $K : C[0,1] \to C[0,1]$ defined as

$$ \begin{aligned} K(u)(t) = {} & \frac{1}{\Gamma(\nu)}\int_0^T |t-s|^{\nu-1}F(s,u(s))\,ds + \frac{T-2t}{T\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}F(s,u(s))\,ds \\ & - \frac{T-t}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}F(s,u(s))\,ds + (1-\frac{t}{T})g(u) + \frac{t}{T}u_T. \qquad (3.7) \end{aligned} $$

We now state and prove the first existence result by using Banach contraction principle.

Theorem 3.2. *Assume that the following assumptions hold*

A1. *The function F is Lipschitz continuous in the second variable, that is, there is a positive constant C_1 such that*

$$|F(t,z)-F(t,y)|\le C_1|z-y|,\quad \textit{for each}\quad t\in J,\quad \textit{and}\quad \forall z,y\in\mathbb{R}.$$

A2. *The function g is Lipschitz continuous, that is, there is a positive constant C_2 such that*

$$|g(z)-g(y)|\le C_2|z-y|,\quad \textit{for each}\quad t\in J,\quad \textit{and}\quad \forall z,y\in C(J).$$

Assume also that

$$\frac{(3+\nu)T^\nu}{\Gamma(\nu+1)}C_1+C_2<1. \tag{3.8}$$

Then the problem (1.1) has a unique solution.

Proof. Obviously, the solutions of the problems (1.1) are the fixed point of the operator K. We will show that the operator K is a contraction. To this end, let $u,v\in C(J)$. Then for $t\in J$ we get

$$\begin{aligned}
|K(u)(t)-K(v)(t)| &\le \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}|F(s,u(s))-F(s,v(s))|\,ds\\
&+\frac{1}{\Gamma(\nu)}\int_t^T (s-t)^{\nu-1}|F(s,u(s))-F(s,v(s))|\,ds\\
&+\frac{1}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}|F(s,u(s))-F(s,v(s))|\,ds\\
&+\frac{T}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}|F(s,u(s))-F(s,v(s))|\,ds+|g(u)-g(v)|\\
&\le\frac{C_1\|u-v\|}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}\,ds+\frac{C_1\|u-v\|}{\Gamma(\nu)}\int_t^T (s-t)^{\nu-1}\,ds\\
&+\frac{C_1\|u-v\|}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}\,ds+\frac{TC_1\|u-v\|}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}\,ds+C_2\|u-v\|\\
&\le\Big(\frac{C_1t^\nu}{\Gamma(\nu+1)}+\frac{C_1(T-t)^\nu}{\Gamma(\nu+1)}+\frac{C_1T^\nu}{\Gamma(\nu+1)}+\frac{C_1T^\nu}{\Gamma(\nu)}+C_2\Big)\|u-v\|\\
&\le\Big(\frac{(3+\nu)T^\nu}{\Gamma(\nu+1)}C_1+C_2\Big)\|u-v\|.
\end{aligned}$$

Therefore we arrive at

$$\|K(u)-K(v)\|\le\frac{(3+\nu)T^\nu}{\Gamma(\nu+1)}C_1+C_2\Big)\|u-v\|.$$

This shows that K is a contraction operator. Banach fixed point theorem implies that K has at least one fixed point u which is a unique solution of the problem (1.1). □

Next we present the second existence theorem in the next theorem.

Theorem 3.3. *Assume that the following conditions are satisfied*

A3 *$F\in C([0,T]\times\mathbb{R})$, that is, F is a continuous function.*

A4 *There is a positive constant* L_1 *and* $\beta \in (0, \sigma)$ *for some* $0 < \sigma < \nu - 1$ *such that*

$$|F(t, z)| \leq L_1(1 + |z|^{\beta}) \quad \textit{for each} \quad t \in J \quad \textit{and} \quad \forall z \in \mathbb{R}.$$

A5 *There exists a positive constant* L_2 *such that*

$$|g(z)| \leq L_2 \quad \forall z \in C[0, T].$$

Then the problem (1.1) has at least one solution on J.

Proof. We will show that K has a fixed point by using the Schaefer fixed point theorem. We first show K is continuous operator. To show this, consider a sequence $\{u_n\}$ with the limit $u_n \to u \in C[0, 1]$. Then for $t \in J$, we get

$$\begin{aligned}
&|K(u_n)(t) - K(u)(t)| \leq \frac{1}{\Gamma(\nu)} \int_0^t (t - s)^{\nu - 1} |F(s, u_n(s)) - F(s, u(s))| \, ds \\
&+ \frac{1}{\Gamma(\nu)} \int_t^T (s - t)^{\nu - 1} |F(s, u_n(s)) - F(s, u(s))| \, ds \\
&+ \frac{1}{\Gamma(\nu)} \int_0^T (T - s)^{\nu - 1} |F(s, u_n(s)) - F(s, u(s))| \, ds \\
&+ \frac{T}{\Gamma(\nu - 1)} \int_0^T (T - s)^{\nu - 2} |F(s, u_n(s)) - F(s, u(s))| \, ds + |g(u_n) - g(u)| \\
\leq& \Big(\frac{t^{\nu}}{\Gamma(\nu + 1)} + \frac{(T - t)^{\nu}}{\Gamma(\nu + 1)} + \frac{T^{\nu}}{\Gamma(\nu + 1)} + \frac{T^{\nu}}{\Gamma(\nu)} \Big) \|F(s, u_n(s)) - F(s, u(s))\| \\
&+ \|g(u_n) - g(u)\| \leq \frac{(3 + \nu) T^{\nu}}{\Gamma(\nu + 1)} \|F(s, u_n(s)) - F(s, u(s))\| + \|g(u_n) - g(u)\|.
\end{aligned}$$

The continuity of the functions F and g yields

$$\|K(u_n) - K(u)\| \to 0, \quad \textit{as} \quad n \to \infty,$$

which shows that K is continuous.

Secondly we will show that K transforms bounded sets to bounded sets in $C[0, T]$. Let $M_\ell = \{u \in C[0, T] : \|u\| \leq \ell\}$ be a bounded subset of $C[0, T]$. Our goal is to show that $\|K(z)\| \leq m$ for some constant m. For each $t \in J$ and $u \in M_\ell$ we have

$$\begin{aligned}
&|K(u)(t)| \\
&\leq \frac{1}{\Gamma(\nu)} \int_0^T |t - s|^{\nu - 1} |F(s, u(s))| \, ds + \frac{T - 2t}{T\Gamma(\nu)} \int_0^T (T - s)^{\nu - 1} |F(s, u(s))| \, ds \\
&+ \frac{T - t}{\Gamma(\nu - 1)} \int_0^T (T - s)^{\nu - 2} |F(s, u(s))| \, ds + |(1 - \frac{t}{T}) g(u)| + |\frac{t}{T} u_T|
\end{aligned}$$

$$
\begin{aligned}
\leq & \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |F(s,u(s))|\, ds + \frac{1}{\Gamma(\nu)} \int_t^T (s-t)^{\nu-1} |F(s,u(s))|\, ds \\
& + \frac{1}{\Gamma(\nu)} \int_0^T (T-s)^{\nu-1} |F(s,u(s))|\, ds \\
& + \frac{T}{\Gamma(\nu-1)} \int_0^T (T-s)^{\nu-2} |F(s,u(s))|\, ds + 2|g(u)| + |u_T| \\
\leq & \Big(\frac{t^\nu}{\Gamma(\nu+1)} + \frac{(T-t)^\nu}{\Gamma(\nu+1)} + \frac{T^\nu}{\Gamma(\nu+1)} + \frac{T^\nu}{\Gamma(\nu)} \Big) L_1(1+\ell^\beta) + 2L_2 + |u_T| \\
\leq & \frac{(3+\nu)T^\nu}{\Gamma(\nu+1)} L_1(1+\ell^\beta) + 2L_2 + |u_T|.
\end{aligned}
$$

Therefore we get

$$
\|K(z)\| \leq \frac{(3+\nu)T^\nu}{\Gamma(\nu+1)} L_1(1+\ell^\beta) + 2L_2 + |u_T| := m,
$$

which proves the desired result.

Finally we will show that K transforms bounded sets to equicontinuous sets in $C[0,T]$. Again, let $M_\ell = \{u \in C[0,T] : \|u\| \leq \ell\}$ be a bounded subset of $C[0,T]$. We give a bound on the derivative of $K(u)'(t)$ for each $t \in J$ and $u \in M_\ell$ as follows

$$
\begin{aligned}
& |K(u)'(t)| \leq \\
& \frac{1}{\Gamma(\nu-1)} \int_0^t (t-s)^{\nu-2} |F(s,u(s))|\, ds + \frac{1}{\Gamma(\nu-1)} \int_t^T (s-t)^{\nu-2} |F(s,u(s))|\, ds \\
& + \frac{2}{T\Gamma(\nu)} \int_0^T (T-s)^{\nu-1} |F(s,u(s))|\, ds \\
& + \frac{1}{\Gamma(\nu-1)} \int_0^T (T-s)^{\nu-2} |F(s,u(s))|\, ds + |g(u)| + \frac{1}{T}|u_T| \\
\leq & \Big(\frac{t^{\nu-1}}{\Gamma(\nu)} + \frac{(T-t)^{\nu-1}}{\Gamma(\nu)} + \frac{2T^{\nu-1}}{\Gamma(\nu+1)} + \frac{T^{\nu-1}}{\Gamma(\nu)} \Big) L_1(1+\ell^\beta) + 2L_2 + \frac{1}{T}|u_T| \\
\leq & \frac{(2+3\nu)T^{\nu-1}}{\Gamma(\nu+1)} L_1(1+\ell^\beta) + L_2 + \frac{1}{T}|u_T|.
\end{aligned}
$$

Set $L := \dfrac{(2+3\nu)T^{\nu-1}}{\Gamma(\nu+1)} L_1(1+\ell^\beta) + L_2 + \frac{1}{T}|u_T|$. Let $t_1, t_2 \in J$ with $t_1 < t_2$, then we have

$$
|K(u)(t_1) - K(u)(t_2)| = \int_{t_1}^{t_2} |K(u)'(s)|\, ds \leq L(t_2 - t_1).
$$

Thus, $K(M_\ell)$ is equicontinuous in $C[0,T]$. Sp far we have shown that the operator K is completely continuous.

Lastly, we will show that the set

$$
\mathcal{E}(K) = \{u \in C[0,T] : u = \mu K(u), \quad \mu \in (0,1)\}
$$

is bounded. Let $u = \mu K(u)$ for $\mu \in (0,1)$. Then we have for $t \in J$

$$
\begin{aligned}
&|u(t)| \leq \\
&\frac{1}{\Gamma(\nu)} \int_0^T |t-s|^{\nu-1} |F(s,u(s))|\, ds + \frac{1}{\Gamma(\nu)} \int_0^T (T-s)^{\nu-1} |F(s,u(s))|\, ds \\
&+ \frac{T}{\Gamma(\nu-1)} \int_0^T (T-s)^{\nu-2} |F(s,u(s))|\, ds + 2|g(u)| + |u_T| \\
\leq &\frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |F(s,u(s))|\, ds + \frac{1}{\Gamma(\nu)} \int_t^T (s-t)^{\nu-1} |F(s,u(s))|\, ds \\
&+ \frac{1}{\Gamma(\nu)} \int_0^T (T-s)^{\nu-1} |F(s,u(s))|\, ds \\
&+ \frac{T}{\Gamma(\nu-1)} \int_0^T (T-s)^{\nu-2} |F(s,u(s))|\, ds + 2|g(u)| + |u_T| \\
\leq &\frac{(3+\nu)T^\nu}{\Gamma(\nu+1)} L_1 + 2L_2 + |u_T| + \frac{L_1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |u(s)^\beta|\, ds \\
&+ \frac{L_1}{\Gamma(\nu)} \int_t^T (s-t)^{\nu-1} |u(s)^\beta|\, ds + \frac{L_1}{\Gamma(\nu)} \int_0^T (T-s)^{\nu-1} |u(s)^\beta|\, ds \\
&+ \frac{L_1 T}{\Gamma(\nu-1)} \int_0^T (T-s)^{\nu-2} |u(s)^\beta|\, ds.
\end{aligned}
$$

Lemma 2.6 implies that there is a positive constant C such that

$$\|u\| \leq C.$$

This concludes that $\mathcal{E}(K)$ is bounded. By using Schaefer's fixed point theorem, we infer that K has a fixed point which is a solution of the problem (1.1). □

Theorem 3.4. *Assume the condition A5 in the previous theorem and the following condition hold*

A6 *There are $\phi \in C[0,T]$ and $\Phi : [0,\infty) \to \mathbb{R}^+$ continuous and increasing functions such that $|F(t,z)| \leq \phi(t)\Phi(|z|)$ for each $t \in J$ and $\forall z \in \mathbb{R}$.*

Assume also that there is a positive constant C_m such that

$$\Big(\frac{(3+\nu)T^\nu}{\Gamma(\nu+1)}\Big) \frac{\phi_s \Phi(C_m)}{C_m} + \frac{2L_2 + |u_T|}{C_m} < 1, \quad \text{where} \quad \phi_s = \sup_{t \in J} \phi(t). \tag{3.9}$$

Then The problem (1.1) has at least one solution on $[0,T]$.

Proof. Define $M_c = \{u \in C(0,T] : \|u\| \leq C_m\}$. Clearly, M_c is closed, convex and bounded subset of $C[0,T]$. For each $t \in J$ and $u \in M_c$ using the assumptions $A5$ and

$A6$ we have

$$
\begin{aligned}
&|Ku(t)| \leq \\
&\frac{1}{\Gamma(\nu)}\int_0^T |t-s|^{\nu-1}|F(s,u(s))|\,ds + \frac{1}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}|F(s,u(s))|\,ds \\
&+ \frac{T}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}|F(s,u(s))|\,ds + 2|g(u)| + |u_T| \\
\leq& \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}|F(s,u(s))|\,ds + \frac{1}{\Gamma(\nu)}\int_t^T (s-t)^{\nu-1}|F(s,u(s))|\,ds \\
&+ \frac{1}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}|F(s,u(s))|\,ds \\
&+ \frac{T}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}|F(s,u(s))|\,ds + 2|g(u)| + |u_T| \\
\leq& \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}\phi(s)\Phi(|u(s)|)\,ds + \frac{1}{\Gamma(\nu)}\int_t^T (s-t)^{\nu-1}\phi(s)\Phi(|u(s)|)\,ds \\
&+ \frac{1}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}\phi(s)\Phi(|u(s)|)\,ds \\
&+ \frac{T}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}\phi(s)\Phi(|u(s)|)\,ds + 2|g(u)| + |u_T| \\
\leq& \frac{(3+\nu)T^\nu}{\Gamma(\nu+1)}\phi_s\Phi(\|u\|) + 2L_2 + |u_T| \\
\leq& \frac{(3+\nu)T^\nu}{\Gamma(\nu+1)}\phi_s\Phi(C_m) + 2L_2 + |u_T| \\
\leq& C_m \qquad \text{(from the condition (3.9))}.
\end{aligned}
$$

We have shown that the operator $K : M_c \to M_c$ is continuous and completely continuous. If there is $u \in \partial M_c$ with $\mu \in (0,1)$ satisfying $u = \mu K u$, then the we would get a contradiction from the discussion above. As a consequence of Leray-Schauder fixed point theorem (see Lemma (2.7)), K has a fixed point $u \in \overline{M_c}$. This implies that there exists one solution of the equation (1.1). Thus we complete the proof. □

4. Numerical examples

This section is devoted to numerical examples to illustrate the application of the results presented in this paper.

Example 4.1. Consider the following differential equation with the Riesz-Caputo fractional derivative of order $\nu \in (1,2]$

$$
\begin{aligned}
{}^{RC}_{\ 0}D^{\nu}_1 u(t) &= \frac{1}{12}u(t) + (1-t)(\frac{1}{6}+t), \quad t \in [0,1], \nu \in (1,2], \\
u(0) &= \frac{1}{2}u(\frac{1}{2}), \quad u(1) = 0,
\end{aligned}
\tag{4.1}
$$

Let $F(t,w) = \frac{1}{12}u(t) + (1-t)(\frac{1}{6}+t)$, $\quad (t,w) \in [0,1]\times\mathbb{R}$, and $g(u) = \frac{1}{2}u(\frac{1}{2})$. Then for any $u, w \in \mathbb{R}$ and $t \in [0,1]$ we have

$$|F(t,u) - F(t,w)| \leq \frac{1}{12}|u-w|.$$

Moreover, we have

$$|g(u) - g(w)| \leq \frac{1}{2}|u-w|.$$

Thus, the conditions $A1$ and $A2$ are satisfied with $C_1 = \frac{1}{12}$ and $C_2 = \frac{1}{10}$. Taking $T = 1$, we observe that

$$\frac{1}{12}\frac{(3+\nu)}{\Gamma(\nu+1)} + \frac{1}{2} < 1$$

if and only if $\dfrac{(3+\nu)}{6} < \Gamma(\nu+1)$ which holds true since $\Gamma(\nu+1) > \frac{5}{6}$ when $\nu \in (1,2]$. So, the condition (3.8) is satisfied. Theorem 3.2 implies that the problem (4.1) has a unique solution in $[0,1]$.

In general, the exact solutions of nonlinear fractional differential equation (even ordinary nonlinear differential equations) are not available. Thus, we use the method in [7] to plot the numerical solutions of problems. We report the numerical solution of problem (4.1) with $\nu = \frac{3}{2}$ in Figure 1.

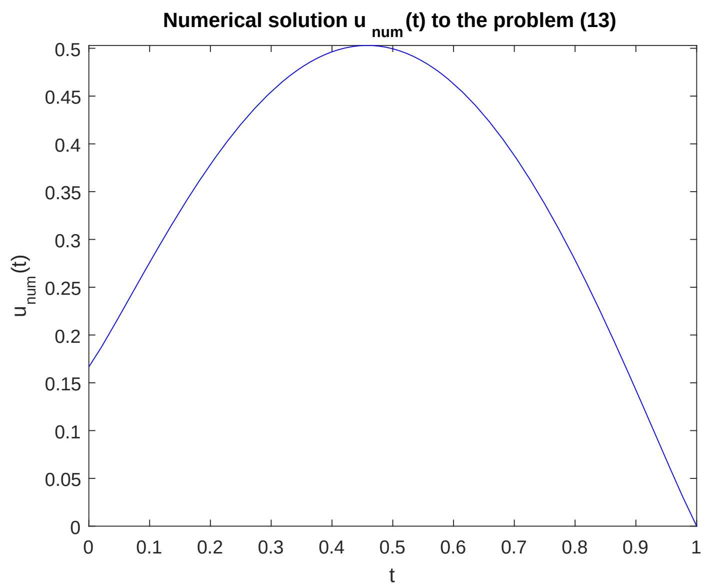

FIGURE 1. The numerical trajectory of the solution for Example 4.1 with $\nu = \frac{3}{2}$.

Example 4.2. Consider the following boundary value problem of the fractional Riezs-Caputo derivative,

$$\begin{aligned} {}^{RC}_{\ 0}D^{\nu}_{1}u(t) &= \frac{|u(t)|}{(8+t^2)(1+|u(t)|)}, \quad t\in[0,1], \nu\in(1,2] \\ u(0) &= \sum_{k=1}^{n} a_k u(t_k), \quad u(1)=0, \end{aligned} \tag{4.2}$$

where $0<t_1<t_2<\cdots<t_n<1$, and $a_k>0, k=0,1\ldots,n$ are constants satisfying

$$\sum_{k=1}^{n} a_k < \frac{1}{4}.$$

Let $F(t,w)=\dfrac{w}{(8+t^2)(1+w)}, \quad (t,w)\in[0,1]\times[0,\infty)$, and

$$g(u)=\sum_{k=1}^{n} a_k u(t_k).$$

Then for any $u,w\in[0,\infty)$ and $t\in[0,1]$ we have

$$\begin{aligned} |F(t,u)-F(t,w)| &= \frac{1}{8+t^2}\left|\frac{u}{1+u}-\frac{w}{1+w}\right| = \frac{1}{8+t^2}\frac{|u-w|}{(1+u)(1+w)} \\ &\le \frac{1}{8}|u-w|. \end{aligned}$$

Moreover, we have

$$|g(u)-g(w)| \le \sum_{k=1}^{n} a_k|u-w|.$$

Thus, the conditions $A1$ and $A2$ are satisfied with $C_1=\frac{1}{10}$ and $C_2\le\frac{1}{4}$. We also have with $T=1$

$$\frac{(3+\nu)}{\Gamma(\nu+1)}\frac{1}{8}+\frac{2}{8} \le \frac{5+\nu}{8\Gamma(\nu+1)} \le \frac{1}{\Gamma(\nu+1)} < 1$$

if and only if $\Gamma(\nu+1)>1$ which holds true when $\nu\in(1,2]$. So, the condition (3.8) is satisfied. Theorem 3.2 implies that the problem (4.2) has a unique solution in $[0,1]$

Example 4.3. Consider the following fractional differential equation of the fractional Riesz-Caputo derivative

$$\begin{aligned} {}^{RC}_{\ 0}D^{\frac{8}{5}}_{1}u(t) &= \frac{|u(t)|^{\frac{1}{5}}}{(1+t^2)(1+|u(t)|)}, \quad t\in[0,1], \\ u(0) &= \sin(2\pi u(\frac{1}{2})), \quad u(1)=0. \end{aligned} \tag{4.3}$$

Let $F(t,w)=\dfrac{|u(t)|^{\frac{1}{5}}}{(1+t^2)(1+|u(t)|)}, \quad (t,w)\in[0,1]\times[0,\infty)$, and $g(u)=\sin(2\pi u(\frac{1}{2}))$ with $\nu=\frac{8}{5}$. Let $\beta=\frac{1}{5}$ and $\sigma=\frac{2}{5}$, then $\beta\in(0,\sigma)$ and $0<\sigma<\nu-1$. Then for any

$u \in [0, \infty)$ and $t \in [0, 1]$ we have

$$|F(t,u)| = \frac{|u(t)|^{\frac{1}{5}}}{(1+t^2)(1+|u(t)|)} \leq \frac{1}{2}(1 + |u(t)|^{\frac{1}{5}}).$$

Additionally, we have

$$|g(u)| \leq 1 = L_2.$$

Thus, the conditions $A3 - A4$ and $A5$ are satisfied. Then we infer from Theorem 3.3 that the problem (4.3) has at least one solution on J.

Let $\phi(t) = \frac{1}{1+t^2}$ and $\Phi(|u|) = |u(t)|^{\frac{1}{5}}$. Then we have $F(t, u(t)) \leq \phi(t)\Phi(|u|)$ with $\phi_s = \frac{1}{2}$. Let $C_m = 81$ and $u_T = 0$. We find that

$$\Big(\frac{(3+\nu)T^\nu}{\Gamma(\nu+1)}\Big)\frac{\phi_s\Phi(C_m)}{C_m} + \frac{2L_2 + |u_T|}{C_m} = \frac{23}{270\Gamma(\frac{13}{5})} + \frac{2}{81} < 1.$$

So the condition (3.9) is satisfied. Theorem 3.4 tells us there exists at least one solution to the problem (4.3).

Example 4.4. Consider the following three-point fractional boundary value problem

$$\begin{aligned} {}^{RC}_{\ 0}D_1^{\frac{3}{2}} u(t) &= \frac{1}{4}t^2u^2(t)e^{-u^2(t)}, \quad t \in [0,1], \\ u(0) &= \frac{1}{32}e^{-u(\eta)}, \quad \eta \in (0,1], \quad u(1) = \frac{1}{16}. \end{aligned} \tag{4.4}$$

We will exhibit that the conditions $A5 - A6$ and (3.9) are satisfied.

Let $F(t, w) = \frac{1}{4}t^2u^2(t)e^{-u^2(t)}$, $\quad (t, w) \in [0, 1]\times\mathbb{R}$, and $g(u) = e^{-u(\eta)}$, $\quad \eta \in (0, 1]$ with $\nu = \frac{3}{2}$. For each $u \in \mathbb{R}$ and $t \in [0, 1]$ we have

$$|F(t,u)| = |\frac{1}{4}t^2u^2(t)e^{-u^2(t)}| \leq \frac{1}{4}t^2u^2(t) = \phi(t)\Phi(u), \quad (t,u) \in [0,1] \times \mathbb{R},$$

where $\phi(t) = \frac{1}{4}t^2$ and $\Phi(u) = u^2$ with $\phi_s = \sup_{t\in[0,1]} |\phi(t)| = \frac{1}{4}$. The function $g(u)$ is bounded, that is,

$$|g(u)| \leq \frac{1}{32} = L_2.$$

Lastly we check the condition (3.9). Let $C_m = 1$ and $u_T = \frac{1}{16}$, then

$$\Big(\frac{(3+\nu)T^\nu}{\Gamma(\nu+1)}\Big)\frac{\phi_s\Phi(C_m)}{C_m} + \frac{2L_2 + |u_T|}{C_m} = \frac{9}{8\Gamma(\frac{5}{2})} + \frac{1}{8} < 1.$$

Again Theorem 3.3 implies that the problem (4.4) has at least one solution on $[0, T]$.

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Şuayip Toprakseven
Faculty of Engineering,
Department of Computer Science,
Artvin Çoruh University,
Artvin, 08100, Turkey
e-mail: `topraksp@artvin.edu.tr`

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Notes on various operators of fractional calculus and some of their implications for certain analytic functions

Hüseyin Irmak

Abstract. The main purpose of this note is firstly to present certain information in relation with some elementary operators created by the well-known fractional calculus, also to determine a number of applications of them for certain complex function analytic in the open unit disc, and then to reveal (*or* point out) some implications of the fundamental results of this research.

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1. Introduction and preliminaries

Fractional Calculus has important roles in both applied studies and theoretical researches. We also know that various operators have been defined by the help of Fractional Calculus. This scientific note is an example for one of such theoretical investigations. In this present investigation, only three elementary operators of fractional calculus, *which* are frequently encountered in the literature, will be considered for determining a number of results relating to certain complex functions. They are well-known operators which are also called as Fractional Integral Operator, Fractional Derivative Operator and Tremblay Operator in the literature. Specially, as we just have indicated just above, these mentioned operators will be taken into consideration for certain analytic functions. For their details and also some extra examples, one

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may refer to the earlier works in [5] and [26]-[28], and also see [1]-[3], [11]-[10], [19], [18], [27] and [13] in the references of this investigation.

Let us now recall certain notations, notions and also some extra information in relation with the mentioned operators in certain domains of the complex plane, *which* there will need for our investigation.

Firstly, *here and also in parallel with this research,* let

$$\mathbb{C} \quad , \quad \mathbb{R} \quad , \quad \mathbb{N} \quad \text{and} \quad \mathbb{U}$$

be the set of *complex numbers,* be the set of *real numbers,* be the set of *positive integers,* and the *open unit* disc, *namely,* the well-known *open* set given by

$$\Big\{ z \ : \ z \in \mathbb{C} \text{ and } |z| < 1 \Big\},$$

respectively.

Also let

$$\mathbb{R}^* := \mathbb{R} - \{0\} \quad \text{and} \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Moreover, by the notation $\mathcal{A}(n)$ denote the family of the functions $f(z)$ normalized by the following Taylor-Maclaurin series:

$$f(z) = z + q_{n+1}z^{n+1} + q_{n+2}z^{n+2} + \cdots \quad \big(q_{n+1} \in \mathbb{C};\ n \in \mathbb{N}\big). \tag{1.1}$$

Secondly, for an analytic function $f(z)$, *the fractional integral of order* λ is then defined by

$$\mathcal{D}_z^{-\lambda}\{f\}(z) = \frac{1}{\Gamma(\lambda)} \int_0^z f(T)\big(z-T\big)^{\lambda-1} dT \quad \big(\lambda > 0\big), \tag{1.2}$$

where the multiplicity of $\big(z-T\big)^{\lambda-1}$ is removed by requiring $log\big(z-T\big)$ to be real *when* $z - T > 0$.

For an analytic function $f(z)$, *the fractional derivative of order* λ is also defined by

$$\mathcal{D}_z^{\lambda}\{f\}(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z f(T)\big(z-T\big)^{-\lambda} dT \quad \big(0 \le \lambda < 1\big), \tag{1.3}$$

where is constrained, and the multiplicity of $\big(z-T\big)^{-\lambda}$ is removed, as in the definition of the fractional integral operator accentuated as (1.2).

Under the hypotheses of the definition of the fractional derivative of order λ, emphasized as (1.3), for an analytic function $f(z)$, *the fractional derivative of order* $m + \lambda$ is defined by

$$\mathcal{D}_z^{m+\lambda}\{f\}(z) = \frac{d^m}{dz^m}\Big(\mathcal{D}_z^{\lambda}\{f\}(z)\Big), \tag{1.4}$$

where $0 \le \lambda < 1$ and $m \in \mathbb{N}_0$.

In the light of the fractional derivative operator, given by (1.2), for an analytic function $f(z)$, the Tremblay operator is also defined by

$$\mathcal{T}_{\tau,\lambda}\{f\}(z) = \frac{\Gamma(\lambda)}{\Gamma(\tau)}\, z^{1-\lambda}\, \mathcal{D}_z^{\tau-\lambda}\big\{z^{\tau-1} f\big\}(z), \tag{1.5}$$

where $0 < \tau \le 1$, $0 < \lambda \le 1$, $0 \le \tau - \lambda < 1$ and $z \in \mathbb{U}$.

In consideration of the fractional integral operator (1.2), fractional derivative operator (1.3) and Tremblay operator (1.4), for an elementary-analytic function given by

$$\varphi := \varphi(z) = z^{\kappa},$$

we remark in passing that the following-special results can be easily assigned as the forms:

$$\mathcal{D}_z^{-\lambda}\{\varphi\} = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\lambda+1)} z^{\kappa+\lambda} \quad \big(\lambda > 0\big), \tag{1.6}$$

$$\mathcal{D}_z^{\lambda}\{\varphi\} = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\lambda+1)} z^{\kappa-\lambda} \quad \big(0 \leq \lambda < 1\big), \tag{1.7}$$

$$\mathcal{D}_z^{m+\lambda}\{\varphi\} = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-m-\lambda+1)} z^{\kappa-m-\lambda} \quad \big(0 \leq \lambda < 1; m \in \mathbb{N}_0\big), \tag{1.8}$$

and

$$\mathcal{T}_{\tau,\lambda}\{\varphi\} = \frac{\Gamma(\kappa+\tau)\Gamma(\lambda)}{\Gamma(\kappa+\lambda)\Gamma(\tau)} z^{\kappa} \quad \big(0 < \tau \leq 1; 0 < \lambda \leq 1; 0 \leq \tau-\lambda < 1\big). \tag{1.9}$$

In terms of our purposes, in special, by means of the assertions presented in (1.6)-(1.9), for a function $f(z)$ belonging to the family $\mathcal{A}(n)$, there is a need to state certain results which are given by the following relations:

$$\mathcal{D}_z^{-\lambda}\{f\}(z) = \frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+\lambda+1)} q_k z^{k+\lambda} \tag{1.10}$$

$$\big(\lambda > 0; z \in \mathbb{U}\big)$$

$$\mathcal{D}_z^{\lambda}\{f\}(z) = \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} q_k z^{k-\lambda} \tag{1.11}$$

$$\big(0 \leq \lambda < 1; z \in \mathbb{U}\big)$$

and

$$\mathcal{T}_{\tau,\lambda}\{f\}(z) = \frac{\tau}{\lambda} z + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+\tau)\Gamma(\lambda)}{\Gamma(k+\lambda)\Gamma(\tau)} q_k z^{k} \tag{1.12}$$

$$\big(0 < \tau \leq 1; 0 < \lambda \leq 1; 0 \leq \tau-\lambda < 1; z \in \mathbb{U}\big),$$

and, from (11) and (12), the following results are easily determined:

$$\mathcal{D}_z^{1+\lambda}\{f\}(z) = \frac{1}{\Gamma(1-\lambda)} z^{-\lambda} + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda)} q_k z^{k-\lambda-1} \tag{1.13}$$

$$\big(0 \leq \lambda < 1; z \in \mathbb{U}\big)$$

and

$$\frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big) = \frac{\tau}{\lambda} + \sum_{k=n+1}^{\infty} \frac{k\Gamma(k+\tau)\Gamma(\lambda)}{\Gamma(k+\lambda)\Gamma(\tau)} q_k z^{k-1} \tag{1.14}$$

$$\big(0 < \tau \leq 1; 0 < \lambda \leq 1; 0 \leq \tau-\lambda < 1; z \in \mathbb{U}\big).$$

The following assertion, namely, Lemma 1.1 (below) will be required for stating and proving of our main results. By considering the well-known result (see [22] and

[24]), it was earlier proven by Nunokawa [23]. In addition, it has been also used by a great number of researchers for their studies. For some of them, for instance, it can be checked some of results given by [13]-[16] in the references. Specially, in the recent time, by making use of the same assertions considered in [17]-[20], they have used those for their earlier results and they have also obtained various results identified by the assertions relating to the mentioned operators and their applications given by (1.12)-(1.16). Moreover, we point that some (special) results can be compared with certain earlier results obtained in [20], [16] and [18].

Lemma 1.1. *Let $p(z)$ be an analytic function in the open set $\mathbb{U}$ with $p(0) = 1$ and also suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$\Re e\Big(p(z)\Big) > 0 \quad when \quad |z| < |z_0| < 1\,, \tag{2.1}$$

$$\Re e\Big(p(z)\big|_{z:=z_0}\Big) = 0 \tag{2.2}$$

and

$$p(z)\big|_{z:=z_0} \neq 0\,. \tag{2.3}$$

Then,

$$p(z)\big|_{z:=z_0} = is \quad \big(s \in \mathbb{R}^*\big) \tag{2.4}$$

and

$$zp'(z)\big|_{z:=z_0} = ic\left(s + s^{-1}\right) p(z)\big|_{z:=z_0} \quad \big(s \in \mathbb{R}^*\big) \tag{2.5}$$

for all c in $\big[1/2, \infty\big)$.

2. The main results and their implications

In this section, by considering certain necessity and sufficiency, which are terms used to describe a conditional (*or* implicational) relationship between two statements in mathematics, various comprehensive theories consisting of some complex-valued-exponential forms constituted by the operator given by (1.3)-(1.5) will be presented and they will be then proven.

Theorem 2.1. *For admissible values of the parameters given by*

$$0 < \tau \leq 1 \quad , \quad 0 < \lambda \leq 1 \quad , \quad 0 \leq \tau - \lambda < 1 \quad and \quad 0 \leq \alpha < 1\,, \tag{2.6}$$

if the following statement:

$$Arg\left(\frac{z\frac{d^2}{dz^2}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)}{\frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)}\right) \notin \begin{cases} \big(-\pi, -\frac{\pi}{2}\big] \cup \big[\frac{\pi}{2}, \pi\big) & if \quad \omega > 0 \\ \big[-\frac{\pi}{2}, 0\big) \cup \big(0, \frac{\pi}{2}\big] & if \quad \omega < 0 \end{cases} \tag{2.7}$$

is true, then

$$\Re e\left\{\left[\frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)\right]^{\omega}\right\} > \alpha\Big(\frac{\tau}{\lambda}\Big)^{\omega} \quad \big(\omega \in \mathbb{R}^*\big) \tag{2.8}$$

is also true, where $\omega \in \mathbb{R}^*$, $z \in \mathbb{U}$ *and* $f(z) \in \mathcal{A}(n)$, *and, here and through the proof of this theorem and its implications, each one of the values of the complex expressions like*

$$\left[\frac{d}{dz}\Big(\mathcal{T}_{\tau,\mu}\{f\}(z)\Big)\right]^{\omega} \qquad (\omega \in \mathbb{R}^*)$$

is taken to be as its principal value.

Proof. By the help of the application (of Tremblay operator in (1.5)) given by (1.14) and under the conditions given in (2.6), for a function $f(z) \in \mathcal{A}(n)$, let us then consider a function $p(z)$ in the implicit form, given in

$$\left[\frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)\right]^{\omega} = \left(\frac{\tau}{\lambda}\right)^{\omega}\big[\alpha + (1-\alpha)p(z)\big]\,, \tag{2.9}$$

where $0 \le \alpha < 1$, $\omega \in \mathbb{R}^*$ and $z \in \mathbb{U}$. By a simple focusing, clearly, the function $p(z)$ satisfies the condition $p(0) = 1$ in the hypothesis of Lemma 1.1.

By differentiating the both sides of the definition in (2.9) with respect to the complex variable z, it can be easily obtained that

$$\omega\, z\,\frac{d^2}{dz^2}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)\left[\frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)\right]^{\omega-1} = (1-\alpha)\left(\frac{\tau}{\lambda}\right)^{\omega} zp'(z)\,, \tag{2.10}$$

and, by combining (2.9) and (2.10), the following statement:

$$\omega\cdot\frac{z\frac{d^2}{dz^2}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)}{\frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)} = \frac{(1-\alpha)zp'(z)}{\alpha + (1-\alpha)p(z)} \tag{2.11}$$

is also received, *where,* of course,

$$0 \le \alpha < 1\ ,\ \omega \in \mathbb{R}^*\ ,\ f(z) \in \mathcal{A}(n)\ \text{ and }\ \frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big) \neq 0\ \ \big(\forall z \in \mathbb{U}\big)\,.$$

For the proof, suppose now that there exists a point z belonging to $\mathbb{U}$, which satisfies the condition:

$$\Re e\big(p(z)\big) = 0 \quad \Big(z_0 \in \mathbb{U}; p\big(z_0\big) \neq 0\Big)\,,$$

indicated by (1.6) of Lemma 1.1. Then, by applying of the assertions of Lemma 1.1, given in (2.4) and (2.5) to the result given by (2.11), the following-special result:

$$\begin{aligned}
&Arg\left(-\omega\cdot\left.\frac{z\frac{d^2}{dz^2}\Big(\mathcal{T}_{\tau,\mu}\{f\}(z)\Big)}{\frac{d}{dz}\Big(\mathcal{T}_{\tau,\mu}\{f\}(z)\Big)}\right|_{z:=z_0}\right) = Arg\left(\left.\frac{\big(\alpha-1\big)zp'(z)}{\alpha + \big(1-\alpha\big)p(z)}\right|_{z:=z_0}\right)\\
&= Arg\Big(\ \big(\alpha-1\big)zp'(z)\big|_{z:=z_0}\Big) - Arg\Big(\ \alpha + (1-\alpha)p(z)\big|_{z:=z_0}\Big)\\
&= Arg\Big(c(1-\alpha)\big(1+a^2\big)\Big) - Arg\Big(\alpha + ia(1-\alpha)\Big)\\
&= -Arg\Big(\alpha + ia(1-\alpha)\Big) \in \begin{cases} \big[-\frac{\pi}{2}, 0\big) & \text{if} \quad a < 0 \\ \big(0, \frac{\pi}{2}\big] & \text{if} \quad a > 0 \end{cases}
\end{aligned}$$

is easily obtained, *which* contradicts the result given in (2.7), of course, after some calculations.

This shows that there is no any point $z_0 \in \mathbb{U}$ satisfying the condition given in (2.2). This means that

$$\Re e\Big(p(z)\Big) > 0 \quad \big(\forall z \in \mathbb{U}\big).$$

Therefore, the special definition in (2.9) immediately yields that the inequality in (2.8). The desired proof is also completed. $\square$

In this section, as we know, an extensive-main result has been constituted by applying one of elementary operators of fractional calculus, which is introduced by (1.5) (*or,* (1.12) and (1.14)), to a function $f(z)$ belonging to the family $\mathcal{A}(n)$. In consideration of the main result, as certain conclusions and recommendations, by considering those extensive information about all operators (together with combining some of them given in (1.2)-(1.4)), it can be easily redetermined several new results like Theorem 2.1 again. With the help of such information, the main theorem can help us to recompose many new-comprehensive results and also to reveal a great number of some important-specific results will be obtained by the possible results. In these determinations *or* constructions, we want to give some suggestions to the relevant researchers for stating and proving of new possible theorems *or* their special results.

As first suggestion, in view of the results determined by (1.6)-(1.14), several new theorems, *which* are similar to Theorem 2.1, can be also reconstituted. For it and its proof, it is enough to redefine a similar type function like $p(z)$, *which* is defined as (2.9) and plays an important role in the creation and the proof of Theorem 2.1. As example, if one takes into account the related function $p(z)$, which also consists of fractional fractional derivative(s), given as the following-implicit form:

$$\left[\frac{d}{dz}\Big(z^{\lambda}\mathcal{D}_z^{\lambda}\{f\}(z)\Big)\right]^{\omega} = \Big(\frac{1}{\Gamma(2-\lambda)}\Big)^{\omega}\big[\alpha + (1-\alpha)p(z)\big]$$

$$\big(0 \leq \alpha < 1; 0 \leq \lambda < 1; \omega \in \mathbb{R}^*; f(z) \in \mathcal{A}(n)\big)$$

and also follows the similar manner in the proof of Theorem 2.1, the following theorem can be then demonstrated. Its detail is excluded here.

Theorem 2.2. *Let* $0 \leq \alpha < 1$, $\omega \in \mathbb{R}^*$ *and* $z \in \mathbb{U}$. *For a function* $f(z) \in \mathcal{A}(n)$, *if the statement:*

$$Arg\left(\frac{z\frac{d^2}{dz^2}\Big(z^{\lambda}\mathcal{D}_z^{\lambda}\{f\}(z)\Big)}{\frac{d}{dz}\Big(z^{\lambda}\mathcal{D}_z^{\lambda}\{f\}(z)\Big)}\right) \notin \begin{cases} \big(-\pi, -\frac{\pi}{2}\big] \cup \big[\frac{\pi}{2}, \pi\big) & \textit{if} \quad \omega > 0 \\ \big[-\frac{\pi}{2}, 0\big) \cup \big(0, \frac{\pi}{2}\big] & \textit{if} \quad \omega < 0 \end{cases}$$

is true, then

$$\Re e\left\{\left[\frac{d}{dz}\Big(z^{\lambda}\mathcal{D}_z^{\lambda}\{f\}(z)\Big)\right]^{\omega}\right\} > \alpha\big[\Gamma(2-\lambda)\big]^{-\omega} \quad \big(\omega \in \mathbb{R}^*\big)$$

is also true, where the value of the complex power given by

$$\left[\frac{d}{dz}\Big(z^{\lambda}\mathcal{D}_z^{\lambda}\{f\}(z)\Big)\right]^{\omega} \quad \big(\omega \in \mathbb{R}^*\big)$$

is considered to be as its principal value.

As second suggestion, it will be the suggested determinations of both the new theorems to be determined by the researchers and the specific results of the main results determined by us. In order to reveal them, it will be sufficient to select the suitable values of the related parameters. We also leave to reveal the others to the attentions of the interested researchers. For this, as examples, we want also to present only two of them as propositions.

The first special result is one of the main result, *which* is Proposition 2.3 (below). It can be also constituted by choosing the value of ω as $\omega := 1$ in Theorem 2.1.

Proposition 2.3. *For a function $f(z) \in \mathcal{A}(n)$, if the statement*

$$Arg\left(\frac{z\frac{d^2}{dz^2}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)}{\frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)}\right) \in \Big(-\frac{\pi}{2},\frac{\pi}{2}\Big) \cup \{\pm\pi\}$$

holds, then

$$\Re e\left[\frac{d}{dz}\Big(\mathcal{T}_{\tau,\lambda}\{f\}(z)\Big)\right] > \alpha\frac{\tau}{\lambda}$$

holds for all $z \in \mathbb{U}$ and also for some of the admissible values of the parameters given by $0 < \tau \leq 1$, $0 < \lambda \leq 1$, $0 \leq \tau - \lambda < 1$ and $0 \leq \alpha < 1$.

By taking the values of the parameters ω, τ and λ as $\omega := 1$, $\tau := 1$ and $\mu := 1$ in Theorem 2.1 (*or,* by selecting the values of the parameters of τ and λ as $\tau := 1$ and $\lambda := 1$ in Proposition 2.3), for a function $f(z) \in \mathcal{A}(n)$, the second special result is then received. In this case, as a special consequence of the main result, *which* relates to (Analytic and) Geometric Function Theory (see, for details, [6]), it can be easily identified by the following-well-known result (Proposition 2.4 below).

Proposition 2.4. *Let the function $f(z)$ be in $\mathcal{A}(n)$. Then, the following statement is true:*

$$Arg\left(\frac{zf''(z)}{f'(z)}\right) \in \Big(-\frac{\pi}{2},\frac{\pi}{2}\Big) \cup \{\pm\pi\} \quad \Rightarrow \quad \Re e\Big(f'(z)\Big) > \alpha.$$

It also shows that the function $f(z)$ is a close-to-convex of order α $(0 \leq \alpha < 1)$ in the open disc $\mathbb{U}$.

As concluding remark, all other results (and, of course, their possible consequences), which will be new (*or* known) for the literature and are also omitted in this scientific note, are presented to reveal to the attention of the researchers who have been working on the topics of this investigation. In particular, for the related researchers, we believe that it would be useful to focus on the results determined in the papers given by the references in [17-20], in terms of highlighted results and even their specific implications. As an example in relation with geometric properties of our works, Proposition 2.4 (above) has been presented. In the same time, extra simple examples can be also determined for those results (and also their special forms). These are also left to interested researchers. In addition, as was indicated before, some examples of certain applications of fractional calculations in different disciplines are especially emphasized in the references.

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Hüseyin Irmak
Çankırı Karatekin University,
Faculty of Science, Department of Mathematics,
Uluyazı Campus, TR-18100, Çankırı, Turkey
e-mail: `hirmak@karatekin.edu.tr, hisimya@yahoo.com`

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Implicit Caputo-Fabrizio fractional differential equations with delay

Salim Krim, Saïd Abbas, Mouffak Benchohra and Juan J. Nieto

Abstract. This article deals with some existence and uniqueness results for several classes of implicit fractional differential equations with delay. Our results are based on some fixed point theorems. To illustrate our results, we present some examples in the last section.

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1. Introduction

Functional differential equations and inclusions of fractional order have recently been applied in various areas of sciences; see the monographs [1, 2, 3, 20, 24, 28, 25], the papers [5, 8, 26, 27], and the references therein.

The study of implicit differential equations has received great attention in the last years; see [1, 5, 6, 8, 9, 10, 7, 22, 27].

Functional differential equations with delay have received significant attention in recent years. Several authors studied differential equations with delay [1, 4, 8, 13, 14, 15, 16, 17, 18, 19].

In this paper, first we investigate the following class of Caputo-Fabrizio fractional differential equation with finite delay

$$\begin{cases} \wp(t) = \zeta(t); \ t \in [-h, 0], \\ ({}^{CF}D_0^r \wp)(t) = f(t, \wp_t, ({}^{CF}D_0^r \wp)(t)); \ t \in I := [0, T], \end{cases} \tag{1.1}$$

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where $h > 0,\ T > 0,\ \zeta \in \mathcal{C}, f : I \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, ${}^{CF}D_0^r$ is the Caputo-Fabrizio fractional derivative of order $r \in (0, 1]$, and $\mathcal{C} := C([-h, 0], \mathbb{R})$ is the space of continuous functions on $[-h, 0]$. Here, for any $t \in I$, we define $\wp_t$ by

$$\wp_t(s) = \wp(t + s);\ \text{for } s \in [-h, 0].$$

Next, we consider the following infinite delay problem

$$\begin{cases} \wp(t) = \zeta(t);\ t \in \mathbb{R}_- := (-\infty, 0], \\ ({}^{CF}D_0^r \wp)(t) = f(t, \wp_t, ({}^{CF}D_0^r \wp)(t));\ t \in I, \end{cases} \tag{1.2}$$

where $\zeta : \mathbb{R}_- \to \mathbb{R}, f : I \times \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and $\mathcal{B}$ is called a phase space that will be specified later. In this case, for any $t \in I$, we let $\wp_t \in \mathcal{B}$ be such that

$$\wp_t(s) = \wp(t + s);\ \text{for } s \in \mathbb{R}_-.$$

In the third section, we investigate the following state-dependent finite delay problem

$$\begin{cases} \wp(t) = \zeta(t);\ t \in [-h, 0], \\ ({}^{CF}D_0^r \wp)(t) = f(t, \wp_{\rho(t,\wp_t)}, (({}^{CF}D_0^r \wp)(t));\ t \in I, \end{cases} \tag{1.3}$$

where $\zeta \in \mathcal{C},\ \rho : I \times \mathcal{C} \to \mathbb{R},\ f : I \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Finally, we study the following class of Caputo-Fabrizio fractional differential equations with state dependent infinite delay

$$\begin{cases} \wp(t) = \zeta(t);\ t \in \mathbb{R}_-, \\ ({}^{CF}D_0^r \wp)(t) = f(t, \wp_{\rho(t,\wp_t)}, (({}^{CF}D_0^r \wp)(t));\ t \in I, \end{cases} \tag{1.4}$$

where $\zeta : \mathbb{R}_- \to \mathbb{R},\ f : I \times \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

In the last section, we present some examples illustrating our presented results.

2. Preliminaries

Let $(C(I), \|\cdot\|_\infty)$ be the Banach space of continuous real functions on I with

$$\|\xi\|_\infty := \sup_{t \in I} |\xi(t)|.$$

As usual, $AC(I)$ denotes the space of absolutely continuous real functions on I, and by $L^1(I)$ we denote the space of measurable real functions on I which are Lebesgue integrable with the norm

$$\|\xi\|_1 = \int_I |\xi(t)| dt.$$

Definition 2.1. [11, 23] The Caputo-Fabrizio fractional integral of order $0 < r < 1$ for a function $w \in L^1(I)$ is defined by

$${}^{CF}I^r w(\tau) = \frac{2(1-r)}{M(r)(2-r)} w(\tau) + \frac{2r}{M(r)(2-r)} \int_0^\tau w(x)dx, \quad \tau \geq 0$$

where $M(r)$ is normalization constant depending on r.

Definition 2.2. [11, 23] The Caputo-Fabrizio fractional derivative for a function $w \in C^1(I)$ of order $0 < r < 1$, is defined by

$$^{CF}D^r w(\tau) = \frac{(2-r)M(r)}{2(1-r)} \int_0^\tau \exp(-\frac{r}{1-r}(\tau - x))w'(x)dx; \ \tau \in I.$$

Note that $(^{CF}D^r)(w) = 0$ if and only if w is a constant function.

Example 2.3. [11]
1- For $h(t) = t$ and $0 < r \leq 1$, we have

$$(^{CF}D^r h)(t) = \frac{M(r)}{r}\left(1 - \exp\left(-\frac{r}{1-r}t\right)\right).$$

2- For $g(t) = e^{\lambda t}$, $\lambda \geq 0$ and $0 < r \leq 1$, we have

$$(^{CF}D^r g)(t) = \frac{\lambda M(r)}{r + \lambda(1-r)} e^{\lambda t}\left(1 - \exp\left(-\lambda - \frac{r}{1-r}t\right)\right).$$

Lemma 2.4. [21] *Let $h \in L^1(I)$. Then the linear problem*

$$\begin{cases} (^{CF}D_0^r \wp)(t) = h(t); \ t \in I := [0, T] \\ \wp(0) = \wp_0, \end{cases} \tag{2.1}$$

has a unique solution given by

$$\wp(t) = \wp_0 - a_r h(0) + a_r h(t) + b_r \int_0^t h(s)ds, \tag{2.2}$$

where

$$a_r = \frac{2(1-r)}{(2-r)M(r)}, \ b_r = \frac{2r}{(2-r)M(r)}.$$

3. Existence of solutions with finite delay

In this section, we establish the existence results for problem (1.1). Consider the Banach space

$$C = \{\wp : (-h, T] \to \mathbb{R}, \ \wp|_{[-h,T]} \equiv \zeta, \ \wp|_I \in C(I)\}.$$

with the norm

$$\|\wp\|_C = \max\{\|\zeta\|_{[-h,0]}, \|\wp\|_\infty\}.$$

Definition 3.1. By a solution of problem (1.1), we mean a function $\wp \in C$ such that

$$\wp(t) = \begin{cases} \zeta(t); \ t \in [-h, 0], \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds; \ t \in I, \end{cases}$$

where $g \in C(I)$ with $g(t) = f(t, \wp_t, g(t))$.

The following hypotheses will be used in the sequel.

- (H_1) There exist constants $\omega_1 > 0$, $0 < \omega_2 < 1$ such that:

  $$|f(t, \wp_1, \Im_1) - f(t, \wp_2, \Im_2)| \leq \omega_1 \|\wp_1 - \wp_2\|_{[-h,0]} + \omega_2 |\Im_1 - \Im_2|,$$

  for any $\wp_1, \wp_2 \in \mathcal{C}$, $\Im_1, \Im_2 \in \mathbb{R}$, and each $t \in I$.

- (H_2) For any bounded set $B \subset \mathcal{C}$, the set:
$$\{t \mapsto f(t, \wp_t, ({}^{CF}D_0^r \wp)(t)) : \wp \in B\};$$
is equicontinuous in C.

Theorem 3.2. *If (H_1) holds, and*
$$\ell := \frac{\omega_1(2a_r + Tb_r)}{1-\omega_2} < 1, \tag{3.1}$$
then problem (1.1) has a unique solution on $[-h, T]$.

Proof. Consider the operator $N : C \to C$ defined by:
$$(N\wp)(t) = \begin{cases} \zeta(t); \ t \in [-h, 0], \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds; \ t \in I, \end{cases} \tag{3.2}$$
where $g \in C(I)$ such that $g(t) = f(t, \wp_t, g(t))$.
Let $u, v \in C(I)$. Then, for each $t \in [-h, 0]$, we have
$$|(N\wp)(t) - (N\Im)(t)| = 0,$$
and for each $t \in I$, we have
$$\begin{aligned} |(N\wp)(t) - (N\Im)(t)| \ \leq \ & a_r|g(0) - h(0)| + a_r|g(t) - h(t)| \\ & + b_r \int_0^t |g(s) - h(s)|ds \end{aligned}$$
where $g, \ h \in C(I)$ such that
$$g(t) = f(t, \wp_t, g(t)) \quad and \quad h(t) = f(t, \Im_t, h(t)).$$
From (H_1), we have
$$\begin{aligned} |g(t) - h(t)| &= |f(t, \wp_t, g(t)) - f(t, \Im_t, h(t))| \\ &\leq \omega_1 \|\wp_t - \Im_t\|_{[-h,0]} + \omega_2|g(t) - h(t)|. \end{aligned}$$
This gives,
$$|g(t) - h(t)| \leq \frac{\omega_1}{1-\omega_2} \|\wp_t - \Im_t\|_{[-h,0]}.$$
Thus, for each $t \in I$, we get
$$\begin{aligned} |(N\wp)(t) - (N\Im)(t)| &\leq 2a_r \tfrac{\omega_1}{1-\omega_2} \|\wp_t - \Im_t\|_{[-h,0]} \\ &\quad + b_r \int_0^t \tfrac{\omega_1}{1-\omega_2} \|\wp_s - \Im_s\|_{[-h,0]} ds \\ &\leq 2a_r \tfrac{\omega_1}{1-\omega_2} \|\wp - \Im\|_C + Tb_r \tfrac{\omega_1}{1-\omega_2} \|\wp - \Im\|_C \\ &\leq \tfrac{\omega_1(2a_r + Tb_r)}{1-\omega_2} \|\wp - \Im\|_C \\ &\leq \ell \|\wp - \Im\|_C. \end{aligned}$$
Hence, we get
$$\|N(\wp) - N(\Im)\|_C \leq \ell \|\wp - \Im\|_C.$$
Since $\ell < 1$, the Banach contraction principle implies that problem (1.1) has a unique solution.

Theorem 3.3. *If* (H_1) *and* (H_2) *hold, and*

$$\frac{\omega_1}{1-\omega_2}(2a_r + Tb_r) < 1,$$

then problem (1.1) *has at least one solution on* $[-h, T]$.

Proof. Consider the operator $N: \ C \to C$ defined in (3.2).
Let $R > 0$ such that

$$R \geq \max\left\{\|\zeta\|_{C([-h,0])}, \frac{|\zeta(0)| + \frac{f^*}{1-\omega_2}(2a_r + Tb_r)}{1 - \frac{\omega_1}{1-\omega_2}(2a_r + Tb_r)}\right\}, \tag{3.3}$$

where $f^* := \sup\limits_{t\in I} |f(t, 0, 0)|$.
Define the ball

$$B_R = \{x \in C(I, \mathbb{R}), \ \|x\|_C \leq R\}.$$

Step 1. N is continuous .
Let $\{\wp_n\}_n$ be a sequence such that $\wp_n \to \wp$ on B_R. For each $t \in [-h, 0]$, we have

$$|(N\wp_n)(t) - (N\wp)(t)| = 0,$$

and for each $t \in I$, we have

$$\begin{aligned} |(N\wp_n)(t) - (N\wp)(t)| \quad &\leq \quad a_r|g_n(0) - g(0)| + a_r|g_n(t) - g(t)| \\ &\quad\ +b_r \int_0^t |g_n(s) - g(s)|ds, \end{aligned} \tag{3.4}$$

where $g_n, \ g \in C(I)$ such that

$$g_n(t) = f(t, \wp_{nt}, g_n(t)) \quad and \quad g(t) = f(t, \wp_t, g(t)).$$

Since $\|\wp_n - \wp\|_C \to 0$ as $n \to \infty$ and f, g and g_n are continuous, then the Lebesgue dominated convergence theorem, implies that

$$\|N(\wp_n) - N(\wp)\|_C \to 0 \quad \text{as } n \to \infty.$$

Hence, N is continuous.
Step 2. $N(B_R) \subset B_R$.
Let $\wp \in B_R$, If $t \in [-h, 0]$ then $\|(N\wp)(t)\| \leq \|\zeta\|_C \leq R$. From (H_1), for each $t \in I$, we have

$$\begin{aligned} |g(t)| \quad &= \quad |f(t, \wp_t, g(t))| \\ &\leq \quad |f(t, 0, 0)| + \omega_1\|\wp_t\|_{[-h,0]} + \omega_2|g(t)| \\ &\leq \quad f^* + \omega_1\|\wp\|_C + \omega_2\|g\|_\infty \\ &\leq \quad f^* + \omega_1 R + \omega_2\|g\|_\infty. \end{aligned}$$

Then

$$\|g\|_\infty \leq \frac{f^* + \omega_1 R}{1 - \omega_2}.$$

Thus,

$$\begin{aligned} |(N\wp)(t)| \quad &\leq \quad |\zeta(0) - a_r g(0) + a_r g(t) + b_r \textstyle\int_0^t g(s)ds| \\ &\leq \quad |\zeta(0)| + a_r|g(0)| + a_r|g(t)| + b_r \textstyle\int_0^t |g(s)|ds \\ &\leq \quad |\zeta(0)| + \tfrac{f^*+\omega_1 R}{1-\omega_2}(2a_r + b_r \textstyle\int_0^t ds) \\ &\leq \quad |\zeta(0)| + \tfrac{f^*+\omega_1 R}{1-\omega_2}(2a_r + Tb_r) \\ &\leq \quad R. \end{aligned}$$

Hence

$$\|N(\wp)\|_C \leq R.$$

Consequently, $N(B_R) \subset B_R$.

Step 3. $N(B_R)$ is equicontinuous.
For $1 \leq t_1 \leq t_2 \leq T$, and $u \in B_R$, we have

$$\begin{array}{rcl} |N(\wp)(t_1) - N(\wp)(t_2)| & \leq & a_r|g(t_2) - g(t_1)| + b_r \int_{t_1}^{t_2} |g(s)|ds| \\ & \leq & a_r|g(t_2) - g(t_1)| + \frac{RKb_r}{1-L}(t_2 - t_1). \end{array}$$

Thus, from (H_2), $a_r|g(t_2) - g(t_1)| + \frac{RKb_r}{1-L}(t_2 - t_1) \to 0$; as $t_2 \to t_1$. This gives the equicontinuity of $N(B_R)$.

From the above steps and the Arzelá-Ascoli theorem, we conclude that N is continuous and compact. Consequently, from Schauder's theorem [12] we deduce that problem (1.1) has at least one solution.

4. Existence of solutions with infinite delay

In this section, we establish some existence results for problem (1.2). Let the space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, T]$ into $\mathbb{R}$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato [13] for ordinary differential functional equations:

(A_1). If $\wp : (-\infty, T] \to \mathbb{R}$, and $\wp_0 = \zeta(0) \in \mathcal{B}$, then there exist constants $L, M, H > 0$, such that for each $t \in I$; we have:
- (i). $\wp_t$ *is in* $\mathcal{B}$,
- (ii). $\|\wp_t\|_{\mathcal{B}} \leq K\|\wp_0\|_{\mathcal{B}} + M \sup_{s\in[0,t]} |\wp(s)|$,
- (iii). $\|\wp(t)\| \leq H\|\wp_t\|_{\mathcal{B}}$.

(A_2). For the function $\wp(\cdot)$ *in* $(A1)$, u_t is a $\mathcal{B}-$ valued continuous function on I.

(A_3). The space $\mathcal{B}$ is complete.

Consider the space

$$\Omega = \{\wp : (-\infty, T] \to \mathbb{R},\ \wp|_{\mathbb{R}_-} \in \mathcal{B},\ \wp|_I \in C(I)\}.$$

Definition 4.1. By a solution of problem (1.2), we mean a continuous function $\wp \in \Omega$

$$\wp(t) = \begin{cases} \zeta(t);\ t \in \mathbb{R}_-, \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds;\ t \in I, \end{cases} \tag{4.1}$$

where $g \in C$ such that $g(t) = f(t, \wp_t, g(t))$.

Let us introduce the following hypotheses:

- (H_{01}) The function f satisfies the Lipschitz condition:

$$|f(t, \wp_1, \Im_1) - f(t, \wp_2, \Im_2)| \leq b_1\|\wp_1 - \wp_2\|_{\mathcal{B}} + b_2|\Im_1 - \wp_2|,$$

for any $\wp_1, \Im_1 \in \mathcal{B}$, $\wp_2, \Im_2 \in \mathbb{R}$, and each $t \in I$, where $b_1 > 0$ and $0 < b_2 < 1$.

- (H_{02}) For any bounded set $B_1 \subset \Omega$, the set:
$$\{t \mapsto f(t, \wp_t, ({}^{CF}D_0^r \wp)(t)) : \wp \in B_1\};$$
is equicontinuous in Ω.

First, we prove an existence and uniqueness result by using the Banach's fixed point theorem.

Theorem 4.2. *Assume that the hypothesis (H_{01}) holds. If*
$$\lambda := (2a_r + Tb_r)\frac{b_1}{1-b_2} < 1, \tag{4.2}$$
then problem (1.2) has a unique solution on $(-\infty, T]$.

Proof. Consider the operator $N_1 : \ \Omega \to \Omega$ defined by:
$$(N_1\wp)(t) = \begin{cases} \zeta(t); \ t \in \mathbb{R}_-, \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds; \ t \in I, \end{cases} \tag{4.3}$$
where $g \in C(I)$ such that $g(t) = f(t, \wp_t, g(t))$.
Let $x(\cdot) : (-\infty, T] \to \mathbb{R}$ be a function defined by
$$x(t) = \begin{cases} \zeta(t); & t \in \mathbb{R}_-, \\ \zeta(0)- & t \in I. \end{cases}$$
Then $x_0 = \zeta$, For each $z \in C(I)$, with $z(0) = 0$, we denote by $\overline{z}$ the function defined by
$$\overline{z} = \begin{cases} 0; & t \in t \in \mathbb{R}_-, \\ z(t), & t \in I. \end{cases}$$
If $\wp(\cdot)$ satisfies the integral equation
$$\wp(t) \quad = \quad \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds.$$
We can decompose $\wp(\cdot)$ as $\wp(t) = \overline{z}(t) + x(t)$; for $t \in I$, which implies that $\wp_t = \overline{z}_t + x_t$ for every $t \in I$, and the function $z(\cdot)$ satisfies
$$z(t) = -a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds,$$
where
$$g(t) = f(t, \overline{z}_t + x_t, g(t)); \ t \in I.$$
Set
$$C_0 = \{z \in C(I); \ z_0 = 0\},$$
and let $\|\cdot\|_T$ be the norm in C_0 defined by
$$\|z\|_T = \|z_0\|_{\mathcal{B}} + \sup_{t \in I} |z(t)| = \sup_{t \in I} |z(t)|; \ z \in C_0.$$
C_0 is a Banach space with norm $\|\cdot\|_T$. Define the operator $P : C_0 \to C_0$; by
$$(Pz)(t) = -a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds, \tag{4.4}$$

where

$$g(t) = f(t, \overline{z}_t + x_t, g(t));\ t \in I.$$

We shall show that $P : C_0 \to C_0$ is a contraction map. Let $z, z' \in C_0$, then we have for each $t \in I$

$$|P(z)(t) - P(z')(t)| \quad \leq \quad a_r|g(0) - h(0)| + a_r|g(t) - h(t)| + b_r \int_0^t |g(s) - h(s)|ds, \tag{4.5}$$

where $g,\ h \in C(I)$ such that

$$g(t) = f(t, \overline{z}_t + x_t, g(t)) \quad and \quad h(t) = f(t, \overline{z'}_t + x_t, h(t)).$$

Since, for each $t \in I$, we have

$$|g(t) - h(t)| \leq \frac{b_1}{1 - b_2}\|\overline{z}_t - \overline{z'}_t\|_{\mathcal{B}}.$$

Then, for each $t \in I$; we get

$$\begin{array}{rcl} |P(z)(t) - P(z')(t)| & \leq & (2a_r + b_r \int_0^t ds)\frac{b_1}{1-b_2}\|\overline{z}_t - \overline{z'}_t\|_{\mathcal{B}} \\ & \leq & (2a_r + Tb_r)\frac{b_1}{1-b_2}\|\overline{z}_t - \overline{z'}_t\|_{\mathcal{B}} \\ & = & \lambda\|\overline{z} - \overline{z'}\|_T. \end{array}$$

Thus, we get

$$\|P(z)(t) - P(z')(t)\|_T \leq \lambda\|\overline{z} - \overline{z'}\|_T.$$

Hence, from the Banach contraction principle, the operator P has a unique fixed point. Consequently, N has a unique fixed point which is the unique solution of problem (1.2).

Now, we prove an existence result by using Schaefer's fixed point theorem.

Theorem 4.3. *Assume that the hypotheses* (H_{01}) *and* H_{02} *hold. Then problem (1.2) has at least one solution on* $(-\infty, T]$.

Proof. Let $P : C_0 \to C_0$ defined as in (4.4), For each given $R > 0$, we define the ball

$$B_R = \{x \in C_0,\ \|x\|_T \leq R\}.$$

Step 1. N is continuous.
Let z_n be a sequence such that $z_n \to z$ in C_0. For each $t \in I$, we have

$$\begin{array}{rl} |(Pz_n)(t) - (Pz)(t)| & \leq a_r|g_n(0) - g(0)| + a_r|g_n(t) - g(t)| \\ & +b_r \int_0^t |g_n(s) - g(s)|ds, \end{array} \tag{4.6}$$

where $g_n,\ g \in C(I)$ such that

$$g_n(t) = f(t, \overline{z}_{nt} + x_t, g_n(t)) \quad and \quad g(t) = f(t, \overline{z}_t + x_t, g(t)).$$

Since $\|z_n - z\|_T \to 0$ as $n \to \infty$ and f, g and g_n are continuous, then

$$\|P(\wp_n) - P(\wp)\|_T \to 0 \quad \text{as } n \to \infty.$$

Hence, P is continuous.

Step 2. $P(B_R)$ is bounded.
Let $z \in B_R$, for each $t \in I$, we have

$$\begin{array}{rcl} |g(t)| & \leq & |f(t, \overline{z}_t + x_t, g(t))| \\ & \leq & |f(t,0,0)| + b_1\|\overline{z}_t + x_t\|_{\mathcal{B}} + b_2|g(t)| \\ & \leq & f^* + b_1\left[\|\overline{z}_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}}\right] + b_2\|g\|_\infty \\ & \leq & f^* + b_1 MR + b_1 K\|\zeta\|_{\mathcal{B}} + b_2\|g\|_\infty. \end{array}$$

Then

$$\|g\|_\infty \leq \frac{f^* + b_1 MR + b_1 K\|\zeta\|_{\mathcal{B}}}{1 - b_2}.$$

Thus,

$$\begin{array}{rcl} |(Pz)(t)| & \leq & a_r|g(0)| + a_r|g(t)| + b_r\int_0^t |g(s)|ds \\ & \leq & (2a_r + b_r\int_0^t ds)\frac{f^*+b_1MR+b_1K\|\zeta\|_{\mathcal{B}}}{1-b_2} \\ & \leq & (2a_r + Tb_r)\frac{f^*+b_1MR+b_1K\|\zeta\|_{\mathcal{B}}}{1-b_2} \\ & := & \ell. \end{array}$$

Hence

$$\|P(z)\|_T \leq \ell.$$

Consequently, P maps bounded sets into bounded sets in C_0.
Step 3. $P(B_R)$ is equicontinuous.
For $1 \leq t_1 \leq t_2 \leq T$, and $z \in B_R$, we have

$$\begin{array}{rcl} |P(z)(t1) - P(z)(t2)| & \leq & a_r|g(t_2) - g(t_1)| + b_r\int_{t_1}^{t_2} |g(s)|ds \\ & \leq & a_r|g(t_2) - g(t_1)| + b_r(t_2 - t_1)\frac{f^*+b_1MR+b_1K\|\zeta\|_{\mathcal{B}}}{1-b_2}. \end{array}$$

By (H_{02}), as $t_2 \to t_1$ the right-hand side of the above inequality tends to zero, we conclude that P maps bounded sets into equicontinuous sets in C_0.
Step 4. The priori bounds.
We prove that the set

$$\mathcal{E} = \{\wp \in C_0 : \Im = \lambda P(\wp);\ for\ some\ \lambda \in (0,1)\}$$

is bounded. Let $z \in C_0$. Let $u \in C_0$, such that $z = \lambda P(z);\ for\ some\ \lambda \in (0,1)$. Then for each $t \in I$, we have

$$z(t) = \lambda(Pz)(t) = \lambda\zeta(0) + \lambda a_r(g(t) - g(0)) + \lambda b_r\int_0^t g(s)ds.$$

From (H_{01}) we have

$$\begin{array}{rcl} |g(t)| & \leq & |f(t, \overline{z}_t + x_t, g(t))| \\ & \leq & f^* + b_1\|\overline{z}_t + x_t\|_{\mathcal{B}} + b_2|g(t)| \\ & \leq & f^* + b_1\left[\|\overline{z}_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}}\right] + b_2\|g\|_\infty \\ & \leq & f^* + b_1 M\|z\|_T + b_1 K\|\zeta\|_{\mathcal{B}} + b_2\|g\|_\infty. \end{array}$$

This gives,

$$\|g\|_\infty \leq \frac{f^* + b_1 M\|z\|_T + b_1 K\|\zeta\|_{\mathcal{B}}}{1 - b_2} := \eta.$$

Thus, for each $t \in I$, we obtain

$$\begin{array}{rcl} |z(t)| & \leq & |\zeta(0)| + a_r|g(0)| + a_r g(t) + b_r \int_0^t |g(s)|ds \\ & \leq & |\zeta(0)| + \eta(2a_r + Tb_r) \\ & := & \eta'. \end{array}$$

Hence

$$\|z\|_T \leq \eta'.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Schaefer's theorem [12], the operator N has a fixed point which is a solution of problem (1.2).

5. Existence results with state-dependent delay

5.1. The finite delay case

In this section, we establish the existence results for problem (1.3).

Definition 5.1. By a solution of problem (1.3), we mean a continuous function $\wp \in C$ such that

$$\wp(t) = \begin{cases} \zeta(t); \ t \in [-h, 0], \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds; \ t \in I, \end{cases}$$

where $g \in C(I)$ with $g(t) = f(t, \wp_{\rho(t,\wp_t)}, g(t))$.

- (H_4) The function f satisfies the Lipschitz condition:

  $$|f(t, \wp_1, \Im_1) - f(t, \wp_2, \Im_2)| \leq \omega_3\|\wp_1 - \wp_2\|_{[-h,0]} + \omega_4|\Im_1 - \Im_2|,$$

  for any $\wp_1, \Im_1 \in \mathcal{C}, \ \wp_2, \Im_2 \in \mathbb{R}$, and each $t \in I$,
  where $\omega_3 > 0, \ 0 < \omega_4 < 1$.
- (H_5) For any bounded set $B_2 \subset \mathcal{C}$, the set:

  $$\{t \mapsto f(t, \wp_t, ({}^{CF}D_0^r\wp)(t)) : \wp \in B_2\};$$

  is equicontinuous in C.

As in Theorems 3.2 and 3.3, we give without prove, the following results:

Theorem 5.2. *Assume that the hypothesis* (H_4) *holds. If*

$$(2a_r + Tb_r)\frac{\omega_3}{1 - \omega_4} < 1,$$

then problem (1.2) has a unique solution on $[-h, T]$.

Theorem 5.3. *Assume that the hypotheses* (H_4) *and* (H_5) *hold. If*

$$\frac{a_1}{1 - a_2}(2a_r + Tb_r) < 1,$$

then problem (1.3) has at least one solution on $[-h, T]$.

5.2. The infinite delay case

Now, we establish the last problem (1.4).

Definition 5.4. By a solution of problem (1.4), we mean a continuous $\wp \in \Omega$

$$\wp(t) = \begin{cases} \zeta(t);\ t \in \mathbb{R}_-, \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds;\ t \in I, \end{cases} \tag{5.1}$$

where $g \in C(I)$ such that $g(t) = f(t, \wp_{\rho(t,\wp_t)}, g(t))$.

Set

$$R' := R'_{\rho^-} = \{\rho(t,\wp) : t \in I,\ \wp \in \mathcal{B}\ \rho(t,\wp) < 0\}$$

We always assume that $\rho : I \times \mathcal{B} \to \mathbb{R}$ is continuous and the function $t \to \wp_t$ is continuous from R' into $\mathcal{B}$. We will need the following hypothesis:
(H_ζ) There exists a continuous bounded function $L : R'_{\rho^-} \to (0, \infty)$ such that

$$\|\zeta_t\|_{\mathcal{B}} \leq L(t)\|\zeta\|_{\mathcal{B}},\ \textit{for any}\ t \in R'.$$

Lemma 5.5. *If $\wp \in \Omega$ then*

$$\|\wp_t\|_{\mathcal{B}} = (M + L')\|\zeta\|_{\mathcal{B}} + K \sup_{\theta \in [0, \max\{0,t\}]} \|\wp(\theta)\|,$$

where

$$L' = \sup_{t \in R'} L(t).$$

- (H_{04}) The function f satisfies the Lipschitz condition:

  $$|f(t, \wp_1, \Im_1) - f(t, \wp_2, \Im_2)| \leq b_3\|\wp_1 - \wp_2\|_{\mathcal{B}} + b_4|\Im_1 - \Im_2|,$$

  for any $\wp_1, \Im_1 \in \mathcal{B}$, $\wp_2, \Im_2 \in \mathbb{R}$, and each $t \in I$, where $b_3 > 0$ and $0 < b_4 < 1$.
- (H_{05}) For any bounded set $B_2 \subset \Omega$, the set:

  $$\{t \mapsto f(t, \wp_t, ({}^{CF}D_0^r \wp)(t)) : u \in B_2\};$$

  is equicontinuous in Ω.

As in Theorems 4.2 and 4.3, we give without prove, the following results:

Theorem 5.6. *Assume that the hypothesis (H_{04}) holds. If*

$$(2a_r + Tb_r)\frac{b_3}{1 - b_4} < 1,$$

then problem (1.4) has a unique solution on $(-\infty, T]$.

Theorem 5.7. *Assume that the hypotheses (H_ζ), (H_{04}) and (H_{05}) hold. Then problem (1.4) has at least one solution on $(-\infty, T]$.*

6. Some examples

Example 6.1. Consider the following problem

$$\begin{cases} \wp(t) = 1 + t^2; \ t \in [-1, 0], \\ ({}^{CF}D_0^{1/2}\wp)(t) = \frac{\varsigma}{90(1+\|\wp_t\|)} + \frac{1}{30\left(1+|({}^{CF}D_0^{1/2}\wp(t))|\right)}; \ t \in [0, 2], \end{cases} \tag{6.1}$$

where $\varsigma < \frac{87}{2a_{\frac{1}{2}} + 2b_{\frac{1}{2}}}$.

Set

$$f(t, \wp, \Im) = \frac{\varsigma}{90\,(1 + \|\wp\|)} + \frac{1}{30\,(1 + |\Im|)}; \ t \in [1, e], \ \wp \in \mathcal{C}, \ \Im \in \mathbb{R}.$$

Clearly, the function f is continuous. For any $\wp, \widetilde{\wp} \in \mathcal{C}$, $\wp, \widetilde{\wp} \in \mathbb{R}$, and $t \in [0, 2]$, we have

$$|f(t, \wp, \Im) - f(t, \widetilde{\wp}, \widetilde{\Im})| \leq \frac{\varsigma}{90}\|\wp - \widetilde{\wp}\|_{[-1,0]} + \frac{1}{30}|\Im - \widetilde{\Im}|.$$

Hence hypothesis (H_1) is satisfied with

$$\omega_1 = \frac{\varsigma}{90} \quad and \quad \omega_2 = \frac{1}{30}.$$

Next, condition (3.1) is satisfied with $T = 2$ and $r = \frac{1}{2}$. Indeed,

$$\begin{aligned} \tfrac{\omega_1(2a_r + Tb_r)}{1-\omega_2} &= \tfrac{\varsigma\left(2a_{\frac{1}{2}} + 2b_{\frac{1}{2}}\right)}{87} \\ &< 1. \end{aligned}$$

Theorem 3.2 implies that problem (6.1) has a unique solution defined on $[-1, 2]$.

Example 6.2. Consider now the following problem

$$\begin{cases} \wp(t) = t; \ t \in \mathbb{R}_-, \\ ({}^{CF}D_0^{2/3}\wp)(t) = \frac{\wp_t e^{-\gamma t + t}}{180(e^t - e^{-t})(1+\|\wp_t\|)} + \frac{\wp(t)e^{-\gamma t + t}}{60(e^t - e^{-t})\left(1+|({}^{CF}D_0^{2/3}\wp(t))|\right)}; \ t \in [0, 1]. \end{cases} \tag{6.2}$$

Let γ be a positive real constant and

$$B_\gamma = \{\wp \in C((-\infty, 1], \mathbb{R},) : \lim_{\theta \to -\infty} e^{\gamma\theta\theta}\wp(\theta) \ exists \ in \ \mathbb{R}\}. \tag{6.3}$$

The norm of B_γ is given by

$$\|\wp\|_\gamma = \sup_{\theta \in (-\infty, 1]} e^{\gamma\theta}|\wp(\theta)|.$$

Let $\wp : \mathbb{R}_- \to \mathbb{R}$ be such that $\wp_0 \in B_\gamma$. Then

$$\begin{aligned} \lim_{\theta \to -\infty} e^{\gamma\theta}\wp_t(\theta) &= \lim_{\theta \to -\infty} e^{\gamma\theta}\wp(t + \theta - 1) = \lim_{\theta \to -\infty} e^{\gamma(\theta - t + 1)}\wp(\theta) \\ &= e^{\gamma(-t+1)} \lim_{\theta \to -\infty} e^{\gamma(\theta)}\wp_1(\theta) < \infty. \end{aligned}$$

Hence $\wp_t \in B_\gamma$. Finally we prove that

$$\|\wp_t\|_\gamma \leq K\|\wp_1\|_\gamma + M \sup_{s \in [0,t]} |\wp(s)|,$$

where $K = M = 1$ and $H = 1$. We have

$$\|\wp_t(\theta)\| = |\wp(t+\theta)|.$$

If $t+\theta \leq 1$, we get

$$\|\wp_t(\beta)\| \leq \sup_{s\in\mathbb{R}_-} |\wp(s)|.$$

For $t+\theta \geq 0$, then we have

$$\|\wp_t(\beta)\| \leq \sup_{s\in[0,t]} |\wp(s)|.$$

Thus for all $t+\theta \in I$, we get

$$\|\wp_t(\beta)\| \leq \sup_{s\in\mathbb{R}_-} |\wp(s)| + \sup_{s\in[0,t]} |\wp(s)|.$$

Then

$$\|\wp_t\|_\gamma \leq \|\wp_0\|_\gamma + \sup_{s\in[0,t]} |\wp(s)|.$$

It is clear that $(B_\gamma, \|\cdot\|)$ is a Banach space. We can conclude that B_γ a phase space. Set

$$f(t,\wp,\Im) = \frac{e^{-\gamma t+t}}{180\,(e^t - e^{-t})\left(1+\|\wp\|_{B_\gamma}\right)} + \frac{e^{-\gamma t+t}}{60\,(e^t - e^{-t})\,(1+|\Im|)};$$
$$t \in [0,1],\ \ \wp \in B_\gamma, \Im \in \mathbb{R}.$$

We can verify that the hypothesis (H_{01}) is satisfied with

$$B_1 = \frac{1}{180} \quad \text{and} \quad B_2 = \frac{1}{60}.$$

Theorem 4.3 ensures that problem (6.2) has a solution defined on $(-\infty, 1]$.

Example 6.3. We consider the following problem

$$\begin{cases} \wp(t) = 1 + t^2;\ t \in [-1,0], \\ ({}^{CF}D_0^{1/2}\wp)(t) = \frac{1}{90(1+|\wp(t-\sigma(\wp(t)))|)} + \frac{1}{30\left(1+|({}^{CF}D_0^{1/2}\wp(t))|\right)};\ t\in[0,1], \end{cases} \tag{6.4}$$

where $\sigma \in C(\mathbb{R},[0,1])$. Set

$$\rho(t,\zeta) = t - \sigma(\zeta(0)), \quad (t,\zeta) \in [0,e] \times C([-1,0],\mathbb{R}),$$

$$f(t,\wp,\Im) = \frac{1}{90(1+|\wp(t-\sigma(\wp(t)))|)} + \frac{1}{30\,(1+|\Im(t)|)};\ t \in [1,e],\ \wp \in \mathcal{C},\ \Im \in \mathbb{R}.$$

Clearly, the function f is jointly continuous. For any $\wp, \widetilde{\wp} \in \mathcal{C},\ \Im, \widetilde{\Im} \in \mathbb{R}$ and $t \in [0,1]$, we have

$$|f(t,\wp,\Im) - f(t,\widetilde{\wp},\widetilde{\Im})| \leq \frac{1}{90}\|\wp - \widetilde{\wp}\|_{[-1,0]} + \frac{1}{30}|\Im - \widetilde{\Im}|.$$

Hence hypothesis (H_{04}) is satisfied with

$$\omega_3 = \frac{1}{90} \quad \textit{and} \quad \omega_4 = \frac{1}{30}.$$

From Theorem 5.2, problem (6.4) has a unique solution on $[-1,1]$.

Example 6.4. Consider now the problem

$$\begin{cases} \wp(t) = t^2;\ t \in \mathbb{R}_-, \\ ({}^{CF}D_0^{1/4}\wp)(t) = \frac{\wp(t-\lambda(\wp(t)))e^{-\gamma t+t}}{180(e^t-e^{-t})(1+|\wp(t-\sigma(\wp(t))|)} + \frac{\wp(t)e^{-\gamma t+t}}{60(e^t-e^{-t})\left(1+|({}^{CF}D_0^{1/4}\wp(t))|\right)};\ t \in [0,3]. \end{cases} \tag{6.5}$$

Let γ be a positive real constant and the phase space B_γ defined in Example 6.2. Define

$$\rho(t,\zeta) = t - \lambda(\zeta(0)), \quad (t,\zeta) \in [0,3] \times B_\gamma,$$

and set

$$f(t,\wp,\Im) = \frac{e^{-\gamma t+t}}{180\left(e^t - e^{-t}\right)\left(1 + \|\wp\|_{B_\gamma}\right)} + \frac{e^{-\gamma t+t}}{60\left(e^t - e^{-t}\right)\left(1 + |\Im|\right)};$$

$$t \in [0,3],\ \wp \in B_\gamma,\ \Im \in \mathbb{R}.$$

By Theorem 4.3, problem (6.5) has a solution defined on $(-\infty, 3]$.

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Salim Krim
Laboratory of Mathematics, University of Saïda–Dr. Moulay Tahar,
P.O. Box 138, EN-Nasr, 20000 Saïda, Algeria
e-mail: salimsalimkrim@gmail.com, salim.krim@univ-saida.dz

Saïd Abbas
Department of Mathematics, University of Saïda–Dr. Moulay Tahar,
P.O. Box 138, EN-Nasr, 20000 Saïda, Algeria
e-mail: abbasmsaid@yahoo.fr, abbas.said@univ-saida.dz

Mouffak Benchohra
Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès,
P.O. Box 89, Sidi Bel-Abbès 22000, Algeria
e-mail: benchohra@yahoo.com

Juan J. Nieto
Departamento de Estatistica, Análise Matemática e Optimización,
Instituto de Matemáticas, Universidade de Santiago de Compostela,
Santiago de Compostela, Spain
e-mail: juanjose.nieto.roig@usc.es

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Radius problems for certain classes of analytic functions

Yao Liang Chung, Maisarah Haji Mohd and Shamani Supramaniam

Abstract. Radius constants for functions in three classes of analytic functions to be a starlike function of order α, parabolic starlike function, starlike function associated with lemniscate of Bernoulli, exponential function, cardioid, sine function, lune, a particular rational function, and reverse lemniscate are obtained. One of these classes are characterized by the condition $\operatorname{Re} g/(ze^z) > 0$. The other two classes are defined by using the function g and they consist respectively of functions f satisfying $\operatorname{Re} f/g > 0$ and $|f/g - 1| < 1$.

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1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions f in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with normalization $f(0) = 0$ and $f'(0) = 1$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. Let $\mathcal{P}$ be the class of functions with positive real part consisting of all analytic functions $p : \mathbb{D} \to \mathbb{C}$ satisfying $p(0) = 1$ and $\operatorname{Re}(p(z)) > 0$. For $0 \leq \alpha < 1$, let $\mathcal{S}^*(\alpha)$ be the subclasses of $\mathcal{S}$ consisting of starlike functions of order α. Analytically, we have $f \in \mathcal{S}^*(\alpha)$ if and only if $\operatorname{Re}(zf'(z)/f(z)) > \alpha$. For $\alpha = 0$, we have $\mathcal{S}^*(0) := \mathcal{S}^*$ which is the starlike functions. For analytic functions f and g on $\mathbb{D}$, we say that f is subordinate to g, denoted $f \prec g$, if there exists a Schwarz function ω in $\mathbb{D}$ such that $f(z) = g(\omega(z)), z \in \mathbb{D}$. Several subclasses of starlike functions defined by subordination were discussed in the literature. We shall be interested in the following classes:

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- $\mathcal{S}_L^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{1+z},\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}_p^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2,\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}_e^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec e^z,\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}_c^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2,\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}_{\sin}^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \sin z,\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}_m^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec z + \sqrt{1+z^2},\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}_R^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{z}{k} \left(\frac{k+z}{k-z} \right),\ k = \sqrt{2}+1,\ z \in \mathbb{D} \right\}$,
- $\mathcal{S}_{RL}^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}},\ z \in \mathbb{D} \right\}$.

For more information on the subclasses, refer [1, 2, 4, 6, 10, 11, 12, 13, 17, 18].

The radius problems is an important area of study in geometric function theory (see [1, 9]). Let F and G be two subclasses of $\mathcal{A}$. If for every $f \in F, r^{-1}f(rz) \in G$ for $r \leq r_0$, and r_0 is the largest number for which this holds, then r_0 is the G radius (or the radius of the property connected to G) in F. For example, the radius of starlikeness for the class $\mathcal{S}$ is $\tanh(\pi/4)$. Recently, Asha and Ravichandran [14] consider some analytic functions and obtained the radii for these functions to belong to various subclasses of starlike functions. See also [3, 5, 7, 8]. Motivated by the aforementioned works, three subclasses of analytic functions are introduced below:

$$\begin{aligned}
E_1 &= \{f \in \mathcal{A} :\ f/g \in \mathcal{P} \text{ for some } g \in \mathcal{A} \text{ with } g/(ze^z) \in \mathcal{P}\},\\
E_2 &= \{f \in \mathcal{A} :\ |f/g - 1| < 1 \text{ for some } g \in \mathcal{A} \text{ with } g/(ze^z) \in \mathcal{P}\},\\
E_3 &= \{f \in \mathcal{A} :\ f/(ze^z) \in \mathcal{P}\}.
\end{aligned}$$

The main objective of the paper is to compute radius constants of the above functions for several subclasses of $\mathcal{A}$ such as starlike functions of order α, parabolic starlike functions, starlike functions associated with lemniscate of Bernoulli, exponential function, cardioid, sine function, lune, a particular rational function, and reverse lemniscate.

2. Main results

Our first theorem gives several radius results for the class E_1. Recall that E_1 is defined by

$$E_1 = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{f(z)}{g(z)} > 0 \text{ for some } g \in \mathcal{A} \text{ with } \operatorname{Re} \frac{g(z)}{ze^z} > 0,\ z \in \mathbb{D} \right\}.$$

The function $f_1 : \mathbb{D} \to \mathbb{C}$ defined by

$$f_1(z) = \left(\frac{1+z}{1-z} \right)^2 ze^z \tag{2.1}$$

belongs to E_1 and acts as an extremal function.

Theorem 2.1. *For the class E_1, the following results hold:*

(i) For $0 \leq \alpha < 1$, the $\mathcal{S}^*_\alpha$ radius is the smallest positive real root of the equation

$$r^3 - \alpha r^2 - 5r + \alpha = 0.$$

(ii) The $\mathcal{S}^*_L$-radius is the smallest positive real root of the equation

$$r^3 + (1-\sqrt{2})r^2 - 5r + \sqrt{2} - 1 = 0, \ \ i.e.\ R_{\mathcal{S}^*_L} \approx 0.0824.$$

(iii) The $\mathcal{S}^*_p$-radius is the smallest positive real root of the equation

$$2r^3 - r^2 - 10r + 1 = 0 \ \ i.e.\ R_{\mathcal{S}^*_p} \approx 0.09921.$$

(iv) The $\mathcal{S}^*_e$-radius is the smallest positive root of the equation

$$er^3 + (1-e)r^2 - 5er + e - 1 = 0 \ \ i.e.\ R_{\mathcal{S}^*_e} \approx 0.1248.$$

(v) The $\mathcal{S}^*_c$-radius is the smallest positive root of the equation

$$3r^3 - 2r^2 - 15r + 2 = 0 \ \ i.e.\ R_{\mathcal{S}^*_c} \approx 0.13148.$$

(vi) The $\mathcal{S}^*_{\sin}$-radius is the smallest positive root of the equation

$$r^3 - r^2 \sin 1 - 5r + \sin 1 = 0 \ \ i.e.\ R_{\mathcal{S}^*_{\sin}} \approx 0.1646.$$

(vii) The $\mathcal{S}^*_m$-radius is the smallest positive root of the equation

$$r^3 - r^2(2-\sqrt{2}) - 5r + 2 - \sqrt{2} = 0 \ \ i.e.\ R_{\mathcal{S}^*_m} \approx 0.1159.$$

(viii) The $\mathcal{S}^*_R$-radius is the smallest positive root of the equation

$$r^3 - r^2(2-2\sqrt{2}) - 5r + 3 - 2\sqrt{2} = 0 \ \ i.e.\ R_{\mathcal{S}^*_R} \approx 0.0345.$$

(ix) The $\mathcal{S}^*_{RL}$-radius is $R_{\mathcal{S}^*_{RL}}$ which is root of the equation

$$\frac{(5r-r^3)^2}{(1-r^2)^2} = (1-(\sqrt{2}-(1+r^2)/(1-r^2))^2)^{1/2} - (1-(\sqrt{2}-(1+r^2)/(1-r^2))^2).$$

Proof. Let $f \in E_1$ and $g : \mathbb{D} \to \mathbb{C}$ be chosen such that

$$\operatorname{Re} \frac{f(z)}{g(z)} > 0 \quad \text{and} \quad \operatorname{Re} \frac{g(z)}{ze^z} > 0 \quad \text{for all } z \in \mathbb{D}. \tag{2.2}$$

Define the functions $p_1, p_2 : \mathbb{D} \to \mathbb{C}$ by

$$p_1(z) = \frac{f(z)}{g(z)} \quad \text{and} \quad p_2(z) = \frac{g(z)}{ze^z}. \tag{2.3}$$

By equations (2.2) and (2.3), we have p_1 and p_2 are in $\mathcal{P}$. Also, equation (2.3) yields

$$f(z) = ze^z p_1(z) p_2(z).$$

Further computations then yields

$$\frac{zf'(z)}{f(z)} = 1 + z + \frac{zp_1'(z)}{p_1(z)} + \frac{zp_2'(z)}{p_2(z)}. \tag{2.4}$$

For $p \in \mathcal{P}(\alpha) := \{p \in \mathcal{P} : \operatorname{Re}(p(z)) > \alpha,\ z \in \mathbb{D}\}$, by [15, Lemma 2], we have

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2(1-\alpha)r}{(1-r)(1+(1-2\alpha)r)},\ |z| \leq r. \tag{2.5}$$

By using (2.4) and setting $\alpha = 0$ in (2.5), we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{5r - r^3}{1-r^2}. \tag{2.6}$$

Hence, by (2.6), we have

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \frac{1-5r-r^2+r^3}{1-r^2} \geq 0.$$

Thus the function $f \in E_1$ is starlike in $|z| \leq 0.1939$. Hence, all the radius estimate here will be less than 0.1939.

(i) The function $m(r) = (1-5r-r^2+r^3)(1-r^2)^{-1},\ 0 \leq r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}^*(\alpha)}$ be the smallest positive root of the equation $m(r) = \alpha$. From (2.6), it follows that

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \frac{1-5r-r^2+r^3}{1-r^2} = m(r) \geq m(\varrho) = \alpha.$$

This shows that $R_{\mathcal{S}^*(\alpha)}$ is at least ϱ. At $z = R_{\mathcal{S}^*(\alpha)} = \varrho$, the function f_1 defined in (2.1) satisfies

$$\operatorname{Re}\frac{zf_1'(z)}{f_1(z)} = \frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2} = \alpha.$$

Thus the radius is sharp.

(ii) The function $m(r) = (5r-r^3)(1-r^2)^{-1}+1,\ 0 \leq r < 1$ is an increasing function. Let $\varrho = R_{\mathcal{S}^*_L}$ be the root of the equation $m(r) = \sqrt{2}$. For $0 < r \leq R_{\mathcal{S}^*_L}$, we have $m(r) \leq \sqrt{2}$. That is,

$$\frac{5r-r^3}{1-r^2} + 1 \leq \sqrt{2} = m(\varrho).$$

For the class E_1, the centre of the disc in (2.6) is 1. Using [1, Lemma 2.2], the disc obtained in (2.6) is contained in the region bounded by lemniscate. For the function f_1 defined in (2.1), at $z = R_{\mathcal{S}^*_L} = -\rho$,

$$\left|\left(\frac{zf_1'(z)}{f_1(z)}\right)^2 - 1\right| = \left|\left(\frac{1+5\rho-\rho^2+\rho^3}{1-\rho^2}\right)^2 - 1\right| = \left|(\sqrt{2})^2 - 1\right| = 1.$$

(iii) The function $m(r) = (1-5r-r^2+r^3)(1-r^2)^{-1},\ 0 \leq r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}^*_p}$ be the root of the equation $m(r) = 1/2$. For $0 < r \leq R_{\mathcal{S}^*_p}$, we have $m(r) \geq 1/2$. That is,

$$\frac{5r-r^3}{1-r^2} \leq \frac{1}{2} = m(\rho).$$

Using [16, Lemma 1], we see that the disc obtained in (2.6) is contained in the region bounded by parabola. For the function f_1 defined in (2.1), at $z = R_{\mathcal{S}^*_p} = \rho$,

$$\operatorname{Re} \frac{zf_1'(z)}{f_1(z)} = \frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2} = \frac{1}{2} = \left| \frac{zf_1'(z)}{f_1(z)} - 1 \right|.$$

(iv) The function $m(r) = (1-5r-r^2+r^3)(1-r^2)^{-1}$, $0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}^*_e}$ be the root of the equation $m(r) = 1/e$. For $0 < r \le R_{\mathcal{S}^*_e}$, we have $m(r) \ge 1/e$. That is,

$$\frac{5r-r^3}{1-r^2} \le 1 - \frac{1}{e}.$$

Using [11, Lemma 2.2], the disc obtained in (2.6) is contained in the region bounded by exponential function. For the function f_1 defined in (2.1), at $z = R_{\mathcal{S}^*_e} = \rho$,

$$\left| \log \frac{zf_1'(z)}{f_1(z)} \right| = \left| \log \frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2} \right| = 1.$$

(v) The function $m(r) = (1-5r-r^2+r^3)(1-r^2)^{-1}$, $0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}^*_c}$ be the root of the equation $m(r) = 1/3$. For $0 < r \le R_{\mathcal{S}^*_c}$, we have $m(r) \ge 1/3$. That is,

$$\frac{5r-r^3}{1-r^2} \le 1 - \frac{1}{3}.$$

Using [17, Lemma 2.5], the disc obtained in (2.6) is contained in the region bounded by the cardioid. For the function f_1 defined in (2.1), at $z = R_{\mathcal{S}^*_c} = \rho$,

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2} = \frac{1}{3} = h_c(-1),$$

where $h_c(z) = 1 + (4/3)z + (2/3)z^2$ is the superordinate function in the class $\mathcal{S}^*_c$.

(vi) The function $m(r) = (1-5r-r^2+r^3)(1-r^2)^{-1}$, $0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}^*_{sin}}$ be the root of the equation $m(r) = 1 - \sin 1$. For $0 < r \le R_{\mathcal{S}^*_{sin}}$, we have $m(r) \ge 1 - \sin 1$. That is,

$$\frac{5r-r^3}{1-r^2} \le \sin 1.$$

Using [2, Lemma 3.3], the disc obtained in (2.6) is contained in the region Ω_s bounded by the sine function. For the function f_1 defined in (2.1), at $z = -R_{\mathcal{S}^*_{sin}} = -\rho$,

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2} = 1 + \sin 1 = h_s(1),$$

where $h_s(z) = 1 + \sin z$ is the superordinate function in the class $\mathcal{S}^*_{sin}$.

(vii) The function $m(r) = (1-5r-r^2+r^3)(1-r^2)^{-1},\ 0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}^*_m}$ be the root of the equation $m(r) = \sqrt{2}-1$. For $0 < r \le R_{\mathcal{S}^*_m}$, we have $m(r) \ge \sqrt{2}-1$. That is,

$$\frac{5r-r^3}{1-r^2} \le 2-\sqrt{2}.$$

Using [4, Lemma 2.1], the disc obtained in (2.6) is contained in the region bounded by the intersection of disk $\{w : |w-1| < \sqrt{2}\}$ and $\{w : |w+1| < \sqrt{2}\}$. For the function f_1 defined in (2.1), at $z = -R_{\mathcal{S}^*_m} = -\rho$,

$$\left|\left(\frac{zf_1'(z)}{f_1(z)}\right)^2 - 1\right| = \left|\left(\frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2}\right)^2 - 1\right| = 2\left|\frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2}\right|.$$

(viii) The function $m(r) = (1-5r-r^2+r^3)(1-r^2)^{-1},\ 0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}^*_R}$ be the root of the equation $m(r) = 2(\sqrt{2}-1)$. For $0 < r \le R_{\mathcal{S}^*_R}$, we have $m(r) \ge 2(\sqrt{2}-1)$. That is,

$$\frac{5r-r^3}{1-r^2} \le 1-2(\sqrt{2}-1).$$

Using [6, Lemma 2.2], the disc obtained in (2.6) is contained in the region bounded by the rational function. For the function f_1 defined in (2.1), at $z = -R_{\mathcal{S}^*_R} = -\rho$,

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2} = 2(\sqrt{2}-1) = h_R(-1)$$

where $h_R(z) = 1+(zk+z^2)/(k^2-kz)$, $k = 1+\sqrt{2}$ is the superordinate function in the class $\mathcal{S}^*_R$.

(ix) The function $m(r) = ((5r-r^3)(1-r^2)^{-1})+1,\ 0 \le r < 1$ is an increasing function. Let $\varrho = R_{\mathcal{S}^*_{RL}}$ be the root of the equation

$$m(r) = ((1-(\sqrt{2}-1)^2)^{1/2} - (1-(\sqrt{2}-1)^2))^{1/2}.$$

Using [10, Lemma 3.2], the disc obtained in (2.6) is contained in the region

$$\{w : |(w-\sqrt{2})^2 - 1| < 1\}.$$

For the function f_1 defined in (2.1), at $z = -R_{\mathcal{S}^*_{RL}} = -\rho$,

$$\left|\left(\frac{zf_1'(z)}{f_1(z)}\right)^2 - 1\right| = \left|\left(\frac{1-5\rho-\rho^2+\rho^3}{1-\rho^2} - \sqrt{2}\right)^2 - 1\right| = 1. \qquad \square$$

Recall that the class E_2 was defined by

$$E_2 = \left\{ f \in \mathcal{A} : \left|\frac{f(z)}{g(z)} - 1\right| < 1 \text{ for some } g \in \mathcal{A} \text{ with } \operatorname{Re}\frac{g(z)}{ze^z} > 0,\ z \in \mathbb{D} \right\}.$$

The function f_2 defined by

$$f_2(z) = \frac{(1+z)^2}{1-z} ze^z \tag{2.7}$$

belongs to the class E_2 and is an extremal function.

Theorem 2.2. *For the class E_2, the following results hold:*

(i) For $0 \leq \alpha < 1$, the $\mathcal{S}^*_\alpha$-radius is the smallest positive real root of the equation

$$r^3 - (\alpha+1)r^2 - 4r + \alpha = 0.$$

(ii) The $\mathcal{S}^*_L$-radius is the smallest positive root of the equation

$$r^3 + r^2(2-\sqrt{2}) - 4r + \sqrt{2} - 1 = 0 \text{ i.e. } R_{\mathcal{S}^*_L} \approx 0.1055.$$

(iii) The $\mathcal{S}^*_p$-radius is the smallest positive root of the equation

$$2r^3 - 3r^2 - 8r + 1 = 0 \text{ i.e. } R_{\mathcal{S}^*_p} \approx 0.1200.$$

(iv) The $\mathcal{S}^*_e$-radius is the smallest positive root of the equation

$$er^3 + r^2(1-2e) - 4er + e - 1 = 0 \text{ i.e. } R_{\mathcal{S}^*_e} \approx 0.1497.$$

(v) The $\mathcal{S}^*_C$-radius is the smallest positive root of the equation

$$3r^3 - 5r^2 - 12r + 2 = 0 \text{ i.e. } R_{\mathcal{S}^*_C} \approx 0.1573.$$

(vi) The $\mathcal{S}^*_{\sin}$-radius the smallest positive root of the equation

$$r^3 - r^2 \sin 1 - 5r + \sin 1 = 0 \text{ i.e. } R_{\mathcal{S}^*_{\sin}} \approx 0.00349.$$

(vii) The $\mathcal{S}^*_m$-radius is the smallest positive root of the equation

$$r^3 - r^2(3-\sqrt{2}) - 4r + 2 - \sqrt{2} = 0 \text{ i.e. } R_{\mathcal{S}^*_m} \approx 0.1394.$$

(viii) The $\mathcal{S}^*_R$-radius is the smallest positive root of the equation

$$r^3 - r^2(3-2\sqrt{2}) - 4r + 3 - 2\sqrt{2} = 0 \text{ i.e. } R_{\mathcal{S}^*_R} \approx 0.0428.$$

(ix) The $\mathcal{S}^*_{RL}$-radius is $R_{\mathcal{S}^*_{RL}}$ which is root of the equation

$$\frac{(r^2+4r-r^3)^2}{(1-r^2)^2} = ((1-(\sqrt{2}-(1+r^2)/(1-r^2)^2))^{1/2} - (1-(\sqrt{2}-(1+r^2)/(1-r^2))^2).$$

Proof. Let $f \in E_2$ and $g : \mathbb{D} \to \mathbb{C}$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 \quad \text{and} \quad \operatorname{Re} \frac{g(z)}{ze^z} > 0. \tag{2.8}$$

Using the fact $|w-1| < 1$ if and only if $\operatorname{Re}(1/w) > 1/2$, it follows that $\operatorname{Re}(g(z)/f(z)) > 1/2$. Define the functions $p_1, p_2 : \mathbb{D} \to \mathbb{C}$ by

$$p_1(z) = \frac{g(z)}{ze^z} \quad \text{and} \quad p_2(z) = \frac{g(z)}{f(z)}. \tag{2.9}$$

By (2.8) and (2.9), we have $p_1 \in \mathcal{P}$ and $p_2 \in \mathcal{P}(1/2)$. Also, from (2.9), we have

$$f(z) = \frac{ze^z p_1(z)}{p_2(z)}.$$

and eventually

$$\frac{zf'(z)}{f(z)} = 1 + z + \frac{zp_1'(z)}{p_1(z)} - \frac{zp_2'(z)}{p_2(z)}.$$

Hence,

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{r^2+4r-r^3}{1-r^2} \tag{2.10}$$

and

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \frac{1-4r-2r^2+r^3}{1-r^2} \geq 0.$$

Thus the function $f \in E_2$ is starlike in $|z| \leq 0.2271$. Hence, all the radius estimate here will be less than 0.2271.

(i) The function $m(r) = (1-4r-2r^2+r^3)(1-r^2)^{-1},\ 0 \leq r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}^*(\alpha)}$ is the smallest positive root of the equation $m(r) = \alpha$. From (2.10), it follows that

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \frac{1-4r-2r^2+r^3}{1-r^2} = m(r) \geq m(\varrho) = \alpha.$$

This shows that $R_{\mathcal{S}^*(\alpha)}$ is at least ϱ. At $z = R_{\mathcal{S}^*(\alpha)} = \varrho$, the function f_2 defined in (2.7) satisfies

$$\operatorname{Re}\frac{zf_2'(z)}{f_2(z)} = \frac{1-4\rho-2\rho^2+\rho^3}{1-\rho^2} = \alpha.$$

Thus the radius is sharp.

(ii) The function $m(r) = (4r+r^2-r^3)(1-r^2)^{-1}+1,\ 0 \leq r < 1$ is an increasing function. Let $\varrho = R_{\mathcal{S}^*_L}$ be the root of the equation $m(r) = \sqrt{2}$. For $0 < r \leq R_{\mathcal{S}^*_L}$, we have $m(r) \leq \sqrt{2}$. That is,

$$\frac{4r+r^2-r^3}{1-r^2}+1 \leq \sqrt{2} = m(\varrho).$$

For the class E_2, the centre of the disc in (2.10) is 1. Using [1, Lemma 2.2], the disc obtained in (2.10) is contained in the region bounded by lemniscate. For the function f_2 defined in (2.7), at $z = R_{\mathcal{S}^*_L} = -\rho$,

$$\left|\left(\frac{zf_2'(z)}{f_2(z)}\right)^2-1\right| = \left|\left(\frac{1+4\rho-2\rho^2+\rho^3}{1-\rho^2}\right)^2-1\right| = |(\sqrt{2})^2-1| = 1.$$

(iii) The function $m(r) = (1-4r-2r^2+r^3)(1-r^2)^{-1},\ 0 \leq r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}^*_p}$ be the root of the equation $m(r) = 1/2$. For $0 < r \leq R_{\mathcal{S}^*_p}$, we have $m(r) \geq 1/2$. That is,

$$\frac{4r+r^2-r^3}{1-r^2} \leq \frac{1}{2} = m(\rho).$$

Using [16, Lemma 1], we see that the disc obtained in (2.10) is contained in the region bounded by parabola. For the function f_2 defined in (2.7), at $z = R_{\mathcal{S}^*_p} = \rho$,

$$\operatorname{Re}\frac{zf_2'(z)}{f_2(z)} = \frac{1-4\rho-2\rho^2+\rho^3}{1-\rho^2} = \frac{1}{2} = \left|\frac{zf_2'(z)}{f_2(z)}-1\right|.$$

(iv) The function $m(r) = (1-4r-2r^2+r^3)(1-r^2)^{-1},\ 0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}_e^*}$ be the root of the equation $m(r) = 1/e$. For $0 < r \le R_{\mathcal{S}_e^*}$, we have $m(r) \ge 1/e$. That is,

$$\frac{4r+r^2-r^3}{1-r^2} \le 1 - \frac{1}{e}.$$

Using [11, Lemma 2.2], it follow that the disc obtained in (2.10) is contained in the region bounded by exponential function. For the function f_2 defined in (2.7), at $z = R_{\mathcal{S}_e^*} = \rho$,

$$\left|\log \frac{zf_2'(z)}{f_2(z)}\right| = \left|\log \frac{1-4\rho-2\rho^2+\rho^3}{1-\rho^2}\right| = 1.$$

(v) The function $m(r) = (1-4r-2r^2+r^3)(1-r^2)^{-1},\ 0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}_c^*}$ be the root of the equation $m(r) = 1/3$. For $0 < r \le R_{\mathcal{S}_c^*}$, we have $m(r) \ge 1/3$. That is,

$$\frac{4r+r^2-r^3}{1-r^2} \le 1 - \frac{1}{3}.$$

Using [17, Lemma 2.5], we see that the disc obtained in (2.10) is contained in the region bounded by the cardioid. For the function f_2 defined in (2.7), at $z = R_{\mathcal{S}_c^*} = \rho$,

$$\frac{zf_2'(z)}{f_2(z)} = \frac{1-4\rho-2\rho^2+\rho^3}{1-\rho^2} = \frac{1}{3} = h_c(-1),$$

where $h_c(z) = 1+(4/3)z+(2/3)z^2$ is the superordinate function in the class $\mathcal{S}_c^*$.

(vi) The function $m(r) = (1-4r-2r^2+r^3)(1-r^2)^{-1},\ 0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}_{sin}^*}$ be the root of the equation $m(r) = 1-\sin 1$. For $0 < r \le R_{\mathcal{S}_{sin}^*}$, we have $m(r) \ge 1-\sin 1$. That is,

$$\frac{4r+r^2-r^3}{1-r^2} \le \sin 1.$$

Using [2, Lemma 3.3], the disc obtained in (2.10) is contained in the region Ω_s bounded by the sine function. For the function f_2 defined in (2.7), at $z = -R_{\mathcal{S}_{sin}^*} = -\rho$,

$$\frac{zf_2'(z)}{f_2(z)} = \frac{1-4\rho-2\rho^2+\rho^3}{1-\rho^2} = 1+\sin 1 = h_s(1),$$

where $h_s(z) = 1+\sin z$ is the superordinate function in the class $\mathcal{S}_{sin}^*$.

(vii) The function $m(r) = (1-4r-2r^2+r^3)(1-r^2)^{-1},\ 0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}_m^*}$ be the root of the equation $m(r) = \sqrt{2}-1$. For $0 < r \le R_{\mathcal{S}_m^*}$, we have $m(r) \ge \sqrt{2}-1$. That is,

$$\frac{4r+r^2-r^3}{1-r^2} \le 2 - \sqrt{2}.$$

Using [4, Lemma 2.1], the disc obtained in (2.10) is contained in the region bounded by the intersection of disks $\{w : |w-1| < \sqrt{2}\}$ and $\{w : |w+1| < \sqrt{2}\}$. For the function f_2 defined in (2.7), at $z = -R_{\mathcal{S}^*_m} = -\rho$,

$$\left|\left(\frac{zf_2'(z)}{f_2(z)}\right)^2 - 1\right| = \left|\left(\frac{1-4\rho-2\rho^2+\rho^3}{1-\rho^2}\right)^2 - 1\right| = 2\left|\frac{1-4\rho-2\rho^2+\rho^3}{1-\rho^2}\right|.$$

(viii) The function $m(r) = (1-4r-2r^2+r^3)(1-r^2)^{-1},\ 0 \le r < 1$ is a decreasing function. Let $\varrho = R_{\mathcal{S}^*_R}$ be the root of the equation $m(r) = 2(\sqrt{2}-1)$. For $0 < r \le R_{\mathcal{S}^*_R}$, we have $m(r) \ge 2(\sqrt{2}-1)$. That is,

$$\frac{4r+r^2-r^3}{1-r^2} \le 1 - 2(\sqrt{2}-1).$$

Using [6, Lemma 2.2], the disc obtained in (2.10) is contained in the region bounded by the rational function. For the function f_2 defined in (2.7), at $z = -R_{\mathcal{S}^*_R} = -\rho$,

$$\left|\frac{zf_2'(z)}{f_2(z)}\right| = \left|\frac{1-4\rho-2\rho^2+\rho^3}{1-\rho^2}\right| = 2(\sqrt{2}-1) = h_R(-1),$$

where $h_R(z) = 1 + (zk+z^2)/(k^2-kz)$, $k = 1+\sqrt{2}$ is the superordinate function in the class $\mathcal{S}^*_R$.

(ix) The function $m(r) = ((4r+r^2-r^3)(1-r^2)^{-1}) + 1,\ 0 \le r < 1$ is an increasing function. Let $\varrho = R_{\mathcal{S}^*_{RL}}$ be the root of the equation

$$m(r) = ((1-(\sqrt{2}-1)^2)^{1/2} - (1-(\sqrt{2}-1)^2))^{1/2}.$$

Using [10, Lemma 3.2], the disc obtained in (2.10) is contained in the region $\{w : |(w-\sqrt{2})^2 - 1| < 1\}$. For the function f_2 defined in (2.7), at $z = -R_{\mathcal{S}^*_{RL}} = -\rho$,

$$\left|\left(\frac{zf_2'(z)}{f_2(z)}\right)^2 - 1\right| = \left|\left(\frac{1-4\rho-2\rho^2+\rho^3}{1-\rho^2} - \sqrt{2}\right)^2 - 1\right| = 1. \qquad \square$$

Recall that the class E_3 is defined by

$$E_3 = \left\{ f : \mathcal{A} : \operatorname{Re} \frac{f(z)}{ze^z} > 0,\ z \in \mathbb{D} \right\}.$$

An extremal function in the class E_3 is

$$f(z) = \frac{ze^z(1+z)}{1-z}.$$

For this class E_3, we have the following result:

Theorem 2.3. *For the class E_3, the following results hold:*

(i) For $0 \le \alpha < 1$, the $\mathcal{S}^*_\alpha$-radius is the smallest positive real root of the equation

$$r^3 + (\alpha-1)r^2 - 4r + \alpha = 0.$$

(ii) The $\mathcal{S}^*_L$-radius is the smallest positive root of the equation

$$r^3 + r^2(1-\sqrt{2}) - 3r + \sqrt{2} - 1 = 0 \text{ i.e. } R_{\mathcal{S}^*_L} \approx 0.1363.$$

(iii) The $\mathcal{S}_p^*$-radius is the smallest positive root of the equation

$$2r^3 - r^2 - 6r + 1 = 0 \text{ i.e. } R_{\mathcal{S}_p^*} \approx 0.1637.$$

(iv) The $\mathcal{S}_e^*$-radius is the smallest positive root of the equation

$$er^3 + r^2(1-e) - 3er + e - 1 = 0 \text{ i.e. } R_{\mathcal{S}_e^*} \approx 0.2047.$$

(v) The $\mathcal{S}_C^*$-radius is the smallest positive root of the equation

$$3r^3 - 2r^2 - 9r + 2 = 0 \text{ i.e. } R_{\mathcal{S}_C^*} \approx 0.2153.$$

(vi) The $\mathcal{S}_{\sin}^*$-radius the smallest positive root of the equation

$$r^3 - r^2 \sin 1 - 3r + \sin 1 = 0 \text{ i.e. } R_{\mathcal{S}_{\sin}^*} \approx 0.005817.$$

(vii) The $\mathcal{S}_m^*$-radius is the smallest positive root of the equation

$$r^3 - r^2(2-\sqrt{2}) - 3r + 2 - \sqrt{2} = 0 \text{ i.e. } R_{\mathcal{S}_m^*} \approx 0.1905.$$

(viii) The $\mathcal{S}_R^*$-radius is the smallest positive root of the equation

$$r^3 - r^2(2-2\sqrt{2}) - 3r + 3 - 2\sqrt{2} = 0 \text{ i.e. } R_{\mathcal{S}_R^*} \approx 0.0428.$$

(ix) The $\mathcal{S}_{RL}^*$-radius is $R_{\mathcal{S}_{RL}^*}$ which is root of the equation

$$\frac{(3r-r^3)^2}{(1-r^2)^2} = ((1-(\sqrt{2}-(1+r^2)/(1-r^2)^2))^{1/2} - (1-(\sqrt{2}-(1+r^2)/(1-r^2))^2).$$

Proof. We can conclude the hypothesis appropriately adopting the similar technique as in the previous proof. □

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Yao Liang Chung
School of Mathematical Sciences,
Universiti Sains Malaysia,
11800 Penang, Malaysia
e-mail: chungyaoliang@gmail.com

Maisarah Haji Mohd
School of Mathematical Sciences,
Universiti Sains Malaysia,
11800 Penang, Malaysia
e-mail: maisarah_hjmohd@usm.my

Shamani Supramaniam
School of Mathematical Sciences,
Universiti Sains Malaysia,
11800 Penang, Malaysia
e-mail: shamani@usm.my

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Some classes of Janowski functions associated with conic domain and a shell-like curve involving Ruscheweyh derivative

Kadhavoor Ragavan Karthikeyan,
Seetharam Varadharajan and Sakkarai Lakshmi

Abstract. Making use of Ruscheweyh derivative, we define a new class of starlike functions of complex order subordinate to a conic domain impacted by Janowski functions. Coefficient estimates and Fekete-Szegö inequalities for the defined class are our main results. Some of our results generalize the related work of some authors.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f analytic in the open unit disk

$$\mathcal{U} = \{ \, z \in \mathbb{C} : |z| < 1\}$$

and satisfying the normalization condition

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Thus, the functions in $\mathcal{A}$ are represented by the Taylor-Maclaurin series expansion given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathcal{U}). \tag{1.1}$$

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Let $\mathcal{S} \subset \mathcal{A}$ be the class of functions which are univalent. We let $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ to denote the well known classes of starlike, convex and close-to-convex (normalized) function respectively. For $0 \leq \alpha < 1$, $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ symbolize the classes of starlike functions of order α and convex functions of order α respectively. Also let $\mathcal{P}$ denote the class of functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ that are analytic in $\mathcal{U}$ and such that $Re\,(p(z)) > 0$ for all z in $\mathcal{U}$.

For arbitrary fixed numbers A, B, $-1 < A \leq 1$, $-1 \leq B < A$, we denote by $\mathcal{P}(A,\, B)$ the family of functions $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$ analytic in the unit disc and $p(z) \in \mathcal{P}(A,\, B)$ if and only if

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)},$$

where $w(z)$ is the Schwartz function. Geometrically, $p(z) \in \mathcal{P}(A,\, B)$ if and only if $p(0) = 1$ and $p(U)$ lies inside an open disc centered with center $\frac{1-AB}{1-B^2}$ on the real axis having radius $\frac{A-B}{1-B^2}$ with diameter end points $p_1(-1) = \frac{1-A}{1-B}$ and $p_1(1) = \frac{1+A}{1+B}$. On observing that $w(z) = \frac{p(z)-1}{p(z)+1}$ for $p(z) \in \mathcal{P}$, we have $P(z) \in \mathcal{P}(A,\, B)$ if and only if for some $p(z) \in \mathcal{P}$

$$P(z) = \frac{(1+A)p(z) + 1 - A}{(1+B)p(z) + 1 - B}. \tag{1.2}$$

For detailed study on the class of Janowski functions, we refer [3].

The function $p_{k,\alpha}(z)$ plays the role of an extremal functions those related to these conic domain $\mathcal{D}_k = \left\{u + iv : \; u > k\sqrt{(u-1)^2 + v^2}\right\}$ and is given by

$$\hat{p}_{k,\alpha}(z) = \begin{cases} \frac{1+(1-2\alpha)z}{1-z}, & \text{if } k = 0, \\ 1 + \frac{2(1-\alpha)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & \text{if } k = 1, \\ 1 + \frac{2(1-\alpha)}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos k\right) arc\tanh\sqrt{z}\right], & \text{if } 0 < k < 1, \\ 1 + \frac{2(1-\alpha)}{1-k^2}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{t}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right) + \frac{1}{k^2-1}, & \text{if } k > 1, \end{cases} \tag{1.3}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0,\, 1)$ and t is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, with $R(t)$ is Legendres complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$. Clearly, $\hat{p}_{k,\alpha}(z)$ is in $\mathcal{P}$ with the expansion of the form

$$\hat{p}_{k,\alpha}(z) = 1 + \delta_1 z + \delta_2 z^2 + \cdots, \qquad (\delta_j = p_j(k,\, \alpha),\; j = 1,\, 2,\, 3, \ldots), \tag{1.4}$$

we get

$$\delta_1 = \begin{cases} \frac{8(1-\alpha)(\arccos k)^2}{\pi^2(1-k^2)}, & \text{if } 0 \leq k < 1, \\ \frac{8(1-\alpha)}{\pi^2}, & \text{if } k = 1, \\ \frac{\pi^2(1-\alpha)}{4\sqrt{t}(k^2-1)R^2(t)(1+t)}, & \text{if } k > 1. \end{cases} \tag{1.5}$$

Noor in [8, 9] replaced $p(z)$ in (1.2) with $\hat{p}_{k,\alpha}(z)$ and studied the impact of Janowski function on conic regions.

Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$, if there exists an Schwartz function $w(z)$ in $\mathcal{U}$ such that

$|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Using the concept of subordination for holomorphic functions, Ma and Minda [6] introduced the classes

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf^{'}(z)}{f(z)} \prec \phi \right\} \quad \text{and} \quad \mathcal{C}(\phi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi \right\}$$

where $\phi \in \mathcal{P}$ with $\phi^{'}(0) > 0$ maps $\mathcal{U}$ onto a region starlike with respect to 1 and symmetric with respect to real axis. By choosing ϕ to map unit disc on to some specific regions like parabolas, cardioid, lemniscate of Bernoulli, booth lemniscate in the right-half plane of the complex plane, various interesting subclasses of starlike and convex functions can be obtained. Raina and Sokół [10] studied the class $\mathcal{S}^*(\phi)$ for $\phi(z) = z + \sqrt{1+z^2}$ and found some interesting coefficient inequalities. The function $\phi(z) = z + \sqrt{1+z^2}$ maps the unit disc $\mathcal{U}$ onto a shell shaped region on the right half plane and it is analytic and univalent on $\mathcal{U}$. For detailed study of starlike functions related to shell shaped region, refer to a recent work of Murugusundaramoorthy and Bulboacă [7]. Khatter et al. [5] studied the convex combination of constant function $f(z) = 1$ with e^z and $\sqrt{1+z}$. Recently, Gandhi in [2] studied a class $\mathcal{S}^*(\phi)$ with $\phi = \beta e^z + (1-\beta)(1+z)$, $0 \le \delta \le 1$ a convex combination of two starlike functions.

Definition 1.1. [12] For $f \in \mathcal{A}$ of the form (1.1) and $\lambda \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the operator R^λ is defined by $R^\lambda : \mathcal{A} \to \mathcal{A}$,

$$\begin{aligned} R^0 f(z) &= f(z), \\ R^1 f(z) &= zf'(z), \\ &\vdots \\ (\lambda+1)R^{\lambda+1} f(z) &= z\left(R^\lambda f(z)\right)' + \lambda R^\lambda f(z), \quad z \in \mathcal{U}. \end{aligned}$$

Remark 1.2. If $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$, then for $\lambda > -1$

$$R^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) = z + \sum_{n=2}^{\infty} \varphi_n(\lambda) a_n z^n,$$

where

$$\varphi_n(\lambda) = \frac{[\lambda+1]_{n-1}}{(n-1)!}, \tag{1.6}$$

$$[t]_n = \begin{cases} 1, & n = 0, \\ (t)(t+1)(t+2)\dots(t+n-1), & n \in \mathbb{N}. \end{cases}$$

is a Pochhammer symbol,

$$\Gamma(t+1) = \begin{cases} 1, & t = 1, \\ [t]\Gamma(t), & t > 0. \end{cases}$$

is a gamma function. The symbol “ $*$ ” stands for Hadamard product.

Motivated by Gandhi [2], we introduce the following new subclasses of analytic functions using Ruscheweyh differential operator.

Definition 1.3. For $\hat{p}_{k,\alpha}(z)$, $(k \geq 0,\ 0 \leq \alpha < 1)$ is defined as in (1.3), $-1 \leq B < A \leq 1$, $\lambda > -1$, $\mid t \mid \leq 1, t \neq 1$ and for some $b \in \mathbb{C}\backslash\{0\}$, we let $k - \mathcal{SL}(A, B, \alpha, \beta, \lambda, t, b)$ to be the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$1 + \frac{1}{b}\left(\frac{(1-t)R^{\lambda+1}f(z)}{R^{\lambda}f(z) - R^{\lambda}f(tz)} - 1\right) \prec \frac{(A+1)h(z) - (A-1)}{(B+1)h(z) - (B-1)}, \quad (z \in \mathcal{U}) \tag{1.7}$$

where

$$h(z) = \beta\left[\hat{p}_{k,\alpha}(z)\right] + (1-\beta)\left[z + \sqrt{1+z^2}\right],\ 0 \leq \beta \leq 1. \tag{1.8}$$

Remark 1.4. Note that $\hat{p}_{k,\alpha}(z)$ is not univalent but belongs to $\mathcal{P}$, whereas $z + \sqrt{1+z^2}$ is univalent in $\mathcal{U}$. Since the linear combination of two convex function is not convex in $|z| < 1$, $h(z)$ is not convex univalent in $\mathcal{U}$.

The following definition is motivated by the Alexander transform relationship between convex and starlike functions.

Definition 1.5. For $\hat{p}_{k,\alpha}(z)$, $(k \geq 0,\ 0 \leq \alpha < 1)$ is defined as in (1.3), $-1 \leq B < A \leq 1$, $\lambda > -1$, $\mid t \mid \leq 1, t \neq 1$ and for some $b \in \mathbb{C}\backslash\{0\}$, we let $k - \mathcal{CL}(A, B, \alpha, \beta, \lambda, t, b)$ to be the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$1 + \frac{1}{b}\left(\frac{(1-t)\left(R^{\lambda+1}f(z)\right)'}{\left(R^{\lambda}f(z) - R^{\lambda}f(tz)\right)'} - 1\right) \prec \frac{(A+1)h(z) - (A-1)}{(B+1)h(z) - (B-1)}, \quad (z \in \mathcal{U}) \tag{1.9}$$

where $h(z)$ is defined as in (1.8).

We let $k - \mathcal{CL}(A, B, \lambda, t, b)$ and $k - \mathcal{CL}(A, B, \alpha, 1, \lambda, t, b)$ to denote the special cases of the function class $k - \mathcal{CL}(A, B, \alpha, \beta, \lambda, t, b)$ obtained by letting $\beta = 0$ and $\beta = 1$ respectively.

Remark 1.6. The versatility of classes $k - \mathcal{SL}(A, B, \alpha, \beta, \lambda, t, b)$ and $k - \mathcal{CL}(A, B, \alpha, \beta, \lambda, t, b)$ is that it unifies the study of starlike and convex functions with respect to symmetric points. Here we list just a few special cases.

1. If we let $b = 1$, $t = 0$, $\alpha = 0$, $\beta = 1$ and $\lambda = 0$ in the definition of the function class $k - \mathcal{SL}(A, B, \alpha, \beta, \lambda, t, b)$ and $k - \mathcal{CL}(A, B, \alpha, \beta, \lambda, t, b)$, we get the classes $k - \mathcal{SL}(A, B)$ and $k - \mathcal{CL}(A, B)$ introduced and studied by Noor and Malik in [9].
2. For $b = 1$, $\beta = 1$ and $\lambda = 0$, the class $k - \mathcal{SL}(A, B, \alpha, \beta, \lambda, t, b)$ reduces to the respective classes $k - \mathcal{SL}(A, B, \alpha, 1, 0, t, 1)$ studied by Arif et al. in [1].

Unless otherwise mentioned, we assume throughout this paper that the function $0 \leq \alpha < 1$, $0 \leq \beta \leq 1$, $\lambda > -1$, $k \geq 0$, $-1 \leq B < A \leq 1$, $|t| \leq 1$, $t \neq 1$, $b \in \mathbb{C}\backslash\{0\}$ and $z \in \mathcal{U}$.

2. Fekete-Szegö inequalities for the starlike class $k-\mathcal{SL}(A,B,\alpha,\beta,\lambda,t,b)$

Many extremal problems within the class of univalent functions are solved by the Koebe function. On the other hand, the Koebe function satisfies

$$\left|a_3-\lambda a_2^2\right|=|3-4\lambda|$$

whereas Fekete and Szegö showed

$$\max_{f\in\mathcal{S}}\left|a_3-\lambda a_2^2\right|=|3-4\lambda|=1+2e^{-2\lambda/(1-\lambda)}$$

for $\lambda\in[0,1]$. In this section, we obtain the Fekete-Szegö for the class $k-\mathcal{SL}(A,B,\alpha,\beta,\lambda,t,b)$. We need the following lemma to establish our main result.

Lemma 2.1. [6] *Let $p(z)\in\mathcal{P}$ and also let v be a complex number, then*

$$|c_2-vc_1^2|\le 2\max\{1,|2v-1|\},\tag{2.1}$$

the result is sharp for functions given by

$$p(z)=\frac{1+z^2}{1-z^2},\qquad p(z)=\frac{1+z}{1-z}.$$

Theorem 2.2. *If $f(z)\in k-\mathcal{SL}(A,B,\alpha,\beta,\lambda,t,b)$ then for $\mu\in\mathbb{C}$ we have*

$$|\,a_3-\mu a_2^2\,|\le\frac{|\,b\,|\,|\beta(\delta_1-1)+1|(A-B)}{2[\varphi_3(\lambda+1)-u_3\varphi_3(\lambda)]}\max\{1,|2v-1|\},\tag{2.2}$$

where

$$\begin{aligned}v=&\frac{1}{2}-\frac{\beta(2\delta_2-1)+1}{4\left[\beta(\delta_1-1)+1\right]}+\frac{\left[\beta(\delta_1-1)+1\right](B+1)}{4}\\&-\frac{b\varphi_2(\lambda)[\beta(\delta_1-1)+1](A-B)}{4[\varphi_2(\lambda+1)-u_2\varphi_2(\lambda)]}\left(u_2-\mu\frac{\varphi_3(\lambda+1)-u_3\varphi_3(\lambda)}{\varphi_2(\lambda)[\varphi_2(\lambda+1)-u_2\varphi_2(\lambda)]}\right)\end{aligned}\tag{2.3}$$

and $u_n=1+t+t^2+\cdots+t^{n-1}$. The result is sharp.

Proof. Let $p(z)\in\mathcal{P}$ be of the form $1+\sum\limits_{n=1}^{\infty}p_nz^n$, we consider

$$p(z)=\frac{1+w(z)}{1-w(z)},$$

where $w(z)$ is such that $w(0)=0$ and $|\,w(z)\,|<1$. On simple computation, we have

$$\begin{aligned}w(z)&=\frac{p(z)-1}{p(z)+1}=\frac{p_1z+p_2z^2+p_3z^3+\cdots}{2+p_1z+p_2z^2+p_3z^3+\cdots}\\&=\frac{1}{2}p_1z+\frac{1}{2}\left(p_2-\frac{1}{2}p_1^2\right)z^2+\frac{1}{2}\left(p_3-p_1p_2+\frac{1}{4}p_1^3\right)z^3+\cdots.\end{aligned}\tag{2.4}$$

Using (2.4) in $h(z) = 1 + [\beta(\delta_1 - 1) + 1]\, z + \frac{1}{2}\, [\beta(2\delta_2 - 1) + 1]\, z^2 + \cdots$, we have

$$\begin{aligned} h(w(z)) &= 1 + [\beta(\delta_1 - 1) + 1]\, w(z) + \frac{1}{2}\, [\beta(2\delta_2 - 1) + 1]\, [w(z)]^2 + \cdots \\ &= 1 + [\beta(\delta_1 - 1) + 1] \left[\frac{1}{2} p_1 z + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right) z^2 + \frac{1}{2}\left(p_3 - p_1 p_2 + \frac{1}{4}p_1^3\right) z^3 + \cdots\right] \\ &+ \frac{1}{2}\, [\beta(2\delta_2 - 1) + 1] \left[\frac{1}{2} p_1 z + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right) z^2 + \frac{1}{2}\left(p_3 - p_1 p_2 + \frac{1}{4}p_1^3\right) z^3 + \cdots\right]^2 + \cdots \\ &= 1 + \frac{[\beta(\delta_1 - 1) + 1]\, p_1}{2} z + \frac{[\beta(\delta_1 - 1) + 1]}{2} \left[p_2 - \frac{p_1^2}{2}\left(1 - \frac{\beta(2\delta_2 - 1) + 1}{2\, [\beta(\delta_1 - 1) + 1]}\right)\right] z^2 + \cdots . \end{aligned}$$

As $f(z) \in k - \mathcal{SL}(A, B, \alpha, \beta, \lambda, t, b)$, by (1.7) we have

$$1 + \frac{1}{b}\left(\frac{(1-t)R^{\lambda+1}f(z)}{R^\lambda f(z) - R^\lambda f(tz)} - 1\right) = p(z), \tag{2.5}$$

where

$$\begin{aligned} p(z) &= \frac{(A+1)h(w(z)) - (A-1)}{(B+1)h(w(z)) - (B-1)} \\ &= \frac{2 + \frac{(A+1)[\beta(\delta_1-1)+1]p_1}{2} z + \frac{(A+1)[\beta(\delta_1-1)+1]}{2}\left[p_2 - \frac{p_1^2}{2}\left(1 - \frac{\beta(2\delta_2-1)+1}{2[\beta(\delta_1-1)+1]}\right)\right] z^2 + \cdots}{2 + \frac{(B+1)[\beta(\delta_1-1)+1]p_1}{2} z + \frac{(B+1)[\beta(\delta_1-1)+1]}{2}\left[p_2 - \frac{p_1^2}{2}\left(1 - \frac{\beta(2\delta_2-1)+1}{2[\beta(\delta_1-1)+1]}\right)\right] z^2 + \cdots} \\ &= 1 + \frac{[\beta(\delta_1 - 1) + 1]\,(A - B)p_1}{4} z + \frac{[\beta(\delta_1 - 1) + 1]\,(A - B)}{4} \\ &\quad \left[p_2 - \frac{p_1^2}{2}\left(1 - \frac{\beta(2\delta_2 - 1) + 1}{2\,[\beta(\delta_1 - 1) + 1]} + \frac{[\beta(\delta_1 - 1) + 1]\,(B + 1)}{2}\right)\right] z^2 + \cdots . \end{aligned} \tag{2.6}$$

From (2.5), we obtain

$$\begin{aligned} 1 + \frac{1}{b}\left(\frac{(1-t)R^{\lambda+1}f(z)}{R^\lambda f(z) - R^\lambda f(tz)} - 1\right) &= 1 + \frac{1}{b}\Bigg([\varphi_2(\lambda+1) - u_2\varphi_2(\lambda)]a_2 z \\ &+ \Big[[\varphi_3(\lambda+1) - u_3\varphi_3(\lambda)]a_3 - [\varphi_2(\lambda+1) - u_2\varphi_2(\lambda)]u_2\varphi_2(\lambda)a_2^2\Big] z^2 + \cdots\Bigg). \end{aligned} \tag{2.7}$$

From (2.6) and (2.7), the coefficients of z and z^2 are given by

$$a_2 = \frac{b\,[\beta(\delta_1 - 1) + 1]\,(A - B)p_1}{4[\varphi_2(\lambda+1) - u_2\varphi_2(\lambda)]}$$

and

$$\begin{aligned} a_3 = \frac{b\,[\beta(\delta_1 - 1) + 1]\,(A - B)}{4[\varphi_3(\lambda+1) - u_3\varphi_3(\lambda)]}\Bigg[p_2 - \frac{p_1^2}{2}\Bigg(1 - \frac{\beta(2\delta_2 - 1) + 1}{2\,[\beta(\delta_1 - 1) + 1]} + \frac{[\beta(\delta_1 - 1) + 1]\,(B + 1)}{2} \\ - \frac{b u_2\varphi_2(\lambda)\,[\beta(\delta_1 - 1) + 1]\,(A - B)}{2[\varphi_2(\lambda+1) - u_2\varphi_2(\lambda)]}\Bigg)\Bigg]. \end{aligned}$$

Therefore, we have

$$| a_3 - \mu a_2^2 | \leq \frac{| b | |\beta(\delta_1 - 1) + 1|(A - B)}{2[\varphi_3(\lambda + 1) - u_3\varphi_3(\lambda)]} | p_2 - vp_1^2 |, \tag{2.8}$$

where

$$v = \frac{1}{2} - \frac{\beta(2\delta_2 - 1) + 1}{4\,[\beta(\delta_1 - 1) + 1]} + \frac{[\beta(\delta_1 - 1) + 1]\,(B + 1)}{4}$$
$$- \frac{b\varphi_2(\lambda)[\beta(\delta_1 - 1) + 1](A - B)}{4[\varphi_2(\lambda + 1) - u_2\varphi_2(\lambda)]} \left(u_2 - \mu \frac{\varphi_3(\lambda + 1) - u_3\varphi_3(\lambda)}{\varphi_2(\lambda)[\varphi_2(\lambda + 1) - u_2\varphi_2(\lambda)]} \right). \tag{2.9}$$

Taking the modules for both sides of the above relation, with the aid of the inequality (2.1) of Lemma 2.1, we easily get the required estimate. The result is sharp for the functions

$$1 + \frac{1}{b} \left(\frac{(1 - t)R^{\lambda+1} f(z)}{R^\lambda f(z) - R^\lambda f(tz)} - 1 \right) = p(z)$$

and

$$1 + \frac{1}{b} \left(\frac{(1 - t)R^{\lambda+1} f(z)}{R^\lambda f(z) - R^\lambda f(tz)} - 1 \right) = p(z^2).$$

where $p(z)$ is given by the equation (2.6). Hence the proof of the Theorem 2.2 is complete. □

If $\beta = 0$ in the Theorem 2.2, we get the following corollary.

Corollary 2.3. *If $f(z) \in k - \mathcal{CL}(A, B, \lambda, t, b)$ then for $\mu \in \mathbb{C}$ we have*

$$| a_3 - \mu a_2^2 | \leq \frac{| b | \,(A - B)}{12[\varphi_3(\lambda + 1) - u_3\varphi_3(\lambda)]} \max\{1, |2v - 1|\}, \tag{2.10}$$

where

$$v = \frac{B + 1}{4} - \frac{b\varphi_2(\lambda)(A - B)}{4[\varphi_2(\lambda + 1) - u_2\varphi_2(\lambda)]} \left(u_2 - \mu \frac{3[\varphi_3(\lambda + 1) - u_3\varphi_3(\lambda)]}{4\varphi_2(\lambda)[\varphi_2(\lambda + 1) - u_2\varphi_2(\lambda)]} \right) \tag{2.11}$$

and $u_n = 1 + t + t^2 + \cdots + t^{n-1}$. The result is sharp.

If $\beta = 1$ in the Theorem 2.2, we get the following corollary.

Corollary 2.4. *If $f(z) \in k - \mathcal{CL}(A, B, \alpha, 1, \lambda, t, b)$ then for $\mu \in \mathbb{C}$ we have*

$$| a_3 - \mu a_2^2 | \leq \frac{| b | \,|\delta_1|(A - B)}{12[\varphi_3(\lambda + 1) - u_3\varphi_3(\lambda)]} \max\{1, |2v - 1|\}, \tag{2.12}$$

where

$$v = \frac{1}{2} - \frac{\delta_2}{2\delta_1} + \frac{\delta_1(B + 1)}{4}$$
$$- \frac{b\varphi_2(\lambda)\delta_1(A - B)}{4[\varphi_2(\lambda + 1) - u_2\varphi_2(\lambda)]} \left(u_2 - \mu \frac{3[\varphi_3(\lambda + 1) - u_3\varphi_3(\lambda)]}{4\varphi_2(\lambda)[\varphi_2(\lambda + 1) - u_2\varphi_2(\lambda)]} \right), \tag{2.13}$$

δ_1 is defined as in (1.5) and $u_n = 1 + t + t^2 + \cdots + t^{n-1}$. The result is sharp.

If $t = -1$ in the Theorem 2.2, we get the following corollary.

Corollary 2.5. *If $f \in k-\mathcal{SL}(A,B,\alpha,\beta,\lambda,-1,b)$ then for $\mu \in \mathbb{C}$ we have*

$$| a_3 - \mu a_2^2 | \leq \frac{| b | \, |\beta(\delta_1 - 1) + 1|(A-B)}{2[\varphi_3(\lambda+1) - \varphi_3(\lambda)]} \max\{1, |2v-1|\}, \tag{2.14}$$

where

$$\begin{aligned} v = \frac{1}{2} - \frac{\beta(2\delta_2 - 1) + 1}{4\,[\beta(\delta_1 - 1) + 1]} + \frac{[\beta(\delta_1 - 1) + 1]\,(B+1)}{4} \\ + \frac{\mu b[\varphi_3(\lambda+1) - \varphi_3(\lambda)]}{[\varphi_2(\lambda+1)]^2} \frac{[\beta(\delta_1 - 1) + 1]\,(A-B)}{4}. \end{aligned} \tag{2.15}$$

The result is sharp.

If $t = -1$, $A = 1$, $B = -1$, $\alpha = 0$, $\beta = 1$, $\lambda = 0$ and $b = 1$ in the Theorem 2.2, we get the following corollary of [4].

Corollary 2.6. *If $f(z) \in \mathcal{M}_s(p_k)$ then we have*

$$a_2 = \frac{\delta_1 p_1}{4}, \quad a_3 = \frac{\delta_1}{4}\left[p_2 - \frac{p_1^2}{2}\left(1 - \frac{\delta_2}{\delta_1}\right)\right]$$

and for any complex number μ,

$$| a_3 - \mu a_2^2 | \leq \frac{\delta_1}{2} \max\left\{1, \left|\frac{\delta_2}{\delta_1} - \frac{\mu\delta_1}{2}\right|\right\}.$$

If $t = 0$, $A = 1$, $B = -1$, $\alpha = 0$, $\beta = 0$, $\lambda = 0$ and $b = 1$ in the Theorem 2.2, we get the following corollary.

Corollary 2.7. [10] *If $f(z) \in \mathcal{SL}_q$ then $|a_2| \leq 1$, $|a_3| \leq \frac{3}{4}$ and*

$$|a_3 - \mu a_2^2| \leq \max\left\{\frac{1}{2}, \left|\mu - \frac{3}{4}\right|\right\}.$$

3. Coefficient estimates for the convex classes $k-\mathcal{CL}(A,B,\lambda,t,b)$ and $k-\mathcal{CL}(A,B,\alpha,1,\lambda,t,b)$

To find the coefficient estimates, we need the following lemmas.

Lemma 3.1. [11] *Let $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$ be an analytic and $g(z) = \sum\limits_{n=1}^{\infty} b_n z^n$ is an analytic and convex in $\mathcal{U}$. If $f(z) \prec g(z)$, then $|a_n| \leq |b_1|$, for $n = 1, 2, \ldots$.*

Remark 3.2. Since Lemma 3.1 can be applied only if $g(z)$ is convex in $\mathcal{U}$. But the right hand side in (1.7) namely $\frac{(A+1)h(z)-(A-1)}{(B+1)h(z)-(B-1)}$ (where $h(z)$ is given as in (1.8)) is not convex in $\mathcal{U}$. So we find the coefficient inequalities for the fixed values of $\beta = 0$ and $\beta = 1$.

The following result was obtained by Noor and Malik in [9].

Lemma 3.3. [9] *Let the function* $\hat{p}_{k,\alpha}(z)$ *be defined as in (1.4) and let* $p(z) \in \mathcal{P}$ *satisfy the condition*

$$p(z) \prec \frac{(A+1)\hat{p}_{k,\alpha}(z) - (A-1)}{(B+1)\hat{p}_{k,\alpha}(z) - (B-1)}. \tag{3.1}$$

Then

$$\mid p_n \mid \leq \frac{|\delta_1|(A-B)}{2}, \quad (n \geq 1). \tag{3.2}$$

Remark 3.4. Similar result fails if $\frac{(A+1)k(z)-(A-1)}{(B+1)k(z)-(B-1)}$, as $k(z) = z + \sqrt{1+z^2}$ is starlike but not convex.

Theorem 3.5. *Let* $k - \mathcal{CL}(A, B, \alpha, 1, \lambda, t, b)$*, then for* $n \geq 2$

$$\mid a_n \mid \leq \prod_{j=1}^{n-1} \frac{\left| b\, j\, u_j \varphi_j(\lambda)\delta_1(A-B) - 2j\left[\varphi_j(\lambda+1) - u_j\varphi_j(\lambda)\right] B \right|}{2(j+1)[\varphi_{j+1}(\lambda+1) - u_{j+1}\varphi_{j+1}(\lambda)]}, \tag{3.3}$$

where δ_1 *is defined as in (1.5) and* $u_n = 1 + t + t^2 + \cdots + t^{n-1}$.

Proof. By the definition of $k - \mathcal{CL}(A, B, \alpha, \lambda, t, b)$, we have

$$1 + \frac{1}{b}\left(\frac{(1-t)\left(R^{\lambda+1}f(z)\right)'}{\left(R^{\lambda}f(z) - R^{\lambda}f(tz)\right)'} - 1 \right) = p(z), \tag{3.4}$$

where $p(z) \in \mathcal{P}$ and satisfies the subordination condition

$$p(z) \prec \frac{(A+1)\hat{p}_{k,\alpha}(z) - (A-1)}{(B+1)\hat{p}_{k,\alpha}(z) - (B-1)}.$$

Equivalently (3.4) can be rewritten as

$$\begin{aligned}
&1 + \frac{1}{b}\left(\frac{\sum\limits_{n=2}^{\infty} n\left[\varphi_n(\lambda+1) - u_n\varphi_n(\lambda)\right] a_n z^{n-1}}{1 + \sum\limits_{n=2}^{\infty} n u_n \varphi_n(\lambda) a_n z^{n-1}} - 1 \right) \\
&= 1 + \sum_{n=1}^{\infty} p_n z^n \sum_{n=2}^{\infty} n\left[\varphi_n(\lambda+1) - u_n\varphi_n(\lambda)\right] a_n z^{n-1} \\
&= b\left(1 + \sum_{n=2}^{\infty} n u_n \varphi_n(\lambda) a_n z^{n-1}\right) \sum_{n=1}^{\infty} p_n z^n.
\end{aligned}$$

Equating the coefficients of z^{n-1} on both sides of the above equation, we have

$$n\left[\varphi_n(\lambda+1) - u_n\varphi_n(\lambda)\right] a_n = b\sum_{j=1}^{n-1}(n-j)u_{n-j}\varphi_{n-j}(\lambda)\, a_{n-j} p_j$$

which implies that

$$n\left[\varphi_n(\lambda+1) - u_n\varphi_n(\lambda)\right] \mid a_n \mid \leq b\sum_{j=1}^{n-1}(n-j)u_{n-j}\varphi_{n-j}(\lambda) \mid a_{n-j} \mid \mid p_j \mid. \tag{3.5}$$

Since $p \in \mathcal{P}$, by Lemma 3.3, we obtain

$$| p_j | \leq \frac{|\delta_1|A - B}{2}.$$

Following the steps as in Theorem 2.6 of Noor and Malik [9], we can establish the assertion of the Theorem 3.5. □

If $b = 1$, $t = 0$, $\alpha = 0$ and $\lambda = 0$ in the Theorem 3.5, we get the following result.

Corollary 3.6. [9] *Let $f \in k - \mathcal{CL}(A, B)$, then*

$$| a_n | \leq \frac{1}{n} \prod_{j=0}^{n-2} \frac{|\delta_1(A - B) - 2jB|}{2(j + 1)} \quad (n \geq 2).$$

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Kadhavoor Ragavan Karthikeyan
Department of Applied Mathematics and Science,
National University of Science & Technology
(By Merger of Caledonian College of Engineering and Oman Medical College),
Sultanate of Oman
e-mail: kr_karthikeyan1979@yahoo.com

Seetharam Varadharajan
Mathematics Section, Department of IT,
University of Technology and Applied Sciences – Al Mussanah,
Sultanate of Oman
e-mail: svrajanram@gmail.com

Sakkarai Lakshmi
Mathematics Section, Department of IT,
University of Technology and Applied Sciences – Al Mussanah,
Sultanate of Oman
e-mail: laxmirmk@gmail.com

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On certain properties of some subclasses of univalent functions

Milutin Obradović and Nikola Tuneski

Abstract. In this paper we determine the disks $|z| < r \leq 1$ where for different classes of univalent functions, we have the property

$$\operatorname{Re}\left\{2\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (|z| < r).$$

Mathematics Subject Classification (2010): 30C45.

Keywords: Univalent, inverse functions, coefficients, sharp bound.

Let $\mathcal{A}$ denote the family of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ satisfying the normalization $f(0) = 0 = f'(0) - 1$.

Further, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{D}$, and $\mathcal{S}^\star$ and $\mathcal{K}$ be the subclasses of $\mathcal{A}$ of functions that are starlike and convex in $\mathbb{D}$, respectively. Next, let $\mathcal{U}$ denote the set of all $f \in \mathcal{A}$ satisfying the condition

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1 \qquad (z \in \mathbb{D}).$$

More on this class can be found in [4, 5, 9].

Next, by $\mathcal{G}$ we denote the class of all $f \in \mathcal{A}$ in $\mathbb{D}$ satisfying the condition

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{3}{2} \qquad (z \in \mathbb{D}).$$

More about the class $\mathcal{G}$ one can find in [2] and [7].

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In their paper ([3]) Miller and Mocanu introduced the classes of α-convex functions $f \in \mathcal{A}$ by the next condition:

$$\operatorname{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > 0 \qquad (z \in \mathbb{D}), \tag{1}$$

where $\frac{f(z)f'(z)}{z} \neq 0$ for all $z \in \mathbb{D}$, and $\alpha \in \mathbb{R}$. Those classes they denoted by $\mathcal{M}_\alpha$ and proved the next

Theorem A.
(a) $\mathcal{M}_\alpha \subseteq \mathcal{S}^\star$ *for every* $\alpha \in \mathbb{R}$*;*
(b) $\mathcal{M}_1 = \mathcal{K} \subseteq \mathcal{M}_\alpha \subseteq \mathcal{S}^\star$ *for* $0 \leq \alpha \leq 1$*;*
(c) $\mathcal{M}_\alpha \subset \mathcal{M}_1 = \mathcal{K}$ *for* $\alpha > 1$.

In [8] the authors proved
Theorem B.
(a) $\mathcal{M}_\alpha \subset \mathcal{U}$ *for* $\alpha \leq -1$*;*
(b) $\mathcal{M}_\alpha$ *is not subset of* $\mathcal{U}$ *for* $0 \leq \alpha \leq 1$.

Choosing $\alpha = -1$ in Theorem A(a) and Theorem B(a), from (1), we have that the condition

$$\operatorname{Re}\left\{2\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} > 1 \qquad (z \in \mathbb{D}) \tag{2}$$

implies $f \in \mathcal{S}^\star \cap \mathcal{U}$, i.e., the above inequality is sufficient for univalence in the unit disc. As expected, it is not necessary condition for univalence, i.e., univalent functions does not necessarily have property (2). See functions f_2 and f_3 analysed bellow.

But, is the following weaker inequality necessary for univalence

$$\operatorname{Re}\left\{2\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in \mathbb{D})?$$

The answer is also negative. Even more, it is not necessary condition even for starlikeness, nor for the classes $\mathcal{U}$ and $\mathcal{G}$.

Namely, let consider the differential operator

$$D(f;z) := 2\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} \tag{3}$$

and the functions

$$k(z) = \frac{z}{(1-z)^2}, \; f_1(z) = \frac{z}{1-z^2}, \; f_2(z) = -\log(1-z),$$

and

$$f_3(z) = \frac{z(1-\frac{1}{\sqrt{2}}z)}{1-z^2}.$$

Then, we have, respectively,

$$
\begin{aligned}
D(k;z) &= 1 + \frac{1+z^2}{1-z^2},\\
D(f_1;z) &= 1 + \frac{1-z^2}{1+z^2},\\
D(f_2;z) &= -\frac{z(2+\log(1-z))}{(1-z)\log(1-z)},\\
D(f_3;z) &= \frac{-\sqrt{2}z^3+3z^2-3\sqrt{2}z+2}{(1-\frac{1}{\sqrt{2}}z)(1-\sqrt{2}z+z^2)}.
\end{aligned}
$$

From the previous remark we easily conclude that the functions k and f_1 belong to the class $\mathcal{S}^\star \cap \mathcal{U}$, but for the function f_2 (which is convex) for $z = r$, $0 \le r < 1$, we have

$$D(f_2;r) = -\frac{r(2+\log(1-r))}{(1-r)\log(1-r)} < 0$$

if $1 - e^{-2} = 0.86466\ldots \le r < 1$. Also, we note that $f_2 \notin \mathcal{U}$.

For the function f_3, in [6], the authors showed that it is close-to-convex and univalent in $\mathbb{D}$, but not in $\mathcal{U}$. Additionally, $\mathrm{Re}[D(f;z)] > 0$ does not hold on the unit disk. Indeed, let we put

$$D(f_3;z) =: \frac{g(z)}{h(z)}, \tag{4}$$

where

$$g(z) = -\sqrt{2}z^3 + 3z^2 - 3\sqrt{2}z + 2$$

and

$$h(z) = \left(1 - \frac{1}{\sqrt{2}}z\right)(1-\sqrt{2}z+z^2),$$

and use $z = r$, $0 \le r < 1$. Then it is evident that $h(r) > 0$ for all $0 \le r < 1$. Also we have $g'(r) = -3(\sqrt{2}r^2 - 2r + \sqrt{2}) < 0$ for all $r \in [0,1)$, which implies that $g(r)$ is a decreasing function on the interval $[0,1)$. Thus, $2 = g(0) \ge g(r) > g(1/\sqrt{2}) = 0$ for $0 \le r < 1/\sqrt{2}$, and $g(r) \le 0$ for $\frac{1}{\sqrt{2}} \le r < 1$. Now, from (4), we easily conclude that the condition $\mathrm{Re}[D(f;z)] > 0$ is not satisfied for the function f_3 in the disc $|z| < r$, where $\frac{1}{\sqrt{2}} \le r < 1$, i.e., for close-to-convex functions, $\mathrm{Re}[D(f;z)] > 0$ on a disk with radius smaller then $\frac{1}{\sqrt{2}} = 0.7071\ldots$.

The above analysis raises the question of finding radius r_* for each of the classes defined above, such that $\mathrm{Re}[D(f;z)] > 0$ at least in the disc $|z| < r_*$. The next theorem answers this question. We don't know if the values for r_* are the best possible.

Theorem 1. *Let $D(f;z)$ be defined by* (3). *Then*

$$\mathrm{Re}[D(f;z)] > 0 \qquad (|z| < r_*)$$

in each of the following cases:

(*i*) *$f \in \mathcal{U}$ and $r_* = r_1 = 0.839\ldots$ is the root of the equation $r^3 + 2r^2 - 2 = 0$;*

(ii) $f \in \mathcal{S}^\star(1/2)$ *and* $r_* = r_2 = \sqrt{\frac{\sqrt{5}-1}{2}} = 078615\ldots$;
(iii) $f \in \mathcal{G}$ *and* $r_* = r_3 = \frac{2}{3} = 0.666\ldots$;
(iv) $f \in \mathcal{S}^\star$ *and* $r_* = r_4 = \frac{1}{2} = 0.5$;
(v) $f \in \mathcal{S}$ *and* $r_* = r_5 = \frac{1}{4} = 0.25$.

Proof. (i) First, from the definition of the class $\mathcal{U}$, we easily conclude that $f \in \mathcal{U}$ if, and only if, there exists a function ϕ, analytic in $\mathbb{D}$ with $|\phi(z)| \le 1$ in $\mathbb{D}$, such that

$$\left[\frac{z}{f(z)}\right]^2 f'(z) = 1 + z^2\phi(z). \tag{5}$$

From (5), after some calculations, we obtain that

$$2\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} = 2\frac{1-\frac{1}{2}z^3\phi'(z)}{1+z^2\phi(z)}. \tag{6}$$

Since $|\phi(z)| \le 1$, then

$$|\phi'(z)| \le \frac{1-|\phi(z)|^2}{1-|z|^2}, \tag{7}$$

(see [1, p.198]) and from here

$$\left|\frac{1}{2}z^3\phi'(z)\right| \le \frac{|z^3|}{2(1-|z|^2)}(1-|\phi(z)|^2) < 1-|\phi(z)|^2, \tag{8}$$

because $\frac{|z^3|}{2(1-|z|^2)} < 1$ for $|z| < r_1$. Also,

$$|z^2\phi(z)| < |r_1^2\phi(z)| < \frac{1}{\sqrt{2}}|\phi(z)|, \tag{9}$$

since

$$r_1^2 = 0.7044\ldots < \frac{1}{\sqrt{2}} = 0.7071\ldots.$$

Finally, by using (7),(8) and (9), we have

$$\begin{aligned}\left|\arg\left[2\frac{1-\frac{1}{2}z^3\phi'(z)}{1+z^2\phi(z)}\right]\right| &\le |\arg\left[1-\frac{1}{2}z^3\phi'(z)\right]| + |\arg(1+z^2\phi(z))| \\ &< \arcsin(1-|\phi(z)|^2) + \arcsin\left(\frac{1}{\sqrt{2}}|\phi(z)|\right) \\ &= \arcsin\sqrt{1-\frac{1}{2}|\phi(z)|^2} \\ &\le \arcsin 1 \\ &= \frac{\pi}{2},\end{aligned}$$

which implies $\mathrm{Re}[D(f;z)] > 0$.

(ii) Since $f \in \mathcal{S}^\star(1/2)$, we can put

$$\frac{zf'(z)}{f(z)} = \frac{1}{1-\omega(z)},$$

where ω is analytic in $\mathbb{D}$, $\omega(0)=0$ and $|\omega(z)|<1$ for all $z\in\mathbb{D}$. From here we have that

$$\frac{zf''(z)}{f'(z)}=\frac{z\omega'(z)}{1-\omega(z)}+\frac{1}{1-\omega(z)}-1,$$

and so

$$D(f;z)=2-\frac{z\omega'(z)-\omega(z)}{1-\omega(z)}.$$

Since $\omega(0)=0$ and $|\omega(z)|<1$, $z\in\mathbb{D}$, implies that $\left|\frac{\omega(z)}{z}\right|\le 1$, $z\in\mathbb{D}$, then by using the estimate (7) (with $\frac{\omega(z)}{z}$ in stead of ϕ), we obtain

$$|z\omega'(z)-\omega(z)|\le\frac{r^2-|\omega(z)|^2}{1-r^2}, \tag{10}$$

(where $|z|=r$ and $|\omega(z)|\le r$). Further, we have

$$\begin{aligned}\mathrm{Re}[D(f;z)]&\ge 2-\frac{|z\omega'(z)-\omega(z)|}{1-|\omega(z)|}\\&\ge 2-\frac{1}{1-r^2}\frac{r^2-|\omega(z)|^2}{1-|\omega(z)|}\\&=2-\frac{1}{1-r^2}\varphi(t),\end{aligned}$$

where we put $|\omega(z)|=t$, $0\le t\le r$ and $\varphi(t)=\frac{r^2-t^2}{1-t}$. By elementary calculation we obtain that $\varphi(t)\le 2(1-\sqrt{1-r^2})$ for $t\in[0,r]$. This implies that

$$\mathrm{Re}[D(f;z)]\ge 2-\frac{2(1-\sqrt{1-r^2})}{1-r^2}=2\frac{\sqrt{1-r^2}-r^2}{1-r^2}>0,$$

since $|z|=r<\sqrt{\frac{\sqrt{5}-1}{2}}=r_2$.

(iii) For $f\in\mathcal{G}$ in [2] is proven that

$$\frac{zf'(z)}{f(z)}\prec\frac{1-z}{1-\frac{z}{2}},$$

i.e., that

$$\frac{zf'(z)}{f(z)}=\frac{1-\omega(z)}{1-\frac{\omega(z)}{2}},$$

where ω is analytic in $\mathbb{D}$ such that $\omega(0)=0$ and $|\omega(z)|<1$ for $z\in\mathbb{D}$. From the last relation we easily obtain

$$D(f;z)=2-\frac{\omega(z)}{2-\omega(z)}+\frac{z}{(1-\omega(z))(2-\omega(z))}\omega'(z)$$

and from here

$$\mathrm{Re}[D(f;z)]\ge 2-\frac{|\omega(z)|}{2-|\omega(z)|}-\frac{|z|}{(1-|\omega(z)|)(2-|\omega(z)|)}|\omega'(z)|.$$

Applying the inequality (7), we give

$$\mathrm{Re}[D(f;z)] \geq 2 - \frac{|\omega(z)|}{2-|\omega(z)|} - \frac{|z|}{(1-|\omega(z)|)(2-|\omega(z)|)}\frac{1-|\omega(z)|^2}{1-|z|^2},$$

or, if we use $|\omega(z)| \leq r$, where $|z| = r$:

$$\mathrm{Re}[D(f;z)] \geq 2 - \frac{r}{1-r} = \frac{2-3r}{1-r} > 0,$$

since $r < \frac{2}{3} = r_3$.

(iv) We can use relation (7) and the same method as in the previous cases. Namely, now we can put

$$\frac{zf'(z)}{f(z)} = \frac{1+\omega(z)}{1-\omega(z)},$$

where ω is analytic in $\mathbb{D}$, $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. Then,

$$\frac{zf''(z)}{f'(z)} = 2\frac{z\omega'(z)}{1-\omega^2(z)} + \frac{1+\omega(z)}{1-\omega(z)} - 1$$

and after that

$$D(f;z) = \frac{2}{1-\omega(z)} - 2\frac{z\omega'(z)}{1-\omega^2(z)}.$$

Finally, we have (using $|\omega(z)| \leq r$, where $|z| = r$):

$$\begin{aligned}
\mathrm{Re}[D(f;z)] &\geq \mathrm{Re}\frac{2}{1-\omega(z)} - 2\frac{|z||\omega'(z)|}{1-|\omega(z)|^2} \\
&\geq \frac{2}{1+|\omega(z)|} - 2\frac{|z|}{1-|\omega(z)|^2}\frac{1-|\omega(z)|^2}{1-|z|^2} \\
&\geq \frac{2}{1+r} - \frac{2r}{1-r^2} \\
&= 2\frac{1-2r}{1-r^2} > 0,
\end{aligned}$$

if $|z| = r < \frac{1}{2} = r_4$.

(v) If $f \in \mathcal{S}$ then, from the classical result (see [1, p.32]), we have

$$\left|\log\frac{zf'(z)}{f(z)}\right| \leq \log\frac{1+r}{1-r}, \quad |z| = r < 1.$$

If we put $w = \log\frac{zf'(z)}{f(z)}$ and $R = \log\frac{1+r}{1-r}$, then we have $\frac{zf'(z)}{f(z)} = e^w$, where $|w| \leq R$. If we choose $r \leq \tanh\frac{1}{2} = \frac{e-1}{e+1} = 0.46...$, then we have $R \leq 1$. For such R the function e^w is convex with positive real coefficients that maps the unit disk onto a region that is symmetric with respect to the real axes ($\overline{e^w} = e^{\overline{w}}$), with diameter end points for $w = -1$ and $w = 1$. This implies that

$$\mathrm{Re}\frac{zf'(z)}{f(z)} = \mathrm{Re}(e^w) \geq e^{-R} = \frac{1-r}{1+r}.$$

Also, from the relation for functions from the class $\mathcal{S}$ (see [1, Theorem 2.4, p.32]) we have

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \le \frac{4r}{1-r^2}$$

and from here

$$\mathrm{Re}\frac{zf''(z)}{f'(z)} \le \frac{2r^2}{1-r^2} + \frac{4r}{1-r^2} = 2\frac{2r+r^2}{1-r^2}.$$

Finally,

$$\mathrm{Re}[D(f;z)] \ge 2\frac{1-r}{1+r} - 2\frac{2r+r^2}{1-r^2} = 2\frac{1-4r}{1-r^2} > 0$$

if $|z| = r < \frac{1}{4} = r_5$. □

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Milutin Obradović
Department of Mathematics, Faculty of Civil Engineering, University of Belgrade,
Bulevar Kralja Aleksandra 73, 11000, Belgrade, Serbia
e-mail: `obrad@grf.bg.ac.rs`

Nikola Tuneski
Department of Mathematics and Informatics, Faculty of Mechanical Engineering,
Ss. Cyril and Methodius University in Skopje,
Karpoš II b.b., 1000 Skopje, Republic of North Macedonia
e-mail: `nikola.tuneski@mf.edu.mk`

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Differential subordination implications for certain Carathéodory functions

Meghna Sharma, Sushil Kumar and Naveen Kumar Jain

Abstract. In this article, we wish to establish some first order differential subordination relations for certain Carathéodory functions with nice geometrical properties. Moreover, several implications are determined so that the normalized analytic function belongs to various subclasses of starlike functions.

Mathematics Subject Classification (2010): 30C45.

Keywords: Differential subordination, Carathéodory function, starlike functions, sufficient conditions.

1. Introduction

Denote the collection of all functions f which are analytic on the open unit disc by $\mathscr{H}$. Let $\mathcal{A} \subset \mathscr{H}$ be the subclass consisting of analytic functions given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and normalised by the conditions $f(0) = 0$ and $f'(0) - 1 = 0$. Further, let $\mathcal{S}^*$ and $\mathcal{C}$ denote the subclasses of univalent function consisting of starlike and convex functions, characterized by the quantities $zf'(z)/f(z)$ and $1+zf''(z)/f'(z)$ lying in the interior of the right half plane respectively. Let f and g be members of $\mathscr{H}$. We say f is subordinate to g (written as $f \prec g$) if there exists a function $w \in \mathscr{H}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Equivalently, if g is univalent in $\mathbb{D}$, then the conditions $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$ together gives $f \prec g$. For more details, see [15]. The unified class of starlike functions $\mathcal{S}^*_{\varphi} := \{f \in \mathcal{A} : zf'(z)/f(z) \prec \varphi(z);\ \text{for all } z \in \mathbb{D}\}$ where φ is analytic, univalent, $\varphi(\mathbb{D})$ is starlike with respect to $\varphi(0)=1$ and $\operatorname{Re}(\varphi) > 0$, was introduced and studied by Ma and Minda [13]. Various subclasses of starlike functions have been studied by considering

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different choices of φ in recent years. For $\varphi(z) := (1+Az)/(1+Bz), (-1 \leq B < A \leq 1)$, the class $\mathcal{S}^*_\varphi$ reduces to the class $\mathcal{S}^*[A, B]$, introduced by Janowski [9]. A function $f \in \mathscr{H}$ is said to be a Carathéodory function if $f(0) = 1$ and $\mathrm{Re}(f(z)) > 0$. The class of such functions is denoted by $\mathcal{P}$. On taking some Carathéodory functions $\varphi(z) := e^z$, $\phi_q(z)$, $\phi_0(z), \phi_c(z)$, $\phi_{\lim}(z)$, $\mathcal{Q}(z)$, ϕ_{SG}, $\phi_s(z)$, the class $\mathcal{S}^*_\varphi$ reduce to subclasses $\mathcal{S}^*_e$ [14], $\mathcal{S}^*_q$ [19], $\mathcal{S}^*_R$ [11], $\mathcal{S}^*_c$ [20], $\mathcal{S}^*_{LC}$ [22], $\mathcal{S}^*_B$ [5], $\mathcal{S}^*_{SG}$ [8], $\mathcal{S}^*_s$ [4] respectively, where

$$\phi_q(z) := z + \sqrt{1+z^2},\ \phi_0(z) := 1 + \frac{z}{k}.\frac{k+z}{k-z}; \quad k = 1+\sqrt{2},$$

$$\phi_c(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3},\ \phi_{\lim}(z) := 1 + \sqrt{2}z + \frac{z^2}{2},\ \phi_s(z) := 1 + \sin z.$$

Recently, Kumar *et al.*[5] introduced and studied differential subordination relations and radius estimates for the class $\mathcal{S}^*_B := \mathcal{S}^*(\mathcal{Q}(z)$, where

$$\mathcal{Q}(z) := e^{e^z - 1} \tag{1.1}$$

In 2020, Goel and Kumar [8] studied the subclass $\mathcal{S}^*_{SG} := \mathcal{S}^*(\phi_{SG})$, where

$$\phi_{SG}(z) = 2/(1+e^{-z}) \text{ for all z } \in \mathbb{D}. \tag{1.2}$$

These subclasses of starlike functions are well associated with the right half plane of the complex plane.

In 1989, for $p \in \mathcal{P}$, Nunokawa *et al.* [17] proved that the differential subordination $1 + zp'(z) \prec 1 + z$ implies $p(z) \prec 1 + z$. Further, authors [18] established sufficient conditions for starlike functions discussed by Silverman [21] to be strongly convex and strongly starlike in $\mathbb{D}$. In 2006, Kanas [10] determined the conditions for the functions to map $\mathbb{D}$ onto hyperbolic and parabolic regions using the concept of differential subordination. In 2007, Ali *et al.* [2] obtained conditions on $\beta \in \mathbb{R}$ for which $1 + \beta zp'(z)/p^j(z) \prec (1+Dz)/(1+Ez)$, $j = 0, 1, 2$ implies $p(z) \prec (1+Az)/(1+Bz)$, where $A, B, D, E \in [-1, 1]$. Later, Kumar and Ravichandran [12] determined sharp upper bounds on β such that $1 + \beta zp'(z)/p^j(z), j = 0, 1, 2$ is subordinate to some Carathéodory functions like $e^z, \phi_0(z)$ etc. implies $p(z) \prec e^z$ and $(1+Az)/(1+Bz)$. For more such results, we refer [1, 3, 7, 6].

In the present paper, we determine sharp estimate on β so that $p(z) \prec \phi_q(z)$, $\mathcal{Q}(z)$, $\phi_c(z)$, $\phi_0(z)$, $\phi_{\lim}(z)$, $\phi_s(z)$, $\phi_{SG}(z)$ whenever $1 + \beta zp'(z)/p^j(z) \prec \mathcal{Q}(z)$ and $\phi_{SG}(z)$; $(j = 0, 1, 2)$. Further the best possible bound on β is computed such that $p(z) \prec \mathcal{Q}(z)$ whenever $1 + \beta zp'(z)/p^j(z) \prec \phi_c(z)$; $(j = 0, 1, 2)$. At last, the upper bound on β is estimated so that the subordination $1+\beta zp'(z)/p^j(z) \prec \phi_0(z)$ and $\phi_c(z)$ implies $p(z) \prec \phi_{SG}(z)$. Moreover, sufficient conditions are obtained for an analytic function f to be a member of a certain subclass of starlike function.

2. Main results

First, we recall following lemma which plays a vital role in our proofs.

Lemma 2.1. [16, Theorem 3.4h, p.132] *Let $q : \mathbb{D} \to \mathbb{C}$ be analytic, and ψ and v be analytic in a domain $U \supseteq q(\mathbb{D})$ with $\psi(w) \neq 0$ whenever $w \in q(\mathbb{D})$. Set*

$$Q(z) := zq'(z)\psi(q(z)) \quad \textit{and} \quad h(z) := v(q(z)) + Q(z), z \in \mathbb{D}.$$

Suppose that

(i) *either $h(z)$ is convex, or $Q(z)$ is starlike univalent in $\mathbb{D}$ and*

(ii) $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0, z \in \mathbb{D}$.

If p is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subset U$ and

$$v(p(z)) + zp'(z)\psi(p(z)) \prec v(q(z)) + zq'(z)\psi(q(z))$$

then $p \prec q$, and q is the best dominant.

Throughout this paper, the following notations will be used:

$$\Psi_\beta(z, p(z)) = 1 + \beta zp'(z),\ \Lambda_\beta(z, p(z)) = 1 + \beta\frac{zp'(z)}{p(z)},\ \text{and}$$

$$\Theta_\beta(z, p(z)) = 1 + \beta\frac{zp'(z)}{p^2(z)}.$$

Theorem 2.2. *Let $\mathcal{Q}(z) \in \mathcal{P}$ be defined by* (1.1) *and further*

$$\mathcal{L} = \int_{-1}^{0} \frac{e^{e^t-1}-1}{t}dt \quad \text{and} \quad \mathfrak{U} = \int_{0}^{1} \frac{e^{e^t-1}-1}{t}dt. \tag{2.1}$$

Assume p to be an analytic function in $\mathbb{D}$ with $p(0) = 1$. If $\Psi_\beta(z, p(z)) \prec \mathcal{Q}(z)$, then

(a) $p(z) \prec \phi_q(z)$ *for* $\beta \geq \frac{1}{\sqrt{2}}\mathfrak{U} \approx 1.49762$.

(b) $p(z) \prec \mathcal{Q}(z)$ *for* $\beta \geq \frac{1}{1-e^{(e^{-1}-1)}}\mathcal{L} \approx 1.446103$.

(c) $p(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{1}{2}\mathfrak{U} \approx 1.05898$.

(d) $p(z) \prec \phi_0(z)$ *for* $\beta \geq \left(3 + 2\sqrt{2}\right)\mathcal{L} \approx 3.94906$.

(e) $p(z) \prec \phi_{\lim}(z)$ *for* $\beta \geq \frac{2}{2\sqrt{2}+1}\mathfrak{U} \approx 1.10643$.

(f) $p(z) \prec \phi_s(z)$ *for* $\beta \geq \frac{1}{\sin 1}\mathfrak{U} \approx 2.51696$.

(g) $p(z) \prec \phi_{SG}(z)$ *for* $\beta \geq \frac{e+1}{e-1}\mathfrak{U} \approx 4.583145$.

The bounds in each case are sharp.

Proof. The analytic function $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ defined by

$$q_\beta(z) = 1 + \frac{1}{\beta}\int_0^z \frac{e^{e^t-1}-1}{t}dt$$

is a solution of the first order linear differential equation $1 + \beta zq'_\beta(z) = e^{e^z-1}$. For $w \in \mathbb{C}$, define the functions $v(w) = 1$ and $\psi(w) = \beta$. Now, the function $Q : \overline{\mathbb{D}} \to \mathbb{C}$ defined by

$$Q(z) = zq'_\beta(z)\psi(q_\beta(z)) = \beta zq'_\beta(z) = e^{e^z-1} - 1$$

is starlike in $\mathbb{D}$. Also, note that by analytic characterization of starlike functions, the function $h : \overline{\mathbb{D}} \to \mathbb{C}$ defined by $h(z) := v(q_\beta(z)) + Q(z)$ satisfies the inequality

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) > 0.$$

Therefore, the subordination $1+\beta zp'(z) \prec 1+\beta zq'_\beta(z)$ implies $p \prec q_\beta$ by Lemma 2.1. For suitable $\mathcal{P}(z)$, as $r \to 1$, $q_\beta(z) \prec \mathcal{P}(z)$ holds if the following inequalities holds:

$$\mathcal{P}(-1) < q_\beta(-1) < q_\beta(1) < \mathcal{P}(1). \tag{2.2}$$

By the transitivity property, the required subordination $p(z) \prec \mathcal{P}(z)$ holds if $q_\beta(z) \prec \mathcal{P}(z)$. The condition (2.2) turns out to be both necessary and sufficient for the subordination $p \prec \mathcal{P}$ to hold.

(a) Consider $\mathcal{P}(z) = \phi_q(z)$. Then the inequalities

$$q_\beta(-1) > -1 + \sqrt{2} \text{ and } q_\beta(1) < 1 + \sqrt{2}$$

reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{1}{2-\sqrt{2}}\mathcal{L} \quad \text{and} \quad \beta_2 = \frac{1}{\sqrt{2}}\mathfrak{U}$$

respectively.
Thus, the subordination $q_\beta \prec \phi_q$ holds whenever $\beta \geq \max\{\beta_1, \beta_2\} = \beta_2$.

(b) For $\mathcal{P}(z) = \mathcal{Q}(z)$, the inequalities $q_\beta(-1) > \mathcal{Q}(-1)$ and $q_\beta(1) < \mathcal{Q}(1)$ give $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{1}{1-e^{e^{-1}-1}}\mathcal{L} \quad \text{and} \quad \beta_2 = \frac{1}{e^{e-1}-1}\mathfrak{U}$$

respectively. Therefore, $q_\beta \prec \mathcal{Q}$ whenever $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(c) On taking $\mathcal{P}(z) = \phi_c(z)$, a simple calculation shows that the inequalities $q_\beta(-1) > \phi_c(-1)$ and $q_\beta(1) < \phi_c(1)$ give $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where $\beta_1 = (3/2)\mathcal{L}$ and $\beta_2 = (1/2)\mathfrak{U}$ respectively. Therefore, $q_\beta \prec \phi_c$ holds whenever $\beta \geq \max\{\beta_1, \beta_2\} = \beta_2$.

(d) On substituting $\mathcal{P}(z) = \phi_0(z)$, the inequalities

$$q_\beta(-1) > \phi_0(-1) \text{ and } q_\beta(1) < \phi_0(1)$$

give $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where $\beta_1 = (1/(3-2\sqrt{2}))\mathcal{L}$ and $\beta_2 = \mathfrak{U}$ respectively. Therefore, the subordination $q_\beta \prec \phi_0$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(e) Take $\mathcal{P}(z) = \phi_{\lim}(z)$. Then the inequalities

$$q_\beta(-1) > \frac{3}{2} - \sqrt{2} \text{ and } q_\beta(1) < \frac{3}{2} + \sqrt{2}$$

reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = (2/(2\sqrt{2}-1)\mathcal{L} \text{ and } \beta_2 = 2/(2\sqrt{2}+1)\mathfrak{U}$$

respectively.
Thus, the required subordination $q_\beta \prec \phi_{\lim}$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_2$.

(f) Take $\mathcal{P}(z) = \phi_s(z)$. Then the inequalities

$$q_\beta(-1) > 1 + \sin(-1) \text{ and } q_\beta(1) < 1 + \sin(1)$$

give $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where $\beta_1 = \mathcal{L}/\sin 1$ and $\beta_2 = \mathfrak{U}/\sin 1$ respectively. This shows that the subordination $q_\beta \prec \phi_s$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_2$.

(g) Set $\mathcal{P}(z) = 2/(1+e^{-z})$. Then $q_\beta(-1) > 2/(e+1)$ and $q_\beta(1) < 2e/(e+1)$ gives

$$\beta_1 = \frac{e+1}{e-1}\mathcal{L} \quad \text{and} \quad \beta_2 = \frac{e+1}{e-1}\mathfrak{U}.$$

Hence, the subordination holds true for $\beta \geq \beta_2$ since $\max\{\beta_1, \beta_2\} = \beta_2$.

Thus, we get the required result. $\square$

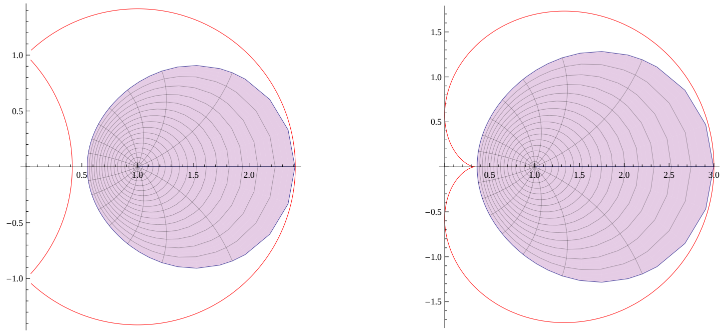

FIGURE 1. Sharpness for the case (a) and (b).

As an application of Theorem 2.2, we have the following sufficient conditions for starlikeness:

Corollary 2.3. *Set* $\mathfrak{M}(z) := 1 - zf'(z)/f(z) + zf''(z)/f'(z)$. *If the function* $f \in \mathcal{A}$ *satisfies* $1 + \beta\frac{zf'(z)}{f(z)}\mathfrak{M}(z) \prec \mathcal{Q}(z)$, *then*

(a) $f \in \mathcal{S}^*_q$ *if* $\beta \geq \left(1/\sqrt{2}\right)\mathfrak{U}$,
(b) $f \in \mathcal{S}^*_B$ *if* $\beta \geq \left(1/(1 - e^{(e^{-1}-1)})\right)\mathcal{L}$,
(c) $f \in \mathcal{S}^*_c$ *if* $\beta \geq (1/2)\,\mathfrak{U}$,
(d) $f \in \mathcal{S}^*_R$ *if* $\beta \geq \left(3 + 2\sqrt{2}\right)\mathcal{L}$,
(e) $f \in \mathcal{S}^*_{LC}$ *if* $\beta \geq \left(2/(2\sqrt{2}+1)\right)\mathfrak{U}$,
(f) $f \in \mathcal{S}^*_s$ *if* $\beta \geq (1/(\sin 1))\,\mathfrak{U}$
(g) $f \in \mathcal{S}^*_{SG}$ *if* $\beta \geq ((e+1)/(e-1))\,\mathfrak{U}$,

where $\mathfrak{U}$ *and* $\mathcal{L}$ *are given by* (2.1).

Theorem 2.4. *Let* $\mathfrak{U}$ *and* $\mathcal{L}$ *be given by* (2.1) *and* $\mathcal{Q}(z)$ *be given by* (1.1). *Let* p *be an analytic function in* $\mathbb{D}$ *with* $p(0) = 1$. *If* $\Lambda_\beta(z, p(z)) \prec \mathcal{Q}(z)$, *then*

(a) $p(z) \prec \phi_q(z)$ *for* $\beta \geq \frac{1}{\log(1+\sqrt{2})}\mathfrak{U} \approx 2.40301$.
(b) $p(z) \prec \mathcal{Q}(z)$ *for* $\beta \geq \frac{1}{e-1}\mathfrak{U} \approx 1.23260$.
(c) $p(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{1}{\log 3}\mathfrak{U} \approx 1.92784$.
(d) $p(z) \prec \phi_0(z)$ *for* $\beta \geq \frac{1}{\log\left(\frac{1+\sqrt{2}}{2}\right)}\mathcal{L} \approx 3.59966$.
(e) $p(z) \prec \phi_{\lim}(z)$ *for* $\beta \geq \frac{1}{\log(\sqrt{2}+3/2)}\mathfrak{U} \approx 1.98013$.
(f) $p(z) \prec \phi_s(z)$ *for* $\beta \geq \frac{1}{\log(1+\sin 1)}\mathfrak{U} \approx 3.4688$.
(g) $p(z) \prec \phi_{SG}(z)$ *for* $\beta \geq \frac{1}{1+\log 2-\log(1+e)}\mathfrak{U} \approx 5.57523$.

The bounds on β *are best possible.*

Proof. Consider the first order differential equation given by

$$1 + \beta\frac{z\breve{q}'_\beta(z)}{\breve{q}_\beta(z)} = e^{e^z-1}. \tag{2.3}$$

It is easy to verify that the analytic function $\breve{q}_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ defined by

$$\breve{q}_\beta(z) = \exp\left(\frac{1}{\beta}\int_0^z \frac{e^{e^t-1}-1}{t}dt\right)$$

is a solution of differential equation (2.3). On taking $v(w) = 1$ and $\psi(w) = \beta/w$, the functions $Q, h : \overline{\mathbb{D}} \to \mathbb{C}$ reduces to

$$Q(z) = z\breve{q}'_\beta(z)\psi(\breve{q}_\beta(z)) = \beta z\breve{q}'_\beta(z)/\breve{q}_\beta(z) = e^{e^z-1} - 1$$

and

$$h(z) = v(\breve{q}_\beta(z)) + Q(z) = 1 + Q(z) = e^{e^z-1}.$$

It is seen that the function Q is starlike and $\operatorname{Re}(zh'(z)/Q(z)) > 0$, for $z \in \mathbb{D}$. Hence,

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \beta\frac{z\breve{q}'_\beta(z)}{\breve{q}_\beta(z)} \quad \text{implies} \quad p(z) \prec \breve{q}_\beta(z)$$

which follows from Lemma 2.1. Proceeding as Theorem 2.2, proof is completed. □

Theorem 2.5. *Let* $\mathfrak{U}$ *and* $\mathcal{L}$ *be given by* (2.1) *and* $\mathcal{Q}(z)$ *be given by* (1.1)*. Assume* p *to be an analytic function in* $\mathbb{D}$ *with* $p(0) = 1$*. If* $\Theta_\beta(z, p(z)) \prec \mathcal{Q}(z)$*, then each of the following subordination holds:*

(a) $p(z) \prec \phi_q(z)$ *for* $\beta \geq \frac{1}{2-\sqrt{2}}\mathfrak{U} \approx 3.61556$.
(b) $p(z) \prec \mathcal{Q}(z)$ *for* $\beta \geq \frac{e^{e-1}}{e^{e-1}-1}\mathfrak{U} \approx 2.58089$.
(c) $p(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{3}{2}\mathfrak{U} \approx 3.17692$.
(d) $p(z) \prec \phi_0(z)$ *for* $\beta \geq 2\mathfrak{U} \approx 4.2359$.
(e) $p(z) \prec \phi_{\lim}(z)$ *for* $\beta \geq \frac{5+4\sqrt{2}}{7}\mathfrak{U} \approx 3.22438$.
(f) $p(z) \prec \phi_s(z)$ *for* $\beta \geq \frac{1+\sin 1}{\sin 1}\mathfrak{U} \approx 4.63491$.
(g) $p(z) \prec \phi_{SG}(z)$ *for* $\beta \geq \frac{2e}{e-1}\mathfrak{U} \approx 6.7011$.

The estimates on β *cannot be improved further.*

Proof. The function

$$\hat{q}_\beta(z) = \left(1 - \frac{1}{\beta}\int_0^z \frac{e^{e^t-1}-1}{t}dt\right)^{-1}$$

is the analytic solution of the differential equation

$$\beta\frac{z\hat{q}'_\beta(z)}{\hat{q}^2_\beta(z)} = e^{e^z-1} - 1.$$

Consider the functions $v(w) = 1$ and $\psi(w) = \beta/w^2$. Moreover, the function $Q(z) = z\hat{q}'_\beta(z)\psi(\hat{q}_\beta(z)) = e^{e^z-1} - 1$ is starlike in $\mathbb{D}$. Simple computation shows that the function $h(z) := 1 + Q(z)$ satisfies the inequality $\operatorname{Re}(zh'(z)/Q(z)) > 0, (z \in \mathbb{D})$. Now, by Lemma 2.1, we see that the subordination

$$1 + \beta\frac{zp'(z)}{p^2(z)} \prec 1 + \beta\frac{z\hat{q}'_\beta(z)}{\hat{q}^2_\beta(z)}$$

implies $p(z) \prec \hat{q}_\beta(z)$. Proceeding as Theorem 2.2, we conclude the proof. □

Theorem 2.6. *Let ϕ_{SG} be given by* (1.2) *and further*

$$I_{-} = \int_{-1}^{0} \frac{e^t - 1}{t(e^t+1)} dt \quad \textit{and} \quad I_{+} = \int_{0}^{1} \frac{e^t - 1}{t(e^t+1)} dt. \tag{2.4}$$

Assume p to be an analytic function in $\mathbb{D}$ with $p(0) = 1$. If the subordination

$$\Psi_\beta(z, p(z)) \prec \phi_{SG}(z)$$

holds, then each of the following subordination inclusion hold:

(a) $p(z) \prec \phi_q(z)$ for $\beta \geq \frac{1}{2-\sqrt{2}} I_- \approx 0.83117$.
(b) $p(z) \prec \phi_c(z)$ for $\beta \geq \frac{3}{2} I_- \approx 0.730335$.
(c) $p(z) \prec \phi_0(z)$ for $\beta \geq (3 + 2\sqrt{2}) I_- \approx 2.837797$.
(d) $p(z) \prec \mathcal{Q}(z)$ for $\beta \geq \frac{1}{1-e^{e^{-1}-1}} I_- \approx 1.039170$.
(e) $p(z) \prec \phi_{\lim}(z)$ for $\beta \geq \frac{2}{2\sqrt{2}-1} I_- \approx 0.53257$.
(f) $p(z) \prec \phi_s(z)$ for $\beta \geq \frac{1}{\sin 1} I_- \approx 0.578616$
(g) $p(z) \prec \phi_{SG}(z)$ for $\beta \geq \frac{e+1}{e-1} I_- \approx 1.05361$.

The bounds in each of the above case are sharp.

Proof. Consider the functions v and ψ defined as in Theorem 2.2. Define the function $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ by

$$q_\beta(z) = 1 + \frac{1}{\beta} \int_0^z \frac{e^t - 1}{t(e^t+1)} dt$$

Note that the function $q_\beta(z)$ is analytic solution of the differential equation

$$1 + \beta z q'_\beta(z) = 2/(1 + e^{-z}).$$

The function $Q(z) = zq'_\beta(z)\psi(q_\beta(z)) = (e^z - 1)/(e^z + 1)$ is starlike in $\mathbb{D}$ and $h(z) = 1 + Q(z)$ satisfies the inequality $\operatorname{Re}(zh'(z)/Q(z)) > 0$, $z \in \mathbb{D}$. Thus, applying Lemma 2.1, it follows that the subordination $1 + \beta z p'(z) \prec 1 + \beta z q'_\beta(z)$ implies $p(z) \prec q_\beta(z)$. Each of the subordination $p(z) \prec \mathcal{P}(z)$, for appropriate $\mathcal{P}$, from (a) to (g) holds if $q_\beta(z) \prec \mathcal{P}(z)$ holds. This subordination holds provided

$$\mathcal{P}(-1) < q_\beta(-1) < q_\beta(1) < \mathcal{P}(1).$$

These inequalities yield necessary and sufficient condition for the required subordination.

(a) Take $\mathcal{P}(z) = \phi_q(z)$. Then, the inequalities $q_\beta(-1) > -1+\sqrt{2}$ and $q_\beta(1) < 1+\sqrt{2}$ reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{1}{2-\sqrt{2}} I_- \quad \text{and} \quad \beta_2 = \frac{1}{\sqrt{2}} I_+$$

respectively. Therefore, $q_\beta \prec \phi_q$ whenever $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(b) Consider $\mathcal{P}(z) = \phi_c(z)$. A simple calculation shows that the inequalities $q_\beta(-1) > \phi_c(-1)$ and $q_\beta(1) < \phi_c(1)$ gives $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{3}{2} I_- \quad \text{and} \quad \beta_2 = \frac{1}{2} I_+$$

respectively.
Therefore, the subordination $q_\beta \prec \phi_c$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(c) On taking $\mathcal{P}(z) = \phi_0(z)$, the inequalities $q_\beta(-1) > \phi_0(-1)$ and $q_\beta(1) < \phi_0(1)$ give $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{1}{3 - 2\sqrt{2}} I_- \quad \text{and} \quad \beta_2 = I_+$$

respectively. Therefore, $q_\beta \prec \phi_0$ if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(d) Consider $\mathcal{P}(z) = \mathcal{Q}(z)$. From the inequalities $q_\beta(-1) > \mathcal{Q}(-1)$ and $q_\beta(1) < \mathcal{Q}(1)$, we note that $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{1}{1 - e^{e^{-1}-1}} I_- \quad \text{and} \quad \beta_2 = \frac{1}{e^{e-1} - 1} I_+$$

respectively. Thus, $q_\beta \prec \mathcal{Q}$ if $\beta \geq \max\{\beta_1, \beta_2\}$.

(e) Take $\mathcal{P}(z) = \phi_{\lim}(z)$. Then, the inequalities $q_\beta(-1) > \frac{3}{2} - \sqrt{2}$ and $q_\beta(1) < \frac{3}{2} + \sqrt{2}$ reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{2}{2\sqrt{2} - 1} I_- \quad \text{and} \quad \beta_2 = \frac{2}{2\sqrt{2} + 1} I_+$$

respectively. Thus, $q_\beta \prec \phi_{\lim}$ whenever $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(f) Take $\mathcal{P}(z) = \phi_s(z)$. Then, the inequalities $q_\beta(-1) > 1 + \sin(-1)$ and $q_\beta(1) < 1 + \sin(1)$ give $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{1}{\sin 1} I_- \quad \text{and} \quad \beta_2 = \frac{1}{\sin 1} I_+$$

respectively.
Therefore, the subordination $q_\beta \prec \phi_s$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(g) Let $\mathcal{P}(z) = \phi_{SG}(z)$. On simplifying the inequalities $q_\beta(-1) > 2/(e+1)$ and $q_\beta(1) < 2e/(e+1)$, we get β_1 and β_2, where

$$\beta_1 = \frac{e+1}{e-1} I_- \quad \text{and} \quad \beta_2 = \frac{e+1}{e-1} I_+$$

respectively and thus, $q_\beta \prec \phi_{SG}$ whenever $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

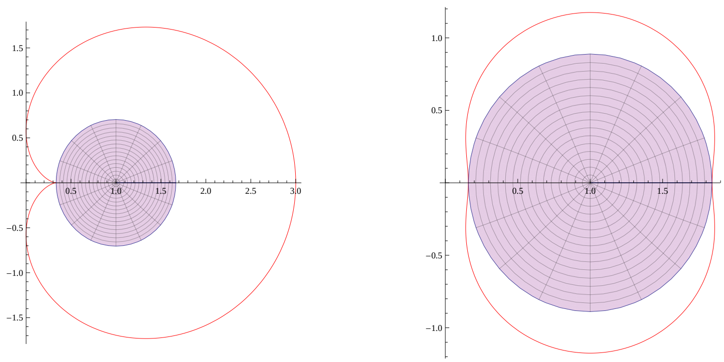

FIGURE 2. Sharpness for the case (b) and (f).

Hence, the result holds. $\square$

As an application of Theorem 2.6, we have the following sufficient conditions for starlikeness:

Corollary 2.7. *Let $f \in \mathcal{A}$ be analytic function which satisfies*

$$1 + \beta \frac{zf'(z)}{f(z)} \mathfrak{M}(z) \prec \phi_{SG}(z).$$

Then,

(a) $f \in \mathcal{S}_q^*$ *if* $\beta \geq \left(1/(2-\sqrt{2})\right) I_-$,
(b) $f \in \mathcal{S}_c^*$ *if* $\beta \geq (3/2)\, I_-$,
(c) $f \in \mathcal{S}_R^*$ *if* $\beta \geq (3+2\sqrt{2}) I_-$,
(d) $f \in \mathcal{S}_B^*$ *if* $\beta \geq \left(1/(1-e^{e^{-1}-1})\right) I_-$,
(e) $f \in \mathcal{S}_{LC}^*$ *if* $\beta \geq \left(2/(2\sqrt{2}-1)\right) I_-$,
(f) $f \in \mathcal{S}_s^*$ *if* $\beta \geq (1/(\sin 1))\, I_-$,

where $\mathfrak{M}(z)$ *is defined in Corollary 2.3.*

Theorem 2.8. *Let I_+ and I_- be given by 2.4 and ϕ_{SG} be given by* (1.2). *Assume p to be an analytic function in $\mathbb{D}$ with $p(0) = 1$. If $\Lambda_\beta(z, p(z)) \prec \phi_{SG}(z)$, then each of the following holds.*

(a) $p(z) \prec \phi_q(z)$ *for* $\beta \geq \frac{1}{\log(1+\sqrt{2})} I_- \approx 0.55242$.
(b) $p(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{1}{\log 3} I_- \approx 0.443185$.
(c) $p(z) \prec \phi_0(z)$ *for* $\beta \geq \frac{1}{\log\left(\frac{1+\sqrt{2}}{2}\right)} I_- \approx 2.58671$.
(d) $p(z) \prec \mathcal{Q}(z)$ *for* $\beta \geq \frac{1}{1-e^{-1}} I_- \approx 0.77024$.
(e) $p(z) \prec \phi_{\lim}(z)$ *for* $\beta \geq \frac{1}{\log(\sqrt{2}+3/2)} I_+ \approx 0.455206$.
(f) $p(z) \prec \phi_s(z)$ *for* $\beta \geq \frac{1}{\log(1+\sin 1)} I_+ \approx 0.79744$.
(g) $p(z) \prec \phi_{SG}(z)$ *for* $\beta \geq \frac{1}{1+\log 2-\log(1+e)} I_+ \approx 1.28167$.

The estimates on β are best possible.

Proof. Let the functions v and ψ be defined as in Theorem 2.4. Define the analytic function $\breve{q}_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ by

$$\breve{q}_\beta(z) = \exp\left(\frac{1}{\beta}\int_0^z \frac{e^t-1}{t(e^t+1)}\,dt\right),$$

which satisfies the differential equation

$$\frac{d\breve{q}'_\beta(z)}{dz} = \frac{1}{\beta z}\left(\frac{1-e^{-z}}{1+e^{-z}}\right)\breve{q}_\beta(z).$$

Now, observe that the function $Q(z) = z\breve{q}'_\beta(z)\psi(\breve{q}_\beta(z)) = \frac{1-e^{-z}}{1+e^{-z}}$ is starlike in $\mathbb{D}$. Also, it can be easily seen that the function h defined by $h(z) := v(\breve{q}_\beta(z)) + Q(z) = 1 + Q(z)$ satisfies the inequality $\operatorname{Re}(zh'(z)/Q(z)) > 0$, $z \in \mathbb{D}$. Therefore, the Lemma 2.1 states that the subordination $1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \beta\frac{z\breve{q}'_\beta(z)}{\breve{q}_\beta(z)}$ implies $p(z) \prec \breve{q}_\beta(z)$. As in the proof of Theorem 2.6, we conclude the result. $\square$

Theorem 2.9. *Let I_+ and I_- be given by* (2.4)*. Assume p to be an analytic function in $\mathbb{D}$ with $p(0)=1$. If $\Theta_\beta(z,p(z)) \prec \phi_{SG}(z)$, then*

(a) $p(z) \prec \phi_q(z)$ *for* $\beta \geq \frac{1}{2-\sqrt{2}} I_+ \approx 0.83117$.
(b) $p(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{3}{2} I_+ \approx 0.73033$.
(c) $p(z) \prec \phi_0(z)$ *for* $\beta \geq (2+2\sqrt{2}) I_- \approx 2.35090$.
(d) $p(z) \prec \mathcal{Q}(z)$ *for* $\beta \geq \frac{e^{e-1}}{e^{e-1}-1} I_+ \approx 0.59331$.
(e) $p(z) \prec \phi_{\lim}(z)$ *for* $\beta \geq \frac{5+4\sqrt{2}}{7} I_+ \approx 0.74124$.
(f) $p(z) \prec \phi_s(z)$ *for* $\beta \geq \frac{1+\sin 1}{\sin 1} I_+ \approx 1.06550$.
(g) $p(z) \prec \phi_{SG}(z)$ *for* $\beta \geq \frac{2e}{e-1} I_+ \approx 1.54049$.

All these estimates are sharp.

Proof. The function $\hat{q}_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ defined by

$$\hat{q}_\beta(z) = \left(1 - \frac{1}{\beta}\int_0^z \frac{e^t-1}{t(e^t+1)}dt\right)^{-1}$$

is clearly analytic in $\mathbb{D}$. It is noted that the function $\hat{q}_\beta(z)$ is a solution of the differential equation

$$1+\beta\frac{z\hat{q}'_\beta(z)}{\hat{q}^2_\beta(z)} = \frac{2}{1+e^{-z}}.$$

We take the functions v and ψ as in Theorem 2.5. Note that the function Q defined by $Q(z) = z\hat{q}'_\beta(z)\psi(\hat{q}_\beta(z)) = (1-e^{-z})/(1+e^{-z})$ is starlike in $\mathbb{D}$ and the function h defined as $h(z) := v(\hat{q}_\beta(z)) + Q(z) = 1 + Q(z)$ follows the inequality $\operatorname{Re}(zh'(z)/Q(z)) = \operatorname{Re}(zQ'(z)/Q(z)) > 0$. Therefore, as in view of Lemma 2.1, the subordination

$$1+\beta\frac{zp'(z)}{p^2(z)} \prec 1+\beta\frac{z\hat{q}'_\beta(z)}{\hat{q}^2_\beta(z)}$$

implies $p(z) \prec \hat{q}_\beta(z)$. Proceeding as in Theorem 2.6, proof is completed. □

Theorem 2.10. *Let p be an analytic function in $\mathbb{D}$ with $p(0)=1$. Then each of the following subordination implies $p(z) \prec \mathcal{Q}(z) := e^{e^z-1}$:*

(a) $\Psi_\beta(z,p(z)) \prec \phi_c(z)$ *if* $\beta \geq \frac{1}{1-e^{(e^{-1}-1)}} \approx 2.13430$.
(b) $\Lambda_\beta(z,p(z)) \prec \phi_c(z)$ *if* $\beta \geq \frac{e}{e-1} \approx 1.581976$.
(c) $\Theta_\beta(z,p(z)) \prec \phi_c(z)$ *if* $\beta \geq \frac{5e^{e-1}}{3(e^{e-1}-1)} \approx 2.030970$.

The bounds in each case are sharp.

Proof. (a) Define the analytic function $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ by

$$q_\beta(z) = 1 + \frac{1}{\beta}\left(\frac{4z}{3}+\frac{z^2}{3}\right)$$

It is easy to see that the function q_β satisfies the differential equation $\beta zq'(z) = \phi_c(z) - 1$. Proceeding as similar lines in Theorem 2.2, the required subordination holds if and only if,

$$e^{e^{-1}-1} < q_\beta(-1) < q_\beta(1) < e^{e-1}. \tag{2.5}$$

Simplifying the condition (2.5), we obtain the inequalities

$$\beta \geq \frac{1}{1-e^{(e^{-1}-1)}} = \beta_1 \quad \text{and} \quad \beta \geq \frac{5e}{3\,(e^e - e)} = \beta_2.$$

Thus, the required subordination holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(b) Define the analytic function $\breve{q}_\beta(z)$ by,

$$\breve{q}_\beta(z) = \exp\left(\frac{1}{\beta}\left(\frac{4z}{3} + \frac{z^2}{3}\right)\right)$$

which is a solution of the equation

$$\frac{d\breve{q}'_\beta(z)}{dz} = \frac{2(2+z)}{3\beta}\breve{q}_\beta(z).$$

Proceeding as similar lines in Theorem 2.4, the subordination $p(z) \prec e^{e^z-1}$ holds if $\beta \geq \max\{\breve{\beta}_1, \breve{\beta}_2\}$, where

$$\breve{\beta}_1 = \frac{e}{e-1} \text{ and } \breve{\beta}_2 = \frac{5}{3(e-1)}$$

are obtained from the inequalities $\breve{q}_\beta(-1) > e^{e^{-1}-1}$ and $\breve{q}_\beta(1) < e^{e-1}$ respectively.

(c) The differential equation

$$\frac{d\hat{q}'_\beta(z)}{dz} = \frac{2(2+z)}{3\beta}\hat{q}^2_\beta(z)$$

has an analytic solution

$$\hat{q}_\beta(z) = \left(1 - \frac{1}{\beta}\left(\frac{4z}{3} + \frac{z^2}{3}\right)\right)^{-1}$$

in $\mathbb{D}$. Therefore, proceeding as in Theorem 2.5, the required subordination $p(z) \prec e^{e^z-1}$ holds if $\beta \geq \max\{\hat{\beta}_1, \hat{\beta}_2\} = \hat{\beta}_2$, where

$$\hat{\beta}_1 = \frac{e^{\frac{1}{e}-1}}{1-e^{\frac{1}{e}-1}} \text{ and } \hat{\beta}_2 = \frac{5e^{e-1}}{3(e^{e-1}-1)}.$$

□

Corollary 2.11. *Let $f \in \mathcal{A}$ be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. If one of the following subordinations holds*

(a) $1 + \beta\frac{zf'(z)}{f(z)}\mathfrak{M}(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{1}{1-e^{(e^{-1}-1)}}$,

(b) $1 + \beta\mathfrak{M}(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{e}{e-1}$,

(c) $1 + \beta\left(\frac{zf'(z)}{f(z)}\right)^{-1}\mathfrak{M}(z) \prec \phi_c(z)$ *for* $\beta \geq \frac{5e^{e-1}}{3(e^{e-1}-1)}$,

then $f \in \mathcal{S}^*_B$, where $\mathfrak{M}(z)$ is defined in Corollary 2.3.

The next results provide best possible bound on β so that the subordination $1+\beta zp'(z)/p^j(z) \prec \phi_c(z)$, $\phi_0(z)(j = 0, 1, 2)$ implies the subordination $p(z) \prec \phi_{SG}(z)$. Proofs of the following results are omitted as similar to the previous Theorem 2.10.

Theorem 2.12. *Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Then the following subordinations hold for $p(z) \prec \phi_{SG}(z) := 2/(1 + e^{-z})$.*

(a) $\Psi_\beta(z, p(z)) \prec \phi_0(z)$ *if* $\beta \geq \frac{(e+1)(1-\sqrt{2}-2\log(2-\sqrt{2}))}{e-1} \approx 1.418226.$

(b) $\Lambda_\beta(z, p(z)) \prec \phi_0(z)$ *if* $\beta \geq \frac{1-\sqrt{2}-2\log(2-\sqrt{2})}{1+\log 2-\log(1+e)} \approx 1.725221.$

(c) $\Theta_\beta(z, p(z)) \prec \phi_0(z)$ *if* $\beta \geq \frac{2e(1-\sqrt{2}-2\log(2-\sqrt{2}))}{e-1} \approx 2.073612.$

The bounds on β in each case are sharp.

Theorem 2.13. *Let p be an analytic function in $\mathbb{D}$ which satisfies $p(0) = 1$. Then each of the following subordination is sufficient for $p(z) \prec \phi_{SG}(z)$.*

(a) $\Psi_\beta(z, p(z)) \prec \phi_c(z)$ *if* $\beta \geq \frac{5(e+1)}{3(e-1)} \approx 3.60659.$

(b) $\Lambda_\beta(z, p(z)) \prec \phi_c(z)$ *if* $\beta \geq \frac{5}{3(1+\log 2-\log(1+e))} \approx 4.387286.$

(c) $\Theta_\beta(z, p(z)) \prec \phi_c(z)$ *if* $\beta \geq \frac{10e}{3(e-1)} \approx 5.27326.$

The bounds on β in each case are sharp.

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Meghna Sharma
Department of Mathematics, University of Delhi,
Delhi-110 007, India
e-mail: `meghnasharma203@gmail.com`

Sushil Kumar
Bharati Vidyapeeth's College of Engineering,
Delhi-110063, India
e-mail: `sushilkumar16n@gmail.com`

Naveen Kumar Jain
Department of Mathematics, Aryabhatta College,
Delhi-110021, India
e-mail: `naveenjain05@gmail.com`

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Certain geometric properties of generalized Bessel-Maitland function

Amit Soni and Deepak Bansal

Abstract. In the present study, we first introduce Generalized Bessel-Maitland function $\mathbb{J}_{\zeta,a}^{\xi}(z)$ and then derive sufficient conditions under which the Generalized Bessel-Maitland function $\mathbb{J}_{\zeta,a}^{\xi}(z)$ have geometric properties like univalency, starlikeness and convexity in the open unit disk $\mathscr{D}$.

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1. Introduction and preliminaries

Let $\mathscr{H}$ denote the class of all functions analytic in the open unit disk

$$\mathscr{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and $\mathscr{A}$ be the class of all functions $f \in \mathscr{H}$ which are normalized by $f(0) = 0$ and $f'(0) = 1$. Each $f(z) \in \mathscr{A}$ has a Maclaurin series expansion of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... \tag{1.1}$$

Let $g, h \in \mathscr{H}$, we say that g is subordinated to h in $\mathscr{D}$, and write $g(z) \prec h(z)$, if there exists a function $\omega \in \mathscr{H}$ with $|\omega(z)| < |z|,\ z \in \mathscr{D}$, such that $g(z) = h(\omega(z))$ in $\mathscr{D}$. In particular, if h is univalent in $\mathscr{D}$, then we have:

$$g(z) \prec h(z) \iff g(0) = h(0) \text{ and } g(\mathscr{D}) \subset h(\mathscr{D}).$$

For a given $0 \leq \beta < 1$, a function $g \in \mathscr{A}$ is called starlike function of order β, if $\Re\left(zg'(z)/g(z)\right) > \beta,\ z \in \mathscr{D}$ class of such functions denoted by $\mathscr{S}^*(\beta)$. Similarly, for $0 \leq \beta < 1$, a function $g \in \mathscr{A}$ is called convex function of order β if

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$\Re\left(1+zg''(z)/g'(z)\right) > \beta$, $z \in \mathscr{D}$, class of such function denoted by $\mathscr{K}(\beta)$. It is customary that $\mathscr{S}^*(0)=\mathscr{S}^*$ and $\mathscr{K}(0)=\mathscr{K}$. Moreover, a function $g \in \mathscr{A}$ is said to be close-to-convex with respect to a fixed starlike function h, denoted by $\mathcal{C}_h$, if $\Re\left(zg'(z)/h(z)\right) > 0$, $z \in \mathscr{D}$. For more details one can refer [6].

In the present perusal, we study some geometric properties of Generalized Bessel-Maitland function (see, e.g., [9], Eq.(8.3)), $J_\zeta^\xi(z)$. This function is defined by the following series representation:

$$J_\zeta^\xi(z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma(\xi n+\zeta+1)}\ (\Re(\xi)\geq 0,\ \Re(\zeta)\geq -1 \text{ and } z\in\mathscr{D}). \tag{1.2}$$

It has many application in various research fields of Science and Engineering. For a comprehensive description of applications of Bessel functions and its generalization, the reader may be referred to [20]. Here in the present paper, we define a new (probably) generalization of Bessel-Maitland function called generalized Bessel-Maitland function $J_{\zeta,c}^\xi(z)$, given by:

$$J_{\zeta,a}^\xi(z)=\sum_{n=0}^{\infty}\frac{(-a)^n z^n}{n!\Gamma(\xi n+\zeta+1)}\ (a\in\mathbb{C}-\{0\},\xi>0,\zeta>-1 \text{ and } z\in\mathscr{D}). \tag{1.3}$$

It can be easily seen that

$$J_{\zeta,-1}^\xi(z)=W_{\xi,\zeta+1}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\xi n+\zeta+1)} \tag{1.4}$$

where $W_{\xi,\zeta+1}(z)$ is called Wright function and

$$J_{\zeta,1}^\xi(z)=J_\zeta^\xi(z)=\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!\Gamma(\xi n+\zeta+1)}. \tag{1.5}$$

Observe that the Generalized Bessel-Maitland function $J_{\zeta,a}^\xi(z)\notin\mathscr{A}$. We can consider the following two types of normalization of the Generalized Bessel-Maitland function:

$$\mathbb{J}_{\zeta,a}^\xi(z)=z\Gamma(\zeta+1)J_{\zeta,a}^\xi(z)=z+\sum_{n=1}^{\infty}\frac{(-a)^n\Gamma(\zeta+1)z^{n+1}}{n!\Gamma(\xi n+\zeta+1)} \tag{1.6}$$

and

$$\begin{aligned}\mathscr{J}_{\zeta,a}^\xi(z)&=\frac{\Gamma(\xi+\zeta+1)}{(-a)}\left(J_{\zeta,a}^\xi(z)-\frac{1}{\Gamma(\zeta+1)}\right)\\&=\sum_{n=0}^{\infty}\frac{(-a)^n\Gamma(\xi+\zeta+1)z^{n+1}}{(n+1)!\Gamma(\xi n+\xi+\zeta+1)}.\end{aligned} \tag{1.7}$$

$$(\xi>0,\ \xi+\zeta>-1,\ a\in\mathbb{C}-\{0\},\ z\in\mathscr{D})$$

Also note that

$$\mathbb{J}_{\zeta,1}^1(z)=\mathbb{J}_\zeta(z)=\Gamma(\zeta+1)z^{1-\zeta/2}J_\zeta(2\sqrt{z})=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\zeta+1)z^{n+1}}{n!\Gamma(n+\zeta+1)}. \tag{1.8}$$

where $J_\zeta(z)$ is well known Bessel function of order ζ and $\mathbb{J}_\zeta(z)$ is the normalized Bessel function, studied recently for the various geometric properties (see [14]-[18]). Conversely, it can be easily seen that

$$J_\zeta(z) = \frac{1}{\Gamma(\zeta+1)}\left(\frac{z}{2}\right)^{\zeta-2}\mathbb{J}^1_{\zeta,1}\left(\frac{z^2}{4}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n+\zeta}}{n!\Gamma(n+\zeta+1)}.$$

Additionally, we observe that

$$\begin{aligned}\mathbb{V}^\xi_{\zeta,a}(z) = \frac{\mathbb{J}^\xi_{\zeta,a}(z)}{z} &= \frac{1}{z}\left[z + \sum_{n=1}^{\infty}\frac{(-a)^n\Gamma(\zeta+1)z^{n+1}}{n!\Gamma(\xi n+\zeta+1)}\right]\\ &= 1 + \sum_{n=1}^{\infty}\frac{(-a)^n\Gamma(\zeta+1)z^n}{n!\Gamma(\xi n+\zeta+1)}\end{aligned}$$

and

$$z(\mathbb{V}^\xi_{\zeta,a}(z))' = \sum_{n=1}^{\infty}\frac{(-a)^n\Gamma(\zeta+1)nz^n}{n!\Gamma(\xi n+\zeta+1)}.$$

The following identity relations can be easily established:

$$\xi z(\mathbb{J}^\xi_{\zeta,a}(z))' = (\zeta+1)\mathbb{J}^\xi_{\zeta,a}(z) + (\xi-\zeta-1)\mathbb{J}^\xi_{\zeta+1,a}(z) \tag{1.9}$$

$$z(\mathcal{J}^\xi_{\zeta,a}(z))' = \mathbb{J}^\xi_{\xi+\zeta,a}(z) \tag{1.10}$$

and

$$\left(\mathbb{V}^\xi_{\zeta,a}(z)\right)' = \frac{(-a)\Gamma(\zeta+1)}{\Gamma(\xi+\zeta+1)}\mathbb{V}^\xi_{\xi+\zeta,a}(z). \tag{1.11}$$

Lately, several researchers have studied innumerable special functions belonging to class $\mathscr{A}$ and found sufficient conditions such that the special functions belonging to class $\mathscr{A}$ have certain properties like univalency, starlikeness or convexity in $\mathscr{D}$. For the generalized hypergeometric functions one can refer [10, 13, 16], Bessel functions [3, 1, 2, 4] and Wright function [15]. In the present paper, we derive sufficient conditions for the same geometric properties for the functions $\mathbb{J}^\xi_{\zeta,a}(z)$ and $\mathcal{J}^\xi_{\zeta,a}(z)$.

2. Lemmas

To prove main results, we requisite the following results:

Lemma 2.1. (see [7]). *Let $g \in \mathscr{A}$ satisfy the inequality*

$$|(g(z)/z) - 1| < 1 \ (z \in \mathscr{D})$$

then g is starlike in the disk $\mathscr{D}_{1/2} = \{z : |z| < 1/2\}$.

Lemma 2.2. (see [8]). *Let $g \in \mathscr{A}$ satisfy the inequality*

$$|g'(z) - 1| < 1 \ (z \in \mathscr{D})$$

then g is convex in the disk $\mathscr{D}_{1/2}$.

Lemma 2.3. (see [11]). *Let $g \in \mathscr{A}$ satisfy*

$$|g'(z) - 1| < 2/\sqrt{5}\ (z \in \mathscr{D})$$

then g is starlike in the disk $\mathscr{D}$.

Lemma 2.4. (see [21]). *Let $g \in \mathscr{A}$ satisfy the inequality*

$$\left|\frac{zg'(z)}{g(z)} - 1\right| < L, \quad z \in \mathscr{D},$$

where L is solution of the equation $\cos L = L$, then $\Re(g'(z)) > 0$.

Lemma 2.5. (see [12]). *Let $\delta \in \mathbb{C}$ with $\Re(\delta) > 0, d \in \mathbb{C}$ with $|d| \leq 1,\ d \neq -1$. If $h \in \mathscr{A}$ satisfies*

$$\left| d|z|^{2\delta} + (1 - |z|^{2\delta}) \frac{zh''(z)}{\delta h'(z)} \right| \leq 1, \quad z \in \mathscr{D}$$

then the integral operator

$$\mathcal{C}_\delta(z) = \left\{ \delta \int_0^z t^{\delta - 1} h'(t) dt \right\}^{1/\delta}, \ z \in \mathscr{D}$$

is analytic and univalent in $\mathscr{D}$.

For $\delta = 1$ and $d = 0$, Lemma 2.5 is equivalent to Becker's criterion for univalency [5], which shows that, if $f \in \mathscr{A}$ satisfy the inequality $(1 - |z|^2)|zf''(z)/f'(z)| \leq 1$ for each $z \in \mathscr{D}$, then f is one-to-one (univalent) in $\mathscr{D}$.

3. Main results

Theorem 3.1. (i)*Let $\xi \geq 1$ and $\zeta \geq \dfrac{3(|a| - 1) + \sqrt{9|a|^2 + 2|a| + 1}}{2}$, then $\mathbb{J}^{\xi}_{\zeta,a}$ is starlike in $\mathscr{D}$.*

(ii) *Let $\xi \geq 1$ and $\xi + \zeta \geq \dfrac{3(|a| - 1) + \sqrt{9|a|^2 + 2|a| + 1}}{2}$, then $\mathbb{V}^{\xi}_{\zeta,a}$ is convex in $\mathscr{D}$.*

(iii) *Let $\xi \geq 1$ and $\xi + \zeta \geq \dfrac{3(|a| - 1) + \sqrt{9|a|^2 + 2|a| + 1}}{2}$, then $\mathscr{J}^{\xi}_{\zeta,a}$ is convex in $\mathscr{D}$.*

Proof. Let $q(z)$ be a function defined by the equality $q(z) = z(\mathbb{J}^{\xi}_{\zeta,a}(z))'/\mathbb{J}^{\xi}_{\zeta,a}(z)\ \ z \in \mathscr{D}$. Since $\mathbb{J}^{\xi}_{\zeta,a}(z)/z \neq 0, \ \ z \in \mathscr{D}$, the function q is analytic in $\mathscr{D}$ and $q(0) = 1$. To prove the result, we need to show that $\Re(q(z)) > 0$ which follows if we show $|q(z) - 1| < 1$.

For $\xi \geq 1$, it is easy to see that $\Gamma(\zeta + n + 1) \leq \Gamma(\xi n + \zeta + 1)$, $n \in \mathbb{N}$, holds and is equivalent to

$$\frac{1}{(\zeta + 1)(\zeta + 2)...(\zeta + n)} \geq \frac{\Gamma(\zeta + 1)}{\Gamma(\xi n + \zeta + 1)}, \quad n \in \mathbb{N}. \tag{3.1}$$

If $z \in \mathscr{D}$, then using (1.6) and (3.1), we obtain

$$
\begin{aligned}
\left| (\mathbb{J}^{\xi}_{\zeta,a}(z))' - \frac{\mathbb{J}^{\xi}_{\zeta,a}(z)}{z} \right| &= \left| \sum_{n=1}^{\infty} \frac{(-a)^n n z^n \Gamma(\zeta+1)}{n!\Gamma(\xi n+\zeta+1)} \right| \\
&\leq \sum_{n=1}^{\infty} \frac{|a|^n n}{n!(\zeta+1)(\zeta+2)(\zeta+n)} \\
&< \frac{|a|}{(\zeta+1)} \sum_{n=0}^{\infty} \left(\frac{|a|}{\zeta+2} \right)^n \\
&= \frac{|a|(\zeta+2)}{(\zeta+1)(\zeta+2-|a|)} \qquad (3.2)
\end{aligned}
$$

and

$$
\begin{aligned}
\left| \frac{\mathbb{J}^{\xi}_{\zeta,a}(z)}{z} \right| &\geq 1 - \left| \sum_{n=1}^{\infty} \frac{(-a)^n z^n \Gamma(\zeta+1)}{n!\Gamma(\xi n+\zeta+1)} \right| \\
&\geq 1 - \sum_{n=1}^{\infty} \frac{|a|^n}{n!(\zeta+1)(\zeta+2)...(\zeta+n)} \\
&> 1 - \frac{|a|}{(\zeta+1)} \sum_{n=0}^{\infty} \left(\frac{|a|}{\zeta+2} \right)^n \\
&= 1 - \frac{|a|(\zeta+2)}{(\zeta+1)(\zeta+2-|a|)} \\
&= \frac{(\zeta+1)(\zeta+2-|a|) - |a|(\zeta+2)}{(\zeta+1)(\zeta+2-|a|)}. \qquad (3.3)
\end{aligned}
$$

From (3.2) and (3.3), we have for $z \in \mathscr{D}$

$$
\begin{aligned}
|q(z)-1| &= \left| \frac{z(\mathbb{J}^{\xi}_{\zeta,a}(z))'}{\mathbb{J}^{\xi}_{\zeta,a}(z)} - 1 \right| = \left| \frac{(\mathbb{J}^{\xi}_{\zeta,a}(z))' - \frac{\mathbb{J}^{\xi}_{\zeta,a}(z)}{z}}{\frac{\mathbb{J}^{\xi}_{\zeta,a}(z)}{z}} \right| \\
&< \frac{|a|(\zeta+2)}{(\zeta+1)(\zeta+2-|a|) - |a|(\zeta+2)}. \qquad (3.4)
\end{aligned}
$$

This implies that if $(\zeta+1)(\zeta+2-|a|) - |a|(\zeta+2) \geq |a|(\zeta+2)$, then $\Re(q(z)) > 0$, hence $\mathbb{J}^{\xi}_{\zeta,a}(z)$ is starlike in $\mathscr{D}$, but the inequality $(\zeta+1)(\zeta+2-|a|) - 2|a|(\zeta+2) \geq 0$ is a consequence of the hypothesis $\zeta \geq \dfrac{3(|a|-1) + \sqrt{9|a|^2+2|a|+1}}{2}$. This shows that $\mathbb{J}^{\xi}_{\zeta,a}(z)$ is starlike in $\mathscr{D}$. □

(ii) In view of the hypothesis the inequality $\Gamma(n+\xi+\zeta+1) \leq \Gamma(\xi n+\xi+\zeta+1)$ holds. If $z \in \mathscr{D}$, then a calculation similar to (3.2) and (3.3) gives

$$
\left| z\left(\mathbb{V}^{\xi}_{\xi+\zeta,a}(z)\right)' \right| < \frac{|a|(\xi+\zeta+2)}{(\xi+\zeta+1)(\xi+\zeta+2-|a|)}
$$

and
$$\left|\mathbb{V}^{\xi}_{\xi+\zeta,a}(z)\right| > \frac{(\xi+\zeta+1)(\xi+\zeta+2-|a|)-|a|(\xi+\zeta+2)}{(\xi+\zeta+1)(\xi+\zeta+2-|a|)}.$$
From these inequalities and (1.11), we obtain
$$\left|\frac{z\left(\mathbb{V}^{\xi}_{\zeta,a}(z)\right)''}{\left(\mathbb{V}^{\xi}_{\zeta,a}(z)\right)'}\right| = \left|\frac{z\left(\mathbb{V}^{\xi}_{\xi+\zeta,a}(z)\right)'}{\left(\mathbb{V}^{\xi}_{\xi+\zeta,a}(z)\right)}\right|$$
$$< \frac{|a|(\xi+\zeta+2)}{(\xi+\zeta+1)(\xi+\zeta+2-|a|)-|a|(\xi+\zeta+2)} \ (z \in \mathscr{D}).$$
This means that, if $(\xi+\zeta+1)^2+(\xi+\zeta+1)(1-3|a|)-2|a| \geq 0$, then by definition $\mathbb{V}^{\xi}_{\zeta,a}$ is convex in $\mathscr{D}$. But this inequality is true under the condition
$$\xi+\zeta > \frac{3(|a|-1)+\sqrt{9|a|^2+2|a|+1}}{2}.$$
Hence, $\mathbb{V}^{\xi}_{\zeta,a}$ is convex in $\mathscr{D}$.
(iii) The function $\mathscr{J}^{\xi}_{\zeta,a}(z)$ is convex iff $z(\mathscr{J}^{\xi}_{\zeta,a}(z))'$ is starlike, but from (1.10)
$$z(\mathscr{J}^{\xi}_{\zeta,a}(z))' = \mathbb{J}^{\xi}_{\xi+\zeta,a}(z).$$
This in view of part (i) of the theorem completes the proof.

Remark 3.2. If we put $a = 1$ and $\xi = 1$, then we obtain part(a) and part(b) of Corollary 2.8 of [15]. Similarly if we put $a = -1$ and $\xi = 1$ and using Lemma 2.4, we obtain part(c) of Corollary 2.8 of [15].

Theorem 3.3. (i) *Let $\xi \geq 1$ and $\zeta \geq \frac{-3+2|a|+\sqrt{1+4|a|^2}}{2}$, then $\mathbb{J}^{\xi}_{\zeta,a}(z)$ is starlike in the disk $\mathscr{D}_{1/2}$.*
(ii) *Let $\xi \geq 1$ and $\zeta \geq \dfrac{3(|a|-1)+\sqrt{9|a|^2+2|a|+1}}{2}$, then $\mathbb{J}^{\xi}_{\zeta,a}(z)$ is convex in the disk $\mathscr{D}_{1/2}$.*
(iii) *Let $\xi \geq 1$ and $\zeta > \zeta^*$, where ζ^* is positive root of the equation*
$$\zeta^2+\zeta(3-(1+\sqrt{5})|a|)+(2-(1+2\sqrt{5})|a|) = 0,$$
then $\mathbb{J}^{\xi}_{\zeta,a}(z)$ is starlike in the disk $\mathscr{D}$.

Proof. On performing calculations, we have
$$\left|\frac{\mathbb{J}^{\xi}_{\zeta,a}(z)}{z}-1\right| = \left|\frac{1}{z}\left\{z+\sum_{n=1}^{\infty}\frac{(-a)^n\Gamma(\zeta+1)z^{n+1}}{n!\Gamma(\xi n+\zeta+1)}\right\}-1\right|$$
$$\leq \sum_{n=1}^{\infty}\frac{|a|^n}{n!}\frac{1}{(\zeta+1)(\zeta+2)...(\zeta+n)} < \frac{|a|}{(\zeta+1)}\sum_{n=0}^{\infty}\left(\frac{|a|}{\zeta+2}\right)^n$$
$$= \frac{|a|}{(\zeta+1)}\frac{(\zeta+2)}{(\zeta+2-|a|)}$$

In view of Lemma 2.1, $\mathbb{J}^{\xi}_{\zeta,a}$ is starlike in $\mathscr{D}_{1/2}$, if $(\zeta+1)^2+(\zeta+1)(1-2|a|)-|a|\geq 0$, but this is true in view of the hypothesis. Hence, the result is proved.

(ii) Using Lemma 2.2, we obtain

$$\left|(\mathbb{J}^{\xi}_{\zeta,a}(z))'-1\right| = \left|\sum_{n=1}^{\infty}\frac{(-a)^n(n+1)\Gamma(\zeta+1)z^n}{n!\Gamma(\xi n+\zeta+1)}\right| \tag{3.5}$$

$$\leq \sum_{n=1}^{\infty}\frac{|a|^n n\Gamma(\zeta+1)}{n!\Gamma(\xi n+\zeta+1)}+\sum_{n=1}^{\infty}\frac{|a|^n\Gamma(\zeta+1)}{n!\Gamma(\xi n+\zeta+1)}$$

$$\leq \sum_{n=1}^{\infty}\frac{2|a|^n}{(\zeta+1)(\zeta+2)...(\zeta+n)} < \frac{2|a|(\zeta+2)}{(\zeta+1)(\zeta+2-|a|)} \leq 1. \tag{3.6}$$

This shows that $\mathbb{J}^{\xi}_{\zeta,a}(z)$ is convex in $\mathscr{D}_{1/2}$.
(iii) Using the Lemma 2.3 and equation (3.6), we see that $\mathbb{J}^{\xi}_{\zeta,a}(z)$ is starlike in $\mathscr{D}_{1/2}$, if

$$\zeta^2+\zeta(3-(1+\sqrt{5})|a|)+(2-(1+2\sqrt{5})|a|)>0.$$

This proves the result. □

Remark 3.4. Setting $\xi=1$, and $a=1$ in Theorem 3.3, we obtain Part (a), (b) and (c) of Corollary 2.10 of [15].

Theorem 3.5. *Let $\xi\geq 1$ and $0\leq\eta<1$. Suppose also that*

$$\psi(\eta)=\frac{(3|a|-1)-\eta(2|a|-1)+\sqrt{\eta^2(4|a|^2+1)-2\eta(6|a|^2+|a|+1)+(9|a|^2+2|a|+1)}}{2(1-\eta)}.$$

(i)*Let $\zeta\geq\psi(\eta)$, then $\mathbb{J}^{\xi}_{\zeta,a}(z)\in\mathscr{S}^*(\eta)$.*
(ii) *Let $\zeta+\xi\geq\psi(\eta)$, then $\mathbb{V}^{\xi}_{\zeta,a}(z)\in\mathscr{K}^*(\eta)$.*
(iii) Let $\zeta+\xi\geq\psi(\eta)$, then $\mathcal{J}^{\xi}_{\zeta,a}(z)\in\mathscr{K}^*(\eta)$.

Proof. Following the proof of Theorem 3.1, we find that $\mathbb{J}^{\xi}_{\zeta,a}(z)\in\mathscr{S}^*(\eta)$ii, if

$$\frac{|a|(\zeta+2)}{(\zeta+1)(\zeta+2-|a|)-|a|(\zeta+2)}\leq(1-\eta),$$

but this inequality is a direct consequence of hypothesis. Hence the result. Remaining part can be shown similarly. □

Theorem 3.6. *Let $\xi\geq 1$ and ζ^* be the positive root of the cubic equation*

$$\zeta^3+\zeta^2(2-3|a|)-3\zeta(1+2|a|)-(6+|a|)\geq 0,$$

then $\mathbb{J}^{\xi}_{\zeta,a}$ is close-to-convex with respect to $\mathbb{J}_{\zeta}$ in $\mathscr{D}$, provided $\zeta>\max\{\zeta^*,|a|-2,\sqrt{3}\}$.

Proof. Using definition, we need to show $\exists h\in\mathcal{S}^*$, such that

$$\Re\left(\frac{z(\mathbb{J}^{\xi}_{\zeta,a}(z))'}{h(z)}\right)>0 \quad (z\in\mathscr{D}),$$

this can be easily shown by proving

$$\left|\frac{z(\mathbb{J}^{\xi}_{\zeta,a}(z))'}{h(z)}-1\right|<1,\quad z\in\mathscr{D}.$$

If $z\in\mathscr{D}$, then a computation gives

$$\begin{aligned}\left|(\mathbb{J}^{\xi}_{\zeta,a}(z))'-\frac{\mathbb{J}_{\xi}(z)}{z}\right| &\leq \sum_{n=1}^{\infty}\frac{\Gamma(\zeta+1)}{n!}\left|\frac{(-a)^n(n+1)}{\Gamma(\xi n+\zeta+1)}-\frac{(-1)^n}{\Gamma(n+\zeta+1)}\right|\\ &\leq \sum_{n=1}^{\infty}\frac{\Gamma(\zeta+1)}{n!}\left|\frac{(a)^n(n+1)}{\Gamma(\xi n+\zeta+1)}+\frac{1}{\Gamma(n+\zeta+1)}\right|\\ &\leq \sum_{n=1}^{\infty}\frac{\Gamma(\zeta+1)}{n!}\left[\frac{|a|^n(n+1)+1}{\Gamma(n+\zeta+1)}\right]\\ &\leq \sum_{n=1}^{\infty}\frac{2|a|^n}{(\zeta+1)(\zeta+2)...(\zeta+n)}+\sum_{n=1}^{\infty}\frac{1}{(\zeta+1)(\zeta+2)...(\zeta+n)}\\ &< \frac{1}{(\zeta+1)}\sum_{n=0}^{\infty}\frac{(2|a|^{n+1}+1)}{(\zeta+2)^n}=\frac{(\zeta+2)[|a|(2\zeta+1)+\zeta+2]}{(\zeta+2-|a|)(\zeta+1)^2}\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}\left|\frac{\mathbb{J}_{\zeta}(z)}{z}\right| &\geq 1-\sum_{n=1}^{\infty}\frac{1}{n!(\zeta+1)(\zeta+2)...(\zeta+n)}\\ &> 1-\frac{1}{\zeta+1}\sum_{n=0}^{\infty}\left(\frac{1}{\zeta+2}\right)^n=\frac{\zeta^2+\zeta-1}{(\zeta+1)^2}.\end{aligned}\tag{3.8}$$

From (3.7) and (3.8)

$$\begin{aligned}\left|\frac{z(\mathbb{J}^{\xi}_{\zeta,a}(z))'}{\mathbb{J}_{\zeta}(z)}-1\right| &= \left|\frac{(\mathbb{J}^{\xi}_{\zeta,a}(z))'-\dfrac{\mathbb{J}_{\xi}(z)}{z}}{\dfrac{\mathbb{J}_{\zeta}(z)}{z}}\right|\\ &< \frac{(\zeta+2)}{(\zeta+2-|a|)(\zeta^2+\zeta-1)}[|a|(2\zeta+1)+\zeta+2]\leq 1,\quad z\in\mathscr{D}.\end{aligned}$$

This shows that $\Re\left(z(\mathbb{J}^{\xi}_{\zeta,a}(z))'/\mathbb{J}_{\zeta}(z)\right)>0$ and hence $\mathbb{J}^{\xi}_{\zeta,a}$ is close-to-convex in $\mathscr{D}$. Starlikeness of $\mathbb{J}_{\zeta}$ can be deduced from Theorem 3.1 for $a=1$ and it comes out $\xi\geq 1$ and $\zeta\geq\sqrt{3}$. □

For a non-zero complex number δ, we define an integral operator $\mathcal{F}_{\delta}:\mathscr{D}\to\mathbb{C}$, by

$$\mathcal{F}_{\delta}(z)=\left\{\delta\int_0^z t^{\delta-2}\mathbb{J}^{\xi}_{\zeta,a}(t)dt\right\}^{\frac{1}{\delta}},\quad z\in\mathscr{D}.\tag{3.9}$$

Note that $\mathcal{F}_{\delta}(z)\in\mathscr{A}$. In the next theorem, we find conditions so that $\mathcal{F}_{\delta}$ is univalent in $\mathscr{D}$.

Theorem 3.7. *Let* $\xi > -1$, $\zeta > -1$, $\kappa = \dfrac{|a|(\zeta+2)}{(\zeta+1)(\zeta+2-|a|)-|a|(\zeta+2)}$ *and* $L \in \mathbb{R}^+$ *such that* $\left|\mathbb{J}^{\xi}_{\zeta,a}(z)\right| \le L$ *in* $\mathscr{D}$*, then following results holds*
(i) *If* $\kappa + |\delta-1| + L/\delta \le 1$*, then* $\mathcal{F}_\delta$ *is univalent in* $\mathscr{D}$*.*
(ii) *If* $d \in \mathbb{C}$ *with* $|d| \le 1$, $d \ne -1$ *and* $|d| + \kappa/|\delta| \le 1$*, then* $\mathcal{F}_\delta$ *is univalent in* $\mathscr{D}$*.*

Proof. (i) A simple calculation gives us

$$\frac{z\mathcal{F}''_\delta(z)}{\mathcal{F}'_\delta(z)} = \frac{z(\mathbb{J}^{\xi}_{\zeta,a}(z))'}{\mathbb{J}_\zeta(z)} + \frac{z^{\delta-1}}{\delta}\mathbb{J}^{\xi}_{\zeta,a}(z) + \delta - 2, \quad z \in \mathscr{D}. \tag{3.10}$$

Since $\mathbb{J}^{\xi}_{\zeta,a} \in \mathscr{A}$, so using Schwarz Lemma, we obtain $\left|\mathbb{J}^{\xi}_{\zeta,a}(z)\right| \le L|z|$ in $\mathscr{D}$.
Now using (3.4) and the triangle inequality ($|z_1+z_2| \le |z_1|+|z_2|$), we obtain

$$\begin{aligned}(1-|z|^2)\left|\frac{z\mathcal{F}''_\delta(z)}{\mathcal{F}'_\delta(z)}\right| &\le (1-|z|^2)\left\{|\delta-1| + \left|\frac{z(\mathbb{J}^{\xi}_{\zeta,a}(z))'}{\mathbb{J}^{\xi}_{\zeta,a}(z)} - 1\right| + \frac{|z|^{\Re(\delta)}}{|\delta|}\left|\frac{\mathbb{J}^{\xi}_{\zeta,a}(z)}{z}\right|\right\}\\ &< (1-|z|^2)\left\{\zeta + |\delta-1| + \frac{L}{|\delta|}\right\} \le 1.\end{aligned}$$

This implies that $\mathcal{F}_\delta$ satisfy Becker's criterion for univalence, hence $\mathcal{F}_\delta$ is univalent in $\mathscr{D}$.
(ii) Let us consider the function

$$\mathcal{G}(z) = \int_0^z \frac{\mathbb{J}^{\xi}_{\zeta,a}(t)}{t}dt, \quad z \in \mathscr{D}.$$

Observe that, $\mathcal{G} \in \mathscr{A}$. Using (3.4) and the triangle inequality, we get

$$\begin{aligned}\left|d|z|^{2\delta} + (1-|z|^{2\delta})\frac{z\mathcal{G}''(z)}{\delta\mathcal{G}'(z)}\right| &\le \left|d|z|^{2\delta} + (1-|z|^{2\delta})\frac{1}{\delta}\left(\frac{z(\mathbb{J}^{\xi}_{\zeta,a}(z))'}{\mathbb{J}_\zeta(z)} - 1\right)\right|\\ &\le |d| + \frac{\zeta}{|\delta|} \le 1 \quad \text{(using the hypothesis of Theorem 3.7).}\end{aligned}$$

This in view of Lemma 2.5, implies that $\mathcal{F}_\delta(z)$ defined by

$$\mathcal{F}_\delta(z) = \left\{\delta\int_0^z t^{\delta-1}\mathcal{G}'(t)dt\right\}^{1/\delta} = \left\{\delta\int_0^z t^{\delta-2}\mathbb{J}^{\xi}_{\zeta,a}(t)dt\right\}^{1/\delta} \ (z \in \mathscr{D}). \tag{3.11}$$

is univalent in $\mathscr{D}$. □

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Amit Soni
Department of Mathematics, Govt. Engineering College, Bikaner, 334004, Rajasthan, India
e-mail: aamitt1981@gmail.com

Deepak Bansal
Department of Mathematics, College of Engg. and Technology, Bikaner, 334004, Rajasthan, India
e-mail: deepakbansal_79@yahoo.com

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On generalized close-to-convexity related with strongly Janowski functions

Khalida Inayat Noor and Shujaat Ali Shah

Abstract. Strongly Janowski functions are used to define certain classes of analytic functions which generalize the concepts of close-to-convexity and bounded boundary rotation. Coefficient results, a necessary condition, distortion bounds, Hankel determinant problem and several other interesting properties of these classes are studied. Some significant well known results are derived as special cases.

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1. Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. If the functions f and g are analytic in E, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwartz function w in $\mathbb{D}$ such that $f(z) = g(w(z))$. Furthermore, if the function g is univalent in $\mathbb{D}$, then we have the following equivalence

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

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Let f be given by (1.1) and $g \in \mathcal{A}$ is of the form $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$. Then the convolution (Hadamard product) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}.$$

Let $\mathcal{S} \subset \mathcal{A}$ be the class of univalent functions in $\mathbb{D}$ and let $\mathcal{C}$, $\mathcal{S}^*$ and $\mathcal{K}$ be the subclasses of $\mathcal{S}$ consisting of convex, starlike and close-to-convex functions, respectively. Also, let p be analytic in $\mathbb{D}$ with $p(0) = 1$. Then the function p is known a strongly Janowski type functions of order α if

$$p(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\alpha}, \quad \alpha \in (0,1], \ -1 \leq B < A \leq 1 \text{ and } z \in \mathbb{D}.$$

We note that, when $\alpha = 1$, $A = 1$ and $B = -1$, then p is a Carathéodory function of positive real part.

Definition 1.1. Let p be analytic in $\mathbb{D}$ with $p(0) = 1$ and let ϕ be convex univalent in $\mathbb{D}$. Then $p \in \mathcal{P}_m(\phi)$, $m \geq 2$, if and only if there exists functions p_i with $p_i(0) = 1$, $i = 1, 2$ such that

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z), \tag{1.2}$$

where $p_i \prec \phi$.

Special cases:

Let $\phi(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha}$, $\alpha \in (0,1]$, $-1 \leq B < A \leq 1$. Then the series representation of $\phi(z)$ is given by

$$\phi(z) = 1 + \alpha\,(A-B)\,z + \left[-\alpha\,(A-B)\,B + \frac{1}{2}\alpha\,(\alpha-1)\,(A-B)^2\right] z^2 + \dots .$$

On differentiating we get

$$\phi'(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha} \frac{\alpha\,(A-B)}{(1+Az)\,(1+Bz)}.$$

Now, for $-1 \leq B < A \leq 1$ and $z \in \mathbb{D}$, we have

$$\Re\left(\phi'(z)\right) \geq \left\{\alpha\,|A-B|\,\frac{(1-|A|)^{\alpha-1}}{(1-|B|)^{\alpha+1}}\right\} > 0,$$

and by simple calculations we can easily prove that

$$\Re\left\{\frac{(z\phi'(z))'}{\phi'(z)}\right\} \geq 0.$$

This implies that $\phi(z)$ is convex univalent function in $\mathbb{D}$.
Thus we have

$$\mathcal{P}_m\left(\left(\frac{1+Az}{1+Bz}\right)^{\alpha}\right) = \mathcal{P}_{m,\alpha}\left[A,B\right] \subset \mathcal{P}_m(\rho),$$

where $\rho = \left(\frac{1-A}{1-B}\right)^{\alpha}$. Also, we note that $\mathcal{P}_{m,1}[1,-1] = \mathcal{P}_m$, see [19]. Moreover, $\mathcal{P}_{2,1}[1,-1] = \mathcal{P}$ is the well-known class of Carathéodory functions of positive real part. When $m=2$, then $p \in \mathcal{P}_{2,\alpha}[1,-1]$ implies $|\arg p(z)| \leq \frac{\alpha\pi}{2}$. When $m=2$, $\alpha=1$, $A=1-2\beta$ and $B=-1$, we obtain the class $\mathcal{P}(\beta)$, $\beta \in (0,1]$, of functions with real part greater than β.

For the class $\mathcal{P}_m(\rho)$, we refer to [18]. It is worth noting that $\mathcal{P}_{2,\alpha}[1,-1] = \mathcal{P}_\alpha$ and the class $\mathcal{P}_{\frac{1}{2}}[1,0] = \pounds\mathcal{P}$ is associated with the right-half of the Lemniscate of Bernoulli $\partial\pounds$ (see [11]) enclosing the region

$$\pounds = \left\{w \in \mathbb{C} : \Re(w) > 0,\ \left|w^2 - 1\right| < 1\right\},$$

where $\pounds \subset \left\{w \in \mathbb{C} : |\arg w| < \frac{\pi}{4}\right\}$.

The well-known hypergeometric function $G(a,b,c;z)$ is of the form

$$\begin{aligned} G(a,b,c;z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+b)\Gamma(b+n)}{\Gamma(c+n)} \cdot \frac{z^n}{n!} \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-zu)^{-b}\,du, \end{aligned}$$

where $\Re(a) > 0$, $\Re(c-a) > 0$ and Γ represents notation for Gamma function.

Definition 1.2. Let $f \in \mathcal{A}$. Then $f \in \mathcal{R}_{m,\alpha}[A,B]$ if and only if

$$\frac{zf'(z)}{f'(z)} \in \mathcal{P}_{m,\alpha}[A,B].$$

Definition 1.3. Let $f \in \mathcal{A}$. Then $f \in \mathcal{V}_{m,\alpha}[A,B]$ if and only if

$$\frac{(zf'(z))'}{f'(z)} \in \mathcal{P}_{m,\alpha}[A,B].$$

We note the following special cases.

(i) $\mathcal{R}_{2,1}[A,B] = \mathcal{S}^*[A,B]$ and $\mathcal{V}_{2,1}[A,B] = \mathcal{C}[A,B]$, see [9].

(ii) $\mathcal{R}_{m,1}[1,-1] = \mathcal{R}_m$ and $\mathcal{V}_{m,1}[1,-1] = \mathcal{V}_m$, the class of functions with bounded radius and bounded boundary rotations, respectively; see [2, 19].

(iii) $\mathcal{V}_{2,\alpha}[A,B] = \mathcal{C}_\alpha[A,B] \subset \mathcal{C}(\rho) \subset \mathcal{C}$, with $\rho = \left(\frac{1-A}{1-B}\right)^{\alpha}$, where $\mathcal{C}$ is the class of convex functions.

(iv) $\mathcal{R}_{m,\alpha}[A,B] \subset \mathcal{R}_m(\rho)$ and $\mathcal{V}_{m,\alpha}[A,B] \subset \mathcal{V}_m(\rho)$, see [18].

It is observed that

$$f \in \mathcal{V}_{m,\alpha}[A,B] \Leftrightarrow zf' \in \mathcal{R}_{m,\alpha}[A,B].$$

Definition 1.4. Let $f \in \mathcal{A}$. Then $f \in \mathcal{T}_{m,\alpha}[A,B]$ if and only if

$$\frac{f'(z)}{g'(z)} \in \mathcal{P}_\alpha[A,B],$$

for some $g \in \mathcal{V}_{m,\alpha}[1,-1]$.

The class $\mathcal{T}_{m,1}[1,-1] = \mathcal{T}_m$ has been introduced and studied in [17], and $\mathcal{T}_{2,1}[1,-1] = \mathcal{K}$, the class of close-to-convex functions, see [10].

In the present work, we derive coefficient inequalities and distortion results for certain subclasses of analytic functions. Further, necessary condition and radius problem are discussed. Also, the Hankel determinant problem is estimated. We need the following results in our investigations.

Lemma 1.5. [24] *If $f \in \mathcal{C}$, $g \in \mathcal{S}^*$, then for each h analytic in $\mathbb{D}$ with $h(0) = 1$,*

$$\frac{(f * hg)(\mathbb{D})}{(f * g)(\mathbb{D})} \subset \overline{CO}h(\mathbb{D}),$$

where $\overline{CO}h(\mathbb{D})$ denotes the closed convex hull of $h(\mathbb{D})$.

Using well-known distortion results for the class $\mathcal{P}$, we can easily prove:

Lemma 1.6. *Let $p(z)$ be analytic in $\mathbb{D}$ with $p(0) = 1$. Let*

$$p(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\alpha}, \ \alpha \in (0,1], \ -1 \le B < A \le 1.$$

Then

$$\left(\frac{1-Ar}{1-Br}\right)^{\alpha} \le |p(z)| \le \left(\frac{1+Ar}{1+Br}\right)^{\alpha}$$

and

$$\left|\frac{zp'(z)}{p(z)}\right| \le \frac{\alpha(A-B)r}{(1-Ar)(1-Br)}.$$

Lemma 1.7. [21] *Let $\theta_1 < \theta_2 < \cdots < \theta_l < \theta_1 + 2n\pi$ and $\lambda \ge \lambda_j$ $(j = 1, 2, \cdots, l)$. If*

$$\begin{aligned} \Psi(z) &= \prod_{j=1}^{l}\left(1 - e^{-i\theta_j}z\right)^{-\lambda_j} \qquad (1.3)\\ &= \sum_{n=1}^{\infty} b_n z^n, \end{aligned}$$

then

$$b_n = O(1).n^{\lambda-1}, \ as \ n \to \infty.$$

Lemma 1.8. [8] *Let $p \in \mathcal{P}$ and $z = re^{i\theta}$. Then*

$$\int_0^{2\pi} \left|p(re^{i\theta})\right|^{\eta} d\theta < c(\eta)\frac{1}{(1-r)^{\eta-1}},$$

where $\eta > 1$ and $c(\eta)$ is a constant depending on η only.

2. Main results

This section presents our main investigations. In the following theorem we derive the coefficient inequalities. Here, we use terminology Schlicht disc d by the disc d contained in the image of $\mathbb{D}$ under univalent function f.

Theorem 2.1. *Let* $\frac{zf'}{g} \in \mathcal{P}_{m,\alpha}[A,B]$, $g \in \mathcal{V}_2$, *and let* f *be given by* (1.1). *Then*

$$|a_n| \leq \frac{\{m\alpha|A-B|(n-1)+4\}}{4n}.$$

Proof. Let $g \in \mathcal{V}_2$ be of the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

and since g is convex univalent in $\mathbb{D}$, so we have $|b_n| \leq 1$, for all n.

Let

$$\frac{zf'}{g} = p(z) \in \mathcal{P}_{m,\alpha}[A,B], \tag{2.1}$$

where $p(z)$ be of the form $p(z) = 1 + \sum\limits_{n=2}^{\infty} c_n z^n$. We write $p(z)$ as given in (1.2) with $p_i(z) = 1 + \sum\limits_{n=2}^{\infty} c_{n,i} z^n$, $i = 1, 2$. Then $|c_{n,i}| \leq \alpha|A-B|$ by using a result due to Rogosinski [22]. From this, it easily follows that

$$|c_n| \leq \frac{m\alpha|A-B|}{2}, \quad (n \geq 1). \tag{2.2}$$

Now, using the expansions of $f(z)$, $g(z)$ and $p(z)$ in (2.1) to get

$$z + \sum_{n=2}^{\infty} n a_n z^n = \left(z + \sum_{n=2}^{\infty} b_n z^n\right)\left(1 + \sum_{n=2}^{\infty} c_n z^n\right).$$

On simplification and equating the coefficients of z^n $(n \geq 2)$, we have

$$n|a_n| \leq \sum_{k=1}^{n-1} |b_k||c_{n-k}| + |b_n|,$$

using $|b_n| \leq 1$ together with (2.2), we obtain

$$\begin{aligned} n|a_n| &\leq \frac{m\alpha|A-B|}{2} \sum_{k=1}^{n-1} k + 1 \\ &= \frac{m\alpha|A-B|}{2}\left[\frac{n(n-1)}{2}\right] + 1 \\ &= \frac{m\alpha|A-B|\, n(n-1)}{4} + 1, \end{aligned}$$

and this implies

$$|a_n| \leq \frac{m\alpha|A-B|(n-1)}{4} + \frac{1}{n}.$$

This proves our required result.
In particular, we have

$$|a_2| \leq \frac{m\alpha |A-B|}{4} + \frac{1}{2}$$

and

$$|a_3| \leq \frac{m\alpha |A-B|}{2} + \frac{1}{3}.$$

□

Corollary 2.2. *Let* $\frac{zf'}{g} \in \mathcal{P}_{2,\alpha}[A,B]$*,* $g \in \mathcal{V}_2$*, and let* f *be given by* (1.1)*. Then* $f(\mathbb{D})$ *contains the disc* d *such that*

$$d = \left\{ w : w < \frac{2}{5+\alpha |A-B|} \right\}.$$

Proof. Let w_0 $(w_0 \neq 0)$ be any complex number such that $f(z_0) \neq w_0$ for $z \in E$. Then, the function

$$F(z) = \frac{w_0 f(z)}{w_0 - f(z)} = z + \left(a_2 + \frac{1}{w_0} \right) z^2 + ...$$

is analytic and univalent in E, see [7]. Now, using the well known Bieberbach theorem for the best bound of second coefficient of univalent functions, we have

$$\begin{aligned} \frac{1}{|w_0|} &\leq 2 + |a_2| \leq \frac{\alpha |A-B| + 1}{2} + 2 \\ &= \frac{5 + \alpha |A-B|}{2}. \end{aligned}$$

this implies

$$|w_0| \geq \frac{2}{5 + \alpha |A-B|}.$$

Thus, $f(\mathbb{D})$ contains the disc d such that

$$d = \left\{ w : w < \frac{2}{5+\alpha |A-B|} \right\}.$$

□

Theorem 2.3. *Let* $\frac{f'}{g'} \in \mathcal{P}_\alpha$ *for* $g \in \mathcal{V}_{m,\alpha}[A,B]$*. Then*

$$\begin{aligned} \frac{2^{(2\alpha-1)} r_1^{\xi}}{\xi} \left[G(a,b,c,-1) - G(a,b,c,-r_1) \right] &\leq |f(z)| \\ &\leq \frac{2^{(2\alpha-1)} r_1^{-\xi}}{\xi} \left[G(a,b,c,-1) - G\left(a,b,c,-r_1^{-1}\right) \right], \end{aligned}$$

where $\xi = \left[(1-\varrho)\left(\frac{m}{2} - 1\right) + \alpha + 1 \right]$ *with* $\varrho = ((1-A)/(1-B))^{\alpha}$*,* $r_1 = \frac{1-r}{1+r}$*,* G *is hypergeometric function and* a, b, c *are given in* (2.9).

Proof. If $\frac{f'}{g'} \in \mathcal{P}_\alpha$ for $g \in \mathcal{V}_{m,\alpha}[A,B]$, then we can write

$$f'(z) = g'(z)\, p(z), \quad p \in \mathcal{P}_\alpha. \tag{2.3}$$

Since $g \in \mathcal{V}_{m,\alpha}[A,B] \subset \mathcal{V}_m(\varrho)$, with $\varrho = ((1-A)/(1-B))^\alpha$ implies

$$g'(z) = (g_1'(z))^{1-\varrho}, \text{ for } g_1 \in \mathcal{V}_m, \text{ (see [18])}.$$

Therefore, by using distortion results of $\mathcal{V}_m$ [2, 19], we have

$$\left[\frac{(1-r)^{\frac{m}{2}-1}}{(1+r)^{\frac{m}{2}+1}}\right]^{(1-\varrho)} \leq |g'(z)| \leq \left[\frac{(1+r)^{\frac{m}{2}-1}}{(1-r)^{\frac{m}{2}+1}}\right]^{(1-\varrho)}. \tag{2.4}$$

Also, for $p \in \mathcal{P}_\alpha$, we have

$$\left(\frac{1-r}{1+r}\right)^\alpha \leq |p(z)| \leq \left(\frac{1+r}{1-r}\right)^\alpha. \tag{2.5}$$

Therefore, from (2.3) to (2.5), it follows that

$$\frac{(1-r)^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}}}{(1+r)^{\left\{(1-\varrho)\left(\frac{m}{2}+1\right)+\alpha\right\}}} \leq |f'(z)| \leq \frac{(1+r)^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}}}{(1-r)^{\left\{(1-\varrho)\left(\frac{m}{2}+1\right)+\alpha\right\}}}. \tag{2.6}$$

Let $d_r = |f(z)|$ denote the radius of the largest Schlicht disc centered at the origin and contained in the image of $|z| < r$ under $f(z)$. Then there is a point z_0, $|z_0| = r$ such that $|f(z_0)| = d_r$.
Thus, we have

$$\begin{aligned} d_r &= |f(z_0)| = \int_C |f'(z)|\,|dz| \\ &\geq \int_C \frac{(1-|z|)^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}}}{(1+|z|)^{\left\{(1-\varrho)\left(\frac{m}{2}+1\right)+\alpha\right\}}}\,|dz| \\ &\geq \int_0^{|z|} \frac{(1-s)^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}}}{(1+s)^{\left\{(1-\varrho)\left(\frac{m}{2}+1\right)+\alpha\right\}}}\,ds \\ &= \int_0^{|z|} \left(\frac{1-s}{1+s}\right)^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}} \frac{ds}{(1+s)^{2(1-\varrho)}}. \end{aligned} \tag{2.7}$$

Let $\frac{1-s}{1+s} = t$. Then $\frac{-2}{(1+s)^2}ds = dt$ and we can write (2.7) as

$$|f(z_0)| \geq 2^{2\varrho-1} \int_{\frac{1-|z|}{1+|z|}}^1 t^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}} (1+t)^{-2\varrho}\, dt. \tag{2.8}$$

Now, let $\frac{1-r}{1+r} = r_1$ and $t = r_1 u$. Then, from (2.8), we get

$$\begin{aligned} |f(z_0)| &\geq 2^{2\varrho-1} \int_{r_1}^1 (r_1 u)^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}} (1+r_1 u)^{-2\varrho} (r_1 du) \\ &= 2^{2\varrho-1} r_1^{\left\{(1-\varrho)\frac{m}{2}+2\alpha+1\right\}} [I_1 - I_2], \end{aligned}$$

with

$$
\begin{aligned}
I_2 &= \int_0^{r_1} u^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}}\left(1+r_1 u\right)^{-2\varrho}(du) \\
&= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} G\left(a, b, c, -r_1\right),
\end{aligned} \tag{2.9}
$$

where $G(a, b, c, z)$ represents hypergeometric function and

$$
a=(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+1,\ b=2\varrho,\ c=a+1.
$$

Therefore,

$$
I_2=\frac{1}{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+2}\left[G\left(a, b, c,-r_1\right)\right].
$$

Also,

$$
\begin{aligned}
I_1 &= \int_0^{1} u^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha\right\}}\left(1+r_1 u\right)^{-2\varrho}(du) \\
&= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} G(a, b, c, -1).
\end{aligned} \tag{2.10}
$$

Thus

$$
\begin{aligned}
\left|f\left(z_0\right)\right| \geq 2^{2\varrho-1} r_1^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+1\right\}} & \frac{1}{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+2} \times \\
& \left[G(a, b, c,-1)-G\left(a, b, c,-r_1\right)\right].
\end{aligned}
$$

For the upper bound, we use (2.6) with similar method and routine computations and have

$$
\begin{aligned}
& 2^{2\varrho-1} r_1^{\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+1\right\}} \frac{1}{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+2} \times \\
& \qquad \left[G(a, b, c,-1)-G\left(a, b, c,-r_1\right)\right] \\
& \leq |f(z)| \leq 2^{2\varrho-1} r_1^{-\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+1\right\}} \frac{1}{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+2} \times \\
& \qquad \left[G(a, b, c,-1)-G\left(a, b, c,-r_1^{-1}\right)\right].
\end{aligned}
$$

$\square$

Corollary 2.4. (*Covering result*) *Let $r \to 1$ and f satisfy the condition of Theorem 2.3. Then $f(\mathbb{D})$ contains the Schlicht disc $|z|<\frac{2^{2\varrho-1}}{\xi}$, $\xi=\left\{(1-\varrho)\left(\frac{m}{2}-1\right)+\alpha+1\right\}$.*

As special cases, we note that the radius of this disc is

(i) $\frac{1}{m+2}$, when $A=1$, $B=-1$ and $\alpha=1$, (see [15]).

(ii) $\frac{2^{2\varrho-1}}{2(1+\alpha)-\varrho}$, when $m=2$ and for $\varrho=\alpha=\frac{1}{2}$, it is $\frac{2}{5}$.

(ii) $m=4$ gives $\frac{2^{2\alpha-1}}{2(\alpha-\varrho)+3}$ and for $\varrho=\alpha=\frac{1}{2}$, it is $\frac{1}{3}$.

Theorem 2.5. *Let* $\frac{f'}{g'} \in \mathcal{P}_\alpha$ *for* $g \in \mathcal{V}_{m,\alpha}[A,B]$ *and let* $f(z)$ *be given by* (1.1). *Then, for* $m > 2 + \frac{2-\alpha}{1-\rho}$, . *Thus, by taking* $r = \left(1 - \frac{1}{n}\right)$, $n \to \infty$, *it follows that*

$$a_n = O(1)n^\beta \quad \textit{with} \quad \beta = \left\{(1-\rho)\left(\frac{m}{2} - 1\right) + \alpha\right\},$$

where $O(1)$ *is a constant depending only on* α, m, A, B *and* $\rho = \left(\frac{1-A}{1-B}\right)^\alpha$.

Proof. We can write

$$f'(z) = g'(z)p(z), \;\; g \in \mathcal{V}_{m,\alpha}[A,B] \subset \mathcal{V}_m(\rho),$$

where $\rho = \left(\frac{1-A}{1-B}\right)^\alpha$ and $p \in \mathcal{P}_\alpha$ implies, for $z \in \mathbb{D}$,

$$p(z) = (p_1(z))^\alpha, \quad p_1 \in \mathcal{P}. \tag{2.11}$$

For $g \in \mathcal{V}_m(\rho)$, it is well known that there exists $g_1 \in \mathcal{V}_m$ such that

$$g'(z) = (g_1'(z))^{1-\rho}, \quad z \in \mathbb{D}. \tag{2.12}$$

Also, it is known [3] that, for $g_1 \in \mathcal{V}_m$,

$$g_1'(z) = s(z)h^{\frac{m}{2}-1}(z),\; m > 2,\; s \in \mathcal{S}^*,\; h \in \mathcal{P}. \tag{2.13}$$

From (2.11), (2.12), (2.13) and Cauchy theorem, we have

$$\begin{aligned}
n|a_n| &\le \frac{1}{2\pi r^n}\int_0^{2\pi} |s(z)|^{1-\rho} |h(z)|^{\left(\frac{m}{2}-1\right)(1-\rho)} |p(z)|^\alpha \, d\theta \\
&\le \frac{1}{r^n}\left(\frac{r}{(1-r)^2}\right)^{1-\rho} \left[\frac{1}{2\pi}\int_0^{2\pi} |h(z)|^{\left\{\left(\frac{m}{2}-1\right)(1-\rho)\right\}\frac{2}{2-\alpha}} d\theta\right]^{\frac{2-\alpha}{2}} \times \\
&\qquad \left[\frac{1}{2\pi}\int_0^{2\pi} |p(z)|^2 \, d\theta\right]^{\frac{\alpha}{2}} \\
&\le C(\rho,m,\alpha)\left\{\frac{1}{(1-r)}\right\}^{\left(\frac{m}{2}-1\right)(1-\rho)+\alpha+1},
\end{aligned}$$

where we have used distortion result for starlike functions, Holder's inequality and a result for the class $\mathcal{P}$, due to Hayman [8], with

$$m > 2 + \frac{2-\alpha}{1-\rho}, \qquad \rho = \left(\frac{1-A}{1-B}\right)^\alpha.$$

Thus, by taking $r = \left(1 - \frac{1}{n}\right)$, $n \to \infty$, it follows that

$$a_n = O(1).n^{\left\{(1-\rho)\left(\frac{m}{2}-1\right)+\alpha\right\}},\; (n \to \infty).$$

□

Special cases:

(i) We note that, for $m = 4$, we have

$$a_n = O(1).n^{(1-\rho+\alpha)}.$$

(ii) $A = 1$, $B = -1$ gives us $\rho = 0$ and with $\alpha = \frac{1}{2}$, $m = 5$, we get $\beta = 2$. Therefore, in this case

$$a_n = O(1).n^2, \ (n \to \infty).$$

(iii) Choosing ρ in such a way that $\rho = \alpha$ and $m = 4$, we have

$$a_n = O(1).n, \ (n \to \infty).$$

Theorem 2.6. *Let $\frac{f'}{g'} \in \mathcal{P}_\alpha$ for $g \in \mathcal{V}_{m,\alpha}[A, B]$. Then $f(z)$ is a convex function of order ρ for $|z| < r_*$, where*

$$r_* = \frac{2}{m_1 + \sqrt{m_1^2 - 4}}, \qquad \textit{with} \quad m_1 = m + \frac{2\alpha}{1-\rho}.$$

Proof. We have

$$f'(z) = g'(z)p(z), \ \ p \in \mathcal{P}_\alpha. \tag{2.14}$$

Since $\mathcal{V}_{m,\alpha}[A, B] \subset \mathcal{V}_m(\rho)$ with $\rho = \left(\frac{1-A}{1-B}\right)^\alpha$, so

$$g'(z) = \left(g_1'(z)\right)^{1-\rho}, \ \ g_1 \in \mathcal{V}_m.$$

Also, for $g_1 \in \mathcal{V}_m$, it is known [3] that there exists a starlike function s such that

$$g_1'(z) = \left(\frac{s(z)}{z}\right)(h(z))^{\left(\frac{m}{2}-1\right)}, \ \ m > 2, \ \ h \in \mathcal{P}. \tag{2.15}$$

From (2.14) and (2.15), we can write

$$f'(z) = \left(\frac{s(z)}{z}\right)^{1-\rho}(h(z))^{(1-\rho)\left(\frac{m}{2}-1\right)}(p_1(z))^\alpha, \ \ p_1 \in \mathcal{P}. \tag{2.16}$$

Logarithmic differentiation of ([19]) yields to us

$$\frac{zf''(z)}{f'(z)} = (1-\rho)\left(\frac{zs'(z)}{s(z)} - 1\right)(1-\rho)\left(\frac{m}{2} - 1\right)\frac{zh'(z)}{h(z)} + \alpha\frac{zp'(z)}{p(z)}.$$

Now, for h, p and h_1 in $\mathcal{P}$, we have

$$1 + \frac{zf''(z)}{f'(z)} = \rho + (1-\rho)\left\{h_1(z) + \left(\frac{m}{2} - 1\right)\frac{zh'(z)}{h(z)}\right\} + \alpha\frac{zp'(z)}{p(z)}.$$

That is,

$$\begin{aligned}
\Re\left[\frac{(zf'(z))'}{f'(z)} - \rho\right] &\geq (1-\rho)\left[\Re\left(h_1(z)\right) - \left(\frac{m}{2} - 1\right)\left|\frac{zh'(z)}{h(z)}\right| - \alpha\left|\frac{zp'(z)}{p(z)}\right|\right] \\
&\geq (1-\rho)\left[\frac{1-r}{1+r} - \left(\frac{m}{2} - 1\right)\frac{2r}{1-r^2}\right] - \frac{2\alpha r}{1-r^2} \\
&= (1-\rho)\left[\frac{1 - 2r + r^2 - (m-2)\,r}{1-r^2}\right] - \frac{2\alpha r}{1-r^2},
\end{aligned}$$

where we have used Lemma 1.6 with $A = 1$ and $B = -1$. Therefore, we get

$$\frac{1}{(1-\rho)}\Re\left[\frac{(zf'(z))'}{f'(z)} - \rho\right] \geq \frac{1 - \left(m + \frac{2\alpha}{1-\rho}\right)r + r^2}{1-r^2} = \frac{T(r)}{1-r^2}.$$

We note $T(0)=1>0$ and $T(1)=1-m-\frac{2\alpha}{1-\rho}+1=2-\left(m+\frac{2\alpha}{1-\rho}\right)<0$. This shows $r_*\in(0,1)$. Solving $T(r)=0$ gives us the value of r_* which is

$$r_*=\frac{2}{\left(m+\frac{2\alpha}{1-\rho}\right)+\sqrt{\left(m+\frac{2\alpha}{1-\rho}\right)^2-4}}.\qquad\square$$

When $A=1$, $B=-1$, $\alpha=1$, then $\rho=0$ and $g\in\mathcal{V}_m$. This gives radius of convexity for $f\in\mathcal{T}_m$ for $|z|<r_*=\frac{2}{(m+2)+\sqrt{m^2+4m}}$. Furthermore, the case $m=2$ gives us $r_*=\frac{1}{2+\sqrt{3}}$ and this is the well-known radius of convexity for the class $\mathcal{K}$ of close-to-convex functions, see [7]. By assigning other permissible values to the parameters α, A, B and m, we obtain several new and known results.

Theorem 2.7. *Let $f\in\mathcal{T}_{2,\alpha}[A,B]$. Let, for $b>-1$,*

$$F(z)=\frac{b+1}{z^b}\int_0^z t^{b-1}f(t)dt. \tag{2.17}$$

Then $F\in\mathcal{T}_{2,\alpha}[A,B]$ in $\mathbb{D}$.

Proof. Since $f\in\mathcal{T}_{2,\alpha}[A,B]$, $\frac{f'}{g'}\prec\left(\frac{1+Az}{1+Bz}\right)^\alpha$, for some $g\in\mathcal{V}_{2,\alpha}[1,-1]$. We can write (2.17) as

$$F(z)=\phi_b(z)*f(z),$$

where $*$ represents convolution and $\phi_b(z)=\sum\limits_{n=1}^{\infty}\frac{b+1}{b+n}z^n$, see [23].
We define

$$G(z)=\frac{b+1}{z^b}\int_0^z t^{b-1}g(t)dt,\ \ g\in\mathcal{V}_{2,\alpha}[1,-1].$$

Then

$$\begin{aligned}G(z)&=\phi_b(z)*g(z)\\ zG'(z)&=\phi_b(z)*zg'(z)\\ z\left(zG'(z)\right)'&=\phi_b(z)*\frac{\left(zg'(z)\right)'}{g'(z)}.zg'(z).\end{aligned}$$

So

$$\frac{\left(zG'(z)\right)'}{G'(z)}=\frac{\phi_b(z)*\frac{\left(zg'(z)\right)'}{g'(z)}.zg'(z)}{\phi_b(z)*.zg'(z)}.$$

Since $g\in\mathcal{V}_{2,\alpha}[1,-1]$, this implies $zg'\in\mathcal{R}_{2,\alpha}[1,-1]\subset\mathcal{S}^*$, we use Lemma 1.5 and it follows that $G\in\mathcal{V}_{2,\alpha}[1,-1]$.
Now,

$$\frac{F'}{G'}=\frac{\phi_b(z)*\frac{f'(z)}{g'(z)}.zg'(z)}{\phi_b(z)*zg'(z)},$$

and this proves $F'(z)/G'(z)\prec\left((1+Az)/(1+Bz)\right)^\alpha$. Hence the class $\mathcal{T}_{2,\alpha}[A,B]$ is preserved under the integral operator given by (2.17). This operator is known as Bernardi operator, see [1]. $\square$

Theorem 2.8. *Let $f \in \mathcal{T}_{m,\alpha}[0,-1]$. Then, for $z = re^{i\theta}$, $0 \le \theta_1 < \theta_2 \le 2\pi$*

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta > \beta\pi,$$

where $\beta = (1-\rho_1)(m/2-1)+\alpha$, with $\rho_1 = (1/2)^{\alpha}$.

Proof. It can easily be seen that

$$\mathcal{V}_{m,\alpha}[0,-1] \subset \mathcal{V}_m(\rho_1), \quad \text{for } \rho_1 = (1/2)^{\alpha}.$$

So, for $g \in \mathcal{V}_m(\rho_1)$, there exists $g_1 \in \mathcal{V}_m$ such that

$$g'(z) = (g_1'(z))^{1-\rho_1}. \tag{2.18}$$

Also, for $g_1 \in \mathcal{V}_m$, we have

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{(zg_1'(z))'}{g_1'(z)}\right\} d\theta > -\left(\frac{m}{2}-1\right)\pi. \tag{2.19}$$

We have $h \in \mathcal{P}_\alpha$ which implies $h(z) \prec ((1+z)/(1-z))^{\alpha}$ and so $h(z) = (h_1(z))^{\alpha}$, $h_1 \in P$.
We observe, for $h_1 \in \mathcal{P}$

$$\begin{aligned}\frac{\partial}{\partial\theta}\arg h_1(re^{i\theta}) &= \frac{\partial}{\partial\theta}\Re\left\{-i\ln h_1(re^{i\theta})\right\}\\ &= \Re\left\{\frac{re^{i\theta}h_1'(re^{i\theta})}{h_1(re^{i\theta})}\right\}.\end{aligned}$$

Therefore

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{re^{i\theta}h_1'(re^{i\theta})}{h_1(re^{i\theta})}\right\} d\theta = \arg h_1(re^{i\theta_2}) - \arg h_1(re^{i\theta_1}),$$

and

$$\max_{h_1\in P}\left|\int_{\theta_1}^{\theta_2} \Re\left\{\frac{re^{i\theta}h_1'(re^{i\theta})}{h_1(re^{i\theta})}\right\} d\theta\right| = \max_{h_1\in P}\left|\arg h_1(re^{i\theta_2}) - \arg h_1(re^{i\theta_1})\right|.$$

Since $h_1 \in \mathcal{P}$, it is known [25] that

$$\left|h_1(z) - \frac{1+r^2}{1-r^2}\right| \le \frac{2r}{1-r^2},$$

and so

$$|\arg h_1(z)| \le \sin^{-1}\left(\frac{2r}{1-r^2}\right).$$

This gives us

$$\begin{aligned}\max_{h_1\in P}\left|\int_{\theta_1}^{\theta_2} \Re\left\{\frac{re^{i\theta}h_1'(re^{i\theta})}{h_1(re^{i\theta})}\right\} d\theta\right| &\le 2\sin^{-1}\left(\frac{2r}{1-r^2}\right)\\ &\le \pi - 2\cos^{-1}\left(\frac{2r}{1-r^2}\right).\end{aligned} \tag{2.20}$$

For $f \in \mathcal{T}_{m,\alpha}[0,-1]$ we can write

$$f'(z) = \left(g_1'(z)\right)^{1-\rho_1}\left(h_1(z)\right)^{\alpha}, \quad \rho_1 = \left(\frac{1}{2}\right)^{\alpha}, \; g_1 \in \mathcal{V}_m, \, h_1 \in \mathcal{P}. \tag{2.21}$$

Hence, from (2.18), (2.19), (2.20) and (2.21) together with some computations, it follows that

$$\max_{h_1 \in P}\left|\int_{\theta_1}^{\theta_2} \Re\left\{\frac{zf'(z)}{f'(z)}\right\} d\theta\right| > -\left\{(1-\rho_1)\left(\frac{m}{2}-1\right)+\alpha\right\}\pi, \;\; z = re^{i\theta}, \; (r \to 1)\,. \tag{2.22}$$

□

Remark 2.9. It has been proved in [10] by Kaplan that f satisfying (2.22) is close-to-convex in $\mathbb{D}$ if and only if $\beta = \left\{(1-\rho_1)\left(\frac{m}{2}-1\right)+\alpha\right\} \leq 1$. Thus $f \in \mathcal{T}_{m,\alpha}[0,-1]$ is univalent in $\mathbb{D}$ for $2 \leq m \leq 2 + \frac{2(1-\alpha)}{(1-\rho_1)}$, with $\rho_1 = \left(\frac{1}{2}\right)^{\alpha}$.

We shall now discuss the rate of growth of qth Hankel determinant $L_q(n)$ of $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n \in \mathcal{T}_{m,\alpha}[0,B]$, $B \in [-1,0)$, $\alpha \in (0,1]$, and $L_q(n)$, $q \geq 1$, $n \geq 1$ is defined as

$$L_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \tag{2.23}$$

Hankel determinant problem has been studied by several prominent researchers in the past, see [4, 5, 12, 13, 14, 16, 20, 21].

Now, we prove

Theorem 2.10. *Let f given by $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ and let $\frac{f'}{g'} \in \mathcal{P}_\alpha[0,B]$, $B \in [-1,0)$ with $g \in \mathcal{V}_{m,\alpha}[0,B]$, $m > 2$. Then, for $k = 0,1,2,\cdots$, there are numbers γ_k and $c_{k\mu}$ $(\mu = 0,1,2,\cdots,k)$ that satisfy $|c_{k0}| = |c_{kk}| = 1$ and*

$$\sum_{l=0}^{\infty} \gamma_l \leq 3, \;\; 0 \leq \gamma_k \leq \frac{2}{k+1} \tag{2.24}$$

such that

$$\sum_{\mu=0}^{\infty} c_{k\mu} a_{n+\mu} = O(1) n^{\beta_1}, \;\; \beta_1 = \gamma_k + \left(\frac{m}{2}-1\right)(1-\rho_1) + \alpha - 2, \quad (n \to \infty)\,.$$

The bounds (2.24) are the best possible.

Proof. We can write

$$f'(z) = g'(z)\, h(z)\,,$$

where $g \in \mathcal{V}_m[0,B]$ with $B \in [-1,0)$ and $h(z) \prec \left(\frac{1}{1+Bz}\right)^{\alpha}$. Since $g \in \mathcal{V}_{m,\alpha}[0,B]$ implies $g \subset \mathcal{V}_m(\rho_1)$, where $\rho_1 = \left(\frac{1}{1-B}\right)^{\alpha}$.

Thus, we have

$$f'(z) = \left(g_1'(z)\right)^{1-\rho_1} \left(h_1(z)\right)^{\alpha}, \quad g_1 \in \mathcal{V}_m,\, h_1 \in \mathcal{P}. \tag{2.25}$$

It is shown [3] that, for all $m > 2$, there exists a starlike function s and $p \in \mathcal{P}$ such that

$$zg_1'(z) = s(z)(p(z))^{\left(\frac{m}{2}-1\right)}. \tag{2.26}$$

From (2.25) and (2.26), it follows that

$$f'(z) = \left[\frac{s(z)}{z}(p(z))^{\left(\frac{m}{2}-1\right)}\right]^{(1-\rho_1)} \left(h_1(z)\right)^{\alpha}. \tag{2.27}$$

Now $s(z)$ can be represented by as

$$s(z) = z \exp \int_0^{2\pi} \log \frac{1}{1 - ze^{-it}} dv(t),$$

where $v(t)$ is an increasing function and $v(2\pi) - v(0) = 2$. We here note the jumps of $v(t)$ as $\alpha_1 \geq \alpha_2 \geq \cdots$ at $t = t_1, t_2, \ldots$ and assume $t_1 = 0$. Then $\alpha_1 + \alpha_2 + \ldots \leq 2$ also $\alpha_1 + \alpha_2 + \ldots + \alpha_q = 2$, for some q, if and only if $s(z)$ is of the form

$$s(z) = z \prod_{j=1}^{q} \left(1 - e^{it_j} z\right)^{\frac{-2}{q}}. \tag{2.28}$$

Following the similar arguments given in [21], we define

$$\phi_k(z) = \prod_{\mu=1}^{k} \left(1 - e^{it_\mu} z\right)^{\frac{-2}{q}} = \sum_{\mu=0}^{k} C_{k\mu} z^{k-\mu},$$

and consider three cases. It is shown in [21] that the bounds (2.24) are the best possible.

We use Lemma 1.7 to complete the proof. We write

$$\phi_k . zf'(z) = \sum_{n=0}^{k} b_{kn} z^{n+k} + \sum_{n=1}^{\infty} (n+k)\, a_{kn} z^{n+k}, \tag{2.29}$$

where

$$b_{kn} = \sum_{v=0}^{n} (n+v)\, C_{k-v} a_{n-v},$$

$$a_{kn} = \sum_{\mu=0} C_{k\mu} a_{n+\mu}, \quad |C_{kn}| = |C_{kk}| = 1.$$

Let $s(z)$ in (2.27) be not of the form (2.28). Then $\alpha_1 + \alpha_2 + \ldots + \alpha_q < 2$ for $q \geq 1$ and in particular $\alpha_1 < 2$

$$0 \leq \gamma_k < \frac{2}{1+l},\, \eta_0 + \eta_1 + \ldots < 3.$$

It can easily be shown [21] that, in each of three cases considered in [21],

$$\underset{|z|=r}{Max} \left|\phi_k . s(z)\right| = O(1)\,(1-r)^{-\eta_k - \delta_k}, \tag{2.30}$$

where

$$\eta_k < \frac{2}{1+k}, \ \ \eta_1 + \eta_2 + \ldots < 3$$

and

$$\delta_k = \frac{1}{3}\min\left\{\frac{2}{1+k} - \eta_k, \frac{1}{2^{1+k}}\left(3 - \sum_{j=0}^{k}\eta_j\right)\right\},$$

Thus, from (2.27), (2.29) and Cauchy integral formula, we proceed with $m > \left(2 + \frac{2-\alpha}{1-\rho_1}\right)$ for $\rho_1 = \left(\frac{1}{1-B}\right)$ and $B \in [-1, 0)$.

$$\begin{aligned}
(k+n)\left|a_{kn}\right| &\leq \frac{1}{r^{n+k}}\left[\frac{1}{2\pi}\int_0^{2\pi}\left|\phi_k.\left(s(z)\right)^{1-\rho_1}\right|\left|p(z)\right|^{\left(\frac{m}{2}-1\right)(1-\rho_1)}\left|h(z)\right|^{\alpha}d\theta\right] \\
&\leq \frac{4^{\rho_1}}{r^{n+k}}\max\left|\phi_k.s(z)\right|\left[\frac{1}{2\pi}\int_0^{2\pi}\left|p(z)\right|^{\left(\frac{(m-2)(1-\rho_1)}{2-\alpha}\right)}\right]^{\frac{2-\alpha}{2}}\times \\
&\left(\frac{1}{2\pi}\int_0^{2\pi}\left|h_1(z)\right|^2 d\theta\right)^{\frac{\alpha}{2}}. \qquad (2.31)
\end{aligned}$$

Where we have used distortion result for starlike function $s(z)$ along with the Holder's inequality. Now using Lemma 1.8 and (2.30), we obtain from (2.31)

$$(l+n)\left|a_{kn}\right| \leq C(m,\alpha)\left(1-r\right)^{\left\{-\eta_k-\gamma_k-\left(\frac{m}{2}-1\right)(1-\rho_1)-1+\alpha\right\}}, \quad (r \to 1),$$

where $C(m,\alpha)$ is a constant $m > \left(2 + \frac{2-\alpha}{1-\rho_1}\right)$ with $\rho_1 = \left(\frac{1}{1-B}\right)^{\alpha}$.

This implies, with $r = 1 - \frac{1}{n}$, $n \to \infty$

$$a_{kn} = O(1).n^{\left\{\gamma_k + \left(\frac{m}{2}-1\right)(1-\rho_1)+\alpha-2\right\}},$$

where $O(1)$ represents a constant.

The case when $s(z)$ is of the form (2.28) follows on similar lines. □

We can now easily prove the following.

Theorem 2.11. *Let the function f satisfy the conditions given in Theorem 2.10. Then, for $q \geq 1$, $n \geq 1$ and $m > 2 + \frac{2-\alpha}{1-\rho_1}$ with $\rho_1 = \left(\frac{1}{1-B}\right)^{\alpha}$.*

$$L_q(n) = O(1).n^{2+\left\{\left(\frac{m}{2}-1\right)(1-\rho_1)+\alpha-2\right\}q}.$$

We note some special cases:

(i) $B = -1$, $\rho_1 = \left(\frac{1}{1-B}\right)^{\alpha} = \left(\frac{1}{2}\right)^{\alpha}$, $\alpha = 1$. Then, for $m > 4$

$$L_q(n) = O(1).n^{2+\left\{\left(\frac{m}{4}-\frac{1}{2}\right)-1\right\}q}.$$

(ii) Also $L_1(n) = a_n$ and, from Theorem 2.5, we have

$$L_1(n) = O(1).n^{\left\{\left(\frac{m}{2}-1\right)(1-\rho_1)+\alpha\right\}},$$

for $m > \left(2 + \frac{2-\alpha}{1-\rho_1}\right)$ with $\rho_1 = \left(\frac{1}{1-B}\right)^{\alpha}$.

For the case $m = 2$, we solve this problem separately as follows.

Corollary 2.12. *Let* $\frac{f'}{g'} \in \mathcal{P}_\alpha$ *with* $g \in \mathcal{V}_{2,\alpha}[-1,0]$, $\alpha \in (\frac{1}{2},1]$. *Then, for* $q \geq 1$, $n \geq 1$ *and* $m = 2$,

$$L_q(n) = O(1).n^{2+(\alpha-2)q}.$$

Proof. Let $\frac{f'}{g'} \in \mathcal{P}_\alpha$ with $g \in \mathcal{V}_{2,\alpha}[-1,0]$, $\alpha \in (\frac{1}{2},1]$. Then

$$f'(z) = (g_1'(z))^{1-\rho_2} h^\alpha(z), \;\; g_1 \in \mathcal{V}_2, \;\; h \in \mathcal{P}.$$

We take $\frac{s(z)}{z} = g_1'(z)$, and $s(z)$ of the form (2.28) and in the case $\alpha_1 + \alpha_2 + \cdots = 2$, $\sum\limits_{l=0}^{\infty} \gamma_l \leq 3$, $0 \leq \gamma_k \leq \frac{2}{k+1}$. Also $\gamma_k = \frac{2}{k+1}$ implies that $k = q-1$, $\alpha_1 = \alpha_2 = \cdots = \alpha_q$. So using distortion result for $s(z)$ together with Cauchy's theorem, we can write

$$(k+n)\left|a_{kn}\right| \leq \frac{4^{\rho_2}}{2\pi r^{n+k}} \int_0^{2\pi} \left|\phi_k.s(z)\right| \left|h(z)\right|^\alpha d\theta,$$

by Holder's inequality, this implies

$$(k+n)\left|a_{kn}\right| \leq \frac{4^{\rho_2}}{r^{n+k}} \left[\frac{1}{2\pi}\int_0^{2\pi} \left|\phi_k.s(z)\right|^2 d\theta\right]^{\frac{1}{2}} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|h(z)\right|^{2\alpha} d\theta\right)^{\frac{1}{2}}. \tag{2.32}$$

When we write $\left|\phi_k.s(z)\right|^2$ in the form (1.3) the exponent $(-\lambda_j)$ satisfy

$$\lambda_j \leq 2\gamma_k, \;\; (k = 1,2,\ldots,q : k > 0).$$

Hence, using Lemma 1.7, we have

$$\int_0^{2\pi} \left|\phi_k.s(z)\right|^2 d\theta \leq C_1 n^{2\gamma_k - 1}, \;\; (n \to \infty). \tag{2.33}$$

Also, for $\alpha \in (\frac{1}{2},1]$, it follows from Lemma 1.8

$$\int_0^{2\pi} \left|h_1(z)\right|^{2\alpha} d\theta \leq C_2 n^{2\alpha-1}, \;\; (n \to \infty). \tag{2.34}$$

Hence, from (2.32) to (2.34), we obtain

$$(n+k)\left|a_{kn}\right| \leq C_3 n^{\gamma_k + \alpha - 1}. \tag{2.35}$$

From (2.35), we have

$$a_{kn} = O(1).n^{\gamma_k+\alpha-2}, \;\; (n \to \infty).$$

Thus, for $q \geq 1$, $n \geq 1$

$$L_q(n) = O(1).n^{2+(\alpha-2)q}. \qquad \square$$

Particularly, when $\alpha = 1$, $L_q(n) = O(1).n^{2-q}$, and the exponent $(2-q)$ is best possible, see [13]. C_i, $(i = 1,2,3)$, $O(1)$ represents constants, and f is close-to-convex in $\mathbb{D}$.

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Khalida Inayat Noor
COMSATS University Islamabad, Department of Mathematics,
Park Road, Islamabad, 44000, Pakistan
e-mail: khalidan@gmail.com

Shujaat Ali Shah
COMSATS University Islamabad, Department of Mathematics,
Park Road, 44000, Islamabad, Pakistan
and
Quaid-i-Awam University of Engineering Science and Technology,
Department of Mathematics and Satatistics,
Sakrand Road, Nawabshah, 67480, Pakistan
e-mail: shahglike@yahoo.com, shujaatali@quest.edu.pk

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Oscillatory behavior of a fifth-order differential equation with unbounded neutral coefficients

John R. Graef, Hakan Avcı, Osman Özdemir and Ercan Tunç

Abstract. The authors study the oscillatory behavior of solutions to a class of fifth-order differential equations with unbounded neutral coefficients. The results are obtained by a comparison with first-order delay differential equations whose oscillatory characters are known. Two examples illustrating the results are provided, one of which is applied to Euler type equations.

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1. Introduction

In this paper, we are concerned with the oscillatory behavior of all solutions of the fifth-order neutral differential equation

$$z^{(5)}(t) + q(t)x(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $z(t) = x(t) + p(t)x(\tau(t))$, and the following conditions are assumed to hold throughout:

(C1) p, $q : [t_0, \infty) \to \mathbb{R}$ are continuous functions with $p(t) \geq 1$, $p(t) \not\equiv 1$ for all large t, $q(t) \geq 0$, and $q(t)$ is not identically zero for all large t;

(C2) τ, $\sigma : [t_0, \infty) \to \mathbb{R}$ are continuous functions such that $\tau(t) \leq t$, $\sigma(t) \leq t$, τ is strictly increasing, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

(C3) $h(t) := \tau^{-1}(\sigma(t)) \leq t$ and $\lim_{t\to\infty} h(t) = \infty$, where τ^{-1} is the inverse function of τ.

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By a *solution* of equation (1.1), we mean a function $x \in C([t_x,\infty),\mathbb{R})$ for some $t_x \geq t_0$ such that $z \in C^5([t_x,\infty),\mathbb{R})$ and x satisfies (1.1) on $[t_x,\infty)$. We only consider those solutions of (1.1) that exist on some half-line $[t_x,\infty)$ and satisfy the condition

$$\sup\{|x(t)| : T_1 \leq t < \infty\} > 0 \text{ for any } T_1 \geq t_x,$$

and moreover, we tacitly assume that (1.1) possesses such solutions. Such a solution $x(t)$ of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x,\infty)$, i.e., for any $t_1 \in [t_x,\infty)$ there exists $t_2 \geq t_1$ such that $x(t_2) = 0$; otherwise it is called *nonoscillatory*, i.e., if it is eventually positive or eventually negative. Equation (1.1) is termed oscillatory if all its solutions are oscillatory.

Recently there has been a great deal of work on the oscillation of solutions of neutral differential equations. A neutral differential equation is a differential equation in which the highest order derivative of the unknown function is evaluated both at the present state t and at one or more past or future states. Besides its theoretical interest, the study of neutral equations has some importance in applications; for example, see Hale's monograph [15] for some applications in science and technology.

Among numerous papers dealing with the oscillation of the solutions of third and higher odd-order neutral differential equations, we refer the reader to the papers [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 13, 16, 17, 21, 22, 23, 25, 26, 27, 28, 29, 30] and the references cited therein as examples of recent results on this topic. However, except for the papers [3, 4, 14, 30] in which third order equations are studied, the results obtained in these other papers are for the case where p is bounded, i.e., the cases $0 \leq p(t) \leq p_0 < 1$, $-1 < p_0 \leq p(t) \leq 0$, or $0 \leq p(t) \leq p_0 < \infty$. To the best of our knowledge, there appears to be no results for fifth and/or higher odd-order differential equations with unbounded neutral coefficients. The aim of the present paper is to initiate the study of the oscillatory behavior of (1.1) and to provide new results that can be applied not only to the case where $p(t) \to \infty$ as $t \to \infty$ but also to the case where $p(t)$ is a bounded function. Since the equation considered here is linear, it is possible to extend our results to more general differential equations (see Remark 2.8 below). It is our belief that the present paper will contribute significantly to the study of oscillatory behavior of solutions of fifth and higher odd-order differential equations with unbounded neutral coefficients.

In the sequel, all functional inequalities are supposed to hold for all t large enough. Without loss of generality, we deal only with positive solutions of (1.1), since if $x(t)$ is a solution of (1.1), then $-x(t)$ is also a solution.

2. Main results

We begin with the following auxiliary lemmas that are essential in the proofs of our main results.

Lemma 2.1 ([1, Lemma 2.2.3]). *Let* $f \in C^n([t_0,\infty),(0,\infty))$ *such that* $f^{(n)}(t)f^{(n-1)}(t) \leq 0$ *for* $t \geq t_x$ *for some* $t_x \geq t_0$, *and assume that* $\lim_{t\to\infty} f(t) \neq 0$. *Then for every*

$\lambda \in (0,1)$, *there exists a* $t_\lambda \in [t_x,\infty)$ *such that, for all* $t \in [t_\lambda,\infty)$,

$$f(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} \left| f^{(n-1)}(t) \right| .$$

Lemma 2.2. *(Kiguradze and Chanturia* [19]*). Let the function* f *satisfy* $f^{(i)}(t) > 0$, $i = 0,1,2,\ldots,m$ *and* $f^{(m+1)}(t) \leq 0$ *eventually. Then, for every* $l \in (0,1)$,

$$\frac{f(t)}{f'(t)} \geq \frac{lt}{m}$$

eventually.

To prove our results we will make use of the additional hypothesis:

(C4) There exist real numbers l_1, $l_2 \in (0,1)$ such that

$$\psi_1(t) := \frac{1}{p(\tau^{-1}(t))}\left[1 - \left(\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)}\right)^{4/l_1} \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\right] \geq 0, \tag{2.1}$$

$$\psi_2(t) := \frac{1}{p(\tau^{-1}(t))}\left[1 - \left(\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)}\right)^{2/l_2} \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\right] \geq 0, \tag{2.2}$$

and

$$\psi_3(t) := \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\right) \geq 0, \tag{2.3}$$

for all sufficiently large t.

The following lemma is a consequence of a well known result of Kiguradze [18].

Lemma 2.3. *Let conditions (C1)–(C3) be satisfied and assume that* x *is an eventually positive solution of equation* (1.1)*. Then, there exists* $t_1 \in [t_0,\infty)$ *such that the corresponding function* z *satisfies one of the following three cases:*

(I) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) > 0$, $z''''(t) > 0$, *and* $z^{(5)}(t) \leq 0$,
(II) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) < 0$, $z''''(t) > 0$, *and* $z^{(5)}(t) \leq 0$,
(III) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, $z'''(t) < 0$, $z''''(t) > 0$, *and* $z^{(5)}(t) \leq 0$,

for $t \geq t_1$.

Theorem 2.4. *Let conditions (C1)–(C4) hold and assume that there exists a function* $\eta \in C([t_0,\infty),\mathbb{R})$ *such that* $h(t) \leq \eta(t) \leq t$ *for* $t \geq t_0$*. If there exist constants* $\lambda_1, \lambda_2 \in (0,1)$ *such that the first-order delay differential equations*

$$w'(t) + \frac{\lambda_1}{24} q(t)\psi_1(\sigma(t))h^4(t)w(h(t)) = 0, \tag{2.4}$$

$$y'(t) + \frac{\lambda_2}{24} q(t)\psi_2(\sigma(t))h^4(t)y(h(t)) = 0, \tag{2.5}$$

and

$$\varphi'(t) + \frac{1}{24} q(t)\psi_3(\sigma(t))(\eta(t) - h(t))^4 \varphi(\eta(t)) = 0 \tag{2.6}$$

are oscillatory, then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then, from Lemma 2.3, $z(t)$ satisfies one of cases (I)-(III) for $t \geq t_1$.

First, we consider case (I). From the definition of z, we have

$$\begin{aligned} x(t) &= \frac{1}{p(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - x(\tau^{-1}(t))\right] \\ &\geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))}z(\tau^{-1}(\tau^{-1}(t))). \end{aligned} \tag{2.7}$$

Now $\tau(t) \leq t$ and τ is strictly increasing, so τ^{-1} is increasing and $t \leq \tau^{-1}(t)$. Thus,

$$\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t)). \tag{2.8}$$

In view of (I) and Lemma 2.2 with $m = 4$, there exists $t_2 \in [t_1, \infty)$ such that, for every $l_1 \in (0, 1)$,

$$\frac{z(t)}{z'(t)} \geq l_1 \frac{t}{4} \quad \text{for } t \geq t_2,$$

which yields

$$\left(\frac{z(t)}{t^{4/l_1}}\right)' = \frac{z'(t) - \frac{4}{l_1 t}z(t)}{t^{4/l_1}} \leq 0,$$

i.e, $z(t)/t^{4/l_1}$ is nonincreasing for $t \geq t_2$. Using the monotonicity of $z(t)/t^{4/l_1}$, it follows from (2.8) that

$$z\left(\tau^{-1}(\tau^{-1}(t))\right) \leq \frac{\left(\tau^{-1}(\tau^{-1}(t))\right)^{4/l_1} z(\tau^{-1}(t))}{\left(\tau^{-1}(t)\right)^{4/l_1}}. \tag{2.9}$$

Using (2.9) in (2.7) yields

$$x(t) \geq \psi_1(t) z(\tau^{-1}(t)) \quad \text{for } t \geq t_2. \tag{2.10}$$

Since $\lim_{t\to\infty} \sigma(t) = \infty$, we can choose $t_3 \geq t_2$ such that $\sigma(t) \geq t_2$ for all $t \geq t_3$. Thus, from (2.10) we have

$$x(\sigma(t)) \geq \psi_1(\sigma(t)) z(\tau^{-1}(\sigma(t))) \quad \text{for } t \geq t_3. \tag{2.11}$$

Using (2.11) in (1.1) gives

$$z^{(5)}(t) + q(t)\psi_1(\sigma(t)) z(\tau^{-1}(\sigma(t))) \leq 0. \tag{2.12}$$

Now $z(t) > 0$ and $z'(t) > 0$ on $[t_3, \infty) \subseteq [t_2, \infty)$, so

$$\lim_{t\to\infty} z(t) \neq 0,$$

and hence by Lemma 2.1 with $n = 5$ and case (I), for every λ, $0 < \lambda < 1$, there exists $t_\lambda \geq t_3$ such that

$$z(t) \geq \frac{\lambda}{24} t^4 z''''(t) \quad \text{for } t \geq t_\lambda, \tag{2.13}$$

from which we see that

$$z(\tau^{-1}(\sigma(t))) \geq \frac{\lambda}{24} (\tau^{-1}(\sigma(t)))^4 z''''(\tau^{-1}(\sigma(t))) \quad \text{for } t \geq t_5, \tag{2.14}$$

where $\tau^{-1}(\sigma(t)) \geq t_\lambda$ for $t \geq t_5$ for some $t_5 \geq t_\lambda$. Using (2.14) in (2.12) yields

$$z^{(5)}(t) + \frac{\lambda}{24} q(t)\psi_1(\sigma(t))(\tau^{-1}(\sigma(t)))^4 z''''(\tau^{-1}(\sigma(t))) \leq 0, \tag{2.15}$$

for every λ with $0 < \lambda < 1$. Letting $w(t) = z''''(t)$, we see that w is a positive solution of the first-order delay differential inequality

$$w'(t) + \frac{\lambda}{24} q(t)\psi_1(\sigma(t))h^4(t)w(h(t)) \leq 0 \quad \text{for } t \geq t_5. \tag{2.16}$$

It follows from [24, Theorem 1] that the delay differential equation (2.4) corresponding to (2.16) also has a positive solution for all $\lambda_1 \in (0,1)$, but this contradicts our assumption on Eq. (2.4).

Next, we consider case (II). Since $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, and $z'''(t) < 0$, by Lemma 2.2 with $m = 2$, there exists $t_2 \in [t_1, \infty)$ such that, for every $l_2 \in (0,1)$,

$$\frac{z(t)}{z'(t)} \geq l_2 \frac{t}{2} \quad \text{for } t \geq t_2,$$

which yields

$$\left(\frac{z(t)}{t^{2/l_2}}\right)' = \frac{z'(t) - \frac{2}{l_2 t} z(t)}{t^{2/l_2}} \leq 0,$$

i.e, $z(t)/t^{2/l_2}$ is nonincreasing for $t \geq t_2$. Using the fact that $z(t)/t^{2/l_2}$ is nonincreasing, it follows from (2.8) that

$$z\left(\tau^{-1}(\tau^{-1}(t))\right) \leq \frac{\left(\tau^{-1}(\tau^{-1}(t))\right)^{2/l_2} z(\tau^{-1}(t))}{\left(\tau^{-1}(t)\right)^{2/l_2}}. \tag{2.17}$$

Using (2.17) in (2.7) yields

$$x(t) \geq \psi_2(t) z(\tau^{-1}(t)). \tag{2.18}$$

Using (2.18) in (1.1) gives

$$z^{(5)}(t) + q(t)\psi_2(\sigma(t)) z(\tau^{-1}(\sigma(t))) \leq 0 \tag{2.19}$$

for $t \geq t_3$ for some $t_3 \geq t_2$. Now $z(t) > 0$ and $z'(t) > 0$ on $[t_3, \infty) \subseteq [t_2, \infty)$, so

$$\lim_{t\to\infty} z(t) \neq 0,$$

and hence by Lemma 2.1 with $n = 5$ and case (II), for every λ, $0 < \lambda < 1$, there exists $t_\lambda \geq t_3$ such that

$$z(t) \geq \frac{\lambda}{24} t^4 z''''(t) \quad \text{for } t \geq t_\lambda, \tag{2.20}$$

so

$$z(\tau^{-1}(\sigma(t))) \geq \frac{\lambda}{24} (\tau^{-1}(\sigma(t)))^4 z''''(\tau^{-1}(\sigma(t))) \quad \text{for } t \geq t_5, \tag{2.21}$$

where $\tau^{-1}(\sigma(t)) \geq t_\lambda$ for $t \geq t_5$ for some $t_5 \geq t_\lambda$. Using (2.21) in (2.19) gives

$$z^{(5)}(t) + \frac{\lambda}{24} q(t)\psi_2(\sigma(t))(\tau^{-1}(\sigma(t)))^4 z''''(\tau^{-1}(\sigma(t))) \leq 0, \tag{2.22}$$

for every λ with $0 < \lambda < 1$. Letting $y(t) = z''''(t)$, we see that y is a positive solution of the first-order delay differential inequality

$$y'(t) + \frac{\lambda}{24} q(t)\psi_2(\sigma(t))h^4(t)y(h(t)) \leq 0 \quad \text{for } t \geq t_5. \tag{2.23}$$

As in case (I), we conclude that there exists a positive solution $y(t)$ of (2.5) for all $\lambda_2 \in (0,1)$, which contradicts the fact that equation (2.5) is oscillatory.

Finally, we consider case (III). Since $z'(t) < 0$, it follows from (2.8) that

$$z(\tau^{-1}(t)) \geq z(\tau^{-1}(\tau^{-1}(t))),$$

and so inequality (2.7) takes the form

$$x(t) \geq \psi_3(t)z(\tau^{-1}(t)). \tag{2.24}$$

Using (2.24) in (1.1) gives

$$z^{(5)}(t) + q(t)\psi_3(\sigma(t))z(h(t)) \leq 0 \tag{2.25}$$

for $t \geq t_2$ for some $t_2 \geq t_1$. Since $(-1)^k z^{(k)}(t) > 0$ for $k = 0,1,2,3,4$ and $z^{(5)}(t) \leq 0$, for $t_2 \leq u \leq v$, we can easily see that

$$z(u) \geq \frac{(v-u)^4}{24} z''''(v). \tag{2.26}$$

Letting $u = h(t)$ and $v(t) = \eta(t)$ in (2.26), we obtain

$$z(h(t)) \geq \frac{(\eta(t) - h(t))^4}{24} z''''(\eta(t)),$$

and using this in (2.25), we arrive at

$$z^{(5)}(t) + \frac{1}{24} q(t)\psi_3(\sigma(t))(\eta(t) - h(t))^4 z''''(\eta(t)) \leq 0.$$

With $\varphi(t) = z''''(t)$, we see that φ is a positive solution of the first-order delay differential inequality

$$\varphi'(t) + \frac{1}{24} q(t)\psi_3(\sigma(t))(\eta(t) - h(t))^4 \varphi(\eta(t)) \leq 0. \tag{2.27}$$

As before, we conclude that equation (2.6) has a positive solution, which is a contradiction. This completes the proof of the theorem. □

It is well known from [20] (see also [1, Lemma 2.2.9] that if

$$\liminf_{t\to\infty} \int_{g(t)}^{t} a(s)ds > \frac{1}{e}, \tag{2.28}$$

then the first-order delay differential equation

$$x'(t) + a(t)x(g(t)) = 0 \tag{2.29}$$

is oscillatory, where $a, g \in C([t_0,\infty),\mathbb{R})$ with $a(t) \geq 0$, $g(t) < t$, and $\lim_{t\to\infty} g(t) = \infty$. Thus, from Theorem 2.4, we have the following oscillation result for equation (1.1).

Corollary 2.5. *Let conditions (C1)–(C4) hold and assume that there exists a function* $\eta \in C([t_0,\infty),\mathbb{R})$ *such that* $h(t) < \eta(t) < t$ *for* $t \geq t_0$*. If*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} q(s)\psi_1(\sigma(s))h^4(s)ds > \frac{24}{e}, \tag{2.30}$$

$$\liminf_{t\to\infty} \int_{h(t)}^{t} q(s)\psi_2(\sigma(s))h^4(s)ds > \frac{24}{e}, \tag{2.31}$$

and

$$\liminf_{t\to\infty} \int_{\eta(t)}^{t} q(s)\psi_3(\sigma(s))(\eta(s)-h(s))^4ds > \frac{24}{e}, \tag{2.32}$$

then equation (1.1) *is oscillatory.*

Proof. From (2.30), one can choose a positive constant λ_1 with $0 < \lambda_1 < 1$ such that

$$\liminf_{t\to\infty} \lambda_1 \int_{h(t)}^{t} q(s)\psi_1(\sigma(s))h^4(s)ds > \frac{24}{e}. \tag{2.33}$$

Now, in view of (2.28)–(2.29), inequality (2.33) ensures that equation (2.4) is oscillatory. Again, in view of (2.28)–(2.29), inequalities (2.31) and (2.32) guarantee that equations (2.5) and (2.6) are oscillatory, respectively. So, by Theorem 2.4, the conclusion of Corollary 2.5 holds. □

We conclude this paper with the following examples and remarks to illustrate the above results. Our first example is concerned with an equation with bounded neutral coefficients in the case where p is a constant function; the second example is for an equation with unbounded neutral coefficients where $p(t) \to \infty$ as $t \to \infty$.

Example 2.6. Consider the fifth-order differential equation of Euler type

$$[x(t) + 128x(t/2)]^{(5)} + \frac{q_0}{t^5}x(t/6) = 0, \quad t \geq 1. \tag{2.34}$$

Here $p(t) = 128$, $q(t) = q_0/t^5$, $\tau(t) = t/2$, and $\sigma(t) = t/6$. Then, it is easy to see that conditions (C_1)–(C_3) hold, and

$$\tau^{-1}(t) = 2t,\ \tau^{-1}(\tau^{-1}(t)) = 4t, \ \text{ and } \ h(t) = t/3.$$

Choosing $l_1 = l_2 = 2/3$, we see that

$$\psi_1(t) = 1/2^8, \ \ \psi_2(t) = 15/2^{11} \ \text{ and } \ \psi_3(t) = 127/2^{14},$$

i.e., condition (C4) holds. With $\eta(t) = t/2$, we have $h(t) < \eta(t) < t$ for $t \geq 1$. Then, by Corollary 2.5, Eq. (2.34) is oscillatory for

$$q_0 > \max\left\{\frac{2^{11}3^5}{e\ln 3}, \frac{2^{14}3^4}{5e\ln 3}, \frac{2^{21}3^5}{127e\ln 2}\right\} = \frac{2^{21}3^5}{127e\ln 2} \approx 2.1297\times 10^6.$$

Example 2.7. Consider the equation

$$[x(t) + tx(t/2)]^{(5)} + \frac{q_0}{t^4}x(t/4) = 0, \quad t \geq 128. \tag{2.35}$$

Here $p(t) = t$, $q(t) = q_0/t^4$, $\tau(t) = t/2$, and $\sigma(t) = t/4$. Then, it is easy to see that conditions $(C1)$–$(C3)$ hold, and

$$\tau^{-1}(t) = 2t,\ \tau^{-1}(\tau^{-1}(t)) = 4t, \quad \text{and} \quad h(t) = t/2.$$

Choosing $l_1 = l_2 = 1/2$, we see that

$$\psi_1(t) \geq 1/4t, \quad \psi_2(t) \geq 31/64t \quad \text{and} \quad \psi_3(t) \geq 511/2^{10}t,$$

so (C4) holds. With $\eta(t) = 2t/3$, it is easy to see that all conditions of Corollary 2.5 hold, and so Eq. (2.35) is oscillatory if

$$q_0 > \max\left\{\frac{3\cdot 2^7}{e\ln 2}, \frac{3\cdot 2^{11}}{31e\ln 2}, \frac{3^5\cdot 2^{15}}{511e\ln\frac{3}{2}}\right\} = \frac{2^{15}\cdot 3^5}{511e\ln\frac{3}{2}} \approx 14138.$$

Remark 2.8. The results of this paper can be extended to the fifth-order differential equation with unbounded neutral coefficients

$$\left(r(t)\left(z''''(t)\right)^{\gamma}\right)' + q(t)x^{\beta}(\sigma(t)) = 0, \quad t \geq t_0 > 0,$$

under each of the conditions

$$\int_{t_0}^{\infty} r^{-1/\gamma}(t)dt = \infty$$

or

$$\int_{t_0}^{\infty} r^{-1/\gamma}(t)dt < \infty,$$

where $r \in C([t_0,\infty),(0,\infty))$, γ and β are the ratios of odd positive integers, and the other functions in the equation are defined as in this paper.

Remark 2.9. Since it is known that $p(t) \equiv -1$ is a bifurcation point for the behavior of solutions of neutral differential equations (see [12, 13]), it would be of interest to study the oscillatory behavior of all solutions of (1.1) for $p(t) \leq -1$ with $p(t) \not\equiv -1$ for large t.

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John R. Graef
University of Tennessee at Chattanooga,
Department of Mathematics,
Chattanooga, TN 37403, USA
e-mail: John-Graef@utc.edu

Hakan Avcı
Samsun University,
Department of Basic Sciences,
Faculty of Engineering,
Samsun, Turkey
e-mail: hakan.avci@samsun.edu.tr

Osman Özdemir
Tokat Gaziosmanpaşa University,
Department of Mathematics,
Faculty of Arts and Sciences,
60240, Tokat, Turkey
e-mail: osman.ozdemir@gop.edu.tr

Ercan Tunç
Tokat Gaziosmanpaşa University,
Department of Mathematics,
Faculty of Arts and Sciences,
60240, Tokat, Turkey
e-mail: ercantunc72@yahoo.com

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Existence results for some classes of differential systems with ”maxima”

Elisabetta Mangino and Eduardo Pascali

Dedicated to the memory of Prof. Antonio Avantaggiati

Abstract. Local existence of solutions of initial value problems associated with a new type of systems of differential equations with “maxima” are investigated.

Mathematics Subject Classification (2010): 34K07.

Keywords: Systems of differential equations with ”maxima”, initial value problems, local existence of solutions.

1. Introduction

In this short note we consider a class of functional differential equations (and systems) that can be used to describe complex evolutionary phenomena in which the future behaviour depends not only on the present state but also on the past history. The model problem is an initial value problem (IVP) associated with a modified logistic equation which contains the maximum of the square of the unknown function over a past interval:

$$\begin{cases} \dot{x}(t) = x(t) - \max_{[0,t]} x^2(s) \quad t \geq 0; \\ x(0) = x_0 \end{cases} \tag{1.1}$$

where $x_0 \in \mathbb{R}$.

As it is emphasized in the book [3], the application of the classical logistic equations in the setting of experimental sciences entails two order of difficulties: on one hand the necessity of experimentally setting some of the parameters appearing in the equation, and on the other hand the fact that the derivative changes sign exactly when a certain value of the function is reached. To tackle with the second problem,

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often an apriori set delay τ is considered in the equation. It is evident that there are situations in which neither the delay nor the parameters can be determined on an experimental base. The problem (1.1) seems to be more appropriate to deal with those cases.

Analizing (1.1), it is obvious that, if $x_0 = 0$ (resp. $x_0 = 1$), then the constant function $x \equiv 0$ (resp. $x \equiv 1$) is a solution. Moreover, if $x \in C^1([0,T])$ is a solution of (1.1), we observe that:

- if $x_0 < 0$ or $x_0 > 1$, then $\dot{x}(0) < 0$. Therefore, in a neighbourhood of 0, $\dot{x}(t) < 0$ and the equation reduces to $\dot{x}(t) = x(t) - x_0^2$.
- if $0 < x_0 < 1$, then $\dot{x}(0) > 0$. Therefore, in a neighbourhood of 0, $\dot{x}(t) > 0$ and the equation reduces to the well know equation $\dot{x}(t) = x(t) - x^2(t)$.

These easy considerations show that the problem (1.1) somehow "contains" two different types of problems, on the basis of the initial value.

Moreover the IVP (1.1) features also the following strange behaviour. Let $t_0 > 0$ and assume that $0 < x_1 < 1$: then a solution of the following IVP

$$\dot{x}(t) = x(t) - \max_{[0,t]} x^2(s) \quad t_0 \leq t; \quad x(t_0) = x_1 \tag{1.2}$$

could be an extension of a solution either of the IVP

$$\dot{x}(t) = x(t) - \max_{[0,t]} x^2(s) \quad 0 \leq t; \quad x(0) = y_0 \tag{1.3}$$

or of the IVP

$$\dot{x}(t) = x(t) - \max_{[0,t]} x^2(s) \quad 0 \leq t; \quad x(0) = z_0 \tag{1.4}$$

for suitable $0 < y_0 < 1, \quad 1 < z_0$. This "uncertainty" situation for a solution $x = x(t)$ could appear at all time $t > 0$ for which $0 < x(t) < 1$.

More generally, we are going to consider the system

$$\begin{cases} \dot{x}(t) = f\Big(t, x(t), \max_{s\in[0,t]} g_1(x_1(s)), \dots, \max_{s\in[0,t]} g_m(x_m(s))\Big), & t \geq 0 \\ x(0) = x_0 \end{cases} \tag{1.5}$$

where $x_0 \in \mathbb{R}^m$, $f \in C([0,+\infty[\times\mathbb{R}^{2m}, \mathbb{R}^m)$ and is locally Lipschitz with respect to the second variable and the functions $g_i \in C(\mathbb{R})$ are locally Lipschitz on $\mathbb{R}$, for every $i = 1, \dots, m$.

This type of systems belongs to the class of systems of differential equations with "maxima". Much attention has been paid to this type of equations and systems in the last years. Without any pretensions to being exhaustive, we recall only the recent papers [1, 4, 6, 5, 7], while we refer to the monograph [2] for a survey of motivations and techniques on the subject. In particular, Section 3.3 of [2] is devoted to the study of IVP associated with scalar differential equations of the type

$$\begin{cases} \dot{x}(t) = f(t, x(t), \max_{s\in[0,t]} x(s)) & t \geq 0 \\ x(0) = x_0 \end{cases} \tag{1.6}$$

Clearly, even in the scalar case, the class of problems (1.5) is wider than (1.6).

Our aim is to provide first, via fixed point theory, a local esistence and uniqueness result for the general system (1.5). Afterwards, in particular situations as (1.1) and for systems of the type

$$\begin{cases} \dot{x}(t) = x(t) - \max_{s\in[0,t]} y(s) \\ \dot{y}(t) = y(t) - \max_{s\in[0,t]} x(s) \end{cases} \quad \text{or} \quad \begin{cases} \dot{x}(t) = x(t) - \max_{s\in[0,t]} y^2(s) \\ \dot{y}(t) = y(t) - \max_{s\in[0,t]} x^2(s) \end{cases},$$

we will provide more precise existence results by the use of Peano-Picard's approximation.

2. Local existence results via contraction theorem

We start with two remarks that will help along the proofs of our results.

Remark 2.1. Let $g, h \in C([a,b])$. Then

$$|\max_{[a,b]} g - \max_{[a,b]} h| \leq \max_{[a,b]} |g - h|.$$

Indeed, assume that $\max_{[a,b]} g \geq \max_{[a,b]} h$ and let $x_0 \in [a,b]$ such that

$$\max_{[a,b]} g = g(x_0).$$

Then,

$$|\max_{[a,b]} g - \max_{[a,b]} h| = g(x_0) - \max_{[a,b]} h \leq g(x_0) - h(x_0) = |g(x_0) - h(x_0)| \leq \max_{[a,b]} |h - g|.$$

Remark 2.2. Let $g \in C([a,b])$. Then the function

$$h(s) = \max_{\tau\in[0,s]} g(\tau), \quad s \in [a,b]$$

is continuous. Indeed let $s_0 \in [a,b]$. Fix $\varepsilon > 0$ and consider $\delta > 0$ such that

$$|g(\tau) - g(s)| < \varepsilon \text{ if } |\tau - s| < \delta.$$

For any $s_0 < s < s_0 + \delta$, it can happen that $h(s) = h(s_0)$ or that $h(s) = g(\overline{\tau})$ for some $\overline{\tau} \in [s_0, s]$. In the first case obviously $h(s) - h(s_0) < \varepsilon$, while in the second case

$$|h(s) - h(s_0)| = h(s) - h(s_0) \leq g(\overline{\tau}) - g(s_0) < \varepsilon.$$

Therefore $\lim_{s\to s_0^+} h(s) = h(s_0)$.

If $s_0 - \delta < s < s_0$, then $h(s_0) = h(s)$ or $h(s_0) = g(\overline{\tau})$ for some $\overline{\tau} \in [s, s_0]$. In the last case,

$$|h(s) - h(s_0)| = h(s_0) - h(s) \leq g(\overline{\tau}) - g(s) < \varepsilon.$$

So we get that $\lim_{s\to s_0^-} h(s) = h(s_0)$.

Theorem 2.3. *Let $x_0 \in \mathbb{R}^m$, $f \in C([0, +\infty[\times\mathbb{R}^{2m}, \mathbb{R}^m)$ and locally Lipschitz with respect to the second variable and $g_i \in C(\mathbb{R})$ locally Lipschitz on $\mathbb{R}$, for every $i = 1, \ldots, m$.*

Given $\alpha > 0$ and $T > 0$, set

$$M_{\alpha,T} :=$$
$$= \max\left\{ ||f(t,u,v)|| : t \in [0,T], u \in [x_0-\alpha, x_0+\alpha]^m, v \in \prod_{i=1}^m g_i([x_0-\alpha, x_0+\alpha]) \right\}$$

and assume $M_{\alpha,T} > 0$. Let $L_{\alpha,T} > 0$ and $L_\alpha > 0$ be such that for every $t \in [0,T]$, $u_1, u_2 \in [x_0-\alpha, x_0+\alpha]^m$, $v_1, v_2 \in \prod_{i=1}^m g_i([x_0-\alpha, x_0+\alpha])$ and for every $x, y \in [x_0-\alpha, x_0+\alpha]$

$$||f(t,u_1,v_1) - f(t,u_2,v_2)|| \le L_{\alpha,T}(||u_1-v_1|| + ||u_2-v_2||)$$
$$|g_i(x) - g_i(y)| \le L_\alpha |x-y|.$$

Then, for every

$$0 < \overline{T} < \min\left\{ \frac{\alpha}{M_{\alpha,T}}, \frac{1}{L_{\alpha,T}(1+L_\alpha\sqrt{m})}, T \right\}$$

there exists $x \in C^1([0,\overline{T}];\mathbb{R}^m)$ unique solution of the IVP (1.5).

Proof. We will apply the Banach Fixed Point Theorem.

Indeed, observe first that the existence of a C^1 solution of problem (1.5) is equivalent to the existence of a continuous solution of the integral problem

$$x(t) = x_0 + \int_0^t f\Big(s, x(s), \max_{\tau\in[0,s]} g_1(x_1(\tau)), \ldots, \max_{\tau\in[0,s]} g_m(x_m(\tau))\Big) ds. \tag{2.1}$$

Fix

$$0 < \overline{T} < \min\left\{ \frac{\alpha}{M_{\alpha,T}}, \frac{1}{L_{\alpha,T}L_\alpha\sqrt{m}}, T \right\}$$

and consider the map $F : C([0,\overline{T}];\mathbb{R}^m) \to C([0,\overline{T}];\mathbb{R}^m)$ defined by

$$F(x)(t) = x_0 + \int_0^t f\Big(s, x(s), \max_{\tau\in[0,s]} g_1(x_1(\tau)), \ldots, \max_{\tau\in[0,s]} g_m(x_m(\tau))\Big) ds$$

and the ball

$$X := \Big\{ x \in C([0,\overline{T}];\mathbb{R}^m) \mid ||x(t) - x_0|| \le \alpha \ \ \forall t \in [0,\overline{T}] \Big\}.$$

Clearly X is a complete metric space, with respect the the distance induced by the norm of $C([0,\overline{T}];\mathbb{R}^m)$:

$$||x||_\infty := \sup_{t\in[0,\overline{T}]} ||x(t)||, \qquad x \in C([0,\overline{T}];\mathbb{R}^m).$$

If $x \in X$, then

$$\begin{aligned} &||F(x) - x_0||_\infty \\ &\le \sup_{0\le t\le \overline{T}} \int_0^t \left\| f\Big(s, x(s), \max_{\tau\in[0,s]} g_1(x_1(\tau)), \ldots, \max_{\tau\in[0,s]} g_m(x_m(\tau))\Big) \right\| ds \\ &\le \overline{T} M_{\alpha,T} \le \alpha. \end{aligned}$$

Hence $F(X) \subseteq X$. On the other hand, for every $x, y \in X$, it holds

$$\begin{aligned}
&||F(x) - F(y)||_\infty \\
&\leq \overline{T} L_{\alpha,T}(||x-y||_\infty \\
&+ \max_{s\in[0,\overline{T}]} ||\Big(\max_{\tau\in[0,s]} g_1(x_1(\tau)), \ldots, \max_{\tau\in[0,s]} g_m(x_m(\tau))\Big) \\
&- \Big(\max_{\tau\in[0,s]} g_1(y_1(\tau)), \ldots, \max_{\tau\in[0,s]} g_m(y_m(\tau))\Big)|| \\
&\leq \overline{T} L_{\alpha,T}\Big(||x-y||_\infty + \sqrt{m} \max_{s\in[0,\overline{T}]} \max_{i=1}^{m} |\max_{\tau\in[0,s]} g_i(x_i(\tau))) - \max_{\tau\in[0,s]} g_i(y_i(\tau)))|\Big) \\
&\leq \overline{T} L_{\alpha,T}\Big(||x-y||_\infty + \sqrt{m} \max_{i=1}^{m} \max_{\tau\in[0,\overline{T}]} |g_i(x_i(\tau)) - g_i(y_i(\tau))|\Big) \\
&\leq \overline{T} L_{\alpha,T}(1 + \sqrt{m} L_\alpha)||x-y||_\infty.
\end{aligned}$$

Therefore F is a contraction on X and it has a unique fixed point. □

Remark 2.4. The previous result applies, for example, to the following types of problems

$$\dot{x}(t) = \alpha(t)x(t) - \beta(t) \max_{s\in[0,t]} x^2(s) \quad 0 \leq t; \quad x(0) = x_0;$$

$$\dot{x}(t) = \alpha(t)x(t) - \beta(t) \max_{s\in[0,t]} x(s) \quad 0 \leq t; \quad x(0) = x_0;$$

$$\dot{x}(t) = \alpha(t)x(t) - \beta(t) \max_{s\in[0,t]} |x(s)| \quad 0 \leq t; \quad x(0) = x_0.$$

under suitable conditions on the functions α, β.

3. Existence proofs with approximations

Theorem 3.1. *Consider the following problem*

$$\begin{cases} \dot{x}(t) = x(t) - \max_{s\in[0,t]} x^2(s) \quad t \geq 0; \\ x(0) = x_0 \end{cases} \tag{3.1}$$

with $x_0 \neq 1$. Let $\alpha > 1$ and

$$0 < T^* < \frac{\alpha - 1}{\alpha(1 + \alpha|x_0|)}.$$

Then there exists a unique solution $x \in C^1([0, T^*])$ of (3.1).

Proof. We prove the existence of a solution via Peano-Picard's approximations. Set $x_0(t) = x_0$ for every $t \geq 0$ and define

$$x_n(t) = x_0 + \int_0^t x_{n-1}(s)ds - \int_0^t \max_{\eta\in[0,s]} x_{n-1}^2(\eta)ds \quad t \in [0, T^*], \qquad n \geq 1$$

It immediate to prove that

$$x_n(t) = x_0 g_n(t) \quad \forall n \in N, t \geq 0 \tag{3.2}$$

where $g_0 \equiv 1$ and

$$g_n(t) = [1 + \int_0^t g_{n-1}(s)ds - x_0 \int_0^t \max_{[0,s]}[g_{n-1}(\eta)]^2 ds].$$

By induction, using the choice of T^*, we easily get that

$$\forall n \in \mathbb{N}, t \in [0, T^*] \quad |g_n(t)| \leq \alpha,$$

and, as a consequence, that

$$|g_{n+1}(t) - g_n(t)| \leq \frac{|1 - x_0|}{1 + 2\alpha|x_0|} \frac{(1 + 2\alpha|x_0|)^{n+1} t^{n+1}}{(n+1)!}.$$

Then the sequence $(g_n)_n$ is uniformly convergent on $[0, T^*]$ and therefore also the sequence $(x_n)_n$ is uniformly convergent on $[0, \overline{T}]$. It is immediate that its uniform limit is the solution of the problem (3.1). □

Remark 3.2. It is worth noticing that

$$T^* < \max_{\alpha \geq 1} \frac{\alpha - 1}{\alpha(1 + \alpha|x_0|)}.$$

We consider now the following system

$$\begin{cases} \dot{x}(t) = x(t) - \max_{s\in[0,t]} y(s) \\ \dot{y}(t) = y(t) - \max_{s\in[0,t]} x(s) \\ x(0) = x_0 \quad y(0) = y_0 \end{cases} \tag{3.3}$$

with $x_0, y_0 \in \mathbb{R}$. We remark that (3.3) is equivalent to the functional system

$$\begin{cases} x(t) = x_0 + \int_0^t x(s)ds - \int_0^t \max_{\tau\in[0,s]} y(\tau)ds, \\ y(t) = y_0 + \int_0^t y(s)ds - \int_0^t \max_{\tau\in[0,s]} x(\tau)ds, \end{cases} \tag{3.4}$$

The following theorem holds.

Theorem 3.3. *Assume that $x_0 > 0$, $y_0 > 0$ and $x_0 \neq y_0$. Then for all $T > 0$ there exists a unique solution $(x(t), y(t)) \in C^1([0,T])^2$ of the system* (3.3).

Proof. Assume $0 < y_0 < x_0$ and consider the sequences of functions (x_n) and (y_n) defined on $[0, +\infty[$ by

$$x_0(t) = x_0 \quad y_0(t) = y_0$$

$$x_{n+1}(t) = x_0 + \int_0^t (x_n(s) - y_0)ds$$

$$y_{n+1}(t) = y_0 + \int_0^t (y_n(s) - x_n(s))ds$$

It holds that, for every $n \in \mathbb{N}$ and for every $t \geq 0$, $x_n(t) \geq y_0$ and $y_n(t) \leq x_n(t)$. Indeed, the assertion is obviously true if $n = 0$. Assuming that $x_n(t) \geq y_0$ and

$y_n(t) \leq x_n(t)$ for every $t \geq 0$, we get that

$$x_{n+1}(t) - y_0 = x_0 - y_0 + \int_0^t (x_n(s) - y_0)ds \geq 0,$$

$$x_{n+1}(t) - y_{n+1}(t) = x_0 - y_0 + \int_0^t (x_n(s) - y_n(s))ds \leq 0.$$

As a consequence we get that, for every $n \in \mathbb{N}$, $\dot{x}_n \geq 0$ and $\dot{y}_n \leq 0$ and consequently

$$\max_{[0,s]} x(\tau) = x(s), \quad \max_{[0,s]} y_n(\tau) = y_n(0) = y_0.$$

Therefore, for the sequences (x_n) and (y_n), it holds that

$$x_{n+1}(t) = x_0 + \int_0^t x_n(s)ds - \int_0^t \max_{[0,s]} y_n(\tau)ds,$$

$$y_{n+1}(t) = y_0 + \int_0^t y_n(s)ds - \int_0^t \max_{[0,s]} x_n(\tau)ds,$$

By induction, one can prove that for every $n \in \mathbb{N}$ and every $t \geq 0$

$$|x_{n+1}(t) - x_n(t)| \leq |x_0 - y_0| \frac{t^{n+1}}{(n+1)!}$$

$$|y_{n+1}(t) - y_n(t)| \leq |x_0 - y_0| T \frac{t^{n+1}}{n!}$$

Hence the sequences (x_n) and (y_n) are uniformly convergent on $[0,T]$ to continuous functions $x_\infty = x_\infty(t)$ and $y_\infty = y_\infty(t)$ and the couple (x_∞, y_∞) is the unique solution of the functional system (3.4). $\square$

Remark 3.4. It is worth observing that the proof fails if $x_0 = y_0$. Moreover the proof highlights the difference with the system

$$\begin{cases} \dot{x}(t) = x(t) - y(t) \\ \dot{y}(t) = y(t) - x(t). \end{cases}$$

Remark 3.5. More interesting seems to be the study of the following general system

$$\begin{cases} \dot{x}(t) = a(t)x(t) - b(t)\max_{s\in[0,t]} y(s) \\ \dot{y}(t) = c(t)y(t) - d(t)\max_{s\in[0,t]} x(s) \\ x(0) = x_0 > 0, \quad y(0) = y_0 > 0 \end{cases}$$

where the functions a, b, c, d are continuous, non negative and defined on the interval $[0,T]$.

If the functions a, b, c, d are constant, one can prove the following partial results.

If $A = ax_0 - by_0 < 0$, $B = cy_0 - dx_0 < 0$ and $a > 0, \quad c > 0$, then a solution is the following couple of functions

$$x(t) = x_0 + A\frac{1}{a}[e^{at} - 1] \quad y(t) = y_0 + B\frac{1}{c}[e^{ct} - 1].$$

and therefore more information follow. For example we have that

$$x(t) = 0 \Leftrightarrow t = \frac{1}{a}\log\frac{by_0}{|A|} \qquad\qquad y(t) = 0 \Leftrightarrow t = \frac{1}{c}\log\frac{dx_0}{|B|}.$$

For different situations, such as $A > 0, B < 0$, or $A < 0, B > 0$, or $A > 0, B > 0$ an explicit representation for the solution is not available.

Next we consider the following problem, for $t \geq 0$

$$\begin{cases} \dot{x}(t) = x(t) - \max_{s\in[0,t]} y^2(s) \\ \dot{y}(t) = y(t) - \max_{s\in[0,t]} x^2(s) \\ x(0) = x_0 > 0, \\ y(0) = y_0 > 0. \end{cases} \tag{3.5}$$

Theorem 3.6. *If* $T, c_0 > 0$ *satisfy*

$$|x_0| + |x_0 - y_0^2|T \leq c_0, \quad |y_0| + |y_0 - x_0^2|T \leq c_0;$$
$$|x_0| + c_0T + c_0^2T \leq c_0, \quad |y_0| + c_0T + c_0^2T \leq c_0.$$

then there exists $(x, y) \in C^1([0, T]; \mathbb{R}^2)$ *unique solution of the IVP* (3.5)

Proof. The initial problem (3.5) is equivalent to the following functional system:

$$\begin{cases} x(t) = x_0 + \displaystyle\int_0^t x(s)ds - \int_0^t \max_{\eta\in[0,s]} y^2(\eta)ds; \\ y(t) = y_0 + \displaystyle\int_0^t y(s)ds - \int_0^t \max_{\eta\in[0,s]} x^2(\eta)ds. \end{cases}$$

As usual, we define the sequences of functions (x_n) and (y_n) on $[0, +\infty[$ by:

$$x_0(t) = x_0, \quad y_0(t) = y_0$$
$$x_{n+1} = x_0 + \int_0^t x_n(s)ds - \int_0^t \max_{\eta\in[0,s]} y_n^2(\eta)ds$$
$$y_{n+1} = x_0 + \int_0^t x_n(s)ds - \int_0^t \max_{\eta\in[0,s]} x_n^2(\eta)ds.$$

Under the assumptions, it is immediate to prove by induction that

$$|x_n(t)| \leq c_0, \quad |y_n(t)| \leq c_0 \quad \forall n \in N, \quad t \geq 0.$$

Consequently

$$|x_{n+1}(t) - x_n(t)| \leq \frac{c_0}{T}(1 + 2c_0)^n \frac{t^{n+1}}{(n+1)!};$$
$$|y_{n+1}(t) - y_n(t)| \leq \frac{c_0}{T}(1 + 2c_0)^n \frac{t^{n+1}}{(n+1)!}.$$

Indeed, the last assertion is immediately true if $n = 0$. Assuming it for n, we get that

$$
\begin{aligned}
&|x_{n+1}(t) - x_n(t)| \\
\leq &\int_0^t |x_n(t) - x_{n-1}(t)| dt + \int_0^t \left| \max_{\eta\in[0,s]} y_n^2(\eta) - \max_{\eta\in[0,s]} y_{n-1}^2(\eta) \right| ds \\
\leq &\frac{c_0}{T}(1+2c_0)^{n-1} \frac{t^{n+1}}{(n+1)!} + \int_0^t \max_{[0,s]} \left| y_n^2 - y_{n-1}^2 \right| ds \\
\leq &\frac{c_0}{T}(1+2c_0)^{n-1} \frac{t^{n+1}}{(n+1)!} + 2c_0 \int_0^t \max_{[0,s]} |y_n - y_{n-1}| ds \\
\leq &\frac{c_0}{T}(1+2c_0)^{n} \frac{t^{n+1}}{(n+1)!}.
\end{aligned}
$$

Hence the sequences (x_n) and (y_n) are uniformly convergent to continuous functions x_∞, y_∞ defined in the interval $[0,T]$, that solve the functional system. □

Remark 3.7. The methods we have considered could also be applied to investigate a version of Lotka-Volterra systems (see [8, 9] for the first steps in the study of the classical situation) with "maxima", namely

$$
\dot{x}(t) = x(t) - \max_{s\in[0,t]} x(s)y(s); \qquad \dot{y}(t) = y(t) + \max_{s\in[0,t]} x(s)y(s).
$$

or other analogous equations and systems.

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Elisabetta Mangino
Dipartimento di Matematica e Fisica “Ennio De Giorgi”,
Università del Salento,
Via per Arnesano, I-73100 Lecce, Italy
e-mail: elisabetta.mangino@unisalento.it

Eduardo Pascali
Dipartimento di Matematica e Fisica “Ennio De Giorgi”,
Università del Salento,
Via per Arnesano, I-73100 Lecce, Italy
e-mail: eduardo.pascali@unisalento.it

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Global nonexistence of solutions to a logarithmic nonlinear wave equation with infinite memory and delay term

Abdeldjabar Bourega and Djamel Ouchenane

Abstract. As a continuity to the study by M. Kafini [24], we consider a logarithmic nonlinear wave condition with delay term. We obtain a blow-up result of solutions under suitable conditions.

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Keywords: Logarithmic source, blow up, wave equation, negative, initial energy, delay term.

1. Introduction

In this paper, we are concerned with the blow-up in finite-time of solutions for the initial boundary value problem:

$$\begin{gathered} u_{tt} - \Delta u + \int_0^\infty g(t-s)\Delta u(x,s)ds + \mu_1 u_t(x,t) \\ +\mu_2 u_t(x,t-\tau) = u|u|^{p-2}\ln|u|^k, \quad \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

and the initial conditions

$$\begin{gathered} u_t(x,t-\tau) = f_0(x,t-\tau), \quad \text{in } (0,\tau), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \end{gathered}$$

where $u = u(x,t)$, $t \geq 0$, $x \in \Omega$, Δ means the Laplacian administrator regarding the x variable, Ω is an ordinary and limited area of $\mathbb{R}^n$, $n \geq 1$, $p \geq 2$, k, μ_1, are positive constants, μ_2 is a genuine number, $\tau > 0$ speak to the time delay. The capacity $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function, the unwinding capacity exposed to

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conditions to be determined and u_0, u_1, f_0 are given capacities having a place with reasonable spaces.

Presenting the defer term $\mu_2 u_t(x, t-\tau)$ makes the issue unique in relation to those considered in the writing.

In [24] the nonappearance of the viscoelastic term ($g = 0$), the issue has been widely examined and numerous outcomes concerning neighborhood presence result has been set up utilizing the semigroup hypothesis. Likewise, for negative introductory energy, a limited time explode result is demonstrated. For example, for the condition

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = u|u|^{p-2} \ln |u|^k, \quad \text{in } \Omega \times (0, \infty). \tag{1.2}$$

In [22], Han studied the global existence of weak solutions for the initial boundary value problem

$$\begin{gathered} u_{tt} - \Delta u + u - u \ln |u|^2 + u_t + u|u|^2 = 0, \quad in\ \Omega \times (0, T), \\ u(x,t) = 0, \quad x \in \partial\Omega, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \end{gathered} \tag{1.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$. The model (1.1) is closely related to the following equation with logarithmic nonlinearity

$$\begin{gathered} u_{tt} - u_{xx} + u - \varepsilon u \ln |u|^2 + u_t = 0, \text{ in } O \times (0, T), \\ u(x,t) = 0, \quad x \in \partial O, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in O. \end{gathered} \tag{1.4}$$

where $O = [a, b]$, the parameter $\varepsilon \in [0, 1]$ [22].

The remainder of our paper is coordinated as follows. In section 2, we review the documentation, speculations, and some fundamental primers. In section 3, we demonstrate the globale nonexistence result utilizing the semigroup hypothesis [24]. In section 4, we present the statement and the proof of our main blow-up result.

2. Preliminaries and assumptions

In this section, we give notations, hypotheses, $(.,.)$ and $\|.\|_p$ denote the inner prodution in the space $L(\Omega)$ and the norm of the space $L^p(\Omega)$, respectively. For breviy, we denote $\|.\|_2$ by $\|.\|$.
For the relaxation function g we assume the following.
(G) : We assume that the function $g : \mathbb{R}^+ \to \mathbb{R}$ is of class C^1 satisfying:

$$1 - \int_0^\infty g(s)ds = l > 0, \quad g(t) \geq 0, \quad g'(t) \leq 0.$$

and under the assumption

$$\mu_1 \geq |\mu_2|.$$

By using the direct calculations, we have

$$\begin{aligned} \int_0^\infty g(t-s) \left(\nabla u_t(t), \nabla u(s)\right) ds = &-\frac{1}{2} g(t) \|u(t)\|_2^2 + \frac{1}{2}(g' \circ \nabla u)(t) \\ &- \frac{1}{2}\frac{d}{dt}\left[(g \circ \nabla u)(t) - \left(\int_0^\infty g(s)ds\right) \|\nabla u(t)\|_2^2\right] \end{aligned}$$

where

$$(g \circ u)(t) = \int_0^\infty g(t-s) \left\| u(t) - u(s) \right\|_2^2 ds.$$

3. Local existence

We introduce the variable

$$z(x, \rho, t) = u_t(x, t - \tau\rho), \ (x, \rho, t) \in \ \Omega \times (0,1) \times (0, \infty).$$

Consequently, we have

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \ (x, \rho, t) \in \ \Omega \times (0,1) \times (0, \infty).$$

Therefore, problem (1.1) is equivalent to:

$$\begin{gathered} u_{tt}(x,t) - \Delta u(x,t) + \textstyle\int_0^\infty g(t-s)\Delta u(x,s)ds + \mu_1 u_t(x,t) \\ +\mu_2 z(x,1,t) = u(x,t)|u(x,t)|^{p-2} ln|u(x,t)|^k, \quad \text{in } \Omega \times (0,\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad x \in \Omega, \quad \rho \in (0,1), \ t > 0, \end{gathered} \tag{3.1}$$

and the initial conditions

$$\begin{gathered} z(x,1,t) = f_0(x, t-\tau), \quad \text{in } \Omega \times (0,1), \\ u(x,t) = 0, \quad x \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega. \end{gathered}$$

Let $\upsilon = u_t$ and denote by

$$\Phi = (u, \upsilon, z)^T, \quad \Phi(0) = \Phi_0 = \left(u_0, u_1, f_0(., -\rho\tau)\right)^T,$$

Then Φ satisfies the problem

$$\begin{gathered} \partial_t \Phi + A\Phi = J(\Phi) \\ \Phi(0) = \Phi_0, \end{gathered} \tag{3.2}$$

where the operator $A : D(A) \longrightarrow \mathcal{H}$ is defined by

$$A\Phi = \begin{pmatrix} -\upsilon \\ -\Delta u + \mu_1 \upsilon + \mu_2 z(1,.) + \int_0^\infty g(t-s)\Delta u(x,s)ds \\ \frac{1}{\tau} z_\rho \end{pmatrix}$$

and

$$J(\Phi) = \left(0, \ u|u|^{p-2} ln|u|^k, \ 0\right)^T.$$

We introduce the following Hilbert space:

$$\mathcal{H} = \left(H_0^1(\Omega) \cap L_g^2(\mathbb{R}^+, H_0^1(\Omega))\right) \times L^2(\Omega) \times L^2(\Omega \times (0,1)),$$

where $L_g^2(\mathbb{R}^+, H_0^1(\Omega))$ denotes the Hilbert space of H_0^1-valued functions on $\mathbb{R}^+$, endowed with the inner product

$$\langle \phi, \vartheta \rangle_{L_g^2(\mathbb{R}^+, H_0^1(\Omega))} = \int_\Omega \int_0^\infty g\,(t-s)\, \nabla\phi(x,s) \nabla\vartheta(x,s) ds dx.$$

We define the inner product in the energy space $\mathcal{H}$,

$$\left\langle \Phi, \widetilde{\Phi} \right\rangle_{\mathcal{H}} = \int_{\Omega} (\bigtriangledown u \bigtriangledown \widetilde{u} + \upsilon \widetilde{\upsilon}) dx + \tau |\mu_2| \int_0^1 \int_{\Omega} z\widetilde{z} dx d\rho + \int_{\Omega} \int_0^{\infty} g(t-s) \bigtriangledown u \bigtriangledown \widetilde{u} ds dx,$$

for all $\Phi = (u, \upsilon, z)^T$ and $\widetilde{\Phi} = (\widetilde{u}, \widetilde{\upsilon}, \widetilde{z})^T$ in $\mathcal{H}$. The domain of A is

$$D(A) = \left\{ \begin{array}{c} \Phi \in H : \; u \in H^2(\Omega) \cap L_g^2(\mathbb{R}^+, H_0^1(\Omega)), \; \upsilon \in H_0^1(\Omega), \; z(1,.) \in L^2(\Omega), \\ z, \; z_\rho \in L^2 (\Omega \times (0,1)), \; z(0,.) = \upsilon. \end{array} \right\}.$$

Lemma 3.1. [24] *For every ε, there exists $A > 0$, such that the real function*

$$j(s) = |s|^{p-2} ln|s|, \quad p > 2,$$

satisfies

$$|j(s)| \leq A + |s|^{p-2+\varepsilon}.$$

We have the following existence and uniqueness result:

Theorem 3.2. *Assume that $\mu_1 \geq |\mu_2|$ and p be such that*

$$\left\{ \begin{array}{lll} 2 < p < \infty & & if \;\; n = 1, 2, \\ 2 < p < \frac{2(n-1)}{n-2} & & if \;\; n \geq 3. \end{array} \right. \tag{3.3}$$

Then for any $\Phi_0 \in \mathcal{H}$, problem (3.2) has a unique weak solution $\Phi \in C([0,T]; \mathcal{H})$.

Proof. First, for all $\Phi \in D(A)$, we have

$$\begin{aligned} \langle A\Phi, \Phi \rangle_{\mathcal{H}} &= \left\langle \left(\begin{array}{c} -\upsilon \\ -\Delta u + \mu_1 \upsilon + \mu_2 z(1,.) + \int_0^\infty g(t-s) \Delta u(x,s) ds \\ \frac{1}{\tau} z_\rho \end{array} \right), \left(\begin{array}{c} u \\ \upsilon \\ z \end{array} \right) \right\rangle \\ &= -\int_{\Omega} \bigtriangledown u \bigtriangledown \upsilon dx + \int_{\Omega} \upsilon \left[-\Delta u + \mu_1 \upsilon + \mu_2 z(1,.) + \int_0^\infty g(t-s) \Delta u(x,s) ds \right] dx \\ &+ |\mu_2| \int_0^1 \int_{\Omega} z z_\rho dx d\rho + \int_{\Omega} \int_0^\infty g(t-s) \bigtriangledown u \bigtriangledown \upsilon ds dx \\ &= \mu_1 \int_{\Omega} |\upsilon|^2 dx + \mu_2 \int_{\Omega} \upsilon z(1,.) dx + \frac{|\mu_2|}{2} \int_{\Omega} |z(1,.)|^2 dx - \frac{|\mu_2|}{2} \int_{\Omega} |\upsilon|^2 dx \\ &+ \int_{\Omega} \upsilon(x,t) \left(\int_0^\infty g(t-s) \Delta u(x,s) ds \right) dx \\ &+ \int_{\Omega} \int_0^\infty g(t-s) \bigtriangledown u(x,s) \bigtriangledown \upsilon(x,s) ds dx. \end{aligned} \tag{3.4}$$

Looking now at the last term on the right-hand side of (3.4), we have

$$\begin{aligned} |\mu_2| \int_0^1 \int_{\Omega} z_\rho(x,\rho) z(x,\rho) dx d\rho &= |\mu_2| \int_{\Omega} \int_0^1 \frac{1}{2} \frac{\partial}{\partial \rho} z^2(x,\rho) d\rho dx \\ &= \frac{|\mu_2|}{2} \int_{\Omega} \left(z^2(x,1) - z^2(x,0) \right) dx. \\ &= \frac{|\mu_2|}{2} \int_{\Omega} \left(z^2(x,1) - \upsilon^2 \right) dx. \end{aligned}$$

Using Young's inequality, estimate (3.4) becomes

$$-\mu_2 vz \leq \frac{|\mu_2|}{2}|v|^2 + \frac{|\mu_2|}{2}|z|^2.$$

By combining all the estimates,

$$\begin{aligned}\langle A\Phi, \Phi\rangle_{\mathcal{H}} &\geq (\mu_1 - |\mu_2|)\int_\Omega |v|^2 dx + \frac{1}{2}\int_\Omega g(t)\left(\nabla u(x,t)\right)^2 dx \\ &\quad -\frac{1}{2}\int_\Omega\int_0^\infty g'(t-s)\left(\nabla u(x,t) - \nabla u(x,s)\right)^2 dxds \\ &\geq 0.\end{aligned}$$

Therefore, A is a monotone operator.
Next, we prove the operator A is maximal. It is sufficient to show that the operator $(I+A)$ is subjective. Indeed, for any $F = (f_1, f_2, f_3)^T \in \mathcal{H}$, we prove that there exists a unique $V = (u, v, z)^T \in D(A)$ such that

$$(I+A)V = F.$$

Or, equivalently

$$\begin{aligned} u - v &= f_1 \\ v - \Delta u + \mu_1 v + \mu_2 z(1,.) + \int_0^\infty g(t-s)\Delta u(x,s)ds &= f_2 \\ \tau z + z_\rho &= \tau f_3. \end{aligned} \tag{3.5}$$

Noting that $v = u - f_1$, we deduce, from $(3.5)_3$, that

$$z(\rho,.) = (u - f_1)e^{-\rho\tau} + \tau e^{-\rho\tau}\int_0^t f_3(\tau,.)e^{\gamma\tau}d\gamma. \tag{3.6}$$

Substituting (3.6) in $(3.5)_2$, we obtain

$$\sigma u - \Delta u + \int_0^\infty g(t-s)\Delta u(x,s)ds = f_2, \tag{3.7}$$

where,

$$\sigma = 1 + \mu_1 + \mu_2 e^{-\tau} > 0,\ G = f_2 + \sigma f_1 - \tau\mu_2 e^{-\tau}\int_0^1 f_3(\tau,.)e^{\gamma\tau}d\gamma \in L^2(\Omega). \tag{3.8}$$

Now we define, over $H_0^1(\Omega)$, the bilinear and linear forms

$$B(u,w) = \sigma\int_\Omega uw + \left(1 - \int_0^\infty g(s)ds\right)\int_\Omega \nabla u.\nabla w,\quad L(w) = \int_\Omega f_2 w.$$

Thus, for some $\alpha > 0$

$$B(u,u) \geq \alpha\|u\|^2_{H_0^1(\Omega)}$$

Thus B is coercive and L is continuous on $H_0^1(\Omega)$. According to Lax-Milgram Theorem, we can easily obtain unique

$$u \in H_0^1(\Omega),$$

satisfying

$$B(u,w) = L(w),\quad \forall w \in H_0^1(\Omega). \tag{3.9}$$

Consequently, $\upsilon = u - f_1 \in H_0^1(\Omega)$, $\upsilon = u - f_1 \in H_0^1(\Omega)$ and, $z_\rho \in L^2(\Omega \times (0,1))$. Thus, $V \in H$. Using (3.9), we get

$$\sigma \int_\Omega uwdx + \left(1 - \int_0^\infty g(s)ds\right) \int_\Omega \nabla u.\nabla wdx = \int_\Omega Gwdx, \quad w \in H_0^1(\Omega).$$

The standard elliptic regularity theory, gives $u \in H^2(\Omega)$. And using Green's formula and $(3.5)_2$, we obtain

$$\int_\Omega \left[(1+\mu_1)\,\upsilon - \Delta u + \int_0^\infty g(t-s)\Delta u(x,s)ds + \mu_2 z(1,.) - f_2\right] wdx = 0,\ \forall w \in H_0^1(\Omega).$$

Hence,

$$(1+\mu_1)\,\upsilon - \Delta u + \int_0^\infty g(t-s)\Delta u(x,s)ds + \mu_2 z(1,.) = f_2 \in L^2(\Omega)$$

Therefore,

$$V = (u, \upsilon, z)^T \in D(A).$$

Consequently, $I + A$ is surjective and then A is maximal.
We prove that $J : H \to H$ is locally Lipschitz. So, if we set

$$F(s) = |s|^{p-2} s ln|s|^k \text{ then } F'(s) = k\left[1 + (p-1)ln|s|\right]|s|^{p-2}.$$

Therefore,

$$\begin{aligned}
\left\|J(\Phi) - J(\widetilde{\Phi})\right\|_H^2 &= \left\|(0, u|u|^{p-2}ln|u|^k - \widetilde{u}|\widetilde{u}|^{p-2}ln|\widetilde{u}|^k, 0)\right\|_{\mathcal{H}}^2 \\
&= \left\|u|u|^{p-2}ln|u|^k - \widetilde{u}|\widetilde{u}|^{p-2}ln|\widetilde{u}|^k\right\|_{L^2(\Omega)}^2 \\
&= \left\|F(u) - F(\widetilde{u})\right\|_{L^2(\Omega)}^2 . \qquad (3.10)
\end{aligned}$$

Consequently, using value theorem, we have, for $0 \le \theta \le 1$,

$$\begin{aligned}
|F(u) - F(\widetilde{u})| &= |F'(\theta u + (1-\theta)\widetilde{u})(u - \widetilde{u})| \\
&\le k\left[1 + (p-1)ln(\theta u + (1-\theta)\widetilde{u})\right]|\theta u + (1-\theta)\widetilde{u}|^{p-2}|u - \widetilde{u}| \\
&\le k|\theta u + (1-\theta)\widetilde{u}|^{p-2}|u - \widetilde{u}| + k(p-1)|j(\theta u + (1-\theta)\widetilde{u})||u - \widetilde{u}|
\end{aligned}$$

By Lemma 3.1, we find

$$\begin{aligned}
|F(u) - F(\widetilde{u})| \le{} & k|\theta u + (1-\theta)\widetilde{u}|^{p-2}|u - \widetilde{u}| + k(p-1)A|u - \widetilde{u}| \\
& + k(p-1)|\theta u + (1-\theta)\widetilde{u}|^{p-2+\varepsilon}|u - \widetilde{u}| \\
\le{} & k(|u| + |\widetilde{u}|)^{p-2}|u - \widetilde{u}| + k(p-1)A|u - \widetilde{u}| \\
& + k(p-1)(|u| + |\widetilde{u}|)^{p-2+\varepsilon}|u - \widetilde{u}| . \qquad (3.11)
\end{aligned}$$

As u, $\widetilde{u} \in H^1(\Omega)$, we then applying Hölder's inequality and the Sobolev embedding

$$H_0^1(\Omega) \hookrightarrow L^r(\Omega), \quad \forall\, 1 \le r \le \frac{2n}{n-2},$$

to get

$$\begin{aligned}
&\int_{\Omega}\left[(|u|+|\widetilde{u}|)^{p-2}|u-\widetilde{u}|\right]^{2}dx\\
&\quad = \int_{\Omega}(|u|+|\widetilde{u}|)^{2(p-2)}|u-\widetilde{u}|^{2}dx\\
&\quad \leq C\left(\int_{\Omega}(|u|+|\widetilde{u}|)^{2(p-1)}dx\right)^{\frac{(p-2)}{(p-1)}}\times\left(\int_{\Omega}|u-\widetilde{u}|^{2(p-1)}dx\right)^{\frac{1}{(p-1)}}\\
&\quad \leq C\left[\|u\|_{L^{2(p-1)}(\Omega)}^{2(p-1)}+\|\widetilde{u}\|_{L^{2(p-1)}(\Omega)}^{2(p-1)}\right]^{\frac{(p-2)}{(p-1)}}\times\|u-\widetilde{u}\|_{H_0^1(\Omega)}^{2}. \qquad (3.12)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{\Omega}\left[(|u|+|\widetilde{u}|)^{p-2+\varepsilon}|u-\widetilde{u}|\right]^{2}dx\\
&\quad = \int_{\Omega}(|u|+|\widetilde{u}|)^{2(p-2+\varepsilon)}|u-\widetilde{u}|^{2}dx\\
&\quad \leq \left(\int_{\Omega}(|u|+|\widetilde{u}|)^{\frac{2(p-2+\varepsilon)(p-1)}{(p-2)}}dx\right)^{\frac{(p-2)}{(p-1)}}\times\left(\int_{\Omega}|u-\widetilde{u}|^{2(p-1)}dx\right)^{\frac{1}{(p-1)}}\\
&\quad \leq \left(\int_{\Omega}(|u|+|\widetilde{u}|)^{\frac{2(p-2+\varepsilon)(p-1)}{(p-2)}}dx\right)^{\frac{(p-2)}{(p-1)}}\times\|u-\widetilde{u}\|_{L^{2(p-1)}(\Omega)}^{2}. \qquad (3.13)
\end{aligned}$$

Since, $p<\frac{2(n-1)}{(n-2)}$, we can choose $\varepsilon>0$ so small that

$$p^{*}=2(p-1)+\frac{2\varepsilon(p-1)}{p-2}\leq\frac{2n}{n-2}.$$

Therefore, we have

$$\begin{aligned}
&\int_{\Omega}(|u|+|\widetilde{u}|)^{2(p-2+\varepsilon)}|u-\widetilde{u}|^{2}dx\\
&\qquad \leq C\left[\|u\|_{L^{p^*}(\Omega)}^{p^*}+\|\widetilde{u}\|_{L^{p^*}(\Omega)}^{p^*}\right]^{\frac{(p-2)}{(p-1)}}\times\|u-\widetilde{u}\|_{L^{2(p-1)}(\Omega)}^{2}\\
&\qquad \leq C\left[\|u\|_{H_0^1(\Omega)}^{p^*}+\|\widetilde{u}\|_{H_0^1(\Omega)}^{p^*}\right]^{\frac{(p-2)}{(p-1)}}\times\|u-\widetilde{u}\|_{H_0^1(\Omega)}^{2}. \qquad (3.14)
\end{aligned}$$

A combination with (3.9) – (3.14) gives

$$\begin{aligned}
&\left\|J(\Phi)-J(\widetilde{\Phi})\right\|_{H}^{2}\\
&\leq\left[k^{2}(p-1)^{2}A^{2}\right]\|u-\widetilde{u}\|_{H_0^1(\Omega)}^{2}\\
&+C\left[\left(\|u\|_{H_0^1(\Omega)}^{2(p-1)}+\|\widetilde{u}\|_{H_0^1(\Omega)}^{2(p-1)}\right)^{\frac{(p-2)}{(p-1)}}+\left(\|u\|_{H_0^1(\Omega)}^{p^*}+\|\widetilde{u}\|_{H_0^1(\Omega)}^{p^*}\right)^{\frac{(p-2)}{(p-1)}}\right]\times\|u-\widetilde{u}\|_{H_0^1(\Omega)}^{2}\\
&\leq C\left(\|u\|_{H_0^1(\Omega)}+\|\widetilde{u}\|_{H_0^1(\Omega)}\right)\|u-\widetilde{u}\|_{H_0^1(\Omega)}^{2},
\end{aligned}$$

since

$$\left\| J(\Phi) - J(\widetilde{\Phi}) \right\|_H^2 \leq K \left\| u - \widetilde{u} \right\|_H^2 .$$

Hence, J is locally Lipschitz. See [25]. This completes the proof of Theorem 3.2. $\square$

Remark 3.3. The weak solution is taken in the sense of [29]. That is, a function

$$\Phi = (u, u_t, z) \in C([0,T); H),$$

satisfying, for a.e $x \in \Omega$,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u_t(x,t) w\,(x) + &\int_\Omega \nabla u(x,t).\nabla w\,(x)\,dx \\
&- \int_\Omega \left[\left(\int_0^\infty g(t-s)\nabla u(x,s).\nabla w\,(x)\,ds\right)\right] dx \\
&+ \mu_1 \frac{d}{dt}\int_\Omega u_t(x,t) w\,(x)\,dx + \mu_2 \int_\Omega z(x,1,t) w\,(x)\,dx \\
&= \int_\Omega u(x,t)|u(x,t)|^{p-2} \ln |u(x,t)|^k w\,(x)\,dx. \qquad (3.15)
\end{aligned}$$

for all $(w, \psi) \in H_0^1(\Omega) \times L^2(\Omega \times (0,1))$.

4. Main result

Our main blow-up result reads as follows.

Lemma 4.1. *Now, we introduce the energy functional defined by*

$$\begin{aligned}
E(t) := \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^\infty g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)\,(t) + \frac{k}{p}\|\nabla u\|_p^p \\
+ \frac{\xi}{2}\int_\Omega\int_0^1 |z(x,\rho,t)|^2\,dxd\rho - \frac{1}{p}\int_\Omega |u|^p \ln |u|^k dx,
\end{aligned}$$

where

$$\tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|), \quad \mu_1 > |\mu_2|, \qquad (4.1)$$

satisfies the estimate

$$\begin{aligned}
E'(t) \leq -C_0 &\left[\int_\Omega \left(|u_t|^2 + |z(x,1,t)|^2\right) dx\right] \\
&- \frac{1}{2}g(t)\int_\Omega |\nabla u\,(t,x)|^2\,dx + \frac{1}{2}(g' \circ \nabla u)(t) \leq 0. \qquad (4.2)
\end{aligned}$$

Proof. We approximate the initial data $(u_0, u_1, f_0(., -\rho\tau))$ by a sequence

$$(u_0^v, u_1^v, f_0^v) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega) \times C_0^\infty(\Omega \times (0,1)).$$

Then problem (3.1) has a unique classical solution $(u^{\upsilon}, u_t^{\upsilon}, z^{\upsilon})$ such that (3.15) takes the form

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega} u_t^{\upsilon}(x,t)w\left(x\right) &+ \int_{\Omega} \nabla u^{\upsilon}(x,t).\nabla w\left(x\right)dx \\
&- \int_{\Omega}\left[\left(\int_0^{\infty} g(t-s)\nabla u^{\upsilon}(x,s).\nabla w\left(x\right)ds\right)\right]dx \\
&+ \mu_1\frac{d}{dt}\int_{\Omega} u^{\upsilon}(x,t)w\left(x\right)dx \\
&+ \mu_2\int_{\Omega} z^{\upsilon}(x,1,t)w\left(x\right)dx \\
&= \int_{\Omega} u^{\upsilon}(x,t)|u^{\upsilon}(x,t)|^{p-2}\ln|u^{\upsilon}(x,t)|^{k}w\left(x\right)dx,
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega}\int_0^1 \tau z^{\upsilon}(x,\rho,t)\psi\left(x,\rho\right)dxd\rho &+ \int_{\Omega} z^{\upsilon}(x,\rho,t)\psi\left(x,\rho\right)dx \\
&= \int_{\Omega} u_t^{\upsilon}(x,t)\psi\left(x,\rho\right)dx.
\end{aligned} \tag{4.4}$$

By replacing w by u_t^{υ} and ψ by z^{υ} and integrating over $(0,\infty)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega}\left(\left|u_t^{\upsilon}(x,t)\right|^2 + \left|\nabla u^{\upsilon}(x,t)\right|^2\right)dx \\
&\qquad +\frac{1}{2}\left[(g\circ\nabla u^{\upsilon})(t) - \left(\int_0^{\infty} g(s)ds\right)\left\|u^{\upsilon}\left(t\right)\right\|_2^2\right] + \mu_1\int_{\Omega}\int_0^1 \left|u_t^{\upsilon}(x,s)\right|^2 ds \\
&= \frac{1}{2}\int_{\Omega}\left(\left|u_1^{\upsilon}(x)\right|^2 + \left|\nabla u_0^{\upsilon}(x)\right|^2\right)dx - \frac{1}{2}\int_0^{\infty} g(s)\left\|u^{\upsilon}\left(s\right)\right\|_2^2 ds + \frac{1}{2}(g'\circ\nabla u^{\upsilon})(t) \\
&= \frac{1}{2}\int_{\Omega}\left(\left|u_1^{\upsilon}(x)\right|^2 + \left|\nabla u_0^{\upsilon}(x)\right|^2\right)dx \\
&\qquad -\frac{1}{2}\int_0^{\infty} g(s)\left\|u^{\upsilon}\left(s\right)\right\|_2^2 ds + \frac{1}{2}(g'\circ\nabla u^{\upsilon})(t) \\
&\qquad -\mu_2\int_{\Omega}\int_0^1 z^{\upsilon}(x,1,s)u_t^{\upsilon}(x,s)dxds \\
&\qquad +\frac{1}{p}\int_{\Omega}\left[(u^{\upsilon}(x,t))^p\ln|u^{\upsilon}(x,t)|^k - k|u^{\upsilon}(x,t)|^p\right]dx \\
&\qquad -\frac{1}{p}\int_{\Omega}\left[(u_0^{\upsilon}(x))^p\ln|u_0^{\upsilon}(x)|^k - k|u_0^{\upsilon}(x)|^p\right]dx
\end{aligned} \tag{4.5}$$

and

$$\tau z_t(x,\rho,t)z(x,\rho,t) + z_{\rho}(x,\rho,t)z(x,\rho,t) = 0$$

integrating over $(0,\infty)$ and $\rho\in(0,1)$, then

$$\begin{aligned}\frac{\xi}{2}\int_0^1\int_\Omega |z^\upsilon(x,\rho,t)|^2\,dxd\rho &= \frac{\xi}{2}\int_0^1\int_\Omega |f_0(x,-\rho\tau)|^2\,dxd\rho\\ &+\frac{\xi}{2\tau}\int_0^\infty\int_\Omega |u_t^\upsilon(x,t)|^2\,dxd\rho\\ &-\frac{\xi}{2\tau}\int_0^\infty\int_\Omega |z^\upsilon(x,1,s)|^2\,dxds \qquad (4.6)\end{aligned}$$

where $\xi>0$ is defined in (4.1). Also integration by parts, we get

$$\begin{aligned}&\int_\Omega\left[u_t(x,t)\left(\int_0^\infty g(t-s)\Delta u(x,s)ds\right)\right]dx\\ &\qquad= -\int_0^\infty\left[g(t-s)\left(\int_\Omega \nabla u_t(x,t).\nabla u(x,s)dx\right)\right]ds, \qquad (4.7)\end{aligned}$$

and using

$$\begin{aligned}-\nabla u_t(x,t).\nabla u(x,s) &= \frac{1}{2}\frac{d}{dt}\left\{|\nabla u(x,s)-\nabla u(x,t)|^2\right\}\\ &-\frac{1}{2}\frac{d}{dt}\left\{|\nabla u(x,t)|^2\right\}, \qquad (4.8)\end{aligned}$$

then

$$\begin{aligned}&-\int_0^\infty\left[g(t-s)\left(\int_\Omega \nabla u_t(x,t).\nabla u(x,s)dx\right)\right]ds\\ &=\frac{1}{2}\int_0^\infty\left[g(t-s)\left(\int_\Omega\frac{d}{dt}\left\{|\nabla u(x,s)-\nabla u(x,t)|^2\right\}dx\right)\right]ds\\ &-\frac{1}{2}\int_0^\infty\left[g(t-s)\left(\frac{d}{dt}\left\{|\nabla u(x,t)|^2\right\}dx\right)\right]ds. \qquad (4.9)\end{aligned}$$

Using the direct account and (G), we find

$$\begin{aligned}&\frac{1}{2}\int_0^\infty\left[g(t-s)\left(\int_\Omega\frac{d}{dt}\left\{|\nabla u(x,s)-\nabla u(x,t)|^2\right\}dx\right)\right]ds\\ &=\frac{1}{2}\int_0^\infty\left[\left(\frac{d}{dt}\left(g(t-s)\left(\int_\Omega|\nabla u(x,s)-\nabla u(x,t)|^2\,dx\right)\right)\right)\right]ds\\ &-\frac{1}{2}\int_0^\infty g'(t-s)\int_\Omega\left(\int_\Omega|\nabla u(x,s)-\nabla u(x,t)|^2\,dx\right)ds\\ &=\frac{1}{2}\frac{d}{dt}\left[\int_0^\infty g(t-s)\int_\Omega|\nabla u(x,s)-\nabla u(x,t)|^2\,dxds\right]\\ &-\frac{1}{2}\int_0^\infty g'(t-s)\left(\int_\Omega|\nabla u(x,s)-\nabla u(x,t)|^2\,dx\right)ds\\ &=\frac{1}{2}\frac{d}{dt}\left\{(g\circ\nabla u)(t)\right\}-\frac{1}{2}(g'\circ\nabla u)(t); \qquad (4.10)\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{2}\int_0^\infty \left[g(t-s)\left(\frac{d}{dt}\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx\right)\right]ds \\
&= -\frac{1}{2}\left(\int_0^\infty g(t-s)ds\right)\left(\frac{d}{dt}\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx\right) \\
&= -\frac{1}{2}\left(\int_0^\infty g(s)ds\right)\left(\frac{d}{dt}\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx\right) \\
&= -\frac{1}{2}\frac{d}{dt}\left[\left(\int_0^\infty g(s)ds\right)\left(\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx\right)\right] \\
&+ \frac{1}{2}g(t)\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx. \qquad (4.11)
\end{aligned}
$$

By replacement of (4.6) – (4.10), we get

$$
\begin{aligned}
& \int_\Omega \left[u_t(x,t)\left(\int_0^\infty g(t-s)\Delta u(x,s)ds\right)\right]dx \\
&= \frac{1}{2}\frac{d}{dt}\left\{(g\circ\nabla u)(t)\right\} - \frac{1}{2}(g'\circ\nabla u)(t) \\
&- \frac{1}{2}\frac{d}{dt}\left[\left(\int_0^\infty g(s)ds\right)\left(\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx\right)\right] \\
&+ \frac{1}{2}g(t)\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx \\
&= \frac{d}{dt}\left\{\frac{1}{2}(g\circ\nabla u)(t) - \frac{1}{2}\left(\int_0^\infty g(s)ds\right)\left(\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx\right)\right\} \\
&- \frac{1}{2}(g'\circ\nabla u)(t) + \frac{1}{2}g(t)\int_\Omega \left\{|\nabla u(x,t)|^2\right\}dx. \qquad (4.12)
\end{aligned}
$$

Combining (4.6) and (4.10) we get

$$
\begin{aligned}
& \int_\Omega\left(\frac{1}{2}\left|u_t^\upsilon(x,t)\right|^2 + \frac{1}{2}\left|\nabla u^\upsilon(x,t)\right|^2 - \int_\Omega \frac{1}{p}\left(u^\upsilon(x,t)\right)^p \ln|u^\upsilon(x,t)|^k dx + \frac{k}{p^2}|u^\upsilon(x,t)|^p\right)dx \\
&+ \frac{1}{2}\left[(g\circ\nabla u^\upsilon)(t) - \left(\int_0^\infty g(s)ds\right)\|u^\upsilon\,(t)\|_2^2\right] \\
&+ \frac{\xi}{2}\int_\Omega\int_0^1 \left|z^\upsilon(x,\rho,t)\right|^2 d\rho dx \\
&= -\mu_1\int_0^\infty\int_\Omega \left|u_t^\upsilon(x,s)\right|^2 dxds - \mu_2\int_0^\infty\int_\Omega z^\upsilon(x,1,s)u_t^\upsilon(x,s)dxds \\
&+ \frac{1}{2}\int_\Omega\left(|u_1^\upsilon(x)|^2 + |\nabla u_0^\upsilon(x)|^2\right)dx \\
&- \frac{1}{2}\int_0^\infty g(s)\,\|u^\upsilon\,(s)\|_2^2\,ds + \frac{1}{2}\int_0^\infty (g'\circ\nabla u^\upsilon)(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\xi}{2}\int_0^1\int_\Omega |f_0(x,-\rho\tau)|^2\,dxd\rho \\
&-\frac{1}{p}\int_\Omega \left[(u_0^\upsilon(x))^p \ln|u_0^\upsilon(x)|^k - \frac{k}{p}|u_0^\upsilon(x)|^p\right]dx \\
&+\frac{\xi}{2\tau}\int_0^t\int_\Omega |u_t^\upsilon(x,t)|^2\,dxd\rho - \frac{\xi}{2\tau}\int_0^\infty\int_\Omega |z^\upsilon(x,1,s)|^2\,dxds. \qquad (4.13)
\end{aligned}
$$

Repeating the steps (3.6) – (3.11) of [2], we conlude that for any $\upsilon \in \mathbb{N}$,

$$
\begin{aligned}
&(u^\upsilon) \text{ is uniformly bounded in } L^\infty((0,T);H_0^1(\Omega)) \\
&(u_t^\upsilon) \text{ is uniformly bounded in } L^\infty((0,T);H_0^1(\Omega)) \\
&(z^\upsilon) \text{ is uniformly bounded in } L^\infty((0,T);L^2(\Omega\times(0,1))).
\end{aligned}
$$

thus, we get

$$
\begin{aligned}
u^\upsilon &\rightharpoonup u \text{ weakly star in } L^\infty((0,T);H_0^1(\Omega)) \\
u_t^\upsilon &\rightharpoonup u_t \text{ weakly star in } L^\infty((0,T);L^2(\Omega)) \\
z^\upsilon &\rightharpoonup z \text{ weakly star in } L^\infty((0,T);L^2(\Omega\times(0,1)))
\end{aligned}
$$

and by using Loins-Aubin theorem,

$$
u^\upsilon \rightharpoonup u \text{ in } L^2(\Omega\times(0,T)) \text{ and for a.e } (x,t) \text{ in } \Omega\times(0,T).
$$

By integrating (4.3) over $(0,\infty)$, we arrive at

$$
\begin{aligned}
\int_\Omega u_t^\upsilon(x,t)w(x)\,dx \quad + \quad & \mu_1\int_\Omega u^\upsilon(x,t)w(x)\,dx \\
= \quad & -\int_0^\infty\int_\Omega \nabla u^\upsilon(x,s).\nabla w(x)\,dxds \\
& +\int_\Omega\left[\int_0^\infty\left(\int_0^\xi g(\xi-s)\nabla u^\upsilon(x,s).\nabla w(x)\,ds\right)d\xi\right]dx \\
& -\mu_2\int_0^\infty\int_\Omega z^\upsilon(x,1,s)u_t^\upsilon(x,s)dxds \\
& +\int_\Omega\int_0^\infty u^\upsilon(x,t)|u^\upsilon(x,t)|^{p-2}\ln|u^\upsilon(x,t)|^k w(x)\,dsdx \\
& -\int_\Omega u_1^\upsilon(x,t)w(x)\,dx - \mu_1\int_\Omega u_0^\upsilon(x,t)w(x)\,dx, \qquad (4.14)
\end{aligned}
$$

and

$$
\begin{aligned}
\tau\int_\Omega z^\upsilon(x,\rho,t)\psi(x,\rho)\,dx = \quad & -\quad \int_0^\infty\int_\Omega z_\rho^\upsilon(x,\rho,t)\psi(x,\rho)\,dxd\rho \\
& +\quad \tau\int_\Omega f_0(x,-\rho\tau)\psi(x,\rho)\,dx. \qquad (4.15)
\end{aligned}
$$

By passing to the limit, we get

$$
\begin{aligned}
\int_\Omega u_t(x,t)w(x)\,dx \quad &+ \quad \mu_1 \int_\Omega u(x,t)w(x)\,dx \\
&= \quad -\int_0^\infty \int_\Omega \nabla u(x,s).\nabla w(x)\,dxds \\
&\quad\; + \int_0^\infty \int_\Omega \left[\left(\int_0^\xi g(\xi-s)\nabla u(x,s).\nabla w(x)\,ds\right)\right] dxd\xi \\
&\quad\; -\mu_2 \int_0^\infty \int_\Omega z(x,1,s)u_t^v(x,s)dxds \\
&\quad\; + \int_0^\infty \int_\Omega u(x,t)|u(x,s)|^{p-2}\ln|u(x,s)|^k w(x)\,dxds \\
&\quad\; - \int_\Omega u_1(x,t)w(x)\,dx - \mu_1 \int_\Omega u_0(x,t)w(x)\,dx, \qquad (4.16)
\end{aligned}
$$

and

$$
\begin{aligned}
\tau \int_\Omega z(x,\rho,t)\psi(x,\rho)\,dx = \quad &- \quad \int_0^\infty \int_\Omega z_\rho(x,\rho,t)\psi(x,\rho)\,dxd\rho \\
&+ \quad \tau \int_\Omega f_0(x,-\rho\tau)\psi(x,\rho)\,dx, \qquad (4.17)
\end{aligned}
$$

for all $(w,\psi) \in H_0^1(\Omega) \times L^2(\Omega \times (0,1)))$.

Notice that the right hand sides of (4.16) and (4.17) are absolutely continuous. So, by differentiating, we obtain, for a.e $x \in \Omega$,

$$
\begin{aligned}
&\int_\Omega u_{tt}(x,t)w(x)\,dx + \int_\Omega \nabla u(x,t).\nabla w(x)\,dx \\
&\qquad - \quad \int_\Omega \left[\left(\int_0^\infty g(t-s)\nabla u(x,s).\nabla w(x)\,ds\right)\right] dx \\
&\qquad + \quad \mu_1 \int_\Omega u_t(x,t)w(x)\,dx + \mu_2 \int_\Omega z_\rho(x,1,t)\psi(x,\rho)\,dx \\
&\qquad = \quad \int_\Omega u(x,t)|u(x,t)|^{p-2}\ln|u(x,t)|^k w(x)\,dx \qquad (4.18)
\end{aligned}
$$

$$
\tau \int_\Omega z_t(x,\rho,t)\psi(x,\rho)\,dx + \int_\Omega z_\rho(x,\rho,t)\psi(x,\rho)\,dx = 0 \qquad (4.19)
$$

for all $(w,\psi) \in H_0^1(\Omega) \times L^2(\Omega \times (0,1))$.

We use the density of $H_0^1(\Omega)$ in $L^2(\Omega)$ to replace (w,ψ) by (u_t,z) in (4.18) and (4.19). Then we integrate (4.18) over $(0,t)$ and (4.19) over $(0,t)\times(0,1)$, we obtain

$$\begin{aligned} E(t) =& -\left(\mu_1-\frac{\xi}{2\tau}\right)\int_0^\infty\int_\Omega |u_t(x,s)|^2\,dxds \\ &+\frac{1}{2}\left(1-\int_0^\infty g(s)ds\right)\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t) \\ &-\frac{\xi}{2\tau}\int_0^t\int_\Omega |z^\upsilon(x,1,s)|^2\,dxds-\mu_2\int_0^\infty\int_\Omega z(x,1,t)u_t(x,t)dxds+E(0). \end{aligned}$$

Therefore,

$$\begin{aligned} E'(t) =& -\left(\mu_1-\frac{\xi}{2\tau}\right)\int_\Omega |u_t(x,s)|^2\,dx+\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}g(t)\int_\Omega |\nabla u\,(t,x)|^2\,dx \\ &-\frac{\xi}{2\tau}\int_\Omega |z(x,1,t)|^2\,dx-\mu_2\int_\Omega z(x,1,t)u_t(x,t)dx. \end{aligned} \tag{4.20}$$

for a.e $t\in(0,T)$.
Using Young's inequality, we estimate

$$-\mu_2\int_\Omega z(x,1,t)u_t(x,t)dx \le \frac{|\mu_2|}{2}\int_\Omega\left(|u_t(x,s)|^2+|z(x,1,t)|^2\right)dx.$$

Hence, from (4.20), we obtain

$$\begin{aligned} E'(t) \le& -\left(\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_\Omega |u_t(x,s)|^2\,dx-\left(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_\Omega |z(x,1,t)|^2\,dx \\ &+(g'\circ\nabla u)(t)-g(t)\int_\Omega |\nabla u\,(t,x)|^2\,dx. \end{aligned} \tag{4.21}$$

Using (4.1), we have, for some $C_0>0$,

$$\begin{aligned} E'(t) \le \;& - \; C_0\left[\int_\Omega\left(|u_t|^2+|z(x,1,t)|^2\right)dx\right] \\ & - \; \frac{1}{2}g(t)\int_\Omega |\nabla u\,(t,x)|^2\,dx+\frac{1}{2}(g'\circ\nabla u)(t)\le 0. \end{aligned} \tag{4.22}$$

where $C_0=\min\left\{\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2},\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right\}$, which is positive by (4.1). $\square$

Lemma 4.2. *There exists a positive constant $C>0$ such that*

$$\left(\int_\Omega |u|^p\ln|u|^k dx\right)^{\frac{s}{p}} \le C\left[\int_\Omega |u|^p\ln|u|^k dx+\|\nabla u\|_2^2\right].$$

For any $u\in L^{p+1}(\Omega)$ and $2\le s\le p$, provided that $\int_\Omega |u|^p\ln|u|^k dx\ge 0$.

Proof. If $\int_\Omega |u|^p\ln|u|^k dx>1$ then

$$\left(\int_\Omega |u|^p\ln|u|^k dx\right)^{\frac{s}{p}} \le \int_\Omega |u|^p\ln|u|^k dx. \tag{4.23}$$

If $\int_\Omega |u|^p \ln|u|^k dx \le 1$ then we set

$$\Omega_1 = \{x \in \Omega \ \ | \ |u| > 1\}$$

and, for any $\beta \le 2$, we have

$$\begin{aligned} \left(\int_\Omega |u|^p \ln|u|^k dx\right)^{\frac{s}{p}} &\le \left(\int_\Omega |u|^p \ln|u|^k dx\right)^{\frac{\beta}{p}} \le \left(\int_{\Omega_1} |u|^p \ln|u|^k dx\right)^{\frac{\beta}{p}} \\ &\le \left(\int_{\Omega_1} |u|^{p+1} dx\right)^{\frac{\beta}{p}} \le \left(\int_\Omega |u|^{p+1} dx\right)^{\frac{\beta}{p}} = \|u\|_{P+1}^{\frac{\beta(p+1)}{P}}. \end{aligned}$$

We choose $\beta = \frac{2p}{p+1} < 2$ to get

$$\left(\int_\Omega |u|^p \ln|u|^k dx\right)^{\frac{s}{p}} \le \|u\|_{P+1}^2 \le C\,\|\nabla u\|_2^2. \tag{4.24}$$

Combining (4.23) and (4.24), we get the desired result. □

Lemma 4.3. *There exists a positive constant $C > 0$ such that for any $u \in L^p(\Omega)$ we have*

$$\|u\|_P^p \le C\left[\int_\Omega |u|^p \ln|u|^k dx + \|\nabla u\|_2^2\right]. \tag{4.25}$$

provided that $\int_\Omega |u|^p \ln|u|^k dx \ge 0$.

Proof. We set

$$\Omega_+ = \{x \in \Omega \ ||u| > e\} \text{ and } \ \Omega_- = \{x \in \Omega \ ||u| \le e\}.$$

Therefore,

$$\begin{aligned} \|u\|_P^p &= \int_{\Omega_+} |u|^p dx + \int_{\Omega_-} |u|^p dx \\ &\le \int_{\Omega_+} |u|^p \ln|u|^k dx + \int_{\Omega_-} e^p |\frac{u}{e}|^p dx \\ &\le \int_{\Omega_+} |u|^p \ln|u|^k dx + e^p \int_{\Omega_-} |\frac{u}{e}|^2 dx \\ &\le \int_{\Omega_+} |u|^p \ln|u|^k dx + e^{p-2} \int_{\Omega_-} |u|^2 dx \\ &\le C\left(\int_{\Omega_+} |u|^p \ln|u|^k dx + \|\nabla u\|_2^2\right). \end{aligned}$$

□

Corollary 4.4. *There exists a positive constant $C > 0$ such tha*

$$\|u\|_2^2 \le C\left[\left(\int_\Omega |u|^p \ln|u|^k dx\right)^{\frac{2}{p}} + \|\nabla u\|_2^{\frac{4}{p}}\right] \tag{4.26}$$

provided that $\int_\Omega |u|^p \ln|u|^k dx \ge 0$.

Lemma 4.5. *There exists a positive constant C such that for any $u \in L^p(\Omega)$ and $2 \leq s \leq p$, we have*

$$\|u\|_p^s \leq C\left[\|u\|_p^p + \|\nabla u\|_2^2\right]. \tag{4.27}$$

Proof. If $\|u\|_p \geq 1$ then

$$\|u\|_p^s \leq \|u\|_p^p.$$

If $\|u\|_p \leq 1$ then, $\|u\|_p^s \leq \|u\|_p^2$. Using Sobolev embedding theorems, we have

$$\|u\|_p^s \leq \|u\|_p^2 \leq C\|\nabla u\|_2^2.$$

Now we are ready to state and prove your main result. For this purpose, we define

$$\begin{aligned} H(t) = -E(t) = {} & \frac{1}{p}\int_\Omega |u(t)|^p \ln|u(t)|^k dx - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|u\|_2^2 - \frac{1}{2}\|\nabla u\|_2^2 \\ & -\frac{1}{2}\left(1 - \int_0^\infty g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{2}(g \circ \nabla u)(t) \\ & -\frac{k}{p}\|\nabla u\|_p^p - \frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx. \end{aligned}$$

□

Theorem 4.6. *Suppose that* (4.1) *and* (3.3) *hold. Assume further that*

$$\begin{aligned} E(0) \quad = \quad & \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\|u_1\|_2^2 + \frac{k}{p}\|u_0\|_p^p \\ & +\frac{\xi}{2}\int_\Omega\int_0^1 |f_0(x,-\rho\tau)|^2 d\rho dx - \frac{1}{p}\int_\Omega |u_0|^p \ln|u_0|^k dx < 0. \end{aligned} \tag{4.28}$$

Then the solution of (3.1) *blows up in finite time.*

Proof. As $E(t)$ is a nonincreasing function, we have

$$E(0) \geq E(t).$$

A differentiation of $H(t)$ gives

$$\begin{aligned} H'(t) \quad & = \quad -E'(t) \\ & \geq \quad C_0\left[\int_\Omega \left(|u_t|^2 + |z(x,1,t)|^2\right)dx\right] + \frac{1}{2}g(t)\int_\Omega |\nabla u(t,x)|^2 dx \\ & \quad\;\; -\frac{1}{2}\int_\Omega (g' \circ \nabla u)(t)dx \\ & \geq \quad C_0\int_\Omega z^2(x,1,t)dx \geq 0. \end{aligned} \tag{4.29}$$

and

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\int_\Omega u^p \ln|u_0|^k dx. \tag{4.30}$$

We set

$$L(t) = H^{1-\alpha}(t) + \varepsilon\int_\Omega u_t u dx + \varepsilon\frac{u_t}{2}\int_\Omega u^2 dx, \quad t \geq 0,$$

where $\varepsilon > 0$ to be specified later and

$$0 < \frac{2(p-2)}{p^2} < \alpha < \frac{p-2}{p^2} < 1. \tag{4.31}$$

Differentiating $L(t)$ we easily obtain

$$\begin{aligned} L'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|_2^2 - \varepsilon \|\nabla u\|_2^2 \\ &\quad + \varepsilon \int_0^\infty g(t-s) \int_\Omega \nabla u(s,x) . \nabla u(t,x) ds \\ &\quad - \varepsilon \mu_2 \int_\Omega u z(x,1,t) dx + \frac{\varepsilon}{p} \int_\Omega u^p \ln |u|^k dx. \end{aligned} \tag{4.32}$$

Using Young's inequality, we estimate

$$\begin{aligned} &-\varepsilon \mu_2 \int_\Omega u z(x,1,t) dx \\ &\qquad \geq -\varepsilon |\mu_2| \left(\delta \int_\Omega u^2 dx + \frac{1}{4\delta} \int_\Omega z^2(x,1,t) dx \right), \quad \forall \delta > 0. \end{aligned} \tag{4.33}$$

and Cachy-Schwarz and Young inequalities, we have

$$\begin{aligned} &\int_0^\infty g(t-s) \int_\Omega \nabla u(s,x) . \nabla u(t,x) dx ds \\ &\quad = \int_0^\infty g(t-s) \int_\Omega \nabla u(t,x) . (\nabla u(s,x) - \nabla u(t,x)) dx ds \\ &\qquad + \int_0^\infty g(t-s) \|\nabla u\|_2^2 ds \\ &\quad \geq \left(1 - \frac{1}{4\delta}\right) \left(\int_0^\infty g(s) ds \right) \|\nabla u\|_2^2 - \delta (g \circ \nabla u)(t), \quad \forall \delta > 0. \end{aligned}$$

We get, from (4.32),

$$\begin{aligned} L'(t) \geq &\left[(1-\alpha) H^{-\alpha}(t) - \frac{\varepsilon |\mu_2|}{4\delta C_0} \right] H'(t) \\ &+ \varepsilon \left(1 - \frac{1}{4\delta}\right) \left(\int_0^\infty g(s) ds \right) \|\nabla u(t,x)\|_2^2 \\ &- \varepsilon \delta (g \circ \nabla u)(t) + \varepsilon \|u_t\|_2^2 - \varepsilon \|\nabla u\|_2^2 \\ &- \varepsilon \delta |\mu_2| \|u\|_2^2 + \varepsilon \int_\Omega |u|^p \ln |u|^k dx. \end{aligned} \tag{4.34}$$

Of course (4.34) remains valid even if δ is time dependent. Therefore by taking δ so that

$$\frac{|\mu_2|}{4\delta C_0} = \kappa H^{-\alpha}(t),$$

for large κ to be specified later, and substituting in (4.34) we arrive at

$$\begin{aligned}
L'(t) &\geq \left[(1-\alpha)-\varepsilon\kappa\right] H^{-\alpha}(t)H'(t)+\varepsilon \left\|u_t\right\|_2^2-\varepsilon\left\|\nabla u\right\|_2^2 \\
&+\varepsilon\left(1-\frac{1}{4\delta}\right)\left(\int_0^\infty g(s)ds\right)\left\|\nabla u\right\|_2^2-\varepsilon\delta(g\circ\nabla u)(t) \\
&-\frac{\varepsilon|\mu_2|^2}{4\kappa C_0}H^\alpha(t)\left\|u\right\|_2^2+\varepsilon\int_\Omega |u|^p\ln|u|^k dx.
\end{aligned}$$

For $0<a<1$, we have

$$\begin{aligned}
L'(t) &\geq \left[(1-\alpha)-\varepsilon\kappa\right] H^{-\alpha}(t)H'(t)+\frac{\varepsilon a}{p}\int_\Omega |u|^p\ln|u|^k dx+\varepsilon\frac{p(1-a)+2}{p}\left\|u_t\right\|_2^2 \\
&+\varepsilon\left(\frac{p(1-a)}{2}-\left(\frac{p(1-a)-2}{2}+\frac{1}{4\delta}\right)\int_0^\infty g(s)ds\right)\left\|\nabla u\right\|_2^2 \\
&+\varepsilon\left(\frac{p(1-a)}{2}-\delta\right)(g\circ\nabla u)(t)+\varepsilon k(1-a)\left\|u\right\|_p^p-\frac{\varepsilon|\mu_2|^2}{4\kappa C_0}H^\alpha(t)\left\|u\right\|_2^2 \\
&+\varepsilon p(1-a)H(t)+\frac{\varepsilon(1-a)p\xi}{2}\int_\Omega\int_0^1 z^2(x,1,t)d\rho dx. \qquad (4.35)
\end{aligned}$$

Using (4.26), (4.30) and Young's inequality, we find

$$\begin{aligned}
H^\alpha(t)\left\|u\right\|_2^2 &\leq \left(\int_\Omega |u|^p\ln|u|^k dx\right)^\alpha \left\|u\right\|_2^2 \\
&\leq C\left[\left(\int_\Omega |u|^p\ln|u|^k dx\right)^{\alpha+\frac{2}{p}}+\left(\int_\Omega |u|^p\ln|u|^k dx\right)^\alpha\left\|\nabla u\right\|_2^{\frac{4}{p}}\right] \\
&\leq C\left[\left(\int_\Omega |u|^p\ln|u|^k dx\right)^{\frac{p\alpha+2}{p}}+\left\|\nabla u\right\|_2^2+\left(\int_\Omega |u|^p\ln|u|^k dx\right)^{\frac{\alpha p}{p-2}}\right].
\end{aligned}$$

Exploiting (4.31), we have

$$2<\alpha p+2\leq p \text{ and } 2<\frac{\alpha p^2}{p-2}\leq p.$$

Thus, lemma 4.2 yields

$$H^\alpha(t)\left\|u\right\|_2^2\leq C\left[\int_\Omega |u|^p\ln|u|^k dx+\left\|\nabla u\right\|_2^2\right]. \qquad (4.36)$$

Combining (4.35) and (4.36), we obtain

$$\begin{aligned}
L'(t) &\geq \left[(1-\alpha) - \varepsilon\kappa\right] H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{a}{p} - \frac{\varepsilon|\mu_2|^2}{4\kappa C_0}\right)\int_\Omega |u|^p \ln|u|^k dx \\
&+ \varepsilon\left(\frac{p\,(1-a)}{2} - \frac{\varepsilon|\mu_2|^2}{4\kappa C_0} - \left(\frac{p\,(1-a)-2}{2} + \frac{1}{4\delta}\right)\int_0^\infty g(s)ds\right)\|\nabla u\|_2^2 \\
&\varepsilon\left(\frac{p\,(1-a)}{2} - \delta\right)(g\circ\nabla u)\,(t) + \varepsilon k\,(1-a)\,\|u\|_p^p + \varepsilon\frac{p\,(1-a)+2}{p}\,\|u_t\|_2^2 \\
&+ \varepsilon p\,(1-a)\,H(t) + \frac{\varepsilon\,(1-a)\,p\xi}{2}\int_\Omega\int_0^1 z^2(x,1,t)d\rho dx. \qquad (4.37)
\end{aligned}$$

We choose $a > 0$ so small that

$$\frac{p(1-a)-2}{2} > 0,\ \frac{p\,(1-a)}{2} - \delta > 0,$$

and κ so large that

$$\frac{p\,(1-a)}{2} - \frac{\varepsilon|\mu_2|^2}{4\kappa C_0} - \left(\frac{p\,(1-a)-2}{2} + \frac{1}{4\delta}\right)\int_0^\infty g(s)ds > 0 \text{ and } \frac{a}{p} - \frac{\varepsilon|\mu_2|^2}{4\kappa C_0} > 0.$$

We pick ε so small so that

$$(1-\alpha) - \varepsilon\kappa > 0,\ H(0) + \varepsilon\int_\Omega u_0u_1dx > 0.$$

Next, for some $\lambda > 0$, estimate (4.37) becomes

$$\begin{aligned}
L'(t) &\geq \lambda\left[H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + \|u\|_p^p\right] \\
&+ \lambda\left[(g\circ\nabla u)\,(t) + \int_\Omega\int_0^1 z^2(x,1,t)d\rho dx + \int_\Omega |u|^p\ln|u|^k dx\right] \qquad (4.38)
\end{aligned}$$

and

$$L(t) \geq L(0) > 0,\ t \geq 0. \qquad (4.39)$$

Using Hölder's inequality and the embedding $\|u\|_2 \leq C\,\|u\|_p$, we have

$$\int_\Omega u_tudx \leq \|u\|_2\,\|u_t\|_2 \leq C\,\|u\|_p\,\|u_t\|_2\,,$$

and exploiting Young's inequality, we obtain

$$\left|\int_\Omega u_tudx\right|^{\frac{1}{1-\alpha}} \leq C\left(\|u\|_p^{\frac{\mu}{1-\alpha}} + \|u_t\|_2^{\frac{\theta}{1-\alpha}}\right), \quad \text{for } \frac{1}{\mu} + \frac{1}{\theta} = 1. \qquad (4.40)$$

To be able to use Lemma 4.2, we take $\theta = 2(1-\alpha)$ which gives $\frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \leq p$. Consequently, for $s = \frac{2}{1-2\alpha}$, estimate (4.40) gives

$$\left|\int_\Omega u_tudx\right|^{\frac{1}{1-\alpha}} \leq C\left(\|u_t\|_2^2 + \|u\|_p^s\right).$$

Hence, Lemma 4.2 gives

$$\left|\int_{\Omega} u_t u dx\right|^{\frac{1}{1-\alpha}} \leq C\left(\|\nabla u\|_2^2 + \|u_t\|_2^2 + \|u\|_p^p\right). \tag{4.41}$$

Consequently,

$$\begin{aligned}
L^{\frac{1}{1-\alpha}}(t) &= \left(H^{1-\alpha}(t) + \varepsilon \int_{\Omega} u_t u dx + \frac{u_t \varepsilon}{2} \int_{\Omega} u^2 dx\right)^{\frac{1}{1-\alpha}} \\
&\leq C\left[H(t) + (g \circ \nabla u)(t) + \left|\int_{\Omega} u_t u dx\right|^{\frac{1}{1-\alpha}} + \|u\|_2^{\frac{1}{1-\alpha}}\right] \\
&\leq C\left[H(t) + (g \circ \nabla u)(t) + \left|\int_{\Omega} u_t u dx\right|^{\frac{1}{1-\alpha}} + \|u\|_2^{\frac{2}{1-\alpha}}\right] \\
&\leq C\left[H(t) + (g \circ \nabla u)(t) + \|\nabla u\|_2^2 + \|u_t\|_2^2 + \|u\|_p^p\right]. \qquad (4.42)
\end{aligned}$$

Combining (4.38) and (4.42), we obtain

$$L'(t) \geq \Lambda L^{\frac{1}{1-\alpha}}(t), \text{ for } t \geq 0. \tag{4.43}$$

where Λ is a positive constant.
A direct integration over $(0, t)$ of (4.43) then yields

$$L^{\frac{1}{1-\alpha}}(t) \geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \Lambda \frac{\alpha t}{1-\alpha}}, \text{ for } t \geq 0.$$

Therefore, $L(t)$ blows up in time

$$T \leq T^* = \frac{1-\alpha}{\Lambda \alpha L^{\frac{\alpha}{1-\alpha}}(0)}.$$

This completes the proof. □

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Abdeldjabar Bourega
Laboratoire de Mathématiques Fondamentales et Appliquées d'Oran (LMFAO),
Université Oran 1. B.P. 1524, El Mnaouer, Oran, Algerie
e-mail: `jabourega@gmail.com Or a.bourega@lagh-univ.dz`

Djamel Ouchenane
Laboratory of Pure and Applied Mathematics, Amar Teledji University,
Laghouat, Algeria
e-mail: `ouchenanedjamel@gmail.com Or d.ouchenane@lagh-univ.dz`

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Nonlinear elliptic equations by topological degree in Musielak-Orlicz-Sobolev spaces

Mustapha Ait Hammou and Badr Lahmi

Abstract. We prove by using the topological degree theory the existence of at least one weak solution for the nonlinear elliptic equation

$$- \operatorname{div}\, a_1(x, \nabla u) + a_0(x, u) = f(x, u, \nabla u)$$

with homogeneous Dirichlet boundary condition in Musielak-Orlicz-Sobolev spaces.

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1. Introduction

Recently, there has been an increasing interest in the study of elliptic and parabolic mathematical problems in Musielak-Orlicz-Sobolev spaces. This setting includes and generalizes variable exponent, anisotropic and classical Orlicz settings.

The interest brought to the study of such differential equations comes for example from applications to non-Newtonian fluids (see [12, 13] for a wide expository) and other physics phenomena. We refer to some results on existence of solutions for Leray-Lions problems studied in variable exponent Sobolev (see, e.g., [3, 19, 23]) or Orlicz-Sobolev spaces (see, e.g., [1, 10]).

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$. let us suppose that the boundary of Ω denoted $\partial\Omega$ is $\mathcal{C}^1$. We consider a class of nonlinear Dirichlet problems of the form:

$$\begin{cases} - \operatorname{div}\, a_1(x, \nabla u) + a_0(x, u) = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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The right-hand side f is a Carathéodory function which depend on the solution u and on its gradient ∇u satisfying a growth condition and where $a_1 : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ and $a_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying Leray-Lions-like conditions which generate an operator of the monotone type $-\operatorname{div} a_1(x, \nabla u) + a_0(x, u)$ defined on $W_0^1 L_\Phi(\Omega)$ with values in its dual $(W_0^1 L_\Phi(\Omega))'$. Here Φ is a Musielak-Orlicz function satisfying Some sufficient conditions, namely Δ_2-condition which assure the reflexivity of such spaces.

The authors in [7] studied the problem (1.1) and proved the existence of weak solutions by using a linear functional analysis and sub-supersolution methods. In the case when $a_0 = 0$, the authors in [20] obtained the existence of weak solutions for (1.1). Bisedes, for $a_0 \neq 0$ verifying suitable conditions, the author in [8] proved the existence of weak solution with homogeneous Neumann or Dirichlet boundary condition by a sub-supersolution method.

The aim of this paper is to prove the existence result that is found in [7] by using a different approach opening new perspectives: we apply the degree theory in [4, 16] to give a result about existence of nonzero solutions of operator equations of the abstract Hammerstein equation in reflexive Banach spaces X

$$u + STu = 0, \quad u \in X,$$

where $S : X' \to X$ and $S : X \to X'$ two mappings [4, 16].

The approach considered here require the reflexivity of the spaces. For that, we suppose that the Musielak-Orlicz functions satisfy suitable conditions (see condition (E) below). The principal prototype that we have in mind is the Φ-Laplacian equation, i.e.

$$\text{-div}\left(\frac{a(x, \nabla u)}{|\nabla u|} \cdot \nabla u\right) = f(x, u, \nabla u)$$

The Musielak-Orlicz setting generalize both Sobolev with variable exponent and Orlicz spaces. Typical examples of equations involving the Musielak-Orlicz setting include models of electrorheological fluids [22], elasticity [17], non-Newtonian fluids [11], the theory of potential [14] and harmonic analysis [6].

The plan of paper is as follows: in section 2, some fundamental properties concerning the Musielak and Musielak-Orlicz-Sobolev spaces spaces are given. Section 3 deals with the properties and the existence of the topological degree for some classes of operators. In section 4, we give some auxiliary results and the main result and its proof.

2. Musielak and Musielak-Orlicz-Sobolev spaces

Standard references on Musielak-Orlicz-Sobolev spaces and their properties include [15, 21, 9] and references therein.

Definition 2.1. Let Ω be an open subset of $\mathbb{R}^N$. A function $M : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is called a Musielak-Orlicz function if

1. $M(x,\cdot)$ is an N-function, i.e. convex, nondecreasing, continuous, $M(x,0)=0, M(x,t)>0$ $(\forall t>0)$,

$$\lim_{t\to 0^+}\sup_{x\in\Omega}\frac{M(x,t)}{t}=0 \text{ and } \lim_{t\to+\infty}\inf_{x\in\Omega}\frac{M(x,t)}{t}=+\infty,$$

2. $M(\cdot,t)$ is a measurable function.

For each $x\in\Omega$, the inverse of function $M(x,\cdot)$ is denoted by $M_x^{-1}(x,\cdot)$ or for simplicity $M^{-1}(x,\cdot)$ and then $M^{-1}(x,\Phi(x,s))=s$ and $M(x,M^{-1}(x,s))=s$ for all $s\geq 0$.

Remark 2.2. M admits the representation

$$M(x,t)=\int_0^t m(x,s)\,ds, \text{ for all } t\geq 0,$$

where $m(x,\cdot)$ is the right-hand derivative of $M(x,\cdot)$ for a fixed $x\in\Omega$. We recall that for every x in Ω, the function $m(x,\cdot)$ is a right-continuous and nondecreasing verifying for all $s\geq 0$: $m(x,0)=0$, $m(x,s)>0$ for $s>0$, $\lim\limits_{s\to+\infty}\inf\limits_{x\in\Omega} m(x,s)=+\infty$ and $M(x,s)\leq sm(x,s)\leq M(x,2s)$.

The complementary function $\overline{M}$ to a Musielak-Orlicz function M is defined as follows:

$$\overline{M}(x,r)=\sup_{s\geq 0}\big(sr-M(x,s)\big),\qquad \text{for } x\in\Omega, r\geq 0.$$

Note that $\overline{M}$ is a Musielak-Orlicz function which admits a similar representation where $\overline{m}$ is defined as above or by

$$\overline{m}(x,s)=\sup\{\delta;\ m(x,\delta)\leq s\}.$$

We recall *Young's inequality*

$$r\cdot s\leq M(x,s)+\overline{M}(x,r),\quad \forall r,s\in\mathbb{R}^+, x\in\Omega,$$

Note that when $\overline{M}$ satisfy the Δ_2-condition, a variant of Young's inequality holds, i.e.,

$$r\cdot s\leq \varepsilon M(x,s)+c(\varepsilon)\overline{M}(x,r),\quad \forall r,s\in\mathbb{R}^+, x\in\Omega,$$

where $\varepsilon\in]0,1[$ and $c(\varepsilon)$ a constant depending of ε.
For $u:\Omega\to\mathbb{R}$ measurable function, we define the modular $\varrho_{M,\Omega}$ or ϱ_M induced by the positive Musielak-Orlicz function M as

$$\varrho_M(u)=\int_\Omega M(x,|u(x)|)\,dx$$

Let us consider the *Musielak-Orlicz class*

$$K_M(\Omega)=\{u:\Omega\to\mathbb{R} \text{ measurable; } \varrho_M(u)<\infty\}.$$

The *Orlicz space* $L_M(\Omega)$ is defined as the linear hull of $K_M(\Omega)$ and it is a Banach space with respect to the *Luxemburg norm*

$$\|u\|_M=\inf\left\{k>0;\ \int_\Omega M\left(x,\frac{|u(x)|}{k}\right)\leq 1\right\}.$$

Or the equivalent norm called *Orlicz norm*

$$\|u\|_{(M)} = \sup\left\{\left|\int_\Omega u(x)v(x)\,dx\right|; v \in K_{\overline{M}}(\Omega), \varrho_{\overline{M}}(v) \le 1\right\}.$$

One has a *Hölder's type inequality*: if $u \in L_M(\Omega)$ and $v \in L_{\overline{M}}(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\left|\int_\Omega u(x)v(x)\,dx\right| \le 2\|u\|_M\|v\|_{\overline{M}}.$$

The closure in $L_M(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. It is a separable space and $(E_{\overline{M}}(\Omega))' = L_M(\Omega)$. Generally $K_M(\Omega) \subset L_M(\Omega)$ but we can obtain $E_M(\Omega) = L_M(\Omega) = K_M(\Omega)$ if and only if M satisfies the Δ_2-condition, i.e. there is a constant $k > 1$ independent of $x \in \Omega$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$M(x, 2s) \le kM(x, s) + h(x), \qquad \text{for all } s \ge 0, \text{ a.e. } x \in \Omega.$$

Note also that under this condition, the space $L_M(\Omega)$ is reflexive.
Let M and P two Musielak-Orlicz functions, $M \preceq P$ means that M is weaker than P, i.e. there is two positive constants k_1 and k_2 and a nonnegative function $H \in L^1(\Omega)$ such that

$$M(x, s) \le k_1P(x, k_2s) + H(x), \qquad \text{for all } s \ge 0, \text{ a.e. } x \in \Omega.$$

Remark 2.3. [21, 15]
Let M and P two Musielak-Orlicz functions such that $M \preceq P$. Then $\overline{P} \preceq \overline{M}$, $L_P(\Omega) \hookrightarrow L_M(\Omega)$ and $L_{\overline{M}}(\Omega) \hookrightarrow L_{\overline{P}}(\Omega)$.

We say that the sequence $(u_n)_n \subset L_M(\Omega)$ converges to $u \in L_M(\Omega)$ in the modular sense if there exists $\lambda > 0$ such that

$$\varrho_M\left(\frac{u_n - u}{\lambda}\right) \to 0, \qquad \text{when } n \to +\infty.$$

In any Musielak-Orlicz space, norm convergence implies the modular convergence and the modular convergence implies the weak convergence.

Proposition 2.4. [21, 15, 8] *Let M be a Museilak-Orlicz function satisfy Δ_2-condition. Let $u \in L_M(\Omega)$ and $(u_n)_n \subset L_M(\Omega)$. Then the following assertions hold.*

1. $\int_\Omega M(x, u_n)\,dx > 1$ (*resp* $= 1; < 1$) $\Leftrightarrow \|u\|_M > 1$ (*resp* $= 1; < 1$),
2. $\int_\Omega M(x, u_n)\,dx \underset{n\to\infty}{\to} 0$ (*resp* $= 1; +\infty$) $\Leftrightarrow \|u_n\|_M \underset{n\to\infty}{\to} 0$ (*resp* $= 1; +\infty$),
3. $u_n \underset{n\to\infty}{\to} u$ *in* $L_M(\Omega) \Rightarrow \int_\Omega M(x, u_n)\,dx \underset{n\to\infty}{\to} \int_\Omega M(x, u)\,dx$,
4. $\|u\|_M \le \varrho_M(u) + 1$,
5. $m(\cdot, u(\cdot)) \in L_{\overline{M}}(\Omega)$ *(the function m is defined in remark 2.2).*

The Musielak-Orlicz-Sobolev space $W^1L_M(\Omega)$ is the space of all $u \in L_M(\Omega)$ whose distributional derivatives $D^\alpha u$ are in $L_M(\Omega)$ for any α, with $|\alpha| \leq 1$. Let

$$\varrho_{1,M} = \sum_{|\alpha|\leq 1} \varrho_M(D^\alpha u)$$

the convex modular on $W^1L_M(\Omega)$. The space $W^1L_M(\Omega)$ equipped with the norm

$$\|u\|_{1,(M)} := \|u\|_{W^1L_M(\Omega)} = \inf\left\{\lambda > 0; \varrho_{1,M}\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

or the equivalent norm

$$\|u\|_{1,M} := \|u\|_M + \|\nabla u\|_M.$$

This space is a Banach space if and only if there is a constant c such that $\inf_{x\in\Omega} M(x,1) > c$ (see [21]). The space $W_0^1L_M(\Omega)$ is defined as the norm-closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$. Moreover if this condition is satisfied, then $W^1L_M(\Omega)$ and $W_0^1L_M(\Omega)$ are separable Banach spaces and $W_0^1L_M(\Omega) \hookrightarrow W^1L_M(\Omega) \hookrightarrow W^{1,1}(\Omega)$.
We say that the sequence $(u_n)_n \subset L_M(\Omega)$ converges to $u \in W^1L_M(\Omega)$ in the modular sense if there exists $\lambda > 0$ such that

$$\varrho_{1,M}\left(\frac{u_n - u}{\lambda}\right) \to 0, \qquad \text{when } n \to +\infty.$$

Suppose also that

$$\lim_{t\to 0}\int_t^1 \frac{M_x^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\,d\tau < \infty, \quad \lim_{t\to\infty}\int_1^t \frac{M_x^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\,d\tau = \infty. \tag{2.1}$$

With (2.1) satisfied, we define the *Sobolev conjugate* M_* of M as the reciprocal function of F with respect to t where

$$F(x,t) = \int_0^t \frac{M_x^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\,d\tau, t \geq 0.$$

Proposition 2.5. [2] *If the Musielak-Orlicz function M satisfies* (2.1)*, then*

$$W_0^1L_M(\Omega) \hookrightarrow L_{\overline{M}}(\Omega).$$

Moreover, if Ω_0 is a bounded subdomain of Ω, then the imbeddings

$$W_0^1L_M(\Omega) \hookrightarrow\hookrightarrow L_P(\Omega_0)$$

exist and are compact for any Musielak-Orlicz function P increasing essentially more slowly than $\overline{M}$ near infinity (see proof of Theorem 4. in [2] *for more informations).*

We have the following result:

Lemma 2.6. *Let Ω be a bounded domain in $\mathbb{R}^N$. Let ν a Musielak-Orlicz function locally integrable satisfy Δ_2-condition such that $\inf\limits_{x\in\Omega} \nu(x,1) = c_1 > 0$. If $(u_n)_n \subset L_\nu(\Omega)$ with $u_n \to u$ in $L_\nu(\Omega)$, then there exists $\widetilde{w} \in L_\nu(\Omega)$ and a subsequence $(u_{n_k})_{n_k}$ such that:*

$$|u_{n_k}(x)| \leq \widetilde{w}(x), \qquad \text{and} \qquad u_{n_k}(x) \to u(x) \text{ a.e. in } \Omega.$$

Proof. Let $(u_n) \subset L_\nu(\Omega)$ such that $u_n \to u$ in $L_\nu(\Omega)$, we can suppose that

$$\|(u_n - u)\|_M \leq \frac{1}{2},$$

then by proposition 2.4

$$\begin{aligned}\int_\Omega \nu(x, 2(u_n(x) - u(x))\, dx &\leq 2\|(u_n - u)\|_M \int_\Omega \nu(x, \frac{u_n(x) - u(x)}{\|(u_n - u)\|_M})\, dx.\\ &\leq 2\|(u_n - u)\|_M.\end{aligned}$$

Therefore $\|\varrho_\nu(u_n - u)\|_{L^1(\Omega)} \to 0$ as $n \to \infty$. On other hand, since Ω has a finite measure, the continous embedding $L_\nu(\Omega) \hookrightarrow L^1(\Omega)$ hold (by using the generalized Hölder's inequality) then $u_n \to u$ in $L^1(\Omega)$. We deduce that there exists $w \in L^1(\Omega)$ and a subsequence $(u_{n_k})_{n_k}$ such that $u_{n_k}(x) \to u(x)$ a.e. in Ω and $\nu(x, u_{n_k}(x) - u(x)) \leq w(x)$ a.e. in Ω. Since ν_x^{-1} is a nondecreasing function, we obtain

$$| u_{n_k}(x) | \leq | u(x) | + \nu_x^{-1}(x, w(x)).$$

Let $\widetilde{w}(x) = | u(x) | + \nu^{-1}(x, w(x))$, then

$$\int_\Omega \nu(x, \widetilde{w}(x))\, dx \leq \frac{1}{2} \int_\Omega \nu(x, 2 \mid u(x) \mid) + \int_\Omega w(x)\, dx.$$

Thus $\widetilde{w} \in K_\nu(\Omega) = L_\nu(\Omega)$. □

2.1. Functional setting

Let Φ and Ψ are two Musielak-Orlicz functions defined on $\Omega \times \mathbb{R}^+$.
We say that Φ and Ψ satisfy the condition (E) if:

- E_1. Φ, Ψ, $\overline{\Phi}$ and $\overline{\Psi}$ are locally integrable, uniformly convex and satisfy Δ_2-condition,
- E_2. Φ satisfy the condition (2.1),
- E_3. $\Phi \preceq \Psi$ and the embedding $W_0^1 L_\Phi(\Omega) \hookrightarrow L_\Psi(\Omega)$ is compact,
- E_4. Φ satisfies the following coerciveness condition: there is a function ζ defined on $(0; +\infty)$ such that $\lim\limits_{s \to +\infty} \zeta(s) = +\infty$ and $\Phi(x, ts) \geq \zeta(s) s \Phi(x, t)$ for $x \in \Omega, s > 0$ and $t \in \mathbb{R}^+$.
- E_5. there is a constant c_1 such that $\inf\limits_{x\in\Omega} \Phi(x, 1) = c_1 > 0$ and for every $t_0 > 0$ there exists $c_2 = c_2(t_0)$ such that $\inf\limits_{x\in\Omega} \dfrac{\overline{\Phi}(x, t)}{t} = c_2 > 0$ for every $t \geq t_0$.

Note that under the condition (E), the spaces $L_\Phi(\Omega), L_\Psi(\Omega), W_0^1 L_\Phi(\Omega)$ and $W^1 L_\Phi(\Omega)$ are separable reflexive Banach spaces [21].

3. Topological degree

Degree theory has been developed as a tool for checking the solution existence of nonlinear equations. A number of degree theories for various combinations of nonlinear operators have been developed by various authors. References that contain the theory of topological degree and historical information on the development of this theory

include [4, 5, 16] and references therein.
Let X and Y be two real Banach spaces and Γ a nonempty subset of X.
An operator $F : X \to Y$ is said to be bounded if it takes any bounded set into a bounded set.
F is said to be demicontinuous if for each $u \in \Gamma$ and any sequence $\{u_n\}$ in Γ, $u_n \to u$ imply that $F(u_n) \rightharpoonup F(u)$.
F is said to be compact if is continuous and the image of any bounded set is relatively compact. Let X be a real reflexive Banach space with dual X'.
We say that an operator $F : \Gamma \subset X \to X'$ satisfies condition (S_+) if for any sequence (u_n) in Γ with $u_n \rightharpoonup u$ and $\limsup\langle Fu_n, u_n - u\rangle \leq 0$ we have $u_n \to u$.
F is said to be quasimonotone if for any sequence (u_n) in Γ with $u_n \rightharpoonup u$, we have $\limsup\langle Fu_n, u_n - u\rangle \geq 0$.
For any operator $F : \Gamma \subset X \to X$ and any bounded operator
$T : \Gamma_1 \subset X \to X'$ such that $\Gamma \subset \Gamma_1$, we say that F satisfies condition $(S_+)_T$ if for any sequence(u_n) in Γ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup\langle Fu_n, y_n - y\rangle \leq 0$, we have $u_n \to u$.

For any $\Gamma \subset X$, we consider the following classes of operators:

$$\mathcal{F}_1(\Gamma) := \{F : \Gamma \to X' \mid F \text{ is bounded, demicontinuous and satisfies condition } (S_+)\},$$
$$\mathcal{F}_T(\Gamma) := \{F : \Gamma \to X \mid F \text{ is demicontinuous and satisfies condition } (S_+)_T\}.$$

For any $\Omega \subset D_F$, where D_F denotes the domain of F, and any $T \in F_1(\Omega)$, let

$$\mathcal{F}(X) := \{F \in \mathcal{F}_T(\bar{G}) \mid G \in \mathcal{O}, T \in \mathcal{F}_1(\bar{G})\},$$

where $\mathcal{O}$ denotes the collection of all bounded open set in X. Here, $T \in \mathcal{F}_1(\bar{G})$ is called an essential inner map to F.

Lemma 3.1. [16, Lemma 2.3][4, Lemma 2.2] *Suppose that $T \in \mathcal{F}_1(\bar{G})$ is continuous and $S : D_S \subset X' \to X$ is demicontinuous such that $T(\bar{G}) \subset D_s$, where G is a bounded open set in a real reflexive Banach space X. Then the following statement are true:*

- **(i).** *If S is quasimonotone, then $I + SoT \in \mathcal{F}_T(\bar{G})$, where I denotes the identity operator.*
- **(ii).** *If S satisfies condition (S_+), then $SoT \in \mathcal{F}_T(\bar{G})$*

As in [16] and in [4], we introduce a suitable topological degree for the class $\mathcal{F}(X)$:

Theorem 3.2. *Let*

$$M = \{(F, G, h) | G \in \mathcal{O}, T \in \mathcal{F}_1(\bar{G}), F \in \mathcal{F}_T(\bar{G}), h \notin F(\partial G)\}.$$

There exists a unique degree function $d : \mathcal{M} \to \mathbb{Z}$ that satisfies the following properties:

1. *(Existence) if $d(F, G, h) \neq 0$, then the equation $Fu = h$ has a solution in G,*
2. *(Additivity) Let $F \in \mathcal{F}_T(\bar{G})$. If G_1 and G_2 are two disjoint open subset of G such that $h \notin F(\bar{G} \setminus (G_1 \cup G_2))$, then we have*

$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h),$$

3. *(Homotopy invariance) Suppose that* $H : [0,1] \times \bar{G} \to X$ *is an admissible affine homotopy with a common continuous essential inner map and* $h : [0,1] \to X$ *is a continuous path in* X *such that* $h(t) \notin H(t, \partial G)$ *for all* $t \in [0,1]$ *,then the value of* $d(H(t,.), G, h(t))$ *is constant for all* $t \in [0,1]$,

4. *(Normalization) For any* $h \in G$, *we have*

$$d(I, G, h) = 1,$$

5. *(Boundary dependence) If* $F, S \in \mathcal{F}_T(\bar{G})$ *coincide on* ∂G *and* $h \notin F(\partial G)$, *then*

$$d(F, G, h) = d(S, G, h).$$

4. Main result

4.1. Basic assumptions and technical lemmas

Let Φ and Ψ satisfying the condition (E) and $a_1 : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $a_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ Carathéodory functions which satisfies the growth, the coercivity and the monotony conditions: for a.e. $x \in \Omega$, for every ξ, $\xi' \in \mathbb{R}^N$ and $t, t' \in \mathbb{R}$ there is two positive constants C an C', a nonnegative function g in $L_{\overline{\Phi}}(\Omega)$ and a nonnegative function h in $L^1(\Omega)$ such that

$$|a_1(x,\xi)| \leq C\overline{\Phi}^{-1}(x, \Phi(x, |\xi|)) + g(x), \tag{4.1}$$

$$a_1(x,\xi).\xi \geq C'\Phi(x, |\xi|) - h(x), \tag{4.2}$$

$$\big(a_1(x,\xi) - a_1(x,\xi')\big) \cdot \big(\xi - \xi'\big) > 0, \qquad \xi \neq \xi', \tag{4.3}$$

and

$$|a_0(x,t)| \leq C\overline{\Phi}^{-1}(x, \Phi(x, |t|)) + g(x), \tag{4.4}$$

$$a_0(x,t)t \geq C'\Phi(x, |t|) - h(x), \tag{4.5}$$

$$\big(a_0(x,t) - a_0(x,t')\big)\big(t - t'\big) > 0, \qquad t \neq t', \tag{4.6}$$

$f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function verifying the following growth condition: there is a function q in $L_{\overline{\Phi}}(\Omega)$ and two positives constants α and β such that

$$|f(x,t,\xi)| \leq q(x) + \alpha\overline{\Phi}^{-1}\Phi(x, |t|) + \beta\overline{\Phi}^{-1}\Phi(x, |\xi|) \tag{4.7}$$

for all $t \in \mathbb{R}$, $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$.
The Nemytsky operator F defined by f is given by

$$F(u)(x) = f(x, u(x), \nabla u(x)), \qquad x \in \Omega.$$

Lemma 4.1. *Let* Φ *a Musielak-Orlicz function such that both* Φ *and* $\overline{\Phi}$ *satisfy the* Δ_2-*condition. Assume* (4.7). *Then* $F(W_0^1 L_\Phi(\Omega)) \subset L_{\overline{\Psi}}(\Omega)$ *and moreover,* F *is continuous from* $W_0^1 L_\Phi(\Omega)$ *into* $L_{\overline{\Psi}}(\Omega)$ *and maps bounded sets into bounded sets.*

Proof. Let $u \in W_0^1 L_\Phi(\Omega)$. For $\lambda > max(3\alpha; 3\beta)$ we have

$$\begin{aligned}
&\int_\Omega \overline{\Phi}(x, \frac{F(u)(x)}{\lambda})\, dx \qquad (4.8)\\
&= \int_\Omega \overline{\Phi}(x, \frac{f(x, u(x), \nabla u(x))}{\lambda})\, dx\\
&\leq \int_\Omega \overline{\Phi}(x, \frac{1}{\lambda}[q(x) + \alpha\overline{\Phi}^{-1}\Phi(x, |u(x)|) + \beta\overline{\Phi}^{-1}(x, \Phi(x, |\xi|)])\, dx\\
&\leq \int_\Omega \frac{1}{3}\overline{\Phi}(x, \frac{3q(x)}{\lambda}) + \frac{1}{3}\Phi(x, |u(x)|) + \frac{1}{3}\Phi(x, |\nabla u(x)|)\, dx\\
&< +\infty.
\end{aligned}$$

By condition (E) we have $\Phi \prec \Psi$ then $\overline{\Psi} \prec \overline{\Phi}$ and by consequent there is $\lambda' > 0$ such that

$$\int_\Omega \overline{\Psi}(x, \frac{F(u)(x)}{\lambda'})\, dx < +\infty.$$

For the continuity of F, let us consider a sequence $(u_n)_n \subset W_0^1 L_\Phi(\Omega)$ such that $\|u_n - u\|_{1,\Phi} \to 0$ as $n \to +\infty$ in $W^1 L_\Phi(\Omega)$(we mean by $\|.\|_{1,\Phi}$ the norm of $W_0^1 L_\Phi(\Omega)$ defined as the norm-closure of $\mathcal{D}(\Omega)$). Then $\|u_n - u\|_\Phi \to 0$ and $\|\nabla u_n - \nabla u\|_\Phi \to 0$ as $n \to +\infty$. Applying Lemma 2.6 we can find $w \in L_\Phi(\Omega)$ and extract a subsequence of $(u_n)_n$ still denoted $(u_n)_n$ such that

$$\begin{aligned}
&|u_n(x)| \leq w(x), \quad u_n(x) \to u(x) \quad \text{a.e. in } \Omega, \qquad (4.9)\\
&|\nabla u_n(x)| \leq w(x), \quad \nabla u_n(x) \to \nabla u(x) \quad \text{a.e. in } \Omega.
\end{aligned}$$

Since f is a Carathéodory function, we obtain that

$$f(x, u_n, \nabla u_n) \to f(x, u, \nabla u) \quad \text{a.e. in } \Omega \quad \text{as } n \to +\infty$$

therefore,

$$\overline{\Phi}(x, F(u_n)(x) - F(u)(x)) \to 0 \quad \text{a.e. in } \Omega \quad \text{as } n \to +\infty.$$

By using (4.7), (4.9) and a similar argument to that in (4.8), there is a positive constant such that

$$\begin{aligned}
&\int_\Omega \overline{\Phi}(x, F(u_n)(x) - F(u)(x))\, dx\\
&\leq c \int_\Omega \overline{\Phi}(x, q(x)) + \Phi(x, w(x)) + \Phi(x, |u(x)|) + \Phi(x, |\nabla u(x)|)\, dx
\end{aligned}$$

The right term of this inequality belongs to $L^1(\Omega)$, then by applying Lebesgue's dominated convergence theorm it follows that

$$\lim_{n \to +\infty} \int_\Omega \overline{\Phi}(x, F(u_n)(x) - F(u)(x))\, dx = 0$$

which implies by the continuous embedding $L_{\overline{\Phi}} \hookrightarrow L_{\overline{\Psi}}$ that

$$\lim_{n \to +\infty} \int_\Omega \overline{\Psi}(x, F(u_n)(x) - F(u)(x))\, dx = 0$$

therefore the subsequence $F(u_n)$ converges to $F(u)$ in $L_{\overline{\Psi}}(\Omega)$ for the modular convergence. By applying proposition 2.4 we deduce that the sequence $F(u_n)$ converges in norm to $F(u)$ in $L_{\overline{\Psi}}(\Omega)$. The limit $F(u)$ is independent of the subsequence, by consequent this convergence hold true for the sequence $(u_n)_n$. Thus F is continuous from $W_0^1L_\Phi(\Omega)$ into $L_{\overline{\Psi}}(\Omega)$.
The functions Ψ and $\overline{\Psi}$ satisfy Δ_2-condition, then modular boundedness is equivalent to the norm boundedness. Using arguments similar to those above, F maps bounded sets of $W_0^1L_\Phi(\Omega)$ into bounded sets of $L_{\overline{\Psi}}(\Omega)$. □

Define A_1 and $A_0 : W_0^1L_\Phi(\Omega) \to (W_0^1L_\Phi(\Omega))'$ respectively for all $u, v \in W_0^1L_\Phi(\Omega)$ by

$$\langle A_1u, v\rangle = \int_\Omega a_1(x, \nabla u)v\, dx,$$

$$\langle A_0u, v\rangle = \int_\Omega a_0(x, u)v\, dx.$$

By the same way like in the proof of Theorem 2.2. an Theorem 2.3. in [8] we can proof the following lemma

Lemma 4.2. *Under the assumptions* (E), (4.1),(4.2), (4.3), (4.4),(4.5) *and* (4.6) *the mapping* $A := A_1 + A_0$ *is bounded, continuous and strictly monotone homeomorphism of type* (S^+).

Lemma 4.3. *Suppose that the assupmtions* (E),(4.2) *and* (4.5) *hold. Then* A *is coercive, i.e.,*

$$\frac{\langle Au, u\rangle}{\|u\|_{1,\Phi}} \to +\infty \text{ as } \|u\|_{1,\Phi} \to +\infty.$$

Proof. Let $u \in W_0^1L_\Phi(\Omega)$ $(u \neq 0)$ such that Φ verify the coerciveness condition (see condition (E) below), by using (4.2) and (4.5) we have

$$\begin{aligned}
\langle Au, u\rangle &= \int_\Omega a_1(x, \nabla u)\cdot\nabla u + a_0(x, u)u\, dx \\
&\geq 2C'\big(\int_\Omega \Phi(x, |\nabla u|) + \Phi(x, |u|) - h(x)\, dx\big) \\
&\geq 2C'\big(\int_\Omega \Phi(x, \frac{\|u\|_{1,\Phi}|\nabla u|}{\|u\|_{1,\Phi}}) + \Phi(x, \frac{\|u\|_{1,\Phi}|u|}{\|u\|_{1,\Phi}})\, dx\big) - 2\|h\|_{L^1(\Omega)} \\
&\geq 2C'\zeta(\|u\|_{1,\Phi})\|u\|_{1,\Phi}(\int_\Omega \Phi(x, \frac{|\nabla u|}{\|u\|_{1,\Phi}}) + \Phi(x, \frac{|u|}{\|u\|_{1,\Phi}})\, dx) - 2\|h\|_{L^1(\Omega)}
\end{aligned}$$

We have

$$\left\|\frac{|\nabla u|}{\|u\|_{1,\Phi}}\right\|_{1,\Phi} \leq 1, \quad \left\|\frac{|u|}{\|u\|_{1,\Phi}}\right\|_{1,\Phi} \leq 1$$

and

$$\lim_{\|u\|_{1,\Phi}\to+\infty} \zeta(\|u\|_{1,\Phi}) = +\infty,$$

therefore $\dfrac{\langle Au, u\rangle}{\|u\|_{1,\Phi}} \to +\infty$ as $\|u\|_{1,\Phi} \to +\infty$. □

By applying Minty-Browder theorem (or Lemma 4.3 and Lemma 5.2. in [4]), we deduce that the inverse operator $T : (W_0^1 L_\Phi(\Omega))' \to W_0^1 L_\Phi(\Omega)$ of A is also bounded, continuous and of type (S^+). On other hand, by the condition (E), the embedding $I : W_0^1 L_\Phi(\Omega) \to L_\Psi(\Omega)$ is compact, by consequent the adjoint operator $I^* :\to L_{\overline{\Psi}}(\Omega) \to (W_0^1 L_\Phi(\Omega))'$ is also compact. On other hand, the continuity and boundedness of Nemytsky operator F proved in Lemma 4.1 implies that the composition $S := -I^* oF$ is compact. Consequently we have the following lemma

Lemma 4.4. *The mapping* $S : W_0^1 L_\Phi(\Omega) \to (W_0^1 L_\Phi(\Omega))'$ *is continuous and compact, in particular it is quasimonotone.*

4.2. Existence result

Let us give a definition of a weak solution of problem (1.1):

Definition 4.5. A function u is called weak solution for (1.1) if $u \in W_0^1 L_\Phi(\Omega)$, $F(u) \in L_{\overline{\Psi}}(\Omega)$ and

$$\int_\Omega a_1(x, \nabla u) v \, dx + \int_\Omega a_0(x, u) v \, dx = \int_\Omega f(x, u, \nabla u) v \, dx, \text{ for all } v \in W_0^1 L_\Phi(\Omega). \tag{4.10}$$

Theorem 4.6. *Let Φ and Ψ satisfy the condition* (E). *Suppose that the assumptions* (4.1)–(4.7) *hold true. Then there exists at least one weak solution of problem* (1.1).

Proof. The weak formulation (4.10) is equivalent to the abstract Hammerstein equation

$$(I + SoT)v = 0, \qquad \text{and} \qquad u = Tv. \tag{4.11}$$

and T are the maps defined in Lemme 4.2 and Lemma 4.4. To solve equation 4.11, We can proceed with degree theoretic arguments, it suffices to prove the boundedness of solution set of the homotopy equation

$$v + tSoTv = 0, \quad v \in (W_0^1 L_\Phi(\Omega))', \quad t \in [0, 1],$$

Let

$$B = \{v \in (W_0^1 L_\Phi(\Omega))'; v + tSoTv = 0, \quad v \in X, \quad \text{for some } t \in [0, 1]\}$$

let $v \in B$ and $u \in W_0^1 L_\Phi(\Omega)$ such that $Tv = u$, we have for some t in $[0, 1]$

$$\begin{aligned} \langle v, Tv \rangle &= \langle Au, u \rangle \\ &= -t \langle SoTv, Tv \rangle \\ &= t \int_\Omega f(x, u, \nabla u) u \, dx \\ &\leq \int_\Omega |f(x, u, \nabla u)||u| \, dx. \end{aligned}$$

As in the proof of Lemma 4.3, there two positive constants C and $\widetilde{C}$ such that

$$\langle Au, u \rangle \geq C\zeta(\|u\|_{1,\Phi})\|u\|_{1,\Phi} - \widetilde{C} \tag{4.12}$$

Let $\lambda > max(3\alpha; 3\beta)$. Since Φ satisfy the Δ_2-condition, then by using proposition 2.3 in [7], there is a function $\gamma \in L^1(\Omega)$ and a constant c such that

$$\Phi(x, \lambda|u(x)|) \leq c\Phi(x, |u(x)|) + \gamma(x)$$

which implies, by using the young's inequality, that

$$\begin{aligned}|f(x,u,\nabla u)||u| &\leq \overline{\Phi}(x,\frac{|f(x,u,\nabla u)|}{\lambda})+\Phi(x,\lambda|u|)\\ &\leq \frac{1}{3}\overline{\Phi}(x,\frac{3q(x)}{\lambda})+\frac{1}{3}\Phi(x,|u(x)|)\\ &+ \frac{1}{3}\Phi(x,|\nabla u(x)|)+c\Phi(x,|u(x)|)+\gamma(x).\end{aligned}$$

by combining (4.12) and (4.13) we can find two constants C' and $\widetilde{C'}$ such that

$$\|u\|_{1,\Phi}(\zeta(\|u\|_{1,\Phi})-C')\leq\widetilde{C'}$$

which implies that $u=Tv$ remain bounded in $W_0^1L_\Phi(\Omega)$, consequently, there exists $R>0$ such that

$$\|v\|_{(W_0^1L_\Phi(\Omega))'}\leq R\quad\forall v\in B.$$

We deduce that for all $t\in[0,1]$,

$$v+tSoTv\neq 0,\quad\forall v\in\partial B_R(0).$$

According to Lemma 3.1, the Hammersein operator $I+SoT$ belongs to the class $\mathcal{F}_T(\overline{B_R(0)})$.
Let us consider the homotopy $\mathcal{H}:[0,1]\times\overline{B_R(0)}\to(W_0^1L_\Phi(\Omega))'$ defined by

$$\mathcal{H}(t,v)=v+tSoTv.$$

By invariance and normalisation properties of the degree d of the class $\mathcal{F}_T$ (see Theorem 3.2) we deduce that

$$d(I+SoT,B_R(0),0)=d(I,B_R(0),0)=1.$$

By Theorem 3.2 we conclude that there is at least one $\overline{v}\in B_R(0)$ verifying

$$\overline{v}+SoT\overline{v}=0.$$

Thus $\overline{u}=T\overline{v}$ is a weak solution of problem (1.1). □

Example 4.7. Let $x\in\Omega$ and $t\in\mathbb{R}^+$. Set

$$\Phi(x,t)=\Psi(x,t)=\frac{1}{p(x)}t^{p(x)},$$

then $\varphi(x,t)=t^{p(x)-1}$ where $p:\Omega\to\mathbb{R}$ is a measurable function such that

$$2\leq p^-\leq p(x)\leq p^+<N.$$

Put

$$a_1(x,\xi)=\varphi(x,|\xi|)\frac{\xi}{|\xi|}=|\xi|^{p(x)-2}\xi,\quad a_0(x,t)=\varphi(x,|t|)=|t|^{p(x)-1}$$

and

$$f(x,t,\xi)=\alpha|t|^{p(x)-2}t+\beta|\xi|^{p(x)-1}$$

for $x\in\Omega, t\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$ where α and β are two positives constants. So, the problem (1.1) becomes

$$\begin{cases}-\Delta_{p(x)}u+|u|^{p(x)-1}=\alpha|u|^{p(x)-2}u+\beta|\nabla u|^{p(x)-1} & \text{in }\Omega,\\ u=0 & \text{on }\partial\Omega,\end{cases}\tag{4.13}$$

where $\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplace operator.

- It is clear that the assumptions (4.1)–(4.7) are verified.
- E_1 and E_5 are verified as in example 3.1 in [7].
- We have

$$\lim_{t\to 0}\int_t^1 \frac{\phi_x^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\,d\tau = \frac{p(x)^{\frac{1}{p(x)}}}{\frac{1}{p(x)}-\frac{1}{N}} < \frac{p^{+\frac{1}{p^-}}}{\frac{1}{p^+}-\frac{1}{N}} < \infty$$

  and

$$\lim_{t\to\infty}\int_1^t \frac{\phi_x^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\,d\tau = \lim_{t\to\infty}\frac{p(x)^{\frac{1}{p(x)}}}{\frac{1}{p(x)}-\frac{1}{N}}(t^{\frac{1}{p(x)}-\frac{1}{N}}-1) = \infty$$

  because $p^+ < N$, then E_2 is verified.
- Since Φ satisfies the Δ_2-condition, then there is a constant $k > 1$ independent of $x \in \Omega$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$\Phi(x,s) \le k\Phi(x,\frac{1}{2}s) + h(x) = k\Psi(x,\frac{1}{2}s) + h(x)$$

  for all $s \ge 0$, a.e. $x \in \Omega$. Therefore $\Phi \preceq \Psi$.
  Furthermore we have $W_0^1L_\Phi(\Omega) = W_0^{1,p(x)}(\Omega)$ and $L^{p(x)}(\Omega) = L_\Psi(\Omega)$. Since $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ with compact embedding (see [18]), then we have the compact embedding $W_0^1L_\Phi(\Omega) \hookrightarrow L_\Psi(\Omega)$. So E_3 is verified.
- Finally, E_4 is verified for $\zeta(s) = s^{p(x)-1}$.

We deduce that the problem (4.13) admits at least one weak solution.

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Mustapha Ait Hammou
Sidi Mohamed Ben Abdellah University,
Faculty of Sciences Dhar El Mahraz,
Department of Mathematics, Laboratory LAMA,
Fez, Morocco
e-mail: `mustapha.aithammou@usmba.ac.ma`

Badr Lahmi
Moulay Ismail University, Faculty of Sciences,
Department of Mathematics, Laboratory LMI,
Meknes, Morocco
e-mail: `lahmi.badr@gmail.com`

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Existence of solutions for a biharmonic equation with gradient term

Ahmed Hamydy, Mohamed Massar and Hilal Essaouini

Abstract. In this paper, we mainly study the existence of radial solutions for a class of biharmonic equation with a convection term, involving two real parameters λ and ρ. We mainly use a combination of the fixed point index theory and the Banach contraction theorem to prove that there are $\lambda_0 > 0$ and $\rho_0 > 0$ such the equation admits at least one radial solution for all $(\lambda, \rho) \in [-\lambda_0, \infty[\times [0, \rho_0]$.

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1. Introduction and the main result

In the present paper, we mainly investigate the existence of radial solutions for the following biharmonic problems

$$(P_{\lambda,\rho})\quad \begin{cases} \Delta(\Delta u) + \lambda|\nabla u|^q = \rho f(u) & \text{in} \quad B_1 \\ u = 0, \quad \Delta u = 0 & \text{in} \quad \partial B_1, \end{cases}$$

where $B_1 = \{x \in \mathbb{R}^N : |x| \leq 1\}$ is the unit ball in $\mathbb{R}^N$ $(N \geq 2)$, $(\lambda, \rho) \in \mathbb{R} \times \mathbb{R}^+$, $q \geq 1$ and $f \in C^1(\mathbb{R},]0, \infty[)$. Fourth-order equations are derived as models of different engineering and physical phenomena, such as the motion of fluid, static deflection of an elastic plate in a fluid [2, 4], epitaxial growth of nanoscale thin films [10, 14] and traveling waves in suspension bridges [5, 12]. Due to their several applications, both quasilinear and semilinear biharmonic equations have attracted much attention and many papers appeared in the literature studying existence and the multiplicity

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of solutions, see for instance [9, 15, 14, 6, 7, 11] and the references therein. In [11], L. Kong studied the following boundary value problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \rho g(x)f(u) + h(x) & \text{in} \quad B_1 \\ u = \Delta u = 0 & \text{in} \quad \partial B_1, \end{cases} \tag{1.1}$$

and by Schauder's fixed point, introduced some sufficient conditions for existence of radial solutions. In particular, Guo et al. [7] considered the above problem with $h = 0$, and by using the fixed point index theory and the upper-lower solutions method, proved that for some $\rho^* > 0$, problem (1.1) has no positive radial solution if $\rho > \rho^*$; while if $\rho < \rho^*$, (1.1) has at least two positive radial solutions. Motivated by the above results, especially [7, 11], the purpose of this work is to prove the existence of radial solutions for the biharmonic problem $(P_{\lambda,\rho})$ by combining the fixed point index theory and the Banach contraction theorem. By changing the variable $u(x) = u(|x|)$, $r = |x|$, we transform problem $(P_{\lambda,\rho})$ to the following problem

$$\begin{cases} \mathcal{L}(\mathcal{L}(u)) + \lambda|u'|^q = \rho f(u) \text{ in } (0,1) \\ u(1) = \mathcal{L}(u)(1) = 0, \end{cases} \tag{1.2}$$

where $\mathcal{L}$ denotes the polar form of the Laplacian operator given by

$$\mathcal{L} := \frac{1}{r^{N-1}}\frac{d}{dr}\left(r^{N-1}\frac{d}{dr}\right).$$

We notice that any solution u of the ordinary equation (1.2), $u(|x|)$ is a radial solution of problem $(P_{\lambda,\rho})$. Similar to in [7, Pages 4-5] with $p = q = 2$, we see that problem (1.2) has an integral formulation given by

$$u(t) = \int_0^1\int_0^1 K(t,\tau)K(\tau,s)\left(-\lambda|u'(s)|^q + \rho f(u(s)\right)dsdt, \tag{1.3}$$

where, for $0 \le t, s \le 1$,

$$K(t,s) := \begin{cases} \frac{1}{N-2}s^{N-1}\left(\max\{t,s\}^{2-N} - 1\right), & if N > 2, \\ -s\ln(\max\{t,s\}), & if\, N = 2. \end{cases}$$

Define operators T and $\breve{T}$ in $C^1([0,1])$ as follows

$$T(u)(t) = T_{\lambda,\rho}(u)(t) := \int_0^1\int_0^1 K(t,\tau)K(\tau,s)\left(-\lambda|u'(s)|^q + \rho f(u(s))\right)dsdt \tag{1.4}$$

and for $(h,\beta) \in \mathbb{R}^2$,

$$\breve{T}(u)(t) = K_{\beta,h} + \int_0^1\int_0^1 K(1-t,\tau)K(\tau,1-s)\left(-\lambda|u'(s)|^q + \rho f(u(s))\right)dsdt \tag{1.5}$$

where

$$K_{\beta,h}(t) := h + \beta\int_0^1\left(k(t,s) + \frac{t}{N}\right)dt.$$

Remark 1.1. From [7] and [13] , we have

(i) $K(t,s) > 0$ for all $(t,s) \in (0,1)^2$;
(ii) $K(t,s) \le K(s,s)$ for all $(t,s) \in [0,1]^2$.
(iii) $K(t,s) \le K_\infty$, for all $(t,s) \in [0,1]^2$,

with $K_\infty := \frac{1}{e}$ if $n = 2$ and $K_\infty := (n-2)(n-1)^{-\frac{(n-1)}{n-2}}$ if $n \geq 3$.

We are now in position to present the main results.

Theorem 1.2. *Let $f : (-\infty, \infty) \longrightarrow \mathbb{R}^+$ be a nondecreasing continuous function such that* $\inf f > 0$. *Then there are $\lambda_0 > 0$ and $\rho_0 > 0$ such that problem $(P_{\lambda,\rho})$ has at least one radial solution for any $(\lambda, \rho) \in [-\lambda_0, \infty) \times [0, \rho_0]$. Moreover, for all $0 < \rho \leq \rho_0$,*

$$\lambda_\infty := \sup\{\lambda / \, (\lambda, \rho) \in S\} < \infty$$

and for all $0 > \lambda \geq -\lambda_0$,

$$\rho_\infty := \sup\{\rho / \, (\lambda, \rho) \in S\} < \infty$$

where

$$S := \{(\lambda, \rho) \in \mathbb{R}^2 / every\, \sigma, \mu \in \mathbb{R},\ \sigma \geq -\lambda,\ 0 \leq \mu \leq \rho,\ P_{\sigma,\mu}\, has\, a\, radial\, sol\}.$$

2. Preliminary results and proof of Theorem 1.2

We now introduce some basic technical lemmas that will be necessary to prove the main result. Let's start with a result introduced in [3], [7] and [1].

Lemma 2.1. *Let E be a Banach space, and P be a cone in E, and Ω be a boundary open set in E. Suppose that $T : \overline{\Omega \cap P} \to P$ is a completely continuous operator. If $Tu \neq \nu u$, for all $u \in \partial(\Omega \bigcap P)$ and all $\nu > 1$, then the fixed point index $i(T, \Omega, P) = 1$.*

Lemma 2.2. *If $g \in C[0,1]$, we have that there exists $c_a(t) \in [a, 1]$, independent of t, such that*

$$\int_0^t \tau^{n-1} \int_a^1 K(\tau, s)|g(s)|ds d\tau = |g(c_a(t))| \int_0^t \tau^{n-1} \int_a^1 K(\tau, s) ds d\tau \tag{2.1}$$

for all $t \geq a \geq 0$.

Proof. By Fubini's theorem we obtain

$$\int_0^t \tau^{n-1} \int_a^1 K(\tau, s)|g(s)|ds dt = \int_a^1 |g(s)|h(s, t)ds$$

where $h(s, t) := \int_0^t \tau^{n-1} K(\tau, s) d\tau$. It is easy to see that

$$\min_{[a,1]} | g | \leq \frac{\int_a^1 |g(s)|h(s, t)ds}{\int_a^1 h(s, t)ds} \leq \max_{[a,1]} | g |$$

Thus, there exists $a \leq c_a(t) \leq 1$, such that

$$\int_a^1 |g(s)|h(s, t)ds = |g(c_a(t))| \int_a^1 h(s, t)ds.$$

This completes the proof. □

Let us stress that in addition to the properties of function K presented in Remark 1.1, we will give another property in the following lemma.

Lemma 2.3. *Function $K(t,s)$ verifies the following assertion*

$$\int_0^1 K(1-t,s)ds = \frac{2t-1}{2N} + \int_0^1 K(t,s)ds$$

for all $t \in [0,1]$ and $N \geq 2$.

Proof. Let

$$\varphi(t) = \int_0^1 K(1-t,s)ds.$$

Then

$$\varphi(t) = \int_0^{1-t} K(1-t,s)ds + \int_{1-t}^1 K(1-t,s)ds =: \varphi_0(t) + \varphi_1(t).$$

Note that

$$\varphi_1(t) = \int_{1-t}^1 K(1-t,s)ds = \int_{1-t}^1 K(s,s)ds,$$

thus $\varphi_1'(t) = K(1-t,1-t)$. We also have

$$\varphi_0'(t) = \frac{1-t}{N} - K(1-t,1-t).$$

Therefore $\varphi'(t) = \frac{1-t}{N}$. Similarly, we have

$$\psi'(t) = \frac{-t}{N},\ \psi(t) := \int_0^1 K(t,s)ds.$$

If we set $\phi(t) = \varphi(t) - \psi(t) - \frac{t}{N}$, we obtain $\phi'(t) = 0$ for all $t \in [0,1]$, which implies

$$\phi(t) = \phi(0) = -\int_0^1 K(0,s)ds = -\int_0^1 K(s,s)ds = \frac{1}{2N}.$$

this completes the proof of the lemma. □

Lemma 2.4. *Let $(\alpha,\beta) \in \mathbb{R}^* \times \mathbb{R}^*$. Suppose that $\widetilde{T}$ has a fixed point in $C^1([0,1])$. Then the following problem*

$$\left(P^{\alpha,\beta}\right) \quad \begin{cases} \Delta(\Delta u) + \lambda|\nabla u|^q = \rho f(u) & in \quad B_1 \\ u = \alpha, \quad \Delta u = -\beta & in \quad \partial B_1, \end{cases}$$

has at least one solution.

Proof. Let $\overline{u}$ be a fixed point of $\widetilde{T}$ in $C^1([0,1])$ and let $v(r) = \overline{u}(1-r)$ for all $r \in [0,1]$. By the change of variable $\tau = 1-s$, we get

$$v(r) = K_{\beta,h}(r) + \int_0^1\int_0^1 K(r,t)K(t,\tau)\left(-\lambda|v'(\tau)|^q + \rho f(v(\tau)\right)d\tau dt$$

It follows, from Lemma 2.3, that $v(r) = K_\beta(t) + T(v)(r)$. By a straightforward computation, we have

$$\mathcal{L}(\mathcal{L}(T(v))) = -\lambda|v'|^q + f(v).$$

Since $\mathcal{L}(\mathcal{L}(\int_0^1 K(.,t)dt)) = 0$, we deduce that $\mathcal{L}(\mathcal{L}(v)) = -\lambda|v'|^q + \rho f(v)$. Furthermore, we have $v(1) = h + \frac{\beta}{2N}$, $\mathcal{L}(v)(1) = -\beta$. Therefore, by taking $h = \alpha - \frac{\beta}{2N}$, we obtain that $u(x) = v(|x|)$ is a solution of problem $(P^{\alpha,\beta})$. □

Lemma 2.5. *There are $\beta_0, \lambda_0 > 0$ and $\rho_0 > 0$ such that $\breve{T}$ has a fixed point, for all $|\lambda| \le \lambda_0$ and all $|\rho| \le \rho_0$, with*

$$\lambda_0 = \begin{cases} \lambda_0(\beta_0) & \textit{if} \quad |\beta| \le \beta_0 \\ \lambda_0(\beta) & \textit{if} \quad |\beta| > \beta_0 \end{cases} \quad \textit{and} \quad \rho_0 = \begin{cases} \rho_0(\beta_0) & \textit{if} \quad |\beta| \le \beta_0 \\ \rho_0(\beta) & \textit{if} \quad |\beta| > \beta_0. \end{cases}$$

Proof. We argue as [8], to prove the above lemma. Let $c > 0$ be fixed. By the continuity of f' on $[0,1]$, we can find $\lambda_0^{(1)}, \rho_0^{(1)}, \beta_0 > 0$ depended on c and sufficiently small such that

$$\frac{\beta_0}{N} + \left(\rho_0^{(1)} \sup_{0<|t|<\frac{\beta_0}{2N}+|h|} f(t) + \rho_0^{(1)} c \sup_{0<|t|<c+\frac{\beta_0}{2N}+|h|} |f'(t)| + \lambda_0^{(1)} c^q \right) K_\infty < c.$$

Thus for all $|\beta| \le \beta_0$, $|\lambda| \le \lambda_0^{(1)}$ and $|\rho| \le \rho_0^{(1)}$, we have

$$\frac{|\,\beta\,|}{N} + \left(\rho f\left(\frac{\beta}{2N} + h\right) + |\rho| c \sup_{[0,c+\frac{\beta_0}{2N}+|h|]} |f'| + |\lambda| c_\beta^q \right) K_\infty < c. \tag{2.2}$$

Let $|\beta| > \beta_0$, there are $c_\beta, \lambda_\beta, \rho_\beta > 0$ such that for all $|\lambda| \le \lambda_\beta$ and $\rho \le \rho_\beta$,

$$\frac{|\beta|}{N} + \left(\rho f\left(\frac{\beta}{2N} + h\right) + \rho c_\beta \sup_{[0,c_\beta+\frac{|\beta|}{2N}+|h|]} |f'| + |\lambda| c_\beta^q \right) K_\infty < c_\beta. \tag{2.3}$$

Consider

$$E_\beta := \left\{ u \in C([0,1]) : \left\| u - \frac{\beta}{2N} - h \right\| \le M \right\},$$

where $||u|| := \max\{|u|_\infty, |u'|_\infty\}$ and $M = c$ if $|\beta| \le \beta_0$, $M = c_\beta$ if $|\beta| > \beta_0$. For all $u \in E_\beta$, from $\int_0^1 K(r,t)dt \le \int_0^1 K(t,t)dt$ (see Remark 1.1) and as $\int_0^1 K(t,t)dt = \frac{1}{2N}$, we have that

$$\breve{T}(u)(r) - A \ge -\frac{|\beta|}{2N} + \int_0^1 \int_0^1 K(1-r,t)K(t,1-s)F_{\lambda,\rho}(s)dsdt \tag{2.4}$$

where

$$F_{\lambda,\rho}(s) := -\lambda|u'(s)|^q + \rho f(u(s)),\ A := \frac{\beta}{2N} + h$$

. It is easy to check that if $u \in E_\beta$, we have

$$\rho f(u) < \rho f(\frac{\beta}{2N} + h) + \rho L M,$$

with $L := \sup_{|t|<M+\frac{\max\{|\beta|,\beta_0\}}{2N}+|h|} |f'(t)|$. It follows, from $u \in E_\beta$ and (2.4), that

$$\breve{T}(u)(r) - A \ge -\frac{|\beta|}{2N} - C \int_0^1 \int_0^1 K(1-r,t)K(t,1-s)dsdt. \tag{2.5}$$

where $C = |\lambda| M^q + \rho f(A) + \rho L M$. Since

$$0 < K_\infty < 1\, and \int_0^1 \int_0^1 K(1-r,t)K(t,1-s)dsdt \le K_\infty^2 \tag{2.6}$$

we obtain that

$$\breve{T}(u)(r)-\frac{\beta}{2N}-h\geq-\frac{|\beta|}{2N}-\left(|\lambda|\,M^{q}+\rho f(\frac{\beta}{2N}+h)+\rho LM\right)K_{\infty}^{2}>-M. \quad (2.7)$$

From $\int_0^1 K(t,r)dr \leq \int_0^1 K(r,r)dr = \frac{1}{2N}$ and (2.6), we have

$$\breve{T}(u)(r)-\frac{\beta}{2N}-h\leq\frac{|\beta|}{N}+\left(|\lambda|\,M^{q}+\rho f(\frac{\beta}{2N}+h)+\rho LM\right)K_{\infty}^{2}<M. \quad (2.8)$$

It follows that

$$\left|\breve{T}(u)(r)-\frac{\beta}{2N}-h\right|<M,$$

for all $|\lambda|\leq\lambda_0^{(1)}$ and all $0\leq\rho\leq\rho_0^{(1)}$. Now we are able to show that $|\breve{T}(u)'(r)|<M$. Indeed, a straightforward computations show that

$$\begin{aligned}\left|\breve{T}(u)'(r)\right| &= \left|K'_\beta+\int_0^1\int_0^{1-r}\left(\frac{t}{1-r}\right)^{N-1}K(t,1-s)F_{\lambda,\rho}(s)\right|\\ &\leq \left(\frac{|\beta|}{N}+|\lambda|\,M^{q}+\rho f(\frac{\beta}{2N}+h)+\rho LM\right)K_{\infty}.\end{aligned}$$

Since $0<K_\infty<1$, we deduce that $\left|\breve{T}(u)'(r)\right|<M$. On the other hand, for u and $v\in E_\beta$, we obtain that

$$\begin{aligned}\left|\breve{T}(u)(r)-\breve{T}(v)(r)\right| &\leq |\lambda|\int_0^1\int_0^1 K(1-r,t)K(t,1-s)qM^{q-1}\,|v'-u'|\\ &\quad+\rho\sup_{|t|<M+\frac{\beta}{2N}+h}|f'(t)|\int_0^1\int_0^1 K(1-r,t)K(t,1-s)|u-v|dsdt.\end{aligned}$$

We deduce that

$$\left|\breve{T}(u)(r)-\breve{T}(v)(r)\right| \leq D\left|\int_0^1\int_0^1 K(1-r,t)K(t,1-s)dsdt\right|\,||u-v||$$

where

$$D=|\lambda|\,qM^{q-1}+\rho\sup_{|t|<M+\frac{|\beta|}{2N}+|h|}|f'(t)|\,.$$

From (2.6), there are $\lambda_0^{(2)}>0$ and $\rho_0^{(2)}>0$ such that, for all $|\lambda|<\lambda_0^{(2)}$ and all $0\leq\rho<\rho_0^{(2)}$,

$$\begin{aligned}\left|\breve{T}(u)(r)-\breve{T}(v)(r)\right| &\leq K_\infty^2 D_0\,||u-v||\\ &\leq \frac{1}{2}\,||u-v||\end{aligned}$$

with

$$D_0:=|\lambda|\,qM^{q-1}+\rho\sup_{|t|<M+|\frac{\max\{\beta,\beta_0\}}{2N}|+|h|}|f'(t)|\,.$$

On the other hand, we have

$$\begin{aligned}
&\left|\breve{T}'(u)(r) - \breve{T}'(v)(r)\right| \\
&\leq \left|\int_0^1 \int_0^{1-r} \left(\frac{t}{1-r}\right)^{N-1} K(t,1-s)\left(\lambda|v'(s)|^q - \lambda|u'(s)|^q\right)\right| \\
&+ \left|\int_0^1 \int_0^{1-r} \left(\frac{t}{1-r}\right)^{N-1} K(t,1-s)\rho\left(f(u(s)) - f(v(s))\right) dsdt\right|.
\end{aligned}$$

Thus

$$\begin{aligned}
\left|\breve{T}'(u)(r) - \breve{T}'(v)(r)\right| &\leq D_0 K_\infty \|u-v\| \\
&\leq \frac{1}{2}\|u-v\|
\end{aligned}$$

for all $0 \leq \rho < \rho_0^{(2)}$ and $|\lambda| < \lambda_0^{(2)}$.
Therefore, for all

$$|\lambda| < \lambda_0 = \min\left\{\lambda_0^{(2)}, \lambda_0^{(1)}\right\},\ 0 \leq \rho < \rho_0 = \min\left\{\rho_0^{(2)}, \rho_0^{(1)}\right\},$$

we obtain that

$$\|\breve{T}(u)(r) - \breve{T}(v)(r)\| \leq \frac{1}{2}\|u-v\|.$$

According to the Banach contraction theorem, $\breve{T}$ has a fixed point in E_β. □

2.1. Proof of Theorem 1.2

Let P be a cone defined as

$$P := \{u \in C[0,1],\ u \geq 0\}.$$

The proof is done in five steps.
Step 1. Case $-\lambda_0 \leq \lambda \leq 0$. Consider the following operator

$$\widetilde{T}_\beta(u)(t) := K_{\beta,0}(t) + \int_0^1 \int_0^1 K(t,\tau)K(\tau,s)\left(-\lambda|u'(s)|^q + \rho f(u(s))\right) dsdt. \tag{2.9}$$

In view of Lemma 2.5, we obtain that $\breve{T}_\beta$ has a fixed point in $C[0,1]$. Then, by Lemma 2.5, for all $|\beta| < \beta_0$ and $|\lambda| < \lambda_0$ there exists v_β in $C[0,1]$ such that $\breve{T}_\beta(v_\beta) = v_\beta$. Taking $W_\beta := -v_\beta + \frac{t\beta}{N}$, we get

$$\begin{aligned}
W_\beta(t) = &-\int_0^1 \int_0^1 K(t,\tau)K(\tau,s)\left(-\lambda|W'|^q + \rho f\left(-W_\beta + \frac{t\beta}{N}\right)\right) dsd\tau \\
&- \beta\int_0^1 K(t,s)ds \\
&=: \breve{L}(W_\beta)(t).
\end{aligned}$$

Then

$$\begin{aligned} W_\beta' &= \frac{t\beta}{N} + \int_0^t \left(\frac{\tau}{t}\right)^{N-1} \int_0^1 K(\tau,s)\left(-\lambda|W'|^q + \rho f\left(\int_s^1 W_\beta' d\xi + \frac{t\beta}{N}\right)\right) \\ &= \breve{L}(W_\beta)'(t). \end{aligned}$$

Let $X = C\,[0,1]$, with norm $||u|| = |u|_\infty$ and consider

$$L(u)(t) := \begin{cases} \int_0^t \left(\frac{r}{t}\right)^{N-1} \int_0^1 K(r,s)\left(-\lambda|u|^q + \rho f(\int_s^1 u(s)ds)\right) dsdr, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

for $u \in X$. Clearly, $(X, ||.||)$ is a Banach space. On other hand, $L : P \longrightarrow P$ is completely continuous. Indeed, by Hospital's rule, we obtain that for all $u \in X$, $L(u) \in X$. It is easy to see that $L(u) \geq 0$. We deduce that $L(P) \subset P$. By Ascoli-Arzela theorem and absolute continuity of integral, we obtain that L is completely continuous. Let us consider the set $\Omega := \left\{u \in X,\ u < W_\beta^{'}\right\}$. For $u \in \partial\Omega \cap P$, we have

$$-\lambda\,|u|^q \leq -\lambda \left|W_\beta^{'}\right|^q \ and \int_s^1 uds < \int_s^1 W_\beta^{'} ds.$$

Using f is nondecreasing and $u \in \partial\Omega$ and by choosing $\beta > 0$, we have

$$L(u)(t) \leq L(W_\beta^{'})(t) < \breve{L}(W_\beta)^{'}(t) = W_\beta^{'}(t) = u(t)$$

Then $L(u)(t) \neq \nu u(t)$, for all $\nu > 1$. Moreover, from $f(0) > 0$, we have that $L(0)(t) \neq 0$. Then $L(u)(t) \neq \nu u(t)$, for all $u \in \partial(\Omega \bigcap P)$ and for all $\nu > 1$. It follows, from Lemma 2.3, that $i(L, \Omega, P) = 1$. Thus, there exits $u \in \Omega$ such that $L(u) = u$. Let

$$W(r) := \int_r^1 u(s)ds.$$

Then, we have

$$\begin{aligned} W(r) &= \int_r^1 u(s)ds = \int_r^1 L(u)(s)ds \\ &= \left[-\int_0^1 \int_0^1 K(t,\tau)K(\tau,s)\left(-\lambda|u|^q + \rho f(\int_s^1 u(\xi)d\xi)\right) dsdt\right]_r^1 \\ &= \int_0^1 \int_0^1 K(t,\tau)K(\tau,s)\left(-\lambda|W^{'}(s)|^q + \rho f(W(s))\right) dsdt. \end{aligned}$$

This implies that $W = T(W)$. Therefore, the function $W : B(0,1) \to \mathbb{R}$, $x \to W(|x|)$ is a solution of problem $(P_{\lambda,\rho})$, for all $-\lambda_0 \leq \lambda \leq 0$ and for all $0 < \rho \leq \rho_0$.

Step 2. Case $\lambda_0' > \lambda > 0$ (λ_0' will be defined below). By taking $\lambda = 0$ in step 1, we obtain that there exists $V_\beta \in C\,[0,1]$ such that

$$\begin{aligned} V_\beta &= K_{\beta,-\beta/N}(t) + \rho \int_0^1 \int_0^1 K(t,\tau)K(\tau,s)f(V_\beta)dsdt \\ &=: \breve{L}_0(V_\beta). \end{aligned}$$

Then

$$V_\beta' = \beta\frac{(1-t)}{tN} - \int_0^t \left(\frac{r}{t}\right)^{N-1} \int_0^1 K(r,s)\rho f\left(V_\beta\right) dsdr$$
$$= \left(\breve{L_0}(V_\beta)\right)'.$$

Let us consider the set

$$\Omega' := \left\{u \in X,\ u < -V_\beta{}' - \frac{\beta}{N}\right\},$$

for $\beta < 0$. Then, for $u \in \Omega' \cap P$, we have $0 < u < -V_\beta{}' - \frac{\beta}{N}$. This implies that $\|u\| \leq \left\|V_\beta'\right\|$. So, if we take

$$0 \leq \lambda \leq \lambda_0' := \rho \min\left\{\frac{\inf f(t)}{\left\|-V_\beta{}' - \frac{\beta}{N}\right\|^q}, \lambda_0\right\}$$

we obtain

$$-\lambda|u(s)|^q + \rho f\left(\int_r^1 u(s)ds\right) \geq 0.$$

Therefore, $L(\Omega' \cap P) \subset P$. Now, let $u \in \partial\Omega' \cap P$. We have

$$\begin{aligned} L(t) &:= \int_0^t \left(\frac{r}{t}\right)^{N-1} \int_0^1 K(r,s)\left(-\lambda|u(s)|^q + \rho f\left(\int_s^1 u(\xi)d\xi\right)\right) dsdr \\ &< \int_0^t \left(\frac{r}{t}\right)^{N-1} \int_0^1 K(r,s)\rho f\left(-\int_s^1 V_\beta(\xi)'d\xi - \frac{\beta}{N} + \frac{\beta s}{N}\right) dsdr \\ &< \int_0^t \left(\frac{r}{t}\right)^{N-1} \int_0^1 K(r,s)\rho f\left(V_\beta(s) + \frac{\beta s}{N}\right) dsdr \end{aligned}$$

By using $\beta < 0$ and the fact that f is nondecreasing, we get

$$L(t) < -\left(\breve{L_0}(V_\beta)\right)'(t) = -V_\beta'(t) = u(t) + \frac{\beta}{N} < u(t)$$

Then $L(t) \neq \nu u(t)$ for all $\nu > 1$ and for all $u \in \partial\Omega' \cap P$. Moreover, $L(0)(t) \neq 0$. Thus, $L(u)(t) \neq \nu u(t)$ for all $\nu > 1$ and for all $u \in \partial(\Omega' \cap P)$. Therefore, from Lemma 2.1, $i(L, \Omega', P) = 1$. Then, there exists $u \in C[0,1]$ such that $L(u) = u$. We deduce that

$$W : X \to \mathbb{R},\ t \to \int_t^1 u(s)ds$$

satisfies $T(W) = W$. So, we obtain that problem $(P_{\lambda,\rho})$ has a radial solution for all $0 \leq \lambda \leq \lambda_0'$ and for all $0 < \rho \leq \rho_0$.

Step 3. For every $(\lambda, \rho) \in [\lambda_0', \infty[\times[0, \rho_0]$, the problem $(P_{\lambda,\rho})$ has a radial solution. Indeed, let $(\lambda, \rho) \in [\lambda_0', \infty[\times[0, \rho_0]$. From Step 3, problem (P_{0,ρ_0}) has a radial solution. Then, there exists $u_0 \in C[0,1]$ such that $T_{0,\rho_0}(u_0) = u_0$. Consider the cone

$$P := \{u \in X,\ u \geq 0\}$$

and the set $\Omega := \{u \in X,\ u < u_0\}$. Then, we have

$$\Omega \cap P = \{u \in X,\ 0 \leq u < u_0\}.$$

So, $\partial(\Omega \cap P) = \{0\} \cup \{u = u_0\}$. Since f is nondecreasing, we get

$$T_{\lambda,\rho}(u)(t) < T_{0,\rho_0}(u)(t) < T_{0,\rho_0}(u_0)(t) = u_0(t) = u(t)$$

for $u \in \partial\Omega$. We also have $T_{\lambda,\rho}(0)(t) > 0$. Therefore, $T_{\lambda,\rho}(u)(t) \neq \nu u(t)$, for all $\nu \geq 1$ and for all $u \in \partial(\Omega \cap P)$. So, from Lemma 2.1, $i(T_{\lambda,\rho}, \Omega, P) = 1$. Consequently, $(P_{\lambda,\rho})$ has a least one radial solution.

Step 4. $\lambda_\infty(\rho) < \infty$ and $\rho_\infty(\lambda) < \infty$. Let $0 \leq \rho \leq \rho_0$. Suppose that $\lambda_\infty(\rho) = -\infty$. Then, there exits $(\lambda_n, \rho) \in S$, with $\lambda_n \to -\infty$ and let u_n be a solution radial of problem $(P_{\lambda_n,\rho})$. Then

$$u_n(t)' = -\int_0^t \left(\frac{\tau}{t}\right)^{N-1} \int_0^1 K(\tau, s)\left(-\lambda_n|u_n'(s)| + \rho f(u_n(s))\right) ds d\tau < 0 \tag{2.10}$$

since $f > 0$, we get

$$|u_n(t)'| > -\lambda_n \int_0^t \left(\frac{\tau}{t}\right)^{N-1} \int_{\frac{1}{2}}^1 K(\tau, s)|u_n(s)'|^q ds d\tau.$$

In view of Lemma 2.2, there exists $1/2 \leq c_{1/2}(t) \leq 1$ such that

$$|u_n(t)'| > -\lambda_n |u_n(c_{1/2}(t))'|^q \int_0^t \left(\frac{\tau}{t}\right)^{N-1} \int_{\frac{1}{2}}^1 K(\tau, s) ds d\tau.$$

From (2.10), we have $0 < |u_n'(1/2)| \leq |u_n'(c_{1/2}(t))|$. By taking $t = \frac{1}{2}$, we get

$$1 > -\lambda_n |u_n(1/2)'|^{q-1} \int_0^{1/2} (2\tau)^{N-1} \int_0^1 K(\tau, s) ds d\tau.$$

By (2.10) and $\epsilon_0 := \inf f > 0$, we get

$$\begin{aligned} |u_n'(1/2)| &= \int_0^{1/2} (2\tau)^{N-1} \int_0^1 K(\tau, s)\rho f(u_n(s)) ds d\tau \\ &> \epsilon_0 \int_0^{\frac{1}{2}} (2\tau)^{N-1} \int_0^1 K(\tau, s)\rho ds d\tau. \end{aligned}$$

It follows that

$$1 > -\lambda_n (\rho\epsilon_0)^{q-1} \left(\int_0^{1/2} 2\tau^{N-1} \int_{\frac{1}{2}}^1 K(\tau, s) ds d\tau\right)^q.$$

Letting $n \longrightarrow \infty$, we obtain a contradiction. On other hand, let $-\lambda_0 < \lambda \leq 0$. Suppose that $\rho_\infty(\lambda) = \infty$. Then, there exits $(\lambda, \rho_n(\lambda)) \in S$ such that $\rho_n(\lambda) \to \infty$. If we follow the same way as above, we obtain

$$1 > -\lambda(\rho_n\epsilon_0)^{q-1} \left(\int_0^{1/2} 2\tau^{N-1} \int_{\frac{1}{2}}^1 K(\tau, s) ds d\tau\right)^q. \tag{2.11}$$

Letting $n \to \infty$, we obtain a contradiction. This concludes the proof of Theorem 1.2.

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Ahmed Hamydy
"Abdelmalek Essaadi" University, CRMEFTTH of Tetuan,
Department of Mathematics, Morocco
e-mail: `a.hamydy@yahoo.fr`

Mohamed Massar
"Abdelmalek Essaadi" University,
Faculty of Technical Sciences of Alhoceima,
Department of Mathematics, Morocco
e-mail: massarmed@hotmail.com

Hilal Essaouini
"Abdelmalek Essaadi" University,
Faculty of Sciences of Tetuan,
Department of Physics,
Energy Laboratory, Morocco
e-mail: hilal_essaouini@yahoo.fr

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Global solution for a diffusive epidemic model (HIV/AIDS) with an exponential behavior of source

El Hachemi Daddiouaissa

Abstract. We consider the question of global existence and uniform boundedness of nonnegative solutions of a system of reaction-diffusion equations with exponential nonlinearity, without any restriction on initial data, using maximum principle and Lyapunov function techniques.

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Keywords: Reaction-diffusion systems, Lyapunov function, global solution.

1. Introduction

In this paper we consider the following reaction-diffusion system

$$\frac{\partial u}{\partial t} - a\Delta u = \Pi - f(u,v) - \alpha u \quad (x,t) \in \Omega \times \mathbb{R}_+ \tag{1.1}$$

$$\frac{\partial v}{\partial t} - b\Delta v = f(u,v) - \sigma\kappa(v) \quad (x,t) \in \Omega \times \mathbb{R}_+ \tag{1.2}$$

with the boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \quad \text{on } \partial\Omega \times \mathbb{R}_+, \tag{1.3}$$

and the initial data

$$u(0,x) = u_0(x) \geq 0; \quad v(0,x) = v_0(x) \geq 0 \quad \text{in } \Omega, \tag{1.4}$$

where Ω is a smooth open bounded domain in $\mathbb{R}^n$, with boundary $\partial\Omega$ of class C^1 and

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η is the outer normal to $\partial\Omega$. The constants of diffusion a, b are positive and such that $a \neq b$ and Π, α, σ are positive constants, κ and f are nonnegative functions of class $C^1(\mathbb{R}_+)$ and $C^1(\mathbb{R}_+ \times \mathbb{R}_+)$ respectively.

The reaction-diffusion system $(1.1)-(1.4)$ arises in the study of physical, chemical, and various biological processes including population dynamics (especially AIDS, see C. Castillo-Chavez et al. [4], for further details see [6] [11] [17] [21] [22]).

The case $\Pi = 0$, $\alpha = 0, \sigma = 0$ and $f(u,v) = h(u)Q(v)$, with $h(u) = u$ (for simplicity), has been studied by many authors. Alikakos [1] established the existence of global solutions when $Q(v) \leq C(1 + |v|^{(n+2)/n})$. Then Massuda [18] obtained a positive result for the case $Q(v) \leq C(1 + |v|^{\alpha})$ with arbitrary $\alpha > 0$. The question when $Q(v) = e^{\alpha v^{\beta}}$, $0 < \beta < 1$, $\alpha > 0$ was positively answered by Haraux and Youkana [13], using Lyapunov function techniques, see also Barabanova [2] for $\beta = 1$, with some conditions and later on by Kanel [16], using useful properties inherent to the Green function. For $Q(v) = e^{\alpha v^{\beta}}$, $\beta > 1$, Rebiai [3] proved the global existence. The idea behind the Lyapunov functional stems from Zelenyak's article [23], which has also been used by Crandall et al. [5] for other purposes.
The case $\Pi > 0$, $\alpha > 0, \sigma > 0$ L. Melkemi et al. [19] established the existence of global solutions, when $f(\xi, \tau) \leq \psi(\xi)\varphi(\tau)$ such that

$$\lim_{\tau \to +\infty} \frac{\ln(1 + \varphi(\tau))}{\tau} = 0.$$

For $f(v) = e^{\alpha v^{\beta}}$, $\beta > 1$, Djebara et al [9] showed the global existence.
The goal of this work is to generalize the existing result in [7], where it is proved the existence of global solutions with following exponential nonlinearity

$$0 \leq f(\xi, \tau) \leq \varphi(\xi)(\tau + 1)^{\lambda} e^{r\tau}, \tag{1.5}$$

with restriction on initial data

$$\max\left(\| u_0 \|_\infty, \frac{\Pi}{\alpha} \right) < \frac{\theta^2}{2 - \theta} \quad \frac{8ab}{rn(a-b)^2}. \tag{1.6}$$

Hence, the main purpose of this paper is to give a positive answer, concerning the global existence and the uniform boundedness in time, of solutions of system $(1.1) - (1.4)$, with out any restriction on inital data u_0 and v_0 and same exponential nonlinearity, i.e,

(S1) $\forall \tau \geq 0$, $f(0, \tau) = 0$,
(S2) $\forall \xi \geq 0$, $\forall \tau \geq 0$, $0 \leq f(\xi, \tau) \leq \varphi(\xi)(\tau + 1)^{\lambda} e^{r\tau}$,
(S3) $\kappa(\tau) = \tau^{\mu}$, $\mu \geq 1$,

where r, λ are positive constants, such that $\lambda \geq 1$, φ is a nonnegative function of class $C(\mathbb{R}^+)$.
For this end we use maximum principle and Lyapunov function techniques, and an idea inspired from [8].

2. Existence of local solutions

The usual norms in spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and $C(\overline{\Omega})$ are respectively denoted by

$$\| u \|_p^p = \frac{1}{|\Omega|}\int_\Omega | u(x) |^p \, dx, \quad \| u \|_\infty = \max_{x\in\Omega} | u(x) | .$$

Concerning a local existence, we can conclude directly from the theory of abstract semilinear equations (see A. Friedman [10], D. Henry [14], A. Pazy [20]), that for nonnegative functions u_0 and v_0 in $L^\infty(\Omega)$, there exists a unique local nonnegative solution (u, v) of system (1.1) − (1.4) in $C(\overline{\Omega})$ on $]0, T^*[$, where T^* is the eventual blowing-up time.

3. Existence of global solutions

Using the comparison principle, one obtains

$$0 \le u(t,x) \le \max\left(\| u_0 \|_\infty, \frac{\Pi}{\alpha}\right) = M, \tag{3.1}$$

from which it remains to establish the uniform boundedness of v.

According to the results of [12], it is enough to show that

$$\| f(u,v) - \sigma\kappa(v) \|_p \le C \tag{3.2}$$

(where C is a nonnegative constant independent of t) for some $p > \frac{n}{2}$. To reach this goal, let us start with this preliminaries results.
We consider the following reaction-diffusion system:

$$\frac{\partial u_1}{\partial t} - a_1\Delta u_1 = 1 - h(u_1,u_2) - u_1 \quad (x,t) \in \Omega_1 \times \mathbb{R}_+ \tag{3.3}$$

$$\frac{\partial u_2}{\partial t} - (2-\sqrt{3})a_1\Delta u_2 = h(u_1,u_2) - \delta u_2 \quad (x,t) \in \Omega_1 \times \mathbb{R}_+ \tag{3.4}$$

$$\frac{\partial u_1}{\partial \eta} = \frac{\partial u_2}{\partial \eta} = 0 \quad \text{on } \partial\Omega_1 \times \mathbb{R}_+, \tag{3.5}$$

$$u_1(0,x) = u_{1,0}(x) \ge 0; \quad u_2(0,x) = u_{2,0}(x) \ge 0 \quad \text{in } \Omega_1, \tag{3.6}$$

where Ω_1 is a smooth open bounded domain in $\mathbb{R}^2$, with boundary $\partial\Omega_1$ of class C^1 and η is the outer normal to $\partial\Omega_1$ and $a_1 > 0$ is the diffusion constant, δ is a positive constant and $\|u_{1,0}\|_\infty = \frac{1}{2}$, h is differentiable nonnegative function such that:

(A1) $\forall\tau \ge 0, \quad h(0,\tau) = 0,$

(A2) $\forall\xi \ge 0, \forall\tau \ge 0, \quad 0 \le h(\xi,\tau) = \xi\varphi(\tau) \le \xi(\tau + \alpha_1)e^{\frac{1}{16}\tau},$

where φ is differentiable nonnegative function and

$$\alpha_1 = \max\left(\frac{48}{5}, \left(\frac{3}{2}\frac{M}{|\Omega_1|}\right)^{\frac{1}{4}}\right). \tag{3.7}$$

Using the maximum principle, we obtain

$$0 \le u_1(t,x) \le 1. \tag{3.8}$$

To establish the boundness of u_2, we use the results of [14, 15], where it is enough to show that

$$\| h(u_1, u_2) - \delta u_2 \|_4 \leq C, \tag{3.9}$$

where C is a nonnegative constant independent of t. For this end we need the following

Lemma 3.1. *Let ϕ be a nonnegative function of class $C(\mathbb{R}^+)$, such that*

$$\lim_{\tau \to +\infty} \frac{\phi(\tau)}{\tau} = 0$$

and let A be positive constant. Then there exists $\Pi_2 > 0$, such that

$$\left[\frac{\phi(\tau)}{\tau} - A \right] \tau h_1(\tau) \leq \Pi_2, \tag{3.10}$$

for all $\tau > 0$; h_1 is a nonnegative function of class $C(\mathbb{R}^+)$.

Proof. Since

$$\lim_{\tau \to +\infty} \frac{\phi(\tau)}{\tau} = 0,$$

there exists $\tau_0 > 0$, such that for all $\tau > \tau_0$, we have

$$\left[\frac{\phi(\tau)}{\tau} - A \right] \tau h_1(\tau) \leq 0.$$

Now if τ is in the compact interval $[0, \tau_0]$, then the continuous function

$$[\phi(\tau) - A\tau] h_1(\tau)$$

is bounded. □

Lemma 3.2. *Assume that* $(A1)$ *and* $(A2)$ *hold and let* (u_1, u_2) *be a solution of (3.3)-(3.6) on* $]0, T^*[$, *with arbitrary* $u_{2,0}$. *Let*

$$G_1(t) = \int_{\Omega_1} \left(\frac{1}{\frac{3}{2} - u_1}\right)(u_2 + \alpha_1)^4 e^{\frac{1}{4}u_2} dx. \tag{3.11}$$

Then there exist a positive constant Π_1 such that

$$\frac{dG_1}{dt}(t) \leq -\sigma_1 G_1(t) + \Pi_1, \tag{3.12}$$

where σ_1 is a positive constant.

Proof. We put $q(u_1) = \left(\frac{1}{\frac{3}{2} - u_1} \right)$, so that

$$G_1(t) = \int_{\Omega_1} q(u_1)(u_2 + \alpha_1)^4 e^{\frac{1}{4}u_2} dx.$$

Differentiating G_1 with respect to t and a simple use of Green's formula gives

$$G_1'(t) = I_1 + J_1,$$

where

$$\begin{aligned} I_1 &= -a_1 \int_{\Omega_1} q''(u_1)(u_2+\alpha_1)^4 e^{\frac{1}{4}u_2} |\nabla u_1|^2 dx \\ &- (3-\sqrt{3})a_1 \int_{\Omega_1} q'(u)[4+\frac{1}{4}(u_2+\alpha_1)](u_2+\alpha_1)^3 e^{\frac{1}{4}u_2} \nabla u_1 \nabla u_2 dx \\ &- (2-\sqrt{3})a_1 \int_{\Omega_1} q(u)[12+2(u_2+\alpha_1)+\frac{1}{16}(u_2+\alpha_1)^2](u_2+\alpha_1)^2 e^{\frac{1}{4}u_2} |\nabla u_2|^2 dx, \\ J_1 &= \int_{\Omega_1} q'(u_1)(u_2+\alpha_1)^4 e^{\frac{1}{4}u_2} dx - \int_{\Omega_1} q'(u_1)u_1(u_2+\alpha_1)^4 e^{\frac{1}{4}u_2} dx \\ &+ \int_{\Omega_1} \Big(q(u_1)[4+\frac{1}{4}(u_2+\alpha_1)] - q'(u_1)(u_2+\alpha_1) \Big) (u_2+\alpha_1)^3 h(u_1,u_2) e^{\frac{1}{4}u_2} dx \\ &- \int_{\Omega_1} \delta[4+\frac{1}{4}(u_2+\alpha_1)]u_2(u_2+\alpha_1)^3 e^{\frac{1}{4}u_2} dx. \end{aligned}$$

I_1 involves a quadratic form with respect to ∇u_1 and ∇u_2, which is nonnegative if

$$(3-\sqrt{3})^2[4+\frac{1}{4}(u_2+\alpha_1)]^2 - 8(2-\sqrt{3})[12+2(u_2+\alpha_1)+\frac{1}{16}(u_2+\alpha_1)^2]$$

$$= \big[-2[4+\frac{1}{4}(u_2+\alpha_1)]^2+32\big](2-\sqrt{3}) = \big[1-[1+\frac{1}{16}(u_2+\alpha_1)]^2\big]32(2-\sqrt{3}) \le 0.$$

Concerning the second term J_1, we can observe that

$$\begin{aligned} J_1 &\le \int_{\Omega_1} (2-\frac{1}{4}\delta u_2)\frac{1}{\frac{3}{2}-u_1}(u_2+\alpha_1)^4 e^{\frac{1}{4}u_2} dx \\ &+ \int_{\Omega_1} \Big([4+\frac{1}{4}(u_2+\alpha_1)] - \frac{1}{\frac{3}{2}-u_1}(u_2+\alpha_1)\Big)\frac{1}{\frac{3}{2}-u_1}(u_2+\alpha_1)^3 h(u_1,u_2) e^{\frac{1}{4}u_2} dx. \end{aligned}$$

Now we introduce a positive constant σ_1, such that

$$\begin{aligned} J_1 &\le \int_{\Omega_1} -\sigma_1 \frac{1}{\frac{3}{2}-u_1}(u_2+\alpha_1)^4 e^{\frac{1}{4}u_2} + \left(\frac{2+\sigma_1}{u_2} - \frac{1}{4}\delta\right) \frac{1}{\frac{3}{2}-u_1} u_2 (u_2+\alpha_1)^4 e^{\frac{1}{4}u_2} dx \\ &+ \int_{\Omega_1} (4-\frac{5}{12}\alpha_1)\frac{1}{\frac{3}{2}-u_1}(u_2+\alpha_1)^3 h(u_1,u_2) e^{\frac{1}{4}u_2} dx. \end{aligned}$$

using the Lemma 3.1 and the choice in the formula 3.7, let us get

$$J_1 \le -\sigma_1 G_1(t) + \Pi_2 |\Omega_1|.$$

It follows that

$$\frac{dG_1(t)}{dt} \le -\sigma_1 G_1(t) + \Pi_1,$$

where $\Pi_1 = \Pi_2|\Omega_1|$. □

Theorem 3.3. *Under the assumptions* $(A1)$ *and* $(A2)$*, the solutions of* $(3.3)-(3.6)$ *are global and uniformly bounded on* $[0,+\infty[$.

Proof. Multiplying (3.12) by $e^{\sigma_1 t}$ and integrating the inequality on $(0,t)$, it implies the existence of a positive constant $C_3 > 0$ independent of t such that

$$G_1(t) \leq C_3. \tag{3.13}$$

Then we have

$$\int_{\Omega_1} h^4(u_1,u_2)dx \leq \frac{3}{2}G_1(t) \leq \frac{3}{2}C_3. \tag{3.14}$$

□

Remark 3.4. From the choice (3.7) we have for all $t \geq 0$

$$G_1(t) \geq \int_{\Omega_1} \frac{2}{3}\alpha_1^4 dx \geq M. \tag{3.15}$$

3.1. Main result

Now, we will state the main result

Theorem 3.5. *Under the assumptions* $(S1)-(S3)$, *the solutions of (1.1)-(1.4) are global and uniformly bounded on* $[0,+\infty[$.

The key result needed to prove the Theorem 3.5 is the following

Proposition 3.6. *Assume that* $(S1)-(S3)$ *hold and let* (u,v) *be a solution of (1.1)-(1.4) on* $]0,T^*[$, *with arbitrary* v_0 *and* u_0. *Let*

$$G(t) = \int_\Omega \left(\frac{M}{(2-\theta)M - u}\right)^\beta (v+\omega)^{\gamma p} e^{prv} dx + G_1(\psi(t)), \tag{3.16}$$

where ω, β, γ *and* θ *are positive constants such that* $\omega \geq 1$, $\theta < 1$ *and*

$$\beta = \theta\frac{4ab}{(a-b)^2}, \quad \gamma = \max\left(\lambda, \mu, \frac{(\beta+1)(2-\theta)Mr}{\beta\theta(1-\theta)}\right) \tag{3.17}$$

and

$$\psi(t) = \int_0^t \int_\Omega f(u,v)g(u)(v+\omega)^{\gamma p} e^{prv} dx ds. \tag{3.18}$$

Then, there exist $p > n/2$ *and positive constant* Γ *such that*

$$\frac{dG}{dt} \leq -sG + \Gamma, \tag{3.19}$$

where s *is a positive constant.*

It's very important to state this lemma, before proving this proposition,

Lemma 3.7. *For all* $\tau \geq 0$ *we have*

$$\left[\frac{\Pi\beta}{(1-\theta)M} - \sigma p\kappa(\tau)(\frac{\gamma}{\tau+\omega} + r)\right](\tau+\omega)^{\gamma p} e^{pr\tau} \leq -s(\tau+\omega)^{\gamma p} e^{pr\tau} + B_1, \tag{3.20}$$

where B_1 *and* s *are positive constants.*

Proof. Let us put

$$\xi = \frac{\Pi\beta}{(1-\theta)M} + s$$

$$\begin{aligned}
&\frac{\Pi\beta}{(1-\theta)M}(\tau+\omega)^{p\gamma}e^{pr\tau} - \sigma p\kappa(\tau)[\gamma(\tau+\omega)^{\gamma p-1} + r(\tau+\omega)^{\gamma p}]e^{pr\tau} \\
&= \left(\frac{\Pi\beta}{(1-\theta)M} - \xi\right)(\tau+\omega)^{p\gamma}e^{pr\tau} + \left(\frac{\xi}{\kappa(\tau)} - \sigma rp\right)\kappa(\tau)(\tau+\omega)^{\gamma p}e^{pr\tau},
\end{aligned}$$

then, using Lemma 3.1 we can conclude the result. □

Proof. (of Proposition 3.2). Let

$$g(u) = \left(\frac{M}{(2-\theta)M - u}\right)^{\beta},$$

so that

$$G(t) = \int_\Omega g(u)(v+\omega)^{\gamma p}e^{prv}dx + G_1(\psi(t)).$$

Differentiating G with respect to t and a simple use of Green's formula gives

$$G'(t) = I + J,$$

where

$$\begin{aligned}
I = &-a\int_\Omega g''(u)(v+\omega)^{\gamma p}e^{prv}|\nabla u|^2dx \\
&-(a+b)\int_\Omega g'(u)[\gamma p(v+\omega)^{\gamma p-1} + pr(v+\omega)^{\gamma p}]e^{prv}\nabla u\nabla v dx \\
&-b\int_\Omega g(u)[\gamma p(\gamma p-1)(v+\omega)^{\gamma p-2} + 2\gamma p^2 r(v+\omega)^{\gamma p-1} + p^2r^2(v+\omega)^{\gamma p}]e^{prv}|\nabla v|^2dx, \\
J = &\int_\Omega \Pi g'(u)(v+\omega)^{\gamma p}e^{prv}dx - \int_\Omega \alpha g'(u)u(v+\omega)^{\gamma p}e^{prv}dx \\
&+\int_\Omega\left(g(u)\big[\gamma p(v+\omega)^{\gamma p-1} + rp(v+\omega)^{\gamma p}\big] - g'(u)(v+\omega)^{\gamma p}\right)f(u,v)e^{prv}dx \\
&-\int_\Omega \sigma[\gamma p(v+\omega)^{\gamma p-1} + rp(v+\omega)^{\gamma p}]\kappa(v)g(u)e^{prv}dx + \psi'(t)G_1'(\psi(t)).
\end{aligned}$$

We can see that I involves a quadratic form with respect to ∇u and ∇v, which is nonnegative if

$$\begin{aligned}
\delta = &\big(p(a+b)g'(u)[\gamma(v+\omega)^{\gamma p-1} + r(v+\omega)^{\gamma p}]\big)^2 \\
&- 4ab\gamma p(\gamma p-1)g''(u)g(u)(v+\omega)^{2\gamma p-2} \\
&- 4abg''(u)g(u)(v+\omega)^{\gamma p}[2\gamma p^2r(v+\omega)^{\gamma p-1} + p^2r^2(v+\omega)^{\gamma p}] \le 0.
\end{aligned}$$

Indeed

$$\begin{aligned}
\delta = &[(p\gamma)^2(a+b)^2\beta^2 - 4ab\beta(\beta+1)p\gamma(p\gamma-1)]\frac{g(u)^2(v+\omega)^{2p\gamma-2}}{((2-\theta)M-u)^2} \\
&+ [(a+b)^2\beta^2 - 4ab\beta(\beta+1)]\frac{rp^2g(u)^2(v+\omega)^{2p\gamma-1}}{((2-\theta)M-u)^2}[2\gamma + r(v+\omega)],
\end{aligned}$$

the choice of β and γ gives

$$\delta \leq [\beta + 1 - p\gamma(1-\theta)]\frac{4ab\beta p\gamma g(u)^2(v+\omega)^{2p\gamma-2}}{((2-\theta)M-u)^2} + 4ab(\theta-1)\frac{rp\beta g(u)^2(v+\omega)^{2p\gamma-1}}{((2-\theta)M-u)^2}[2+(rp)(v+\omega)] \leq 0,$$

it follows that

$$I \leq 0.$$

Concerning the second term J, we use (3.12), we can observe that

$$\begin{aligned} J \leq & \int_\Omega \Big(\frac{\Pi\beta}{(1-\theta)M} - \sigma p\kappa(v)[\frac{\gamma}{v+\omega}+r]\Big)g(u)(v+\omega)^{p\gamma}e^{prv}dx \\ & + \int_\Omega \Big(p[\frac{\gamma}{v+\omega}+r] - \frac{\beta}{(2-\theta)M-u}\Big)f(u,v)g(u)(v+\omega)^{\gamma p}e^{prv}dx \\ & + \psi'(t)\big(-\sigma_1 G_1(\psi(t)) + \Pi_1\big). \end{aligned}$$

Using Lemma 3.7 and by choosing $\sigma_1 = \frac{1}{M}(rp+\Pi_1)$, we get

$$\begin{aligned} J \leq & \int_\Omega [-s(v+\omega)^{p\gamma}e^{prv} + B_1]g(u)dx \\ & + \int_\Omega \Big(\frac{p\gamma}{v+\omega} - \frac{\theta}{2-\theta}\frac{4ab}{(a-b)^2M}\Big)f(u,v)g(u)(v+\omega)^{\gamma p}e^{prv}dx. \end{aligned}$$

Since f is continuous function, applying the Lemma 3.1, it follows that there exist a positive constant N_1 such that

$$\begin{aligned} J \leq & \int_\Omega [-s(v+\omega)^{p\gamma}e^{prv} + B_1]g(u)dx \\ & + N_1\int_\Omega g(u)dx. \end{aligned}$$

In addition

$$g(u) \leq \Big(\frac{1}{1-\theta}\Big)^\beta,$$

then

$$J \quad \leq \quad -sG(t) + (\mid\Omega\mid B_1 + N_1)\Big(\frac{1}{1-\theta}\Big)^\beta + sC_3,$$

it follows that

$$\frac{dG}{dt} \leq -sG + \Gamma,$$

where $\Gamma = (\mid\Omega\mid B_1 + N_1)\big(\frac{1}{1-\theta}\big)^\beta + sC_3$. □

Proof. (of Theorem 3.5)
Multiplying (3.19) by e^{st} and integrating the inequality, it implies the existence of a positive constant $C_1 > 0$ independent of t such that

$$G(t) \leq C_1.$$

Since

$$g(u) \geq \left(\frac{1}{2-\theta}\right)^{\beta},$$

$$\int_{\Omega}(v+\omega)^{\gamma p}e^{prv}dx \leq (2-\theta)^{\beta}G(t) \leq C_1(2-\theta)^{\beta}.$$

Since $\omega \geq 1$ and (3.17) we have also,

$$\int_{\Omega}(v+1)^{\lambda p}e^{prv}dx \leq \int_{\Omega}(v+\omega)^{\gamma p}e^{prv}dx \leq C_1(2-\theta)^{\beta},$$

$$\int_{\Omega}v^{\mu p}dx \leq \int_{\Omega}(v+\omega)^{\gamma p}dx \leq C_1(2-\theta)^{\beta}.$$

We put

$$A = \max_{0\leq\xi\leq M}\varphi(\xi),$$

according to $(S1)-(S3)$, we have

$$\int_{\Omega}f(u,v)^p dx \leq \int_{\Omega}A^p(v+1)^{\lambda p}e^{prv}dx \leq A^pC_1(2-\theta)^{\beta} = A^pH^p,$$

we conclude

$$\|f(u,v)-\sigma\kappa(v)\|_p \leq \|f(u,v)\|_p + \|\sigma\kappa(v)\|_p \leq H(A+\sigma).$$

By the preliminary remarks (introduction of section 3), we conclude that the solution of (1.1)-(1.4) is global and uniformly bounded on $[0,+\infty[\times\Omega$. □

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El Hachemi Daddiouaissa
Faculty of New Technologies of Information and Communication,
Kasdi Merbah University, UKM,
Route de Ghardaia, BP.511,
Ouargla 30000, Algeria
e-mail: `dmhbsdj@gmail.com`

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Decay rate of solutions to the Cauchy problem for a coupled system of viscoelastic wave equations with a strong delay in $\mathbb{R}^n$

Amina Chaili, Bochra Belhadji and Abderrahmane Beniani

Abstract. Using weighted spaces, we establish a general decay rate properties of solutions as $T \to \infty$ for a coupled system of viscoelastic wave equations in $\mathbb{R}^n$ under some conditions on g_1, g_2, ϕ. We exploit a density function to introduce weighted spaces for solutions and using an appropriate Lyapunov function.

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1. Introduction and statement

Let us consider the following problem

$$\begin{cases} u_1'' + \alpha u_2 + \Delta u_1'(x, t-\tau) = \phi(x)\Delta_x \left(u_1 + \int_0^t g_1(s) u_1(t-s, x) ds \right), x \in \mathbb{R}^n \times \mathbb{R}^+ \\ u_2'' + \alpha u_1 + \Delta u_2'(x, t-\tau) = \phi(x)\Delta_x \left(u_2 + \int_0^t g_2(s) u_2(t-s, x) ds \right), x \in \mathbb{R}^n \times \mathbb{R}^+ \\ u_1'(x, t-\tau) = f_1(x, t-\tau), \quad u_2'(x, t-\tau) = f_2(x, t-\tau) \quad t \in (0, \tau) \\ (u_1(0, x), u_2(0, x)) = (u_{10}(x), u_{20}(x)) \in (\mathcal{H}(\mathbb{R}^n))^2, \\ (u_1'(0, x), u_2'(0, x)) = (u_{11}(x), u_{21}(x)) \in (L^2_\rho(\mathbb{R}^n))^2, \end{cases} \tag{1.1}$$

where the space $\mathcal{H}(\mathbb{R}^n)$ defined in (1.11) and $l, n \geq 2$, $\phi(x) > 0$, $\forall x \in \mathbb{R}^n$, $(\phi(x))^{-1} = \rho(x)$ defined in (**A2**).

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In this paper we are going to consider the solutions in spaces weighted by the density function $\rho(x)$ in order to compensate for the lack of Poincare's inequality which is useful in the proof.

In this framework, (see [5], [9]), it is well known that, for any initial data $(u_{10}, u_{20}) \in (\mathcal{H}(\mathbb{R}^n))^2, (u_{11}, u_{21}) \in (L^l_\rho(\mathbb{R}^n))^2$, then problem (P) has a global solution $(u_1, u_2) \in (C([0,T), \mathcal{H}(\mathbb{R}^n)))^2, (u_1', u_2') \in (C([0,T), L^l_\rho(\mathbb{R}^n))^2$ for T small enough, under hypothesis (**A1**)-(**A2**).

The energy of (u_1, u_2) at time t is defined by

$$E(t) = \frac{1}{2}\sum_{i=1}^{2} \|u_i'\|^2_{L^2_\rho(\mathbb{R}^n)} + \frac{1}{2}\sum_{i=1}^{2}(1 - \int_0^t g_i(s)ds)\|\nabla_x u_i\|_2^2 + \frac{1}{2}\sum_{i=1}^{2}(g_i \circ \nabla_x u_i) + \alpha \int_{\mathbb{R}^n} \rho u_1 u_2 dx. \tag{1.2}$$

When α is sufficiently small, we deduce that:

$$E(t) \geq \frac{1}{2}(1 - |\alpha|\|\rho\|^{-1}_{L^s}) \left[\sum_{i=1}^{2} \|u_i'\|^2_{L^l_\rho} + \sum_{i=1}^{2}(1 - \int_0^t g_i(s)ds)\|\nabla_x u_i\|_2^2 + \sum_{i=1}^{2}(g_i \circ \nabla_x u_i)\right]$$

and the following energy functional law holds, which means that, our energy is uniformly bounded and decreasing along the trajectories.

$$E'(t) = \frac{1}{2}\sum_{i=1}^{2}(g_i' \circ \nabla_x u_i)(t) - \frac{1}{2}\sum_{i=1}^{2} g_i(t)\|\nabla_x u_i(t)\|_2^2, \forall t \geq 0. \tag{1.3}$$

The following notation will be used throughout this paper

$$(\Phi^s \circ \Psi)(t) = \int_0^t \Phi^s(t-\tau)\,\|\Psi(t) - \Psi(\tau)\|_2^2\, d\tau \tag{1.4}$$

For the literature, in $\mathbb{R}^n$ we quote essentially the results of [1], [5], [6], [7], [9], [11]. In [6], authors showed for one equation that, for compactly supported initial data and for an exponentially decaying relaxation function, the decay of the energy of solution of a linear Cauchy problem (1.1) with $l = 2, \rho(x) = 1$is polynomial. The finite-speed propagation is used to compensate for the lack of Poincars inequality. In the case $l = 2$, in [5], author looked into a linear Cauchy viscoelastic equation with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate for the lack of Poincar's inequality. The same problem traited in [5], was considred in [7], where they consider a Cauchy problem for a viscoelastic wave equation. Under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. Conditions used, on the relaxation function g and its derivative g' are different from the usual ones.

The problem (1.1) for the case $l = 2, \rho(x) = 1$, in a bounded domain $\Omega \subset \mathbb{R}^n, (n \geq 1)$ with a smooth boundary $\partial\Omega$ and g is a positive nonincreasing function was considred as equation in [11], where they established an explicit and general decay rate result for relaxation functions satisfying:

$$g'(t) \le -H(g(t)), t \ge 0, H(0) = 0 \tag{1.5}$$

for a positive function $H \in C^1(\mathbb{R}^+)$ and H is linear or strictly increasing and strictly convex C^2 function on $(0, r], 1 > r$. Wich improve the conditions considred recently by Alabau-Boussouira and Cannarsa [1] on the relaxation functions

$$g'(t) \le -\chi(g(t)), \chi(0) = \chi'(0) = 0 \tag{1.6}$$

where χ is a non-negative function, strictly increasing and strictly convex on $(0, k_0], k_0 > 0$. They required that

$$\int_0^{k_0} \frac{dx}{\chi(x)} = +\infty, \int_0^{k_0} \frac{xdx}{\chi(x)} < 1, \lim \inf_{s \to 0^+} \frac{\chi(s)/s}{\chi'(s)} > \frac{1}{2} \tag{1.7}$$

and proved a decay result for the energy of equation (1.1) with $\alpha = 0, l = 2, \rho(x) = 1$ in a bounded domain. In addition to these assumptions, if

$$\lim \sup_{s \to 0^+} \frac{\chi(s)/s}{\chi'(s)} < 1 \tag{1.8}$$

then, in this case, an explicit rate of decay is given.

We omit the space variable x of $u(x,t), u'(x,t)$ and for simplicity reason denote $u(x,t) = u$ and $u'(x,t) = u'$, when no confusion arises. We denote by

$$|\nabla_x u|^2 = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2, \quad \Delta_x u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

The constants c used throughout this paper are positive generic constants which may be different in various occurrences also the functions considered are all real valued, here $u' = du(t)/dt$ and $u'' = d^2u(t)/dt^2$.

The main purpose of this work is to allow a wider class of relaxation functions and improve earlier results in the literature. The basic mechanism behind the decay rates is the relation between the damping and the energy. In section 2, we prove decay estimates of the solution of our problem (1.1) when g_1 and g_2 are of general decay rate. Our approach involves a perturbed energy method and leverages properties of convex functions.

First we recall and make use the following assumptions on the functions ρ and g for $i = 1, 2$ as:

A1: To guarantee the hyperbolicity of the system, we assume that the function $g_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ (for $i = 1, 2$) is of class C^1 satisfying:

$$1 - \int_0^\infty g_i(t)dt \ge k_i > 0, g_i(0) = g_{i0} > 0 \tag{1.9}$$

and there exist nonincreasing continuous functions ξ_1,ξ_2: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

$$g_i'(t) \le -\xi_i g_i(t). \tag{1.10}$$

A2: The function $\rho : \mathbb{R}^n \to \mathbb{R}^*_+, \rho(x) \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0,1)$ and $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = \frac{2n}{2n - qn + 2q}$.

Definition 1.1 ([5], [12]). We define the function spaces of our problem and its norm as follows:

$$\mathcal{H}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla_x f \in L^2(\mathbb{R}^n) \right\} \tag{1.11}$$

and the spaces $L^2_\rho(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(f,h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx.$$

For $1 < p < \infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^q_\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \rho |f|^q dx \right)^{1/q}. \tag{1.12}$$

Corollary 1.2. *The separable Hilbert space $L^2_\rho(\mathbb{R}^n)$ with*

$$(f,f)_{L^2_\rho(\mathbb{R}^n)} = \|f\|^2_{L^2_\rho(\mathbb{R}^n)}.$$

consist of all f for which $\|f\|_{L^q_\rho(\mathbb{R}^n)} < \infty, 1 < q < +\infty$.

The following technical lemma will be pivotal in the next section.

Lemma 1.3. [4] (Lemma 1.1) *For any two functions g, $v \in C^1(\mathbb{R})$ and $\theta \in [0,1]$ we have*

$$\begin{aligned} \int_{\mathbb{R}^n} v'(t) \int_0^t g(t-s) v(s) ds dx &= -\frac{1}{2}\frac{d}{dt}(g \circ v)(t) + \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s) ds \right) \|v(t)\|_2^2 \\ &\quad + \frac{1}{2}(g' \circ v)(t) - \frac{1}{2} g(t) \|v(t)\|_2^2. \end{aligned} \tag{1.13}$$

and

$$\int_{\mathbb{R}^n} \left(\int_0^t g(t-s) |v(s) - v(t)| ds \right)^2 dx \le \left(\int_0^t g^{2(1-\theta)}(s) ds \right) (g^{2\theta} \circ v) \tag{1.14}$$

We are now ready to state and prove our main results

2. Results and proofs

Lemma 2.1. [8] *Let ρ satisfies (**A2**), then for any $u \in \mathcal{H}(\mathbb{R}^n)$*

$$\|u\|_{L^q_\rho(\mathbb{R}^n)} \le \|\rho\|_{L^s(\mathbb{R}^n)} \|\nabla_x u\|_{L^2(\mathbb{R}^n)}, \quad \text{with } s = \frac{2n}{2n - qn + 2q}, 2 \le q \le \frac{2n}{n-2}.$$

Corollary 2.2. *If $q = 2$, then Lemma 2.1. yields*

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \le \|\rho\|_{L^{n/2}(\mathbb{R}^n)} \|\nabla_x u\|_{L^2(\mathbb{R}^n)},$$

where we can assume $\|\rho\|_{L^{n/2}(\mathbb{R}^n)} = C_0 > 0$ to get

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \le C_0 \|\nabla_x u\|_{L^2(\mathbb{R}^n)}. \tag{2.1}$$

Using Cauchy-Schwarz, Poincare's inequalities, the proof of the following Lemma is immediate.

Lemma 2.3. *There exist constants $c, c' > 0$ such that*

$$\int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)(u_i(t)-u_i(s))ds\right)^2 dx \le c(g_i \circ u_i)(t) \le c'(g_i' \circ \nabla u_i)(t) \qquad (2.2)$$

for any $u \in \mathcal{H}(\mathbb{R}^n)$.

To construct a Lyapunov functional L equivalent to E, we introduce the next functionals

$$\psi_1(t) = \sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)u_i|u_i'|^{l-2}u_i'dx \qquad (2.3)$$

$$\psi_2(t) = -\sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)|u_i'|^{l-2}u_i'\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx \qquad (2.4)$$

Lemma 2.4. *Under the assumptions (* **A1**-**A2***), the functional ψ_1 satisfies, along the solution of* (1.1)

$$\psi_1'(t) \le \sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - (k+|\alpha|C_0-\delta-1)\sum_{i=1}^{2}\|\nabla_x u_i\|_2^2 + \frac{(1-k)}{4\delta}\sum_{i=1}^{2}(g_i \circ \nabla_x u_i) \quad (2.5)$$

Proof. From (2.3), integrate by parts over $\mathbb{R}^n$, we have

$$\begin{aligned}
\psi_1'(t) &= \int_{\mathbb{R}^n}\rho(x)u_1'^l dx + \int_{\mathbb{R}^n}\rho(x)u_1\left(|u_1'|^{l-2}u_1'\right)' dx \\
&+ \int_{\mathbb{R}^n}\rho(x)u_2'^l dx + \int_{\mathbb{R}^n}\rho(x)u_2\left(|u_2'|^{l-2}u_2'\right)' dx \\
&= \int_{\mathbb{R}^n}\left(\rho(x)u_1'^l + u_1\Delta_x u_1 - \alpha\rho(x)u_1u_2 - u_1\int_0^t g_1(t-s)\Delta_x u_1(s,x)ds\right)dx \\
&+ \int_{\mathbb{R}^n}\left(\rho(x)u_2'^l + u_2\Delta_x u_2 - \alpha\rho(x)u_1u_2 - u_2\int_0^t g_2(t-s)\Delta_x u_2(s,x)ds\right)dx \\
&\le \sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - \sum_{i=1}^{2}k_i\|\nabla_x u_i\|_2^2 - 2\alpha\int_{\mathbb{R}^n}\rho(x)u_1u_2dx \\
&+ \sum_{i=1}^{2}\int_{\mathbb{R}^n}\nabla_x u_i\int_0^t g_i(t-s)(\nabla_x u_i(s)-\nabla_x u_i(t))dsdx
\end{aligned}$$

Using Young's, Poincare's inequalities, Lemma (2.1) and Lemma (1.3), we obtain

$$\begin{aligned}
\psi_1'(t) &\le \sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - \sum_{i=1}^{2}k_i\|\nabla_x u_i\|_2^2 + (1-|\alpha|\|\rho\|^{-1}_{L^s(\mathbb{R}^n)})\sum_{i=1}^{2}\|\nabla_x u_i\|_2^2 \\
&+ \delta\sum_{i=1}^{2}\|\nabla_x u_i\|_2^2 + \frac{1}{4\delta}\sum_{i=1}^{2}\int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)|\nabla_x u_i(s)-\nabla_x u_i(t)|ds\right)^2 dx \\
&\le \sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - (k+|\alpha|C_0-\delta-1)\sum_{i=1}^{2}\|\nabla_x u_i\|_2^2 + \frac{(1-k)}{4\delta}\sum_{i=1}^{2}(g_i \circ \nabla_x u_i)
\end{aligned}$$

For α small enough and $k = \max\{k_1, k_2\}$. $\square$

Lemma 2.5. *Under the assumptions (* **A1-A2** *), the functional ψ_2 satisfies, along the solution of (P), for any $\sigma \in (0,1)$*

$$\begin{aligned}\psi_2'(t) \quad &\leq \quad \sum_{i=1}^{2}\left(\delta - \int_0^t g_i(s)ds\right)\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} \\ &+ \quad \delta\sum_{i=1}^{2}\|\nabla_x u_i\|_2^2 + \frac{c}{\delta}\sum_{i=1}^{2}(g_i \circ \nabla_x u_i) - c_\delta C_0\sum_{i=1}^{2}(g_i' \circ \nabla_x u_i)^{l/2} \qquad (2.6)\end{aligned}$$

Proof. Exploiting Eq. in (1.1), to get

$$\begin{aligned}\psi_2'(t) = &-\sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)\left(|u_i'|^{l-2}u_i'\right)'\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx \qquad (2.7)\\ &-\sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)|u_i'|^{l-2}u_i'\int_0^t g_i'(t-s)(u_i(t)-u_i(s))dsdx - \sum_{i=1}^{2}\int_0^t g_i(s)ds\|u_i'\|^l_{L^l_\rho}\end{aligned}$$

To simplify the first term in (2.7), we multiply (1.1) by $\displaystyle\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx$ and integrate by parts over $\mathbb{R}^n$. So we obtain

$$\begin{aligned}&-\sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)\left(|u_i'|^{l-2}u_i'\right)'\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx \\ &=\sum_{i=1}^{2}\int_{\mathbb{R}^n}\Delta u_i(x)\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx \\ &-\sum_{i=1}^{2}\int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)(u_i(t)-u_i(s))\int_0^t g_i(t-s)\Delta u_i(s)\right)dx \qquad (2.8)\\ &-\alpha\int_{\mathbb{R}^n}\left[\rho u_2\int_0^t g_1(t-s)(u_1(t)-u_1(s))ds + \rho u_1\int_0^t g_2(t-s)(u_2(t)-u_2(s))ds\right]dx\end{aligned}$$

The first term in the right side of (2.8) is estimated as follows

$$\begin{aligned}&\int_{\mathbb{R}^n}\Delta u_i(x)\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx \\ \leq \quad &-\int_{\mathbb{R}^n}\nabla_x u_i\int_0^t g_i(t-s)(\nabla_x u_i(t)-\nabla_x u_i(s))dsdx \\ \leq \quad &\int_{\mathbb{R}^n}\nabla_x u_i\int_0^t g_i(t-s)(\nabla_x u_i(s)-\nabla_x u_i(t))dsdx \\ \leq \quad &\delta\|\nabla_x u_i\|^2 + \frac{1}{4\delta}\left(\int_0^t g_i(s)\right)(g_i\circ\nabla u_i)(t) \\ \leq \quad &\delta\|\nabla_x u_i\|^2 + \frac{1-k}{4\delta}(g_i\circ\nabla u_i)(t).\end{aligned}$$

while the second term becomes,

$$\begin{aligned}
- & \int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)(u_i(t)-u_i(s))\int_0^t g_i(t-s)\Delta u_i(s)\right)dx\\
= & \int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)(\nabla u_i(t)-\nabla u_i(s))\cdot\int_0^t g_i(t-s)\nabla u_i(s)\right)dx\\
\leq & \ \delta\int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)|\nabla u_i(s)-\nabla u_i(t))+\nabla u_i(t)|\right)^2\\
+ & \ \frac{1}{4\delta}\int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)(\nabla u_i(t)-\nabla u_i(s))\right)^2\\
\leq & \ 2\delta(1-k)^2\|\nabla u_i\|_2^2+\left(2\delta+\frac{1}{4\delta}\right)(1-k)(g_i\circ\nabla u_i)(t).
\end{aligned}$$

Now, using Young's and Poincare's inequalities we estimate

$$\begin{aligned}
- & \ \alpha\int_{\mathbb{R}^n}\rho u_2\int_0^t g_1(t-s)(u_1(t)-u_1(s))dsdx\\
\leq & \ -|\alpha|\delta\|u_2\|_{L^2_\rho}^2-\frac{|\alpha|C_0}{4\delta}(1-k)(g_1\circ\nabla u_1)(t)\\
\leq & \ -|\alpha|\delta C_0\|\nabla u_2\|_{L^2}^2-\frac{|\alpha|C_0}{4\delta}(1-k)(g_1\circ\nabla u_1)(t).
\end{aligned}$$

By Hölder's and Young's inegualities and Lemma (2.1) we estimate

$$\begin{aligned}
- & \int_{\mathbb{R}^n}\rho(x)|u_i'|^{l-2}u_i'\int_0^t g_i'(t-s)(u_i(t)-u_i(s))dsdx\\
\leq & \left(\int_{\mathbb{R}^n}\rho(x)|u_i'|^l dx\right)^{(l-1)/l}\times\left(\int_{\mathbb{R}^n}\rho(x)|\int_0^t -g_i'(t-s)(u_i(t)-u_i(s))ds|^l\right)^{1/l}\\
\leq & \ \delta\|u'\|_{L^l_\rho(\mathbb{R}^n)}^l+\frac{1}{4\delta}\|\rho\|_{L^s(\mathbb{R}^n)}^l\|\int_0^t -g'(t-s)(u(t)-u(s))ds\|_{L^l_\rho(\mathbb{R}^n)}^l\\
\leq & \ \delta\|u'\|_{L^l_\rho(\mathbb{R}^n)}^l-\frac{1}{4\delta}C_0(g'\circ\nabla_x u)^{l/2}(t).
\end{aligned}$$

Using Young's and Poincare's inequalities and Lemma (1.3), we obtain

$$\begin{aligned}
\psi_2'(t) \leq & \sum_{i=1}^2\left(\delta-\int_0^t g_i(s)ds\right)\|u_i'\|_{L^l_\rho(\mathbb{R}^n)}^l\\
+ & \ \delta\sum_{i=1}^2\|\nabla_x u_i\|_2^2+\frac{c}{\delta}\sum_{i=1}^2(g_i\circ\nabla_x u_i)-c_\delta C_0\sum_{i=1}^2(g_i'\circ\nabla_x u_i)^{l/2}.
\end{aligned}$$

□

Our main result reads as follows

Theorem 2.6. *Let* $(u_0, u_1) \in (\mathcal{H}(\mathbb{R}^n(\Omega)) \times L^l_\rho(\mathbb{R}^n)$ *and suppose that* $(\mathbf{A1})-(\mathbf{A2})$ *hold. Then there exist positive constants* α_1, ω *such that the energy of solution given by* (1.1) *satisfies,*

$$E(t) \leq \alpha_1 E(t_0) \exp\left(-\omega \int_{t_0}^{t} \xi(s)ds\right), \forall t \geq t_0 \tag{2.9}$$

where $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}, \quad \forall t \geq 0.$

In order to prove this theorem, let us define

$$L(t) = N_1 E(t) + \psi_1(t) + N_2 \psi_2(t) \tag{2.10}$$

for $N_1, N_2 > 1$. We require the following lemma, indicating an equivalence between the Lyapunov and energy functions

Lemma 2.7. *For* $N_1, N_2 > 1$, *we have*

$$\beta_1 L(t) \leq E(t) \leq L(t)\beta_2, \tag{2.11}$$

holds for two positive constants β_1 *and* β_2.

Proof. By applying Young's inequality to (2.3) and using (2.4) and (2.10), we obtain

$$\begin{aligned} |L(t) - N_1 E(t)| &\leq |\psi_1(t)| + N_2 |\psi_2(t)| \\ &\leq \sum_{i=1}^{2} \int_{\mathbb{R}^n} \left|\rho(x) u_i |u_i'|^{l-2} u_i'\right| dx \\ &+ N_2 \sum_{i=1}^{2} \int_{\mathbb{R}^n} \left|\rho(x) |u_i'|^{l-2} u_i' \int_0^t g_i(t-s)(u_i(t) - u_i(s))ds\right| dx \end{aligned}$$

Thanks to Hölder and Young's inequalities with exponents $\frac{l}{l-1}$, l, since $\frac{2n}{n+2} \geq l \geq 2$, we have by using Lemma 2.1

$$\begin{aligned} \int_{\mathbb{R}^n} \left|\rho(x) u_i |u_i'|^{l-2} u_i'\right| dx &\leq \left(\int_{\mathbb{R}^n} \rho(x) |u_i|^l dx\right)^{1/l} \left(\int_{\mathbb{R}^n} \rho(x) |u_i'|^l dx\right)^{(l-1)/l} \\ &\leq \frac{1}{l}\left(\int_{\mathbb{R}^n} \rho(x)|u_i|^l dx\right) + \frac{l-1}{l}\left(\int_{\mathbb{R}^n} \rho(x)|u_i'|^l dx\right) \\ &\leq c\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + c\|\rho\|^l_{L^s(\mathbb{R}^n)} \|\nabla_x u_i\|_2^l. \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^n} \left|\left(\rho(x)^{\frac{l-1}{l}} |u_i'|^{l-2} u_i'\right)\left(\rho(x)^{\frac{1}{l}} \int_0^t g_i(t-s)(u_i(t) - u_i(s))ds\right)\right| dx \\ &\leq \left(\int_{\mathbb{R}^n} \rho(x)|u_i'|^l dx\right)^{(l-1)/l} \times \left(\int_{\mathbb{R}^n} \rho(x) \left|\int_0^t g_i(t-s)(u_i(t) - u_i(s))ds\right|^l\right)^{1/l} \\ &\leq \frac{l-1}{l}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \frac{1}{l}\left\|\int_0^t g_i(t-s)(u_i(t) - u_i(s))ds\right\|^l_{L^l_\rho(\mathbb{R}^n)} \\ &\leq \frac{l-1}{l}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \frac{1}{l}\|\rho\|^l_{L^s(\mathbb{R}^n)} (g_i \circ \nabla_x u_i)^{l/2}(t). \end{aligned}$$

then, since $l \geq 2$, we have

$$\begin{aligned} |L(t) - N_1 E(t)| &\leq c \sum_{i=1}^{2} \left(\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \|\nabla_x u_i\|_2^l + g_i \circ \nabla_x u_i)^{l/2}(t) \right) \\ &\leq c(E(t) + E^{l/2}(t)) \\ &\leq c(E(t) + E(t) \cdot E^{(l/2)-1}(t)) \\ &\leq c(E(t) + E(t) \cdot E^{(l/2)-1}(0)) \\ &\leq cE(t). \end{aligned}$$

Consequently, (2.11) follows. □

Proof of Theorem 2.6. From (1.3), results of Lemmas (2.4) and (2.5), we have

$$\begin{aligned} L'(t) &= N_1 E'(t) + \psi_1'(t) + N_2 \psi_2'(t) \\ &\leq \left(\frac{1}{2} N_1 - c_\delta C_0 N_2 \right) \sum_{i=1}^{2} (g_i' \circ \nabla_x u_i)^{l/2} + \left(\frac{4\xi_2 c + (1-l)}{4\delta} \right) \sum_{i=1}^{2} (g_i \circ \nabla_x u_i) \\ &- M_1 \sum_{i=1}^{2} \|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - M_2 \sum_{i=1}^{2} \|\nabla_x u_i\|_2^2 \end{aligned}$$

At this point, we choose ξ_2 large enough so that

$$M_1 := \left(N_2 \left(\int_0^{t_1} g(s) ds - \delta \right) - 1 \right) > 0,$$

We choose δ so small that $N_1 > 2c_\delta \|\rho\|^l_{L^s(\mathbb{R}^n)} N_2$. Given that δ is fixed, we can choose ξ_1, ξ_2 large enough so that

$$M_2 := \left(-N_2 \sigma + \frac{1}{2} N_1 g(t_1) + (l - \sigma) \right) > 0,$$

and

$$\left(\frac{1}{2} N_1 - c_\delta C_0 N_2 \right) > 0.$$

which yields

$$L'(t) \leq M_0 \sum_{i=1}^{2} (g_i \circ \nabla_x u_i) - mE(t), \quad \forall t \geq t_1. \tag{2.13}$$

Multiplying (2.13) by $\xi(t)$ gives

$$\xi(t) L'(t) \leq -m\xi(t) E(t) + M_0 \xi(t) \sum_{i=1}^{2} (g_i \circ \nabla_x u_i) \tag{2.14}$$

The last term can be estimated, using (**A1**), as follows

$$
\begin{aligned}
M_0\xi(t)\sum_{i=1}^{2}(g_i\circ\nabla_x u_i) &\le M_0\sum_{i=1}^{2}\xi_i(t)\int_{\mathbb{R}^n}\int_0^t g_i(t-s)|u_i(t)-u_i(s)|^2\\
&\le M_0\sum_{i=1}^{2}\int_{\mathbb{R}^n}\int_0^t \xi_i(t-s)g_i(t-s)|u_i(t)-u_i(s)|^2\\
&\le -M_0\sum_{i=1}^{2}\int_{\mathbb{R}^n}\int_0^t g_i'(t-s)|u_i(t)-u_i(s)|^2\\
&\le -M_0\sum_{i=1}^{2}g_i'\circ\nabla u_i\le -M_0E'(t). \qquad (2.15)
\end{aligned}
$$

Thus, (2.13) becomes

$$\xi(t)L'(t)+M_0E'(t) \quad\le\quad -m\xi(t)E(t) \quad \forall t\ge t_0. \qquad (2.16)$$

Using the fact that ξ is a nonincreasing continuous function as ξ_1 and ξ_2 are nonincreasing, and so ξ is differentiable, with $\xi'(t)\le 0$ for a.e t, then

$$(\xi(t)L(t)+M_0E(t))' \quad\le\quad \xi(t)L'(t)+M_0E'(t)\le -m\xi(t)E(t) \quad \forall t\ge t_0. \qquad (2.17)$$

Since, using (2.11)

$$F=\xi L+M_0E\sim E, \qquad (2.18)$$

we obtain, for some positive constant ω

$$F'(t)\le -\omega\xi(t)F(t) \quad \forall t\ge t_0. \qquad (2.19)$$

Integration over (t_0,t) leads to, for some constant $\omega>0$ such that

$$F(t)\le \alpha_1F(t_0)\exp\left(-\omega\int_{t_0}^{t}\xi(s)ds\right), \forall t\ge t_0 \qquad (2.20)$$

Recalling (2.18), estimate (2.20) yields the desired result (2.9). This completes the proof of Theorem 2.6.

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Amina Chaili
University of Ain Temouchent,
Department of Mathematics,
Engineering and Sustainable Development Laboratory,
Ain Temouchent 46000, Algeria
e-mail: `amina.chaili@univ-temouchent.edu.dz`

Bochra Belhadji
University Ghardaia, Department of Mathematics,
BP 455, 47000 Ghardaia, Algeria
e-mail: `belhadji.bochra@univ-ghardaia.dz`

Abderrahmane Beniani
University of Ain Temouchent,
Department of Mathematics,
Engineering and Sustainable Development Laboratory,
Ain Temouchent 46000, Algeria
e-mail: `abderrahmane.beniani@univ-temouchent.edu.dz`

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Triangular ideal relative convergence on modular spaces and Korovkin theorems

Selin Çınar and Sevda Yıldız

Abstract. In this paper, we introduce the concept of triangular ideal relative convergence for double sequences of functions defined on a modular space. Based upon this new convergence method, we prove Korovkin theorems. Then, we construct an example such that our new approximation results work. Finally, we discuss the reduced results which are obtained by special choices.

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1. Introduction

Let e_r denote the continuous real functions on $[a, b]$ defined by $e_r(s) = s^r$, $r = 0, 1, 2$. The Korovkin theorem establishes the uniform convergence in the space $C[a, b]$ for a sequence of positive linear operators $\{L_i\}$ on $C[a, b]$ via the convergence only on the test functions e_r where $C[a, b]$ is the space of all continuous real functions defined on the interval $[a, b]$ ([21]). A more general framework for the Korovkin theorems can be obtained by using different convergence methods. Gadjiev and Orhan [18] developed these theorems by considering statistical convergence ([17], [31]) instead of ordinary convergence in 2002. After these developments Demirci and Dirik [15] have carried this convergence for double sequences of positive linear operators.

The concept of relative uniform convergence given by Moore [25] in 1910, was later investigated in detail by Chittenden [8]. In consideration of these studies, statistical relative convergence for single sequences was defined by Demirci and Orhan [13] and recently this convergence was given for double sequences by Şahin and Dirik [32]

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(see also [14]). Also, Korovkin theorem has been studied on various function spaces via different convergence methods ([5], [7], [11], [16]). Several forms of Korovkin theorems have been examined in modular spaces including as particular case the L_p spaces, Orlicz and Musielak-Orlicz spaces ([12], [13], [14], [20], [30], [34]).

Recently, Bardaro et al. introduced the triangular $A-$statistical convergence which cannot be compared with statistical convergence ([1], [2]) and then, with the help of this definition, triangular $A-$statistical relative uniform convergence has been defined in [9].

Kostyrko et al. [22] presented the definition of ideal convergence which is a more overall method than statistical convergence and it is based on the notion of the ideal I of subsets of the set $\mathbb{N}$, the natural numbers.

In the present paper, we introduce a new form of convergence for double sequence, called triangular ideal relative modular convergence. We will compare this new convergence with triangular statistical modular convergence and obtain more general results.

We now recall some definitions and notations on modular space.

Let $S=[a,b]$ be a bounded interval of the real line $\mathbb{R}$ provided with the Lebesgue measure. Then, we will denote by $X\left(S^2\right)$ the space of all real-valued measurable functions on $S^2=[a,b]\times[a,b]$ provided with equality $a.e.$. A functional

$$\rho : X\left(S^2\right) \to [0,+\infty]$$

is called a modular on $X\left(S^2\right)$ provided that below conditions hold:

(i) $\rho(h)=0$ if and only if $h=0$ $a.e.$ in S^2,

(ii) $\rho(-h)=\rho(h)$ for every $h\in X\left(S^2\right)$,

(iii) $\rho(\alpha h+\beta g)\le \rho(h)+\rho(g)$ for every $h,g\in X(S^2)$ and for any $\alpha,\beta\ge 0$ with $\alpha+\beta=1$.

A modular ρ is called $N-$quasi convex if there exists a constant $N\ge 1$ such that $\rho(\alpha h+\beta g)\le N\alpha\rho(Nh)+N\beta\rho(Ng)$ holds for every $h,g\in X\left(S^2\right)$, $\alpha,\beta\ge 0$ with $\alpha+\beta=1$. In particular, if $N=1$, then ρ is said to be convex. A modular ρ is called $N-$quasi semiconvex if there exists a constant $N\ge 1$ such that $\rho(ah)\le Na\rho(Nh)$ holds for every $h\in X\left(S^2\right)$ and $a\in(0,1]$. Note that if $\beta=0$, then every $N-$quasi convex modular is $N-$quasi semiconvex (see for details, [5, 6]).

Now, we recall vector subspaces of $X(S^2)$ defined via a modular functional by:

The modular spaces $L^\rho\left(S^2\right)$ generated by ρ is defined by

$$L^\rho\left(S^2\right) := \left\{ h\in X(S^2) : \lim_{\lambda\to 0^+}\rho(\lambda h)=0 \right\},$$

and the space of the finite elements of $L^\rho\left(S^2\right)$ is given by

$$E^\rho\left(S^2\right) := \left\{ h\in L^\rho\left(S^2\right) : \rho(\lambda h)<+\infty \text{ for all } \lambda>0 \right\}.$$

Recognize that if ρ is $N-$quasi semiconvex, then the space

$$\left\{ h\in X\left(S^2\right) : \rho(\lambda h)<+\infty \text{ for some } \lambda>0 \right\}$$

coincides with $L^\rho\left(S^2\right)$. The notions about modulars are introduced in [27] and developed in [6] (see also [23, 28]).

Bardaro and Mantellini [4] introduced some Korovkin theorems through the notions of modular convergence and strong convergence. Afterwards Karakuş et al. [20] investigated the modular Korovkin theorem via statistical convergence and then, Orhan and Demirci [30] extended these type of approximation for double sequences of positive linear operators on modular space. In [14], Demirci and Orhan presented the notion of statistical relative modular (or strong) convergence for double sequences.

Let's first express the concept of statistical convergence given for double sequences by Moricz in [27].

Let $A \subseteq \mathbb{N}^2$ be a two-dimensional subset of positive integers, then A_{ij} denotes the set $\{(m,n) \in A : m \leq i,\ n \leq j\}$ and $|A_{ij}|$ denotes the cardinality of A_{ij}. The double natural density of A is given by

$$\delta_2(A) := P - \lim_{i,j} \frac{1}{ij} |A_{ij}|,$$

if it exists. The number sequence $x = \{x_{i,j}\}$ is said to be statistically convergent to l provided that for every $\varepsilon > 0$, the set

$$A_{m,n}(\varepsilon) := \{m \leq i,\ n \leq j : |x_{i,j} - l| \geq \varepsilon\}$$

has natural density zero; in that case, we write $st_2 - \lim\limits_{i,j} x_{i,j} = l$ (see [27]).

Now we recall the above mentioned convergence methods on modular spaces:

Definition 1.1. [14] Let $\{h_{i,j}\}$ be a double function sequence whose terms belong to $L^\rho(S^2)$. Then, $\{h_{i,j}\}$ is statistically relatively modularly convergent to a function $h \in L^\rho(S^2)$ if there exists a function σ, called a scale function $\sigma \in X(S^2)$, $|\sigma(s,t)| \neq 0$ such that

$$st_2 - \lim_{i,j} \rho\left(\lambda_0\left(\frac{h_{i,j} - h}{\sigma}\right)\right) = 0, \quad \text{for some } \lambda_0 > 0. \tag{1.1}$$

Also, $\{h_{i,j}\}$ is statistically relatively $F-$norm convergent (or, statistically relatively strongly convergent) to h if

$$st_2 - \lim_{i,j} \rho\left(\lambda\left(\frac{h_{i,j} - h}{\sigma}\right)\right) = 0, \quad \text{for every } \lambda > 0. \tag{1.2}$$

It is known from [14] that (1.1) and (1.2) are equivalent if and only if the modular ρ satisfies the Δ_2-condition, i.e., there exists a constant $M > 0$ such that $\rho(2h) \leq M\rho(h)$ for every $h \in X(S^2)$.

2. Triangular ideal relative modular convergence

In this section, we introduce the notion of the triangular ideal relative modular (or strong) convergence for double sequences. Let us first recall the notion of ideal convergence and some of its main features that are required for this article.

If K is a non-empty set, a class I of subsets of K is called an ideal in K if

i) $\varnothing \in I$,

ii) $A, B \in I$ implies $A \cup B \in I$,

iii) for each $A \in I$ and $B \subset A$ we have $B \in I$ ([22]).

The ideal I is called non-trivial if $I \neq \{\varnothing\}$ and $K \notin I$. A non-trivial ideal I is called admissible if $\{x\} \in I$ for each $x \in K$.

A sequence $\{x_i\}$ is said to $I-$convergent to l if for any $\varepsilon > 0$,

$$A(\varepsilon) = \{i \in \mathbb{N} : \ |x_i - l| \geq \varepsilon\} \in I.$$

We write $I - \lim_i x_i = l$ ([22]).

Now, we introduce the following ideal type convergence.

Definition 2.1. The double sequence $x = \{x_{i,j}\}$ is triangular ideal convergent to l provided that for every $\varepsilon > 0$ the set

$$B_i(\varepsilon) := \{j \in \mathbb{N} : \ j \leq i, \ |x_{i,j} - l| \geq \varepsilon\} \in I.$$

We set $I^T - \lim_i x_{i,j} = l$.

It is worthwhile to point out that, the triangular density defined $\mathbb{N}$ [1] as follows.

Let $B \subset \mathbb{N}^2$ be a nonempty set, and for every $i \in \mathbb{N}$, let $B_i = \{j \in \mathbb{N} : j \leq i\}$. Let $|B_i|$ be the cardinality of B_i. The triangular density of B is defined by

$$\delta^T(B) = \lim_i \frac{1}{i} |B_i|$$

provided that the limit on the right-hand side exists in $\mathbb{R}$.

Let $I_\delta^T = \{B : \ \delta^T(B) = 0\}$. I_δ^T is a non-trivial admissible ideal in $\mathbb{N}$ then I_δ^T-convergence coincides with the triangular statistical convergence in [1], [2]. Also, it is clear that $I_\delta^T \subset I$.

Similar to [10], the triangular ideal limit superior and inferior can be define. Given a double sequence $x = \{x_{i,j}\}$, put

$$A_x := \{a \in \mathbb{R} : \{j \in \mathbb{N} : j \leq i, \ x_{i,j} < a\} \notin I\},$$
$$C_x := \{c \in \mathbb{R} : \{j \in \mathbb{N} : j \leq i, \ x_{i,j} > c\} \notin I\},$$

and define

$$I^T - \limsup_i \ x_{i,j} = \begin{cases} \sup C_x, & \text{if } C_x \neq \varnothing, \\ -\infty, & \text{if } C_x = \varnothing, \end{cases}$$
$$I^T - \liminf_i \ x_{i,j} = \begin{cases} \inf A_x, & \text{if } A_x \neq \varnothing, \\ +\infty, & \text{if } A_x = \varnothing. \end{cases}$$

We also have the following theorem from [10]:

Theorem 2.2. *i)If $\beta = I^T - \limsup_i x_{i,j}$ is finite, then for every positive number ε*

$$\{j : \ j \leq i, \ x_{i,j} > \beta - \varepsilon\} \notin I \text{ and } \{j : \ j \leq i, \ x_{i,j} > \beta + \varepsilon\} \in I. \tag{2.1}$$

Conversely, if (2.1) holds for every positive ε, then $\beta = I^T - \limsup_i x_{i,j}$.

ii) If $\alpha = I^T - \liminf_i x_{i,j}$ is finite, then for every positive number ε

$$\{j : \ j \leq i, \ x_{i,j} < \alpha + \varepsilon\} \notin I \text{ and } \{j : \ j \leq i, \ x_{i,j} < \alpha - \varepsilon\} \in I. \tag{2.2}$$

Conversely, if (2.2) holds for every positive ε, then $\alpha = I^T - \liminf_i x_{i,j}$.

We can now introduce our new convergence methods:

Definition 2.3. Let $\{h_{i,j}\}$ be a double function sequence whose terms belong to $L^{\rho}\left(S^2\right)$. Then, $\{h_{i,j}\}$ is triangular ideal relatively modularly convergent to a function $h \in L^{\rho}\left(S^2\right)$ if there exists a scale function σ such that

$$I^T - \lim_i \rho\left(\lambda_0\left(\frac{h_{i,j}-h}{\sigma}\right)\right) = 0, \text{ for some } \lambda_0 > 0. \tag{2.3}$$

And also, $\{h_{i,j}\}$ is triangular ideal relatively modularly strongly convergent (or, triangular ideal relatively $F-$ norm convergent) to a function $h \in L^{\rho}\left(S^2\right)$ if

$$I^T - \lim_i \rho\left(\lambda\left(\frac{h_{i,j}-h}{\sigma}\right)\right) = 0, \text{ for every } \lambda > 0. \tag{2.4}$$

It is worthwhile to point out that (2.3) and (2.4) are equivalent if and only if the modular ρ satisfies the Δ_2-condition.

Below we present an interesting example of a double sequence which is triangular ideal relatively modularly convergent but not triangular statistically modularly convergent.

Example 2.4. Take $S = [0,1]$ and let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous function for which the following conditions hold:

- *φ is convex,*
- *$\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim\limits_{u\to\infty} \varphi(u) = \infty$.*

Let be functional ρ^{φ} on $X(S^2)$ defined by

$$\rho^{\varphi}(h) := \int_0^1 \int_0^1 \varphi\left(|h(s,t)|\right) dsdt \text{ for } \ h \in X\left(S^2\right). \tag{2.5}$$

Then, ρ^{φ} is a convex modular on $X\left(S^2\right)$, which satisfies all the assumptions stated previous section. Let us consider the Orlicz space generated by φ as follows:

$$L^{\rho}_{\varphi}(S^2) := \left\{h \in X\left(S^2\right) : \rho^{\varphi}\left(\lambda h\right) < +\infty \quad \text{for some } \lambda > 0\right\}.$$

Let $I = I^T_{\delta}$ and $B := \{(i,j) : \ j \leq i\}$ be a infinite set. For each $(i,j) \in \mathbb{N}^2$ define $g_{i,j} : [0,1] \times [0,1] \to \mathbb{R}$ by

$$g_{i,j}(s,t) = \begin{cases} 1, & i \text{ and } j \text{ are square}, \\ i^3 j^3 st, & \begin{array}{c}(i,j) \in B, \ i \text{ and } j \text{ are not square}, \\ (s,t) \in \left(0, \frac{1}{i}\right) \times \left(0, \frac{1}{j}\right)\end{array} \\ 0, & \text{otherwise}. \end{cases} \tag{2.6}$$

If $\varphi(x) = x^p$ for $1 \leq p < \infty$, $x \geq 0$, then $L^{\rho}_{\varphi}\left(S^2\right) = L_p\left(S^2\right)$. Moreover, we have for any function $h \in L^{\rho}_{\varphi}\left(S^2\right)$

$$\rho^{\varphi}\left(h\right) = \|h\|^p_{L_p}.$$

We can verify that $\{g_{i,j}\}$ does not converge triangular statistically modularly however converges to $g = 0$ triangular statistically modularly relatively to the scale function

$$\sigma(s,t) = \begin{cases} \frac{1}{s^2t^2}, & \text{if } \ (s,t) \in (0,1] \times (0,1], \\ 1, & \text{otherwise}, \end{cases}$$

on $L_1(S)$. Indeed, for some $\lambda_0 > 0$, when we take $p = 1$, we have $\rho^{\varphi}(.) = \|.\|_{L_1}$,

$$\begin{aligned} \rho\left(\lambda_0\left(g_{i,j}-g\right)\right) &= \left\|\lambda_0\left(g_{i,j}-g\right)\right\|_{L_1} \\ &= \lambda_0 \begin{cases} 1, & i \text{ and } j \text{ are square,} \\ \frac{ij}{4}, & (i,j) \in B\ i \text{ and } j \text{ are not square,} \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.7}$$

For every $\varepsilon \in \left(0, \frac{1}{9}\right]$, we have

$$\lim_i \frac{1}{i} \left|\left\{j \in \mathbb{N}: \ j \le i,\ \rho\left(\lambda_0\left(g_{i,j}-g\right)\right) \ge \varepsilon\right\}\right| = 1$$

Clearly, $\left\{j \in \mathbb{N}: \ j \le i,\ \rho\left(\lambda_0\left(g_{i,j}-g\right)\right) \ge \varepsilon\right\} \notin I_\delta^T$. So, $\{g_{i,j}\}$ does not converge triangular statistically modularly to $g = 0$ (see details, [2]). Using the scale function σ,

$$\rho\left(\lambda_0\left(\frac{g_{i,j}-g}{\sigma}\right)\right) = \lambda_0 \begin{cases} \frac{1}{9}, & i \text{ and } j \text{ are square,} \\ \frac{1}{16ij}, & (i,j) \in B\ i \text{ and } j \text{ are not square,} \\ 0, & \text{otherwise.} \end{cases}$$

for every $\varepsilon \in \left(0, \frac{1}{9}\right]$, and since

$$\lim_i \frac{1}{i} \left|\left\{j \in \mathbb{N}: \ j \le i,\ \rho\left(\lambda_0\left(\frac{g_{i,j}-g}{\sigma}\right)\right) \ge \varepsilon\right\}\right| = 0,$$

then we get,

$$I_\delta^T - \lim_i \rho\left(\lambda_0\left(\frac{g_{i,j}-g}{\sigma}\right)\right) = 0.$$

Prior to expressing the next theorem, we will need below assumptions on a modular ρ :

(a) ρ is monotone, i.e., $\rho(h) \le \rho(g)$ whenever $|h(s,t)| \le |g(s,t)|$ for any $(s,t) \in S^2$ and $h, g \in X\left(S^2\right)$. Further, ρ is finite if the characteristic function $\chi_B \in L^\rho\left(S^2\right)$ whenever B is measurable subset of S^2.

(b) ρ is absolutely finite i.e., ρ is finite and for every $\varepsilon > 0$, $\lambda > 0$, there exists a $\delta > 0$ such that $\rho\left(\lambda\chi_B\right) < \varepsilon$ for any measurable subset $B \subset S^2$ with $\mu(B) < \delta$. Also, we say that ρ is strongly finite, i.e., $\chi_{S^2} \in E^\rho\left(S^2\right)$.

(c) ρ is absolutely continuous, i.e. there exists $\alpha > 0$ such that for every h in $X\left(S^2\right)$, with $\rho(h) < +\infty$, the following condition holds: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\rho\left(\alpha h \chi_B\right) < \varepsilon$ for any measurable subset $B \subset S^2$ with $\mu(B) < \delta$.

As usual, let $C\left(S^2\right)$ be the space of all continuous real-valued functions, and $C^\infty\left(S^2\right)$ be the space of all infinitely differentiable functions on S^2. Based upon the above concepts (see [4, 5]) if a modular ρ is monotone and finite, then we have $C(S^2) \subset L^\rho\left(S^2\right)$. Similarly, if ρ is monotone and strongly finite, then $C(S^2) \subset E^\rho\left(S^2\right)$. Also, if ρ is monotone, absolutely finite and absolutely continuous, then $\overline{C^\infty\left(S^2\right)} = L^\rho\left(S^2\right)$. (For more details see [3, 6, 24, 28]).

Here and in the sequel, we use I as a non-trivial admissible ideal on $\mathbb{N}$.

3. Korovkin theorems

In this section, we apply our definition of triangular ideal relative modular convergence for double sequences of positive linear operators to prove the Korovkin type approximation theorems.

Let ρ be a monotone and finite modular on $X\left(S^2\right)$. Assume that D is a set satisfying $C^{\infty}\left(S^2\right) \subset D \subset L^{\rho}\left(S^2\right)$. Assume further that $\mathbb{L} := \{L_{i,j}\}$ is a sequence of positive linear operators from D into $X\left(S^2\right)$ for which there exists a subset $X_{\mathbb{L}} \subset D$ containing $C^{\infty}\left(S^2\right)$ and $\sigma \in X\left(S^2\right)$ is an unbounded function satisfying $|\sigma(s,t)| \neq 0$ such that

$$I^T - \limsup_i \rho\left(\lambda\left(\frac{L_{i,j}(h)}{\sigma}\right)\right) \leq R\rho(\lambda h) \tag{3.1}$$

holds for every $h \in X_{\mathbb{L}}$, $\lambda > 0$ and for an absolutely positive constant R.
Let $\mathbb{L}$ be a linear operator from $C\left(S^2\right)$ into itself . It is called positive, if $L_{i,j}(h) \geq 0$, for all $h \geq 0$. Also, we denote the value of $L_{i,j}(h)$ at a point $(s,t) \in S^2$ by $L_{i,j}(h; s,t)$.

Now we have the following Korovkin theorem for triangular ideal relative modular convergence that is our main theorem.

Theorem 3.1. *Let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semiconvex modular on $X\left(S^2\right)$. Let $\mathbb{L} := \{L_{i,j}\}$ be a double sequence of positive linear operators from D into $X\left(S^2\right)$ satisfying (3.1) and suppose that σ_r is an unbounded function satisfying $|\sigma_r(s,t)| \geq \alpha_r > 0$ $(r = 0,1,2,3)$. Assume that*

$$I^T - \lim_i \rho\left(\lambda\left(\frac{L_{i,j}(e_r) - e_r}{\sigma_r}\right)\right) = 0, \text{ for every } \lambda > 0 \text{ and } r = 0,1,2,3, \tag{3.2}$$

where $e_0(s,t) = 1$, $e_1(s,t) = s$, $e_2(s,t) = t$, $e_3(s,t) = s^2 + t^2$. Now let h be any function belonging to $L^{\rho}\left(S^2\right)$ such that $h - g \in X_{\mathbb{L}}$ for every $g \in C^{\infty}\left(S^2\right)$. Then, we have

$$I^T - \lim_i \rho\left(\lambda_0\left(\frac{L_{i,j}(h) - h}{\sigma}\right)\right) = 0, \text{ for some } \lambda_0 > 0 \tag{3.3}$$

where $\sigma(s,t) = \max\{|\sigma_r(s,t)| : r = 0,1,2,3\}$.

Proof. We first claim that

$$I^T - \lim_i \rho\left(\eta\left(\frac{L_{i,j}(g) - g}{\sigma}\right)\right) = 0 \text{ for every } g \in C\left(S^2\right) \cap D \text{ and every } \eta > 0. \tag{3.4}$$

To see this, assume that g belongs to $g \in C\left(S^2\right) \cap D$. By the continuity of g on S^2, given $\varepsilon > 0$, there exists a number $\delta > 0$ such that for all $(u,v), (s,t) \in S^2$ satisfying $|u - s| < \delta$ and $|v - t| < \delta$ we have

$$|g(u,v) - g(s,t)| < \varepsilon. \tag{3.5}$$

Also we obtain for all $(u,v), (s,t) \in S^2$ satisfying $|u - s| > \delta$ and $|v - t| > \delta$ that

$$|g(u,v) - g(s,t)| \leq \frac{2M}{\delta^2}\left\{(u-s)^2 + (v-t)^2\right\} \tag{3.6}$$

where $M := \sup_{(s,t)\in S^2} |g(s,t)|$. Combining (3.5) and (3.6) we have for $(u,v), (s,t) \in S^2$ that

$$|g(u,v) - g(s,t)| < \varepsilon + \frac{2M}{\delta^2}\left\{(u-s)^2 + (v-t)^2\right\}.$$

Namely,

$$\begin{aligned} & -\varepsilon - \frac{2M}{\delta^2}\left\{(u-s)^2 + (v-t)^2\right\} \\ < \;& g(u,v) - g(s,t) < \varepsilon + \frac{2M}{\delta^2}\left\{(u-s)^2 + (v-t)^2\right\}. \end{aligned} \tag{3.7}$$

Since $L_{i,j}$ is linear and positive, by applying $L_{i,j}$ to (3.7) for every $i,j \in \mathbb{N}$ we get

$$\begin{aligned} & -\varepsilon L_{i,j}(e_0; s,t) - \frac{2M}{\delta^2} L_{i,j}\left((u-s)^2 + (v-t)^2; s,t\right) \\ < \;& L_{i,j}(g; s,t) - g(s,t) L_{i,j}(e_0; s,t) \\ < \;& \varepsilon L_{i,j}(e_0; s,t) + \frac{2M}{\delta^2} L_{i,j}\left((u-s)^2 + (v-t)^2; s,t\right) \end{aligned}$$

and hence,

$$\begin{aligned} |L_{i,j}(g; s,t) - g(s,t)| \le \;& |L_{i,j}(g; s,t) - g(s,t) L_{i,j}(e_0; s,t)| \\ & + |g(s,t) L_{i,j}(e_0; s,t) - g(s,t)| \\ \le \;& \varepsilon L_{i,j}(e_0; s,t) + M |L_{i,j}(e_0; s,t) - (e_0; s,t)| \\ & + \frac{2M}{\delta^2} L_{i,j}\left((u-s)^2 + (v-t)^2; s,t\right) \end{aligned}$$

holds for every $s,t \in S$ and $i,j \in \mathbb{N}$. The above inequality implies that

$$\begin{aligned} |L_{i,j}(g; s,t) - g(s,t)| \le \;& \varepsilon + \left\{\varepsilon + M + \frac{4M}{\delta^2} E^2\right\} |L_{i,j}(e_0; s,t) - (e_0; s,t)| \\ & + \frac{4M}{\delta^2} E |L_{i,j}(e_1; s,t) - (e_1; s,t)| \\ & + \frac{4M}{\delta^2} E |L_{i,j}(e_2; s,t) - (e_2; s,t)| \\ & + \frac{2M}{\delta^2} E |L_{i,j}(e_3; s,t) - (e_3; s,t)| \end{aligned}$$

where $E := \max\{|t| : \; t \in S\}$. Now, we multiply the both-sides of the above inequality by $\frac{1}{|\sigma(s,t)|}$ and for every $\eta > 0$, the last inequality gives that:

$$\begin{aligned} \eta \left|\frac{L_{i,j}(g; s,t) - g(s,t)}{\sigma(s,t)}\right| \le \;& \frac{\eta\varepsilon}{|\sigma(s,t)|} + K\eta \left\{ \left|\frac{L_{i,j}(e_0; s,t) - (e_0; s,t)}{\sigma(s,t)}\right| \right. \\ & + \left|\frac{L_{i,j}(e_1; s,t) - (e_1; s,t)}{\sigma(s,t)}\right| \\ & + \left|\frac{L_{i,j}(e_2; s,t) - (e_2; s,t)}{\sigma(s,t)}\right| \\ & \left. + \left|\frac{L_{i,j}(e_3; s,t) - (e_3; s,t)}{\sigma(s,t)}\right| \right\}, \end{aligned}$$

where

$$K:=\max\left\{\varepsilon+M+\frac{4M}{\delta^2}E^2,\frac{4M}{\delta^2}E,\frac{2M}{\delta^2}\right\}.$$

Now, applying the modular ρ to both-sides of the above inequality, since ρ is monotone and

$$\sigma\left(s,t\right)=\max\left\{\left|\sigma_r\left(s,t\right)\right|\ ;r=0,1,2,3\right\},$$

we have

$$\begin{aligned}\rho\left(\eta\left(\frac{L_{i,j}\left(g\right)-g}{\sigma}\right)\right) &\leq \rho\left(\eta\frac{\varepsilon}{|\sigma|}+\eta K\left|\frac{L_{i,j}\left(e_0\right)-e_0}{\sigma_0}\right|+\eta K\left|\frac{L_{i,j}\left(e_1\right)-e_1}{\sigma_1}\right|\right.\\ &\quad\left.+\eta K\left|\frac{L_{i,j}\left(e_2\right)-e_2}{\sigma_2}\right|+\eta K\left|\frac{L_{i,j}\left(e_3\right)-e_3}{\sigma_3}\right|\right).\end{aligned}$$

Since ρ is a $N-$quasi semiconvex and strongly finite, also assuming $0<\varepsilon\leq 1$, we can write

$$\begin{aligned}\rho\left(\eta\left(\frac{L_{i,j}\left(g\right)-g}{\sigma}\right)\right) &\leq N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right)+\rho\left(5\eta K\left(\frac{L_{i,j}\left(e_0\right)-e_0}{\sigma_0}\right)\right)\\ &\quad+\rho\left(5\eta K\left(\frac{L_{i,j}\left(e_1\right)-e_1}{\sigma_1}\right)\right)\\ &\quad+\rho\left(5\eta K\left(\frac{L_{i,j}\left(e_2\right)-e_2}{\sigma_2}\right)\right)\\ &\quad+\rho\left(5\eta K\left(\frac{L_{i,j}\left(e_3\right)-e_3}{\sigma_3}\right)\right).\end{aligned}$$

For a given $t>0$, choose an $\varepsilon\in(0,1]$ such that $N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right)<t$. Let's define the following sets:

$$\begin{aligned}D_\eta &:= \left\{j\in\mathbb{N}\ :\ j\leq i,\ \rho\left(\eta\left(\frac{L_{i,j}\left(g\right)-g}{\sigma}\right)\right)>t\right\},\\ D_{\eta,r} &:= \left\{j\in\mathbb{N}\ :\ j\leq i,\ \rho\left(\eta\left(\frac{L_{i,j}\left(e_r\right)-e_r}{\sigma_r}\right)\right)>\frac{t-N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right)}{4}\right\},\end{aligned}$$

where $r=0,1,2,3$. It is a simple matter to see that $D_\eta\subset\bigcup_{r=0}^{3}D_{\eta,r}$. So, by (3.2) we have $D_{\eta,r}\in I$ for $r=0,1,2,3$. Hence, by definition of an ideal $\bigcup_{r=0}^{3}D_{\eta,r}\in I$, $D_\eta\in I$. So we get $I^T-\lim_i\rho\left(\eta\left(\frac{L_{i,j}(g)-g}{\sigma}\right)\right)=0$ which proves our claim (3.4). Obviously (3.4) also holds for every $g\in C^\infty\left(S^2\right)$. Let $h\in L^\rho\left(S^2\right)$ satisfying $h-g\in X_T$ for every $g\in C^\infty\left(S^2\right)$. Since $\mu\left(S^2\right)<\infty$ and ρ is strongly finite and absolutely continuous, we can see that ρ is also absolutely finite on $X\left(S^2\right)$. Using these properties of the modular ρ, it is known from [6, 24] that the space $C^\infty\left(S^2\right)$ is modular dense in $L^\rho\left(S^2\right)$, i.e., there exists a sequence $\{g_{i,j}\}\subset C^\infty\left(S^2\right)$ such that

$$P-\lim_{i,j}\rho\left(3\lambda_0^*\left(g_{i,j}-h\right)\right)=0\text{ for some }\lambda^*>0.$$

This means that, for every $\epsilon > 0$, there is positive number $k_0 = k_0(\varepsilon)$ such that

$$\rho\left(3\lambda_0^*\left(g_{i,j} - h\right)\right) < \varepsilon \text{ for every } i, j \geq k_0. \tag{3.8}$$

Otherwise, by the linearity and positivity of the operators $L_{i,j}$ we can write that

$$\begin{aligned} \lambda_0^*\left|L_{i,j}(h; s,t) - h(s,t)\right| &\leq \lambda_0^*\left|L_{i,j}\left(h - g_{k_0,k_0}; s,t\right)\right| \\ &\quad + \lambda_0^*\left|L_{i,j}\left(g_{k_0,k_0}; s,t\right) - g_{k_0,k_0}(s,t)\right| \\ &\quad + \lambda_0^*\left|g_{k_0,k_0}(s,t) - h(s,t)\right| \end{aligned}$$

holds for every $s, t \in S$ and $i, j \in \mathbb{N}$. Applying the modular ρ in the last enequality and using the monotonicity of ρ and moreover multiplying the both-sides of above inequality by $\frac{1}{|\sigma(s,t)|}$, the last inequality leads to

$$\begin{aligned} \rho\left(\lambda_0^*\left(\frac{L_{i,j}(h) - h}{\sigma}\right)\right) &\leq \rho\left(3\lambda_0^*\frac{L_{i,j}\left(h - g_{k_0,k_0}\right)}{\sigma}\right) \\ &\quad + \rho\left(3\lambda_0^*\left(\frac{L_{i,j}\left(g_{k_0,k_0}\right) - g_{k_0,k_0}}{\sigma}\right)\right) \\ &\quad + \rho\left(3\lambda_0^*\left(\frac{g_{k_0,k_0} - h}{\sigma}\right)\right). \end{aligned}$$

Hence, observing that $|\sigma| \geq \alpha > 0$ ($\alpha = \max\{\alpha_r : r = 0,1,2,3\}$) we can write

$$\begin{aligned} \rho\left(\lambda_0^*\left(\frac{L_{i,j}(h) - h}{\sigma}\right)\right) &\leq \rho\left(3\lambda_0^*\frac{L_{i,j}\left(h - g_{k_0,k_0}\right)}{\sigma}\right) \\ &\quad + \rho\left(3\lambda_0^*\left(\frac{L_{i,j}\left(g_{k_0,k_0}\right) - g_{k_0,k_0}}{\sigma}\right)\right) \\ &\quad + \rho\left(\frac{3\lambda_0^*}{\alpha}\left(g_{k_0,k_0} - h\right)\right). \end{aligned} \tag{3.9}$$

Then, it follows from (3.8) and (3.9) that

$$\begin{aligned} \rho\left(\lambda_0^*\left(\frac{L_{i,j}(h) - h}{\sigma}\right)\right) &\leq \varepsilon + \rho\left(3\lambda_0^*\frac{L_{i,j}\left(h - g_{k_0,k_0}\right)}{\sigma}\right) \\ &\quad + \rho\left(3\lambda_0^*\left(\frac{L_{i,j}\left(g_{k_0,k_0}\right) - g_{k_0,k_0}}{\sigma}\right)\right). \end{aligned} \tag{3.10}$$

So, taking triangular ideal limit superior as $i \to \infty$ in the both-sides of (3.10) and also using the facts that $g_{k_0,k_0} \in C^{\infty}\left(S^2\right)$ and $h - g_{k_0,k_0} \in X_T$, we get from (3.1) that

$$\begin{aligned} I^T - \limsup_i \rho\left(\lambda_0^*\left(\frac{L_{i,j}(h) - h}{\sigma}\right)\right) &\leq \varepsilon + R\rho\left(3\lambda_0^*\left(h - g_{k_0,k_0}\right)\right) \\ &\quad + I^T - \limsup_i \rho\left(3\lambda_0^*\left(\frac{L_{i,j}\left(g_{k_0,k_0}\right) - g_{k_0,k_0}}{\sigma}\right)\right) \end{aligned}$$

which gives

$$\begin{aligned} & I^T - \limsup_i \rho\left(\lambda_0^*\left(\frac{L_{i,j}(h) - h}{\sigma}\right)\right) \\ \leq\ & \varepsilon(R+1) + I^T - \limsup_i \rho\left(3\lambda_0^*\left(\frac{L_{i,j}(g_{k_0,k_0}) - g_{k_0,k_0}}{\sigma}\right)\right). \end{aligned} \tag{3.11}$$

By (3.4), we get

$$I^T - \limsup_i\ \rho\left(3\lambda_0^*\left(\frac{L_{i,j}(g_{k_0,k_0}) - g_{k_0,k_0}}{\sigma}\right)\right) = 0. \tag{3.12}$$

Combining (3.11) with (3.12), from Theorem 2.2 we conclude that

$$I^T - \limsup_i \rho\left(\lambda_0^*\left(\frac{L_{i,j}(h) - h}{\sigma}\right)\right) \leq \varepsilon(R+1).$$

Since $\varepsilon > 0$ is arbitrary, we find

$$I^T - \lim_i \rho\left(\lambda_0^*\left(\frac{L_{i,j}(h) - h}{\sigma}\right)\right) = 0.$$

Thus, the assertion follows. □

Now, we give an example that shows that our triangular ideal relative modular Korovkin theorem is stronger than the Korovkin theorem in [2].

Example 3.2. Take $S = [0,1]$ and $I = I_\delta^T$. Also, φ, σ, ρ^φ, $L_\varphi^\rho(S^2)$ and B be as in Example 2.4. Then consider the following bivariate Bernstein-Kantorovich operator $\mathbb{U} := \{U_{i,j}\}$ on the space $L_\varphi^\rho(S^2)$ which is defined by:

$$\begin{aligned} U_{i,j}(h;s,t) &= \sum_{m=0}^{i}\sum_{n=0}^{j} p_{m,n}^{(i,j)}(s,t)(i+1)(j+1) \\ &\quad \times \int_{m/(i+1)}^{(m+1)/(i+1)} \int_{n/(j+1)}^{(n+1)/(j+1)} h(t,s)\,dsdt \end{aligned} \tag{3.13}$$

for $s,t \in S$, where $p_{m,n}^{(i,j)}(s,t)$ defined by

$$p_{m,n}^{(i,j)}(s,t) = \binom{i}{m}\binom{j}{n} s^m t^n (1-s)^{i-m}(1-t)^{j-n}.$$

Also it is clear that,

$$\sum_{m=0}^{i}\sum_{n=0}^{j} p_{m,n}^{(i,j)}(s,t) = 1. \tag{3.14}$$

Observe that the operators $U_{i,j}$ maps $L_{\varphi}^{\rho}\left(S^{2}\right)$ into itself. In view of (3.14), as in the proof of Lemma 5.1 [4] and also similar to Example 1 [30], we can use the Jensen inequality in order to obtain that for every $h\in L_{\varphi}^{\rho}\left(S^{2}\right)$ and $i,j\in\mathbb{N}$ there is an absolute constant $M>0$ such that

$$\rho^{\varphi}\left(U_{i,j}(h)\right)\leq M\rho^{\varphi}\left(h\right).$$

It is worthwhile to point out that, for any function $h\in L_{\varphi}^{\rho}\left(S^{2}\right)$ such that $h-g\in X_{\mathbb{L}}$ for every $g\in C^{\infty}\left(S^{2}\right)$, $\{U_{i,j}\}$ is modularly convergent to h. If $\varphi\left(x\right)=x^{p}$ for $1\leq p<\infty$, $x\geq 0$, then $L_{\varphi}^{\rho}\left(S^{2}\right)=L_{p}\left(S^{2}\right)$. Moreover we have $\rho^{\varphi}\left(.\right)=\left\Vert .\right\Vert_{L_{p}}^{p}$. For $p=1$, we have $\rho^{\varphi}\left(.\right)=\left\Vert .\right\Vert_{L_{1}}$. In what follows, using the operators $U_{i,j}$, we can obtain the sequence of positive operators $\mathbb{V}:=\{V_{i,j}\}$ on $L_{1}\left(S^{2}\right)$ as follows:

$$\begin{aligned} V_{i,j}\left(h;s,t\right) &= \left(1+g_{i,j}\left(s,t\right)\right)U_{i,j}\left(h;s,t\right) \\ \text{for } h &\in L_{1}\left(S^{2}\right),\ \ (s,t)\in S^{2} \text{ and } i,j\in\mathbb{N} \end{aligned} \tag{3.15}$$

where $\{g_{i,j}\}$ is the same as in (2.6) and we choose $\sigma_{r}=\sigma$ $\left(r=0,1,2,3\right)$, where

$$\sigma\left(s,t\right)=\begin{cases}\frac{1}{s^{2}t^{2}}, & \text{if}\quad (s,t)\in(0,1]\times(0,1], \\ 1, & \text{otherwise.}\end{cases}$$

As in the proof of Lemma 5.1 [4] and similar to Example 1 [30], we get, for every $h\in L_{1}\left(S^{2}\right)$, $\lambda>0$ and for positive constant C, that

$$I_{\delta}^{T}-\limsup_{i}\left\Vert\lambda\left(\frac{V_{i,j}\left(h\right)}{\sigma}\right)\right\Vert_{L_{1}}\leq C\left\Vert\lambda h\right\Vert_{L_{1}}. \tag{3.16}$$

We now claim that

$$I_{\delta}^{T}-\lim_{i}\left\Vert\lambda\left(\frac{V_{i,j}\left(e_{r}\right)-e_{r}}{\sigma}\right)\right\Vert_{L_{1}}=0,\ \ r=0,1,2,3. \tag{3.17}$$

Indeed, first observe that,

$$\begin{aligned} V_{i,j}\left(e_{0};s,t\right) &= 1+g_{i,j}\left(s,t\right), \\ V_{i,j}\left(e_{1};s,t\right) &= \left(1+g_{i,j}\left(s,t\right)\right)\left(\frac{is}{i+1}+\frac{1}{2\left(i+1\right)}\right), \\ V_{i,j}\left(e_{2};s,t\right) &= \left(1+g_{i,j}\left(s,t\right)\right)\left(\frac{jt}{j+1}+\frac{1}{2\left(j+1\right)}\right), \\ V_{i,j}\left(e_{3};s,t\right) &= \left(1+g_{i,j}\left(s,t\right)\right)\left(\frac{i\left(i-1\right)s^{2}}{\left(i+1\right)^{2}}+\frac{2is}{\left(i+1\right)^{2}}+\frac{1}{3\left(i+1\right)^{2}}\right. \\ &\qquad \left.\frac{j\left(j-1\right)t^{2}}{\left(j+1\right)^{2}}+\frac{2jt}{\left(j+1\right)^{2}}+\frac{1}{3\left(j+1\right)^{2}}\right). \end{aligned}$$

We can easily calculate, for any $\lambda > 0$, that

$$\left\|\lambda\left(\frac{V_{i,j}(e_0)-e_0}{\sigma}\right)\right\|_{L_1} = \lambda\begin{cases}\frac{1}{9}, & \text{if } \ i \text{ and } j \text{ are square,}\\ \frac{1}{16ij}, & \text{if } \ (i,j)\in B \ i \text{ and } j \text{ are not square,}\\ 0, & \text{otherwise.}\end{cases} \tag{3.18}$$

Now, since

$$\lim_i \frac{1}{i}\left|\left\{j\in\mathbb{N}:\ j\le i,\ \left\|\lambda\left(\frac{V_{i,j}(e_0)-e_0}{\sigma}\right)\right\|_{L_1}\ge\varepsilon\right\}\right| = 0,$$

we get

$$I_\delta^T - \lim_i \left\|\lambda\left(\frac{V_{i,j}(e_0)-e_0}{\sigma}\right)\right\|_{L_1} = 0,$$

which guarantees that (3.17) holds true for $r = 0$.
Also, we have

$$\begin{aligned}\left\|\lambda\left(\frac{V_{i,j}(e_1)-e_1}{\sigma}\right)\right\|_{L_1} &= \lambda\int_0^1\int_0^1\left|\frac{V_{i,j}(e_1;s,t)-e_1(s,t)}{\sigma(s,t)}\right|dsdt\\ &\le \lambda\int_0^1\int_0^1\left|\frac{g_{i,j}(s,t)}{\sigma(s,t)}\left(\frac{is}{i+1}+\frac{1}{2(i+1)}\right)\right|dsdt\\ &+\lambda\int_0^1\int_0^1\left|\frac{s^2t^2-2s^3t^2}{2(i+1)}\right|dsdt\\ &< \left\|\lambda\frac{g_{i,j}}{\sigma}\right\|_{L_1}+\frac{\lambda}{36(i+1)},\end{aligned}$$

because of

$$\left\{j\in\mathbb{N}:\ j\le i,\ \left\|\lambda\frac{g_{i,j}}{\sigma}\right\|_{L_1}\ge\varepsilon\right\}\in I_\delta^T$$

and

$$\lim_i\frac{1}{i}\left|\left\{j\in\mathbb{N}:\ j\le i,\ \frac{\lambda}{36(i+1)}\ge\varepsilon\right\}\right| = 0,$$

we get

$$I_\delta^T - \lim_i \left\|\lambda\left(\frac{V_{i,j}(e_1)-e_1}{\sigma}\right)\right\|_{L_1} = 0.$$

Hence (3.17) is valid for $r = 1$. Similarly, we have

$$I_\delta^T - \lim_i \left\|\lambda\left(\frac{V_{i,j}(e_2)-e_2}{\sigma}\right)\right\|_{L_1} = 0.$$

Finally, since

$$\left\|\lambda\left(\frac{V_{i,j}(e_3)-e_3}{\sigma}\right)\right\|_{L_1}=\lambda\int_0^1\int_0^1\left|\frac{V_{i,j}(e_3;s,t)-e_3(s,t)}{\sigma(s,t)}\right|dsdt$$

$$\leq \lambda\int_0^1\int_0^1\left|\frac{g_{i,j}(s,t)}{\sigma(s,t)}\left(\frac{i(i-1)s^2}{(i+1)^2}+\frac{2is}{(i+1)^2}\right.\right.$$

$$\left.\left.+\frac{1}{3(i+1)^2}+\frac{j(j-1)t^2}{(j+1)^2}+\frac{2jt}{(j+1)^2}+\frac{1}{3(j+1)^2}\right)\right|dsdt$$

$$+\lambda\int_0^1\int_0^1\left|\frac{(3i+1)s^4t^2}{(i+1)^2}+\frac{(3j+1)s^2t^4}{(j+1)^2}+\frac{2is^3t^2}{(i+1)^2}+\frac{2js^3t^2}{(j+1)^2}\right.$$

$$\left.+s^2t^2\left(\frac{1}{3(i+1)^2}+\frac{1}{3(j+1)^2}\right)\right|dsdt$$

$$< 6\left\|\lambda\frac{g_{i,j}}{\sigma}\right\|_{L_1}+\frac{\lambda(3i+1)}{15(i+1)^2}+\frac{\lambda(3j+1)}{15(j+1)^2}+\frac{\lambda i}{6(i+1)^2}+\frac{\lambda j}{6(j+1)^2}$$

$$+\frac{\lambda}{9}\left(\frac{1}{3(i+1)^2}+\frac{1}{3(j+1)^2}\right),$$

then we have

$$I_\delta^T-\lim_i\left\|\lambda\left(\frac{V_{i,j}(e_3)-e_3}{\sigma}\right)\right\|_{L_1}=0.$$

So, our claim (3.17) is valid for each $i=0,1,2,3$ and for any $\lambda>0$. Also, from (3.16) and (3.17), we observe that our sequence $\mathbb{V}=\{V_{i,j}\}$ defined by (3.15) satisfies all assumptions of Theorem 3.1 and

$$I_\delta^T-\lim_i\left\|\lambda\left(\frac{V_{i,j}(h)-h}{\sigma}\right)\right\|_{L_1}=0,$$

holds for any $h\in L_1(S^2)$ such that $h-g\in X_T=L_1(S^2)$ for every $g\in C^\infty(S^2)$. However, in view of (2.7), since

$$\lim_i\frac{1}{i}\left\{j\in\mathbb{N}:\ j\leq i,\ \|\lambda(V_{i,j}(e_0)-e_0)\|_{L_1}\ \geq\varepsilon\right\}=1,$$

$(V_{i,j}(e_0)-e_0)$ does not triangular statistically modularly convergent. The Korovkin theorem in [2], does not work for the sequence $\mathbb{V}=\{V_{i,j}\}$.

As indicated earlier, if the modular ρ satisfies the Δ_2−condition then the space $C^\infty(S^2)$ is dense in $L^\rho(S^2)$ ([4]). Hence, we get the following result from Theorem 3.1.

Theorem 3.3. *Let $\mathbb{L}:=\{L_{i,j}\}$, ρ and σ be the same as in Theorem 3.1. If ρ satisfies the Δ_2-condition, then the following statements are equivalent:*

(a) $I^T-\lim_i\rho\left(\lambda\left(\frac{L_{i,j}(e_r)-e_r}{\sigma_r}\right)\right)=0$, *for every* $\lambda>0$, $r=0,1,2,3$,

(b) $I^T - \lim_i \rho\left(\lambda\left(\frac{L_{i,j}(h)-h}{\sigma}\right)\right) = 0$, *for every* $\lambda > 0$, *provided that* h *is any fuction belonging to* $L^\rho\left(S^2\right)$ *such that* $h - g \in X_{\mathbb{L}}$ *for every* $g \in C^\infty\left(S^2\right)$.

If one replaces the scale function by nonzero constant, then the condition (3.1) reduces to

$$I^T - \limsup_i \rho\left(\lambda\left(L_{i,j}\left(h\right)\right)\right) \leq R\rho\left(\lambda h\right) \tag{3.19}$$

for every $h \in X_{\mathbb{L}}$, $\lambda > 0$ and for an absolute positive constant R. In this case, the following results immediately follows from our Theorem 3.1 and Theorem 3.3.

Corollary 3.4. *Let* ρ *be a monotone, strongly finite, absolutely continuous and* $N-$*quasi semiconvex modular on* $X\left(S^2\right)$. *Let* $\mathbb{L} := \{L_{i,j}\}$ *be a double sequence of positive linear operators from* D *into* $X\left(S^2\right)$ *satisfying (3.19). If* $\{L_{i,j}\left(e_r\right)\}$ *is triangular ideal strongly convergent to* e_r *for each* $r = 0, 1, 2, 3$, *then* $\{L_{i,j}h\}$ *triangular ideal modularly convergent to* h *provided that* h *is any function belonging to* $L^\rho\left(S^2\right)$ *such that* $h - g \in X_{\mathbb{L}}$ *for every* $g \in C^\infty\left(S^2\right)$.

Corollary 3.5. $\mathbb{L} := \{L_{i,j}\}$ *and* ρ *be the same as in Corollary 3.4. If* ρ *satisfies the* Δ_2-*condition, then the following statements are equivalent:*

(a) $\{L_{i,j}\left(e_r\right)\}$ *is triangular ideal strongly convergent to* e_r *for each* $r = 0, 1, 2, 3$,

(b) $\{L_{i,j}\left(h\right)\}$ *is triangular ideal strongly convergent to* h *provided that* h *is any fuction belonging to* $L^\rho\left(S^2\right)$ *such that* $h - g \in X_{\mathbb{L}}$ *for every* $g \in C^\infty\left(S^2\right)$.

If we take $I = I_\delta^T$, then the condition (3.1) reduces to

$$st^T - \limsup_i \rho\left(\lambda\left(\frac{L_{i,j}\left(h\right)}{\sigma}\right)\right) \leq R\rho\left(\lambda h\right) \tag{3.20}$$

for every $h \in X_{\mathbb{L}}$, $\lambda > 0$ and for an absolute positive constant R. In this case the following results immediately follows from our Theorem 3.1 and Theorem 3.3.

Corollary 3.6. *Let* ρ *be a monotone, strongly finite, absolutely continuous and* $N-$*quasi semiconvex modular on* $X\left(S^2\right)$. *Let* $\mathbb{L} := \{L_{i,j}\}$ *be a double sequence of positive linear operators from* D *into* $X\left(S^2\right)$ *satisfying (3.20). Moreover suppose that* σ_r *is an unbounded function satisfying* $\left|\sigma_r\left(s,t\right)\right| \geq \alpha_r > 0$ $\left(r = 0, 1, 2, 3\right)$. *If* $\{L_{i,j}\left(e_r\right)\}$ *is triangular statistically relatively strongly convergent to* e_r *for each* $r = 0, 1, 2, 3$, *then* $\{L_{i,j}\left(h\right)\}$ *triangular statistically relatively modularly convergent to* h *provided that* h *is any function belonging to* $L^\rho\left(S^2\right)$ *such that* $h - g \in X_{\mathbb{L}}$ *for every* $g \in C^\infty\left(S^2\right)$.

Corollary 3.7. $\mathbb{L} := \{L_{i,j}\}$, ρ *and* σ_r $\left(r = 0, 1, 2, 3\right)$ *be the same as in Corollary 3.6. If* ρ *satisfies the* Δ_2-*condition, then the following statements are equivalent:*

(a) $\{L_{i,j}\left(e_r\right)\}$ *is triangular statistically relatively strongly convergent to* e_r *for each* $r = 0, 1, 2, 3$,

(b) $\{L_{i,j}\left(h\right)\}$ *is triangular statistically relatively strongly convergent to* h *provided that* h *is any fuction belonging to* $L^\rho\left(S^2\right)$ *such that* $h - g \in X_{\mathbb{L}}$ *for every* $g \in C^\infty\left(S^2\right)$.

4. Concluding remarks

Now, we give some reduced results showing the importance of Theorem 3.1 and Theorem 3.3 in approximation theory with special choices:

1. If we take $I = I_{\delta}^{T}$ and the scale function is a non-zero constant, triangular ideal relative modular convergence given in the Definition 2.1 reduces to the triangular statistical modular convergence form in [2]. So, from Theorem 3.1 and Theorem 3.3 we immediately get the triangular statistical modular Korovkin theorems for double sequences in [2].

2. As it is well known, if $(X, \|.\|)$ is a normed space, then $\rho(.) = \|.\|$ is a convex modular in X. So, by choosing $\rho(.) = \|.\|$, then from Theorem 3.1 and Theorem 3.3, the followings are obtained on normed spaces:

i) We get the triangular ideal relative convergence for double sequences on normed spaces by choosing $\rho(.) = \|.\|$.

ii) If we take $I = I_{\delta}^{T}$, then we immediately get the triangular statistical relative convergence for double sequences on normed spaces and in addition, we immediately get the triangular statistical relative Korovkin theorems for double sequences on normed spaces in [9].

iii) If we take $I = I_{\delta}^{T}$ and the scale function is a non-zero constant, then we get triangular statistical convergence for double sequences on normed spaces and in addition, we immediately get the triangular statistical Korovkin theorems for double sequences on normed spaces in [1].

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Selin Çınar
Sinop University, Faculty of Sciences and Arts,
Department of Mathematics,
Sinop, Turkey
e-mail: `scinar@sinop.edu.tr`

Sevda Yıldız
Sinop University, Faculty of Sciences and Arts,
Department of Mathematics,
Sinop, Turkey
e-mail: `sevdaorhan@sinop.edu.tr`

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Coincidence point theorems in some generalized metric spaces

Alexandru-Darius Filip

Dedicated to Prof. Adrian Petruşel on the occasion of his 60^{th} *anniversary*

Abstract. Let (X,d) be a complete dislocated metric space, (Y,ρ) be a semimetric space and $f,g:X\to Y$ be two mappings. We give some metric conditions which imply that the coincidence point set,

$$C(f,g):=\{x\in X\mid f(x)=g(x)\}\neq\varnothing.$$

Several coincidence point results are obtained for singlevalued and multivalued mappings.

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Keywords: Dislocated metric space, semimetric space, singlevalued and multivalued mapping, comparison function, comparison pair, lower semi-continuity, coincidence point displacement functional, iterative approximation of coincidence point, weakly Picard mapping, pre-weakly Picard mapping.

1. Introduction and preliminaries

Let X be a nonempty set and $d:X\times X\to\mathbb{R}_+$ be a functional. Then the pair (X,d) is called (see [3], [5], [6], ...):

(i) semimetric space, if the following assumptions on d hold:
- (i_1) $d(x,y)=0\Leftrightarrow x=y$;
- (i_2) $d(x,y)=d(y,x)$, $\forall\ x,y\in X$.

(ii) dislocated metric space, if the following assumptions on d hold:
- (ii_1) $d(x,y)=d(y,x)=0\Rightarrow x=y$;
- (ii_2) $d(x,y)=d(y,x)$, $\forall\ x,y\in X$;

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(ii_3) $d(x,y) \leq d(x,z) + d(z,y)$, $\forall\ x,y,z \in X$.

Let (X,d) be a dislocated metric space. By definition (for the standard metric space, see [10]), a mapping $f : X \to X$ is a *pre-weakly Picard mapping* (*pre-WPM*) if the sequence of successive approximations $\{f^n(x)\}_{n\in\mathbb{N}}$ is a convergent sequence, for all $x \in X$.

If $f : X \to X$ is pre-WPM, then we consider the mapping $f^\infty : X \to X$, defined by $f^\infty(x) := \lim\limits_{n\to\infty} f^n(x)$.

By definition, if $f : X \to X$ is pre-WPM with

$$f^\infty(x) \in F_f := \{x \in X \mid f(x) = x\},\ \forall\ x \in X,$$

then f is a *weakly Picard mapping* (*WPM*).

In the paper [10] the author gives some coincidence point results in a metric space. The aim of our paper is to extend some of these results in the case of dislocated metric spaces.

Throughout the paper we shall use the notations and the terminology from [2], [6] and [11].

2. Main results

We start this section with the following notions given in [10].

Let $M \in]0,+\infty]$. A function $\varphi : [0,M[\to [0,M[$ is called *comparison function on* $[0,M[$ if φ is increasing on $[0,M[$ and $\varphi^n(t) \to 0$ as $n \to \infty$, $\forall\ t \in [0,M[$.

Let $\varphi : [0,M[\to [0,M[$ and $\psi : [0,M[\to \mathbb{R}_+$ be two functions. By definition, the pair (φ,ψ) is a *comparison pair on* $[0,M[$ if:

(1) φ is a comparison function on $[0,M[$;
(2) ψ is increasing, $\psi(0) = 0$ and ψ is continuous in 0;
(3) $\sum\limits_{i=0}^{\infty} \psi(\varphi^i(t)) < +\infty$, $\forall\ t \in [0,M[$.

Our main result is the following.

Theorem 2.1. *Let (X,d) be a complete dislocated metric space, (Y,ρ) be a semimetric space, $f,g : X \to Y$ be two mappings, $M \in]0,+\infty]$. We suppose that:*

(1) $X_M := \{x \in X \mid \rho(f(x),g(x)) < M\} \neq \varnothing$;
(2) *The coincidence point displacement functional,*
$$\rho_{f,g} : X_M \to \mathbb{R}_+,\ \rho_{f,g}(x) := \rho(f(x),g(x)), \forall\ x \in X_M,$$
is lower semi-continuous (l.s.c.) on X_M;
(3) *There exists a comparison pair, (φ,ψ), on $[0,M[$ with respect to which, for each $x \in X_M$, there exists $x_1 \in X_M$ such that:*
  (a) $\rho(f(x_1),g(x_1)) \leq \varphi(\rho(f(x),g(x)))$;
  (b) $d(x,x_1) \leq \psi(\rho(f(x),g(x)))$.

Then there exists a pre-WPM, $h : X_M \to X_M$ such that:

(i) $h^\infty(x) \in C(f,g)$, $\forall\ x \in X_M$;

(ii) $d(x, h^\infty(x)) \le \sum_{i=0}^{\infty} \psi(\varphi^i(\rho(f(x), g(x))))$, $\forall\, x \in X_M$.

Proof. From the assumption (3), we can define an operator $h : X_M \to X_M$, by $h(x) = x_1$ such that

$$\rho(f(h(x)), g(h(x))) \le \varphi(\rho(f(x), g(x))),\ \forall\, x \in X_M \tag{2.1}$$

and

$$d(x, h(x)) \le \psi(\rho(f(x), g(x))),\ \forall\, x \in X_M. \tag{2.2}$$

From (2.1), we have

$$\begin{aligned}
\rho(f(h(x)), g(h(x))) &\le \varphi(\rho(f(x), g(x)))\\
\rho(f(h^2(x)), g(h^2(x))) &\le \varphi(\rho(f(h(x)), g(h(x)))) \le \varphi^2(\rho(f(x), g(x)))\\
&\ \ \vdots\\
\rho(f(h^n(x)), g(h^n(x))) &\le \varphi(\rho(f(h^{n-1}(x)), g(h^{n-1}(x))))\\
&\le \varphi(\varphi(\rho(f(h^{n-2}(x)), g(h^{n-2}(x)))))\\
&\le \ldots \le \varphi^n(\rho(f(x), g(x))),\ \text{for all } n \in \mathbb{N}^*.
\end{aligned}$$

Notice that since φ is a comparison function, it follows that

$$\rho(f(h^n(x)), g(h^n(x))) \le \varphi^n(\rho(f(x), g(x))) \to 0 \text{ as } n \to \infty,\ \forall\, x \in X_M. \tag{2.3}$$

From (2.2), we have

$$\begin{aligned}
d(x, h(x)) &\le \psi(\rho(f(x), g(x)))\\
d(h(x), h^2(x)) &\le \psi(\rho(f(h(x)), g(h(x)))) \le \psi(\varphi(\rho(f(x), g(x))))\\
&\ \ \vdots\\
d(h^n(x), h^{n+1}(x)) &\le \psi(\rho(f(h^n(x)), g(h^n(x))))\\
&\le \psi(\varphi^n(\rho(f(x), g(x)))),\ \text{for all } n \in \mathbb{N}^*.
\end{aligned}$$

Since (φ, ψ) is a comparison pair on $[0, M[$, we have

$$\sum_{n\in\mathbb{N}} d(h^n(x), h^{n+1}(x)) \le \sum_{n\in\mathbb{N}} \psi(\varphi^n(\rho(f(x), g(x)))) < +\infty. \tag{2.4}$$

This implies that $\{h^n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete dislocated metric space, it follows that $\{h^n(x)\}_{n\in\mathbb{N}}$ is convergent in (X, d), for all $x \in X_M$. So, h is a pre-WPM. Thus, $h^\infty(x) := \lim_{n\to\infty} h^n(x)$.

On the other hand, from the assumption (2) and by (2.3), we have

$$0 \le \rho(f(h^\infty(x)), g(h^\infty(x))) \le \underline{\lim}_{n\to\infty} \rho(f(h^n(x)), g(h^n(x))) = 0.$$

Since ρ is a semimetric, we get $f(h^\infty(x)) = g(h^\infty(x))$, i.e., $h^\infty(x) \in C(f, g)$, $\forall\, x \in X_M$. Since d satisfies the triangle inequality and taking into account the first inequality of

(2.4), we have

$$d(x,h^\infty(x)) \le \sum_{i=0}^{\infty} d(h^i(x),h^{i+1}(x)) \le \sum_{i=0}^{\infty} \psi(\varphi^i(\rho(f(x),g(x)))),\ \forall\ x\in X_M. \qquad \square$$

Remark 2.2. In general, a semimetric is not continuous (see L.M. Blumenthal [3, p. 9]). That is why we have considered the assumption (2) in the above theorem. It would be of great interest to find conditions that imply the lower semi-continuity of the coincidence point displacement functional $\rho_{f,g}$.

Remark 2.3. In Theorem 2.1, if we consider $Y := X$, $g := 1_X$, $\varphi(t) := lt$, where $0 < l < 1$ and $\psi(t) := kt$, with $k > 0$, for all $t \in [0, M[$, we obtain the following result:

Theorem 2.4. *Let (X, d) be a complete dislocated metric space, ρ be a semimetric on X and $f : X \to X$ be a mapping. We suppose that:*

$(2')$ *The coincidence point displacement functional,*
$\rho_f : (X,d) \to \mathbb{R}_+$, $\rho_f(x) := \rho(x, f(x))$, $\forall\ x \in X$, *is l.s.c. on* X;

$(3')$ *There exists $0 < l < 1$ and $k > 0$ w.r.t. which, for each $x \in X$, there exists $x_1 \in X$ such that:*
- (a') $\rho(x_1, f(x_1)) \le l\rho(x, f(x))$;
- (b') $d(x, x_1) \le k\rho(x, f(x))$.

Then there exists a pre-WPM, $h : (X, d) \to (X, d)$ such that:

(i') $h^\infty(x) \in F_f$, $\forall\ x \in X$, *i.e.*, $F_f \ne \varnothing$;

(ii') $d(x, h^\infty(x)) \le \frac{k}{1-l}\rho(x, f(x))$, $\forall\ x \in X$.

Remark 2.5. In the context of Theorem 2.4, the triple $(X, \xrightarrow{d}, \rho)$ is a Kasahara space. Several results given in [5] can be proved using this theorem.

Remark 2.6. If in Theorem 2.4 we take, $\rho := d$ and f an l-graphic contraction, then we have:

Theorem 2.7. *Let (X, d) be a complete dislocated metric space and $f : X \to X$ be an l-graphic contraction. If the coincidence point displacement functional, $d_f : X \to \mathbb{R}_+$, $x \mapsto d(x, f(x))$ is l.s.c. on X, then f is a WPM and*

$$d(x, f^\infty(x)) \le \frac{1}{1-l} d(x, f(x)),\ \forall\ x \in X.$$

Proof. We apply Theorem 2.4, by considering $h(x) := f(x)$, for all $x \in X$. $\square$

Remark 2.8. If in Theorem 2.4 we take, $\rho := d$ and f an l-contraction, then we have the following variant of contraction principle:

Theorem 2.9. *Let (X, d) be a complete dislocated metric space and $f : X \to X$ be an l-contraction. Then we have that:*

(i) $F_f = F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}^*$;

(ii) $f^n(x) \to x^*$ *as* $n \to \infty$, $\forall\ x \in X$;

(iii) $d(x, x^*) \le \frac{1}{1-l} d(x, f(x))$, $\forall\ x \in X$.

Remark 2.10. For similar results given in a metric space, see: [4], [10], [11], [1], [7].

3. The case of multivalued mappings

Throughout this section we follow the notations and terminology given in [9] and [10]. We will use in our result the gap functional between two sets, recalled bellow.

Let (X, d) be a dislocated metric space.

The functional $D : P_{cl}(X) \times P_{cl}(X) \to \mathbb{R}_+ \cup \{+\infty\}$, defined by

$$D(A, B) := inf\{d(a, b) \mid a \in A,\ b \in B\}$$

for all $A, B \in P_{cl}(X)$, is called the gap functional between the sets A and B.

For this operator we have the following property:

If $A \in P_{cl}(X)$ and $x \in X$ then $D(x, A) = 0 \Leftrightarrow x \in A$.

The basic result of this section is the following:

Theorem 3.1. *Let (X, d) be a complete dislocated metric space, (Y, ρ) be a semimetric space, $T, S : X \to P_{cl}(Y)$ be two multivalued mappings, $M \in]0, +\infty]$, (φ, ψ) be a comparison pair on $[0, M[$. We suppose that:*

(1) $X_M := \{x \in X \mid D(T(x), S(x)) < M\} \neq \varnothing$;

(2) *The D-coincidence point displacement functional,* $D_{T,S} : X_M \to \mathbb{R}_+$,

$$D_{T,S}(x) := D(T(x), S(x)),\ \forall\ x \in X_M,$$

is l.s.c. on X_M*;*

(3) *For each $x \in X_M$ there exists $x_1 \in X_M$ such that:*

  (a) $D(T(x_1), S(x_1)) \leq \varphi(D(T(x), S(x)))$*;*

  (b) $d(x, x_1) \leq \psi(D(T(x), S(x)))$.

Then there exists a pre-WPM, $h : X_M \to X_M$ such that:

(i) $D(T(h^\infty(x)), S(h^\infty(x))) = 0,\ \forall\ x \in X_M$*;*

(ii) $d(x, h^\infty(x)) \leq \sum_{i=0}^{\infty} \psi(\varphi^i(D(T(x), S(x)))),\ \forall\ x \in X_M$*;*

(iii) *If in addition, for $A, B \in P_{cl}(Y)$, $D(A, B) = 0$ implies that:*

  (iii_1) $A \cap B \neq \varnothing$,

  then, $C(T, S) := \{x \in X \mid T(x) \cap S(x) \neq \varnothing\} \neq \varnothing$*;*

  (iii_2) $A = B$,

  then, $C(T, S) \neq \varnothing$ *and* $T(h^\infty(x)) = S(h^\infty(x)),\ \forall\ x \in X_M$*;*

  (iii_3) $A = B = \{y^*\}$,

  then $C(T, S) \neq \varnothing$ *and* $T(h^\infty(x)) = S(h^\infty(x)) = \{y^*_x\}$.

Proof. If we take, $h(x) := x_1$, then we have that:

$$D(T(h(x)), S(h(x))) \leq \varphi(D(T(x), S(x))),\ \forall\ x \in X_M,$$

and

$$d(x, h(x)) \leq \psi(D(T(x), S(x))),\ \forall\ x \in X_M.$$

These imply that,

$$D(T(h^n(x)), S(h^n(x))) \to 0 \text{ as } n \to \infty,$$

and h is a pre-WPM, and

$$d(x, h^\infty(x)) \le \sum_{i=0}^{\infty} \psi(\varphi^i(D(T(x), S(x)))),\ \forall\, x \in X_M.$$

Since, $D_{T,S}$ is l.s.c., it follows that,

$$\begin{aligned} 0 \le D(T(h^\infty(x)), S(h^\infty(x))) &\le \varliminf_{n\to\infty} D(T(h^n(x)), S(h^n(x))) \\ &= \lim_{n\to\infty} D(T(h^n(x)), S(h^n(x))) = 0. \end{aligned}$$

So, we have the conclusions (i), (ii) and (iii). □

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Alexandru-Darius Filip
Babeş-Bolyai University,
Faculty of Economics and Business Administration,
Department of Statistics-Forecasts-Mathematics,
Teodor Mihali Street, No. 58-60,
400591 Cluj-Napoca, Romania
e-mail: darius.filip@econ.ubbcluj.ro

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Multiplicity results for nonhomogenous elliptic equation involving the generalized Paneitz-Branson operator

Kamel Tahri

Dedicated to the Memory of the Professor **Tahar Mourid**

Abstract. Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ without boundary ∂M, we consider the multiplicity result of solutions of the following nonhomogenous fourth order elliptic equation involving the generalized Paneitz-Branson operator,

$$P_g(u) = f(x)\,|u|^{2^\sharp - 2}\,u + h(x).$$

Under some conditions and using critical points theory, we prove the existence of two distinct solutions of the above equation. At the end, we give a geometric example when the equation has negative and positive solutions.

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1. Introduction and statement of the main result

Let (M, g) be an $n\,(n \geq 5)$ $-$dimensional compact Riemannian manifold without boundary ∂M. In this decade, there has been extensive analyze of the relationship between the conformally covariant operators which satisfy some invariance properties under conformal change of metric on M and their associated partial differential equations. However, in 1983, Paneitz in [10] has introduced a conformally convariant differential operator on 4$-$dimensional Riemannian manifolds. Branson in [4] has generalized the definition to $n-$dimensional Riemannian manifolds.

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Moreover, for any Riemannian metric g on M, there exists a local differential operator called Paneitz-Branson operator defined by:

$$P_g^n : \quad C^\infty(M) \to C^\infty(M)$$

such that for all $u \in C^\infty(M)$:

$$P_g^n(u) := \Delta_g^2(u) + \operatorname{div}_g\left[\left(a_n S_g g - b_n Ric_g\right)^\sharp du\right] + \frac{(n-4)}{2} Q_g^n u,$$

where $\Delta_g := -\operatorname{div}_g(\nabla_g)$ is the Laplace-Beltrami operator and

$$a_n := \frac{(n-2)^2 + 4}{2(n-2)(n-1)}, b_n := \frac{4}{(n-2)},$$

the symbol stands for the musical isomorphism (index are raised with the metric), and

$$Q_g^n := \frac{2}{n-4} P_g^n(1).$$

This operator has a pertinent geometric behavior in the sense that: if $\tilde{g} := \varphi^{\frac{4}{n-4}} g$ is a conformal metric to g, then for all $\varphi \in C^\infty(M)$,

$$P_g^n(\varphi u) = \varphi^{\frac{n+4}{n-4}} . P_{\tilde{g}}^n(u).$$

Taking account $u = 1$, we find that

$$P_g^n(\varphi) = \frac{(n-4)}{2} Q_{\tilde{g}}^n \varphi^{2^\sharp - 1},$$

such that $2^\sharp = \frac{2n}{n-4}$. We are then naturally led to study extensions to the Paneitz-Branson operator with general coefficients as an operator of the form:

$$P_g(u) := \Delta_g^2(u) + \operatorname{div}_g\left(A^\sharp du\right) + Bu,$$

where $A \in \Lambda^\infty_{(2,0)}(M)$ a smooth symmetric $(2,0)$ $-$tensor field, and $B \in C^\infty(M)$.

In this paper, we consider the multiplicity results of solutions of the following nonhomogenous fourth order elliptic equation involving the generalized Paneitz-Branson operator:

$$P_g(u) = f(x)\,|u|^{2^\sharp - 2} u + h(x), \tag{1.1}$$

where f is a C^∞-function on M with $f > 0$ and h belongs to $L^m(M)$ such that

$$m := \frac{2^\sharp}{2^\sharp - 1} = \frac{2n}{n+4}.$$

The main goal of this paper is to establish the existence and multiplicity of solutions throughout the Ekeland's Variational Principle in [8] and the Mountain-Pass Theorem in [1] in the critical theory. This article is organized as follows: in Section 2, we present some essential mathematical materials. In section 3, we recall some auxiliary lemmas which are important for main theorem result. And in section 4, we give the proof of the main result and at the end, we give a geometric application on Einsteinian Riemannian compact manifold. We prove the following theorem:

Theorem 1.1. *Let (M, g) be an $n\,(n \geq 5)-$dimensional compact Riemannian manifold without boundary ∂M. Let f is a C^{∞}-function on M with $f > 0$ and $h \in L^{m}(M)$ such that $h \neq 0$ satisfying $\|h\|_{m} < m_{o}$ and supposing that the operator $P_g(u)$ is coercive. Then, the equation (1.1) has at least two nontrivial solutions $v, w \in H_2^2(M)$ satisfying:*

$$J(v) < 0 < J(w).$$

2. Preliminaries

We let $H_2^2\,(M)$ be the standard Sobolev space consisting of the functions in $L^2\,(M)$ whose derivatives up the second order are in $L^2\,(M)$. The Sobolev embedding theorem asserts that $H_2^2\,(M)$ is continuously embedded in $L^m\,(M)$ $1 < m \leq 2^{\sharp}$, with the property of this embedding is compact when $m < 2^{\sharp}$. We know from the work [9] that K_0 is the sharp and the best constant of the embedding $H_2^2\,(\mathbb{R}^n)$ in $L^{\frac{2n}{n-4}}\,(\mathbb{R}^n)$ by

$$K_0 := \frac{16}{n\,(n^2-4)\,(n-4)\,(w_n)^{\frac{4}{n}}},$$

where w_n is the volume of the unit $n-$sphere (S^n, h). Moreover, the Euclidian Sobolev embedding has obtained by the extremal functions

$$u_{\lambda}\,(x) := \eta \left(\frac{\lambda}{1 + \lambda^2 \left| x - x_o \right|^2} \right)^{\frac{n-4}{2}}$$

where $\lambda > 0, \eta \in \mathbb{R}^{*}$ and $x_o \in \mathbb{R}^n$.

3. Auxiliary and useful lemmas

Throughout this section, we consider the energy functional J, for each $u \in H_2^2\,(M)$,

$$J(u) = \frac{1}{2} \int_M P_g(u).ud\mu(g) - \int_M h(x).ud\mu(g) - \frac{1}{2_k^{\sharp}} \int_M f(x).\left| u \right|^{2_k^{\sharp}} d\mu(g)$$

Define:

$$\Phi(u) := \langle \nabla J(u), u \rangle$$

$$\Phi(u) = \int_M P_g(u).ud\mu(g) - \int_M h(x).ud\mu(g) - \int_M f(x).\left| u \right|^{2^{\sharp}} d\mu(g)$$

and

$$\langle \nabla \Phi(u), u \rangle = 2 \int_M P_g(u).ud\mu(g) - \int_M h(x).ud\mu(g) - 2^{\sharp} \int_M f(x).\left| u \right|^{2^{\sharp}} d\mu(g)$$

It is well known that the solutions of (1.1) can be seen as critical points of the functional $J(u)$. We assume in what follows that P_g is coercive, in the since that there exists $\Lambda > 0$ such that for all $u \in H_2^2\,(M)$:

$$\int_M P_g(u).ud\mu(g) \geq \Lambda \int_M u^2 d\mu(g).$$

Now, we use the following Sobolev inequalities proved in [7].

Lemma 3.1. *Let (M,g) be an $n\,(n\geq 5)-$dimensional compact Riemannian manifold without boundary ∂M. Then for any $\epsilon>0$, there exists $A_\epsilon\in\mathbb{R}$ such that for all $u\in H_2^2(M)$:*

$$\left(\int_M |u|^{2^\sharp}\,d\mu(g)\right)^{\frac{2}{2^\sharp}}\leq (K_0+\epsilon)\int_M\left[(\Delta_g u)^2+(\nabla_g u)^2\right]d\mu(g)+A_\epsilon\int_M u^2 d\mu(g).$$

The main tool to prove our result is the Montain-Pass Theorem of Ambrossetti-Rabinowitz given by the following theorem:

Theorem 3.2. *Let $J\in C^1\left(H_2^2(M);\mathbb{R}\right)$ satisfies $(P.S)_c$ condition. We suppose:*
(1). *There exist $\alpha>0$, $\rho>0$ such that*

$$J(u)\mid_{\partial B(0;\beta)}\geq J(0)+\alpha$$

Where

$$B_\rho=\left\{u\in H_2^2(M):\|u\|_{H_2^2(M)}\leq\rho\right\}$$

(2). *There is an $e\in H_2^2(M)$ and $\|\, e\,\|_{H_2^2(M)}>\rho$ such that:*

$$J(e)\leq J(0)$$

Then, $J(.)$ has a critical value c which can be characterized as

$$c:=\inf_{\gamma\in\Gamma}\max_{t\in[0;1]}J(\gamma(t))$$

Where

$$\Gamma:=\left\{\gamma\in C([0;1];H_2^2(M)):\gamma(0)=0 \text{ and } \gamma(1)=e\right\}.$$

Then there is a sequence $(u_m)_m$ in $H_2^2(M)$ such that:

$$\begin{cases} J(u_m)\to c \quad in \quad \mathbb{R}\\ \nabla J(u_m)\to 0 \quad in \quad \left(H_2^2(M)\right)^*.\end{cases}$$

Now, to prove theorem 1, we need the following version of Ekeland Principle which is the key for the existence of solution with bounded below functional J.

Lemma 3.3. *(Ekeland Principle-weak form) Let (X,d) be a complete metric space. Let $J:X\to\mathbb{R}\cup\{+\infty\}$ be lower semicontinuous and bounded below. Then given any $\epsilon>0$ there exists $u_\epsilon\in X$ such that*

$$J\left(u_\epsilon\right)\leq\inf_X J+\epsilon,$$

and

$$J\left(u_\epsilon\right)<J\left(u\right)+\epsilon d\left(u,u_\epsilon\right),\ \text{for all } u\in X \text{ and } u\neq u_\epsilon.$$

First, we have the following lemma whose proof is easy and can be found in [8].

Lemma 3.4. *The quantity $\|u\|_{P_g}:=\left(\int_M P_g(u).ud\mu(g)\right)^{\frac{1}{2}}$ is an equivalent norm of the usual one of $H_2^2\left(M\right)$ if only if the operator P_g is coercive.*

Our working norm as follow: for all $u\in H_2^2\left(M\right)$:

$$\|u\|_{P_g}:=\left(\int_M P_g(u).ud\mu(g)\right)^{\frac{1}{2}}.$$

Lemma 3.5. *Let (M,g) be an $n\,(n \geq 5)-$dimensional compact Riemannian manifold without boundary ∂M. Let f is a C^{∞}-function on M with $f>0$ and $h \in L^q(M)$ such that $h \neq 0$, then there exists some constants α, ρ and $m_o > 0$ such that $J(u) \geq \alpha > 0$ with $\|u\|_{P_g} = \rho$ for all $u \in H_2^2(M)$ and h satisfying $\|h\|_q < m_o$.*

Proof. Let $u \in H_2^2(M)$:

$$J(u) = \frac{1}{2}\|u\|_{P_g}^2 - \frac{1}{2^{\sharp}}\int_M f(x).\,|u|^{2^{\sharp}}\,d\mu(g) - \int_M h(x).ud\mu(g)$$

Using Hölder inequality, we have:

$$J(u) \geq \frac{1}{2}\|u\|_{P_g}^2 - \frac{1}{2^{\sharp}}\max_{x\in M} f(x)\,\|u\|_{2^{\sharp}}^{2^{\sharp}} - \|h\|_q.\|u\|_{2^{\sharp}}$$

Using Sobolev inequality, we deduce:

$$J(u) \geq \frac{1}{2}\|u\|_{P_g}^2 - \frac{1}{2^{\sharp}}\max_{x\in M} f(x).\left(\max\left(\left(K_0+\epsilon\right), A_{\epsilon}\right)\right)^{\frac{2^{\sharp}}{2}}.\|u\|_{H_2^2(M)}^{2^{\sharp}}$$

$$-\|h\|_q.\left(\max\left(\left(K_0+\epsilon\right), A_{\epsilon}\right)\right)^{\frac{1}{2}}.\|u\|_{H_2^2(M)}$$

Again the coercivity of P_g implies that there is $\Lambda > 0$, such that:

$$J(u) \geq \frac{1}{2}\|u\|_{P_g}^2 - \frac{1}{2^{\sharp}}\max_{x\in M} f(x).\left(\frac{\max\left(\left(K_0+\epsilon\right), A_{\epsilon}\right)}{\Lambda}\right)^{\frac{2^{\sharp}}{2}}.\|u\|_{P_g}^{2^{\sharp}}$$

$$-\|h\|_q.\left(\frac{\max\left(\left(K_0+\epsilon\right), A_{\epsilon}\right)}{\Lambda}\right)^{\frac{1}{2}}.\|u\|_{P_g}$$

Thus,

$$J(u) \geq \Bigg[\frac{1}{2}\|u\|_{P_g} - \frac{1}{2^{\sharp}}\max_{x\in M} f(x).\left(\frac{\max\left(\left(K_0+\epsilon\right), A_{\epsilon}\right)}{\Lambda}\right)^{\frac{2^{\sharp}}{2}}.\|u\|_{P_g}^{2^{\sharp}-1}$$

$$-\|h\|_q.\left(\frac{\max\left(\left(K_0+\epsilon\right), A_{\epsilon}\right)}{\Lambda}\right)^{\frac{1}{2}}\Bigg].\|u\|_{P_g}$$

Setting for $t \geq 0$:

$$F(t) := \frac{1}{2}t - \frac{1}{2^{\sharp}}\max_{x\in M} f(x).\left(\frac{\max\left(\left(K_0+\epsilon\right), A_{\epsilon}\right)}{\Lambda}\right)^{\frac{2^{\sharp}}{2}}.t^{2^{\sharp}-1}.$$

By continuity argument of the function $F(.)$, we see that

$$\max_{t\geq 0} F(t) = F(\rho) > 0 \text{ where } \rho^{2^{\sharp}-2} := \frac{1}{2.\left(2^{\sharp}-1\right)}\left(\frac{\Lambda}{\max\left(\left(K_0+\epsilon\right), A_{\epsilon}\right)}\right)^{\frac{2^{\sharp}}{2}}. \quad (3.1)$$

Then, it follows from (3.1) that if $\|h\|_q < m_o$ such that

$$m_o := \left(\frac{\max\left(\left(K_0+\epsilon\right), A_{\epsilon}\right)}{\Lambda}\right)^{-\frac{1}{2}}.F(\rho).$$

Then, there exists $\alpha > 0$ such that

$$J(u)|_{\|u\|_{P_g}=\rho} \geq \alpha > 0. \qquad \square$$

Lemma 3.6. *Let (M, g) be an $n\,(n \geq 5)-$dimensional compact Riemannian manifold without boundary ∂M. Let f is a C^∞-function on M with $f > 0$ and $h \in L^q(M)$ such that $h \neq 0$ satisfying $\|h\|_q < m_o$. Then there exists a function $v \in H_2^2(M)$ with $\|v\|_{P_g} > \rho$ such that $J(v) < 0$, where ρ is given by the previous lemma.*

Proof. Let $v \in H_2^2(M)$, for any $t > 0$ we have:

$$J(t.v) = \frac{t^2}{2}\|v\|_{P_g}^2 - \frac{t^{2^\sharp}}{2^\sharp}\int_M f(x).\,|v|^{2^\sharp}\,d\mu(g) - t\int_M h(x).vd\mu(g).$$

Since $2^\sharp > 2$, so we deduce that,

$$\lim_{t\to+\infty} J(t.v) = -\infty.$$

Consequently, there exists a point $v \in H_2^2(M)$ with $\|u\|_{P_g} > \rho$ such that $J(v) < 0$. $\square$

Lemma 3.7. *Let (M, g) be an $n\,(n \geq 5)-$dimensional compact Riemannian manifold without boundary ∂M. Let f is a C^∞-function on M with $f > 0$ and $h \in L^q(M)$ such that $h \neq 0$ satisfying $\|h\|_q < m_o$. Assume $(u_m)_m$ is $(P.S)_c$ sequence with*

$$c < \frac{k}{n.K_0^{\frac{n}{4}}.\,(\max f(x))^{\frac{2}{2^\sharp}}}$$

Then, $(u_m)_m$ is bounded in $H_2^2(M)$.

Proof. Consider a sequence $(u_m)_m$ which satisfies

$$J(u_m) \to c$$
$$\nabla J(u_m) \to 0.$$

We obtain,

$$J(u_m) - \frac{1}{2^\sharp}\langle \nabla J(u_m), u_m\rangle = \frac{2^\sharp - 2}{2.2^\sharp}\|u_m\|_{P_g}^2 - \frac{2^\sharp - 1}{2^\sharp}\int_M h(x).u_m d\mu(g) = c + o\,(1)$$

Using Holder and Sobolev's inequalities and by the coercivity of P_g implies that there is $\Lambda > 0$, such that:

$$c + o\,(1) \geq \frac{2^\sharp - 2}{2.2^\sharp}\|u_m\|_{P_g}^2 - \frac{2^\sharp - 1}{2^\sharp}\|h\|_q.\left(\frac{\max\left((K_0 + \epsilon), A_\epsilon\right)}{\Lambda}\right)^{\frac{1}{2}}\|u_m\|_{P_g}.$$

If $\|u_m\|_{P_g} > 1$, then

$$c + o\,(1) \geq \left[\frac{2^\sharp - 2}{2.2^\sharp} - \frac{2^\sharp - 1}{2^\sharp}\|h\|_q.\left(\frac{\max\left((K_0 + \epsilon), A_\epsilon\right)}{\Lambda}\right)^{\frac{1}{2}}\right].\|u_m\|_{P_g}.$$

And since,

$$\|h\|_q < m_o := \frac{2^\sharp - 2}{2.\,(2^\sharp - 1)}\left(\frac{\max\left((K_0 + \epsilon), A_\epsilon\right)}{\Lambda}\right)^{-\frac{1}{2}}$$

Then the sequence $(u_m)_m$ is bounded in $H_2^2(M)$. □

Lemma 3.8. *Let (M, g) be an $n\,(n \geq 5)-$dimensional compact Riemannian manifold without boundary ∂M. Let f is a C^∞-function on M with $f > 0$ and $h \in L^q(M)$ such that $h \neq 0$ satisfying $\|h\|_q < m_o$. Assume $(u_m)_m$ is a bounded Palais-Smale sequence at level c of J with*

$$c < \frac{2}{n.K^{\frac{n}{2k}}(n,k).\,(\max f(x))^{\frac{2}{2^\sharp}}}$$

Then, $(u_m)_m$ has a strongly convergent sub-sequence in $H_2^2(M)$.

Proof. Using the previous lemma, let $(u_m)_m$ be a bounded $(P.S)_c$ in $H_2^2(M)$ and from the reflexivity of $H_2^2(M)$ and the compact embedding theorem, up to a subsequence noted $(u_m)_m$ there exists $u \in H_2^2(M)$ such that
(1). $u_m \to u$ weakly in $H_2^2(M)$.
(2). $u_m \to u$ strongly in $L^p(M)$ for $1 < p < 2^\sharp$.
(3). $u_m \to u$ a.e in M.
Then we deduce that:

$$\begin{aligned}\left|\int_M h(x)\,(u_m - u)\,d\mu(g)\right| &\leq \left(\int_M |h(x)|^2\,d\mu(g)\right)^{\frac{1}{2}}.\left(\int_M (u_m-u)^2\,d\mu(g)\right)^{\frac{1}{2}}\\ &\leq \|h\|_2\,.\,\|u_m - u\|_2 = o(1).\end{aligned}$$

After these preliminaries, we can prove that $w_m := u_m - u$ converges to 0 strongly in $H_2^2(M)$.
Using Brézis-Lieb Lemma in [5], we obtain

$$\|u_m\|_{P_g}^2 - \|u\|_{P_g}^2 = \|w_m\|_{P_g}^2 + o(1)$$

and

$$\int_M f(x)\left(|u_m|^{2^\sharp} - |u|^{2^\sharp}\right)d\mu(g) = \int_M f(x)\,|w_m|^{2^\sharp}\,d\mu(g) + o(1).$$

Then,

$$J(u_m) - J(u) = \frac{1}{2}\,\|w_m\|_{P_g}^2 - \frac{1}{2^\sharp}\int_M f(x)\,|w_m|^{2^\sharp}\,d\mu(g) + o(1).$$

We obtain

$$\langle \nabla J(u_m) - \nabla J(u), (u_m - u)\rangle = \|w_m\|_{P_g}^2 - \int_M f(x)\,|w_m|^{2^\sharp}\,d\mu(g) = o(1).$$

That is to say

$$\|w_m\|_{P_g}^2 = \int_M f(x)\,|w_m|^{2^\sharp}\,d\mu(g) + o(1). \tag{3.2}$$

Put

$$\ell := \limsup_m \|w_m\|_{P_g}.$$

Using Sobolev's inequality, we have for all $w_m \in H_2^2(M)$:

$$\int_M f(x)\left|w_m\right|^{2^\sharp} d\mu(g) \leq \max_{x\in M} f(x). \int_M \left|w_m\right|^{2^\sharp} d\mu(g) = \max_{x\in M} f(x). \left\|w_m\right\|_{2^\sharp}^{2^\sharp}$$
$$\leq \max_{x\in M} f(x). \left[\max\left(\left(K_0+\epsilon\right), A_\epsilon\right)\right]^{\frac{2^\sharp}{2}}. \left\|w_m\right\|_{H_2^2(M)}^{2^\sharp}.$$

Taking account that $P_g : C^\infty(M) \to C^\infty(M)$ is coercive, there exists a constant $\Lambda > 0$ such that:

$$\int_M f(x)\left|w_m\right|^{2^\sharp} d\mu(g) \leq \max_{x\in M} f(x). \left[\Lambda. \max\left(\left(K_0+\epsilon\right), A_\epsilon\right)\right]^{\frac{2^\sharp}{2}}. \left\|w_m\right\|_{P_g}^{2^\sharp}. \tag{3.3}$$

Consequently, we obtain from (3.2) and (3.3) that:

$$\left\|w_m\right\|_{P_g}^2 \leq \max_{x\in M} f(x). \left[\Lambda. \max\left(\left(K_0+\epsilon\right), A_\epsilon\right)\right]^{\frac{2^\sharp}{2}}. \left\|w_m\right\|_{P_g}^{2^\sharp}.$$

Letting $n \to +\infty$, we get

$$\ell \leq \max_{x\in M} f(x). \left[\Lambda. \max\left(\left(K_0+\epsilon\right), A_\epsilon\right)\right]^{\frac{2^\sharp}{2}}.\ell^{2^\sharp}$$

Then,

$$\ell = 0 \quad \text{or..}\ell \geq \frac{1}{\left[\max_{x\in M} f(x)\right]^{\frac{n-2k}{n+2k}}. \left[\Lambda. \max\left(\left(K_0+\epsilon\right), A_\epsilon\right)\right]^{\frac{n}{n+2k}}}.$$

We deduce that: $\ell = 0$ and then $w_n \to 0$ strongly in $H_2^2(M)$.
i.e. $w_n := u_n - u \to 0$ in $H_2^2(M)$. □

4. Main result

The following theorem is our main result.

Theorem 4.1. *Let (M, g) be an $n\,(n \geq 5)-$dimensional compact Riemannian manifold without boundary ∂M. Let f is a C^∞-function on M with $f > 0$ and $h \in L^q(M)$ such that $h \neq 0$ satisfying $\|h\|_q < m_o$ and supposing that the operator $u \to P_g(u)$ is coercive. Then, the equation (1.1) has at least two nontrivial solutions $v, w \in H_2^2(M)$ satisfying:*

$$J(v) < 0 < J(w).$$

The proof is based on The Mountain-Pass Theorem and Ekeland's Variational Principle.

Proof. We prove this theorem, by the following two steps:
Step 1: There exists $w \in H_2^2(M)$ satisfies

$$J(w) > 0 \text{ and } \nabla J(w) = 0.$$

Using Lemmas 2 and 3 and The Mountain-Pass Theorem, there exists a sequence $(u_m)_m \in H_2^2(M)$ satisfying:

$$J(u_m) \to c^+ \text{ and } \nabla J(u_m) = 0.$$

Then, it follows from Lemmas 3 and 4 that there exists $w \in H_2^2(M)$ such that $J(w) = c > 0$ and $\nabla J(w) = 0$ if $\|h\|_q < m_o$.
Consequently, w is a weak solution of the equation (1.1).
Step 2: There exists $v \in H_2^2(M)$ such that: $J(v) < 0$ and $\nabla J(v) = 0$. Since $h \in L^q(M)$ such that $h \neq 0$, we can choose a function $\varphi \in H_2^2(M)$ such that:

$$\int_M h(x).\varphi(x)d\mu(g) > 0$$

Letting $t > 0$, we have:

$$J(t.\varphi) = \frac{t^2}{2}\|\varphi\|_{P_g}^2 - \frac{t^{2^\sharp}}{2^\sharp}\int_M f(x).|\varphi|^{2^\sharp}d\mu(g) - t\int_M h(x).\varphi(x)d\mu(g)$$

Then for $t > 0$ small enough, we get $J(t.\varphi) < 0$.
Put

$$c^- = \inf_{u \in B_\rho} J(u)$$

Where

$$B_\rho := \left\{u \in H_2^2(M) : \|u\|_{P_g} \leq \rho\right\}$$

It seems that:

$$c^- = \inf_{u \in B_\rho} J(u) < 0$$

Now, applying Ekeland's Variational Principle**,** there exists a $(P.S)_{c^-}$ sequence $(v_m)_m \in \bar{B}_\rho$ satisfying:

$$J(v_m) \to c^- \text{ and } \nabla J(v_m) = 0$$

Using Lemmas 2-7, we obtain a sub-sequence of $(v_m)_m$ which converges strongly to $v \in H_2^2(M)$.
Consequently, w is a weak solution of the equation (1.1). □

5. Geometric application of the main theorem

Remark 5.1. When (M, g) is Einstein, the geometric Paneitz-Branson operator has constant coefficient and reduces as:

$$P_g^n(u) := \Delta_g^2(u) + c_n\Delta_g(u) + d_n u,$$

where

$$c_n := \tfrac{n^2-2n-4}{2n(n-1)}S_g \text{ and } \quad d_n := \tfrac{(n-4)\left(n^2-4\right)}{16n(n-1)^2}S_g^2.$$

In particular, when $(M, g) = (S^n, h)$ is the unit $n-$sphere,

$$P_h^n(u) := \Delta_g^2(u) + c_n\Delta_g(u) + d_n u,$$

where

$$c_n := \tfrac{n^2-2n-4}{2} \text{ and } \quad d_n := \tfrac{(n-4)n\left(n^2-4\right)}{16}.$$

Notice that

$$(c_n)^2 - 4d_n = \frac{S_g^2}{n^2(n-1)^2}.$$

Since $(c_n)^2 - 4d_n \geq 0$, then

$$P_g^n(u) = \left(\Delta_g + a^+\right) \circ \left(\Delta_g + a^-\right) u,$$

with

$$a^{\pm} := \frac{c_n \pm \sqrt{(c_n)^2 - 4d_n}}{2}.$$

Remark 5.2. If $S_g > 0$, then P_g^n is coercive.

In this part we consider the elliptic equation with the condition taken above:

$$P_g^n(u) = f(x)\left|u\right|^{2^\sharp - 2} u + h(x), \tag{5.1}$$

Then we have the following result:

Theorem 5.3. *Let (M, g) be an $n\,(n \geq 5)-$dimensional compact Eisteinian Riemannian manifold without boundary ∂M with its scalar curvature $S_g > 0$. Let f is a C^∞-function on M with $f > 0$ and $h \in L^q(M)$ such that $h > 0$ satisfying $\|h\|_q < m_o$ Then, the equation (5.1) has at least two nontrivial solutions $v, w \in H_2^2(M)$ satisfying:*

$$J(u^-) < 0 < J(u^+),$$

where

$$u^- := \min(u, o) \quad u^+ := \max(u, 0).$$

Proof. Define the two functionals in $H_2^2(M)$ by

$$J^+(u) = \frac{1}{2}\|u\|^2_{P_g^n} - \int_M h(x).ud\mu(g) - \frac{1}{2_k^\sharp}\int_M f(x).\left(u^+\right)^{2_k^\sharp} d\mu(g)$$

and

$$J^-(u) = \frac{1}{2}\|u\|^2_{P_g^n} - \int_M h(x).ud\mu(g) - \frac{1}{2_k^\sharp}\int_M f(x).\left(u^-\right)^{2_k^\sharp} d\mu(g)$$

where

$$u^- := \min(u, o) \quad u^+ := \max(u, 0).$$

Applying the coercitivity of P_g^n on Eisteinian manifold (M, g) and using the same technique that relies on Mountain Pass Theorem for the energies J^- and J^+ for solving the elliptic equation

$$P_g^n(u) = f(x)\left|u\right|^{2^\sharp - 2} u + h(x).$$

Since (M, g) has a positive scalar curvature S_g, we have

$$\left(\Delta_g + a^+\right) \circ \left(\Delta_g + a^-\right) u = f(x)\left|u\right|^{2^\sharp - 2} u + h(x),$$

with

$$a^{\pm} := \frac{c_n \pm \sqrt{(c_n)^2 - 4d_n}}{2}.$$

Applying the strong maximum principle two times to show that $u^-, u^+ \in H_2^2(M)$ are two nontrivial solutions satisfying:

$$J(u^-) < 0 < J(u^+). \qquad \square$$

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Kamel Tahri
High School of Management or Abou Bekr Belkaid University,
Tlemcen, Algeria
e-mail: `Tahri_kamel@yahoo.fr`