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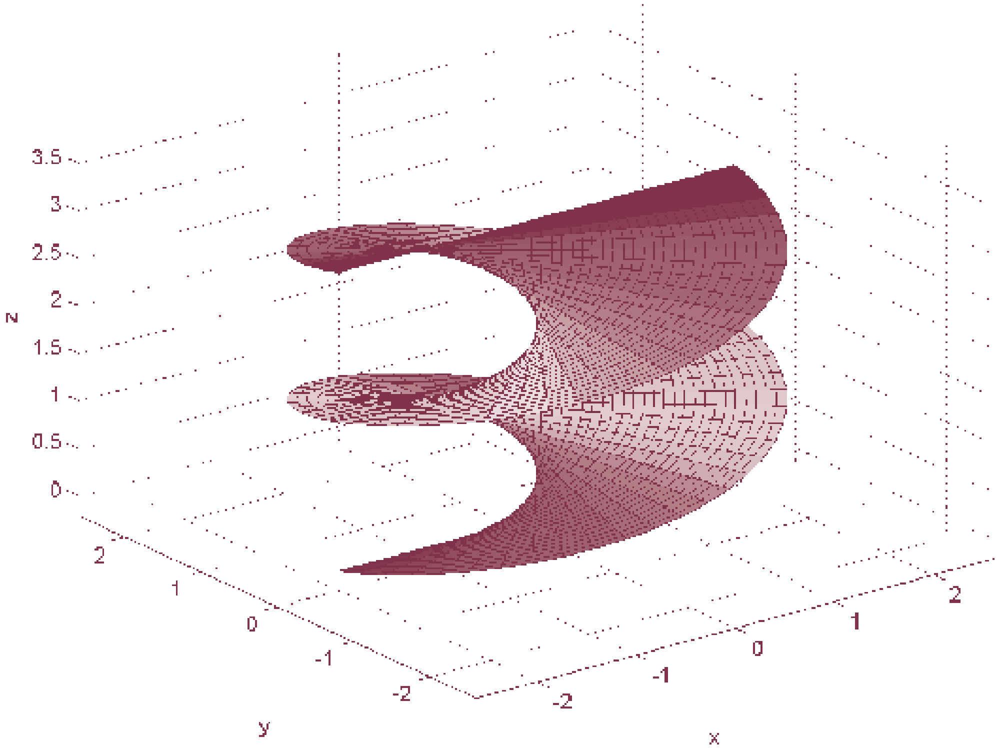

MATHEMATICA

2/2023

STUDIA
UNIVERSITATIS BABEŞ-BOLYAI
MATHEMATICA

2/2023

ISSN (print): 0252-1938
ISSN (online): 2065-961X

YEAR	**(LXVIII) 2023**
MONTH	**JUNE**
ISSUE	**2**

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
2

Redacţia: 400084 Cluj-Napoca, str. M. Kogălniceanu nr. 1
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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 237–247
DOI: 10.24193/subbmath.2023.2.01

On a generalization of the Wirtinger inequality and some its applications

Latifa Agamalieva, Yusif S. Gasimov and Juan E. Nápoles-Valdes

Abstract. In this paper, we present generalized versions of the Wirtinger inequality, which contains as particular cases many of the well-known versions of this classic isoperimetric inequality. Some applications and open problems are also presented in the work.

Mathematics Subject Classification (2010): 26A33, 26Dxx, 35A23.

Keywords: Integral operator, fractional calculus, Wirtinger inequality.

1. Introduction

It is known that Fractional Calculus has a history practically similar to that of Ordinary Calculus, however only in the last 40 years has it become one of the most dynamic areas of Mathematics. Not only the development of the "classic" (global to be more precise) Fractional Calculus has contributed to this, but since the 1960s generalized differential operators, called local fractional ones, began to appear, which have shown their usefulness in different problems of application. However, until 2014 (see [22]) it is that a formalization of these operators is not achieved with the appearance of what is called Conformable Derivative, on the other hand, in 2018, a local derivative of a new type is presented, called Non conformable [13, 34], which comes to consolidate this area as one in constant development.

As we said, between the theoretical development and the multiplicity of applications, a multitude of operators, fractional and generalized, have appeared, making it practically impossible to follow these new operators. In [5] suggests and justifies the idea of a fairly complete classification of the known operators of the Fractional Calculus (global or local), on the other hand, in the work [6] some reasons are presented

Received 27 August 2020; Accepted 22 September 2020.

why new operators linked to applications and developments theorists appear every day. These operators had been developed by numerous mathematicians with a barely specific formulation, for instance, the Riemann-Liouville (RL), the Weyl, Erdelyi-Kober, Hadamard integrals, and the Liouville and Katugampola fractional operators and many authors have introduced new fractional operators generated from general classical local derivatives.

In addition, Chapter 1 of [2] presents a history of differential operators, both local and global, from Newton to Caputo and presents a definition of local derivative with new parameter, providing a large number of applications, with a difference qualitative between both types of operators, local and global. Most importantly, Section 1.4 LIMITATIONS AND STRENGTH ..... concludes “We can therefore conclude that both the Riemann-Liouville and Caputo operators are not derivatives, and then they are not fractional derivatives, but fractional operators. We agree with the result [38] that, the local fractional operator is not a fractional derivative” (p.24). As we said before, they are new tools that have proved their usefulness and potential in the modeling of different processes and phenomena.

Wirtinger’s inequality for real functions was an inequality used in Fourier analysis. It was named after Wilhelm Wirtinger. It was used in 1904 to prove the isoperimetric inequality, one of the versions of this inequality is the following

$$\pi^2 \int_0^b |f(x)|^2 dx \leq b^2 \int_0^b |f'(x)|^2 dx,$$

whenever f is a C^1 function such that $f(0) = f(b) = 0$. In this form, Wirtinger’s inequality is seen as the one-dimensional version of Friedrichs’ inequality. If in the proof of the previous result, the well-known Schwarz Inequality is used, it is reduced to

$$\int_0^b |f(x)|^2 dx \leq b^2 \int_0^b |f'(x)|^2 dx, \tag{1.1}$$

where condition $y(b) = 0$ is not needed. It is worth noting that this inequality is relevant because it gives an estimate of the function f through its derivative.

An interesting survey on Wirtinger’s and related inequalities can be found in a recently published monograph [28], which represents numerous extensions, refinements, variants, discrete analogues and applications of (1.1), and provides more than 200 references on this subject. Some interesting applications of this inequality can be found in [4, 7, 14, 23, 24, 27, 35, 36, 39].

The word calculus comes from the Latin calculus, which means having stones. The Integral Calculus is a branch of mathematics with so many ramifications and applications, that the sole intention of enumerating them makes the task practically impossible. It was used initially by, Aristoteles, Descartes, Newton and Barrow with the contributions of Newton, if we refer only to the case of integral inequalities present in the alliterature, there are different types of these, which involve certain properties of the functions involved, from generalizations of the known Mean Value Theorem of classical Integral Calculus, to varied inequalities in norm, going through the inequality of Wirtinger presented above.

In this article, a Wirtinger-type inequality is studied, in the context of the generalized derivative that we will define in the following section, some remarks will be presented that will show the strength of our results, having as particular cases, several of those reported in the literature. In particular we will deal with real integral operators defined on $\mathbb{R}$.

2. A general integral operator

We assume that the reader is familiar with the classic definition of the Riemann Integral, so we will not present it. In [15] was presented an integral operator generalized, whcih contain as particular cases, many of the well-known integral operators, both integer order and not. First we will present the definition of generalized derivative (see [30]) which was defined in the following way.

Definition 2.1. Given a function $f : [0, +\infty) \to \mathbb{R}$. Then the N-derivative of f of order α is defined by

$$N_F^\alpha f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon F(x, \alpha)) - f(x)}{\varepsilon} \tag{2.1}$$

for all $x > 0$, $\alpha \in (0, 1)$ being $F(\alpha, t)$ is some function. Here we will use some cases of F defined in function of $E_{a,b}(.)$ the classic definition of Mittag-Leffler function with $Re(a), Re(b) > 0$. Also we consider $E_{a,b}(x^{-\alpha})_k$ is the k-nth term of $E_{a,b}(.)$.

If f is $\alpha-$differentiable in some $(0, \alpha)$, and $\lim_{t \to 0^+} N_F^\alpha f(x)$ exists, then define $N_F^\alpha f(0) = \lim_{t \to 0^+} N_F^\alpha f(x)$. Note that if f is differentiable, then $N_F^\alpha f(x) = F(x, \alpha) f'(x)$, where $f'(x)$ is the ordinary derivative.

The function $E_{a,b}(z)$ was defined and studied by Mittag-Leffler in the year 1903. It is a direct generalization of the exponential function. This generalization was studied by Wiman in 1905, Agarwal in 1953 and Humbert and Agarwal in 1953, and others. In this address the reader can check [3, 8, 9, 10, 12, 16-18, 20, 21] where several fractional calculus operators have been introduced and investigated

We consider the following examples:

I) $F(x, \alpha) \equiv 1$, in this case we have the ordinary derivative.

II) $F(x, \alpha) = E_{1,1}(x^{-\alpha})$. In this case we obtain, from Definition 2.1, the non conformable derivative $N_1^\alpha f(x)$ defined in [13] (see also [29]).

III) $F(x, \alpha) = E_{1,1}((1-\alpha)x) = e^{(1-\alpha)x}$, this kernel satisfies that $F(x, \alpha) \to 1$ as $\alpha \to 1$, a conformable derivative used in [11].

IV)$F(x, \alpha) = E_{1,1}(x^{1-\alpha})_1 = x^{1-\alpha}$ with this kernel we have $F(x, \alpha) \to 0$ as $\alpha \to 1$ (see [22]), a conformable derivative.

V)$F(x, \alpha) = E_{1,1}(x^{-\alpha})_1 = x^\alpha$ with this kernel we have $F(x, \alpha) \to x$ as $\alpha \to 1$ (see [31]). It is clear that since it is a non-conformable derivative, the results will differ from those obtained previously, which enhances the study of these cases.

VI) $F(x, \alpha) = E_{1,1}(x^{-\alpha})_1 = x^{-\alpha}$ with this kernel we have $F(x, \alpha) \to x^{-1}$ as $\alpha \to 1$ This is the derivative N_3^α studied in [25]. As in the previous case, the results obtained have not been reported in the literature.

Now, we give the definition of a general fractional integral. Throughout the work we will consider that the integral operator kernel T defined below is an absolutely continuous function.

Let I be an interval $I \subseteq \mathbb{R}$, $a, x \in I$ and $\alpha \in \mathbb{R}$. The integral operator $J^{\alpha}_{T,a}$, right and left, is defined for every locally integrable function f on I as

$$J^{\alpha}_{T,a+}(f)(x) = \int_a^x \frac{f(s)}{T(s,\alpha)} ds, x > a, \tag{2.2}$$

$$J^{\alpha}_{T,b-}(f)(x) = \int_x^b \frac{f(s)}{T(s,\alpha)} ds, b < x. \tag{2.3}$$

Sometimes the kernel of the integral operator is exactly the kernel of the generalized derivative.

Remark 2.2. It is easy to see that the case of the J^{α}_{T} operator defined above contains, as particular cases, the integral operators obtained from conformable and non-conformable local derivatives. However, we will see that it goes much further by containing the cases listed at the beginning of the work. So, we have

1) If $F(x,\alpha) = x^{1-\alpha}$, $\quad T(x,\alpha) = \Gamma(\alpha)F(x-x,\alpha)$, from (2.2) we have the right side Riemann-Liouville fractional integrals $(R^{\alpha}_{a+}f)(x)$, similarly from (2.3) we obtain the left derivative of Riemann-Liouville. Then its corresponding right differential operator is

$$({}^{RL}D^{\alpha}_{a^+}f)(x) = \frac{d}{dx}(R^{1-\alpha}_{a+}f)(x),$$

analogously we obtain the left.

2) With $F(x,\alpha) = x^{1-\alpha}$, $\quad T(x-x,\alpha) = \Gamma(\alpha)F(lnt - lnx,\alpha)t$, we obtain the right Hadamard integral from (2.2), the left Hadamard integral is obtained similarly from (2.3). The right derivative is

$$({}^{H}D^{\alpha}_{a^+}f)(x) = x\frac{d}{dx}(H^{1-\alpha}_{a+}f)(x),$$

in a similar way we can obtain the left.

3) The right Katugampola integral is obtained from (2.2) making

$$F(x,\alpha) = x^{1-\alpha}, \quad e(x) = x^{\varrho}, \quad T(x,\alpha) = \frac{\Gamma(\alpha)}{F(\rho,\alpha)} \frac{F(e(x)-e(x),\alpha)}{e'(x)},$$

analogously for the integral left fractional. In this case, the right derivative is

$$({}^{K}D^{\alpha,\rho}_{a^+}f)(x) = x^{1-\rho}\frac{d}{dx}K^{1-\alpha,\rho}_{a^+}f(x) = F(x,\rho)\frac{d}{dx}K^{1-\alpha,\rho}_{a^+}f(x),$$

and we can obtain the left derivative in the same way.

4) The solution of equation $(-\Delta)^{-\frac{\alpha}{2}}\phi(u) = -f(u)$ called Riesz potential, is given by the expression

$$\phi = C^{\alpha}_{n} \int_{R^n} \frac{f(v)}{|u-v|^{n-\alpha}} dv,$$

where C^{α}_{n} is a constant (see [7, 18, 27]). Obviously, this solution can be expressed in terms of the operator (2.2) very easily.

5) Obviously, we can define the lateral derivative operators (right and left) in the

case of our generalized derivative, for this it is sufficient to consider them from the corresponding integral operator. To do this, just make use of the fact that if f is differentiable, then $N_F^{\alpha} f(x) = F(x,\alpha) f'(x)$ where $f'(x)$ is the ordinary derivative. For the right derivative we have

$$\left(N_{F,a+}^{\alpha} f\right)(x) = N_F^{\alpha}\left[J_{T,a+}^{\alpha}(f)(x)\right] = \frac{d}{dx}\left[J_{T,a+}^{\alpha}(f)(x)\right] F(x,\alpha),$$

similarly to the left.
6) It is clear then, that from our definition, new extensions and generalizations of known integral operators can be defined.
7) We can define the function space $L_{\alpha}^{p}[a,b]$ as the set of functions over $[a,b]$ such that $(J_{F,a+}^{\alpha}[f(x)]^p(b)) < +\infty$.

The following results are generalizations of the known results of the integer order Calculus.

Proposition 2.3. *Let I be an interval $I \subseteq \mathbb{R}$, $a \in I$, $0 < \alpha \leq 1$ and f a α-differentiable function on I such that f' is a locally integrable function on I. Then, we have for all $x \in I$,*

$$J_{F,a+}^{\alpha}\big(N_F^{\alpha}(f)\big)(x) = f(x) - f(a).$$

Proposition 2.4. *Let I be an interval $I \subseteq \mathbb{R}$, $a \in I$ and $\alpha \in (0,1]$.*

$$N_F^{\alpha}\big(J_{F,a+}^{\alpha}(f)\big)(x) = f(x),$$

for every continuous function f on I and $a, t \in I$.

In [22] it is defined the integral operator $J_{F,a}^{\alpha}$ for the choice of the function F given by $F(x,\alpha) = x^{1-\alpha}$, and [22, Theorem 3.1] shows

$$N^{\alpha} J_{x^{1-\alpha},\,a}^{\alpha}(f)(x) = f(x),$$

for every continuous function f on I, $a, x \in I$ and $\alpha \in (0,1]$. Hence, Proposition 2.4 extends to any F this important equality.

Theorem 2.5. *Let I be an interval $I \subseteq \mathbb{R}$, $a, b \in I$ and $\alpha \in \mathbb{R}$. Suppose that f, g are locally integrable functions on I, and $k_1, k_2 \in \mathbb{R}$. Then we have*

(1) $J_{T,a}^{\alpha}\big(k_1 f + k_2 g\big)(x) = k_1 J_{T,a}^{\alpha} f(x) + k_2 J_{T,a}^{\alpha} g(x)$,
(2) *if $f \geq g$, then $J_{T,a}^{\alpha} f(x) \geq J_{T,a}^{\alpha} g(x)$ for every $t \in I$ with $t \geq a$,*
(3) $\left|J_{T,a}^{\alpha} f(x)\right| \leq J_{T,a}^{\alpha}\left|f\right|(x)$ *for every $t \in I$ with $t \geq a$,*
(4) $\int_a^b \frac{f(s)}{T(s,\alpha)} ds = J_{T,a}^{\alpha} f(x) - J_{T,b}^{\alpha} f(x) = J_{T,a}^{\alpha} f(x)(b)$ *for every $t \in I$.*

Let $C^1[a,b]$ be the set of functions f with first ordinary derivative continuous on $[a,b]$, we consider the following norms on $C^1[a,b]$:

$$\|F\|_C = \max_{[a,b]} |f(x)|, \quad \|F\|_{C^1} = \left\{\max_{[a,b]} |f(x)| + \max_{[a,b]} |f'(x)|\right\}.$$

Theorem 2.6. *The fractional derivatives $N_{F,a+}^{\alpha} f(x)$ and $N_{F,b-}^{\alpha} f(x)$ are bounded operators from $C^1[a,b]$ to $C[a,b]$ with*

$$\left|N_{F,a+}^{\alpha} f(x)\right| \leq K \|F\|_C \|f\|_{C^1}, \tag{2.4}$$

$$\left|N^{\alpha}_{F,b-}f(x)\right| \leq K\|F\|_{C}\|f\|_{C^1}, \tag{2.5}$$

where the constant K, may be depend of derivative frame considered.

Remark 2.7. From previous results we obtain that the derivatives $N^{\alpha}_{F,a+}f(x)$ and $N^{\alpha}_{F,b-}f(x)$ are well defined.

Theorem 2.8. *(Integration by parts) Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions and $\alpha \in (0, 1]$. Then, the following property hold*

$$J^{\alpha}_{F,a+}((f)(N^{\alpha}_{F,a+}g(x))) = [f(x)g(x)]^{b}_{a} - J^{\alpha}_{F,a+}((g)(N^{\alpha}_{F,a+}f(x))). \tag{2.6}$$

3. The generalized Wirtinger inequality

In this section, we will state Wirtinger type inequalities using generalized integral operator defined in the previous section.

First, we will give a generalized version of the inequality (1.1).

Theorem 3.1. *For any function f, N-differentiable on $[a, b]$ with $a < b$ such that $f(a) = 0$, $\alpha \in (0, 1]$, we have*

$$J^{\alpha}_{F,a+}(f^2(x))(b) \leq [\mathcal{F}(b)]^2 \quad J^{\alpha}_{F,a+}\left(N^{\alpha}_{F}f(x)\right)(b) \tag{3.1}$$

with $\mathcal{F}(b) = J^{\alpha}_{F,a+}(1)(b)$ and $F(x, \alpha) > 0$.

Proof. From Proposition 2.3 we have $J^{\alpha}_{F,a+}\big(N^{\alpha}_{F}(f)\big)(x) = f(x) - f(a)$, and using the fact that $f(a) = 0$ we have $f(x) = J^{\alpha}_{F,a+}\big(N^{\alpha}_{F}(f)\big)(x)$ and so, from Property (3) of Theorem 2.5

$$|f(x)| = \left|J^{\alpha}_{F,a+}N^{\alpha}_{F}(f)(x)\right| \leq J^{\alpha}_{F,a+}\left|N^{\alpha}_{F}(f)\right|(x) \tag{3.2}$$

for every $x \in [a, b]$. Applying Schwarz inequality to the right side of (3.2) we obtain

$$\begin{aligned} |f(x)| &\leq \left(J^{\alpha}_{F,a+}(1)(x)\right)^{\frac{1}{2}}\left(J^{\alpha}_{F,a+}(N^{\alpha}_{F}(f))^2(x)\right)^{\frac{1}{2}} \\ &\leq (\mathcal{F}(x))^{\frac{1}{2}}\left(J^{\alpha}_{F,a+}(N^{\alpha}_{F}(f))^2(x)\right)^{\frac{1}{2}} \\ &\leq (\mathcal{F}(b))^{\frac{1}{2}}\left(J^{\alpha}_{F,a+}(N^{\alpha}_{F}(f))^2(b)\right)^{\frac{1}{2}}. \end{aligned}$$

From this last inequality, squaring and N-integrating between a and b, we obtain the desired result. □

Remark 3.2. If in the previous result we have $a \equiv 0$, the kernel $F(x, \alpha) \equiv 1$, that is, the classic Riemann integral, we get the inequality (1.1).

Remark 3.3. Similarly, if we use the kernel $F(x, \alpha) = x^{1-\alpha}$, that is, in the case of the conformable derivative T_α of [22], we obtain the inequality of Theorem 3.1 of [1].

Remark 3.4. It's easy to get new versions of the classic Wirtinger Inequality using other kernels, for example, if we take $F(x, \alpha) = x^{\alpha}$ we get, from (3.1) the following

$$J^{\alpha}_{F,a+}f(x)(b) \leq \frac{\left(b^{1-\alpha} - a^{1-\alpha}\right)^2}{(1-\alpha)^2} J^{\alpha}_{F,a+}\left(N^{\alpha}_{F}f(x)\right)(b). \tag{3.3}$$

4. Some applications

In [19] the authors gave a generalized Wirtinger type inequality using an auxiliary function. Thus, the following result is a generalization of this result.

Theorem 4.1. *For any positive continuous funtion $M(x)$ on $[a,b]$ with $N_F^\alpha M(x)>0$ or $N_F^\alpha M(x)<0$ on $[a,b]$, $\alpha\in(0,1]$, we have*

$$J_{F,a+}^{\alpha}\left((N_F^\alpha M(x))\right)(y^2(x))(b)\le 4J_{F,a+}^{\alpha}\left[\left(\frac{M^2(x)}{N_F^\alpha M(x)}\right)(N_F^\alpha y(x))^2\right](b),\tag{4.1}$$

for all continuous function $y(x)$ defined on $[a,b]$ with $y(a)=y(b)=0$.

Proof. We consider the case $N_F^\alpha M(x)>0$ then, N-integrating by parts (see Theorem 2.8), we have

$$\begin{aligned}J_1&=J_{F,a+}^{\alpha}\left((N_F^\alpha M(x))\right)(y^2(x))(b)\\&=M(b)y^2(b)-M(a)y^2(a)-2J_{F,a+}^{\alpha}(M(x))(N_F^\alpha y(x))(y(x))(b).\end{aligned}$$

From here, using the Schwarz inequality we get

$$\begin{aligned}J_1&=J_{F,a+}^{\alpha}\left((N_F^\alpha M(x))\right)(y^2(x))(b)\\&=-2J_{F,a+}^{\alpha}(M(x))(N_F^\alpha y(x))y(x))(b)\\&\le 2J_{F,a+}^{\alpha}\left[\sqrt{\frac{(M(x))^2}{N_F^\alpha M(x)}}\right]|N_F^\alpha y(x)|\sqrt{N_F^\alpha M(x)}\,|y(x)|(b)\\&\le 2\sqrt{J_1J_2}\end{aligned}$$

with

$$J_2=J_{F,a+}^{\alpha}\left[\left(\frac{M^2(x)}{N_F^\alpha M(x)}\right)(N_F^\alpha y(x))^2\right].$$

From the above inequality we have then $J_1=2\sqrt{J_1J_2}$, where the desired conclusion is reached. ☐

Remark 4.2. If in the previous result we have $a\equiv 0$, the kernel $F(x,\alpha)\equiv 1$, that is, the classic Riemann integral, we get the Lemma 1 of [19].

Remark 4.3. Similarly, if we use the kernel $F(x,\alpha)=x^{1-\alpha}$, that is, in the case of the conformable derivative T_α of [22], we obtain the inequality of Theorem 3.2 of [1].

Remark 4.4. Of course we can also generate new generalizations considering other kernels, a simple matter that we leave to the reader.

Theorem 4.5. *For any function f, N-differentiable on $[a,b]$ with $a<b$ such that $f(a)=0$, $p\ge 1$, $\alpha\in(0,1]$, we have*

$$J_{F,a+}^{\alpha}|f(x)|^p(b)\le\frac{[\mathcal{F}(b)]^p}{p}\quad J_{F,a+}^{\alpha}\left(|N_F^\alpha f(x)|^p\right)(b)\tag{4.2}$$

with $\mathcal{F}(b)=J_{F,a+}^{\alpha}(1)(b)$ and $F(x,\alpha)>0$.

Proof. As before, from Proposition 2.3 we have $J^{\alpha}_{F,a+}\big(N^{\alpha}_{F}(f)\big)(x) = f(x) - f(a)$, and using the fact that $f(a) = 0$ we have $f(x) = J^{\alpha}_{F,a+}\big(N^{\alpha}_{F}(f)\big)(x)$ and so, from Property (3) of Theorem 2.5

$$|f(x)| = \left|J^{\alpha}_{F,a+}N^{\alpha}_{F}(f)(x)\right| \leq J^{\alpha}_{F,a+}\left|N^{\alpha}_{F}(f)\right|(x)$$

for every $x \in [a,b]$. Using the Holder inequality with p and $\frac{p}{p-1}$ we obtain

$$|f(x)| \leq \left(J^{\alpha}_{F,a+}(1)(x)\right)^{\frac{p-1}{p}}\left(J^{\alpha}_{F,a+}(N^{\alpha}_{F}(f))^{p}(x)\right)^{\frac{1}{p}},$$

so we have

$$|f(x)|^{p} \leq \left(J^{\alpha}_{F,a+}(1)(x)\right)^{p-1}\left(J^{\alpha}_{F,a+}(N^{\alpha}_{F}(f))^{p}(x)\right),$$

N-integrating by parts (see Theorem 2.8), the conclusion of the Theorem is obtained. □

Remark 4.6. If in the previous result we have $a \equiv 0$, the kernel $F(x,\alpha) \equiv 1$, that is, the classic Riemann integral, we get a light variant of result of [6].

Remark 4.7. Similarly, if we use the kernel $F(x,\alpha) = x^{1-\alpha}$, that is, in the case of the conformable derivative T_{α} of [22], we obtain the inequality of Theorem 6 of [34] (see also Theorem 2.2 of [33]). Theorem 2.2 is obtained in a different way, and its conclusion differs from that presented here, it is necessary that $f(b) = 0$, however the interested reader can obtain it without any difficulty and have, instead of (4.2) the following expression

$$J^{\alpha}_{F,a+}|f(x)|^{p}(b) \leq \frac{[\mathcal{F}(b)]^{p}}{2^{p-1}p} \quad J^{\alpha}_{F,a+}\left(|N^{\alpha}_{F}f(x)|^{p}\right)(b).$$

Remark 4.8. New inequalities of type Wirtinger, generalizations of (4.2), can be obtained using other kernels.

Following the ideas of Theorem 3.1 we can obtain a weighted version of Wirtinger's Inequality as follows.

Theorem 4.9. *Let f and g N-differentiables functions on $[a,b]$ with $f(a) = g(a) = 0$ and $f,g \in L^{2}_{\alpha}[a,b]$ then, we have the following inequality*

$$J^{\alpha}_{F,a+}(|f|\,|g|)(b) \leq \frac{\mathcal{F}^{2}(b)}{2}J^{\alpha}_{F,a+}\left(\left(N^{\alpha}_{F}f\right)^{2} + \left(N^{\alpha}_{F}g\right)^{2}\right).$$

Proof. From assumptions and proceeding as in Theorem 3.1 we get easily

$$J^{\alpha}_{F,a+}(|f|\,|g|)(b) \leq \mathcal{F}^{2}(b)\left[J^{\alpha}_{F,a+}\left(\left(N^{\alpha}_{F}f\right)^{2}\right)\right]^{\frac{1}{2}}\left[J^{\alpha}_{F,a+}\left(\left(N^{\alpha}_{F}g\right)^{2}\right)\right]^{\frac{1}{2}}$$

Applying the known inequality $\sqrt{AB} \leq \frac{A+B}{2}$ with $A, B > 0$, the desired conclusion is obtained. □

5. Conclusion

In this paper, some generalized extensions of classical Wirtinger type inequality are obtained, using less restrictive conditions on the function f, for example, the condition $f(b) = 0$ is not used. In addition, several known results in this topic are obtained as particular cases of our results, in addition to raising several possibilities of future work in this address.

Taking into account the ideas of [26] we can define generalized partial derivatives as follows.

Definition 5.1. Given a real function $f : \mathbb{R}^n \to \mathbb{R}$ and $\overrightarrow{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ a point whose i-th component is positive. Then the generalized partial N-derivative of order α of f en el punto $\overrightarrow{a} = (a_1, \ldots, a_n)$ is defined by

$$N^{\alpha}_{F_i,t_i} f(\overrightarrow{a}) = \lim_{\varepsilon \to 0} \frac{f(a_1, .., a_i + \varepsilon F_i(a_i, \alpha), \ldots, a_n) - f(a_1, \ldots, a_i, ..., a_n))}{\varepsilon} \tag{5.1}$$

if it exists, it is denoted by $N^{\alpha}_{F_i,t_i} f(\overrightarrow{a})$, and is called the i-th generalized partial derivative of order $\alpha \in (0, 1]$ of f in $\overrightarrow{a}$.

Remark 5.2. If a real function f multivariable, has all the partial derivatives of order $\alpha \in (0, 1]$ in $\overrightarrow{a}$, with $a_i > 0$, then the generalized α-gradient of f of order $\alpha \in (0, 1]$ in $\overrightarrow{a}$ is

$$\nabla^{\alpha}_{N} f(\overrightarrow{a}) = (N^{\alpha}_{t_1} f(\overrightarrow{a}), \ldots, N^{\alpha}_{t_n} f(\overrightarrow{a})) \tag{5.2}$$

On this basis, the results of [32] can be generalized to a generalized context without much difficulty.

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 249–260
DOI: 10.24193/subbmath.2023.2.02

Sufficient conditions for univalence obtained by using the Ruscheweyh-Bernardi differential-integral operator

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Abstract. In this paper we introduce the Ruscheweyh-Bernardi differential-integral operator $T^m : A \to A$ defined by

$$T^m[f](z) = (1-\lambda)R^m[f](z) + \lambda B^m[f](z),\ z \in U,$$

where R^m is the Ruscheweyh differential operator (Definition 1.3) and B^m is the Bernardi integral operator (Definition 1.1). By using the operator T^m, the class of univalent functions denoted by $T^m(\lambda,\beta)$, $0 \le \lambda \le 1$, $0 \le \beta < 1$, is defined and several differential subordinations are studied.

Mathematics Subject Classification (2010): 30C20, 30C45.

Keywords: Analytic function, differential operator, integral operator, convex function, univalent function, dominant, best dominant, differential subordination, Briot-Bouquet differential subordination.

1. Introduction and preliminaries

The theory of differential subordinations was introduced by S.S. Miller and P.T. Mocanu in two articles in 1978 [9] and 1981 [10]. This theory subsequently became very popular and its development was broad and fast. Important contributions to this theory can be found in older papers like [5] and newer publications like [13], [18], [4], [14] and [15].

We use the well-known definitions and notations:

Denote by U the unit disc of the complex plane

$$U = \{z \in \mathbb{C} :\ |z| < 1\}.$$

Received 09 July 2020; Accepted 01 June 2021.

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U and let

$$A_n = \{f \in \mathcal{H}(U) : \ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, \ z \in U\},$$

with $A_1 = A$.

Let $S = \{f \in A : \ f \text{ is univalent in } U\}$ be the class of holomorphic and univalent functions in the open unit disc U with the conditions $f(0) = 0$ and $f'(0) = 1$, that is the holomorphic and univalent functions with the following power series development

$$f(z) = z + a_2z^2 + \dots, \ z \in U.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we denote by

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : \ f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \dots, \ z \in U\}.$$

Denote by

$K = \left\{f \in A : \ \operatorname{Re}\left(\dfrac{zf''(z)}{zf'(z)} + 1\right) > 0, \ z \in U\right\}$ the class of normalized convex functions in U and let

$S^* = \left\{f \in A : \ \operatorname{Re}\dfrac{zf'(z)}{f(z)} > 0, \ z \in U\right\}$ denote the class of starlike functions in U.

The core of the theory of differential subordination is found in the monograph published in 2000 by S.S. Miller and P.T. Mocanu [11].

Definition of subordination ([11, p. 4])

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there is a function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$ such that $f(z) = g(w(z))$ for $z \in U$. If g is univalent, then $f \prec g$ or $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition of second-order differential subordination ([11, p. 7])

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the second-order differential subordination

(i) $\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \ z \in U$

then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solutions of the differential subordination, or more simply, a dominant if $p \prec q$ for all p satisfying (i).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (i) is said to be the best dominant of (i). (Note that the best dominant is unique up to a rotation of U).

If we require the more restrictive condition $q \in \mathcal{H}[a, n]$ then p is called an (a, n)-solution, q an (a, n)-dominant and $\widetilde{q}$ the best (a, n)-dominant.

Definition of Briot-Bouquet differential subordination [11, p.80] Let $r, l \in \mathbb{C}$, $r \neq 0$ and let h be a univalent function in U, with $h(0) = a$, and let $p \in \mathcal{H}[a, n]$ satisfy

(ii) $p(z) + \dfrac{zp'(z)}{rp(z) + l} \prec h(z), \ z \in U.$

The first-order differential subordination is called the Briot-Bouquet differential subordination.

In 1969 Bernardi [3] introduced the operator

(iii) $F(z) = \dfrac{\gamma+1}{z^\gamma}\displaystyle\int_0^z f(t)t^{\gamma-1}dt$, for $\gamma = 1, 2, 3, \ldots$,
which generalizes the Libera operator.

Studying subordination properties by using differential and integral operators is a classic topic still of interest at this time, interesting results being currently obtained in forms of criteria for univalence of functions. A recent approach in using operators is to mix a differential and an integral operator as it the case in the very recent papers [1], [16] and [19]. This idea is also used in the present paper for introducing a new differential-integral operator mixing Ruscheweyh differential operator and Bernardi integral operator and by using it, a new class of univalent functions. Some criteria for univalence are derived from proving theorems containing subordination results related to this newly introduced operator.

To prove our main results, we need the following:

Definition 1.1. [17] For $f \in A$, $m \in \mathbb{N}$, $\gamma \in \mathbb{N}^* = \{1, 2, \ldots\}$, the integral operator $B^m : A \to A$ is defined by

$$\begin{aligned}
&B^0[f](z) = f(z)\\
&B^1[f](z) = \frac{\gamma+1}{z^\gamma}\int_0^z B^0[f](t)\cdot t^{\gamma-1}dt = \frac{\gamma+1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt\\
&\vdots\\
&B^m[f](z) = \frac{\gamma+1}{z^\gamma}\int_0^z B^{m-1}[f](t)\cdot t^{\gamma-1}dt. \qquad (1.1)
\end{aligned}$$

Remark 1.2. a) For $m = 1$, $\gamma \in \mathbb{N}^*$, we obtain Bernardi integral operator (iii) defined in [3].

b) For $m = 1$, $\gamma = 1$, we obtain Libera integral operator defined in [7].

c) For $m = 1$, $\gamma = 0$ we obtain Alexander integral operator defined in [2].

d) If $f \in A$ and $f(z) = z + a_2z^2 + a_3z^3 + \ldots = z + \sum_{k=2}^{\infty} a_kz^k$, then

$$B^m[f](z) = z + \sum_{k=2}^{\infty}\frac{(\gamma+1)^m}{(\gamma+k)^m}a_kz^k. \qquad (1.2)$$

e) For $f \in A$, $m \in \mathbb{N}$, $\gamma \in \mathbb{N}^*$. we obtain

$$z(B^m[f](z))' = (\gamma+1)B^{m-1}[f](z) - \gamma B^m[f](z),\ z \in U. \qquad (1.3)$$

Definition 1.3. [20] For $f \in A$, $m \in \mathbb{N}$, the differential operator $R^m : A \to A$ is defined by

$$\begin{aligned}
&R^0[f](z) = f(z)\\
&R^1[f](z) = z(R^0[f](z))' = zf'(z)\\
&\vdots\\
&(m+1)R^{m+1}[f](z) = z(R^m[f](z))' + mR^m[f](z),\ z \in U. \qquad (1.4)
\end{aligned}$$

Remark 1.4. If $f \in A$, $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$R^m[f](z) = z + \sum_{k=2}^{\infty} C_{m+k-1}^{m} a_k z^k = z + \sum_{k=2}^{\infty} \begin{bmatrix} m+k-1 \\ m \end{bmatrix} a_k z^k. \tag{1.5}$$

Lemma A. [13, Th. 10.2.1] *Let $r, l \in \mathbb{C}$, $r \neq 0$, and let h be a convex function that satisfies*

$$\operatorname{Re}[r \cdot h(z) + l] > 0, \ z \in U.$$

If $p \in \mathcal{H}[h(0), n]$, then

$$p(z) + \frac{zp'(z)}{r \cdot p(z) + l} \prec h(z)$$

implies

$$p(z) \prec h(z), \ z \in U.$$

Lemma B. (Hallenbeck and Ruscheweyh [11, Th. 3.1.b, p. 71]) *Let h be a convex function in U, with $h(0) = a$, $\mu \neq 0$ and $\operatorname{Re} \mu \geq 0$. If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\mu} zp'(z) \prec h(z)$$

then

$$p(z) \prec q(z) \prec h(z), \ z \in U,$$

where

$$q(z) = \frac{\frac{\mu}{n}}{z^{\frac{\mu}{n}}} \int_0^z h(t) \cdot t^{\frac{\mu}{n} - 1} dt, \ z \in U.$$

The function q is convex and is the best dominant.

2. Main results

In this paper we define a differential-integral operator $T^m : A \to A$, we define a class of holomorphic univalent functions and study several Briot-Bouquet differential subordinations obtained by using this operator.

Definition 2.1. Let $m \in \mathbb{N}$, $0 \leq \lambda \leq 1$. Denote by $T^m : A \to A$,

$$T^m[f](z) = (1 - \lambda) R^m[f](z) + \lambda B^m[f](z), \ z \in U, \tag{2.1}$$

where R^m is given by (1.4) and B^m is given by (1.1).

Remark 2.2. a) If $f \in A$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, and using (1.2) and (1.5), we have

$$\begin{aligned} T^m[f](z) &= (1 - \lambda) \left(z + \sum_{k=2}^{\infty} \begin{bmatrix} m+k-1 \\ m \end{bmatrix} a_k z^k \right) + \lambda \left(z + \sum_{k=2}^{\infty} \frac{(\gamma+1)^m}{(\gamma+k)^m} a_k z^k \right) \\ &= z + \sum_{k=2}^{\infty} \left\{ \begin{bmatrix} m+k-1 \\ m \end{bmatrix} (1 - \lambda) + \lambda \frac{(\gamma+1)^m}{(\gamma+k)^m} \right\} a_k z^k. \end{aligned} \tag{2.2}$$

b) For $\lambda = 1$, the differential-integral operator T^m coincides with Bernardi integral operator (Definition 1.1).

c) For $\lambda = 0$, the differential-integral operator T^m coincides with R^m, Ruschweyh differential operator (Definition 1.3).

Definition 2.3. If $0 \le \beta < 1$, $0 \le \lambda \le 1$, $m \in \mathbb{N}$, we let $B^m(\lambda, \beta)$ stand for the class of functions $f \in A$, which satisfy the inequality

$$\operatorname{Re}(T^m[f](z))' > \beta, \ z \in U, \tag{2.3}$$

where the differential-integral operator $T^m[f]$ is given by (2.1).

Remark 2.4. a) For $m = 0$, $\beta = 0$, $0 \le \lambda \le 1$, the operator $T^m[f]$ becomes

$$\begin{aligned} T_0[f](z) &= (1-\lambda)R^0[f](z) + \lambda B^0[f](z) \\ &= (1-\lambda)f(z) + \lambda f(z) = f(z), \ z \in U, \end{aligned}$$

then $B^m(\lambda, \beta)$ becomes

$$B^0(\lambda, 0) = R = \{f \in A : \ \operatorname{Re} f'(z) > 0, \ z \in U\},$$

called the class of functions with bounded rotation.

This class of functions was studied by J.W. Alexander [2] and he proved that $R \subset S$. J. Krzyz [6] and P.T. Mocanu [12] have proved that $R \not\subset S^*$. A more systematic study of class R was done by Mac Gregor [8].

b) For $m = 0$, $0 \le \beta < 1$, $0 \le \lambda \le 1$, we have

$$B^0(\lambda, \beta) = M(\beta) = \{f \in A : \ \operatorname{Re} f'(z) > \beta\} \subset R.$$

Theorem 2.5. *The set $B^m(\lambda, \beta)$ is convex.*

Proof. Let the functions

$$f_j(z) = z + \sum_{k=2}^{\infty} \alpha_{kj} z^j, \ j = 1, 2, \ z \in U,$$

where

$$\alpha_{kj} = a_{kj} \left\{ (1-\lambda) \begin{bmatrix} m+k-1 \\ m \end{bmatrix} + \lambda \frac{(\gamma+1)^m}{(\gamma+k)^m} \right\},$$

be in the class $B^m(\lambda, \beta)$. It is sufficient to show that the function

$$h(z) = \mu_1 f_1(z) + \mu_2 f_2(z), \ z \in U,$$

with $\mu_1, \mu_2 \ge 0$ and $\mu_1 + \mu_2 = 1$ is in $B^m(\lambda, \beta)$.
Since $h(z) = \mu_1 f_1(z) + \mu_2 f_2(z)$, $z \in U$, we have

$$T^m[h](z) = z + \sum_{k=2}^{\infty} \left\{ (1-\lambda) \begin{bmatrix} m+k-1 \\ m \end{bmatrix} + \lambda \frac{(\gamma+1)^m}{(\gamma+k)^m} \right\} (\mu_1 a_{k1} + \mu_2 a_{k2}) z^k. \tag{2.4}$$

Differentiating (2.4), we have

$$(T^m[h](z))' = 1 + \sum_{k=2}^{\infty} \left\{ (1-\lambda) \begin{bmatrix} m+k-1 \\ m \end{bmatrix} + \lambda \frac{(\gamma+1)^m}{(\gamma+k)^m} \right\} (\mu_1 a_{k1} + \mu_2 a_{k2}) k z^{k-1}.$$

Hence

$$\mathrm{Re}\,(T^m[h](z))' = 1 + \mathrm{Re}\sum_{k=2}^{\infty}\left\{(1-\lambda)\begin{bmatrix} m+k-1 \\ m \end{bmatrix} + \lambda\frac{(\gamma+1)^m}{(\gamma+k)^m}\right\}\mu_1 a_{k1} k z^{k-1}$$
$$+ \mathrm{Re}\sum_{k=2}^{\infty}\left\{(1-\lambda)\begin{bmatrix} m+k-1 \\ m \end{bmatrix} + \lambda\frac{(\gamma+1)^m}{(\gamma+k)^m}\right\}\mu_2 a_{k2} k z^{k-1}. \quad (2.5)$$

Since $f_1, f_2 \in B^m(\lambda,\beta)$, we have

$$\mu_1 \mathrm{Re}\sum_{k=2}^{\infty}\left\{(1-\lambda)\begin{bmatrix} m+k-1 \\ m \end{bmatrix} + \lambda\frac{(\gamma+1)^m}{(\gamma+k)^m}\right\} a_{k1} k z^{k-1} > \mu_1(\beta-1) \quad (2.6)$$

$$\mu_2 \mathrm{Re}\sum_{k=2}^{\infty}\left\{(1-\lambda)\begin{bmatrix} m+k-1 \\ m \end{bmatrix} + \lambda\frac{(\gamma+1)^m}{(\gamma+k)^m}\right\} a_{k2} k z^{k-1} > \mu_2(\beta-1) \quad (2.7)$$

Using (2.6) and (2.7), we obtain

$$\mathrm{Re}\,(T^m[h](z))' > 1 + \mu_1(\beta-1) + \mu_2(\beta-1)$$

and since $\mu_1 + \mu_2 = 1$, we deduce

$$\mathrm{Re}\,(T^m[h](z))' > \beta,$$

i.e. $B^m(\lambda,\beta)$ is convex.

Theorem 2.6. *Let $0 \le \beta < 1$, $0 \le \lambda \le 1$, $m \in \mathbb{N}$, $f \in A$.*
If $f \in B^n(\lambda,\beta)$, then we have

$$\mathrm{Re}\,\frac{T^m[f](z)}{z} > 2\beta - 1 + 2(1-\beta)\ln 2 = \delta.$$

Proof. We prove that $\delta \in [0,1)$, $\delta = 2\beta(1-\ln 2) + 2\ln 2 - 1$. For $\ln 2 \approx 0,69$,

$$\delta - 2\beta(1-0,69) + 2\cdot 0,69 - 1$$
$$= 2\beta\cdot 0,31 + 0,38 = \beta\cdot 0,62 + 0,38.$$

Hence $0 \le \beta < 1$. We have

$$0 \le \beta\cdot 0,62 < 0,62,$$
$$0,38 \le \beta\cdot 0,62 + 0,38 < 0,62 + 0,38,$$
$$0,38 \le \beta\cdot 0,62 + 0,38 < 1,$$
$$0,38 \le \delta < 1,\ \ \delta \in [0,38,1).$$

Let the convex function

$$h(z) = \frac{1 + (2\beta-1)z}{1+z},\ \ 0 \le \beta < 1,\ \ z \in U. \quad (2.8)$$

For $z \in U$, we have $\mathrm{Re}\, h(z) > \beta$ and $h(0) = 1$.
From the hypothesis we have that $f \in B^m(\lambda,\beta)$, then from Definition 2.3 we have

$$\mathrm{Re}\,(T^m[f](z))' > \beta,\ \ z \in U. \quad (2.9)$$

Let

$$p(z) = \frac{T^m[f](z)}{z},\ \ z \in U. \quad (2.10)$$

Using (2.3) in (2.10) we have

$$p(z) = \frac{z + \sum_{k=2}^{\infty} a_k z^k \left\{ \begin{bmatrix} m+k-1 \\ m \end{bmatrix} (1-\lambda) + \lambda \frac{(\gamma+1)^m}{(k+\gamma)^m} \right\}}{z}$$
$$= 1 + \sum_{k=2}^{\infty} a_k z^{k-1} \left\{ \begin{bmatrix} m+k-1 \\ m \end{bmatrix} (1-\lambda) + \lambda \frac{(\gamma+1)^m}{(k+\gamma)^m} \right\},$$
$$p(0) = 1 \text{ and } p \in \mathcal{H}[1,1].$$

From (2.10), we have

$$T^m[f](z) = zp(z), \ z \in U. \tag{2.11}$$

Differentiating (2.11), we obtain

$$(T^m[f](z))' = p(z) + zp'(z). \tag{2.12}$$

Using (2.12) in (2.9), we have

$$\operatorname{Re}[p(z) + zp'(z)] > \beta, \ z \in U. \tag{2.13}$$

Relation (2.13) can be written as a subordination of the form

$$p(z) + zp'(z) \prec h(z) = \frac{1+(2\beta-1)z}{1+z}, \ z \in U.$$

Using Lemma B, for $\mu = 1$, $n = 1$, we have

$$p(z) \prec q(z),$$

where

$$q(z) = \frac{1}{z}\int_0^z \frac{1+(2\beta-1)t}{t} dt = 2\beta - 1 + 2(1-\beta)\frac{\ln(1+z)}{z}$$

i.e.,

$$\frac{T^m[f](z)}{z} \prec 2\beta - 1 + 2(1-\beta)\frac{\ln(1+z)}{z} = q(z), \ z \in U.$$

The function q is convex and is the best dominant.
Since q is convex function and

$$p(z) \prec q(z) = 2\beta - 1 + 2(1-\beta)\frac{\ln(1+z)}{z}, \ z \in U,$$

we have

$$\operatorname{Re} p(z) > \operatorname{Re} q(1) = 2\beta - 1 + 2(1-\beta)\ln 2 = \delta. \tag{2.14}$$

Using (2.10), the relation (2.14) becomes

$$\operatorname{Re} \frac{T^m[f](z)}{z} > \delta = 2\beta - 1 + 2(1-\beta)\ln 2.$$

From Theorem 2.6 we deduce the following corollary:

Corollary 2.7. *Let $0 \le \lambda \le 1$, $f \in A$, $m \in \mathbb{N}$, $\delta = 2\beta - 1 + 2(1-\beta)\ln 2$.*
If $f \in B^m(\lambda, \delta)$, then

$$\operatorname{Re} \frac{T^m[f](z)}{z} > \delta = 2\beta - 1 + 2(1-\beta)\ln 2.$$

Proof. From the proof of Theorem 2.6, we can see that

$$\frac{T^m[f](z)}{z} \prec q(z) = 2\beta - 1 + 2(1-\beta)\frac{\ln(1+z)}{z}, \ z \in U.$$

Since q is convex function, we have that

$$\operatorname{Re}\frac{T^m[f](z)}{z} > \operatorname{Re} q(1) = \delta = 2\beta - 1 + 2(1-\beta)\ln 2.$$

Theorem 2.8. *Let h be a convex function, with $h(0) = 1$ and*

$$\operatorname{Re} h(z) > 0, \ z \in U.$$

If $f \in A$, $0 \le \lambda \le 1$, $m \in \mathbb{N}$ and satisfies the differential subordination

$$(T^m[f](z))' + \frac{z(T^m[f](z))''}{(T^m[f](z))'} \prec h(z) \tag{2.15}$$

then

$$(T^m[f](z))' \prec h(z), \ z \in U.$$

Proof. We let

$$p(z) = (T^m[f](z))', \ z \in U. \tag{2.16}$$

Using (2.2) in (2.16), we have

$$\begin{aligned} p(z) &= \left(z + \sum_{k=2}^{\infty}\left\{\begin{bmatrix} m+k-1 \\ m \end{bmatrix}(1-\lambda) + \lambda\frac{(\gamma+1)^m}{(\gamma+k)^m}\right\} a_k z^k\right)' \\ &= 1 + \sum_{k=2}^{\infty}\left\{\begin{bmatrix} m+k-1 \\ m \end{bmatrix}(1-\lambda) + \lambda\frac{(\gamma+1)^m}{(\gamma+k)^m}\right\} a_k k z^{k-1} \\ &= 1 + p_1 z + p_2 z^2 + \ldots \end{aligned} \tag{2.17}$$

and $p(0) = 1$, $p \in \mathcal{H}[1,1]$.
Differentiating (2.16), we get

$$\frac{p'(z)}{p(z)} = \frac{(T^m[f](z))''}{(T^m[f](z))'}, \quad \frac{zp'(z)}{p(z)} = \frac{z(T^m[f](z))''}{(T^m[f](z))'} \tag{2.18}$$

and

$$p(z) + \frac{zp'(z)}{p(z)} = (T^m[f](z))' + \frac{z(T^m[f](z))''}{(T^m[f](z))'}. \tag{2.19}$$

Using (2.19), the differential subordination (2.15) becomes

$$p(z) + \frac{zp'(z)}{p(z)} \prec h(z), \ z \in U.$$

Using Lemma A, for $r = 1$, $l = 0$, we obtain

$$p(z) \prec h(z), \ z \in U,$$

i.e.

$$(T^m[f](z))' \prec h(z), \ z \in U.$$

From Theorem 2.8 we deduce the following sufficient conditions for univalent function.

Criterion 2.9. *Let*

$$h(z) = \frac{1+(2\beta-1)z}{1+z}, \ 0 \le \beta < 1,$$

be convex function with $h(0) = 1$ *and* $\operatorname{Re} h(z) > \beta$, $z \in U$.
If $f \in A$, $0 \le \lambda < 1$, $m \in \mathbb{N}$ *and satisfies the differential subordination*

$$(T^m[f](z))' + \frac{z(T^m[f](z))''}{(T^m[f](z))'} \prec \frac{1+(2\beta-1)z}{1+z},$$

then

$$(T^m[f](z))' \prec \frac{1+(2\beta-1)z}{1+z}, \ z \in U, \tag{2.20}$$

where $T^m[f]$ *is defined in (2.1). Hence* f *is an univalent function.*

Proof. Since h is convex, with $h(1) = \beta$, $0 \le \beta < 1$, $\operatorname{Re} h(z) > \beta$, relation (2.20) is equivalent to

$$\operatorname{Re}(T^m[f](z))' > \operatorname{Re} h(1) = \beta.$$

From Definition 2.3, we have $f \in B^m(\lambda, \beta)$, hence f is an univalent function.

Criterion 2.10. *Let*

$$h(z) = \frac{1-z}{1+z},$$

with $h(0) = 1$, $\operatorname{Re} h(z) > 0$, $z \in U$.
If $f \in A$, $0 \le \lambda < 1$, $m \in \mathbb{N}$ *and satisfies the differential subordination*

$$(T^m[f](z))' + \frac{z(T^m[f](z))''}{(T^m[f](z))'} \prec \frac{1-z}{1+z}$$

then

$$(T^m[f](z))' \prec \frac{1-z}{1+z}, \ z \in U, \tag{2.21}$$

where $T^m[f]$ *is defined in (2.1). Hence* f *is an univalent function.*

Proof. Since h is convex, with $h(1) = 0$, $\operatorname{Re} h(z) > 0$, $z \in U$, relation (2.21) becomes

$$\operatorname{Re}(T^m[f](z))' > \operatorname{Re} h(1) > 0, \ z \in U.$$

From Definition 2.3, we have $f \in B^m(\lambda, 0)$, hence f is an univalent function.

Theorem 2.11. *Let* h *be a convex function,* $h(0) = 1$, *with*

$$\operatorname{Re} h(z) > 0, \ z \in U.$$

If $f \in A$, $0 \le \lambda \le 1$, $m \in \mathbb{N}$ *and satisfies the differential subordination*

$$\frac{T^m[f](z)}{z} + \frac{z(T^m[f](z))'}{T^m[f](z)} - 1 \prec h(z), \ z \in U, \tag{2.22}$$

then

$$\frac{T^m[f](z)}{z} \prec h(z), \ z \in U.$$

Proof. We let

$$p(z)=\frac{T^m[f](z)}{z},\ z\in U. \tag{2.23}$$

Using (2.2) in (2.23), we get

$$\begin{aligned}
p(z)&=\frac{z+\sum\limits_{k=2}^{\infty}\left\{\begin{bmatrix} m+k-1\\ m\end{bmatrix}(1-\lambda)+\lambda\dfrac{(\gamma+1)^m}{(\gamma+k)^m}\right\}a_kz^k}{z}\\
&=1+\sum_{k=2}^{\infty}\left\{\begin{bmatrix} m+k-1\\ m\end{bmatrix}(1-\lambda)+\lambda\frac{(\gamma+1)^m}{(\gamma+k)^m}\right\}a_kz^{k-1}\\
&=1+p_1z+p_2z^2+\dots
\end{aligned}$$

and $p(0)=1$, $p\in\mathcal{H}[1,1]$.
From (2.23), we have

$$zp(z)=T^m[f](z),\ z\in U. \tag{2.24}$$

Differentiating (2.24), we get

$$\frac{1}{z}+\frac{p'(z)}{p(z)}=\frac{(T^m[f](z))'}{T[f](z)},$$

$$1+\frac{zp'(z)}{p(z)}=\frac{z(T^m[f](z))'}{T^m[f](z)},$$

and

$$p(z)+\frac{zp'(z)}{p(z)}=\frac{T^m[f](z)}{z}+\frac{z(T^m[f](z))'}{T^m[f](z)}-1. \tag{2.25}$$

Using (2.25), the differential subordination (2.22) becomes

$$p(z)+\frac{zp'(z)}{p(z)}\prec h(z),\ z\in U.$$

Using Lemma A, for $r=1$, $l=0$, we get

$$p(z)\prec h(z),$$

i.e.

$$\frac{T^m[f](z)}{z}\prec h(z),\ z\in U.$$

Example 2.12. Let

$$f(z)=z+\frac{6}{31}z^2,\ m=2,\ k=2,\ \gamma=1,\ \lambda=\frac{1}{2},\ \beta=\frac{1}{3},$$

$$T^2[f](z)=z+\frac{1}{3}z^2,\ h(z)=\frac{1-\frac{1}{3}z}{1+z},\ h(0)=1,\ \operatorname{Re}h(z)>\frac{1}{3},$$

$$(T^2[f](z))'=1+\frac{2}{3}z,\ (T^2[f](z))''=\frac{2}{3}.$$

Using Theorem 2.8, we get:

$$1+\frac{2}{3}z+\frac{2z}{3+2z}\prec\frac{1-\frac{1}{3}z}{1+z},$$

implies

$$1+\frac{2}{3}z\prec\frac{1-\frac{1}{3}z}{1+z},\ z\in U.$$

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 261–268
DOI: 10.24193/subbmath.2023.2.03

An upper bound of the Hankel determinant of third order for the inverse of reciprocal of bounded turning functions

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Abstract. The objective of this paper is to obtain an upper bound of the third order Hankel determinant for the inverse of the function f, when f belongs to the reciprocal of bounded turning functions with new approach.

Mathematics Subject Classification (2010): 30C45, 30C50.

Keywords: Bounded turning function, upper bound, Hankel determinant, positive real function.

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions f of the form

$$f(z) = z + \sum_{n\geq 2} a_n z^n \tag{1.1}$$

in the open unit disc $\mathcal{U}_d = \{z \in \mathbb{C} : |z| < 1\}$ standardized by $f(0) = 0$, and $f'(0) = 1$. Let S be the subclass of $\mathcal{A}$, consisting of univalent functions. In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture also called as Coefficient conjecture, which states that for a univalent function its n^{th}-Taylor's coefficient is bounded by n (see [5]). The bounds of the coefficients for these functions give information about their geometric properties. A typical problem in geometric function theory is to study a functional made up of combination of the coefficients of the original function. The

Received 27 March 2020; Accepted 04 May 2020.

Hankel determinant of order q for the regular mapping f, was defined by Pommerenke [23], as follows.

$$H_{q,t}(f) = \begin{vmatrix} a_t & a_{t+1} & \cdots & a_{t+q-1} \\ a_{t+1} & a_{t+2} & \cdots & a_{t+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{t+q-1} & a_{t+q} & \cdots & a_{t+2q-2} \end{vmatrix}. \tag{1.2}$$

Here $a_1 = 1$, q and t are integers, positive in nature. The determinant given in (1.2) has been investigated by many authors, a few of them are cited here. Ehrenborg [8] studied the Hankel determinant of exponential polynomials. Noor [20] determined the rate of growth of $H_{q,t}$ as $t \to \infty$ for the functions in S with bounded boundary. The Hankel transform of an integer sequence and some of its features were studied by Layman (see [14]). For $q = 2$ and $t = 1$ in (1.2), we obtain $H_{2,1}(f)$, the Fekete-Szegö functional is the classical problem settled by Fekete-Szegö [9] is to find for each $\lambda \in [0, 1]$, the maximum value of the coefficient functional, defined by $\phi_\lambda(f) := |a_3 - \lambda a_2^2|$ over the class S and was proved by using Loewner method. Ali [1] found sharp bounds of the first four coefficients and sharp estimate for the Fekete-Szegö functional $|t_3 - \delta a_2^2|$, where δ is real, for the inverse function of f defined as

$$f^{-1}(w) = w + \sum_{n \geq 2} q_n w^n,$$

when $f^{-1} \in \widetilde{ST}(\alpha)$, the class of strongly starlike functions of order α with $\alpha \in (0, 1]$. In recent years, the research on Hankel determinants has focused on the estimation of $H_{2,2}(f)$, known as the second Hankel determinant obtained for $q = 2 = t$ in (1.2), given by

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

Many authors obtained results associated with estimation of an upper bound of the functional $H_{2,2}(f)$ for various subclasses of univalent and multivalent analytic functions. The exact (sharp) estimates of $H_{2,2}(f)$ for the subclasses of S namely, bounded turning, starlike and convex functions denoted by $\mathcal{R}$, S^* and $\mathcal{K}$ respectively in $\mathcal{U}_d$, i.e., functions satisfying the conditions

$$\mathrm{Re} f'(z) > 0, \ \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \text{ and } \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$$

were proved by Janteng et al. [11, 12] and determined the bounds as 4/9, 1 and 1/8 respectively. For the class of Ma-Minda starlike functions, the sharp bound of the second Hankel determinant was obtained by Lee et al. [16]. Choosing $q = 2$ and $t = p + 1$ in (1.2), we obtain the second Hankel determinant for the p-valent function (see [26]), as follows.

$$H_{2,p+1}(f) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1} a_{p+3} - a_{p+2}^2,$$

The case $q = 3$ appears to be much more difficult than the case $q = 2$. Very few papers have been devoted for the study of third order Hankel determinant denoted by $H_{3,1}(f)$, obtained for $q = 3$ and $t = 1$ in (1.2), namely

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

Expanding the determinant, we have

$$H_{3,1}(f) = a_1(a_3a_5 - a_4^2) + a_2(a_3a_4 - a_2a_5) + a_3(a_2a_4 - a_3^2), \tag{1.3}$$
$$\Leftrightarrow\ H_{3,1}(f) = H_{2,3}(f) + a_2J_2 + a_3H_{2,2}(f),$$

where $J_2 = (a_3a_4 - a_2a_5)$ and $H_{2,3}(f) = (a_3a_5 - a_4^2)$.
The concept of estimation of an upper bound of $H_{3,1}(f)$ was firstly introduced and studied by Babalola [3], who tried to estimate for this functional to the classes $\mathcal{R}$, S^* and $\mathcal{K}$, obtained as follows.

(i) $f \in S^* \Rightarrow |H_{3,1}(f)| \leq 16$.
(ii) $f \in \mathcal{K} \Rightarrow |H_{3,1}(f)| \leq 0.714$.
(iii) $f \in \mathcal{R} \Rightarrow |H_{3,1}(f)| \leq 0.742$.

As a result of this paper, Raza and Malik [24] obtained an upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli. Sudharsan et al. [25] derived an upper bound of the third kind Hankel determinant for a subclass of analytic functions, namely

$$\mathcal{C}_\alpha^\beta = \mathrm{Re}\left\{\frac{(zf'(z) + \alpha z^2 f''(z))'}{f'(z)}\right\} > \beta,$$

where $(0 \leq \alpha \leq 1)$ and $(0 \leq \beta < 1)$. Bansal et al. [4] improved the upper bound of $H_{3,1}(f)$ for some of the classes estimated by Babalola [3] to some extent. Recently, Zaprawa [29] improved all the results obtained by Babalola [3]. Further, Orhan and Zaprawa [21] obtained an upper bound of the third kind Hankel determinant for the classes S^* and $\mathcal{K}$ functions of order α $(0 \leq \alpha < 1)$. Very recently, Kowalczyk et al. [13] calculated sharp upper bound of $H_{3,1}(f)$ for the class $\mathcal{K}$ of convex functions and showed as $|H_{3,1}(f)| \leq \frac{4}{135}$, which is more refined bound than the bound derived by Zaprawa [29]. Lecko et al. [15] determined sharp bound of the third order Hankel determinant for starlike functions of order 1/2. Arif et al. [2] estimated an upper bound of the Fourth Hankel determinant for the family of bounded turning functions. Motivated by the results obtained by different authors mentioned above and who are working in this direction (see [6,26]), in this paper, we are making an attempt to introduce a new subclass of analytic functions and obtain an upper bound of the functional $H_{3,1}(f^{-1})$, where f^{-1} is the inverse function for the function f belonging to this class, defined as follows.

Definition 1.1. A function $f(z) \in \mathcal{A}$ is said to be in the class $\widehat{RT}$, consisting of functions whose reciprocal derivative have a positive real part (also called reciprocal of bounded turning functions) (for the properties of bounded turning functions (see [19]), given by

$$\mathrm{Re}\left\{\frac{1}{f'(z)}\right\} > 0,\ z \in \mathcal{U}_d. \tag{1.4}$$

In proving our result, we require a few sharp estimates in the form of lemmas valid for functions with positive real part.
Let $\mathcal{P}$ denote the class of functions consisting of g, such that

$$g(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = 1 + \sum_{n \geq 1} c_n z^n, \tag{1.5}$$

which are analytic in $\mathcal{U}_d$ and $\text{Re}g(z) > 0$ for $z \in \mathcal{U}_d$. Here g is called the Caratheodory function [7].

Lemma 1.2. ([10]) *If $g \in \mathcal{P}$, then the sharp estimate $|c_n - \mu c_k c_{n-k}| \leq 2$, holds for $n, k \in \mathbb{N} = \{1, 2, 3...\}$, with $n > k$ and $\mu \in [0, 1]$.*

Lemma 1.3. ([18]) *If $g \in \mathcal{P}$, then the sharp estimate $|c_n - c_k c_{n-k}| \leq 2$, holds for $n, k \in \mathbb{N}$, with $n > k$.*

Lemma 1.4. ([22]) *If $g \in \mathcal{P}$ then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the mobious transformation $g(z) = \frac{1+z}{1-z}$, $z \in \mathcal{U}_d$.*

In order to obtain our result, we referred to the classical method devised by Libera and Zlotkiewicz [17], used by several authors.

2. Main result

Theorem 2.1. *If $f \in \widehat{RT}$ and $f^{-1}(w) = w + \sum_{n \geq 2} q_n w^n$ near the origin i.e., $w = 0$ is the inverse function of f, given in (1.1) then*

$$|H_{3,1}(f^{-1})| \leq \frac{527}{540}.$$

Proof. For the function $f \in \widehat{RT}$, by virtue of Definition 1.1, there exists a holomorphic function $g \in \mathcal{P}$ in $\mathcal{U}_d$ with $g(0) = 1$ and $\text{Re}g(z) > 0$ such that

$$\frac{1}{f'(z)} = g(z) \Leftrightarrow \; 1 = g(z)f'(z). \tag{2.1}$$

Replacing f' and g with their series expressions in (2.1), upon simplification, we get

$$\begin{aligned} a_2 &= -\frac{c_1}{2}; \\ a_3 &= -\frac{1}{3}(c_2 - c_1^2); \\ a_4 &= -\frac{1}{4}(c_3 - 2c_1c_2 + c_1^3); \\ a_5 &= -\frac{1}{5}(c_4 - 2c_1c_3 + 3c_1^2c_2 - c_2^2 - c_1^4). \end{aligned} \tag{2.2}$$

According to Koebe's $(\frac{1}{4})^{th}$- theorem, also known as one-quarter theorem every holomorphic and univalent function ϖ in $\mathcal{U}_d$ possesses an inverse denoted by ϖ^{-1}, satisfying

$$z = \left\{\varpi^{-1}(\varpi(z))\right\}, \; z \in \mathcal{U}_d$$

and

$$\varpi\left\{\varpi^{-1}(w)\right\} = w, \quad \left(|w| < \rho_0(f); \rho_0(f) \geq \frac{1}{4}\right).$$

Consider

$$w = \varpi\left\{\varpi^{-1}(w)\right\} = \left\{\varpi^{-1}(w)\right\} + \sum_{n\geq 2} a_n \left\{\varpi^{-1}(w)\right\}^n$$

$$\Leftrightarrow w = \left\{w + \sum_{n\geq 2} q_n w^n\right\} + \sum_{n\geq 2} a_n \left\{w + \sum_{n\geq 2} q_n w^n\right\}^n.$$

By simple computation, we get

$$\Big[(q_2 + a_2)w^2 + (q_3 + 2a_2q_2 + a_3)w^3 + (q_4 + 2a_2q_3 + a_2q_2^2 + 3a_3q_2 + a_4)w^4 + (q_5 + 2a_2q_4 + 2a_2q_2t_3 + 3a_3q_3 + 3a_3q_2^2 + 4a_4q_2 + a_5)w^5 + ...\Big] = 0. \quad (2.3)$$

Equating the coefficients of w^2, w^3, w^4 and w^5 in (2.3), upon simplification, we obtain

$$q_2 = -a_2;\ q_3 = \left\{-a_3 + 2a_2^2\right\};\ q_4 = \left\{-a_4 + 5a_2a_3 - 5a_2^3\right\};$$

$$q_5 = \left\{-a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4\right\}. \quad (2.4)$$

Simplifying the expressions (2.2) and (2.4), we get

$$q_2 = \frac{c_1}{2};\ q_3 = \frac{1}{6}\left\{2c_2 + c_1^2\right\};\ q_4 = \frac{1}{24}\left\{6c_3 + 8c_1c_2 + c_1^3\right\};$$

$$q_5 = \frac{1}{120}\left\{24c_4 + 42c_1c_3 + 22c_1^2c_2 + 16c_2^2 + c_1^4\right\}. \quad (2.5)$$

At this juncture, based on the determinant $H_{3,1}(f)$ given in (1.3), the third order Hankel determinant for the inverse function of f, namely

$$f^{-1}(w) = w + \sum_{n\geq 2} q_n w^n$$

near the origin i.e., $w = 0$, can be defined as

$$H_{3,1}(f^{-1}) = \begin{vmatrix} q_1 & q_2 & q_3 \\ q_2 & q_3 & q_4 \\ q_3 & q_4 & q_5 \end{vmatrix} (q_1 = 1). \quad (2.6)$$

Expanding the determinant, we get

$$H_{3,1}(f^{-1}) = q_1(q_3q_5 - q_4^2) + q_2(q_3q_4 - q_2q_5) + q_3(q_2q_4 - q_3^2). \quad (2.7)$$

Putting the values of q_2, q_3, q_4 and q_5 from (2.5) in the functional given in (2.7), it simplifies to

$$H_{3,1}(f^{-1}) = \Big[\frac{1}{15}c_2c_4 + \frac{1}{135}c_2^3 - \frac{1}{16}c_3^2 - \frac{1}{60}c_1^2c_4 + \frac{1}{30}c_1c_2c_3 - \frac{1}{180}c_1^2c_2^2$$

$$+\frac{1}{720}c_1^4c_2 - \frac{1}{120}c_1^3c_3 - \frac{1}{8640}c_1^6\Big]. \quad (2.8)$$

Upon grouping the terms in the expression (2.8), we have

$$H_{3,1}(f^{-1}) = \Big[\frac{1}{60}c_4(c_2 - c_1^2) - \frac{1}{16}c_3(c_3 - \frac{16}{60}c_1c_2) - \frac{1}{135}c_2(c_4 - c_2^2) - \frac{1}{60}c_2(c_4 - c_1c_3)$$

$$+\frac{1}{720}c_1^4(c_2 - \frac{1}{12}c_1^2) + \frac{2}{27}c_2c_4 - \frac{1}{120}c_1^3c_3 - \frac{1}{180}c_1^2c_2^2\Big]. \quad (2.9)$$

Applying the triangle inequality in (2.9), we obtain

$$\Big|H_{3,1}(f^{-1})\Big| \leq \Big[\frac{1}{60}|c_4||c_2-c_1^2| + \frac{1}{16}|c_3||c_3-\frac{16}{60}c_1c_2| + \frac{1}{135}|c_2||c_4-c_2^2| + \frac{1}{60}|c_2||c_4-c_1c_3|$$

$$+\frac{1}{720}|c_1^4||c_2 - \frac{1}{12}c_1^2| + \frac{2}{27}|c_2||c_4| + \frac{1}{120}|c_1^3||c_3| + \frac{1}{180}|c_1^2||c_2^2|\Big]. \quad (2.10)$$

Upon using the lemmas given in 1.2, 1.3 and 1.4 in the inequality (2.10), after simplifying, we get

$$\Big|H_{3,1}(f^{-1})\Big| \leq \frac{527}{540}. \quad (2.11)$$

□

Remark 2.2. The result, obtained in (2.11) is far better than the result obtained by the authors (see [28]).

Acknowledgements. The authors would like to express sincere gratitude to the esteemed referee(s) for their valuable comments and constructive suggestions, which have helped very much in improving the research paper.

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 269–278
DOI: 10.24193/subbmath.2023.2.04

On the order of convolution consistence of certain classes of harmonic functions with varying arguments

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Abstract. Making use of a modified Hadamard product or convolution of harmonic functions with varying arguments, combined with an integral operator, we study when these functions belong to a given class. Following an idea of U. Bednarz and J. Sokol we define the order of convolution consistence of three classes of functions and determine it for certain classes of harmonic functions with varying arguments defined using a convolution operator.

Mathematics Subject Classification (2010): 30C45, 30C50.

Keywords: Analytic functions with negative coefficients, univalent functions, extreme points, order of convolution consistence, starlikeness, convexity.

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathcal{G}$ if both u and v are real and harmonic in $\mathcal{G}$. In any simply-connected domain $D \subset \mathcal{G}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [5]).
Denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g} \tag{1}$$

which are harmonic, univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f'_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$, the functions h and g analytic in $\mathcal{U}$ can be expressed in the following

Received 29 June 2021; Accepted 14 July 2021.

forms:

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = \sum_{m=1}^{\infty} b_m z^m \quad (0 \le b_1 < 1)\,,$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m} \quad (0 \le b_1 < 1)\,. \tag{2}$$

For functions $f \in \mathcal{H}$ given by (2) and $F \in \mathcal{H}$ given by

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{m=2}^{\infty} A_m z^m + \overline{\sum_{m=1}^{\infty} B_m z^m}, \tag{3}$$

we recall the Hadamard product (or convolution) of f and F by

$$(f * F)(z) = z + \sum_{m=2}^{\infty} a_m A_m z^m + \overline{\sum_{m=1}^{\infty} b_m B_m z^m} \quad (z \in \mathcal{U})\,. \tag{4}$$

In terms of the Hadamard product (or convolution), we choose F as a fixed function in $\mathcal{H}$ such that $(f * F)(z)$ exists for any $f \in \mathcal{H}$, and for various choices of F we get different linear operators which have been studied in recent past.
In [10] it is defined and studied a subclass of $\mathcal{H}$ denoted by $S_{\mathcal{H}}(F;\gamma)$, for $0 \le \gamma < 1$, which involves the convolution (4) and consist of functions of the form (1) satisfying the inequality:

$$\frac{\partial}{\partial\theta}\left(arg\left[(f * F)(z)\right]\right) > \gamma \tag{5}$$

$0 \le \theta < 2\pi$ and $z = re^{i\theta}$. Equivalently

$$Re\left\{\frac{z\left(h\left(z\right) * H\left(z\right)\right)' - \overline{z\left(g\left(z\right) * G\left(z\right)\right)'}}{h\left(z\right) * H\left(z\right) + \overline{g\left(z\right) * G\left(z\right)}}\right\} \ge \gamma \tag{6}$$

where $z \in \mathcal{U}$. We also let $\mathcal{V}_{\mathcal{H}}(F;\gamma) = S_{\mathcal{H}}(F;\gamma) \bigcap V_{\mathcal{H}}$ where $V_{\mathcal{H}}$ is the class of harmonic functions with varying arguments introduced by Jahangiri and Silverman [8], consisting of functions f of the form (1) in $\mathcal{H}$ for which there exists a real number ϕ such that

$$\eta_m + (m-1)\,\phi \equiv \pi\,(mod 2\pi)\,, \quad \delta_m + (m+1)\,\phi \equiv 0\,(mod 2\pi)\,, \quad m \ge 2, \tag{7}$$

where $\eta_m = arg\,(a_m)$ and $\delta_m = arg\,(b_m)$.
Some of the function classes emerge from the function class $S_{\mathcal{H}}(F;\gamma)$ defined above. Indeed, if we specialize the function $F(z)$ we can obtain, respectively, (see [10]) the class of functions defined using: the Wright's generalized operator on harmonic functions ([11], [16]), the Dziok-Srivastava operator on harmonic functions ([1]), the Carlson-Shaffer operator ([4]), the Ruscheweyh derivative operator on harmonic functions ([7], [9], [12]), the Srivastava-Owa fractional derivative operator ([15]), the Sălăgean derivative operator for harmonic functions ([6], [13]).

In the following we suppose that $F(z)$ is of the form

$$F(z) = H(z) + \overline{G(z)} = z + \overline{z} + \sum_{m=2}^{\infty} C_m \left(z^m + \overline{z^m}\right), \tag{8}$$

where $C_m \geq 0\, (m \geq 2)$.
In [10] the following characterization theorem is proved

Theorem 1. *Let $f = h + \overline{g}$ be given by* (2) *with restrictions* (7) *and* $0 \leq b_1 < \dfrac{1-\gamma}{1+\gamma}$, $0 \leq \gamma < 1$. *Then $f \in \mathcal{V}_{\mathcal{H}}(F, \gamma)$ if and only if the inequality*

$$\sum_{m=2}^{\infty} \left(\frac{m-\gamma}{1-\gamma}\,|a_m| + \frac{m+\gamma}{1-\gamma}\,|b_m|\right) C_m \leq 1 - \frac{1+\gamma}{1-\gamma} b_1 \tag{9}$$

holds true.

Let consider the integral operator (for the analytic case see [3], [2], [13])

$$\mathcal{I}^s : f \in \mathcal{V}_{\mathcal{H}}(F, \gamma) \to \mathcal{V}_{\mathcal{H}}(F, \gamma),\ \ s \in \mathbb{R},$$

such that

$$\mathcal{I}^s f(z) = \mathcal{I}^s \left(z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m} \right) = z + \sum_{m=2}^{\infty} \frac{a_m}{m^s} z^m + \overline{\sum_{m=1}^{\infty} \frac{b_m}{m^s} z^m}. \tag{10}$$

Definition 1. The modified Hadamard product or $\circledast$-convolution of two functions f_1 and f_2 in $\mathcal{V}_{\mathcal{H}}$ of the form

$$f_1(z) = z + \sum_{m=2}^{\infty} a_{1,m} z^m + \overline{\sum_{m=1}^{\infty} b_{1,m} z^m} \text{ and } f_2(z) = z + \sum_{m=2}^{\infty} a_{2,m} z^m + \overline{\sum_{m=1}^{\infty} b_{2,m} z^m} \tag{11}$$

is the function $(f \circledast g)$ defined as

$$(f_1 \circledast f_2)(z) = z - \sum_{m=2}^{\infty} a_{1,m} a_{2,m} z^m + \overline{\sum_{m=1}^{\infty} b_{1,m} b_{2,m} z^m}.$$

We note that $(f \circledast g)$ also belongs to $\mathcal{V}_{\mathcal{H}}$.

Definition 2. ([3], [14]) Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be subsets of $\mathcal{V}_{\mathcal{H}}(F; \gamma)$. We say that the three $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is $S_{\circledast}$-closed under the convolution if there exists a number $S_{\circledast}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ such that

$$S_{\circledast}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = min \left\{ s \in \mathbb{R} : \mathcal{I}^s (f \circledast g) \in \mathcal{Z}, \forall f \in \mathcal{X}, \forall g \in \mathcal{Y} \right\} \tag{12}$$

The number $S_{\circledast}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is called the order of $\circledast$-convolution consistence of the three $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$

U. Bednarz and J. Sokol in [3] obtained the order of convolution consistence concerning certain classes of univalent functions (starlike, convex, uniform-starlike or uniform-convex functions) and in [14] it is obtained the order of $\circledast$-convolution consistence for certain classes of analytic functions with negative coefficients. In this paper we obtain similar results, but concerning the class $\mathcal{V}_{\mathcal{H}}(F; \gamma)$ and for $\circledast$-convolution.

Let denote by $\mathcal{V}^1_{\mathcal{H}}(F; \gamma)$ the subset of $\mathcal{V}_{\mathcal{H}}(F; \gamma)$ consisting of functions of the form (2) which satisfy $|a_m| \leq 1, |b_m| \leq 1, \forall m \geq 2$.

Main results

Theorem 2. *Let f_1, f_2 be two functions in $\mathcal{V}^1_{\mathcal{H}}(F;\gamma)$ of the form (1); then $(f_1 \circledast f_2)$ also belongs to $\mathcal{V}^1_{\mathcal{H}}(F;\gamma)$.*

Proof. Since $f_1, f_2 \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma)$, from *Theorem 1* we have

$$\sum_{m=2}^{\infty}\left(\frac{m-\gamma}{1-\gamma}|a_{1,m}|+\frac{m+\gamma}{1-\gamma}|b_{1,m}|\right)C_m \le 1-\frac{1+\gamma}{1-\gamma}b_{1,1}$$

and

$$\sum_{m=2}^{\infty}\left(\frac{m-\gamma}{1-\gamma}|a_{2,m}|+\frac{m+\gamma}{1-\gamma}|b_{2,m}|\right)C_m \le 1-\frac{1+\gamma}{1-\gamma}b_{2,1}$$

and by the Cauchy-Schwarz inequality we deduce

$$\sum_{m=2}^{\infty}\sqrt{\left(\frac{m-\gamma}{1-\gamma}|a_{1,m}|+\frac{m+\gamma}{1-\gamma}|b_{1,m}|\right)\left(\frac{m-\gamma}{1-\gamma}|a_{2,m}|+\frac{m+\gamma}{1-\gamma}|b_{2,m}|\right)}C_m$$
$$\le\sqrt{\left(1-\frac{1+\gamma}{1-\gamma}b_{1,1}\right)\left(1-\frac{1+\gamma}{1-\gamma}b_{2,1}\right)}.$$

In order to prove that $f_1 \circledast f_2 \in \mathcal{V}^1_{\mathcal{H}}(F,\gamma)$ we need to show that

$$\sum_{m=2}^{\infty}\left(\frac{m-\gamma}{1-\gamma}|a_{1,m}||a_{2,m}|+\frac{m+\gamma}{1-\gamma}|b_{1,m}||b_{2,m}|\right)C_m \le 1-\frac{1+\gamma}{1-\gamma}b_{1,1}b_{2,1}.$$

But by using again Cauchy-Schwarz inequality we have

$$\left(\frac{m-\gamma}{1-\gamma}|a_{1,m}||a_{2,m}|+\frac{m+\gamma}{1-\gamma}|b_{1,m}||b_{2,m}|\right)^2$$
$$\le\left[\left(\frac{m-\gamma}{1-\gamma}(|a_{1,m}|)^2+\frac{m+\gamma}{1-\gamma}(|b_{1,m}|)^2\right)\right]\left[\left(\frac{m-\gamma}{1-\gamma}(|a_{2,m}|)^2+\frac{m+\gamma}{1-\gamma}(|b_{2,m}|)^2\right)\right], \forall m\ge 2$$
$$\left[\left(\frac{m-\gamma}{1-\gamma}(|a_{1,m}|)^2+\frac{m+\gamma}{1-\gamma}(|b_{1,m}|)^2\right)\right]\left[\left(\frac{m-\gamma}{1-\gamma}(|a_{2,m}|)^2+\frac{m+\gamma}{1-\gamma}(|b_{2,m}|)^2\right)\right]$$
$$\le\left(\frac{m-\gamma}{1-\gamma}|a_{1,m}|+\frac{m+\gamma}{1-\gamma}|b_{1,m}|\right)\left(\frac{m-\gamma}{1-\gamma}|a_{2,m}|+\frac{m+\gamma}{1-\gamma}|b_{2,m}|\right).$$

On the other hand

$$\sqrt{\left(1-\frac{1+\gamma}{1-\gamma}b_{1,1}\right)\left(1-\frac{1+\gamma}{1-\gamma}b_{2,1}\right)}\le\sqrt{\left(1-\frac{1+\gamma}{1-\gamma}b_{1,1}b_{2,1}\right)\left(1-\frac{1+\gamma}{1-\gamma}b_{1,1}b_{2,1}\right)}$$
$$\le 1-\frac{1+\gamma}{1-\gamma}b_{1,1}b_{2,1}, \quad 0\le b_{1,1}<1, \quad 0\le b_{2,1}<1. \qquad \square$$

Remark. Let the function $F = F_{m_0}, (m_0 \ge 2)$ be of the form (8) with $C_{m_0} = \dfrac{1-\gamma}{m_0-\gamma}$. Then if

$$f_1(z) = f_2(z) = z - \frac{z^{m_0}}{C_0\dfrac{m_0-\gamma}{1-\gamma}} \tag{13}$$

then the condition (9) for f_1 becomes $\frac{m_0-\gamma}{1-\gamma}\left(|a_{1,m_0}|+|b_{1,m_0}|\right)C_{m_0}=1$ and similar for f_2 and this shows that f_1, f_2 belong to $\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)$. For the function $(f_1 \circledast f_2)$ we have

$$\frac{m_0-\gamma}{1-\gamma}\left(|a_{1,m_0}|\,|a_{2,m_0}|+|b_{1,m_0}|\,|b_{2,m_0}|\right)C_{m_0}=\frac{m_0-\gamma}{1-\gamma}\frac{1}{C^2_{m_0}}\left(\frac{1-\gamma}{m_0-\gamma}\right)^2 C_{m_0}=1$$

and this imply that also $(f_1 \circledast f_2) \in \mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)$. This shows that the result is Theorem 2 is sharp when $F=F_{m_0}, (m_0 \geq 2)$.

Corollary 1. *The order of $\circledast$-convolution consistence for the classes $\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)$ is*

$$S_{\circledast}\left(\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma),\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma),\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)\right)=0 \tag{14}$$

Proof. From Theorem 2 we know that if $f_1, f_2 \in \mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)$, then

$$\mathcal{I}^0(f_1 \circledast f_2)=(f_1 \circledast f_2) \in \mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma).$$

This means that

$$S_{\circledast}\left(\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma),\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma),\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)\right)\leq 0$$

But the functions f_1, f_2 given by (13) for which the coefficients of $(f_1 \circledast f_2)$ satisfy the inequalities with equality show that

$$S_{\circledast}\left(\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma),\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma),\mathcal{V}^1_{\mathcal{H}}(F_{m_0};\gamma)\right)\geq 0. \qquad \square$$

Theorem 3. *Let $f_1 \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma_1)$, $f_2 \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma_2)$ be two functions of the form* (1) *then $(f_1 \circledast f_2)$ belongs to $\mathcal{V}^1_{\mathcal{H}}(F;\gamma^*)$, where*

$$\gamma^*=\frac{(2+\gamma_1)(2+\gamma_2)(1-b_{1,1}b_{2,1})-2\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(2+\gamma_1)(2+\gamma_2)(1+b_{1,1}b_{2,1})+\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]},$$

if

$$1-b_{1,1}b_{2,1}-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2)>0$$

or

$$\gamma^*=\frac{(2-\gamma_1)(2-\gamma_2)(1-b_{1,1}b_{2,1})-2\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(2-\gamma_1)(2-\gamma_2)(1+b_{1,1}b_{2,1})-\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]},$$

if

$$1-b_{1,1}b_{2,1}-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2)<0.$$

Proof. Since $f_1 \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma_1)$ and $f_2 \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma_2)$, from *Theorem 1* we have

$$\frac{\displaystyle\sum_{m=2}^{\infty}\left(\frac{m-\gamma_1}{1-\gamma_1}|a_{1,m}|+\frac{m+\gamma_1}{1-\gamma_1}|b_{1,m}|\right)C_m}{1-\dfrac{1+\gamma_1}{1-\gamma_1}b_{1,1}}\leq 1$$

and

$$\frac{\displaystyle\sum_{m=2}^{\infty}\left(\frac{m-\gamma_2}{1-\gamma_2}|a_{2,m}|+\frac{m+\gamma_2}{1-\gamma_2}|b_{2,m}|\right)C_m}{1-\dfrac{1+\gamma_2}{1-\gamma_2}b_{2,1}}\leq 1$$

and by the Cauchy-Schwarz inequality we deduce

$$\frac{\sum\limits_{m=2}^{\infty}\sqrt{\left(\frac{m-\gamma_1}{1-\gamma_1}\left|a_{1,m}\right|+\frac{m+\gamma_1}{1-\gamma_1}\left|b_{1,m}\right|\right)\left(\frac{m-\gamma_2}{1-\gamma_2}\left|a_{2,m}\right|+\frac{m+\gamma_2}{1-\gamma_2}\left|b_{2,m}\right|\right)C_m}}{\sqrt{\left(1-\frac{1+\gamma_1}{1-\gamma_1}b_{1,1}\right)\left(1-\frac{1+\gamma_2}{1-\gamma_2}b_{2,1}\right)}}\leq 1. \tag{15}$$

In order to prove that $f_1 \circledast f_2 \in \mathcal{V}^1_{\mathcal{H}}(F;\gamma^*)$ we need to show that

$$\frac{\sum\limits_{m=2}^{\infty}\left(\frac{m-\gamma^*}{1-\gamma^*}\left|a_{1,m}\right|\left|a_{2,m}\right|+\frac{m+\gamma^*}{1-\gamma^*}\left|b_{1,m}\right|\left|b_{2,m}\right|\right)C_m}{1-\frac{1+\gamma^*}{1-\gamma^*}b_{1,1}b_{2,1}}\leq 1. \tag{16}$$

We note that

$$\frac{\sum\limits_{m=2}^{\infty}\left(\frac{m-\gamma^*}{1-\gamma^*}\left|a_{1,m}\right|\left|a_{2,m}\right|+\frac{m+\gamma^*}{1-\gamma^*}\left|b_{1,m}\right|\left|b_{2,m}\right|\right)C_m}{1-\frac{1+\gamma^*}{1-\gamma^*}b_{1,1}b_{2,1}}$$

$$\leq\frac{\sum\limits_{m=2}^{\infty}\sqrt{\left(\frac{m-\gamma_1}{1-\gamma_1}\left|a_{1,m}\right|+\frac{m+\gamma_1}{1-\gamma_1}\left|b_{1,m}\right|\right)\left(\frac{m-\gamma_2}{1-\gamma_2}\left|a_{2,m}\right|+\frac{m+\gamma_2}{1-\gamma_2}\left|b_{2,m}\right|\right)C_m}}{\sqrt{\left(1-\frac{1+\gamma_1}{1-\gamma_1}b_{1,1}\right)\left(1-\frac{1+\gamma_2}{1-\gamma_2}b_{2,1}\right)}} \tag{17}$$

implies (16).
But by using again Cauchy-Schwarz inequality we have

$$\left(\frac{m-\gamma_1}{1-\gamma_1}\frac{m-\gamma_2}{1-\gamma_2}\left|a_{1,m}\right|\left|a_{2,m}\right|+\frac{m+\gamma_1}{1-\gamma_1}\frac{m+\gamma_2}{1-\gamma_2}\left|b_{1,m}\right|\left|b_{2,m}\right|\right)^2$$
$$\leq\left[\frac{m-\gamma_1}{1-\gamma_1}\left(\left|a_{1,m}\right|\right)^2+\frac{m+\gamma_1}{1-\gamma_1}\left(\left|b_{1,m}\right|\right)^2\right]\left[\frac{m-\gamma_2}{1-\gamma_2}\left(\left|a_{2,m}\right|\right)^2+\frac{m+\gamma_2}{1-\gamma_2}\left(\left|b_{2,m}\right|\right)^2\right]$$
$$\leq\left(\frac{m-\gamma_1}{1-\gamma_1}\left|a_{1,m}\right|+\frac{m+\gamma_1}{1-\gamma_1}\left|b_{1,m}\right|\right)\left(\frac{m-\gamma_2}{1-\gamma_2}\left|a_{2,m}\right|+\frac{m+\gamma_2}{1-\gamma_2}\left|b_{2,m}\right|\right)$$

and using in (17):

$$\frac{\frac{m-\gamma^*}{1-\gamma^*}\left|a_{1,m}\right|\left|a_{2,m}\right|+\frac{m+\gamma^*}{1-\gamma^*}\left|b_{1,m}\right|\left|b_{2,m}\right|}{1-\frac{1+\gamma^*}{1-\gamma^*}b_{1,1}b_{2,1}}$$

$$\leq\frac{\frac{m-\gamma_1}{1-\gamma_1}\frac{m-\gamma_2}{1-\gamma_2}\left|a_{1,m}\right|\left|a_{2,m}\right|+\frac{m+\gamma_1}{1-\gamma_1}\frac{m+\gamma_2}{1-\gamma_2}\left|b_{1,m}\right|\left|b_{2,m}\right|}{1-\frac{1+\gamma_1}{1-\gamma_1}\frac{1+\gamma_2}{1-\gamma_2}b_{1,1}b_{2,1}}.$$

It is sufficient to determine γ^* such that

$$\frac{\frac{m-\gamma^*}{1-\gamma^*}}{1-\frac{1+\gamma^*}{1-\gamma^*}b_{1,1}b_{2,1}} \leq \frac{\frac{m-\gamma_1}{1-\gamma_1}\frac{m-\gamma_2}{1-\gamma_2}}{1-\frac{1+\gamma_1}{1-\gamma_1}\frac{1+\gamma_2}{1-\gamma_2}b_{1,1}b_{2,1}}$$

or equivalently

$$\begin{aligned}&\gamma_1^* = \gamma^*\\ &\leq \frac{(m-\gamma_1)(m-\gamma_2)(1-b_{1,1}b_{2,1}) - m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(m-\gamma_1)(m-\gamma_2)(1+b_{1,1}b_{2,1}) - \left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}\end{aligned} \tag{18}$$

and

$$\frac{\frac{m+\gamma^*}{1-\gamma^*}}{1-\frac{1+\gamma^*}{1-\gamma^*}b_{1,1}b_{2,1}} \leq \frac{\frac{m+\gamma_1}{1-\gamma_1}\frac{m+\gamma_2}{1-\gamma_2}}{1-\frac{1+\gamma_1}{1-\gamma_1}\frac{1+\gamma_2}{1-\gamma_2}b_{1,1}b_{2,1}}$$

or equivalently

$$\begin{aligned}&\gamma_2^* = \gamma^*\\ &\leq \frac{(m+\gamma_1)(m+\gamma_2)(1-b_{1,1}b_{2,1}) - m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(m+\gamma_1)(m+\gamma_2)(1+b_{1,1}b_{2,1}) + \left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}.\end{aligned} \tag{19}$$

$$\begin{aligned}b_{1,1} < \frac{1-\gamma_1}{1+\gamma_1}, b_{2,1} < \frac{1-\gamma_2}{1+\gamma_2} &\Leftrightarrow b_{1,1}b_{2,1} < \frac{1-\gamma_1}{1+\gamma_1}\frac{1-\gamma_2}{1+\gamma_2}\\ &\Leftrightarrow (1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1} > 0\\ &\Leftrightarrow (1-b_{1,1}b_{2,1})(1+\gamma_1\gamma_2)-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2) > 0\end{aligned} \tag{20}$$

From (18) and (19) we choose the smaller one:
1. If

$$1-b_{1,1}b_{2,1}-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2) > 0 \tag{21}$$

then $\gamma_1^* > \gamma_2^*$ or

$$\begin{aligned}&\frac{(m-\gamma_1)(m-\gamma_2)(1-b_{1,1}b_{2,1}) - m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(m-\gamma_1)(m-\gamma_2)(1+b_{1,1}b_{2,1}) - \left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}\\ &> \frac{(m+\gamma_1)(m+\gamma_2)(1-b_{1,1}b_{2,1}) - m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(m+\gamma_1)(m+\gamma_2)(1+b_{1,1}b_{2,1}) + \left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}\end{aligned}$$

or equivalently

$$\begin{aligned}m^2\left[1-b_{1,1}b_{2,1}-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2)\right] - m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]\\ +(1-b_{1,1}b_{2,1})\gamma_1\gamma_2 > 0.\end{aligned}$$

We substitute $m=2$, the smallest value and we make the calculations, we get:

$$(1-b_{1,1}b_{2,1})(2-\gamma_1\gamma_2)-2(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2) > 0 \tag{22}$$

which is true, because if we add (20) with (22) and we divide with 3, we get the (21) condition.

Let us consider the function $E : [2; \infty) \to \mathbb{R}$

$$E(x) = \frac{(x+\gamma_1)(x+\gamma_2)(1-b_{1,1}b_{2,1}) - x[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}]}{(x+\gamma_1)(x+\gamma_2)(1+b_{1,1}b_{2,1}) + [(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}]}.$$

Then its derivative is:

$$E'(x) = \frac{\Delta\left[(1+b_{1,1}b_{2,1})x^2 + (1-b_{1,1}b_{2,1})(2x-1) + (1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2+\gamma_1\gamma_2)\right]}{\{(x+\gamma_1)(x+\gamma_2)(1+b_{1,1}b_{2,1}) + \Delta\}^2} > 0,$$

where $\Delta = [(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}]$.
$E(x)$ is an increasing function. In our case we need $\gamma^* \leq E(m), \forall m \geq 2$ and for this reason we choose

$$\gamma^* = E(2)$$
$$= \frac{(2+\gamma_1)(2+\gamma_2)(1-b_{1,1}b_{2,1}) - 2[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}]}{(2+\gamma_1)(2+\gamma_2)(1+b_{1,1}b_{2,1}) + [(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}]}.$$

2. If

$$1 - b_{1,1}b_{2,1} - (1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2) < 0 \tag{23}$$

then $\gamma_1^* < \gamma_2^*$ or

$$\frac{(m-\gamma_1)(m-\gamma_2)(1-b_{1,1}b_{2,1}) - m[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}]}{(m-\gamma_1)(m-\gamma_2)(1+b_{1,1}b_{2,1}) - [(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}]}$$
$$< \frac{(m+\gamma_1)(m+\gamma_2)(1-b_{1,1}b_{2,1}) - m[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}]}{(m+\gamma_1)(m+\gamma_2)(1+b_{1,1}b_{2,1}) + [(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}]}$$

or equivalently

$$m^2[1-b_{1,1}b_{2,1}-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2)] - m[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}]$$
$$+ (1-b_{1,1}b_{2,1})\gamma_1\gamma_2 < 0.$$

We substitute $m = 2$, the smallest value and we make the calculations, we get:

$$(1-b_{1,1}b_{2,1})(2-\gamma_1\gamma_2) - 2(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2) < 0 \tag{24}$$

which is true, because if we multiply (23) with 2 and add with $-\gamma_1\gamma_2(1-b_{1,1}b_{2,1}) < 0$, we get the (24).

Let us consider the function $E : [2; \infty) \to \mathbb{R}$

$$E(x) = \frac{(x-\gamma_1)(x-\gamma_2)(1-b_{1,1}b_{2,1}) - x[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}]}{(x-\gamma_1)(x-\gamma_2)(1+b_{1,1}b_{2,1}) - [(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}]}.$$

Then its derivative is:

$$E'(x) = \frac{\Delta\left[(1+b_{1,1}b_{2,1})(x-1)^2 + 2b_{1,1}b_{2,1}(x-\gamma_1\gamma_2-\gamma_1-\gamma_2) + 2b_{1,1}b_{2,1}(x-1)\right]}{\{(x-\gamma_1)(x-\gamma_2)(1+b_{1,1}b_{2,1}) - \Delta\}^2} > 0,$$

where

$$\Delta = [(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}].$$

$E(x)$ is an increasing function. In our case we need $\gamma^* \leq E(m), \forall m \geq 2$ and for this reason we choose

$$\gamma^* = E(2) = \frac{(2-\gamma_1)(2-\gamma_2)(1-b_{1,1}b_{2,1}) - 2[(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}]}{(2-\gamma_1)(2-\gamma_2)(1+b_{1,1}b_{2,1}) - [(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}]}.$$

3. If

$$1 - b_{1,1}b_{2,1} - (1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2) = 0$$

or

$$1 - b_{1,1}b_{2,1} = (1 + b_{1,1}b_{2,1})(\gamma_1 + \gamma_2) \tag{25}$$

then $\gamma_1^* = \gamma_2^*$ or

$$\frac{(m-\gamma_1)(m-\gamma_2)(1-b_{1,1}b_{2,1}) - m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(m-\gamma_1)(m-\gamma_2)(1+b_{1,1}b_{2,1}) - \left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}$$

$$= \frac{(m+\gamma_1)(m+\gamma_2)(1-b_{1,1}b_{2,1}) - m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}{(m+\gamma_1)(m+\gamma_2)(1+b_{1,1}b_{2,1}) + \left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]}$$

or equivalently

$$m^2\left[1-b_{1,1}b_{2,1}-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2)\right] - m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right]$$
$$+ (1-b_{1,1}b_{2,1})\gamma_1\gamma_2 = 0.$$

If we use (25) we get

$$-m\left[(1-\gamma_1)(1-\gamma_2)-(1+\gamma_1)(1+\gamma_2)b_{1,1}b_{2,1}\right] + (1-b_{1,1}b_{2,1})\gamma_1\gamma_2 = 0.$$

$$-m\left[(1-b_{1,1}b_{2,1})(1+\gamma_1\gamma_2)-(1+b_{1,1}b_{2,1})(\gamma_1+\gamma_2)\right] + (1-b_{1,1}b_{2,1})\gamma_1\gamma_2 = 0$$

$$\Leftrightarrow -m\left[(1-b_{1,1}b_{2,1})(1+\gamma_1\gamma_2)-(1-b_{1,1}b_{2,1})\right] + (1-b_{1,1}b_{2,1})\gamma_1\gamma_2 = 0$$

$$\Leftrightarrow -m\left[(1-b_{1,1}b_{2,1})\gamma_1\gamma_2\right] + (1-b_{1,1}b_{2,1})\gamma_1\gamma_2 = 0$$

$$\Leftrightarrow (1-m)\left[(1-b_{1,1}b_{2,1})\gamma_1\gamma_2\right] = 0$$

$$\Leftrightarrow m = 1(\text{false}) \text{ or } b_{1,1} = \frac{1}{b_{2,1}} \text{ or } \gamma_1 = 0 \text{ or } \gamma_2 = 0.$$

If we put $b_{1,1} = \dfrac{1}{b_{2,1}}$ in (25) we get $\gamma_2 = -\gamma_1$. If we substitute this in (18) and (19) we get $\gamma_1^* = \gamma_2^* = 0$. □

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 279–293
DOI: 10.24193/subbmath.2023.2.05

Certain sufficient conditions for $\phi-$ like functions in a parabolic region

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Abstract. To obtain the main result of the present paper we use the technique of differential subordination. As special cases of our main result, we obtain sufficient conditions for $f \in \mathcal{A}$ to be $\phi-$like, starlike and close-to-convex in a parabolic region.

Mathematics Subject Classification (2010): 30C80, 30C45.

Keywords: Analytic function, differential subordination, parabolic $\phi-$like function, parabolic starlike function, close-to-convex function.

1. Introduction

Let us denote the class of analytic functions in the unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ by $\mathcal{H}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of the functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots.$$

Let $\mathcal{A}$ be the class of functions f, analytic in the unit disk $\mathbb{E}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

Let $\mathcal{S}$ denote the class of all analytic univalent functions f defined in the open unit disk $\mathbb{E}$ which are normalized by the conditions $f(0) = f'(0) - 1 = 0$. The Taylor series expansion of any function $f \in \mathcal{S}$ is

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots.$$

Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g written as $f \prec g$ in $\mathbb{E}$, if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1$,

Received 21 May 2020; Accepted 21 July 2020.

$\phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that

$$f(z) = g(\phi(z)),\ |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z),\ \Phi(p(0), 0; 0) = h(0). \tag{1.1}$$

A univalent function q is called dominant of the differential subordination (1.1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1), is said to be the best dominant of (1.1). The best dominant is unique up to the rotation of $\mathbb{E}$.

A function $f \in \mathcal{A}$ is said to be starlike in the open unit disk $\mathbb{E}$, if it is univalent in $\mathbb{E}$ and $f(\mathbb{E})$ is a starlike domain. The well known condition for the members of class $\mathcal{A}$ to be starlike is that

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{E}.$$

Let $\mathcal{S}^*$ denote the subclass of $\mathcal{S}$ consisting of all univalent starlike functions with respect to the origin.

A function $f \in \mathcal{A}$ is said to be close-to-convex in $\mathbb{E}$, if there exists a convex function g (not necessarily normalized) such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0,\ z \in \mathbb{E}.$$

In addition, if g is normalized by the conditions $g(0) = 0 = g'(0) - 1$, then the class of close-to-convex functions is denoted by $\mathcal{C}$.

A function $f \in \mathcal{A}$ is called parabolic starlike in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|,\ z \in \mathbb{E}, \tag{1.2}$$

and the class of such functions is denoted by S_P.

A function $f \in \mathcal{A}$ is said to be uniformly close-to-convex in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \left|\frac{zf'(z)}{g(z)} - 1\right|,\ z \in \mathbb{E}, \tag{1.3}$$

for some $g \in \mathcal{S}_P$. Let UCC denote the class of all such functions. Note that the function $g(z) \equiv z \in \mathcal{S}_P$. Therefore, for $g(z) \equiv z$, condition (1.3) becomes:

$$\Re\left(f'(z)\right) > |f'(z) - 1|,\ z \in \mathbb{E}. \tag{1.4}$$

Ronning [6] and Ma and Minda [2] studied the domain Ω and the function $q(z)$ defined below:

$$\Omega = \left\{u + iv : u > \sqrt{(u-1)^2 + v^2}\right\}.$$

Clearly the function

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$

maps the unit disk $\mathbb{E}$ onto the domain Ω. Hence the conditions (1.2) and (1.4) are equivalent to

$$\frac{zf'(z)}{f(z)} \prec q(z),\ z \in \mathbb{E},$$

and

$$f'(z) \prec q(z).$$

Let ϕ be analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0$ and $Re(\phi'(0)) > 0$. Then, the function $f \in \mathcal{A}$ is said to be $\phi-$ like in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > 0,\ z \in \mathbb{E}.$$

This concept was introduced by Brickman [1]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is $\phi-$ like for some analytic function ϕ. Later, Ruscheweyh [7] investigated the following general class of $\phi-$like functions:
Let ϕ be analytic in a domain containing $f(\mathbb{E})$, where $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi(w) \neq 0$ for some $w \in f(\mathbb{E})\backslash\{0\}$, then the function $f \in \mathcal{A}$ is called $\phi-$like with respect to a univalent function q, $q(0) = 1$, if

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z),\ z \in \mathbb{E}.$$

A function $f \in \mathcal{A}$ is said to be parabolic $\phi-$ like in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > \left|\frac{zf'(z)}{\phi(f(z))} - 1\right|,\ z \in \mathbb{E}. \tag{1.5}$$

Equivalently, condition (1.5) can be written as:

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2.$$

In 2005, Ravichandran et al. [5] proved the following result for ϕ-like functions:
Let $\alpha \neq 0$ be a complex number and $q(z)$ be a convex univalent function in $\mathbb{E}$.
Suppose $h(z) = \alpha q^2(z) + (1-\alpha)q(z) + \alpha z q'(z)$ and

$$\Re\left\{\frac{1-\alpha}{\alpha} + 2q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0,\ z \in \mathbb{E}.$$

If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{\phi(f(z))}\left(1 + \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha(f'(z) - (\phi(f(z)))'}{\phi(f(z))}\right) \prec h(z),$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z),\ z \in \mathbb{E},$$

and $q(z)$ is best dominant. Later on, Shanmugam et al. [8] and Ibrahim [4] also obtained the results for ϕ-like functions similar to the above mentioned results of

Ravichandran [5].
In this paper, we investigate the differential operator

$$\left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma}\left[a\frac{zf'(z)}{\phi(g(z))}+b\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta}$$

where $f,\ g\in\mathcal{A}$ and $\beta,\ \gamma$ be complex numbers such that $\beta\neq 0$. Also ϕ is an analytic function in a domain containing $g(\mathbb{E})$ such that $\phi(0)=0=\phi'(0)-1$ and $\phi(w)\neq 0$ for $w\in g(\mathbb{E})\backslash\{0\}$, for real numbers $a, b\ (\neq 0)$. As consequences of our main results, we obtain sufficient conditions for ϕ-like, parabolic ϕ-like, starlike, parabolic starlike, close-to-convex and uniformly close-to-convex functions.
We shall need the following lemma to prove our main result.

Lemma 1.1. ([3], Theorem 3.4h, p. 132) *Let q be univalent in $\mathbb{E}$ and let θ and φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\varphi(w)\neq 0$, when $w\in q(\mathbb{E})$. Set*

$$Q_1(z)=zq'(z)\varphi[q(z)],\ \ h(z)=\theta[q(z)]+Q_1(z)$$

and suppose that either
(i) h is convex, or
(ii) Q_1 is starlike.
In addition, assume that
(iii) $\Re\left(\dfrac{zh'(z)}{Q_1(z)}\right)>0$ for all $z\in\mathbb{E}$.
If p is analytic in $\mathbb{E}$, with $p(0)=q(0),\ p(\mathbb{E})\subset\mathbb{D}$ and

$$\theta[p(z)]+zp'(z)\varphi[p(z)]\prec\theta[q(z)]+zq'(z)\varphi[q(z)],\ \ z\in\mathbb{E},$$

then $p(z)\prec q(z)$ and $q(z)$ is the best dominant.

2. Main results

Theorem 2.1. *Let β and γ be complex numbers such that $\beta\neq 0$. Let $q(z)\neq 0$, be a univalent function in $\mathbb{E}$, such that*

$$\Re\left[1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}\right]>max\left\{0,\ -\frac{a}{b}\left(1+\frac{\gamma}{\beta}\right)\Re(q(z))\right\}\tag{2.1}$$

where a and $b(\neq 0)$ are real numbers. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0)=0=\phi'(0)-1$ and $\phi(w)\neq 0$ for $w\in g(\mathbb{E})\backslash\{0\}$. If $f,\ g\in\mathcal{A},\ \dfrac{zf'(z)}{\phi(g(z))}\neq 0,\ z\in\mathbb{E}$, satisfy the differential subordination

$$\left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma}\left[a\frac{zf'(z)}{\phi(g(z))}+b\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta}$$

$$\prec(q(z))^{\gamma}\left[aq(z)+b\frac{zq'(z)}{q(z)}\right]^{\beta}\tag{2.2}$$

then

$$\frac{zf'(z)}{\phi(g(z))} \prec q(z),\ z \in \mathbb{E},$$

and $q(z)$ is the best dominant.

Proof. On writing $\frac{zf'(z)}{\phi(g(z))} = p(z)$ in (2.2), we obtain:

$$(p(z))^{\gamma}\left(ap(z) + b\frac{zp'(z)}{p(z)}\right)^{\beta} \prec (q(z))^{\gamma}\left(aq(z) + b\frac{zq'(z)}{q(z)}\right)^{\beta}$$

or

$$a(p(z))^{\frac{\gamma}{\beta}+1} + b(p(z))^{\frac{\gamma}{\beta}-1}zp'(z) \prec a(q(z))^{\frac{\gamma}{\beta}+1} + b(q(z))^{\frac{\gamma}{\beta}-1}zq'(z)$$

Let us define the functions θ and ϕ as follows:

$$\theta(w) = aw^{\frac{\gamma}{\beta}+1} \text{ and } \phi(w) = bw^{\frac{\gamma}{\beta}-1}$$

Obviously, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C}\backslash\{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$.
Therefore,

$$Q(z) = \phi(q(z))zq'(z) = b(q(z))^{\frac{\gamma}{\beta}-1}zq'(z)$$

and

$$h(z) = \theta(q(z)) + Q(z) = a(q(z))^{\frac{\gamma}{\beta}+1} + b(q(z))^{\frac{\gamma}{\beta}-1}zq'(z)$$

On differentiating, we obtain

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)}$$

and

$$\frac{zh'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)} + \frac{a}{b}\left(1 + \frac{\gamma}{\beta}\right)q(z).$$

In view of the given condition (2.1), we see that Q is starlike and $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$. Therefore, the proof, now follows from the Lemma [1.1]. $\square$

On taking $g(z) = f(z)$ in Theorem 2.1, we have the following result:

Theorem 2.2. *Let β and γ be complex numbers such that $\beta \neq 0$ and $q(z) \neq 0$, be a univalent function in $\mathbb{E}$, satisfying the condition (2.1) of Theorem 2.1 for real numbers $a, b\ (\neq 0)$. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, satisfy the differential subordination*

$$\left(\frac{zf'(z)}{\phi(f(z))}\right)^{\gamma}\left[a\frac{zf'(z)}{\phi(f(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right)\right]^{\beta} \prec (q(z))^{\gamma}\left[aq(z) + b\frac{zq'(z)}{q(z)}\right]^{\beta}$$

then
$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \ z \in \mathbb{E},$$
and $q(z)$ is the best dominant.

On taking $\phi(z) = z$, $g(z) = f(z)$ in Theorem 2.1, we have the following result:

Theorem 2.3. *Let β and γ be complex numbers such that $\beta \neq 0$ and $q(z) \neq 0$, be a univalent function in $\mathbb{E}$, and satisfies the condition (2.1) of Theorem 2.1 for real numbers a, $b(\neq 0)$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies*
$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left[(a-b)\frac{zf'(z)}{f(z)} + b\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\beta} \prec (q(z))^{\gamma}\left[aq(z) + b\frac{zq'(z)}{q(z)}\right]^{\beta}$$
then
$$\frac{zf'(z)}{f(z)} \prec q(z), \ z \in \mathbb{E},$$
and $q(z)$ is the best dominant.

On selecting $a = 1$ and $b = \alpha$ in Theorem 2.3, we get the following result for the class of α-convex functions.

Theorem 2.4. *Let β and γ be complex numbers such that $\beta \neq 0$. Let α be a non-zero real number and $q(z) \neq 0$, be a univalent function in $\mathbb{E}$, and satisfies the condition (2.1) of Theorem 2.1. If $f \in \mathcal{A}$, $\frac{zf'(z)}{f(z)} \neq 0$ $z \in \mathbb{E}$, satisfies*
$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\beta} \prec (q(z))^{\gamma}\left[q(z) + \alpha\frac{zq'(z)}{q(z)}\right]^{\beta}$$
then
$$\frac{zf'(z)}{f(z)} \prec q(z), \ z \in \mathbb{E},$$
and $q(z)$ is the best dominant.

By defining $\phi(z) = g(z) = z$ in Theorem 2.1, we obtain the following result:

Theorem 2.5. *Let β and γ be complex numbers such that $\beta \neq 0$ and $q(z) \neq 0$, be a univalent function in $\mathbb{E}$, and satisfies the condition (2.1) of Theorem 2.1 for real numbers a, $b(\neq 0)$. If $f \in \mathcal{A}$, $f'(z) \neq 0$, $z \in \mathbb{E}$, satisfies*
$$(f'(z))^{\gamma}\left[af'(z) + b\frac{zf''(z)}{f'(z)}\right]^{\beta} \prec (q(z))^{\gamma}\left(aq(z) + b\frac{zq'(z)}{q(z)}\right)^{\beta}$$
then
$$f'(z) \prec q(z), \ z \in \mathbb{E},$$
and $q(z)$ is the best dominant.

3. Applications

Remark 3.1. When we select the dominant

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,$$

we observed that the condition (2.1) of Theorem 2.1 holds, for real numbers $a, b\ (\neq 0)$ such that $\frac{a}{b} > 0$ and real numbers $\beta(\neq 0),\ \gamma$ such that $\frac{-3}{4} < \frac{\gamma}{\beta} < \frac{3}{2}$. Consequently, we get:

Theorem 3.2. *Let $\beta(\neq 0)$ and γ be real numbers such that $\frac{-3}{4} < \frac{\gamma}{\beta} < \frac{3}{2}$ and a, b $(\neq 0)$ be real numbers having same sign. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A},\ \frac{zf'(z)}{\phi(g(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$\left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma}\left[a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta}$$
$$\prec \left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^{\gamma}$$
$$\left\{a + \frac{2a}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{\frac{4b\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^{\beta}$$

then

$$\frac{zf'(z)}{\phi(g(z))} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z \in \mathbb{E}.$$

On taking $g(z) = f(z)$ in above theorem, we obtain:

Corollary 3.3. *Let $\beta(\neq 0)$ and γ be real numbers such that $\frac{-3}{4} < \frac{\gamma}{\beta} < \frac{3}{2}$ and a, b $(\neq 0)$ be real numbers having same sign. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A},\ \frac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$\left(\frac{zf'(z)}{\phi(f(z))}\right)^{\gamma}\left[a\frac{zf'(z)}{\phi(f(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right)\right]^{\beta}$$
$$\prec \left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^{\gamma}$$

$$\left\{a+\frac{2a}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4b\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^\beta$$

then

$$\frac{zf'(z)}{\phi(f(z))}\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E},$$

and hence $f(z)$ is parabolic ϕ-like.

For $\phi(z)=z$ and $g(z)=f(z)$ in Theorem 3.2, we obtain the following result:

Corollary 3.4. *Let $\beta(\neq 0)$ and γ be real numbers such that $\frac{-3}{4}<\frac{\gamma}{\beta}<\frac{3}{2}$ and a, b $(\neq 0)$ be real numbers having same sign. If $f\in\mathcal{A}$, $\frac{zf'(z)}{f(z)}\neq 0$, $z\in\mathbb{E}$, satisfy*

$$\left(\frac{zf'(z)}{f(z)}\right)^\gamma\left[(a-b)\frac{zf'(z)}{f(z)}+b\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^\beta\prec\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^\gamma$$
$$\left\{a+\frac{2a}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4b\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^\beta$$

then

$$\frac{zf'(z)}{f(z)}\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E},$$

and hence $f(z)$ is parabolic starlike.

Selecting $a=1$ and $b=\alpha$ in above corollary, we get the following result for the class of α-convex functions:

Corollary 3.5. *Let $\beta(\neq 0)$ and γ be real numbers such that $\frac{-3}{4}<\frac{\gamma}{\beta}<\frac{3}{2}$ and α be a non-zero real number. If $f\in\mathcal{A}$, $\frac{zf'(z)}{f(z)}\neq 0$, $z\in\mathbb{E}$, satisfies*

$$\left(\frac{zf'(z)}{f(z)}\right)^\gamma\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^\beta\prec\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^\gamma$$
$$\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^\beta$$

then

$$\frac{zf'(z)}{f(z)}\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E},$$

and hence $f(z)$ is parabolic starlike.

On taking $\phi(z)=g(z)=z$ in Theorem 3.2, we have:

Corollary 3.6. *Let $\beta(\neq 0)$ and γ be real numbers such that $\dfrac{-3}{4} < \dfrac{\gamma}{\beta} < \dfrac{3}{2}$ and a, b $(\neq 0)$ be real numbers having same sign. If $f \in \mathcal{A}$, $f'(z) \neq 0$, $z \in \mathbb{E}$, satisfies*

$$(f'(z))^\gamma \left[af'(z) + b\left(\frac{zf''(z)}{f'(z)}\right)\right]^\beta \prec \left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^\gamma$$

$$\left\{a + \frac{2a}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{\frac{4b\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^\beta$$

then

$$f'(z) \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, \ z \in \mathbb{E},$$

and hence $f(z)$ is uniformly close-to-convex.

Remark 3.7. It is easy to verify that the dominant $q(z) = \dfrac{1+z}{1-z}$, satisfies the condition (2.1) of Theorem 2.1, for real numbers a, b $(\neq 0)$ having same sign and real numbers γ and $\beta(\neq 0)$ such that $\gamma = \beta$ or $\gamma = 0$.
For $\gamma = \beta$, Theorem 2.1 yields:

Theorem 3.8. *Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\setminus\{0\}$. If f, $g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \neq 0$, $z \in \mathbb{E}$, and for real numbers a, b $(\neq 0)$ having same sign, satisfies*

$$a\left(\frac{zf'(z)}{\phi(g(z))}\right)^2 + b\left(\frac{zf'(z)}{\phi(g(z))}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right) \prec a\left(\frac{1+z}{1-z}\right)^2 + \frac{2bz}{(1-z)^2}$$

then

$$\frac{zf'(z)}{\phi(g(z))} \prec \frac{1+z}{1-z}, \ z \in \mathbb{E}.$$

On taking $g(z) = f(z)$ in above theorem, we obtain:

Corollary 3.9. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\setminus\{0\}$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, and for real numbers a, b $(\neq 0)$ having same sign, satisfies*

$$a\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + b\left(\frac{zf'(z)}{\phi(f(z))}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right) \prec a\left(\frac{1+z}{1-z}\right)^2 + \frac{2bz}{(1-z)^2}$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{1+z}{1-z},\ z \in \mathbb{E},$$

i.e. $f(z)$ is $\phi-$like function.

For $\phi(z) = z$ and $g(z) = f(z)$ in Theorem 3.8, we obtain the following result:

Corollary 3.10. *Let $a, b\ (\neq 0)$ be real numbers having same sign. If $f \in \mathcal{A}$,*

$$\frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E},$$

satisfy

$$(a-b)\left(\frac{zf'(z)}{f(z)}\right)^2 + b\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right) \prec a\left(\frac{1+z}{1-z}\right)^2 + \frac{2bz}{(1-z)^2}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z},\ z \in \mathbb{E},$$

and hence $f(z)$ is starlike.

Selecting $a = 1$ and $b = \alpha$ in above corollary, we get the following result for the class of α-convex functions:

Corollary 3.11. *Let α be a non-zero real number. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies*

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)^2 + \alpha\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^2 + \frac{2\alpha z}{(1-z)^2}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z},\ z \in \mathbb{E}.$$

Hence $f(z)$ is starlike.

On taking $\phi(z) = g(z) = z$ in Theorem 3.8, we have:

Corollary 3.12. *Let $a, b\ (\neq 0)$ are real numbers with same sign. If $f \in \mathcal{A}$, $f'(z) \neq 0$, $z \in \mathbb{E}$, satisfies*

$$a\left(f'(z)\right)^2 + bzf''(z) \prec a\left(\frac{1+z}{1-z}\right)^2 + \frac{2bz}{(1-z)^2}$$

then

$$f'(z) \prec \left(\frac{1+z}{1-z}\right),\ z \in \mathbb{E},$$

and hence $f(z)$ is close-to-convex.

For $\gamma = 0$, Theorem 2.1 yields:

Theorem 3.13. *Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0)-1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \neq 0,\ z \in \mathbb{E}$, and for real numbers $a, b\ (\neq 0)$ with same sign, satisfies*

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right) \prec a\left(\frac{1+z}{1-z}\right) + \frac{2bz}{(1-z)^2}$$

then

$$\frac{zf'(z)}{\phi(g(z))} \prec \frac{1+z}{1-z},\ z \in \mathbb{E}.$$

On taking $g(z) = f(z)$ in above theorem, we obtain:

Corollary 3.14. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, and for real numbers $a, b\ (\neq 0)$ with same sign, satisfies*

$$a\frac{zf'(z)}{\phi(f(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right) \prec a\left(\frac{1+z}{1-z}\right) + \frac{2bz}{(1-z)^2}$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{1+z}{1-z},\ z \in \mathbb{E},$$

i.e. $f(z)$ is $\phi-$like function.

For $\phi(z) = z$ and $g(z) = f(z)$ in Theorem 3.13, we obtain the following result:

Corollary 3.15. *Let $a, b\ (\neq 0)$ are real numbers with same sign. If $f \in \mathcal{A}$,*

$$\frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E},$$

satisfy

$$(a-b)\frac{zf'(z)}{f(z)} + b\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec a\left(\frac{1+z}{1-z}\right) + \frac{2bz}{(1-z)^2}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z},\ z \in \mathbb{E},$$

and hence $f(z)$ is starlike.

Selecting $a = 1$ and $b = \alpha$ in above corollary, we get the following result for the class of α-convex functions:

Corollary 3.16. *Let α be a non-zero real number. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{1+z}{1-z} + \frac{2\alpha z}{(1-z)^2}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \ z \in \mathbb{E}.$$

Hence $f(z)$ is starlike.

On taking $\phi(z) = g(z) = z$ in Theorem 3.13, we have:

Corollary 3.17. *Let a, b $(\neq 0)$ are real numbers with same sign. If $f \in \mathcal{A}$, $f'(z) \neq 0$, $z \in \mathbb{E}$, satisfies*

$$af'(z) + b\frac{zf''(z)}{f'(z)} \prec a\left(\frac{1+z}{1-z}\right) + \frac{2bz}{(1-z)^2}$$

then

$$f'(z) \prec \frac{1+z}{1-z}, \ z \in \mathbb{E},$$

and hence $f(z)$ is close-to-convex.

Remark 3.18. When we select the dominant $q(z) = e^z$, then this dominant satisfies the condition (2.1) of Theorem 2.1 for real numbers a, b $(\neq 0)$ with same sign and real numbers $\gamma,\ \beta(\neq 0)$ such that $0 < \dfrac{\gamma}{\beta} \leq 1$. Consequently, we obtain the following result:

Theorem 3.19. *Let a, b $(\neq 0)$ be real numbers with same sign and γ, β $(\neq 0)$ such that $0 < \dfrac{\gamma}{\beta} \leq 1$. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E}) \setminus \{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$\left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma} \left[a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta} \prec e^{\gamma z}\left[ae^z + bz\right]^{\beta}$$

then

$$\frac{zf'(z)}{\phi(g(z))} \prec e^z, \ z \in \mathbb{E}.$$

On choosing $g(z) = f(z)$ in above theorem, we obtain:

Corollary 3.20. *Let a, b $(\neq 0)$ be real numbers with same sign and γ, β $(\neq 0)$ be real numbers such that $0 < \dfrac{\gamma}{\beta} \leq 1$. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$,*

$$\frac{zf'(z)}{\phi(f(z))} \neq 0, \ z \in \mathbb{E},$$

satisfy

$$\left(\frac{zf'(z)}{\phi(f(z))}\right)^{\gamma} \left[a\frac{zf'(z)}{\phi(f(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right)\right]^{\beta} \prec e^{\gamma z}\left[ae^z + bz\right]^{\beta}$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec e^z, \ z \in \mathbb{E},$$

i.e. $f(z)$ *is* $\phi-$*like.*

On selecting $\phi(z) = z$ and $g(z) = f(z)$ in Theorem 3.19, we get:

Corollary 3.21. *Let* $a, b \ (\neq 0)$ *be real numbers with same sign and* $\gamma, \beta \ (\neq 0)$ *be real numbers such that* $0 < \dfrac{\gamma}{\beta} \leq 1$. *If* $f \ \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0, \ z \in \mathbb{E}$, *satisfy the differential subordination*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma} \left[(a-b)\frac{zf'(z)}{f(z)} + b\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\beta} \prec e^{\gamma z} \left[ae^z + bz\right]^{\beta}$$

then

$$\frac{zf'(z)}{f(z)} \prec e^z, \ z \in \mathbb{E},$$

and hence $f(z)$ *is starlike.*

On choosing $a = 1$ and $b = \alpha$ in above corollary, we obtain:

Corollary 3.22. *Let* α *be a non-zero real number and real numbers* $\gamma, \beta \ (\neq 0)$ *such that* $0 < \dfrac{\gamma}{\beta} \leq 1$. *If* $f \ \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0, \ z \in \mathbb{E}$, *satisfies*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma} \left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\beta} \prec e^{\gamma z} \left[e^z + \alpha z\right]^{\beta}$$

then

$$\frac{zf'(z)}{f(z)} \prec e^z, \ z \in \mathbb{E}.$$

Therefore, $f \in S^*$.

For $\phi(z) = g(z) = z$ in Theorem 3.19, we obtain the following result:

Corollary 3.23. *Let* $a, b \ (\neq 0)$ *be real numbers with same sign and* $\gamma, \beta \ (\neq 0)$ *be real numbers such that* $0 < \dfrac{\gamma}{\beta} \leq 1$. *If* $f \ \in \mathcal{A}$, $f'(z) \neq 0, \ z \in \mathbb{E}$, *satisfies*

$$(f'(z))^{\gamma} \left[af'(z) + b\frac{zf''(z)}{f'(z)}\right]^{\beta} \prec e^{\gamma z} \left[ae^z + bz\right]^{\beta}$$

then

$$f'(z) \prec e^z, \ z \in \mathbb{E},$$

and hence $f(z)$ *is close-to-convex.*

Remark 3.24. By selecting the dominant $q(z) = 1 + mz, \ 0 < m \leq 1$, we observed that the Condition (2.1) of Theorem 2.1 holds for all real numbers $a, b \ (\neq 0)$ such that $\dfrac{a}{b} > 0$, and $\gamma = 0$. Thus from Theorem 2.1, we have the following result:

Theorem 3.25. *Let ϕ be analytic function in the domain containing $g(\mathbb{E})$, where $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. Let real numbers $a, b\ (\neq 0)$ be such that $\dfrac{a}{b} > 0$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$\left[a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right] \prec \left[a(1+mz) + \frac{bmz}{1+mz}\right]$$

then

$$\frac{zf'(z)}{\phi(g(z))} \prec 1 + mz,\ \text{where } 0 < m \leq 1,\ z \in \mathbb{E}.$$

Taking $g(z) = f(z)$ in above theorem, we get the following result:

Corollary 3.26. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$, where $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. Let real numbers $a, b\ (\neq 0)$ be such that $\dfrac{a}{b} > 0$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$\left[a\frac{zf'(z)}{\phi(f(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right)\right] \prec \left[a(1+mz) + \frac{bmz}{1+mz}\right]$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + mz,\ \text{where } 0 < m \leq 1,\ z \in \mathbb{E},$$

i.e. $f(z)$ is $\phi-$like.

From Theorem 3.25, for $\phi(z) = z$ and $g(z) = f(z)$, we obtain:

Corollary 3.27. *Let $a, b\ (\neq 0)$ are real numbers having same sign. If $f\ \in \mathcal{A}$,*

$$\frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E},$$

satisfies

$$\left[(a-b)\frac{zf'(z)}{f(z)} + b\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \prec \left[a(1+mz) + \frac{bmz}{1+mz}\right]$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + mz,\ \text{where } 0 < m \leq 1,\ z \in \mathbb{E},$$

and hence $f(z)$ is starlike.

On selecting $a = 1$ and $b = \alpha$ in above corollary, we get the following result:

Corollary 3.28. *Let $\alpha > 0$ be a real number. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies the differential subordination*

$$\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \prec \left[(1+mz) + \frac{\alpha mz}{1+mz}\right]$$

then
$$\frac{zf'(z)}{f(z)} \prec 1 + mz, \; 0 < m \leq 1, \; z \in \mathbb{E},$$
and hence $f(z)$ *is starlike.*

Selecting $\phi(z) = g(z) = z$, in Theorem 3.25, we have:

Corollary 3.29. *Let* $a, b\ (\neq 0)$ *be real numbers having same sign. If* $f \in \mathcal{A}$, $f'(z) \neq 0$, $z \in \mathbb{E}$, *satisfies*
$$\left[af'(z) + b\frac{zf''(z)}{f'(z)}\right] \prec \left[a(1 + mz) + \frac{bmz}{1 + mz}\right]$$
then
$$f'(z) \prec 1 + mz, \; 0 < m \leq 1, \; z \in \mathbb{E},$$
and hence $f(z)$ *is close-to-convex.*

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 295–306
DOI: 10.24193/subbmath.2023.2.06

Growth properties of solutions of linear difference equations with coefficients having φ-order

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Abstract. In this paper, we investigate the relations between the growth of entire coefficients and that of solutions of complex homogeneous and non-homogeneous linear difference equations with entire coefficients of φ-order by using a slow growth scale, the φ-order, where φ is a non-decreasing unbounded function. We extend some precedent results due to Zheng and Tu (2011) [15] and others.

Mathematics Subject Classification (2010): 30D35, 39A10, 39A12.

Keywords: Nevanlinna's theory, linear difference equation, meromorphic solution, φ-order.

1. Introduction and preliminaries

We assume that the readers are familiar with the fundamental results and standard notations of the Nevanlinna's value distribution theory of entire and meromorphic functions. In addition, let us recall some notations such as $m(r,f)$ and $N(r,f)$ (see [8, 10]). Let $n(r,f)$ be the number of poles of a function f (counting multiplicities) in $|z| \leq r$. Then we define the integrated counting function $N(r,f)$ by

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,f) \log r,$$

and we define the proximity function $m(r,f)$ by

$$m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\phi}\right) \right| d\phi,$$

where $\log^+ x = \max\{0, \log x\}$. We should think of $m(r,f)$ as a measure of how close f is to infinity on $|z| = r$. Nevertheless, within that context, we recall that $T(r,f)$ stands

Received 20 September 2020; Accepted 17 November 2020.

for the Nevanlinna characteristic function of the meromorphic function f defined on each positive real value r by

$$T(r,f) = m(r,f) + N(r,f).$$

And $M(r,f)$ stands for the so called maximum modulus function defined for each non-negative real value r by

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

The applications of Nevanlinna's value distribution theory has been developed since 1960's. Recently, the properties of meromorphic solutions of complex linear difference equations have become a subject of great interest from the viewpoint of Nevanlinna's theory and its difference analogues. Since then, many authors investigated the linear difference equations for example, [3, 11, 12]. Moreover, we use notations $\sigma(f)$ for the order of a meromorphic function $f(z)$ and defined as

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

We denote the linear measure for a set $E \subset [0,\infty)$, by $m(E) = \int_E dt$ and logarithmic measure for a set $E \subset (1,\infty)$, by $m_l(E) = \int_E \frac{dt}{t}$. The upper density of a set $E \subset [0,\infty)$ is defined as

$$\overline{dens}\ E = \limsup_{r\to\infty} \frac{m(E\cap[0,r])}{r},$$

and the upper logarithmic density of a set $E \subset (1,\infty)$ is defined as

$$\overline{\log dens}\ (E) = \limsup_{r\to\infty} \frac{m_l(E\cap[1,r])}{\log r}.$$

Proposition 1.1. [1]*For all $H \subset [1,\infty)$ the following statements hold:*
(i) If $m_l(H) = \infty$, then $m(H) = \infty$;
(ii) If $\overline{dens} H > 0$, then $m(H) = \infty$;
(iii) If $\overline{\log dens} H > 0$, then $m_l(H) = \infty$.

In 2008, Chiang and Feng [3] investigated the proximity function and point wise estimates of $\frac{f(z+\eta)}{f(z)}$, which are discrete versions of the classical logarithmic derivative estimates of $f(z)$. They also applied their results to obtain growth estimates of meromorphic solutions to higher order homogeneous and non-homogeneous linear difference equations

$$A_n(z)f(z+n) + \cdots + A_1(z)f(z+1) + A_0(z)f(z) = 0 \tag{1.1}$$

and

$$A_n(z)f(z+n) + \cdots + A_1(z)f(z+1) + A_0(z)f(z) = F(z), \tag{1.2}$$

where the coefficients $A_0(z), ..., A_n(z)$ and $F(z)\ (\not\equiv 0)$ are entire functions and they obtained the following result.

Theorem 1.2. [3] *Let $A_0(z), ..., A_n(z)$ be entire functions such that there exists an integer l $(0 \leq l \leq n)$ such that*

$$\max_{0\leq j\leq n} \{\sigma(A_j); j \neq l\} < \sigma(A_l),$$

then every meromorphic solution of equation (1.1) *satisfies* $\sigma(f) \geq \sigma(A_l) + 1$.

Above results occur when there exists only one dominant coefficient. In the case that there are more than one dominant coefficients, Laine and Yang [11] obtained the following result.

Theorem 1.3. [11] *Let $A_0(z), ..., A_n(z)$ be entire functions of finite order such that among those having the maximal order $\sigma = \max\limits_{0\leq j\leq n} \sigma(A_j)$, exactly one has its type strictly greater than the others. Then for any meromorphic solution $f\ (\not\equiv 0)$ of equation* (1.1)*, we have $\sigma(f) \geq \sigma + 1$.*

Recently, In 2011, Zheng and Tu [15], studied the growth of meromorphic solutions of homogeneous or non-homogeneous linear difference equations and improved the previous results due to Chiang and Feng [3] and Laine and Yang [11]. In the case there are more than one coefficients of equation (1.1) which have the maximal orders Zheng and Tu [15] obtained the following results.

Theorem 1.4. [15] *Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$ and let $A_j(z)$ $(j = 0, 1, ..., n)$ be entire functions satisfying* $\max\{\sigma(A_j), j = 0, 1, ..., n\} \leq \alpha_1$. *If there exists an integer l $(0 \leq l \leq n)$ and a positive constant α_2 $(\alpha_2 < \alpha_1)$ such that for any given $\varepsilon\,(0 < \varepsilon < \alpha_2 - \alpha_1)$, we have*

$$|A_l(z)| \geq \exp\left\{r^{\alpha_1-\varepsilon}\right\}$$

and

$$|A_j(z)| \leq \exp\{r^{\alpha_2}\}, \quad (j \neq l),$$

as $|z| = r \to +\infty$ for $z \in H$, then every meromorphic solution $f\ (\not\equiv 0)$ of equation (1.1) *satisfies* $\sigma(f) \geq \sigma(A_l) + 1$.

Recently, Chyzhykov et al. [4] introduced the definition of φ-order of $f(z)$ in a unit disc, where $\varphi : [0, 1) \to (0, \infty)$ is a non-decreasing unbounded function and $f(z)$ is a meromorphic function in the unit disc and Shen et al. [14], introduced $[p, q]-\varphi$ order of entire and meromorphic functions in the complex plane $\mathbb{C}$ where $\varphi : [0, \infty) \to (0, \infty)$ is a non-decreasing unbounded function. Since then many researchers investigated the growth oscillation of solutions of linear differential equations and linear difference equations {cf. [2, 5, 6, 13]}. Revisiting their ideas of φ-order we would like to prove some results using the concepts of slow growth scale, the φ-order in the complex plane. To investigate the growth of meromorphic solutions of equations (1.1) and (1.2) more precisely, we recall the following definitions.

Definition 1.5. ([14, 4]) Let $\varphi : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function, the φ-order of a meromorphic function f is defined as

$$\sigma(f, \varphi) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log \varphi(r)}.$$

If f is an entire function, then

$$\sigma(f,\varphi)=\limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log\varphi(r)}.$$

Definition 1.6. ([4]) If f be a meromorphic function satisfying $0<\sigma(f,\varphi)=\sigma<\infty$. Then φ-type of f is defined as

$$\tau(f,\varphi)=\limsup_{r\to\infty}\frac{T(r,f)}{\varphi(r)^{\sigma}}.$$

If f is an entire function, then

$$\tau(f,\varphi)=\limsup_{r\to\infty}\frac{\log M(r,f)}{\varphi(r)^{\sigma}}.$$

Remark 1.7. If $\varphi(r)=r$ in the Definitions 1.5 and 1.6, then we obtain the standard definition of the order and type of a function f respectively.

Remark 1.8. Throughout this paper, we assume that $\varphi:[0,\infty)\to(0,\infty)$ is a non-decreasing unbounded function and always satisfies the following two conditions without special instruction:

(i) $\lim\limits_{r\to+\infty}\frac{\log\log r}{\log\varphi(r)}=0.$

(ii) $\lim\limits_{r\to+\infty}\frac{\log\varphi(\alpha r)}{\log\varphi(r)}=1$ for some $\alpha>1$.

Thus, a natural problem arises that: how to express the growth of solutions of homogeneous and non-homogeneous linear difference equations (1.1) and (1.2) when the coefficients $A_j(z)$ $(j=0,1,...,n)$ and $F(z)\,(\not\equiv 0)$ be entire functions of φ-order in a slow growth scale φ-order. The main purpose of this paper is to make use of the concept of φ-order due to Chyzhykov et al. [4] to extend previous results for solutions to equations (1.1) and (1.2) in the complex plane $\mathbb{C}$.

2. Main results

The main purpose of this paper is to used the concept of φ-order in the complex plane $\mathbb{C}$ to investigate the growth of solutions of homogeneous and non-homogeneous linear difference equations (1.1) and (1.2). In this direction we obtain the following results.

The Theorem 2.1 investigate the order of meromorphic solutions of homogeneous linear difference equation (1.1) in the case when there are more than one coefficients which have the maximal orders.

Theorem 2.1. *Let H be a set of complex numbers satisfying $\overline{\log dens}\,\{|z|:z\in H\}>0$ and let $A_j(z)$ $(j=0,1,...,n)$ be entire functions satisfying*

$$\max\{\sigma(A_j,\varphi),j=0,1,...,n\}\le\sigma.$$

If there exists an integer l $(0 \leq l \leq n)$ such that for some constants α and β with $0 \leq \beta < \alpha$ and ε $(0 < \varepsilon < \sigma)$ sufficiently small, we have

$$T(r, A_l) \geq \exp\left\{\alpha (\varphi(r))^{\sigma-\varepsilon}\right\} \tag{2.1}$$

and

$$T(r, A_j) \leq \exp\left\{\beta (\varphi(r))^{\sigma-\varepsilon}\right\}, \quad (j \neq l), \tag{2.2}$$

as $z \to \infty$ for $z \in H$, then every meromorphic solution f $(\not\equiv 0)$ of equation (1.1) *satisfies $\sigma(f, \varphi) \geq \sigma(A_l, \varphi) + 1$.*

Remark 2.2. By the assumptions of Theorem 2.1, we obtain that $\sigma(A_l, \varphi) = \sigma$. Indeed, we have $\sigma(A_l, \varphi) \leq \sigma$, suppose that $\sigma(A_l, \varphi) = \eta < \sigma$. Then by Definition 1.5 of φ-order and (2.1), we have for any given $\varepsilon \left(0 < \varepsilon < \frac{\sigma-\eta}{2}\right)$

$$\exp\left\{\alpha (\varphi(r))^{\sigma-\varepsilon}\right\} \leq T(r, A_l) \leq \exp\left\{(\varphi(r))^{\eta+\varepsilon}\right\}, \tag{2.3}$$

as $|z| = r \to \infty$ for $z \in H$. So by $\varepsilon \left(0 < \varepsilon < \frac{\sigma-\eta}{2}\right)$ we get a contradiction from (2.3) as $r \to \infty$. Hence $\sigma(A_l, \varphi) = \sigma$.

The following example illustrate the sharpness of Theorem 2.1.

Example 2.3. The function $f(z) = e^{z^2-3z}$ satisfies the equation

$$e^{-z} f(z+2) + e^{z} f(z+1) - 2e^{3z-2} f(z) = 0.$$

Here $A_2(z) = e^{-z}$, $A_1(z) = e^{z}$, $A_0(z) = -2e^{3z-2}$, we take $\varphi(z) = z$, then we obtain that $\sigma(A_2, \varphi) = \sigma(A_1, \varphi) = \sigma(A_0, \varphi) = 1$. Now set $H = \{z : \arg z = \pi\}$ and $l = 2$, then it is clear that $\overline{dens}\{|z| = r : z \in H\} = 1 > 0$. Moreover, $A_2(z)$, $A_1(z)$ and $A_0(z)$ satisfy the assumptions (2.1) and (2.2) of Theorem 2.1. Therefore, we get $\sigma(f, \varphi) = 2 = \sigma(A_2, \varphi) + 1$.

Secondly, we consider the growth of entire solutions of non-homogeneous linear difference equation (1.2). Note that the above result may not be applicable to the equation (1.2) to which equation (1.1) is the corresponding homogeneous equation (see the following Example 2.5). But we can obtain similar results with some additional conditions.

Theorem 2.4. *Let $A_j(z)$ $(j = 0, 1, ..., n)$ and $F(z)\,(\not\equiv 0)$ be entire functions such that there exists an integer l $(0 \leq l \leq n)$ satisfying*

$$b = \max\{\sigma(A_j, \varphi), \sigma(F, \varphi), j \neq l,\} < \sigma(A_l, \varphi) < \frac{1}{2}, \tag{2.4}$$

then every nontrivial entire solution f $(\not\equiv 0)$ of equation (1.2) *satisfies $\sigma(f, \varphi) \geq \sigma(A_l, \varphi) + 1$.*

Example 2.5. Take $\varphi(z) = z$ and the function $f(z) = e^{z}$ satisfies the equation

$$f(z+2) - ef(z+1) + f(z) = e^{z}$$

and

$$f(z+2) - ef(z+1) + e^{-z} f(z) = 1.$$

Though there is only one dominant coefficient such that the assumptions in Theorems 2.1 hold, we cannot get similar results in the non-homogeneous equation case.

Theorem 2.6. *Let $A_j(z)$ $(i = 0, 1, ..., n)$ and $F(z)\,(\not\equiv 0)$ entire functions such that there exists an integer $(0 \le l \le n)$ satisfying*

$$b = \max\{\sigma(A_j, \varphi), \sigma(F, \varphi), j \neq l, \} < \sigma(A_l, \varphi) < \infty.$$

Also suppose that $A_l(z) = \sum_{n=1}^{\infty} C_{\lambda_n} z^{\lambda_n}$ satisfies that the sequence of exponents $\{\lambda_n\}$ satisfies the Fabry gap condition $\frac{\lambda_n}{n} \to \infty$ as $n \to \infty$, then every nontrivial entire solution $f\ (\not\equiv 0)$ of equation (1.2) satisfies $\sigma(f, \varphi) \ge \sigma(A_l, \varphi) + 1$.

3. Preliminary lemmas

To prove the above theorems, we need some lemmas as follows.

Lemma 3.1. [3] *Let f be a meromorphic function, η be a non-zero complex number and let $\gamma > 1$ and $\varepsilon > 0$ be given real constants. Then there exist a subset $E_1 \subset (1, +\infty)$ of finite logarithmic measure and a constant A depending only on γ and η, such that for all $|z| = r \notin E_1 \cup [0, 1]$, we have*

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \le A\left(\frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r}\log^{\gamma} r \log^{+} n(\gamma r)\right),$$

where $n(t) = n(t, \infty, f) + n\left(t, \infty, \frac{1}{f}\right)$.

Lemma 3.2. [7] *Let f be a transcendental meromorphic function and let j be a non-negative integer, let a be a value in the extended complex plane and let $\alpha > 1$ be a real constant. Then there exists a constant $R > 0$ such that for all $r > R$, we have*

$$n\left(r, a, f^{(j)}\right) \le \frac{2j+6}{\log\alpha} T(\alpha r, f).$$

Lemma 3.3. *Let f be a meromorphic function and η be a non-zero complex number and let $\varepsilon > 0$ be given real constants. Then there exists a subset $E_2 \subset (1, +\infty)$ of finite logarithmic measure, such that if f has finite φ-order σ, then for all $|z| = r \notin E_2 \cup [0, 1]$, we have*

$$\exp\left\{-\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\} \le \left|\frac{f(z+\eta)}{f(z)}\right| \le \exp\left\{\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\}.$$

Proof. By Lemma 3.1, there exist a subset there exist a subset $E_2 \subset (1, +\infty)$ of finite logarithmic measure and a constant A depending only on γ and η, such that for all $|z| = r \notin E_2 \cup [0, 1]$, we have

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \le A\left(\frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r}\log^{\gamma} r \log^{+} n(\gamma r)\right), \tag{3.1}$$

where $n(t) = n(t, \infty, f) + n\left(t, \infty, \frac{1}{f}\right)$.

Using (3.1) and Lemma 3.2, we obtain that

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \le A\left(\frac{T(\gamma r, f)}{r} + \frac{12}{\log\alpha}\frac{T(\alpha\gamma r, f)}{r}\log^{\gamma} r \log^{+}\left(\frac{12}{\log\alpha}T(\alpha\gamma r, f)\right)\right)$$

$$\leq B\left(\frac{T(\beta r, f)}{r}+\frac{\log^{\beta} r}{r} T(\beta r, f) \log T(\beta r, f)\right), \tag{3.2}$$

for all $|z|=r \notin [0,1] \cup E_2$ with $m_l(E_2)<+\infty$, where $B>0$ is some constant and $\beta=\alpha\gamma>1$.

Again, since f has finite φ-order $\sigma(f,\varphi)=\sigma<+\infty$, so given $\varepsilon\,(0<\varepsilon<2)$, for sufficiently large r, we have

$$T(r, f)<(\varphi(r))^{\sigma+\frac{\varepsilon}{2}}. \tag{3.3}$$

Then by substituting (3.3) into (3.2), we get that

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq B\left(\frac{(\varphi(\beta r))^{\sigma+\frac{\varepsilon}{2}}}{r}+\frac{\log^{\beta} r}{r}(\varphi(\beta r))^{\sigma+\frac{\varepsilon}{2}} \log (\varphi(\beta r))^{\sigma+\frac{\varepsilon}{2}}\right)$$

$$\leq \frac{(\varphi(r))^{\sigma+\varepsilon}}{r}. \tag{3.4}$$

From (3.4), we obtain that

$$\exp\left\{-\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\} \leq\left|\frac{f(z+\eta)}{f(z)}\right| \leq \exp\left\{\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\}.$$

This proves the lemma. □

Lemma 3.4. *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let f be a meromorphic function of finite φ-order σ and let $\varepsilon>0$ be given. Then there exists a subset $E_3 \subset (1,+\infty)$ of finite logarithmic measure such that for all $|z|=r \notin [0,1] \cup E_3$, we have*

$$\exp\left\{-\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\} \leq\left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leq \exp\left\{\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\}.$$

Proof. We can write

$$\left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right|=\left|\frac{f(z+\eta_2+\eta_1-\eta_2)}{f(z+\eta_2)}\right|, \quad (\eta_1 \neq \eta_2).$$

Then by using Lemma 3.3, there exists a subset $E_3 \subset (1,+\infty)$ such that for any $\varepsilon>0$ and all $|z+\eta_2|=R \notin E_3 \cup [0,1]$, with $m_l(E_3)<\infty$, we get

$$\begin{aligned}
\exp\left\{-\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\} & \leq \exp\left\{-\frac{(\varphi(|z|+|\eta_2|))^{\sigma+\frac{\varepsilon}{2}}}{|z+\eta_2|}\right\} \\
& = \exp\left\{-\frac{(\varphi(R))^{\sigma+\frac{\varepsilon}{2}}}{R}\right\} \leq\left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \\
& = \left|\frac{f(z+\eta_2+\eta_1-\eta_2)}{f(z+\eta_2)}\right| \leq \exp\left\{\frac{(\varphi(R))^{\sigma+\frac{\varepsilon}{2}}}{R}\right\} \\
& \leq \exp\left\{\frac{(\varphi(|z|+|\eta_2|))^{\sigma+\varepsilon}}{|z+\eta_2|}\right\} \leq \exp\left\{\frac{(\varphi(r))^{\sigma+\varepsilon}}{r}\right\},
\end{aligned}$$

where $|z| = r \notin [0,1] \cup E_3$.

This proves the lemma. $\square$

Lemma 3.5. [5]*Let* η_1, η_2 *be two arbitrary complex numbers such that* $\eta_1 \neq \eta_2$, *and let* f *be a meromorphic function of finite* φ*-order. Let* σ *be the* φ*-order of* $f(z)$*. Then for each* $\varepsilon > 0$, *we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O\left((\varphi(r))^{\sigma-1+\varepsilon}\right).$$

Lemma 3.6. [9] *Let* $f(z) = \sum_{n=1}^{\infty} C_{\lambda_n} z^{\lambda_n}$ *be an entire function and the sequence of exponents* $\{\lambda_n\}$ *satisfies the Fabry gap condition* $\frac{\lambda_n}{n} \to \infty$ *as* $n \to \infty$*. Then for any given* $\varepsilon > 0$

$$\log L(r,f) > (1-\varepsilon) \log M(r,f),$$

holds outside a set E_4 *of finite logarithmic measure, where* $M(r,f) = \sup_{|z|=r} |f(z)|$ *and* $L(r,f) = \inf_{|z|=r} |f(z)|$.

Lemma 3.7. *Let* $f(z)$ *be an entire function of finite* φ*-order satisfying* $0 < \sigma(f,\varphi) < \infty$, *where* $\varphi(r)$ *only satisfies* $\lim_{r\to+\infty} \frac{\log\varphi(\alpha r)}{\log\varphi(r)} = 1$ *for some* $\alpha > 1$*. Then for any given* $\beta < \sigma(f,\varphi)$, *there exists a set* $E_5 \subset (1,\infty)$ *having infinite logarithmic measure such that for all* $|z| = r \in E_5$ *we have*

$$M(r,f) > \exp\left\{(\varphi(r))^{\beta}\right\}.$$

Proof. By the Definition 1.5 of the φ-order, there exists an increasing sequence $\{r_n\}$ $(r_n \to \infty)$ satisfying $\left(1+\frac{1}{n}\right) r_n < r_{n+1}$ and

$$\sigma(f,\varphi) = \lim_{r_n\to\infty} \frac{\log\log M(r_n,f)}{\log\varphi(r_n)}.$$

Then, there exists a positive integer n_0 such that for all $n \geq n_0$ and for any given $\varepsilon > 0$, we have

$$M(r_n,f) > \exp\left\{(\varphi(r_n))^{\sigma(f,\varphi)-\varepsilon}\right\}. \tag{3.5}$$

Now we have

$$\lim_{n\to\infty} \frac{\log\varphi\left(\left(1+\frac{1}{n}\right)r\right)}{\log\varphi(r)} = 1.$$

Since $\beta < \sigma(f,\varphi)$, then we can choose sufficiently small $\varepsilon > 0$ to satisfy $0 < \varepsilon < \sigma(f,\varphi) - \beta$, so there exists a positive integer n_1 such that for all $n > n_1$, we have

$$\frac{\log\varphi\left(\left(1+\frac{1}{n}\right)r\right)}{\log\varphi(r)} > \frac{\beta}{\sigma(f,\varphi)-\varepsilon},$$

which implies that

$$(\sigma(f,\varphi)-\varepsilon)\log\varphi\left(\left(1+\frac{1}{n}\right)r\right) > \beta\log\varphi(r)$$

$$\Rightarrow \left(\varphi\left(\left(1+\frac{1}{n}\right)r\right)\right)^{(\sigma(f,\varphi)-\varepsilon)} > \varphi(r)^{\beta}. \tag{3.6}$$

Taking $n \geq n_2 = \max\{n_0, n_1\}$ and $E_5 = \bigcup\limits_{n=n_2}^{\infty} I_n$, where $I_n = \left[r_n, \left(1+\frac{1}{n}\right) r_n\right]$. Then by (3.5) and (3.6), we get for $r \in \left[r_n, \left(1+\frac{1}{n}\right) r_n\right]$ that

$$M(r,f) \geq M(r_n,f) > \exp\left\{(\varphi(r_n))^{\sigma(f,\varphi)-\varepsilon}\right\}$$

$$\geq \exp\left\{\left(\varphi\left(\left(1+\frac{1}{n}\right)r\right)\right)^{\sigma(f,\varphi)-\varepsilon}\right\} > \exp\left\{\varphi(r)^{\beta}\right\}.$$

Now we obtain that

$$m_l(E_5) = \sum_{n=n_2}^{\infty} \int_{I_n} \frac{dr}{r} = \sum_{n=n_2}^{\infty} \left(\log \frac{1}{1-r}\right)_{r_n}^{\left(1+\frac{1}{n}\right)r_n} = \sum_{n=n_2}^{\infty} \log \frac{n+1}{n} = \infty.$$

This proves the lemma. $\square$

Lemma 3.8. *Let $f(z) = \sum_{n=1}^{\infty} C_{\lambda_n} z^{\lambda_n}$ be an entire function with $0 < \sigma(f,\varphi) < \infty$ where $\varphi(r)$ only satisfies $\lim\limits_{r\to+\infty} \frac{\log\varphi(\alpha r)}{\log\varphi(r)} = 1$ for some $\alpha > 1$. If the sequence of exponents $\{\lambda_n\}$ satisfies the Fabry gap condition $\frac{\lambda_n}{n} \to \infty$ as $n \to \infty$. Then for any given $\beta < \sigma(f,\varphi)$, there exists a set $E_6 \subset (1,\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_6$ we have*

$$|f(z)| > \exp\left\{\varphi(r)^{\beta}\right\}.$$

Proof. By Lemma 3.6, for any $\varepsilon > 0$, there exists a set E_4 of finite logarithmic measure such that for all $|z| = r \notin E_4$, we have

$$\log L(r,f) > (1-\varepsilon)\log M(r,f),$$

which implies that

$$L(r,f) > [M(r,f)]^{(1-\varepsilon)}.$$

For any given $\beta < \sigma(f,\varphi)$, we can choose $\delta > 0$ such that $\beta < \delta < \sigma(f,\varphi)$ and sufficiently small ε satisfying $0 < \varepsilon < \frac{\delta-\beta}{2}$. Then by Lemma 3.7, there exists a set E_5 of infinite logarithmic measure such that for all $|z| = r \in E_5$, we have

$$|f(z)| > L(r,f) > [M(r,f)]^{(1-\varepsilon)} > \left(\exp\left\{\varphi(r)^{\beta}\right\}\right)^{(1-\varepsilon)} > \exp\left\{\varphi(r)^{\beta}\right\},$$

where $E_6 = E_5 \backslash E_4$ is a set with infinite logarithmic measure.

Thus the lemma is established. $\square$

4. Proof of main results

Proof of Theorem 2.1. By Remark 2.2, we know that $\sigma(A_l,\varphi) = \sigma$. Let $f \not\equiv 0$ be a meromorphic solution of equation (1.1). Now let us suppose that $\sigma(f,\varphi) < \sigma(A_l,\varphi) + 1 = \sigma + 1 < \infty$. From the conditions of Theorem 2.1, there is a set H of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$ such that for $z \in H$, we have (2.1) and (2.2) as $|z| = r \to \infty$. Set $H_1 = \{|z| = r : z \in H\}$, since $\overline{\log dens}\{|z| : z \in H\} > 0$, then by Proposition 1.1, H_1 is a set with $\int_{H_1} \frac{dr}{r} = \infty$.

We divide equation (1.1) by $f(z+l)$ to get

$$-A_l(z)=\sum_{\substack{j=0\\ i\neq l}}^{n}A_j(z)\frac{f(z+j)}{f(z+l)}. \tag{4.1}$$

Since $A_j(z)$ $(j=0,1,...,n)$ are entire functions, then by equation (4.1), we get that

$$\begin{aligned} m(r,A_l)=T(r,A_l) &\leq \sum_{\substack{j=0\\ i\neq l}}^{n}m(r,A_j)+\sum_{\substack{j=0\\ i\neq l}}^{n}m\left(r,\frac{f(z+j)}{f(z+l)}\right)+O(1)\\ &=\sum_{\substack{j=0\\ i\neq l}}^{n}T(r,A_j)+\sum_{\substack{j=0\\ i\neq l}}^{n}m\left(r,\frac{f(z+j)}{f(z+l)}\right)+O(1). \end{aligned} \tag{4.2}$$

Now by Lemma 3.5, for any ε $\left(0<\varepsilon<\frac{\sigma+1-\sigma(f,\varphi)}{2}\right)$, we have

$$m\left(r,\frac{f(z+j)}{f(z+l)}\right)=O\left((\varphi(r))^{\sigma(f,\varphi)-1+\varepsilon}\right). \tag{4.3}$$

Substituting (2.1), (2.2) and (4.3) into (4.2), we get for $|z|=r\to\infty,\ z\in H$ that

$$\begin{aligned} &\exp\left\{\alpha(\varphi(r))^{\sigma-\varepsilon}\right\}\leq n\exp\left\{\beta(\varphi(r))^{\sigma-\varepsilon}\right\}+O\left((\varphi(r))^{\sigma(f,\varphi)-1+\varepsilon}\right)\\ \Rightarrow\ &\exp\left\{(\varphi(r))^{\sigma-\varepsilon}\right\}\{\exp(\alpha)-\exp(\beta)\}\leq O(1)(\varphi(r))^{\sigma(f,\varphi)-1+\varepsilon}. \end{aligned}$$

Since, $(\exp(\alpha)-\exp(\beta))>0$, so it follows that

$$1\leq O(1)(\varphi(r))^{\sigma(f,\varphi)-1+2\varepsilon-\sigma}\to 0 \text{ as } r\to\infty, \tag{4.4}$$

which is a contradiction since $0<\varepsilon<\frac{\sigma+1-\sigma(f,\varphi)}{2}$.
Hence, we get $\sigma(f,\varphi)\geq\sigma(A_l,\varphi)+1$.
This completes the proof of the theorem. □

Proof of Theorem 2.2. If $\sigma(f,\varphi)=\infty$, then the result is trivial. Now let us suppose that $\sigma(f,\varphi)<\sigma(A_l,\varphi)+1<\infty$. We divide equation (1.2) by $f(z+l)$ to get

$$-A_l(z)=\sum_{\substack{j=0\\ i\neq l}}^{n}A_j(z)\frac{f(z+j)}{f(z+l)}-\frac{F(z)}{f(z)}\cdot\frac{f(z)}{f(z+l)},$$

which implies that

$$|A_l(z)|\leq\sum_{\substack{j=0\\ i\neq l}}^{n}|A_j(z)|\left|\frac{f(z+j)}{f(z+l)}\right|+\left|\frac{F(z)}{f(z)}\right|\cdot\left|\frac{f(z)}{f(z+l)}\right|. \tag{4.5}$$

By Lemma 3.4, for any given ε $\left(0<\varepsilon<\frac{\sigma(A_l,\varphi)+1-\sigma(f,\varphi)}{2}\right)$, there exists a subset $E_3\subset(1,\infty)$ of finite logarithmic measure such that for all $r\notin[0,1]\cup E_3$, we have

$$\left|\frac{f(z+j)}{f(z+l)}\right|\leq\exp\left\{\frac{(\varphi(r))^{\sigma(f,\varphi)+\varepsilon}}{r}\right\},\quad (j=0,1,...,n,j\neq l). \tag{4.6}$$

Now by the assumption (2.4), we have that for sufficiently large r,

$$|A_j(z)| \leq \exp\left\{(\varphi(r))^{\sigma(A_l,\varphi)+\varepsilon}\right\}, \quad (j = 0, 1, ..., n, j \neq l), \tag{4.7}$$

and

$$|F(z)| \leq \exp\left\{(\varphi(r))^{\sigma(A_l,\varphi)+\varepsilon}\right\}.$$

Since $M(r, f) > 1$ for sufficiently large r, we have that

$$\frac{|F(z)|}{M(r,f)} \leq |F(z)| \leq \exp\left\{(\varphi(r))^{\sigma(A_l,\varphi)+\varepsilon}\right\}. \tag{4.8}$$

Now by the Definition 1.5 of φ-order and for above $\varepsilon > 0$, we get that

$$|A_l(z)| \geq \exp\left\{(\varphi(r))^{\sigma(A_l,\varphi)-\varepsilon}\right\}. \tag{4.9}$$

Substituting (4.6)-(4.9) into (4.5) for all $r \notin [0,1] \cup E_3$ and $|f(z)| = M(r, f)$, we have

$$\exp\left\{(\varphi(r))^{\sigma(A_l,\varphi)-\varepsilon}\right\} \leq |A_l(z)|$$

$$\leq (n+1)\exp\left\{(\varphi(r))^{\sigma(A_l,\varphi)+\varepsilon}\right\} \cdot \exp\left\{\frac{(\varphi(r))^{\sigma(f,\varphi)+\varepsilon}}{r}\right\}. \tag{4.10}$$

Since $\varepsilon \left(0 < \varepsilon < \frac{\sigma(A_l,\varphi)+1-\sigma(f,\varphi)}{2}\right)$, so we obtain a contradiction from (4.10) by applying the same procedure we applied in (4.4). Hence we get that $\sigma(f,\varphi) \geq \sigma(A_l,\varphi)+1$.

This proves the theorem. □

Proof of Theorem 2.3. If $\sigma(f,\varphi) = \infty$, then the result is trivial. Now let us suppose that $\sigma(f,\varphi) < \sigma(A_l,\varphi) + 1 < \infty$. Now by Lemma 3.8, there exists a set $E_6 \subset (1,\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_6$ we have

$$|A_l(z)| > \exp\left\{(\varphi(r))^{\beta}\right\}. \tag{4.11}$$

Substituting (4.6)-(4.8) and (4.11) into (4.5) for all $r \in E_6 \backslash [0,1] \cup E_3$ and $|f(z)| = M(r, f)$, we have

$$\exp\left\{(\varphi(r))^{\beta}\right\} \leq |A_l(z)| \leq (n+1)\exp\left\{(\varphi(r))^{\sigma(A_l,\varphi)+\varepsilon}\right\} \cdot \exp\left\{\frac{(\varphi(r))^{\sigma(f,\varphi)+\varepsilon}}{r}\right\}. \tag{4.12}$$

We we get a contradiction from (4.12) by applying the same procedure we applied in (4.4). Hence we get that $\sigma(f,\varphi) \geq \sigma(A_l,\varphi) + 1$.

This proves the theorem. □

Acknowledgement. The authors are grateful to the referee for his valuable suggestions which has considerably improved the presentation of the paper.

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 307–317
DOI: 10.24193/subbmath.2023.2.07

Extension operators and Janowski starlikeness with complex coefficients

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Abstract. In this paper, we obtain certain generalizations of some results from [13] and [14]. Let $\Phi_{n,\alpha,\beta}$ be the extension operator introduced in [7] and let $\Phi_{n,Q}$ be the extension operator introduced in [16]. Let $a \in \mathbb{C}$, $b \in \mathbb{R}$ be such that $|1-a| < b \leq \operatorname{Re} a$. We consider the Janowski classes $S^*(a,b,\mathbb{B}^n)$ and $\mathcal{A}S^*(a,b,\mathbb{B}^n)$ with complex coefficients introduced in [4]. In the case $n=1$, we denote $S^*(a,b,\mathbb{B}^1)$ by $S^*(a,b)$ and $\mathcal{A}S^*(a,b,\mathbb{B}^1)$ by $\mathcal{A}S^*(a,b)$. We shall prove that the following preservation properties concerning the extension operator $\Phi_{n,\alpha,\beta}$ hold: $\Phi_{n,\alpha,\beta}(S^*(a,b)) \subseteq S^*(a,b,\mathbb{B}^n)$, $\Phi_{n,\alpha,\beta}(\mathcal{A}S^*(a,b)) \subseteq \mathcal{A}S^*(a,b,\mathbb{B}^n)$. Also, we prove similar results for the extension operator $\Phi_{n,Q}$:

$$\Phi_{n,Q}(S^*(a,b)) \subseteq S^*(a,b,\mathbb{B}^n),\ \Phi_{n,Q}(\mathcal{A}S^*(a,b)) \subseteq \mathcal{A}S^*(a,b,\mathbb{B}^n).$$

Mathematics Subject Classification (2010): 32H02, 30C45.

Keywords: g-Loewner chain, g-parametric representation, g-starlikeness, Janowski starlikeness, Janowski almost starlikeness, extension operator.

1. Preliminaries

Let $\mathbb{C}^n$ be the space of n complex variables equipped with the Euclidean inner product $\langle\cdot,\cdot\rangle$ and the Euclidean norm $\|\cdot\|$. Let $\mathbb{B}^n$ be the open unit ball in $\mathbb{C}^n$ and let U be the unit disc in $\mathbb{C}$. Also, let $H(\mathbb{B}^n)$ be the set of holomorphic mappings from $\mathbb{B}^n$ into $\mathbb{C}^n$. A mapping $f \in H(\mathbb{B}^n)$ is said to be normalized if $f(0)=0$ and $Df(0)=I_n$. Let $J_f(z)$ be the complex Jacobian determinant of the Fréchet derivative $Df(z)$, i.e. $J_f(z) = detDf(z)$. A mapping $f \in H(\mathbb{B}^n)$ is locally biholomorphic mapping on $\mathbb{B}^n$ if $J_f(z) \neq 0$ for all $z \in \mathbb{B}^n$. We denote by $\mathcal{L}S_n$ the set of normalized locally biholomorphic mappings on the unit ball $\mathbb{B}^n$. In the case $n=1$, we use the notation

Received 05 June 2022; Accepted 06 July 2022.

$\mathcal{L}S$ instead of $\mathcal{L}S_1$. Let $S(\mathbb{B}^n)$ be the set of normalized biholomorphic mappings on $\mathbb{B}^n$ and let S be the set of normalized univalent functions on U. Also, let $S^*(\mathbb{B}^n)$ be the set of normalized starlike mappings on $\mathbb{B}^n$.

Let $f, g \in H(\mathbb{B}^n)$. Then we say that $f \prec g$ if there exists a Schwarz mapping φ (i.e. $\varphi \in H(\mathbb{B}^n)$, $\|\varphi(z)\| \leq \|z\|$, $z \in \mathbb{B}^n$) such that $f = g \circ \varphi$ on $\mathbb{B}^n$. Moreover, if g is biholomorphic on $\mathbb{B}^n$, then the subordination condition $f \prec g$ is equivalent with $f(0) = g(0)$ and $f(\mathbb{B}^n) \subseteq g(\mathbb{B}^n)$.

We recall that $f : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ is a Loewner chain if $f(\cdot, t)$ is biholomorphic on $\mathbb{B}^n$, $f(0, t) = 0$, $Df(0, t) = e^t I_n$ for $t \geq 0$ and $f(\cdot, s) \prec f(\cdot, t)$ with $0 \leq s \leq t < \infty$ (see [17], [8]). The subordination condition $f(\cdot, s) \prec f(\cdot, t)$ is equivalent to the following statement: there is a unique biholomorphic Schwarz mapping $v = v(z, s, t)$ such that $f(z, s) = f(v(z, s, t), t)$, $z \in \mathbb{B}^n$, $0 \leq s \leq t$. The mapping $v = v(z, s, t)$ is called the *transition mapping* associated to $f(z, t)$ and satisfies the semigroup property: $v(z, s, u) = v(v(z, s, t), t, u)$, for all $z \in \mathbb{B}^n$, $0 \leq s \leq t \leq u$. In addition, $Dv(0, s, t) = e^{s-t} I_n$, $0 \leq s \leq t$ (see [17], [8]).

We recall that the following class of holomorphic mappings (see [17], [20]; see also [8]):

$$\mathcal{M} = \{h \in H(\mathbb{B}^n) : h(0) = 0, Dh(0) = I_n, \mathrm{Re}\, \langle h(z), z \rangle > 0, z \in \mathbb{B}^n \backslash \{0\}\}$$

is the generalization to higher dimensions ($n \geq 2$) of the Carathéodory class of functions with positive real part on U.

We next give the definition of parametric representation on the unit ball in $\mathbb{C}^n$ (see [5], [8]).

Definition 1.1. We say that a mapping $f \in S(\mathbb{B}^n)$ has *parametric representation* if there exists a Loewner chain $f(z, t)$ such that f can be embedded as the first element of $f(z, t)$ and the family $\{e^{-t} f(\cdot, t)\}_{t \geq 0}$ is normal on $\mathbb{B}^n$.

Let $S^0(\mathbb{B}^n)$ be the family of mappings with parametric representation. This set has been introduced by Graham, Hamada and Kohr in [5]. Various results regarding this class can be found in [5], [9], [10] and the references therein.

In the following we consider a function $g : U \to \mathbb{C}$ which satisfies the following conditions (see [6]):

Assumption 1.2. Let $g : U \to \mathbb{C}$ be such that g is a univalent (i.e. holomorphic and injective) function on U, $g(0) = 1$ and g has positive real part on U.

For example, the function $g : U \to \mathbb{C}$ given by $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, $\zeta \in U$, satisfies the requirements of Assumption 1.2.

In the following, let $g : U \to \mathbb{C}$ be an arbitrary function which satisfies the conditions of Assumption 1.2.

Let $\mathcal{M}_g$ be the following nonempty subset of $\mathcal{M}$ introduced by Graham, Hamada, Kohr and Kohr in [6] (see also [5], where the function g satisfies in addition the relation $g(\overline{\zeta}) = \overline{g(\zeta)}$, $z \in U$, and other conditions):

$$\mathcal{M}_g = \left\{ h \in H(\mathbb{B}^n) : h(0) = 0, Dh(0) = I_n, \left\langle h(z), \frac{z}{\|z\|^2} \right\rangle \in g(U), z \in \mathbb{B}^n \backslash \{0\} \right\}.$$

For $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$, we have that $\mathcal{M}_g = \mathcal{M}$.

Next, we recall the definition of a g-Loewner chain (see [6]; see also [5] and [9], for $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$).

Definition 1.3. Let $f(z,t) : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$. We say that $f(z,t)$ is a *g-Loewner chain* if $f(z,t)$ is a Loewner chain such that the family $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is normal on $\mathbb{B}^n$ and the mapping $h(z,t)$ which occurs in the following Loewner differential equation:

$$\frac{\partial f}{\partial t} = Df(z,t)h(z,t), \text{ a.e. } t \geq 0, \ \forall z \in \mathbb{B}^n,$$

has the property $h(\cdot,t) \in \mathcal{M}_g$, for a.e. $t \geq 0$.

We remark that a normalized holomorphic mapping $f : \mathbb{B}^n \to \mathbb{C}^n$ has g-parametric representation if and only if there exists a g-Loewner chain $f(z,t)$ such that f can be embedded as the first element of the g-Loewner chain (see [6]; see also [5]).

Let $S_g^0(\mathbb{B}^n)$ be the set of mappings with g-parametric representation on $\mathbb{B}^n$. Then $S_g^0(\mathbb{B}^n) \subseteq S^0(\mathbb{B}^n)$ (see [6]).

If $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$, then any g-Loewner chain is a Loewner chain and the set $S_g^0(\mathbb{B}^n)$ becomes $S^0(\mathbb{B}^n)$ (see [6]; see also [5]). In the case $n \geq 2$, there exists Loewner chains that are not g-Loewner chains when $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$. For example, when $n = 2$, the mapping $p(z,t) : \mathbb{B}^2 \times [0,\infty) \to \mathbb{C}^n$ given by

$$p(z,t) = \left(\frac{e^t z_1}{(1-z_1)^2}, \frac{e^t z_2}{(1-z_2)^2} + \frac{e^{2t} z_1^2}{(1-z_1)^4}\right), \ z = (z_1, z_2) \in \mathbb{B}^2, \ t \geq 0,$$

is a Loewner chain, but the family $\{e^{-t}p(\cdot,t)\}_{t\geq 0}$ is not normal on $\mathbb{B}^2$. Thus, $p(\cdot,t)$ is not a g-Loewner chain for $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$ (see [5]).

In the next part, we shall refer to the following univalent function g on U with $g(0) = 1$ and positive real part on U:

Assumption 1.4. Let $g : U \to \mathbb{C}$ be a holomorphic function on U given by

$$g(\zeta) = \frac{1 + A\zeta}{1 + B\zeta}, \ \zeta \in U, \tag{1.1}$$

where $A, B \in \mathbb{C}$, $A \neq B$ and g has positive real part on U.

This function was considered in [4].

Imposing the condition that the function g given by Assumption 1.4 to have positive real part implies certain conditions on the complex parameters A and B. These conditions are illustrated in the following remark due to Curt [4].

Remark 1.5. [4] Let $g : U \to \mathbb{C}$ be a function described by Assumption 1.4. Then one of the following two conditions holds:

$$|B| < 1, \ |A| \leq 1 \text{ and } \mathrm{Re}(1 - A\overline{B}) \geq |A - B|, \tag{1.2}$$

or

$$|B| = 1, \ |A| \leq 1 \text{ and } -1 \leq A\overline{B} < 1. \tag{1.3}$$

In this context, we remark that the function g maps the unit disc onto the open disc of center $a := \frac{1-A\overline{B}}{1-|B|^2}$ and radius $b := \frac{|A-B|}{1-|B|^2}$, for $|B| < 1$. It is immediate that $|1-a| < b \leq \mathrm{Re}\ a$. If $|B| = 1$ then g maps the unit disc onto the half-plane $\{z \in \mathbb{C} : \mathrm{Re} z > \frac{1+A\overline{B}}{2}\}$.

Moreover, we have that g is convex on U.

Next, we present the following subclasses of starlike mappings on $\mathbb{B}^n$ introduced by Curt [4]:

Definition 1.6. Let $a \in \mathbb{C}$, $b \in \mathbb{R}$ be such that $|1-a| < b \leq \mathrm{Re}\ a$. Let

$$S^*(a,b,\mathbb{B}^n) = \left\{ f \in \mathcal{L}S_n : \left| \frac{\|z\|^2}{\langle [Df(z)]^{-1} f(z), z\rangle} - a \right| < b,\ z \in \mathbb{B}^n \backslash \{0\} \right\},$$

be the set of Janowski starlike mappings on $\mathbb{B}^n$ and let

$$\mathcal{A}S^*(a,b,\mathbb{B}^n) = \left\{ f \in \mathcal{L}S_n : \left| \frac{\langle [Df(z)]^{-1} f(z), z\rangle}{\|z\|^2} - a \right| < b,\ z \in \mathbb{B}^n \backslash \{0\} \right\},$$

be the set of Janowski almost starlike mappings on $\mathbb{B}^n$.

For $a \in \mathbb{R}$ (which is equivalent to Re $a = a$), the above sets become the classes mentioned in [3]. In the case $n = 1$, we denote $S^*(a,b,\mathbb{B}^1)$ by $S^*(a,b)$, respectively $\mathcal{A}S^*(a,b,\mathbb{B}^1)$ by $\mathcal{A}S^*(a,b)$.

The following remark provides a connection between Janowski starlikeness, respectively Janowski almost starlikeness with complex coefficients and g-starlikeness on $\mathbb{B}^n$ (see [4]).

Remark 1.7. Let $a \in \mathbb{C}$, $b \in \mathbb{R}$ be such that $|1-a| < b \leq \mathrm{Re}\ a$.

(i) If $g(\zeta) = \frac{1+(\overline{a}-1)/b\zeta}{1+(|a|^2-b^2-a)/b\zeta}$, $\zeta \in U$, then $S^*_g(\mathbb{B}^n)$ becomes $S^*(a,b,\mathbb{B}^n)$.

(ii) If $g(\zeta) = \frac{1+(a-|a|^2+b^2)/b\zeta}{1+(1-\overline{a})/b\zeta}$, $\zeta \in U$, then $S^*_g(\mathbb{B}^n)$ becomes $\mathcal{A}S^*(a,b,\mathbb{B}^n)$.

(iii) If $b = a \in R$ ($b = a > 0$), then we have that

$$\mathcal{A}S^*\left(a,a,\mathbb{B}^n\right) = S^*_{\frac{1}{2a}}(\mathbb{B}^n) \text{ and } S^*\left(a,a,\mathbb{B}^n\right) = \mathcal{A}S^*_{\frac{1}{2a}}(\mathbb{B}^n).$$

Note that the functions mentioned in Remark 1.7(i), (ii) satisfy the conditions of Assumption 1.4.

Next, we consider the following extension operator introduced by Graham, Hamada, Kohr and Suffridge in [7].

Definition 1.8. Let $\alpha \geq 0$, $\beta \geq 0$ and $n \geq 2$. Let $\Phi_{n,\alpha,\beta} : \mathcal{L}S \to \mathcal{L}S_n$ be given by

$$\Phi_{n,\alpha,\beta}(f)(z) = \left(f(z_1), \tilde{z}\left(\frac{f(z_1)}{z_1}\right)^{\alpha} (f'(z_1))^{\beta} \right),\ z = (z_1, \tilde{z}) \in \mathbb{B}^n, \tag{1.4}$$

where

$$\left(\frac{f(z_1)}{z_1}\right)^{\alpha}\Big|_{z_1=0} = 1,\ (f'(z_1))^{\beta}\big|_{z_1=0} = 1.$$

For $\alpha = 0$ and $\beta = 1/2$, the extension operator $\Phi_{n,\alpha,\beta}$ reduces to Roper-Suffridge extension operator $\Phi_n : \mathcal{L}S \to \mathcal{L}S_n$ given by (see [19])

$$\Phi_n(f)(z) = \left(f(z_1), \tilde{z}\sqrt{f'(z_1)}\right), \ z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

where the branch of the square root is chosen such that $\sqrt{f'(z_1)}|_{z_1=0} = 1$.

The extension operator $\Phi_{n,\alpha,\beta}$ satisfies important preservation properties for $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \le 1$. In [7], it was shown that $\Phi_{n,\alpha,\beta}(f)(S) \subseteq S^0(\mathbb{B}^n)$ and $\Phi_{n,\alpha,\beta}(f)(S^*) \subseteq S^*(\mathbb{B}^n)$. In the same paper, the authors proved that $\Phi_{n,\alpha,\beta}$ conserves convexity only if $(\alpha,\beta) = (0,1/2)$. Also, $\Phi_{n,\alpha,\beta}$ conserves starlikeness of order $\gamma \in (0,1)$ (see [11]), spirallikeness of type $\gamma \in (-\pi/2, \pi/2)$ and order $\delta \in (0,1)$ (see [12]; see also [1]) and almost starlikeness of type $\gamma \in (0,1)$ and order $\delta \in [0,1)$ (see [1]). More recent preservation results regarding this extension operator and Bloch mappings, in the case of complex Banach spaces, are obtained in [6].

We next present the definition of the Muir extension operator $\Phi_{n,Q}$ (see [16]).

Definition 1.9. Assume that $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial of degree 2 and $n \ge 2$. Let $\Phi_{n,Q} : \mathcal{L}S \to \mathcal{L}S_n$ be such that

$$\Phi_{n,Q}(f)(z) = (f(z_1) + Q(\tilde{z})f'(z_1), \tilde{z}\sqrt{f'(z_1)}), \ z = (z_1, \tilde{z}) \in \mathbb{B}^n, \tag{1.5}$$

where $\sqrt{f'(z_1)}|_{z_1=0} = 1$.

For $Q \equiv 0$, the extension operator $\Phi_{n,Q}$ reduces to the extension operator Φ_n.

The extension operator $\Phi_{n,Q}$ preserves parametric representation and starlikeness if $\|Q\| \le 1/4$ (see [10]), convexity if $\|Q\| \le 1/2$ (see [16]) and starlikeness of order $\alpha \in (0,1)$ if $\|Q\| \le \frac{1-|2\alpha-1|}{8\alpha}$ (see [21]; see also [2]). In a recent study, there has been investigated results concerning extended Loewner chains and this extension operator, as well as other preservation results (see [15]). Also, modifications of the Muir extension operator were considered in [6].

Assume that $a \in \mathbb{C}$, $b \in \mathbb{R}$ such that $|1-a| < b \le \operatorname{Re} a$. In the next part, we aim to show that the extension operators $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$ map a function $f \in S^*(a,b)$ into a mapping from $S^*(a,b,\mathbb{B}^n)$. Also, $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$ map a function $f \in \mathcal{A}S^*(a,b)$ into a mapping from $\mathcal{A}S^*(a,b,\mathbb{B}^n)$. Therefore, the extension operators $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$ preserve the Janowski starlikeness and Janowski almost starlikeness with complex coefficients from the case of one complex variable to several complex variables.

2. Main results

In [6], I. Graham, H. Hamada, G. Kohr and M. Kohr proved that g-parametric presentation and g-starlikeness is preserved through the extension operators $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$, when the function g is convex on U and satisfies the conditions of Assumption 1.2. This result was obtained in a more general case, namely on the unit ball of a complex Banach space.

All along this section we assume that $n \ge 2$.

We state in the next two results the preservation of g-starlikeness under $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$, when the function g is convex on U satisfying Assumption 1.2.

Theorem 2.1. [6] *Let $g : U \to \mathbb{C}$ be a univalent holomorphic function on U, with $g(0) = 1$, $\mathrm{Re}g(\zeta) > 0$, $\zeta \in U$, and g is convex on U. Also, let $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha+\beta \leq 1$. If $f \in S_g^*$ then $F = \Phi_{n,\alpha,\beta}(f) \in S_g^*(\mathbb{B}^n)$.*

In the next result, let be the distance from 1 to $\partial g(U)$, denoted by $d(1, \partial g(U))$, and equal to $\inf_{\zeta\in\partial g(U)} |\zeta - 1|$.

Theorem 2.2. [6] *Let $g : U \to \mathbb{C}$ be a univalent function on U, with $g(0) = 1$, $\mathrm{Re}g(\zeta) > 0$, $\zeta \in U$, and g is convex on U. Also, let $\|Q\| \leq d(1, \partial g(U))/4$, where Q is a homogeneous polynomial of degree 2 from $\mathbb{C}^{n-1}$ to $\mathbb{C}$. If $f \in S_g^*$ then*

$$F = \Phi_{n,Q}(f) \in S_g^*(\mathbb{B}^n).$$

It is clear that, for the function g defined by Assumption 1.4, the above statements hold.

In addition, we have the following result.

Remark 2.3. Let g be a function satisfying the conditions from Assumption 1.4. Then

$$d(1, \partial g(U)) = \frac{|A-B|}{1+|B|}.$$

Proof. Since the function g satisfies the requirements of Assumption 1.4, then, in view of Remark 1.5, the complex coefficients A and B satisfy one of the following two relations:

$$|B| < 1,\ |A| \leq 1 \text{ and } \mathrm{Re}(1 - A\overline{B}) \geq |A - B|,$$

or

$$|B| = 1,\ |A| \leq 1 \text{ and } -1 \leq A\overline{B} < 1.$$

We shall analyze the above two cases.

• Assume that $|B| = 1$, $|A| \leq 1$ and $\mathrm{Re}(1 - A\overline{B}) \geq |A - B|$. In this case, we have $g(U) = \{z \in \mathbb{C} : \mathrm{Re}\, z > \frac{1+A\overline{B}}{2}\}$. Thus,

$$\partial g(U) = \{z \in \mathbb{C} : z = \frac{1 + A\overline{B}}{2} + iy,\ y \in \mathbb{R}\}.$$

Let $\zeta \in \partial g(U)$. Then $\zeta = \frac{1+A\overline{B}}{2} + iy$, where $y \in \mathbb{R}$. We have that

$$|\zeta - 1| = \left|\frac{1 + A\overline{B}}{2} + iy - 1\right| = \left|\frac{-1 + A\overline{B}}{2} + iy\right|.$$

Using the above relation and the fact that $-1 \leq A\overline{B} < 1$, we have that

$$\inf_{\zeta\in\partial g(U)} |\zeta - 1| = \inf_{y\in\mathbb{R}} \left|\frac{-1 + A\overline{B}}{2} + iy\right| = \inf_{y\in\mathbb{R}} \sqrt{\left(\frac{1 - A\overline{B}}{2}\right)^2 + y^2} = \frac{1 - A\overline{B}}{2}.$$

Note that, for $|B| = 1$ and since $-1 \leq A\overline{B} < 1$, we have the following equivalence:

$$\frac{1 - A\overline{B}}{2} = \frac{|1 - A\overline{B}|}{2} = \frac{\left||B|^2 - A\overline{B}\right|}{1 + |B|} = \frac{|\overline{B}| \cdot |A - B|}{1 + |B|} = \frac{|A - B|}{1 + |B|}.$$

• Assume that $|B| = 1$, $|A| \leq 1$ and $-1 \leq A\overline{B} < 1$. Then

$$g(U) = U\left(\frac{1 - A\overline{B}}{1 - |B|^2}, \frac{|A - B|}{1 - |B|^2}\right).$$

Thus,

$$\partial g(U) = \left\{z \in C : z = \frac{1 - A\overline{B}}{1 - |B|^2} + \lambda\frac{|A - B|}{1 - |B|^2},\ |\lambda| = 1\right\}.$$

Let $\zeta \in \partial g(U)$. Then there exists $\lambda \in C$ with $|\lambda| = 1$ such that

$$\zeta = \frac{1 - A\overline{B}}{1 - |B|^2} + \lambda\frac{|A - B|}{1 - |B|^2}.$$

Further, an elementary computation implies that:

$$\begin{aligned}
|\zeta - 1| &= \left|\frac{1 - A\overline{B}}{1 - |B|^2} + \lambda\frac{|A - B|}{1 - |B|^2} - 1\right| \\
&= \frac{\Big||B|^2 - A\overline{B} + \lambda|A - B|\Big|}{1 - |B|^2} \\
&= \frac{\Big|\lambda|A - B| - \overline{B}(A - B)\Big|}{1 - |B|^2} \\
&\geq \frac{\Big||A - B| - |\overline{B}| \cdot |A - B|\Big|}{1 - |B|^2} \\
&= \frac{|A - B| \cdot |1 - |\overline{B}||}{1 - |B|^2} \\
&= \frac{|A - B| \cdot |1 - |B||}{1 - |B|^2} \\
&= \frac{|A - B|}{1 + |B|}.
\end{aligned}$$

Note that the equality is attained in the above inequality when

$$\lambda_0 = \frac{\overline{B}(A - B)}{|\overline{B}(A - B)|}\ (|\lambda_0| = 1).$$

In this case, we get

$$\inf_{\zeta \in \partial g(U)} |\zeta - 1| = \inf_{|\lambda| = 1}\left|\frac{1 - A\overline{B}}{1 - |B|^2} + \lambda\frac{|A - B|}{1 - |B|^2} - 1\right| = \frac{|A - B|}{1 + |B|}.$$

Taking into account the both cases analyzed above, we conclude that

$$d(1, \partial g(U)) = \inf_{\zeta \in \partial g(U)} |\zeta - 1| = \frac{|A - B|}{1 + |B|}. \qquad \square$$

In view of Theorem 2.1 and Remark 1.7, we deduce the following consequence.

Theorem 2.4. *Let $a \in \mathbb{C}$, $b \in \mathbb{R}$ be such that $|1 - a| < b \leq \operatorname{Re} a$. Also, let $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, $\alpha + \beta \leq 1$. Then the following properties hold:*

(i) *if* $f \in S^*(a,b)$ *then* $\Phi_{n,\alpha,\beta}(f) \in S^*(a,b,\mathbb{B}^n)$,
(ii) *if* $f \in \mathcal{A}S^*(a,b)$ *then* $\Phi_{n,\alpha,\beta}(f) \in \mathcal{A}S^*(a,b,\mathbb{B}^n)$.

Proof. (i) If we take the function g as in Remark 1.7 (i), then $S_g^* = S^*(a,b)$ and $S_g^*(\mathbb{B}^n) = S^*(a,b,\mathbb{B}^n)$. Therefore, in view of Theorem 2.1, we deduce that

$$\Phi_{n,\alpha,\beta}(S^*(a,b)) \subseteq S^*(a,b,\mathbb{B}^n).$$

(ii) Let the function g be given as in Remark 1.7 (ii). In this case, we have that $S_g^* = \mathcal{A}S^*(a,b)$ and $S_g^*(\mathbb{B}^n) = \mathcal{A}S^*(a,b,\mathbb{B}^n)$. From Theorem 2.1, we obtain that

$$\Phi_{n,\alpha,\beta}(\mathcal{A}S^*(a,b)) \subseteq \mathcal{A}S^*(a,b,\mathbb{B}^n).$$

This completes the proof. □

In the case $a,b \in \mathbb{R}$ with $|1-a| < b \le a = \mathrm{Re}\ a$, the above result was obtained in [13].

The next two results are consequences of Theorem 2.2 and Remark 1.7.

Theorem 2.5. *Let* $a \in \mathbb{C}$, $b \in \mathbb{R}$ *be such that* $|1-a| < b \le \mathrm{Re}\ a$. *Let* $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ *be a homogeneous polynomial of degree* 2, *such that*

$$\|Q\| \le \frac{b^2-(1-a)(1-\overline{a})}{4(b+||a|^2-b^2-a|)}.$$

If $f \in S^*(a,b)$, *then* $\Phi_{n,Q}(f) \in S^*(a,b,\mathbb{B}^n)$.

Proof. Let g be the function from Remark 1.7 (i). Thus, we get that S_g^* becomes $S^*(a,b)$ and $S_g^*(\mathbb{B}^n)$ becomes $S^*(a,b,\mathbb{B}^n)$. Then the asserted property of the Muir extension operator $\Phi_{n,Q}$ follows from Theorem 2.1, i.e.

$$\Phi_{n,Q}(S^*(a,b)) \subseteq S^*(a,b,\mathbb{B}^n). \tag{2.1}$$

The function g has the form from Assumption 1.4, where

$$A = \frac{\overline{a}-1}{b} \text{ and } B = \frac{|a|^2-b^2-a}{b}.$$

Moreover, we have that:

$$\begin{aligned}\frac{|A-B|}{4(1+|B|)} &= \frac{\left|\overline{a}-1-|a|^2+b^2+a\right|}{4\left|b+||a|^2-b^2-a|\right|}\\ &= \frac{|b^2-(|a|^2-2\mathrm{Re}\ a+1)|}{4(b+||a|^2-b^2-a|)}\\ &= \frac{|b^2-(1-a)(1-\overline{a})|}{4(b+||a|^2-b^2-a|)}\\ &= \frac{b^2-(1-a)(1-\overline{a})}{4(b+||a|^2-b^2-a|)},\end{aligned}$$

since $|a|^2-2\mathrm{Re}\ a+1 = (1-a)(1-\overline{a}) \in \mathbb{R}$ and $b > |1-a| = |1-\overline{a}|$.

Therefore, the assumption

$$\|Q\| \le \frac{b^2-(1-a)(1-\overline{a})}{4(b+||a|^2-b^2-a|)}$$

shows that the relation (2.1) holds, as asserted. □

If we assume that $a \in \mathbb{R}$ in the hypothesis of the above result, then we deduce the preservation property concerning the extension operator $\Phi_{n,Q}$ and the class $S^*(a,b)$ with real coefficients obtained in [14].

Let us now refer to the Muir extension operator $\Phi_{n,Q}$ and state the following property.

Theorem 2.6. *Let $a \in \mathbb{C}$, $b \in \mathbb{R}$ be such that $|1-a| < b \le \mathrm{Re}\ a$. Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2, such that*

$$\|Q\| \le \frac{b^2-(1-a)(1-\overline{a})}{4(b+|1-\overline{a}|)}.$$

If $f \in \mathcal{A}S^*(a,b)$ then $\Phi_{n,Q}(f) \in \mathcal{A}S^*(a,b,\mathbb{B}^n)$.

Proof. We consider the function g as in Remark 1.7 (ii). It is clear that $S_g^* = \mathcal{A}S^*(a,b)$ and $S_g^*(\mathbb{B}^n) = \mathcal{A}S^*(a,b,\mathbb{B}^n)$. Taking into account Theorem 2.1, we deduce that the following relation is true:

$$\Phi_{n,Q}(\mathcal{A}S^*(a,b)) \subseteq \mathcal{A}S^*(a,b,\mathbb{B}^n). \tag{2.2}$$

The function g can be also written in the form given in Assumption (1.4), where

$$A = \frac{a-|a|^2+b^2}{b} \text{ and } B = \frac{1-\overline{a}}{b}.$$

Next, we evaluate the following quantity:

$$\begin{aligned}\frac{|A-B|}{4(1+|B|)} &= \frac{\left|a-|a|^2+b^2-1+\overline{a}\right|}{4|b+|1-\overline{a}||} \\ &= \frac{|b^2-(|a|^2-2\mathrm{Re}\ a+1)|}{4(b+|1-\overline{a}|)} \\ &= \frac{|b^2-(1-a)(1-\overline{a})|}{4(b+|1-\overline{a}|)} \\ &= \frac{b^2-(1-a)(1-\overline{a})}{4(b+|1-\overline{a}|)},\end{aligned}$$

using the fact that $|a|^2-2\mathrm{Re}\ a+1 = (1-a)(1-\overline{a}) \in \mathbb{R}$ and $b > |1-a| = |1-\overline{a}|$. Consequently, the condition

$$\|Q\| \le \frac{b^2-(1-a)(1-\overline{a})}{4(b+|1-\overline{a}|)}$$

implies that the relation (2.2) holds, as asserted. □

For $a, b \in \mathbb{R}$ where $|1-a| < b \le a = \mathrm{Re}\ a$, the above property was obtained in [14].

Question 2.7. Assume that $n \geq 2$. Let $\Psi_n : \mathcal{L}S_n \to \mathcal{L}S_{n+1}$ be the Pfaltzgraff-Suffridge extension operator given by (see [18]):

$$\Psi_n(f)(z) = \left(f(\tilde{z}), z_{n+1}[J_f(\tilde{z})]^{\frac{1}{n+1}} \right), \; z = (\tilde{z}, z_{n+1}) \in \mathbb{B}^{n+1},$$

were $[J_f(\tilde{z})]^{\frac{1}{n+1}}\Big|_{\tilde{z}=0} = 1$. We wonder if it is possible that Janowski (almost) starlikeness with complex coefficients to be preserved under the extension operator Ψ_n from the unit ball $\mathbb{B}^n$ to the unit ball $\mathbb{B}^{n+1}$. If it is true, under which conditions does this property hold?

Conclusions. In this paper, we have considered g-parametric representation and g-starlikeness on the Euclidean unit ball $\mathbb{B}^n$, when the function $g : U \to \mathbb{C}$ is univalent on U, $g(0) = 1$ and has positive real part on U (see [6]). Then we have referred to the property of preservation of g-starlikeness under the extension operator $\Phi_{n,\alpha,\beta}$, when g is convex on U and $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, $\alpha + \beta \leq 1$ (see [6]). For the same conditions imposed on g, we have stated that the Muir extension operator $\Phi_{n,Q}$ preserves g-starlikeness when $\|Q\| \leq d(1, \partial g(U))/4$ (see [6]).

Assume $a \in \mathbb{C}$, $b \in \mathbb{R}$ such that $|1-a| < b \leq \mathrm{Re}\, a$. Using the connection between the Janowski classes $S^*(a, b)$, $\mathcal{A}S^*(, b)$ and g-starlikeness, for a particular choice of g depending on the parameters a, b, we have proved that $\Phi_{n,\alpha,\beta}$ preserves these classes for $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, $\alpha + \beta \leq 1$. By making use of the same idea, we also prove that $\Phi_{n,Q}$ conserves these classes when $\|Q\| \leq M(a, b)$, where $M(a, b)$ is a constant depending on the parameters a and b. These results generalize the properties obtained in [13, 14], for the Janowski classes with real parameters.

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 319–330
DOI: 10.24193/subbmath.2023.2.08

Necessary and sufficient conditions for oscillation of second-order differential equation with several delays

Shyam Sundar Santra

Abstract. In this paper, necessary and sufficient conditions are establish of the solutions to second-order delay differential equations of the form

$$\Big(r(t)\big(x'(t)\big)^{\gamma}\Big)' + \sum_{i=1}^{m} q_i(t) f_i\big(x(\sigma_i(t))\big) = 0 \text{ for } t \geq t_0,$$

We consider two cases when $f_i(u)/u^{\beta}$ is non-increasing for $\beta < \gamma$, and non-decreasing for $\beta > \gamma$ where β and γ are the quotient of two positive odd integers. Our main tool is Lebesgue's Dominated Convergence theorem. Examples illustrating the applicability of the results are also given, and state an open problem.

Mathematics Subject Classification (2010): 34C10, 34C15, 34K11.

Keywords: Oscillation, nonoscillation, nonlinear, delay argument, second-order differential equation, Lebesgue's dominated convergence theorem.

1. Introduction

In this article we consider the differential equation

$$\Big(r(t)\big(x'(t)\big)^{\gamma}\Big)' + \sum_{i=1}^{m} q_i(t) f_i\big(x(\sigma_i(t))\big) = 0, \quad \text{for } t \geq t_0, \tag{1.1}$$

where γ is the quotient of two positive odd integers, and the functions f_i, p, q_i, r, σ_i are continuous that satisfy the conditions stated below;

(A1) $\sigma_i \in C([0,\infty),\mathbb{R})$, $\sigma_i(t) < t$, $\lim_{t\to\infty} \sigma_i(t) = \infty$.

Received 18 April 2020; Accepted 10 May 2020.

(A2) $r \in C^1([0,\infty),\mathbb{R})$, $q_i \in C([0,\infty),\mathbb{R})$; $0 < r(t)$, $0 \le q_i(t)$, for all $t \ge 0$ and $i = 1,2,\dots,m$; $\sum q_i(t)$ is not identically zero in any interval $[b,\infty)$.
(A3) $f_i \in C(\mathbb{R},\mathbb{R})$ is non-decreasing and $f_i(x)x > 0$ for $x \neq 0$, $i = 1,2,\dots,m$.
(A4) $\int_0^\infty r^{-1/\gamma}(\eta)\,d\eta = \infty$; let $R(t) = \int_0^t r^{-1/\gamma}(\eta)\,d\eta$.

The main feature of this article is having conditions that are both necessary and sufficient for the oscillation of all solutions to (1.1).

In 1978, Brands [9] has proved that for bounded delays, the solutions of

$$x''(t) + q(t)x(t-\sigma(t)) = 0$$

are oscillatory if and only if the solutions of $x''(t) + q(t)x(t) = 0$ are oscillatory. In [10, 12] Chatzarakis *et al.* have considered a more general second-order half-linear differential equation of the form

$$\big(r(x')^\alpha\big)'(t) + q(t)x^\alpha(\sigma(t)) = 0, \tag{1.2}$$

and established new oscillation criteria for (1.2) when

$$\lim_{t\to\infty}\Pi(t) = \infty \text{ and } \lim_{t\to\infty}\Pi(t) < \infty.$$

Wong [32] has obtained the necessary and sufficient conditions for oscillation of solutions of

$$\big(x(t) + px(t-\tau)\big)'' + q(t)f\big(x(t-\sigma)\big) = 0, \quad -1 < p < 0.$$

in which the neutral coefficient and delays are constants. However, we have seen in [5, 13] that the authors Baculīková and Džurina have studied

$$\Big(r(t)\big(z'(t)\big)^\gamma\Big)' + q(t)x^\alpha(\sigma(t)) = 0, \quad z(t) = x(t) + p(t)x(\tau(t)), \quad t \ge t_0, \tag{1.3}$$

and established sufficient conditions for oscillation of solutions of (1.3) using comparison techniques when $\gamma = \alpha = 1$, $0 \le p(t) < \infty$ and $\lim_{t\to\infty}\Pi(t) = \infty$. In same technique, Baculikova and Džurina [6] have considered (1.3) and obtained sufficient conditions for oscillation of the solutions of (1.3) by considering the assumptions $0 \le p(t) < \infty$ and $\lim_{t\to\infty}\Pi(t) = \infty$. In [31], Tripathy *et al.* have studied (1.3) and established several sufficient conditions for oscillations of the solutions of (1.3) by considering the assumptions $\lim_{t\to\infty}\Pi(t) = \infty$ and $\lim_{t\to\infty}\Pi(t) < \infty$ for different ranges of the neutral coefficient p. In [8], Bohner *et al.* have obtained sufficient conditions for oscillation of solutions of (1.3) when $\gamma = \alpha$, $\lim_{t\to\infty}\Pi(t) < \infty$ and $0 \le p(t) < 1$. Grace *et al.* [16] have established sufficient conditions for the oscillation of the solutions of (1.3) when $\gamma = \alpha$ and by considering the assumptions $\lim_{t\to\infty}\Pi(t) < \infty$, $\lim_{t\to\infty}\Pi(t) = \infty$ and $0 \le p(t) < 1$. In [19], Li *et al.* have established sufficient conditions for the oscillation of the solutions of (1.3), under the assumptions $\lim_{t\to\infty}\Pi(t) < \infty$ and $p(t) \ge 0$. Karpuz and Santra [18] have obtained several sufficient conditions for the oscillatory and asymptotic behavior of the solutions of

$$\big(r(t)(x(t) + p(t)x(\tau(t)))'\big)' + q(t)f\big(x(\sigma(t))\big) = 0,$$

by considering the assumptions $\lim_{t\to\infty}\Pi(t) < \infty$ and $\lim_{t\to\infty}\Pi(t) = \infty$ for different ranges of p.

For further work on the oscillation of the solutions to this type of equations, we refer the readers to [1, 2, 3, 4, 7, 11, 14, 16, 21, 22, 23, 20, 24, 25, 26, 27, 28, 35]. Note that the majority of publications consider only sufficient conditions, and and merely a few consider necessary and sufficient conditions. Hence, the objective in this work is to establish both necessary and sufficient conditions for the oscillatory and asymptotic behavior of solutions of (1.1) without using the comparison and the Riccati techniques.

Delay differential equations have several applications in the natural sciences and engineering. For example, they often appear in the study of distributed networks containing lossless transmission lines (see for e.g. [17]). In this paper, we restrict our attention to the study (1.1), which includes the class of functional differential equations of neutral type.

By a solution to equation (1.1), we mean a function $x \in \mathrm{C}([T_x,\infty),\mathbb{R})$, where $T_x \geq t_0$, such that $rx' \in \mathrm{C}^1([T_x,\infty),\mathbb{R})$, satisfying (1.1) on the interval $[T_x,\infty)$. A solution x of (1.1) is said to be proper if x is not identically zero eventually, i.e., $\sup\{|x(t)| : \ t \geq T\} > 0$ for all $T \geq T_x$. We assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise, it is said to be non-oscillatory. (1.1) itself is said to be oscillatory if all of its solutions are oscillatory.

Remark 1.1. When the domain is not specified explicitly, all functional inequalities considered in this paper are assumed to hold eventually, i.e., they are satisfied for all t large enough.

2. Main Results

Lemma 2.1. *Assume* (A1)–(A4)*, and that x is an eventually positive solution of* (1.1)*. Then there exist $t_1 \geq t_0$ and $\delta > 0$ such that*

$$0 < x(t) \leq \delta R(t), \tag{2.1}$$

$$\big(R(t) - R(t_1)\big)\Big[\int_t^\infty \sum_{i=1}^m q_i(\zeta) f_i\big(x(\sigma_i(\zeta))\big)\, d\zeta\Big]^{1/\gamma} \leq x(t)\,, \tag{2.2}$$

for $t \geq t_1$.

Proof. Let x be an eventually positive solution. Then by (A1) there exists a t^* such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma_i(t)) > 0$ for all $t \geq t^*$ and $i = 1, 2, \ldots, m$. From (1.1) it follows that

$$\Big(r(t)\big(x'(t)\big)^\gamma\Big)' = -\sum_{i=1}^m q_i(t) f_i\big(x(\sigma_i(t))\big) \leq 0. \tag{2.3}$$

Therefore, $r(t)\big(x'(t)\big)^\gamma$ is non-increasing for $t \geq t^*$. Next we show the $r(t)\big(x'(t)\big)^\gamma$ is positive. By contradiction assume that $r(t)\big(x'(t)\big)^\gamma \leq 0$ at a certain time $t \geq t^*$. Using that $\sum q_i$ is not identically zero on any interval $[b,\infty)$, and that $f(x) > 0$ for $x > 0$, by (2.3), there exist $t_2 \geq t^*$ such that

$$r(t)\big(x'(t)\big)^\gamma \leq r(t_2)\big(x'(t_2)\big)^\gamma < 0 \quad \text{for all } t \geq t_2.$$

Recall that γ is the quotient of two positive odd integers. Then

$$x'(t) \le \Big(\frac{r(t_2)}{r(t)}\Big)^{1/\gamma} x'(t_2) \quad \text{for } t \ge t_2\,.$$

Integrating from t_2 to t, we have

$$x(t) \le x(t_2) + \big(r(t_2)\big)^{1/\gamma} x'(t_2)\big(R(t) - R(t_2)\big)\,. \tag{2.4}$$

By (A4), the right-hand side approaches $-\infty$; then $\lim_{t\to\infty} x(t) = -\infty$. This is a contradiction to the fact that $x(t) > 0$. Therefore $r(t)\big(x'(t)\big)^{\gamma} > 0$ for all $t \ge t^*$. From $r(t)\big(x'(t)\big)^{\gamma}$ being non-increasing, we have

$$x'(t) \le \Big(\frac{r(t_1)}{r(t)}\Big)^{1/\gamma} x'(t_1) \quad \text{for } t \ge t_1\,.$$

Integrating this inequality from t_1 to t, and using that x is continuous,

$$x(t) \le x(t_1) + \big(r(t_1)\big)^{1/\gamma} x'(t_1)\big(R(t) - R(t_1)\big)\,.$$

Since $\lim_{t\to\infty} R(t) = \infty$, there exists a positive constant δ such that (2.1) holds. Since $r(t)\big(x'(t)\big)^{\gamma}$ is positive and non-increasing, $\lim_{t\to\infty} r(t)\big(x'(t)\big)^{\gamma}$ exists and is non-negative. Integrating (1.1) from t to a, we have

$$r(a)\big(x'(a)\big)^{\gamma} - r(t)\big(x'(t)\big)^{\gamma} + \int_t^{\infty} \sum_{i=1}^{m} q_i(\eta) f_i\big(x(\sigma_i(\eta))\big)\, d\eta = 0$$

Computing the limit as $a \to \infty$,

$$r(t)\big(x'(t)\big)^{\gamma} \ge \int_t^{\infty} \sum_{i=1}^{m} q_i(\eta) f_i\big(x(\sigma_i(\eta))\big)\, d\eta. \tag{2.5}$$

Then

$$x'(t) \ge \Big[\frac{1}{r(t)} \int_t^{\infty} \sum_{i=1}^{m} q_i(\eta) f_i\big(x(\sigma_i(\eta))\big)\, d\eta\Big]^{1/\gamma}.$$

Since $x(t_1) > 0$, integrating the above inequality yields

$$x(t) \ge \int_{t_1}^{\eta} \Big[\frac{1}{r(\eta)} \int_{\eta}^{\infty} \sum_{i=1}^{m} q_i(\zeta) f_i\big(x(\sigma_i(\zeta))\big)\, d\zeta\Big]^{1/\gamma} d\eta$$

Since the integrand is positive, we can increase the lower limit of integration from η to t, and then use the definition of $R(t)$, to obtain

$$x(t) \ge \big(R(t) - R(t_1)\big)\Big[\int_t^{\infty} \sum_{i=1}^{m} q_i(\zeta) f_i\big(x(\sigma_i(\zeta))\big)\, d\zeta\Big]^{1/\gamma},$$

which yields (2.2). □

For the next theorem we assume that there exists a constant β, the quotient of two positive odd integers, with $\beta < \gamma$, such that

$$\frac{f_i(u)}{u^{\beta}} \text{ is non-increasing for } 0 < u,\ i = 1, 2, \dots, m\,. \tag{2.6}$$

For example $f_i(u) = |u|^{\alpha} \operatorname{sgn}(u)$, with $0 < \alpha < \beta$ satisfies this condition.

Theorem 2.2. *Under assumptions* (A1)–(A4) *and* (2.6), *each solution of* (1.1) *is oscillatory if and only if*

$$\int_0^\infty \sum_{i=1}^m q_i(\eta) f_i(\delta R(\sigma_i(\eta)))\, d\eta = \infty \quad \forall \delta > 0\,. \tag{2.7}$$

Proof. We prove sufficiency by contradiction. Initially we assume that a solution x is eventually positive. So, Lemma 2.1 holds, and then there exists $t_1 \geq t_0$ such that

$$x(t) \geq \big(R(t) - R(t_1)\big) w^{1/\gamma}(t) \geq 0 \quad \forall t \geq t_1\,,$$

where

$$w(t) = \int_t^\infty \sum_{i=1}^m q_i(\zeta) f_i\big(x(\sigma_i(\zeta))\big)\, d\zeta\,.$$

Since $\lim_{t\to\infty} R(t) = \infty$, there exists $t_2 \geq t_1$, such that $R(t) - R(t_1) \geq \frac{1}{2} R(t)$ for $t \geq t_2$. Then

$$x(t) \geq \frac{1}{2} R(t) w^{1/\gamma}(t)\,. \tag{2.8}$$

Computing the derivative of w, we have

$$w'(t) = -\sum_{i=1}^m q_i(t) f_i\big(x(\sigma_i(t))\big).$$

Thus w is non-negative and non-increasing. Since $x > 0$, by (A3), $f_i\big(x(\sigma_i(t))\big) > 0$, and by (A2), it follows that $\sum_{i=1}^m q_i(t) f_i\big(x(\sigma_i(t))\big)$ cannot be identically zero in any interval $[b, \infty)$; thus w' cannot be identically zero, and w can not be constant on any interval $[b, \infty)$. Therefore $w(t) > 0$ for $t \geq t_1$. Computing the derivative,

$$\big(w^{1-\beta/\gamma}(t)\big)' = \Big(1 - \frac{\beta}{\gamma}\Big) w^{-\beta/\gamma}(t) w'(t)\,. \tag{2.9}$$

Integrating (2.9) from t_2 to t, and using that $w > 0$, we have

$$\begin{aligned} w^{1-\beta/\gamma}(t_2) &\geq \Big(1 - \frac{\beta}{\gamma}\Big)\Big[-\int_{t_2}^t w^{-\beta/\gamma}(\eta) w'(\eta)\, d\eta\Big] \\ &= \Big(1 - \frac{\beta}{\gamma}\Big)\Big[\int_{t_2}^t w^{-\beta/\gamma}(\eta)\Big(\sum_{i=1}^m q_i(\eta) f_i\big(x(\sigma_i(\eta))\big)\Big)\, d\eta\Big]\,. \end{aligned} \tag{2.10}$$

Next we find a lower bound for the right-hand side of (2.10), independent of the solution x. By (A3), (2.1), (2.6), and (2.8), we have

$$\begin{aligned} f_i(x(t)) &= f_i\big(x(t)\big)\frac{x^\beta(t)}{x^\beta(t)} \geq \frac{f_i\big(\delta R(t)\big)}{(\delta R(t))^\beta} x^\beta(t) \\ &\geq \frac{f_i\big(\delta R(t)\big)}{(\delta R(t))^\beta}\Big(\frac{R(t) w^{1/\gamma}(t)}{2}\Big)^\beta = \frac{f_i\big(\delta R(t)\big)}{(2\delta)^\beta} w^{\beta/\gamma}(t) \quad \text{for } t \geq t_2\,. \end{aligned}$$

Since w is non-increasing, $\beta/\gamma > 0$, and $\sigma_i(\eta) < \eta$, it follows that

$$f_i\big(x(\sigma_i(\eta))\big) \geq \frac{f_i\big(\delta R(\sigma_i(\eta))\big)}{(2\delta)^\beta} w^{\beta/\gamma}(\sigma_i(\eta)) \geq \frac{f_i\big(\delta R(\sigma_i(\eta))\big)}{(2\delta)^\beta} w^{\beta/\gamma}(\eta)\,. \tag{2.11}$$

Going back to (2.10), we have

$$w^{1-\beta/\gamma}(t_2) \geq \frac{\left(1-\frac{\beta}{\gamma}\right)}{(2\delta)^{\beta}}\Big[\int_{t_2}^{t}\sum_{i=1}^{m} q_i(\eta) f_i(\delta R(\sigma_i(\eta)))\,d\eta\Big]\,. \tag{2.12}$$

Since $(1-\beta/\gamma)>0$, by (2.7) the right-hand side approaches ∞ as $t\to\infty$. This contradicts (2.12) and completes the proof of sufficiency for eventually positive solutions.

For an eventually negative solution x, we introduce the variables $y=-x$ and $g_i(y)=-f_i(y)$. Then y is an eventually positive solution of (1.1) with g_i instead of f_i. Note that g_i satisfies (A3) and (2.6) so can apply the above process for the solution y.

Next we show the necessity part by a contrapositive argument. When (2.7) does not hold we find a eventually positive solution that does not converge to zero. If (2.7) does not hold for some $\delta>0$, then for each $\epsilon>0$ there exists $t_1\geq t_0$ such that

$$\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta) f_i(\delta R(\sigma_i(\zeta)))\,d\zeta \leq \epsilon/2 \tag{2.13}$$

for all $\eta\geq t_1$. Note that t_1 depends on δ. We define the set of continuous functions

$$M=\{x\in C([0,\infty)) : (\epsilon/2)^{1/\gamma}\big(R(t)-R(t_1)\big)\leq x(t)\leq \epsilon^{1/\gamma}\big(R(t)-R(t_1)\big),\ t\geq t_1\}\,.$$

We define an operator Φ on M by

$$(\Phi x)(t)=\begin{cases}0 & \text{if } t\leq t_1\\ \int_{t_1}^{t}\Big[\frac{1}{r(\eta)}\Big[\epsilon/2+\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta) f_i(x(\sigma_i(\zeta)))\,d\zeta\Big]\Big]^{1/\gamma}d\eta & \text{if } t>t_1\,.\end{cases}$$

Note that when x is continuous, Φx is also continuous on $[0,\infty)$. If x is a fixed point of Φ, i.e. $\Phi x=x$, then x is a solution of (1.1).
First we estimate $(\Phi x)(t)$ from below. For $x\in M$, we have

$$0\leq \epsilon^{1/\gamma}\big(R(t)-R(t_1)\big)\leq x(t).$$

By (A3), we have $0\leq f_i(x(\sigma_i(\eta)))$ and by (A2) we have

$$(\Phi x)(t)\geq 0+\int_{t_1}^{t}\Big[\frac{1}{r(\eta)}[\epsilon/2+0+0]\Big]^{1/\gamma}d\eta=(\epsilon/2)^{1/\gamma}\big(R(t)-R(t_1)\big)\,.$$

Now we estimate $(\Phi x)(t)$ from above. For x in M, by (A2) and (A3), we have

$$f_i(x(\sigma_i(\zeta)))\leq f_i\big(\delta R(\sigma_i(\zeta))\big).$$

Then by (2.13),

$$\begin{aligned}(\Phi x)(t)&\leq \int_{t_1}^{t}\Big[\frac{1}{r(\eta)}\Big[\epsilon/2+\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta) f_i(\delta R(\sigma_i(\zeta)))\,d\zeta\Big]\Big]^{1/\gamma}d\eta\\ &\leq \epsilon^{1/\gamma}\big(R(t)-R(t_1)\big)\,.\end{aligned}$$

Therefore, Φ maps M to M.

Next we find a fixed point for Φ in M. Let us define a sequence of functions in M by the recurrence relation

$$u_0(t) = 0 \quad \text{for } t \geq t_0,$$

$$u_1(t) = (\Phi u_0)(t) = \begin{cases} 0 & \text{if } t < t_1 \\ \epsilon^{1/\gamma}\big(R(t) - R(t_1)\big) & \text{if } t \geq t_1 \end{cases},$$

$$u_{n+1}(t) = (\Phi u_n)(t) \quad \text{for } n \geq 1,\ t \geq t_1 .$$

Note that for each fixed t, we have $u_1(t) \geq u_0(t)$. Using that f is non-decreasing and mathematical induction, we can show that $u_{n+1}(t) \geq u_n(t)$. Therefore, the sequence $\{u_n\}$ converges pointwise to a function u. Using the Lebesgue Dominated Convergence Theorem, we can show that u is a fixed point of Φ in M. This shows under assumption (2.13), there a non-oscillatory solution that does not converge to zero. This completes the proof. $\square$

In the next theorem, we assume the existence of a differentiable function σ_0 such that

$$0 < \sigma_0(t) \leq \sigma_i(t), \quad \exists \alpha > 0 : \alpha \leq \sigma_0'(t), \quad \text{for } t \geq t_0,\ i = 1, 2, \dots, m. \tag{2.14}$$

Also we assume that there exists a constant β, the quotient of two positive odd integers, with $\gamma < \beta$, such that

$$\frac{f_i(u)}{u^\beta} \text{ is non-decreasing for } 0 < u,\ i = 1, 2, \dots, m . \tag{2.15}$$

For example $f_i(u) = |u|^\alpha \operatorname{sgn}(u)$, with $\beta < \alpha$ satisfies this condition.

Theorem 2.3. *Under assumptions* (A1)–(A4), (2.14), (2.15), *and* $r(t)$ *is non-decreasing, every solution of* (1.1) *is oscillatory if and only if*

$$\int_{t_1}^\infty \Big[\frac{1}{r(\eta)} \int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\, d\zeta\Big]^{1/\gamma} d\eta = \infty . \tag{2.16}$$

Proof. We prove sufficiency by contradiction. Initially assume that x is an eventually positive solution that does not converge to zero. Using the same argument as in Lemma 2.1, there exists $t_1 \geq t_0$ such that: $x(\sigma_i(t)) > 0$, $x(\tau(t)) > 0$, and $r(t)\big(x'(t)\big)^\gamma$ is positive and non-increasing. Since $r(t) > 0$ so $x(t)$ is increasing for $t \geq t_1$. From (A3), $x(t) \geq x(t_1)$ and (2.15), we have

$$f_i(x(t)) \geq \frac{f_i(x(t))}{x^\beta(t)} x^\beta(t) \geq \frac{f_i(x(t_1))}{x^\beta(t_1)} x^\beta(t) .$$

By (A1) there exists a $t_2 \geq t_1$ such that $\sigma_i(t) \geq t_1$ for $t \geq t_2$. Then

$$f_i\big(x(\sigma_i(t))\big) \geq \frac{f_i(x(t_1))}{x^\beta(t_1)} x^\beta(\sigma_i(t)) \quad \forall t \geq t_2 . \tag{2.17}$$

Using this inequality, (2.5), that $\sigma_i(t) \geq \sigma_0(t)$ which is an increasing function, and that z is increasing, we have

$$r(t)\big(x'(t)\big)^\gamma \geq \frac{x^\beta(\sigma_0(t))}{x^\beta(t_1)} \int_t^\infty \sum_{i=1}^m q_i(\eta) f_i(x(t_1))\, d\eta ,$$

for $t \geq t_2$. From $r(t)\big(z'(t)\big)^\gamma$ being non-increasing and $\sigma_0(t) \leq t$, we have

$$r(\sigma_0(t))\big(x'(\sigma_0(t))\big)^\gamma \geq r(t)\big(x'(t)\big)^\gamma.$$

We use this in the left-hand side of the above inequality. Then dividing by $r(\sigma_0(t)) > 0$, raising both sides to the $1/\gamma$ power, and dividing by $z^{\beta/\gamma}(\sigma_0(t)) > 0$, we have

$$\frac{x'(\sigma_0((t))}{x^{\beta/\gamma}(\sigma_0(t))} \geq \Big[\frac{1}{r(\sigma_0(t))x^\beta(t_1)} \int_t^\infty \sum_{i=1}^m q_i(\eta) f_i(x(t_1))\, d\eta\Big]^{1/\gamma}, \text{ for } t \geq t_2.$$

Multiplying the left-hand side by $\sigma_0'(t)/\alpha \geq 1$, and integrating from t_1 to t,

$$\frac{1}{\alpha} \int_{t_1}^t \frac{z'(\sigma_0(\eta))\sigma_0'(\eta)}{z^{\beta/\gamma}(\sigma_0(\eta))}\, d\eta \geq \frac{1}{z^{\beta/\gamma}(t_1)} \int_{t_1}^t \Big[\frac{1}{r(\sigma_0(\eta))} \int_\eta^\infty \sum_{i=1}^m q_i(\zeta) f_i(x(t_1))\, d\zeta\Big]^{1/\gamma} d\eta. \tag{2.18}$$

On the left-hand side, since $\gamma < \beta$, integrating, we have

$$\frac{1}{\alpha(1-\beta/\gamma)} \Big[z^{1-\beta/\gamma}(\sigma_0(\eta))\Big]_{s=t_2}^t \leq \frac{1}{\alpha(\beta/\gamma - 1)} z^{1-\beta/\gamma}(\sigma_0(t_2))\,.$$

On the right-hand side of (2.18), we use that $\min_{1\leq i\leq m} f_i(z(t_1)) > 0$ and that $r(\sigma_0(s)) \leq r(s)$, to conclude that (2.16) implies the right-hand side approaching ∞, as $t \to \infty$. This contradiction implies that the solution x cannot be eventually positive.

For eventually negative solutions, we use the same change of variables as in Theorem 2.2, and proceed as above.

To prove the necessity part we assume that (2.16) does not hold, and obtain an eventually positive solution that does not converge to zero. If (2.16) does not hold, then for each $\epsilon > 0$ there exists $t_1 \geq t_0$ such that

$$\int_{t_1}^\infty \Big[\frac{1}{r(\eta)} \int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\, d\zeta\Big]^{1/\gamma} d\eta < \epsilon/2(f_i(\epsilon))^{1/\gamma} \quad \forall t \geq t_1\,. \tag{2.19}$$

Let us consider the set of continuous functions

$$M = \{x \in C([0,\infty)) : \epsilon/2 \leq x(t) \leq \epsilon \text{ for } t \geq t_1\}$$

Then we define the operator

$$(\Phi x)(t) = \begin{cases} 0 & \text{if } t \leq t_1, \\ \epsilon/2 + \int_{t_1}^t \frac{1}{r(\eta)} \Big[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta) f_i(x(\sigma_i(\zeta)))\, d\zeta\Big]^{1/\gamma} d\eta & \text{if } t > t_1\,. \end{cases}$$

Note that if x is continuous, Φx is also continuous at $t = t_1$. Also note that if $\Phi x = x$, then x is solution of (1.1).

First we estimate $(\Phi x)(t)$ from below. Let $x \in M$. By $0 < \epsilon/2 \leq x$, we have $(\Phi x)(t) \geq \epsilon/2 + 0 + 0$, on $[t_1, \infty)$.

Now we estimate $(\Phi x)(t)$ from above. Let $x \in M$. Then $x \leq \epsilon$ and by (2.19), we have

$$(\Phi x)(t) \leq \epsilon/2 + \big(f_i(\epsilon)\big)^{1/\gamma} \int_{t_1}^t \Big[\frac{1}{r(\eta)} \int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\, d\zeta\Big]^{1/\gamma} d\eta \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore Φ maps M to M. To find a fixed point for Φ in M, we define a sequence of functions by the recurrence relation

$$\begin{gathered} u_0(t) = 0 \quad \text{for } t \geq t_0, \\ u_1(t) = (\Phi u_0)(t) = 1 \quad \text{for } t \geq t_1, \\ u_{n+1}(t) = (\Phi u_n)(t) \quad \text{for } n \geq 1,\ t \geq t_1\,. \end{gathered}$$

Note that for each fixed t, we have $u_1(t) \geq u_0(t)$. Using that f is non-decreasing and mathematical induction, we can prove that $u_{n+1}(t) \geq u_n(t)$. Therefore $\{u_n\}$ converges pointwise to a function u in M. Then u is a fixed point of Φ and a positive solution to (1.1). This completes the proof. $\square$

Example 2.4. Consider the delay differential equations

$$\Big(e^{-t}\big(x'(t)\big)^{11/3}\Big)' + \frac{1}{t+1}(x(t-2))^{1/3} + \frac{1}{t+2}(x(t-1))^{5/3} = 0\,. \tag{2.20}$$

Here

$$\begin{gathered} \gamma = 11/3,\ r(t) = e^{-t},\ \sigma_1(t) = t-2,\ \sigma_2(t) = t-1, \\ R(t) = \int_0^t e^{5s/3}\,ds = \frac{3}{5}\big(e^{5t/3} - 1\big), \\ f_1(x) = x^{1/3} \text{ and } f_2(x) = x^{5/3}. \end{gathered}$$

For $\beta = 7/3$, we have $0 < \max\{\alpha_1, \alpha_2\} < \beta < \gamma$, and

$$f_1(x)/x^\beta = x^{-2} \text{ and } f_2(x)/x^\beta = x^{-2/3}$$

which both are decreasing functions. To check (2.7) we have

$$\begin{aligned} \int_0^\infty \sum_{i=1}^m q_i(\eta) f_i(\delta R(\sigma_i(\eta)))\,d\eta &\geq \int_0^\infty \sum_{i=1}^m q_i(\eta) f_i(\delta R(\sigma_i(\eta)))\,d\eta \\ &\geq \int_0^\infty q_1(\eta) f_1(\delta R(\sigma_1(\eta)))\,d\eta \\ &= \int_0^\infty \frac{1}{\eta+1}\Big(\delta\frac{3}{5}\big(e^{5(\eta-2)/3} - 1\big)\Big)^{1/3} d\eta = \infty \quad \forall \delta > 0, \end{aligned}$$

since the integral approaches $+\infty$ as $\eta \to +\infty$. So, all the conditions of Theorem 2.2 hold, and therefore, each solution of (2.20) is oscillatory or converges to zero.

Example 2.5. Consider the neutral differential equations

$$\Big(\big(x'(t)\big)^{1/3}\Big)' + t(x(t-2))^{7/3} + (t+1)(x(t-1))^{11/3} = 0\,. \tag{2.21}$$

Here

$$\begin{gathered} \gamma = 1/3,\ r(t) = 1,\ \sigma_1(t) = t-2,\ \sigma_2(t) = t-1, \\ f_1(v) = v^{7/3} \text{ and } f_2(v) = v^{11/3}. \end{gathered}$$

For $\beta = 5/3$, we have $\min\{\alpha_1, \alpha_2\} > \beta > \gamma$, and

$$f_1(x)/x^\beta = x^{2/3} \text{ and } f_2(x)/x^\beta = x^2$$

which both are increasing functions. To check (2.16) we have

$$\begin{aligned}\int_{t_0}^{\infty}\Big[\frac{1}{r(\eta)}\int_{\eta}^{\infty}\sum_{i=1}^{m}q_i(\zeta)\,d\zeta\Big]^{1/\gamma}d\eta &\geq \int_{t_0}^{\infty}\Big[\frac{1}{r(\eta)}\int_{\eta}^{\infty}\sum_{i=1}^{m}q_i(\zeta)\,d\zeta\Big]^{1/\gamma}d\eta\\ &\geq \int_{t_0}^{\infty}\Big[\frac{1}{r(\eta)}\int_{\eta}^{\infty}q_1(\zeta)\,d\zeta\Big]^{1/\gamma}d\eta\\ &\geq \int_{2}^{\infty}\Big[\int_{\eta}^{\infty}\zeta\,d\zeta\Big]^{3}d\eta=\infty.\end{aligned}$$

So, all the conditions of of Theorem 2.3 hold. Thus, each solution of (2.21) is oscillatory or converges to zero.

Open Problem

Based on this work and [5, 6, 8, 13, 16, 18, 20, 19, 26, 31] an open problem that arises is to establish necessary and sufficient conditions for the oscillation of the solutions of the second-order nonlinear neutral differential equation

$$\Big(r(t)\big(z'(t)\big)^{\gamma}\Big)'+\sum_{i=1}^{m}q_i(t)f_i\big(x(\sigma_i(t))\big)=0,\ \text{for } t\geq t_0,$$

where $z(t)=x(t)+p(t)x(\tau(t))$ for $p\in C(\mathbf{R},\mathbf{R})$.

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 331–340
DOI: 10.24193/subbmath.2023.2.09

An algorithm for solving a control problem for Kolmogorov systems

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Abstract. In this paper, a numerical algorithm is used for solving control problems related to Kolmogorov systems. It is proved the convergence of the algorithm and by this it is re-obtained, by a numerical approach, the controllability of the investigated problems.

Mathematics Subject Classification (2010): 34H05, 93C15.

Keywords: Kolmogorov system, control problem, numerical algorithm.

1. Introduction

Many real processes must be controlled to drive their evolution completion according to a desired plan. Mathematically, a control problem returns to determination of one or several parameters of the equation or system of equations so that the solution satisfies certain conditions, others than initial or boundary conditions.

The Kolmogorov system was introduced as a generalization of a model given by the mathematician Volterra from population dynamics. It operates at the general per capita rate of two species that interact with each other and has the following form:

$$\begin{cases} x' = xf(x,y) \\ y' = yg(x,y). \end{cases}$$

Here, the rates f and g are given in terms of parameters that cannot be changed, and others that can be modified in order to control the evolution. Kolmogorov systems arises in many areas, such as population dynamics, ecological balance and the spread of epidemics (for such models see [1], [3], [8]).

Received 20 October 2022; Accepted 04 November 2022.

The Lotka-Volterra system, also known as prey-predator system, consists in a pair of nonlinear differential equations dynamically describing the interaction between two species. Populations change over time according to the system of equations

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy \\ \dfrac{dy}{dt} = \delta xy - \gamma y, \end{cases}$$

where $x(t)$ represents the prey population, $y(t)$ predator population, $\dfrac{dx}{dt}, \dfrac{dy}{dt}$ represents the growth rates of the two populations, t time variable and $\alpha, \beta, \delta, \gamma$ are real positive parameters that describe the interaction of the two species.

In mathematical epidemiology, the SIR model (1.1) is well known. Here, $S(t)$ represented the number of susceptible population, $I(t)$ the number of population infected and $R(t)$ this number of recovered. Significant advances have been made by Kermack and McKendrick, where they studied those circumstances (represented by values of certain parameters) when behaviour of susceptible population falls below a threshold value.

The equations governing the SIR model are as follows:

$$\begin{cases} S'(t) = -aS(t)I(t) \\ I'(t) = aS(t)I(t) - bI(t) \\ R'(t) = bI(t). \end{cases} \tag{1.1}$$

The purpose of this paper is to present a numerical algorithm for solving control problems related to Kolmogorov systems. It is proved the convergence of the algorithm and by this it is reobtained, by a numerical approach, the controllability of the problems.

2. Main results

In what follows, we study the dynamics of the growth rates (not the per capita one) in order for certain conditions to be fulfilled. Consider the problem

$$\begin{cases} x'(t) = x(t)f(x(t), y(t)) - \lambda \\ y'(t) = y(t)g(x(t), y(t)) \\ x(0) = x_0, \ y(0) = y_0, \end{cases} \tag{2.1}$$

where λ is constant.
Here, the controllability condition is $\varphi(x, y) = 0$, where $\varphi : C([0, T], \mathbb{R}^2) \to \mathbb{R}$ represents a continuous function (for example, $\varphi(x, y) = \alpha x(T) + \beta y(T) - \gamma$ with $\alpha, \beta, \gamma \in \mathbb{R}$).

The initial value problem (2.1) has a unique solution $(S_1(\lambda), S_2(\lambda))$, for any fixed λ, which is continuous with respect to λ.

Assume the following conditions hold:

(i) $\varphi(x, y) < 0$ for $\lambda = 0$;
(ii) $\varphi(x, y) \geq 0$ for $\lambda = 1$.

The following iterative algorithm is aimed to bring us as close as possible to a value of λ that corresponds to a solution of the control problem.

The algorithm:

Step 1. Initialize $\underline{\lambda}_0 := 0$, $\overline{\lambda}_0 := 1$

Step 2. At any iteration $k \geq 1$, define $\lambda_k := \dfrac{\underline{\lambda}_{k-1} + \overline{\lambda}_{k-1}}{2}$ and solve system (2.1) for $\lambda := \lambda_k$. Obtain the numerical solution

$$(x_k, y_k) = (S_1(\lambda_k), S_2(\lambda_k)).$$

If $\varphi(x_k, y_k) < 0$, then put $\underline{\lambda}_k = \lambda$, $\overline{\lambda}_k = \overline{\lambda}_{k-1}$, otherwise, take $\underline{\lambda}_k = \underline{\lambda}_{k-1}$, $\overline{\lambda}_k = \lambda$, we make $k = k + 1$ and we repeat Step 2.

Step 3. The algorithm stops if

$$|\varphi(x_k, y_k)| < \delta,$$

where $0 < \delta < 1$ is the admitted error.

To demonstrate convergence we need the following two lemmas of continuous dependence on parameter.

Lemma 2.1. *Assume that $f, g : \mathbb{R}^2 \to \mathbb{R}$ are Lipschitz continuous on $\mathbb{R}^2$ and $|f| \leq C_f$, $|g| \leq C_g$. Then for any $\lambda \in \mathbb{R}$, the Cauchy problem (2.1) has a unique solution that depends continuously on the parameter λ.*

Proof. Problem (2.1) is equivalent to the Volterra integral system

$$\begin{cases} x(t) = x_0 + \int_0^t x(s)f(x(s), y(s))ds - \lambda t \\ y(t) = y_0 + \int_0^t y(s)g(x(s), y(s))ds, \end{cases} \tag{2.2}$$

which is a fixed point equation in (x, y), on the space $C([0, T]; \mathbb{R}^2)$.

For the proof we first show the boundedness of solutions.

I. Boundedness of solutions

We have to prove that there exist two constants $C_1, C_2 > 0$ such that

$$|S_1(\lambda)(t)| \leq C_1 \text{ and } |S_2(\lambda)(t)| \leq C_2,$$

for every $\lambda \in [0, 1]$ and $t \in [0, T]$. Since $|f| \leq C_f$, $|g| \leq C_g$, the first equation in (2.2) yields

$$\begin{aligned} |x(t)| &\leq |x_0| + \int_0^t |x(s)||f(x(s), y(s))|ds + T \\ &\leq |x_0| + T + C_f \int_0^t |x(s)|ds. \end{aligned}$$

From Gronwall's inequality(see [2]), we obtain

$$|x(t)| \leq (|x_0| + T)e^{C_f T} =: C_1, \qquad t \in [0, T].$$

Under a similar reasoning we find $C_2 := (|y_0| + T)e^{C_g T}$ such that $|y(t)| \leq C_2$ for $t \in [0, T]$.

II. Existence and uniqueness

Let $\alpha_{ij}, i, j = 1, 2$ be the Lipschitz constants for $f(x, y)$ and $g(x, y)$ with respect x and y.

Denote

$$\begin{aligned} A(x,y)(t) &= x_0 + \int_0^t x(s)f(x(s),y(s))ds - \lambda t, \\ B(x,y)(t) &= y_0 + \int_0^t y(s)g(x(s),y(s))ds. \end{aligned}$$

We prove that the operator $N := (A, B)$ is a contraction on $C\left([0, T]; \mathbb{R}^2\right)$ with respect to the Bielecki norm $\|(x, y)\|_\theta := \|x\|_\theta + \|y\|_\theta$, where

$$\|x\|_\theta := \max_{t\in[0,T]} \left(|x(t)|\, e^{-\theta t}\right), \qquad \|y\|_\theta := \max_{t\in[0,T]} \left(|y(t)|\, e^{-\theta t}\right).$$

We have

$$\begin{aligned} |A(x,y)(t) - A(\bar{x},\bar{y})(t)| &\leq \int_0^t |x(s)f(x(s),y(s)) - \bar{x}(s)f(\bar{x}(s),\bar{y}(s))|ds \\ &\leq \int_0^t |x(s)f(x(s),y(s)) - x(s)f(\bar{x}(s),\bar{y}(s))|ds \\ &\quad + \int_0^t |x(s)f(\bar{x}(s),\bar{y}(s)) - \bar{x}(s)f(\bar{x}(s),\bar{y}(s))|ds \\ &\leq C_1 \int_0^t (\alpha_{11}|x(s)-\bar{x}(s)| + \alpha_{12}|y(s)-\bar{y}(s)|)ds \\ &\quad + C_f \int_0^t |x(s)-\bar{x}(s)|ds. \end{aligned}$$

Next

$$\begin{aligned} & |A(x,y)(t) - A(\bar{x},\bar{y})(t)| \\ \leq\ & C_1 \int_0^t (\alpha_{11}|x(s)-\bar{x}(s)|e^{-\theta s}e^{\theta s} + \alpha_{12}|y(s)-\bar{y}(s)|e^{-\theta s}e^{\theta s})ds \\ & + C_f \int_0^t |x(s)-\bar{x}(s)|e^{-\theta s}e^{\theta s}ds \\ \leq\ & (C_1\alpha_{11} + C_f)\, \|x-\overline{x}\|_\theta \int_0^t e^{\theta s}ds + C_1\alpha_{12}\, \|y-\overline{y}\|_\theta \int_0^t e^{\theta s}ds \\ \leq\ & \frac{C_1\alpha_{11} + C_f}{\theta} \|x-\overline{x}\|_\theta\, e^{\theta t} + \frac{C_1\alpha_{12}}{\theta} \|y-\overline{y}\|_\theta\, e^{\theta t}. \end{aligned}$$

Now, multipling the above relation with $e^{-\theta t}$ and taking the supremum over t, we obtain

$$||A(x,y) - A(\bar{x},\bar{y})||_\theta \leq \frac{C_1\alpha_{11} + C_f}{\theta} \|x-\overline{x}\|_\theta + \frac{C_1\alpha_{12}}{\theta} \|y-\overline{y}\|_\theta\,. \tag{2.3}$$

Similarly

$$||B(x,y)-B(\bar{x},\bar{y})||_\theta \le \frac{C_2\alpha_{21}}{\theta}||x-\bar{x}||_\theta + \frac{C_2\alpha_{22}+C_g}{\theta}||y-\bar{y}||_\theta. \tag{2.4}$$

Further, adding relations (2.3) and (2.4), we deduce

$$\|(A(x,y),B(x,y))-\left(A\left(\overline{x},\overline{y}\right),B(\overline{x},\overline{y})\right)\|_\theta \le \overline{C}_1||x-\bar{x}||_\theta + \overline{C}_2||y-\bar{y}||_\theta,$$

where

$$\overline{C}_1 = \frac{C_1\alpha_{11}+C_f}{\theta} + \frac{C_2\alpha_{21}}{\theta},$$

$$\overline{C}_2 = \frac{C_1\alpha_{12}}{\theta} + \frac{C_2\alpha_{22}+C_g}{\theta}.$$

Therefore,

$$||N(x,y)-N(\bar{x},\bar{y})||_\theta \le L\left(||x-\bar{x}||_\theta + ||y-\bar{y}||_\theta\right) = L||(x,y)-(\bar{x},\bar{y})||_\theta,$$

where $L := \max\{\overline{C}_1, \overline{C}_2\}$.

If we now take a sufficiently large number θ, then $L<1$, and thus the operator $N=(A,B)$ is a contraction on the space $C([0,T];\mathbb{R}^2)$ endowed with the Bielecki norm $||\cdot||_\theta$. Therefore, Banach contraction principle applies and gives the result.

III. Continuous dependence of parameter λ

Using (2.2), where, first $x=S_1(\lambda)$ and $y=S_2(\lambda)$, and next $x=S_1(\mu)$ and $y=S_2(\mu)$, we have

$$\begin{aligned}
& |S_1(\lambda)(t)-S_1(\mu)(t)| \\
\le & \int_0^t |S_1(\lambda)(s)f(S_1(\lambda)(s),S_2(\lambda)(s))-S_1(\mu)(s)f(S_1(\mu)(s),S_2(\mu)(s))|ds \\
& +|\lambda-\mu|T \\
\le & \int_0^t |S_1(\lambda)(s)f(S_1(\lambda)(s),S_2(\lambda)(s))-S_1(\lambda)(s)f(S_1(\mu)(s),S_2(\mu)(s))|ds \\
& +\int_0^t |S_1(\lambda)(s)f(S_1(\mu)(s),S_2(\mu)(s))-S_1(\mu)(s)f(S_1(\mu)(s),S_2(\mu)(s))|ds \\
& +|\lambda-\mu|T \\
\le & \int_0^t |S_1(\lambda)(s)||f(S_1(\lambda)(s),S_2(\lambda)(s))-f(S_1(\mu)(s),S_2(\mu)(s))|ds \\
& +\int_0^t |f(S_1(\mu)(s),S_2(\mu)(s))||S_1(\lambda)(s)-S_1(\mu)(s)|ds+|\lambda-\mu|T.
\end{aligned}$$

Furthermore, using the Lipschitz property of f, g, and their boundedness, we obtain

$$\begin{aligned}
& |S_1(\lambda)(t) - S_1(\mu)(t)| \\
\leq\ & C_1 \int_0^t (\alpha_{11}|S_1(\lambda)(s) - S_1(\mu)(s)| + \alpha_{12}|S_2(\lambda)(s) - S_2(\mu)(s)|)ds \\
& + C_f \int_0^t |S_1(\lambda)(s) - S_1(\mu)(s)|ds + |\lambda - \mu|T \\
\leq\ & C_1 \int_0^t (\alpha_{11}|S_1(\lambda)(s) - S_1(\mu)(s)|e^{-\theta s}e^{\theta s} + \alpha_{12}|S_2(\lambda)(s) - S_2(\mu)(s)|e^{-\theta s}e^{\theta s})ds \\
& + C_f \int_0^t |S_1(\lambda)(s) - S_1(\mu)(s)|e^{-\theta s}e^{\theta s}ds + |\lambda - \mu|T \\
\leq\ & C_1||S_1(\lambda) - S_1(\mu)||_\theta \frac{\alpha_{11}}{\theta}e^{\theta t} + C_1||S_2(\lambda) - S_2(\mu)||_\theta \frac{\alpha_{12}}{\theta}e^{\theta t} \\
& + \frac{C_f}{\theta}||S_1(\lambda) - S_1(\mu)||_\theta e^{\theta t} + |\lambda - \mu|T.
\end{aligned}$$

Thus

$$\begin{aligned}
& |S_1(\lambda)(t) - S_1(\mu)(t)| \\
& \leq \frac{C_1\alpha_{11} + C_f}{\theta}||S_1(\lambda) - S_1(\mu)||_\theta e^{\theta t} + \frac{C_1\alpha_{12}}{\theta}||S_2(\lambda) - S_2(\mu)||_\theta e^{\theta t} \\
& + |\lambda - \mu|T.
\end{aligned}$$

Multiply by $e^{-\theta t}$, go to the maximum and introduce the Bielecki norm, to obtain

$$\begin{aligned}
||S_1(\lambda) - S_1(\mu)||_\theta \leq & \frac{C_1\alpha_{11} + C_f}{\theta}||S_1(\lambda) - S_1(\mu)||_\theta \\
& + \frac{C_1\alpha_{12}}{\theta}||S_2(\lambda) - S_2(\mu)||_\theta + |\lambda - \mu|T.
\end{aligned}$$

Similarly

$$\begin{aligned}
||S_2(\lambda) - S_2(\mu)||_\theta \leq & \frac{C_2\alpha_{21}}{\theta}||S_1(\lambda) - S_1(\mu)||_\theta \\
& + \frac{C_2\alpha_{22} + C_g}{\theta}||S_2(\lambda) - S_2(\mu)||_\theta.
\end{aligned}$$

Summing up, gives

$$\begin{aligned}
& ||S_1(\lambda) - S_1(\mu)||_\theta + ||S_2(\lambda) - S_2(\mu)||_\theta \\
& \leq \frac{C_1\alpha_{11} + C_f + C_2\alpha_{21}}{\theta}||S_1(\lambda) - S_1(\mu)||_\theta \\
& + \frac{C_1\alpha_{12} + C_2\alpha_{22} + C_g}{\theta}||S_2(\lambda) - S_2(\mu)||_\theta + |\lambda - \mu|T,
\end{aligned}$$

whence

$$\begin{aligned}
||(S_1(\lambda), S_2(\lambda)) - (S_1(\mu), S_2(\mu))||_\theta \leq & M_\theta||(S_1(\lambda), S_2(\lambda)) - (S_1(\mu), S_2(\mu))||_\theta \\
& + |\lambda - \mu|T,
\end{aligned}$$

where

$$M_\theta = \max\left\{ \frac{C_1\alpha_{11} + C_f + C_2\alpha_{21}}{\theta},\ \frac{C_1\alpha_{12} + C_2\alpha_{22} + C_g}{\theta} \right\}.$$

Notice that $M_\theta \to 0$, as $\theta \to +\infty$, so if θ is large enough, one has $M_\theta < 1$. Then

$$(1 - M_\theta)||(S_1(\lambda), S_2(\lambda)) - (S_1(\mu), S_2(\mu))||_\theta \leq |\lambda - \mu|T,$$

where since $1 - M_\theta > 0$ we get that

$$||(S_1(\lambda), S_2(\lambda)) - (S_1(\mu), S_2(\mu))||_\theta \leq \frac{1}{1 - M_\theta}|\lambda - \mu|T.$$

So, if $\mu \to \lambda$, then $(S_1(\mu), S_2(\mu)) \to (S_1(\lambda), S_2(\lambda))$, which means that the solution depends continuously on the parameter λ. □

Alternatively, we have

Lemma 2.2. *Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be such that the functions $xf(x, y)$ and $yg(x, y)$ are Lipschitz continuous on the entire $\mathbb{R}^2$. Then for any $\lambda \in \mathbb{R}$, the Cauchy problem (2.2) has a unique solution that depends continuously on the parameter λ.*

Proof. **I. Existence and uniqueness.** Let $\alpha_{ij}, i, j = 1, 2$ be the Lipschitz constants of the functions $xf(x, y)$ and $yg(x, y)$. Hence

$$\begin{aligned} |xf(x,y) - \bar{x}f(\bar{x},\bar{y})| &\leq \alpha_{11}|x - \bar{x}| + \alpha_{12}|y - \bar{y}|, \\ |yg(x,y) - \bar{y}g(\bar{x},\bar{y})| &\leq \alpha_{21}|x - \bar{x}| + \alpha_{22}|y - \bar{y}|. \end{aligned}$$

Using the notations from the proof of Lemma 2.1, we have that the operator $N = (A, B)$ is a contraction on the space $C\left([0,T]; \mathbb{R}^2\right)$ with respect to a suitable Bielecki norm. Indeed, one has

$$\begin{aligned} & |A(x,y)(t) - A(\bar{x},\bar{y})(t)| \\ \leq & \int_0^t |x(s)f(x(s),y(s)) - \bar{x}(s)f(\bar{x}(s),\bar{y}(s))|ds \\ \leq & \int_0^t (\alpha_{11}|x(s) - \bar{x}(s)| + \alpha_{12}|y(s) - \bar{y}(s)|)ds \\ \leq & \int_0^t (\alpha_{11}|x(s) - \bar{x}(s)|e^{-\theta s}e^{\theta s} + \alpha_{12}|y(s) - \bar{y}(s)|e^{-\theta s}e^{\theta s})ds \\ \leq & \frac{\alpha_{11}}{\theta}||x - \bar{x}||_\theta e^{\theta t} + \frac{\alpha_{12}}{\theta}||y - \bar{y}||_\theta e^{\theta t}. \end{aligned}$$

Multiplying by $e^{-\theta t}$, and taking the maximum, we obtain

$$\begin{aligned} ||A(x,y) - A(\bar{x},\bar{y})||_\theta &\leq \frac{\alpha_{11}}{\theta}||x - \bar{x}||_\theta + \frac{\alpha_{12}}{\theta}||y - \bar{y}||_\theta, \\ ||B(x,y) - B(\bar{x},\bar{y})||_\theta &\leq \frac{\alpha_{21}}{\theta}||x - \bar{x}||_\theta + \frac{\alpha_{22}}{\theta}||y - \bar{y}||_\theta. \end{aligned}$$

Summing up gives

$$\begin{aligned} & ||A(x,y) - A(\bar{x},\bar{y})||_\theta + ||B(x,y) - B(\bar{x},\bar{y})||_\theta \\ \leq & \frac{\alpha_{11} + \alpha_{21}}{\theta}||x - \bar{x}||_\theta + \frac{\alpha_{12} + \alpha_{22}}{\theta}||y - \bar{y}||_\theta \end{aligned}$$

So
$$||N(x,y)-N(\bar{x},\bar{y})||_\theta \le L(||x-\bar{x}||_\theta + ||y-\bar{y}||_\theta),$$
where
$$L=\max\left\{\frac{\alpha_{11}+\alpha_{21}}{\theta},\frac{\alpha_{12}+\alpha_{22}}{\theta}\right\}.$$
It turns out that
$$||N(x,y)-N(\bar{x},\bar{y})||_\theta \le L||(x,y)-(\bar{x},\bar{y})||_\theta.$$
Here again if θ is chosen large enough, then $L<1$, and so $N=(A,B)$ is a contraction on $C([0,T];\mathbb{R}^2)$ endowed with the Bielecki norm $\|.\|_\theta$. It follows that the Cauchy problem has a unique solution.

II. Continuous dependence of parameter λ. Using (2.2) where $x=S_1(\lambda)$ and $y=S_2(\lambda)$, we have
$$\begin{aligned}
&|S_1(\lambda)(t)-S_1(\mu)(t)|\\
\le\ & \int_0^t |S_1(\lambda)(s)f(S_1(\lambda)(s),S_2(\lambda)(s))-S_1(\mu)(s)f(S_1(\mu)(s),S_2(\mu)(s))|ds\\
&+|\lambda-\mu|T\\
\le\ & \int_0^t (\alpha_{11}|S_1(\lambda)(s)-S_1(\mu)(s)|+\alpha_{12}|S_2(\lambda)(s)-S_2(\mu)(s)|)ds+|\lambda-\mu|T\\
\le\ & \int_0^t (\alpha_{11}|S_1(\lambda)(s)-S_1(\mu)(s)|e^{-\theta s}e^{\theta s}+\alpha_{12}|S_2(\lambda)(s)-S_2(\mu)(s)|e^{-\theta s}e^{\theta s})ds\\
&+|\lambda-\mu|T.
\end{aligned}$$
Then
$$\begin{aligned}
|S_1(\lambda)(t)-S_1(\mu)(t)| \le\ & \frac{\alpha_{11}}{\theta}||S_1(\lambda)-S_1(\mu)||_\theta e^{\theta t}+\frac{\alpha_{12}}{\theta}||S_2(\lambda)-S_2(\mu)||_\theta e^{\theta t}\\
&+|\lambda-\mu|T.
\end{aligned}$$
It follows that
$$\begin{aligned}
||S_1(\lambda)-S_1(\mu)||_\theta \le\ & \frac{\alpha_{11}}{\theta}||S_1(\lambda)-S_1(\mu)||_\theta+\frac{\alpha_{12}}{\theta}||S_2(\lambda)-S_2(\mu)||_\theta\\
&+|\lambda-\mu|T.
\end{aligned}$$
Similarly
$$||S_2(\lambda)-S_2(\mu)||_\theta \le \frac{\alpha_{21}}{\theta}||S_1(\lambda)-S_1(\mu)||_\theta+\frac{\alpha_{22}}{\theta}||S_2(\lambda)-S_2(\mu)||_\theta.$$
Summing up gives
$$\begin{aligned}
&||S_1(\lambda)-S_1(\mu)||_\theta+||S_2(\lambda)-S_2(\mu)||_\theta\\
&\le \frac{\alpha_{11}+\alpha_{21}}{\theta}||S_1(\lambda)-S_1(\mu)||_\theta+\frac{\alpha_{12}+\alpha_{22}}{\theta}||S_2(\lambda)-S_2(\mu)||_\theta\\
&+|\lambda-\mu|T.
\end{aligned}$$
Thus
$$\begin{aligned}
&||(S_1(\lambda),S_2(\lambda))-(S_1(\mu),S_2(\mu))||_\theta\\
\le\ & m_\theta||(S_1(\lambda),S_2(\lambda))-(S_1(\mu),S_2(\mu))||_\theta+|\lambda-\mu|T,
\end{aligned}$$

where

$$m_\theta := \max\left\{\frac{\alpha_{11}+\alpha_{21}}{\theta},\ \frac{\alpha_{12}+\alpha_{22}}{\theta}\right\}.$$

Then we have

$$(1-m_\theta)||(S_1(\lambda),S_2(\lambda))-(S_1(\mu),S_2(\mu))||_\theta \le |\lambda-\mu|T.$$

Here again, if $\theta \to +\infty$, then $m_\theta \to 0$ and thus one can choose $\theta > 0$ sufficiently large that $m_\theta < 1$. Then

$$||(S_1(\lambda),S_2(\lambda))-(S_1(\mu),S_2(\mu))||_\theta \le \frac{1}{1-m_\theta}|\lambda-\mu|T.$$

So, if $\mu \to \lambda$, then $(S_1(\mu),S_2(\mu)) \to (S_1(\lambda),S_2(\lambda))$, which proves the continuous dependence of the solution of λ. □

Using Lemmas 2.1 and 2.2 we obtain two convergence results regarding the above algorithm.

Theorem 2.3. *Assume that $f,g:\mathbb{R}^2\to\mathbb{R}$ are Lipschitz continuous on $\mathbb{R}^2$ and $|f|\le C_f$, $|g|\le C_g$. Then the algorithm is convergent to a solution of the control problem.*

Proof. For $k\ge 1$ we have the solution (x_k,y_k) corresponding to $\lambda=\lambda_k$. In addition, the algorithm gives an increasing sequence $(\underline{\lambda}_k)$ and a decreasing sequence $(\overline{\lambda}_k)$ with the following properties

$$\varphi(S_1(\underline{\lambda}_k),S_2(\underline{\lambda}_k))<0, \qquad \varphi(S_1(\overline{\lambda}_k),S_2(\overline{\lambda}_k))\ge 0, \tag{2.5}$$

and

$$\overline{\lambda}_k-\underline{\lambda}_k=\frac{1}{2^k}. \tag{2.6}$$

The two sequences being monotone and bounded are convergent. Moreover, from (2.6) they have the same limit λ^*. Using the continuity of φ and of S_1, S_2 with respect to λ, and (2.5) we deduce that

$$\varphi(S_1(\lambda^*),S_2(\lambda^*))=0. \tag{2.7}$$

Finally, denote $x^* := S_1(\lambda^*)$ and $y^* := S_2(\lambda^*)$. The (2.7) shows that (x^*,y^*,λ^*) is a solution the control problem. □

Similarly, using Lemma 2.2, one can prove the follwoing result.

Theorem 2.4. *Let $f,g:\mathbb{R}^2\to\mathbb{R}$ be such that the functions $xf(x,y)$ and $yg(x,y)$ are Lipschitz continuous on the entire $\mathbb{R}^2$. Then the algorithm is convergent to a solution of the control problem.*

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 341–357
DOI: 10.24193/subbmath.2023.2.10

Strongly quasilinear parabolic systems

Farah Balaadich and Elhoussine Azroul

Abstract. Using the theory of Young measures, we prove the existence of solutions to a strongly quasilinear parabolic system

$$\frac{\partial u}{\partial t} + A(u) = f,$$

where $A(u) = -\operatorname{div}\sigma(x,t,u,Du) + \sigma_0(x,t,u,Du)$, $\sigma(x,t,u,Du)$ and $\sigma_0(x,t,u,Du)$ are satisfy some conditions and $f \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$.

Mathematics Subject Classification (2010): 35K55, 35D30, 46E30.

Keywords: Quasilinear parabolic systems, weak solutions, Young measures.

1. Introduction

Let $n \geq 2$ be an integer and Ω be a bounded open subset of $\mathbb{R}^n$. Let Q be $\Omega \times (0,T)$ where $T > 0$ is given. In this work we are concerned with the problem of existence of a weak solution for a class of quasilinear parabolic systems of the form

$$\frac{\partial u}{\partial t} + A(u) = f \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega; \tag{1.3}$$

where $f \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$, $u_0(x)$ is a given function in $L^2(\Omega;\mathbb{R}^m)$ and $A(u) : L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)) \to L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$ is a Leray-Lions operator of the form $A(u) = -\operatorname{div}\sigma(x,t,u,Du) + \sigma_0(x,t,u,Du)$.

The solvability of (1.1)-(1.3) has been discussed in various papers for $m = 1$ and $m > 1$. Brezis and Browder [11] proved the existence and uniqueness of a solution of (1.1)-(1.3) when σ_0 is independent of ∇u. Landes and Mustonen [24, 25] provided

Received 29 April 2020; Accepted 29 June 2020.

structure conditions on a strongly nonlinear operator $A(u)$, under which (1.1) has weak solutions.

S. Demoulini [13] studied the nonlinear parabolic evolution of forward-backward type $u_t = \nabla.q(\nabla u)$ on $Q_\infty \equiv \Omega \times \mathbb{R}^+$. The author used the concept of Young measures as solutions to this kind of problems. Hungerbühler [22] considered the problem (1.1) with $\sigma_0 \equiv 0$ and proved the existence of a weak solution under classical regularity, growth, and coercivity conditions for σ, but with only very mild monotonicity assumptions for some $p \in (2n/(n+2), \infty)$. See [6, 7, 15, 17] for the utilization of Young's measure theory in elliptic case with dual or measure-valued right hand side, and [4, 16] for some kind of p-Laplacian systems.

Misawa [27] studied partial regularity results for evolutional p-Laplacian systems

$$\partial_t u^i - \sum_{\alpha,\beta=1}^{m} D_\alpha\Big(|Du|_g^{p-2} g^{\alpha\beta}(z,u) D_\beta u^i\Big) = f^i(z,u,Du),\ i=1,...,n,$$

with natural growth on the gradient. Dreyfuss and Hungerbühler [18] investigated a class of Navier-Stokes systems

$$\partial_t u - \operatorname{div}\sigma(x,t,u,Du) + u.\nabla u = f - \operatorname{grad} P$$

and obtained an existence result for a weak solution by the same theory as in [22]. Furthermore, the authors discussed the general case of the external force f.

In the setting of weighted Sobolev spaces, Aharouch et al. [2] studied the existence of weak solutions for (1.1) via pseudo-monotonicity, when $m = 1$. Di Nardo et al. [14] proved the existence of a renormalized solution for

$$u_t - \operatorname{div} a(x,t,u,\nabla u) + \operatorname{div} K(x,t,u) + H(x,t,\nabla u) = f - \operatorname{div} g,$$

where the data belongs to $L^1(Q) + L^{p'}(0,T; W^{-1,p'}(\Omega))$. For more results, the reader can see [10, 9, 12, 19].

In [5], we have investigated the problem (1.1)-(1.3) and prove the existence of weak solutions for every $f \in L^{p'}(0,T; W^{-1,p'}(\Omega;\mathbb{R}^m))$, by using the theory of Young measures and weak monotonicity assumptions. Furthermore, we have considered the following coercivity condition

$$\sigma(x,t,s,\xi) : \xi + \sigma_0(x,t,s,\xi).s \geq \beta|\xi|^p - d_2(x,t),$$

with $\beta > 0$ and $d_2 \in L^1(Q)$. The purpose of this paper, is to prove the existence of weak solutions for (1.1) by considering the coercivity condition only over σ, and the nonlinear term $\sigma_0(x,t,u,Du)$ satisfy

$$|\sigma_0(x,t,s,\xi)| \leq b(|s|)\big(d_2(x,t) + |\xi|^p\big),$$

$$\sigma_0(x,t,s,\xi).s \geq 0,$$

with $d_2 \in L^1(Q)$ and $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and increasing function. It should be noted here, in the above first condition, that there is no growth restriction on the perturbation σ_0 as a function of the unknown. This makes the resolution of (1.1) more complicate.

This paper is organized as follows: in Section 2 we recall the definition of Young measure and some its properties. Section 3 contains basic assumptions and the main result, while Section 4 is devoted to the proof of the main result.

2. Necessary facts about Young measures

In [20] it is claimed that weak convergence is a basic tool of nonlinear analysis, because it has the same compactness properties as the convergence in finite dimensional spaces. Moreover, this convergence sometimes does not behave as one desire with respect to nonlinear functionals and operators. In this situation one can use the technics of Young measures.

Consider

$$C_0(\mathbb{R}^m) = \big\{\varphi \in C(\mathbb{R}^m) : \lim_{|\lambda|\to\infty} \varphi(\lambda) = 0\big\}.$$

Its dual is the well known signed Radon measures $\mathcal{M}(\mathbb{R}^m)$ with finite mass. The duality of $\big(\mathcal{M}(\mathbb{R}^m), C_0(\mathbb{R}^m)\big)$ is given by the following integrand

$$\langle \nu, \varphi \rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu(\lambda), \text{ where } \nu : \Omega \to \mathcal{M}(\mathbb{R}^m).$$

Lemma 2.1 ([20]). *Let $(z_k)_k$ be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exist a subsequence (still denoted (z_k)) and a Borel probability measure ν_x on $\mathbb{R}^m$ for a.e. $x \in \Omega$, such that for almost each $\varphi \in C(\mathbb{R}^m)$ we have*

$$\varphi(z_k) \rightharpoonup^* \overline{\varphi}(x) = \langle \nu_x, \varphi \rangle \quad \textit{weakly in } L^\infty(\Omega;\mathbb{R}^m)$$

for a.e. $x \in \Omega$.

Definition 2.2. The family $\nu = \{\nu_x\}_{x\in\Omega}$ is called Young measures associated with (generated by) the subsequence $(z_k)_k$.

In [8], it is shown that if for all $R > 0$

$$\limsup_{L\to\infty} \big|\{x \in \Omega \cap B_R(0) : |z_k(x)| \geq L\}\big| = 0,$$

then for any measurable $\Omega' \subset \Omega$, we have

$$\varphi(x, z_k) \rightharpoonup \langle \nu_x, \varphi(x, .)\rangle = \int_{\mathbb{R}^m} \varphi(x,\lambda) d\nu_x(\lambda) \quad \text{in } L^1(\Omega'),$$

for every Carathéodory function $\varphi : \Omega \times \mathbb{R}^m \to \mathbb{R}$ such that $(\varphi(x, z_k(x)))_k$ is equiintegrable.

The following lemmas are useful for us.

Lemma 2.3 ([21]). *(i) If $|\Omega| < \infty$ and ν_x is the Young measure generated by the (whole) sequence (z_k), then there holds*

$$z_k \longrightarrow z \textit{ in measure } \Leftrightarrow \nu_x = \delta_{z(x)} \quad \textit{for a.e. } x \in \Omega.$$

(ii) If the sequence (v_k) generates the Young measure $\delta_{v(x)}$, then (z_k, v_k) generates the Young measure $\nu_x \otimes \delta_{v(x)}$.

It should be noted that the above properties remain true when $z_k = Dw_k$, with $w_k : \Omega \to \mathbb{R}^m$ and Ω can be repalced by the cylinder Q. We denote by $\mathbb{M}^{m\times n}$ the set of $m \times n$ matrices equipped with the inner product $\xi : \eta = \sum_{i,j} \xi_{ij}\eta_{ij}$.

Lemma 2.4 ([23]). *Let $\varphi : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function and (w_k) be a sequence of measurable functions, where $w_k : Q \to \mathbb{R}^m$, such that $w_k \to w$ in measure and such that Dw_k generates the Young measure $\nu_{(x,t)}$. Then*

$$\liminf_{k\to\infty} \int_Q \varphi(x,t,w_k,Dw_k)dxdt \geq \int_Q \int_{\mathbb{M}^{m\times n}} \varphi(x,t,w,\lambda)d\nu_{(x,t)}(\lambda)dxdt$$

provided that the negative part $\varphi^-(x,t,w_k,Dw_k)$ is equiintegrable.

We conclude this section by recalling the following lemma which describes limits points of gradients sequences by means of the Young measures.

Lemma 2.5 ([5]). *The Young measure $\nu_{(x,t)}$ generated by Dw_k in $L^p(0,T;L^p(\Omega))$ satisfy the following properties:*

(i) *$\nu_{(x,t)}$ is a probability measure, i.e., $\|\nu_{(x,t)}\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ for a.e. $(x,t) \in Q$.*
(ii) *The weak L^1-limit of Dw_k is given by $\langle \nu_{(x,t)}, id\rangle$.*
(iii) *For a.e. $(x,t) \in Q$, $\langle \nu_{(x,t)}, id\rangle = Dw(x,t)$.*

3. Basic assumptions and the main result

Let $Q = \Omega \times (0,T)$, where Ω is a bounded open subset of $\mathbb{R}^n$ and $T > 0$. Consider the problem (1.1)-(1.3) with $f \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$, $p' = p/(p-1)$. To study this problem we assume the following hypothesis.

(H0) $\sigma : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ and $\sigma_0 : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ are Carathéodory functions (i.e., continuous with respect to $(t,s,\xi) \in (0,T) \times \mathbb{R}^m \times \mathbb{M}^{m\times n}$ for a.e. $x \in \Omega$ and measurable with respect to x for all $(t,s,\xi) \in (0,T) \times \mathbb{R}^m \times \mathbb{M}^{m\times n}$). Moreover, the mapping $\xi \to \sigma_0(x,t,s,\xi)$ is linear.

(H1) There exist $\alpha > 0$, $d_1 \in L^{p'}(Q)$ and $d_2 \in L^1(Q)$ such that

$$\big|\sigma(x,t,s,\xi)\big| \leq d_1(x,t) + |s|^{p-1} + |\xi|^{p-1},$$
$$\sigma(x,t,s,\xi) : \xi \geq \alpha|\xi|^p,$$
$$\big|\sigma_0(x,t,s,\xi)\big| \leq b(|s|)\big(d_2(x,t) + |\xi|^p\big),$$
$$\sigma_0(x,t,s,\xi).s \geq 0,$$

where $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and increasing function.

(H2) σ satisfies one of the following (monotonicity) conditions:

(i) for all $(x,t) \in Q$ and all $u \in \mathbb{R}^m$, the map $\xi \mapsto \sigma(x,t,u,\xi)$ is a C^1-function and is monotone, i.e.,

$$\big(\sigma(x,t,u,\xi) - \sigma(x,t,u,\eta)\big) : (\xi - \eta) \geq 0 \quad \forall \xi,\eta \in \mathbb{M}^{m\times n}.$$

(ii) there exists a function $b : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that

$$\sigma(x,t,u,\xi) = \big(\partial b/\partial\xi\big)(x,t,u,\xi) := D_\xi b(x,t,u,\xi),$$

and $\xi \mapsto b(x,t,u,\xi)$ is convex and a C^1-function for all $(x,t) \in Q$ and all $u \in \mathbb{R}^m$.

(iii) σ is strictly monotone, i.e., σ is monotone and

$$\big(\sigma(x,t,u,\xi)-\sigma(x,t,u,\eta)\big):(\xi-\eta)=0 \quad \text{implies} \quad \xi=\eta.$$

(iv) σ is strictly p-quasimonotone, i.e.,

$$\int_Q\int_{\mathbb{M}^{m\times n}}\big(\sigma(x,t,u,\lambda)-\sigma(x,t,u,\overline{\lambda})\big):(\lambda-\overline{\lambda})d\nu_x(x,t)(\lambda)dxdt>0,$$

where $\overline{\lambda}=\langle\nu_{(x,t)},id\rangle$, $\nu=\{\nu_{(x,t)}\}_{(x,t)\in Q}$ is any family of Young measures generated by a sequence in $L^p(Q)$ which are not a single Dirac mass.

In what follows, $\langle .,.\rangle$ denotes the duality pairing between $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$ and $L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$, $Q_\tau=\Omega\times(0,\tau)$ for $\tau\in(0,T]$.

Definition 3.1. A function $u\in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))\cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$ is a weak solution of problem (1.1)-(1.3) if $\sigma_0(x,t,u,Du)\in L^1(Q;\mathbb{R}^m)$, $\sigma_0(x,t,u,Du)u\in L^1(Q;\mathbb{R}^m)$ and

$$\begin{aligned}-\int_Q u\frac{\partial\varphi}{\partial t}dxdt+\int_\Omega u\varphi dx\Big|_0^T+\int_Q\sigma(x,t,u,Du):D\varphi dxdt&\\+\int_Q\sigma_0(x,t,u,Du)\varphi dxdt=\langle f,\varphi\rangle&\end{aligned}$$

holds for all $\varphi\in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))\cap L^\infty(Q;\mathbb{R}^m)$.

Our main result is the following

Theorem 3.2. *Let $f\in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$ and $u_0\in L^2(\Omega;\mathbb{R}^m)$. Assume that (H0)-(H2) are fulfilled. Then there exists a weak solution $u\in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))\cap C(0,T;L^2(\Omega;\mathbb{R}^m))$ of the problem* (1.1)-(1.3) *in the sense of Definition 3.1.*

4. Proof of the main result

We divide the proof into several steps.

Step 1 Galerkin solutions. We choose a sequence of functions

$$\{w_i\}_{i\geq1}\subset C_0^\infty(\Omega;\mathbb{R}^m)$$

orthonormal with respect to $L^2(\Omega;\mathbb{R}^m)$ such that $\cup_{j\geq1}V_j$, where

$$V_j=span\{w_1,...,w_j\}$$

is dense in $H_0^s(\Omega;\mathbb{R}^m)$ with s large enough such as $s>n/2+1$, so that $H_0^s(\Omega;\mathbb{R}^m)$ is continuously embedded in $C^1(\overline{\Omega})$ (see [1]). We define $W_j=C^1(0,T;V_j)$. Therefore, we have $C_0^\infty(Q;\mathbb{R}^m)\subset\overline{\cup_{j\geq1}W_j}^{C^1(Q;\mathbb{R}^m)}$. Note that there exists $u_0^k\subset\cup_{j\geq1}V_j$ such that $u_0^k\to u_0$ in $L^2(\Omega;\mathbb{R}^m)$.

Definition 4.1. A function $u_k\in W_k$ is called Galerkin solution of (1.1)-(1.3) if and only if

$$\int_\Omega\frac{\partial u_k}{\partial t}vdx+\int_\Omega\sigma(x,t,u_k,Du_k):Dvdx+\int_\Omega\sigma_0(x,t,u_k,Du_k).vdx=\int_\Omega f(t)vdx \tag{4.1}$$

for all $v \in V_k$ and all $t \in [0,T]$ with $u_k(x,0) = u_0^k(x)$.

Setting

$$u_k(x,t) = \sum_{i=1}^{k} d_i(t) w_i(x),$$

we then try to look for the coefficients $d_i \in C^1([0,T])$. To do this, we define a vector valued function $y_k : [0,T] \times \mathbb{R}^k \to \mathbb{R}^k$ for $d = (d_1, ..., d_k)$ by

$$\begin{aligned}\Big(y_k(t,d)\Big)_i = &\int_\Omega \sigma\Big(x,t,\sum_{j=1}^{k} d_j(t)w_j(x), \sum_{j=1}^{k} d_j(t)Dw_j(x)\Big) : Dw_i(x)dx \\ &+ \int_\Omega \sigma_0\Big(x,t,\sum_{j=1}^{k} d_j(t)w_j(x), \sum_{j=1}^{k} d_j(t)Dw_j(x)\Big).w_i(x)dx,\end{aligned}$$

for $i = 1, ..., k$. Note that $y_k(t,d)$ is continuous because σ and σ_0 are both Carathéodory functions. Therefore, we obtain the following system of ordinary differential equations

$$\begin{cases} d' + y_k(t,d) &= F, \\ d(0) &= v_k, \end{cases}$$

where

$$\big(F(t)\big)_i = \int_\Omega f(t) w_i dx \text{ and } (v_k)_i = \int_\Omega u_0^k w_i dx, \text{ for } i = 1, ..., k.$$

Multiplying the first equation by $d(t)$ and using (H1) (coercivity of σ and sign condition of σ_0) one gets $y_k(t,d)d \geq 0$. By virtue of the Young inequality, it yields

$$\frac{1}{2}\frac{d}{dt}\big|d(t)\big|^2 \leq \big|F(t)\big|\big|d(t)\big| \leq \frac{1}{2}\big(\big|F(t)\big|^2 + \big|d(t)\big|^2\big).$$

Then, Gronwall's lemma allows to deduce that

$$\big|d(t)\big| \leq C(T).$$

Thus, we get $\big|d(t) - d(0)\big| \leq 2C(T)$. Now, let us define $A_k = \max_{t\in[0,T]} \big|F - y_k(t, d(t))\big|$ and $q = \min\big\{T, \frac{2C(T)}{A_k}\big\}$. By the Cauchy-Peano theorem (cf. [3]) we obtain a local solution in $[0,q]$. Starting with the initial value q, we obtain a local solution in $[q, 2q]$ and so on we get a local solution d_k in $C^1([0,T])$. Therefore, by construction, we know that the function $u_k(x,t) = \sum_{i=1}^{k} d_{ki}(t)w_i(x)$, which belongs to W_k, is a Galerkin solution for (1.1)-(1.3) satisfying

$$\begin{aligned}\int_{Q_\tau} \frac{\partial u_k}{\partial t} v dxdt + &\int_{Q_\tau} \sigma(x,t,u_k,Du_k) : Dvdxdt \\ &+ \int_{Q_\tau} \sigma_0(x,t,u_k,Du_k).vdxdt = \int_{Q_\tau} fvdxdt,\end{aligned} \tag{4.2}$$

for all $v \in W_k$ and all $\tau \in (0,T]$ with $u_k(x,0) = u_0^k(x)$.

Step 2 A priori estimates. In the sequel, C will denotes a positive constant which may change values from line to line and which depends on the parameters of our problem. Let u_k be a Galerkin solution of (1.1)-(1.3). Choosing u_k as test function in (4.2), we get

$$\int_{Q_\tau} \frac{\partial u_k}{\partial t} u_k dxdt + \int_{Q_\tau} \sigma(x,t,u_k,Du_k) : Du_k dxdt \\ + \int_{Q_\tau} \sigma_0(x,t,u_k,Du_k).u_k dxdt = \int_{Q_\tau} f u_k dxdt, \tag{4.3}$$

for every $\tau \in (0,T]$. By virtue of (H1) (coercivity condition) and Hölder's inequality, we can write

$$\frac{1}{2}\|u_k(\tau)\|^2_{L^2(\Omega)} + \alpha \int_{Q_\tau} |Du_k|^p dxdt \\ \leq \|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \|u_k\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}, \tag{4.4}$$

which implies that

$$\alpha \|Du_k\|_p^p \leq \|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \|u_k\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.$$

Therefore

$$\|u_k\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leq C. \tag{4.5}$$

By virtue of (4.4), the sequence (u_k) is bounded in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$. Since

$$\int_{Q_\tau} \left|\sigma(x,t,u_k,Du_k)\right|^{p'} dxdt \leq \int_{Q_\tau} \left(d_1(x,t)^{p'} + |u_k|^p + |Du_k|^p\right) dxdt \leq C,$$

then

$$\left\|\sigma(x,t,u_k,Du_k)\right\|_{L^{p'}(Q;\mathbb{M}^{m\times n})} \leq C.$$

Going back to (4.3), we obtain

$$0 \leq \int_{Q_\tau} \sigma_0(x,t,u_k,Du_k).u_k dxdt \leq C. \tag{4.6}$$

Let $N > 0$ be fixed. By the condition (H1) and above inequality we can write

$$\begin{aligned}
&\int_{Q_\tau} \left|\sigma_0(x,t,u_k,Du_k)\right| dxdt \\
&= \int_{Q_\tau \cap \{|u_k| \leq N\}} \left|\sigma_0(x,t,u_k,Du_k)\right| dxdt + \int_{Q_\tau \cap \{|u_k| > N\}} \left|\sigma_0(x,t,u_k,Du_k)\right| dxdt \\
&\leq \int_{Q_\tau \cap \{|u_k| \leq N\}} \left|\sigma_0(x,t,u_k,Du_k)\right| dxdt + \frac{1}{N}\int_{Q_\tau} \sigma_0(x,t,u_k,Du_k).u_k dxdt \\
&\leq \int_{Q_\tau \cap \{|u_k| \leq N\}} b(|u_k|)\big(d_3(x,t) + |Du_k|^p\big) dxdt + \frac{C}{N} \\
&\leq b(N)\big(\|d_2\|_{L^1(Q_\tau)} + \|Du_k\|^p_{L^p(Q_\tau)}\big) + \frac{C}{N} \leq C. \qquad (4.7)
\end{aligned}$$

Hence, the sequence $\sigma_0(x,t,u_k,Du_k)$ is uniformly bounded in $L^1(Q;\mathbb{R}^m)$. Therefore, for a subsequence still indexed by k and for a measurable functions $u \in L^p(0,T;W^{1,p}(\Omega;\mathbb{R}^m)) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$, $\Sigma \in L^{p'}(Q;\mathbb{M}^{m\times n})$ and $\Sigma_0 \in L^1(Q;\mathbb{R}^m)$

$$\begin{aligned} u_k \rightharpoonup u \quad & \text{weakly in } L^p(0,T;W^{1,p}(\Omega;\mathbb{R}^m)),\\ u_k \rightharpoonup^* u \quad & \text{weakly in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^m)),\\ \sigma(x,t,u_k,Du_k) \rightharpoonup \Sigma \quad & \text{weakly in } L^{p'}(Q;\mathbb{M}^{m\times n}),\\ \sigma_0(x,t,u_k,Du_k) \rightharpoonup \Sigma_0 \quad & \text{weakly in } L^1(Q;\mathbb{R}^m),\\ u_k \longrightarrow u \quad & \text{strongly in } L^1(Q;\mathbb{R}^m). \end{aligned} \tag{4.8}$$

The last property in (4.8) comes from the fact that,

$$\frac{\partial u_k}{\partial t} = f + \mathrm{div}\sigma(x,t,u_k,Du_k) - \sigma_0(x,t,u_k,Du_k)$$

is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m)) + L^1(Q;\mathbb{R}^m)$.

Lemma 4.2. *The sequence (u_k) constructed above satisfy $u_k(.,T) \rightharpoonup u(.,T)$ in $L^2(\Omega;\mathbb{R}^m)$ and $u(.,0) = u_0(.)$.*

Proof. Since (u_k) is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$, up to a subsequence, we have

$$u_k(.,T) \rightharpoonup z \quad \text{in } L^2(\Omega;\mathbb{R}^m) \quad \text{as} \quad k \to \infty.$$

Let us denote $u(.,T)$ as $u(T)$ and $u(.,0)$ as $u(0)$ (for simplicity).
Let $v \in V_j \cap L^\infty(\Omega;\mathbb{R}^m)$, $j \le k$ and $\psi \in C^\infty([0,T])$, then we have (take $\tau = T$)

$$\begin{aligned} \int_Q \frac{\partial u_k}{\partial t} v\psi dxdt + \int_Q \sigma(x,t,u_k,Du_k) : Dv\psi dxdt \\ + \int_Q \sigma_0(x,t,u_k,Du_k).v\psi dxdt = \int_Q fv\psi dxdt. \end{aligned}$$

The integration of the first term allows to write

$$\begin{aligned} \int_\Omega u_k(T)\psi(T)vdx - \int_\Omega u_k(0)\psi(0)vdx + \int_Q \sigma(x,t,u_k,Du_k) : Dv\psi dxdt \\ + \int_Q \sigma_0(x,t,u_k,Du_k).v\psi dxdt = \int_Q fv\psi dxdt + \int_Q u_k v\psi' dxdt. \end{aligned}$$

By virtue to (4.8), we obtain in passing to the limit as $k \to \infty$

$$\begin{aligned} \int_\Omega z\psi(T)vdx - \int_\Omega u_0\psi(0)vdx + \int_Q \Sigma : Dv\psi dxdt + \int_Q \Sigma_0.v\psi dxdt \\ = \int_Q fv\psi dxdt + \int_Q uv\psi' dxdt. \end{aligned} \tag{4.9}$$

Let $\psi(T)=\psi(0)=0$, then

$$\begin{aligned}\int_Q \Sigma : Dv\psi dxdt + \int_Q \Sigma_0.v\psi dxdt &= \int_Q fv\psi dxdt + \int_Q uv\psi dxdt\\ &= \int_Q fv\psi dxdt - \int_Q u'v\psi dxdt.\end{aligned}$$

Going back to (4.9), one has

$$\begin{aligned}\int_\Omega z\psi(T)vdx - \int_\Omega u_0\psi(0)vdx &= \int_Q u'v\psi dxdt + \int_Q uv\psi' dxdt\\ &= \int_\Omega u(T)\psi(T)vdx - \int_\Omega u(0)\psi(0)vdx.\end{aligned}$$

Now, tending j to ∞, if we take $\psi(T)=0$ and $\psi(0)=1$, then we obtain $u(0)=u_0$, if we take $\psi(T)=1$ and $\psi(0)=0$, then $u(T)=z$. Therefore $u_k(.,T)\rightharpoonup u(.,T)$ in $L^2(\Omega;\mathbb{R}^m)$. □

Step 3 div-curl inequality. As stated in the introduction we will use the tool of Young measures to pass to the limit. To this purpose, since (u_k) is bounded in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$, there exists a Young measure $\nu_{(x,t)}$ generated by Du_k in $L^p(0,T;L^p(\Omega))$, by Lemma 2.1. Moreover, $\nu_{(x,t)}$ satisfy the properties of Lemma 2.5.

The crucial point in the proof of this Section is the following lemma, namely div-curl inequality, which allows the passage to the limit in the approximating equations.

Lemma 4.3. *Assume that (H0)-(H2) hold. Then the Young measure $\nu_{(x,t)}$ generated by Du_k satisfies*

$$\int_Q\int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,u,\lambda)-\sigma(x,t,u,Du)\big):(\lambda-Du)d\nu_{(x,t)}(\lambda)dxdt \le 0.$$

Proof. Let us consider the sequence

$$I_k := \big(\sigma(x,t,u_k,Du_k)-\sigma(x,t,u,Du)\big):(Du_k-Du),$$

and let us prove that its negative part I_k^- is equiintegrable on Q. To do this, we write I_k^- in the form

$$\begin{aligned}I_k &= \sigma(x,t,u_k,Du_k):(Du_k-Du)-\sigma(x,t,u,Du):(Du_k-Du)\\ &=: I_{k,1}+I_{k,2}.\end{aligned}$$

Since $d_1\in L^{p'}(Q)$, it follows by (H1) that

$$\int_Q \big|\sigma(x,t,u,Du)\big|^{p'}dxdt \le C.$$

Thus, $\sigma(.,.,u,Du)\in L^{p'}(Q;\mathbb{M}^{m\times n})$ for arbitrary $u\in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$, and Lemma 2.5 allows to write

$$\liminf_{k\to\infty}\int_Q I_{k,2}dxdt = \int_Q \sigma(x,t,u,Du):\Big(\int_{\mathbb{M}^{m\times n}}\lambda d\nu_{(x,t)}(\lambda)-Du\Big)dxdt = 0. \quad (4.10)$$

Let Q' be a measurable subset of Q, by the Hölder inequality and (H1) it follows that

$$\begin{aligned}
&\int_{Q'} \big|\sigma(x,t,u_k,Du_k):Du\big|dxdt\\
&\quad\le \Big(\int_{Q'} \big|\sigma(x,t,u_k,Du_k)\big|^{p'}dxdt\Big)^{\frac{1}{p'}}\Big(\int_{Q'}|Du|^p dxdt\Big)^{\frac{1}{p}}\\
&\quad\le \Big(\int_{Q'} \big|d_1(x,t)\big|^{p'}+|u_k|^p+|Du_k|^p\big)dxdt\Big)^{\frac{1}{p'}}\Big(\int_{Q'}|Du|^p dxdt\Big)^{\frac{1}{p}}.
\end{aligned}$$

The first integral on the right hand side of the above inequality is uniformly bounded, by the boundedness of $(u_k)_k$. The second integral is arbitrary small if the measure of Q' is chosen small enough. Hence, $\big(\sigma(x,t,u_k,Du_k):Du\big)$ is equiintegrable. A similar argument gives the equiintegrability of $\big(\sigma(x,t,u_k,Du_k):Du_k\big)$. Therefore $I_{k,1}$ is equiintegrable, and by virtue of Lemma 2.4

$$I:=\liminf_{k\to\infty}\int_Q I_k dxdt \ge \int_Q\int_{\mathbb{M}^{m\times n}}\sigma(x,t,u,\lambda):(\lambda-Du)d\nu_{(x,t)}(\lambda)dxdt.$$

To deduce the needed inequality, it is sufficient to show that $I\le 0$. We have

$$\begin{aligned}
&\int_Q \frac{\partial u_k}{\partial t}u_k dxdt+\int_Q\sigma(x,t,u_k,Du_k):Du_k dxdt\\
&\qquad\qquad+\int_Q\sigma_0(x,t,u_k,Du_k).u_k dxdt=\int_Q fu_k dxdt,
\end{aligned}$$

then

$$\begin{aligned}
I&=\liminf_{k\to\infty}\int_Q\sigma(x,t,u_k,Du_k):(Du_k-Du)dxdt\\
&=\liminf_{k\to\infty}\Big(\int_Q\sigma(x,t,u_k,Du_k):Du_k dxdt-\int_Q\sigma(x,t,u_k,Du_k):Dudxdt\Big)\\
&=\liminf_{k\to\infty}\Big(\int_Q fu_k dxdt-\int_Q\frac{\partial u_k}{\partial t}u_k dxdt-\int_Q\sigma_0(x,t,u_k,Du_k).u_k dxdt\\
&\qquad\qquad-\int_Q\sigma(x,t,u_k,Du_k):Dudxdt\Big).
\end{aligned}\tag{4.11}$$

Remark first that $\int_Q f(u_k-u)dxdt$ tends to zero as k tends to ∞. By Lemma 4.2 we have

$$\|u_k(.,0)\|_{L^2(\Omega)}\to\|u(.,0)\|_{L^2(\Omega)}\quad\text{and}\quad\|u(.,T)\|_{L^2(\Omega)}\le\liminf_{k\to\infty}\|u_k(.,T)\|_{L^2(\Omega)},$$

which imply

$$\liminf_{k\to\infty}\Big(-\int_Q\frac{\partial u_k}{\partial t}u_k dxdt\Big)\le\frac{1}{2}\|u(.,0)\|^2_{L^2(\Omega)}-\frac{1}{2}\|u(.,T)\|^2_{L^2(\Omega)}.$$

Now, take $\psi \in C^1(0,T;V_j)\cap L^\infty(Q;\mathbb{R}^m)$, $j \le k$, we have

$$\int_Q \frac{\partial u_k}{\partial t}\psi dxdt + \int_Q \sigma(x,t,u_k,Du_k) : D\psi dxdt + \int_Q \sigma_0(x,t,u_k,Du_k).\psi dxdt$$
$$= \int_Q f\psi dxdt.$$

The first integral (after integration) is equal to

$$\int_\Omega u_k(.,T)\psi(.,T)dx - \int_\Omega u_k(.,0)\psi(.,0)dx - \int_Q u_k\frac{\partial \psi}{\partial t}dxdt.$$

By tending k to infinity, one has

$$\int_\Omega u(.,T)\psi(T)dx - \int_\Omega u(.,0)\psi(0)dx - \int_Q u\frac{\partial \psi}{\partial t}dxdt$$
$$+ \int_Q \Sigma : D\psi dxdt + \int_Q \Sigma_0.\psi dxdt = \int_Q f\psi dxdt.$$

Passing j to ∞, it result for all $\psi \in C^1(0,T;C^1(\overline{\Omega}))$ that

$$\int_\Omega u(.,T)\psi(T)dx - \int_\Omega u(.,0)\psi(0)dx - \int_Q u\frac{\partial \psi}{\partial t}dxdt$$
$$+ \int_Q \Sigma : D\psi dxdt + \int_Q \Sigma_0.\psi dxdt = \int_Q f\psi dxdt,$$

i.e.,

$$-\int_Q u\frac{\partial \psi}{\partial t}dxdt + \int_Q \Sigma : D\psi dxdt + \int_Q \Sigma_0.\psi dxdt = \int_Q f\psi dxdt,$$

for all $\psi \in C_0^\infty(Q) \subset C^1(0,T;C_0^\infty(\Omega;\mathbb{R}^m))$. Consequently

$$\frac{\partial u}{\partial t} - \operatorname{div}\Sigma + \Sigma_0 = f.$$

Hence, for $u \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))\cap L^\infty(Q;\mathbb{R}^m)$

$$-\int_Q \Sigma : Dudxdt - \int_Q \Sigma_0.udxdt = -\int_Q fudxdt + \int_Q u\frac{\partial u}{\partial t}dxdt.$$

Gathering the above results in the Eq. (4.11), it result that $I \le 0$. $\square$

Step 4 Passage to the limit. The passage to the limit will be concern the four cases listed in assumption (H2). Remark first that from Lemma 4.3 and monotonicity of the function σ, it follows that

$$\int_Q\int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big) : (\lambda - Du)d\nu_{(x,t)}(\lambda)\otimes dx\otimes dt = 0$$

implies

$$\big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big) : (\lambda - Du) = 0 \quad \text{on supp}\ \nu_{(x,t)}. \tag{4.12}$$

Now, we have all ingredients to pass to the limit in the approximating equations.

Case (i): Let ∇ denotes the derivative of σ with respect to its last variable. We prove that

$$\sigma(x,t,u,\lambda):\xi=\sigma(x,t,u,Du):\xi+\big(\nabla\sigma(x,t,u,Du)\xi\big):(Du-\lambda)$$

holds on $\operatorname{supp}\nu_{(x,t)}$, for all $\xi\in\mathbb{M}^{m\times n}$. Let $\tau\in\mathbb{R}$, from the monotonicity of σ we infer that

$$\big(\sigma(x,t,u,\lambda)-\sigma(x,t,u,Du+\tau\xi)\big):(\lambda-Du-\tau\xi)\geq 0.$$

The above inequality together with (4.12) imply

$$\begin{aligned}-\sigma(x,t,u,&\lambda):\tau\xi\\ &\geq-\sigma(x,t,u,\lambda):(\lambda-Du)+\sigma(x,t,u,Du+\tau\xi):(\lambda-Du-\tau\xi)\\ &=-\sigma(x,t,u,Du):(\lambda-Du)+\sigma(x,t,u,Du+\tau\xi):(\lambda-Du-\tau\xi).\end{aligned}$$

Since $\sigma(x,t,u,Du+\tau\xi)=\sigma(x,t,u,Du)+\nabla\sigma(x,t,u,Du)\tau\xi+o(\tau)$, we get

$$-\sigma(x,t,u,\lambda):\tau\xi\geq\tau\Big(\big(\nabla\sigma(x,t,u,Du)\big)\xi:(\lambda-Du)-\sigma(x,t,u,Du):\xi\Big).$$

The choice of τ to be arbitrary in $\mathbb{R}$ implies the needed equality

$$\sigma(x,t,u,\lambda):\xi=\sigma(x,t,u,Du):\xi+\big(\nabla\sigma(x,t,u,Du)\xi\big):(Du-\lambda).$$

Using the equiintegrability of $\sigma(x,t,u_k,Du_k)$ and above equality to deduce that its weak L^1-limit is

$$\begin{aligned}\overline{\sigma}:=&\int_{\mathbb{M}^{m\times n}}\sigma(x,t,u,\lambda)d\nu_{(x,t)}(\lambda)\\ =&\int_{\operatorname{supp}\nu_{(x,t)}}\sigma(x,t,u,\lambda)d\nu_{(x,t)}(\lambda)\\ =&\int_{\operatorname{supp}\nu_{(x,t)}}\Big(\sigma(x,t,u,Du)+\big(\nabla\sigma(x,t,u,Du)\big):(Du-\lambda)\Big)d\nu_{(x,t)}(\lambda)\\ =&\,\sigma(x,t,u,Du)\underbrace{\int_{\operatorname{supp}\nu_{(x,t)}}d\nu_{(x,t)}(\lambda)}_{=:1}\\ &+\big(\nabla\sigma(x,t,u,Du)\big)^t\underbrace{\int_{\operatorname{supp}\nu_{(x,t)}}(Du-\lambda)d\nu_{(x,t)}(\lambda)}_{=0}\\ =&\,\sigma(x,t,u,Du).\end{aligned}$$

We have $\sigma(x,t,u_k,Du_k)$ is bounded in $L^{p'}(Q;\mathbb{M}^{m\times n})$ reflexive, then $\sigma(x,t,u_k,Du_k)$ is weakly convergent in $L^{p'}(Q;\mathbb{M}^{m\times n})$ and its weak $L^{p'}$-limit is also $\sigma(x,t,u,Du)$.

Case (ii): In this case we prove that, if $\lambda\in\operatorname{supp}\nu_{(x,t)}$ then

$$b(x,t,u,\lambda)=b(x,t,u,Du)+\sigma(x,t,u,Du):(\lambda-Du).$$

Suppose that $\lambda\in\operatorname{supp}\nu_{(x,t)}$, from (4.12) it follows for $\tau\in[0,1]$

$$(1-\tau)\big(\sigma(x,t,u,Du)-\sigma(x,t,u,\lambda)\big):(Du-\lambda)=0.$$

The above expression together with monotonicity of σ allow to write

$$\begin{aligned}0 &\le (1-\tau)\Big(\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big)-\sigma(x,t,u,\lambda)\Big):(Du-\lambda)\\&=(1-\tau)\Big(\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big)-\sigma(x,t,u,Du)\Big):(Du-\lambda).\end{aligned}\tag{4.13}$$

Since σ is monotone, we have

$$\Big(\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big)-\sigma(x,t,u,Du)\Big):\tau(\lambda-Du)\ge 0$$

which implies since $\tau\in[0,1]$

$$\Big(\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big)-\sigma(x,t,u,Du)\Big):(1-\tau)(\lambda-Du)\ge 0.$$

From this inequality and (4.13) we can infer that

$$\Big(\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big)-\sigma(x,t,u,Du)\Big):(\lambda-Du)=0\quad\forall\tau\in[0,1],$$

i.e.,

$$\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big):(\lambda-Du)=\sigma(x,t,u,Du):(\lambda-Du).$$

We know that (by hypothesis)

$$\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big):(\lambda-Du)=\frac{\partial b}{\partial\tau}\big(x,t,u,Du+\tau(\lambda-Du)\big):(\lambda-Du)$$

for $\tau\in[0,1]$. By integration of the above equation over $[0,1]$, it follows that

$$\begin{aligned}b(x,t,u,\lambda)&=b(x,t,u,Du)+\int_0^1\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big):(\lambda-Du)d\tau\\&=b(x,t,u,Du)+\sigma(x,t,u,Du):(\lambda-Du)\end{aligned}$$

as we desired. Let us denotes

$$K_{(x,t)}=\big\{\lambda\in\mathbb{M}^{m\times n}:\ b(x,t,u,\lambda)=b(x,t,u,Du)+\sigma(x,t,u,Du):(\lambda-Du)\big\}.$$

From the above results, $\lambda\in K_{(x,t)}$. Since b is convex, we can write

$$\underbrace{b(x,t,u,\lambda)}_{=:B_1(\lambda)}\ge\underbrace{b(x,t,u,Du)+\sigma(x,t,u,Du):(\lambda-Du)}_{=:B_2(\lambda)}.$$

Since $\lambda\mapsto B_1(\lambda)$ is C^1-function, then for $\xi\in\mathbb{M}^{m\times n}$ and $\tau\in\mathbb{R}$

$$\frac{B_1(\lambda+\tau\xi)-B_1(\lambda)}{\tau}\ge\frac{B_2(\lambda+\tau\xi)-B_2(\lambda)}{\tau}\quad\text{for }\tau>0,$$

$$\frac{B_1(\lambda+\tau\xi)-B_1(\lambda)}{\tau}\le\frac{B_2(\lambda+\tau\xi)-B_2(\lambda)}{\tau}\quad\text{for }\tau<0.$$

Consequently $D_\lambda B_1=D_\lambda B_2$, i.e.,

$$\sigma(x,t,u,\lambda)=\sigma(x,t,u,Du)\quad\text{on }\operatorname{supp}\nu_{(x,t)}\subset K_{(x,t)}.\tag{4.14}$$

Consdier the function $g(x,t,s,\lambda)=|\sigma(x,t,s,\lambda)-\overline{\sigma}(x,t)|$. Then g is a Carathéodoroy function by that of σ. Moreover, since $\sigma(x,t,u_k,Du_k)$ is equiintegrable, thus

$g_k(x,t) := g(x,t,u_k,Du_k)$ is also equiintegrable, hence $g_k \rightharpoonup \overline{g}$ in $L^1(Q)$ (in fact, this convergence is strong since $g_k \geq 0$), where

$$\begin{aligned}\overline{g}(x,t) &= \int_{\mathbb{R}^m\times\mathbb{M}^{m\times n}} \big|\sigma(x,t,s,\lambda) - \overline{\sigma}(x,t)\big| d\delta_{u(x,t)}(s)\otimes d\nu_{(x,t)}(\lambda)\\ &= \int_{\mathbb{M}^{m\times n}} \big|\sigma(x,t,u,\lambda) - \overline{\sigma}(x,t)\big| d\nu_{(x,t)}(\lambda)\\ &= \int_{\operatorname{supp}\nu_{(x,t)}} \big|\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big| d\nu_{(x,t)}(\lambda) = 0\end{aligned}$$

by (4.14).

Case (iii): On the one hand, by Eq. (4.12) we deduce that $\nu_{(x,t)} = \delta_{Du(x,t)}$ for a.e. $(x,t)\in Q$. By virtue of the first property in Lemma 2.3, one gets

$$Du_k \to Du \quad \text{in measure as } k\to\infty.$$

On the other hand, since (u_k) is bounded in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$, up to a subsequence, $u_k \to u$ in measure. Therefore (for a subsequence) $u_k \to u$ and $Du_k \to Du$ almost everywhere for $k\to\infty$. The continuity of the function σ implies

$$\sigma(x,t,u_k,Du_k) \to \sigma(x,t,u,Du) \quad \text{almost everywhere as } k\to\infty.$$

The Vitali convergence theorem implies $\sigma(x,t,u_k,Du_k) \to \sigma(x,t,u,Du)$ in $L^1(Q)$, by the boundedness and equiintegrability of $\sigma(x,t,u_k,Du_k)$.

Case (iv): Assume that $\nu_{(x,t)}$ is not a Dirac measure on a set $(x,t)\in Q'$ of positive measure. We have $\overline{\lambda} = \langle \nu_{(x,t)}, id\rangle = Du(x,t)$, thus

$$\begin{aligned}&\int_Q\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\overline{\lambda}) : (\lambda-\overline{\lambda}) d\nu_{(x,t)}(\lambda)dxdt\\ &= \int_Q\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\overline{\lambda}) : \lambda d\nu_{(x,t)}(\lambda)dxdt\\ &- \int_Q\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\overline{\lambda}) : \overline{\lambda} d\nu_{(x,t)}(\lambda)dxdt\\ &= \int_Q \sigma(x,t,u,\overline{\lambda}) : \Big(\int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda)\Big)dxdt\\ &- \int_Q \sigma(x,t,u,\overline{\lambda}) : \overline{\lambda}\Big(\int_{\mathbb{M}^{m\times n}} d\nu_{(x,t)}(\lambda)\Big)dxdt\\ &= 0.\end{aligned}$$

It follows by the strict p-quasimonotonicity of σ that

$$\int_Q\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\nu_{(x,t)}(\lambda)dxdt > \int_Q\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \overline{\lambda} d\nu_{(x,t)}(\lambda)dxdt.$$

By virtue of Lemma 4.3 (i.e., $I \leq 0$), it result that

$$\begin{aligned}\int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) :&\lambda d\nu_{(x,t)}(\lambda)dxdt \\ &> \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \overline{\lambda} d\nu_{(x,t)}(\lambda)dxdt \\ &\geq \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\nu_{(x,t)}(\lambda)dxdt,\end{aligned}$$

and this is a contradiction. Hence $\nu_{(x,t)}$ is a Dirac measure, i.e., $\nu_{(x,t)} = \delta_{h(x,t)}$ for a.e. $(x,t) \in Q$, thus

$$h(x,t) = \int_{\mathbb{M}^{m\times n}} \lambda d\delta_{h(x,t)}(\lambda) = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda) = Du(x,t).$$

Thus $\nu_{(x,t)} = \delta_{Du(x,t)}$. Owing to Lemma 2.3, we get $Du_k \to Du$ in measure. The remainder of the proof of this case is similar to that in Case (iii).
To complete the proof of the main result, it remains to pass to the limit on the nonlinearity term $\sigma_0(x,t,u_k,Du_k)$. From the convergence in measure of u_k to u and of Du_k to Du, it then follows by the continuity of σ_0, that

$$\sigma_0(x,t,u_k,Du_k) \longrightarrow \sigma_0(x,t,u,Du) \quad \text{almost everywhere in } Q,$$

(for a subsequence). Let Q' be a subset of Q and let $N > 0$. We can write

$$\begin{aligned}&\int_{Q'} |\sigma_0(x,t,u_k,Du_k)|dx \\ &= \int_{Q'\cap\{|u_k|\leq N\}} |\sigma_0(x,t,u_k,Du_k)|dxdt + \int_{Q'\cap\{|u_k|>N\}} |\sigma_0(x,t,u_k,Du_k)|dxdt.\end{aligned}$$

By the third condition in (H1) together with (4.6), we obtain

$$\begin{aligned}\int_{Q'} &|\sigma_0(x,t,u_k,Du_k)|dxdt \\ &\leq b(N)\int_{Q'} d_2(x,t)dxdt + b(N)\int_{Q'} |Du_k|^p dxdt + \frac{C}{N} \leq \epsilon,\end{aligned}$$

for some $\epsilon > 0$. Applying Vitali's theorem, we obtain

$$\sigma_0(x,t,u_k,Du_k) \longrightarrow \sigma_0(x,t,u,Du) \quad \text{strongly in } L^1(Q).$$

In addition, by Fatou's Lemma, we get $\sigma_0(x,t,u,Du)u \in L^1(Q)$.
Now, since σ_0 is linear with respect to its last variable, then

$$\begin{aligned}\sigma_0(x,t,u_k,Du_k) \rightharpoonup &\langle \nu_{(x,t)}, \sigma_0(x,t,u,.)\rangle \\ &= \int_{\mathbb{M}^{m\times n}} \sigma_0(x,t,u,\lambda)d\nu_{(x,t)}(\lambda) \\ &= \sigma_0(x,t,u,.)o\int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda) \\ &= \sigma_0(x,t,u,Du),\end{aligned}$$

in $L^1(Q)$, by the equiintegrability of σ_0.

Taking $\varphi \in C^1(0,T;V_j) \cap L^\infty(Q;\mathbb{R}^m)$, $j \leq k$

$$\int_Q \frac{\partial u_k}{\partial t}\varphi dxdt + \int_Q \sigma(x,t,u_k,Du_k) : D\varphi dxdt + \int_Q \sigma_0(x,t,u_k,Du_k).\varphi dxdt = \int_Q f\varphi dxdt.$$

By integrating the first term and letting $j \to \infty$, it follows from the above results, that for $\varphi \in C^1(0,T;C_0^\infty(\Omega)) \cap L^\infty(Q;\mathbb{R}^m)$

$$-\int_Q u\frac{\partial \varphi}{\partial t}dxdt + \int_\Omega u\varphi dx\Big|_0^T + \int_Q \sigma(x,t,u,Du) : D\varphi dxdt + \int_Q \sigma_0(x,t,u,Du).\varphi dxdt = \int_Q f\varphi dxdt$$

as $k \to \infty$. Hence $u \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$ is in fact a weak solution for (1.1)-(1.3).

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 359–373
DOI: 10.24193/subbmath.2023.2.11

Generalized result on the global existence of positive solutions for a parabolic reaction diffusion model with a full diffusion matrix

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Abstract. In this paper, we study the global existence in time of solutions for a parabolic reaction diffusion model with a full matrix of diffusion coefficients on a bounded domain. The technique used is based on compact semigroup methods and some estimates. Our objective is to show, under appropriate hypotheses, that the proposed model has a global solution with a large choice of nonlinearities.

Mathematics Subject Classification (2010): 35K57, 37L05, 35K55, 35K40.

Keywords: Global existence, compact semigroups, reaction diffusion systems.

1. Introduction

Diffusion reaction systems have been among the most active and developed mathematical subjects for a long time, especially in recent years. The great interest of mathematicians in the study of this type of problems is due to its great importance in all fields of science and technology, where we find many applications in physics, chemistry, environment, biology and other disciplines. Examples include combustion problems, gas dynamics, population dynamics, industrial catalytic processes, chemistry in interstellar media, transport of contaminants in the environment, flame spread, spread of epidemics, pattern formation. We guide the reader to Britton [5], Fife [6], Murray [21], [22] and Pierre [24] where he finds many detailed mathematical models in biology, ecology, and others, and the reader can also find examples and other models in the references mentioned in this article and the references therein.

Received 11 March 2020; Accepted 13 April 2020.

To study reaction diffusion systems, we need a variety of different methods and techniques in many areas of mathematics, such as numerical analysis, semigroup theory, fixed point methods in appropriate spaces, and many others.

In the works of Mesbahi and Alaa [1], [2], [17] and [18], we find new developed methods based on truncation functions, fixed point theorems and compactness, etc.

Other techniques based mainly on invariant regions and Lyapunov functional have been developed by several authors, in some cases, allow to obtain the global existence of their reaction diffusion systems. The reader can see this technique in Kouachi's works, such as [11] and [12].

There is also another very powerful method that relies on compact semigroups, which is the method we will use in this work. For a better understanding, we send the reader to the works of Moumeni and Barrouk [19] and [20].

In recent years, particular attention has been paid to the reaction diffusion systems of two equations with diffusion coefficients and specific reaction functions. This is due to its broad applications in various sciences, particularly in biology and engineering.

In this paper, we study the existence and uniqueness of solutions for a parabolic reaction diffusion model with homogeneous boundary conditions of Neumann or Dirichlet. To answer these questions, we use a technique based on compact semigroups. To get a more complete survey the reader is refered to Lions [14], Pazy [23], Rothe [25] and Smoller [26].

We are therefore interested in the global existence in time of solutions for the following parabolic reaction diffusion model with homogeneous Neumann or Dirichlet boundary conditions

$$\frac{\partial u}{\partial t} - a\Delta u - b\Delta v = f(u,v) \quad , \text{ in } \mathbb{R}^+ \times \Omega \tag{1.1}$$

$$\frac{\partial v}{\partial t} - c\Delta u - d\Delta v = g(u,v) \quad , \text{ in } \mathbb{R}^+ \times \Omega \tag{1.2}$$

with the following boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \quad \text{or} \quad u = v = 0 \quad , \text{ on } \mathbb{R}^+ \times \partial\Omega \tag{1.3}$$

and the initial data

$$u(0,x) = u_0(x) \in L^1(\Omega) \quad , \quad v(0,x) = v_0(x) \in L^1(\Omega) \tag{1.4}$$

where Ω is an open bounded domain of class C^1 in $\mathbb{R}^n$, with boundary $\partial\Omega$, and $\dfrac{\partial}{\partial \eta}$ denotes the outward normal derivative on $\partial\Omega$. The diffusion coefficients a, b, c and d are supposed to be positive such that $a \leq d$, and $(b+c)^2 \leq 4ad$, which ensures the parabolicity of the system and implies that the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is positive definite, that is the eigenvalues λ_1 and λ_2 $(\lambda_1 < \lambda_2)$ of its transposed are positive.

Several authors have studied the problem proposed in the diagonal case, i.e. where $b = c = 0$, see for example Alikakos [3], Masuda [15], Haraux and Youkana [8].

In [19] and [20], Moumeni and Barrouk obtained a global existence result of solutions for reaction diffusion systems with a diagonal and triangular matrix of diffusion coefficents. By combining the compact semigroup methods and some L^1 estimates, we show the global existence of solutions for a large class of nonlinearities f and g.

In [12], Kouachi and Youkana have generalized the method of Haraux and Youkana in [8] to the triangular case, i.e. when $b = 0$.

In the same direction, Kouachi [11] has proved the global existence of solutions for two-component reaction diffusion systems with a general full matrix of diffusion coefficients, nonhomogeneous boundary conditions and polynomial growth conditions on the nonlinear terms and he obtained in [12] the global existence of solutions for the same system with homogeneous Neumann boundary conditions.

Mebarki and Moumeni [16] consider the problem (1.1)-(1.4) with $b > 0$ and $c > 0$, where the function f and g are assumed to satisfy

$$\sup \{|f(r,s)|, |g(r,s)|\} \leq C(r+s+1)^m, \forall r, s \geq 0$$

and by adopting the Lyapunov method combined with some L^p estimates, they established a result of global existence of the solution.

The system (1.1)-(1.2) is a mathematical model describing various chemical and biological phenomena. In this case the components $u(t,x)$ and $v(t,x)$ represent chemical concentrations or biological population densities of wells. The reader can find models similar to this in Britton [5], Fife [6], Murray [21], [22] and the references therein.

The rest of this paper is organized as follows: In the next section, we present some hypotheses on our problem and then state the main result. In the third section, we provide a result on local existence and another on compactness, they are necessary to fully understand the content of this work. We give in the fourth section some results concerning the approximate problem. The last section is devoted to prove the main result.

2. Formulation of the main result

2.1. Assumptions

We consider the problem (1.1)-(1.4) where we assume the following hypotheses: The Initial data are assumed in the following region

$$\Sigma = \left\{ (u_0, v_0) \in \mathbb{R}^2, \text{ such that } \frac{a-\lambda_2}{c} v_0 \leq u_0 \leq \frac{a-\lambda_1}{c} v_0 \right\} \tag{2.1}$$

and

$$\begin{aligned} f\left(\frac{a-\lambda_1}{c}\xi_2, \xi_2\right) &\leq \frac{a-\lambda_1}{c} g\left(\frac{a-\lambda_1}{c}\xi_2, \xi_2\right) \\ \frac{a-\lambda_2}{c} g\left(\frac{a-\lambda_2}{c}\xi_2, \xi_2\right) &\leq f\left(\frac{a-\lambda_2}{c}\xi_2, \xi_2\right) \end{aligned} \tag{2.2}$$

for all $(\xi_1, \xi_2) \in \Sigma$.

There exist nonnegative constants C, C_1 and C_2 independent of (ξ_1, ξ_2) such that

$$g(\xi_1, \xi_2) \leq C\xi_2 \quad , \text{ for all } \ (\xi_1, \xi_2) \in \Sigma \tag{2.3}$$

$$-f(\xi_1, \xi_2) + \frac{a-\lambda_1}{c} g(\xi_1, \xi_2) \leq C_1 \left(\frac{\lambda_2 - \lambda_1}{c} \xi_2 \right)^{C_2}, \quad \text{for all } \ (\xi_1, \xi_2) \in \Sigma \tag{2.4}$$

and

$$f(\xi_1, \xi_2) - \frac{a-\lambda_2}{c} g(\xi_1, \xi_2) \leq C_1 \left(\frac{\lambda_2 - \lambda_1}{c} \xi_2 \right)^{C_2}, \quad \text{for all } \ (\xi_1, \xi_2) \in \Sigma \tag{2.5}$$

2.2. The main result

Multiplying equation (1.2) one time through by $\frac{a-\lambda_1}{c}$ and subtracting equation (1.1) and another time by $-\frac{a-\lambda_2}{c}$ and adding equation (1.1) we get

$$\frac{\partial w}{\partial t} - \lambda_1 \Delta w = F(w, z) \quad , \text{ in } \mathbb{R}^+ \times \Omega \tag{2.6}$$

$$\frac{\partial z}{\partial t} - \lambda_2 \Delta z = G(w, z) \quad , \text{ in } \mathbb{R}^+ \times \Omega \tag{2.7}$$

$$\frac{\partial w}{\partial \eta} = \frac{\partial z}{\partial \eta} = 0 \quad \text{or} \quad w = z = 0 \quad , \text{ on } \mathbb{R}^+ \times \partial\Omega \tag{2.8}$$

$$w(0, x) = w_0(x) \quad \text{and} \quad z(0, x) = z_0(x) \quad , \text{ in } \Omega \tag{2.9}$$

where

$$\begin{aligned} w(t, x) &= -u(t, x) + \frac{a-\lambda_1}{c} v(t, x) \\ z(t, x) &= u(t, x) - \frac{a-\lambda_2}{c} v(t, x) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} F(w, z) &= -f(u, v) + \frac{a-\lambda_1}{c} g(u, v) \\ G(w, z) &= f(u, v) - \frac{a-\lambda_2}{c} g(u, v) \end{aligned}$$

Suppose that the hypotheses (2.1)-(2.5) are satisfied, then the problem (2.6)-(2.9) satisfies the following hypotheses:

$$w_0, z_0 \text{ are nonnegative functions in } L^1(\Omega) \tag{2.11}$$

$$F(0, z) \geq 0 \quad , \quad G(w, 0) \geq 0 \quad , \text{ for all } w, z \geq 0 \tag{2.12}$$

There exist nonnegative constants C, C_1 and C_2 independent of (w, z) such that

$$F(w, z) + G(w, z) \leq C(w + z) \quad , \text{ for all } \ (w, z) \in \mathbb{R}_+^2 \tag{2.13}$$

$$\begin{cases} F(w, z) \leq C_1 (w + z)^{C_2} \ \text{ for all } \ (w, z) \in \mathbb{R}_+^2 \\ G(w, z) \leq C_1 (w + z)^{C_2} \ \text{ for all } \ (w, z) \in \mathbb{R}_+^2 \end{cases} \tag{2.14}$$

The existence of global solutions for the system (2.6)-(2.9) is equivalent to the existence of (w, z) illustrated by the following main Theorem:

Theorem 2.1. *Assume that the hypotheses (2.11)-(2.14) are satisfied, then there exists a positive global solution (w, z) of the problem (2.6)-(2.9) in the following sense:*

$$\begin{cases} w, z \in C\left([0, +\infty[\,, L^1(\Omega)\right) \\ F(w, z)\,,\ G(w, z) \in L^1(Q_T) \ \text{ where } Q_T =]0, T[\times\Omega \text{ for all } T > 0 \\ w(t) = S_1(t)\, w_0 + \int_0^t S_1(t-s)\, F(w(s), z(s))\, ds\ ,\ \forall t \in [0, T[\\ z(t) = S_2(t)\, z_0 + \int_0^t S_2(t-s)\, G(w(s), z(s))\, ds\ ,\ \forall t \in [0, T[\end{cases} \tag{2.15}$$

where $S_1(t)$ and $S_2(t)$ are contraction semigroups in $L^1(\Omega)$ generated, respectively, by $\lambda_1\Delta$ and $\lambda_2\Delta$.

To prove this Theorem, we will use the results which we will present in the following section:

3. Preliminaries

3.1. Local existence

Theorem 3.1. *Let Ω is an open bounded domain in $\mathbb{R}^n$, and $X = L^1(\Omega) \cap H^2(\Omega)$. The operator A defined by*

$$\begin{cases} D(A) = \left\{ u \in L^1(\Omega) \cap H^2(\Omega)\ ,\ \dfrac{\partial u}{\partial \eta} = 0 \ \text{ or } \ u = 0 \quad \text{on } \partial\Omega \right\} \\ Au = \Delta u\ \ ,\ \text{for all } u \in D(A) \end{cases}$$

is m-dissipative in $L^1(\Omega) \cap H^2(\Omega)$.

An important result of functional analysis which ensures the local existence of the solution is the following Lemma:

Lemma 3.2. *Let A be a m-dissipative operator of dense domain in a Banach space X and $S(t)$ a contraction semigroup generated by A, F a locally Lipchitz function, so $\forall u_0 \in X$, there exists $T_{\max} = T(u_0)$ such that the problem*

$$\begin{cases} u \in C([0, T], D(A)) \cap C^1([0, T], X) \\ \dfrac{du}{dt} - Au = F(u(s)) \\ u(0) = u_0 \end{cases} \tag{3.1}$$

admits a unique solution u verifying

$$u(t) = S(t)\, u_0 + \int_0^t S(t-s)\, F(u(s))\, ds\ \ ,\ \ \forall t \in [0, T_{\max}]$$

3.2. Compactness result

In this section, we will give a compactness result of operator L defining the solution of the problem (3.1) in the case where the initial value equals zero, i.e. $u(0)=0$, with

$$L(F)(t)=u(t)=\int_0^t S(t-s)\,F(u(s))\,ds \quad , \quad \forall t\in[0,T]$$

Theorem 3.3. *If for all $t>0$, the operator $S(t)$ is compact, then L is compact of $L^1([0,T],X)$ in $L^1([0,T],X)$.*

Proof. **Step 1.** To show that $S(\lambda)L : F\to S(\lambda)L(F)$ is compact in $L^1([0,T],X)$, it suffices to prove that the set $\{S(\lambda)L(F)(t)\,;\ \|F\|_1\le 1\}$ is relatively compact in $L^1([0,T],X)\,,\,\forall t\in[0,T]$.

Since $S(t)$ is compact then, the application $t\to S(t)$ is continuous of $]0,+\infty[$ in $\pounds(X)$, therefore

$$\forall\varepsilon>0,\ \forall\delta>0,\ \exists\eta>0,\ \forall 0\le h\le\eta,\ \forall t\ge\delta,\ \ \|S(t+h)-S(t)\|_{\pounds(X)}\le\varepsilon$$

By choosing $\lambda=\delta$, we have for $0\le t\le T-h$

$$\begin{aligned}
&S(\lambda)u(t+h)-S(\lambda)u(t)\\
=&\int_0^{t+h} S(\lambda+t+h-s)\,F(u(s))\,ds-\int_0^t S(\lambda+t-s)\,F(u(s))\,ds\\
=&\int_t^{t+h} S(\lambda+t+h-s)\,F(u(s))\,ds+\\
&\int_0^t (S(\lambda+t+h-s)-S(\lambda+t-s))\,F(u(s))\,ds
\end{aligned}$$

We obtain then

$$\|S(\lambda)u(t+h)-S(\lambda)u(t)\|_X\le\int_t^{t+h}\|F(u(s))\|_X\,ds+\varepsilon\int_0^t\|F(u(s))\|_X\,ds$$

We define $v(t)$ by

$$v(t)=\begin{cases} u(t) & \text{if } 0\le t\le T\\ 0 & \text{otherwise}\end{cases}$$

therefore

$$\|S(\lambda)v(t+h)-S(\lambda)v(t)\|_1\le(h+\varepsilon T)\,\|F(u(s))\|_1$$

which implies that $\{S(\lambda)v,\ \ \|F\|_1\le 1\}$ is equi-integrable, then

$$\{S(\lambda)L(F)(t)\,,\ \ \|F\|_1\le 1\}$$

is relatively compact in $L^1([0,T],X)$, which means that $S(\lambda)L$ is compact.

Step 2. We prove that $S(\lambda)L$ converges to L when λ tends to 0 in $L^1([0,T],X)$. We have

$$S(\lambda)u(t)-u(t)=\int_0^t S(\lambda+t-s)\,F(u(s))\,ds-\int_0^t S(t-s)\,F(u(s))\,ds$$

So, for $t \geq \delta$, we have

$$\begin{aligned} \|S(\lambda) u(t) - u(t)\| &\leq \int_\delta^t \|S(\lambda + s) - S(s)\|_{\mathcal{L}(X)} \|F(u(s))\| \, ds \\ &\quad + 2 \int_{t-\delta}^t \|F(u(s))\| \, ds \end{aligned}$$

We choose $0 < \lambda < \eta$, then

$$\|S(\lambda) u(t) - u(t)\| \leq \varepsilon \int_\delta^t \|F(u(s))\| \, ds + 2 \int_{t-\delta}^t \|F(u(s))\| \, ds$$

and for $0 \leq t < \delta$, we have

$$\|S(\lambda) u(t) - u(t)\| \leq 2 \int_0^t \|F(u(s))\| \, ds$$

Since $F \in L^1(0, T, X)$, we obtain

$$\|S(\lambda) u(t) - u(t)\| \leq (\varepsilon T + 2\delta) \|F(u(s))\|_1$$

So if $\lambda \to 0$ then $S(\lambda) u \to u$ in $L^1([0, T], X)$.
The operator L is a uniform limit with compact linear operator between two Banach spaces, then L is compact in $L^1([0, T], X)$. ☐

Remark 3.4. The semigroup $S(t)$ generated by the operator Δ is compact in $L^1(\Omega)$.

Proof. See Pazy [23]. ☐

4. Approximating problem

For all $n > 0$, we define the functions w_{n_0} and z_{n_0} by

$$w_{n_0} = \min\{w_0, n\} \quad \text{and} \quad z_{n_0} = \min\{z_0, n\}$$

It is clear that w_{n_0} and z_{n_0} verify (2.11), i.e.

$$w_{n_0} \text{ and } z_{n_0} \text{ are nonnegative functions in } L^1(\Omega)$$

Now, we suppose the following problem

$$\begin{cases} \dfrac{\partial w_n}{\partial t} - \lambda_1 \Delta w_n = F(w_n, z_n) & \text{in } [0, T[\times \Omega \\ \dfrac{\partial z_n}{\partial t} - \lambda_2 \Delta z_n = G(w_n, z_n) & \text{in } [0, T[\times \Omega \\ \dfrac{\partial w_n}{\partial \eta} = \dfrac{\partial z_n}{\partial \eta} = 0 \ \text{ or } \ w_n = z_n = 0 & \text{on } [0, T[\times \partial\Omega \\ w_n(0, x) = w_{n_0}(x) \quad , \quad z_n(0, x) = z_{n_0}(x) & \text{in } \Omega \end{cases} \tag{4.1}$$

4.1. Local existence of the solution of problem (4.1)

We transform the system (4.1) into a first order system in the Banach space $X = L^1(\Omega) \times L^1(\Omega)$, we obtain

$$\begin{cases} \dfrac{\partial \omega_n}{\partial t} = A\omega_n + \Psi(\omega_n) & , \ t > 0 \\ \omega_n(0) = \omega_{n_0} = (w_{n_0}, z_{n_0}) \in X \end{cases} \tag{4.2}$$

Here $\omega_n = \mathrm{col}(w_n, z_n)$, the operator A is defined as follows

$$A = \begin{pmatrix} \lambda_1 \Delta & 0 \\ 0 & \lambda_2 \Delta \end{pmatrix}$$

where

$$D(A) := \{\omega_n = \mathrm{col}(w_n, z_n) \in X : \mathrm{col}(\Delta w_n, \Delta z_n) \in X\}$$

and the function Ψ is defined by

$$\Psi(\omega_n(t)) = \mathrm{col}(F(\omega_n(t)), G(\omega_n(t)))$$

Therefore, the system (4.2) can be returned to the form of the system (3.1), thus, if (w_n, z_n) is a solution of (4.2) then it checks the integral equations

$$\begin{cases} w_n(t) = S_1(t) w_{n_0} + \int_0^t S_1(t-s) F(w_n(s), z_n(s))\, ds \\ z_n(t) = S_2(t) z_{n_0} + \int_0^t S_2(t-s) G(w_n(s), z_n(s))\, ds \end{cases} \tag{4.3}$$

where $S_1(t)$ and $S_2(t)$ are the contraction semigroups generated, respectively, by $\lambda_1\Delta$ and $\lambda_2\Delta$.

Theorem 4.1. *There exist $T_M > 0$ and (w_n, z_n) a local solution of (4.2) for all $t \in [0, T_M]$.*

Proof. We know that $S_1(t)$, $S_2(t)$ are contraction semigroups and that Ψ is locally Lipschitz in ω_n, then there exists $T_M > 0$ such that (w_n, z_n) is a local solution of (4.2) on $[0, T_M]$. □

4.2. Positivity of the solution of problem (4.1)

Lemma 4.2. *Let (w_n, z_n) be a solution of problem (4.1), then the region*

$$\Sigma = \left\{(w_{n_0}, z_{n_0}) \in \mathbb{R}^2 \text{ such that } w_{n_0} \geq 0,\ z_{n_0} \geq 0\right\} = \mathbb{R}^+ \times \mathbb{R}^+$$

is invariant for system (4.1).

Proof. Let $\bar{w}_n(t,x) = 0$ in $]0, T[\times \Omega$, then $\dfrac{\partial \bar{w}_n}{\partial t} = 0$ and $\Delta \bar{w}_n = 0$.
According to (4.1), we have

$$\frac{\partial w_n}{\partial t} - \lambda_1 \Delta w_n - F(w_n, z_n) = 0 \geq \frac{\partial \bar{w}_n}{\partial t} - \lambda_1 \Delta \bar{w}_n - F(\bar{w}_n, z_n)$$

and

$$w_n(0,x) = w_{n_0}(x) \geq 0 = \bar{w}_n(0,x)$$

By comparison we get

$$w_n(t,x) \geq \bar{w}_n(t,x)$$

which gives us $w_n(t,x) \geq 0$. In the same way, we get $z_n(t,x) \geq 0$. □

4.3. Global existence of the solution of problem (4.1)

To prove the global existence of the solution of problem (4.1), it suffices to find an estimate of the solution for all $t \geq 0$, according to Haraux and Kirane [7], Henry [9] and Rothe [25]. For this, we give the following Lemma:

Lemma 4.3. *Let (w_n, z_n) be a solution of the problem (4.1), then there exists $M(t)$ which only depends on t, such that, for all $0 \leq t \leq T_M$, we have*

$$\|w_n + z_n\|_{L^1(\Omega)} \leq M(t)$$

Proof. From (4.1), it comes

$$\frac{\partial}{\partial t}(w_n + z_n) - \Delta(\lambda_1 w_n + \lambda_2 z_n) = F(w_n, z_n) + G(w_n, z_n)$$

and by taking into account of (2.13), we have

$$\frac{\partial}{\partial t}(w_n + z_n) - \Delta(\lambda_1 w_n + \lambda_2 z_n) \leq \hat{C}(w_n + z_n)$$

By integration on Ω and by applying Green's formula, we find

$$\frac{\partial}{\partial t}\int_\Omega (w_n + z_n)\, dx \leq C \int_\Omega (w_n + z_n)\, dx$$

which give

$$\frac{\frac{\partial}{\partial t}\int_\Omega (w_n + z_n)\, dx}{\int_\Omega (w_n + z_n)\, dx} \leq C$$

By integrating on $[0, t]$, we get

$$\log\left(\int_\Omega (w_n + z_n)\, dx\Big|_0^t\right) \leq Ct$$

which implies

$$\frac{\int_\Omega (w_n + z_n)\, dx}{\int_\Omega (w_{n_0} + z_{n_0})\, dx} \leq \exp(Ct)$$

and for $w_{n_0} \leq w_0$, $z_{n_0} \leq z_0$, we have

$$\int_\Omega (w_n + z_n)\, dx \leq \exp(Ct) \,.\, \int_\Omega (w_0 + z_0)\, dx$$

Since w_n and z_n are positive, we get

$$\|w_n + z_n\|_{L^1(\Omega)} \leq M(t) \quad , \; 0 \leq t \leq T_M$$

with

$$M(t) = \exp(Ct) \,.\, \|w_0 + z_0\|_{L^1(\Omega)}$$

□

We can conclude from this estimate that the solution (w_n, z_n) given by the Theorem 4.1 is a global solution.

Now, we give the following Lemma which shows the existence of an estimate of the solution (w_n, z_n) of the problem (4.1) in $L^1(Q)$.

Lemma 4.4. *For any solution (w_n, z_n) of (4.1), there exists a constant $K(t)$ depends only on t, such that*

$$\|w_n + z_n\|_{L^1(Q_T)} \leq K(t) \,.\, \|w_0 + z_0\|_{L^1(\Omega)}$$

Proof. To prove this Lemma, we will use some of the results demonstrated in the works of Bonafede and Schmitt [4] and Hollis et *al.* [10].

We introduce $\theta \in C_0^\infty(Q_T)$, $\theta \geq 0$, and $\Phi \in C^{1,2}(Q_T)$ a nonnegative solution of the following system

$$\begin{cases} -\dfrac{\partial \Phi}{\partial t} - d_1 \Delta \Phi = \theta & \text{on } Q_T \\ \dfrac{\partial \Phi}{\partial \eta} = 0 & \text{on } [0,T] \times \partial\Omega \\ \Phi(T, \cdot) = 0 & \text{on } \Omega \end{cases} \tag{4.4}$$

According to Ladyzenskaya et *al.* [13], the system (4.4) has a unique nonnegative solution. Moreover, for all $q \in \left]1, +\infty\right[$, there exists a nonnegative constant c independent of θ, such that,

$$\|\Phi\|_{L^q(Q_T)} \leq c\, \|\theta\|_{L^q(Q_T)}$$

According to Bonafede and Schmitt [4], we have

$$\int_{Q_T} S_1(t)\, w_{n_0}(x) \left(-\frac{\partial \Phi}{\partial t} - d_1 \Delta \Phi\right) dxdt = \int_\Omega w_{n_0}(x)\, \Phi(0,x)\, dx$$

and

$$\begin{aligned} & \int_{Q_T} \left(\int_0^t S_1(t-s)\, F(w_n, z_n)\, ds\right) \left(-\frac{\partial \Phi}{\partial t} - d_1 \Delta \Phi\right) dxdt \\ = & \int_{Q_T} F(w_n, z_n)\, \Phi(s,x)\, dxds \end{aligned}$$

where from

$$\int_{Q_T} (S_1(t)\, w_{n_0}(x))\, \theta dxdt = \int_\Omega w_{n_0}(x)\, \Phi(0,x)\, dx \tag{4.5}$$

and

$$\int_{Q_T} \left(\int_0^t S_1(t-s)\, F(w_n, z_n)\, ds\right) \theta dxdt = \int_{Q_T} F(w_n, z_n)\, \Phi(s,x)\, dxds \tag{4.6}$$

We multiply the first equation of (4.3) by θ, we integrate on Q_T, and using (4.5) and (4.6), we obtain

$$\begin{aligned} \int_{Q_T} w_n \theta dxdt &= \int_{Q_T} S_1(t)\, w_{n_0}(x)\, \theta dxdt \\ &\quad + \int_{Q_T} \left(\int_0^t S_1(t-s)\, F(w_n, z_n)\, ds\right) \theta dxdt \\ &= \int_\Omega w_{n_0}(x)\, \Phi(0,x)\, dx + \int_{Q_T} F(w_n, z_n)\, \Phi(s,x)\, dxds \end{aligned}$$

We also find

$$\int_{Q_T} z_n \theta dxdt = \int_{\Omega} z_{n_0}(x)\,\Phi(0,x)\,dx + \int_{Q_T} G(w_n, z_n)\,\Phi(s,x)\,dxds$$

and therefore

$$\begin{aligned}\int_{Q_T} (w_n + z_n)\,\theta dxdt &= \int_{\Omega} \left(w_{n_0}(x) + z_{n_0}(x)\right)\Phi(0,x)\,dx \\ &\quad + \int_{Q_T} \left(F(w_n, z_n) + G(w_n, z_n)\right)\Phi(s,x)\,dxds \\ &\leq \int_{\Omega} \left(w_0(x) + z_0(x)\right)\Phi(0,x)\,dx \\ &\quad + \int_{Q_T} C(w_n + z_n)\,\Phi(s,x)\,dxds\end{aligned}$$

Using Holder's inequality, we deduce

$$\begin{aligned}\int_{Q_T} (w_n + z_n)\,\theta dxdt &\leq \|w_0 + z_0\|_{L^1(\Omega)} \cdot \|\Phi(0,.)\|_{L^\infty(Q_T)} \\ &\quad + C\,\|w_n + z_n\|_{L^1(Q_T)} \cdot \|\Phi\|_{L^\infty(Q_T)} \\ &\leq k_1 \left(\|w_0 + z_0\|_{L^1(\Omega)} + \|w_n + z_n\|_{L^1(Q_T)}\right) \|\theta\|_{L^\infty(Q_T)}\end{aligned}$$

where $k_1 = \max\{c, cC\}$.
Since θ is arbitrary in $C_0^\infty(Q_T)$, this implies

$$\|w_n + z_n\|_{L^1(Q_T)} \leq k_1 \left(\|w_0 + z_0\|_{L^1(\Omega)} + \|w_n + z_n\|_{L^1(Q_T)}\right)$$

If we take $k = \dfrac{k_1(t)}{1 - k_1(t)}$, we find

$$\|w_n + z_n\|_{L^1(Q_T)} \leq k(t)\,.\,\|w_0 + z_0\|_{L^1(\Omega)}$$

□

5. Proof of the main result (Theorem 2.1)

We are now ready to prove the main result of this work:

Proof of theorem 2.1. We define the application L by

$$L : (w_0, h) \mapsto S_d(t)\,w_0 + \int_0^t S_d(t-s)\,h(s)\,ds$$

where $S_d(t)$ is the contraction semigroup generated by the operator $d\Delta$. According to the previous Theorem 3.3 and as $S_d(t)$ is compact, then the application L is the addition of two compact applications in $L^1(Q)$, which shows that L is also compact from $L^1(Q_T) \times L^1(Q_T)$ in $L^1(Q_T)$.

Therefore, there is a subsequence $\left(w_{n_j}, z_{n_j}\right)$ of (w_n, z_n) and (w, z) of $L^1(Q_T) \times L^1(Q_T)$, such that

$$\left(w_{n_j}, z_{n_j}\right) \to (w, z)$$

Let us now show that $\left(w_{n_j}, z_{n_j}\right)$ is a solution of (4.3). We have

$$\begin{cases} w_{n_j}(t,x) = S_1(t)\, w_{n_0} + \int_0^t S_1(t-s)\, F\left(w_{n_j}(s), z_{n_j}(s)\right) ds \\ z_{n_j}(t,x) = S_2(t)\, z_{n_0} + \int_0^t S_2(t-s)\, G\left(w_{n_j}(s), z_{n_j}(s)\right) ds \end{cases} \tag{5.1}$$

It suffices to show that (w, z) satisfies (2.15). It is clear that if $j \to +\infty$, we have the following limits

$$F\left(w_{n_j}, z_{n_j}\right) \to F(w,z) \quad \text{and} \quad G\left(w_{n_j}, z_{n_j}\right) \to G(w,z) \ \text{, a.e.} \tag{5.2}$$

and

$$w_{n_0} \to w_0 \quad , \quad z_{n_0} \to z_0$$

Thus, to show that (w, z) satisfies (2.15), we have to show that

$$F\left(w_{n_j}, z_{n_j}\right) \to F(w,z) \quad \text{and} \quad G\left(w_{n_j}, z_{n_j}\right) \to G(w,z) \ \text{ in } L^1(Q_T)$$

We integrate the two equations of (4.1) on Q_T taking into account that

$$-\lambda_1 \int_{Q_T} \Delta w_{n_j} dxdt = 0 \quad \text{and} \quad -\lambda_2 \int_{Q_T} \Delta z_{n_j} dxdt = 0$$

we have

$$\begin{aligned} \int_\Omega w_{n_j} dx - \int_\Omega w_{n_0} dx &= \int_{Q_T} F\left(w_{n_j}, z_{n_j}\right) dxdt \\ \int_\Omega z_{n_j} dx - \int_\Omega z_{n_0} dx &= \int_{Q_T} G\left(w_{n_j}, z_{n_j}\right) dxdt \end{aligned}$$

which give

$$-\int_{Q_T} F\left(w_{n_j}, z_{n_j}\right) dxdt \le \int_\Omega w_0 dx \tag{5.3}$$

$$-\int_{Q_T} G\left(w_{n_j}, z_{n_j}\right) dxdt \le \int_\Omega z_0 dx \tag{5.4}$$

We denote

$$\begin{aligned} N_n &= C_1\left(w_{n_j} + z_{n_j}\right)^{C_2} - F\left(w_{n_j}, z_{n_j}\right) \\ M_n &= C_1\left(w_{n_j} + z_{n_j}\right)^{C_2} - G\left(w_{n_j}, z_{n_j}\right) \end{aligned}$$

According to (2.13), it is clear that N_n and M_n are positive. From (5.3) and (5.4), we obtain

$$\begin{aligned} \int_{Q_T} N_n dxdt &\le C_1 \int_{Q_T} \left(w_{n_j} + z_{n_j}\right)^{C_2} dxdt + \int_\Omega w_0 dx \\ \int_{Q_T} M_n dxdt &\le C_1 \int_{Q_T} \left(w_{n_j} + z_{n_j}\right)^{C_2} dxdt + \int_\Omega z_0 dx \end{aligned}$$

The Lemma 4.4 gives us

$$\int_{Q_T} N_n dxdt < +\infty \quad \text{and} \quad \int_{Q_T} M_n dxdt < +\infty$$

which implies

$$\int_{Q_T} \left|F\left(w_{n_j}, z_{n_j}\right)\right| dxdt \leq C_1 \int_{Q_T} \left(w_{n_j} + z_{n_j}\right)^{C_2} dxdt + \int_{Q_T} N_n dxdt < +\infty$$

and

$$\int_{Q_T} \left|G\left(w_{n_j}, z_{n_j}\right)\right| dxdt \leq C_1 \int_{Q_T} \left(w_{n_j} + z_{n_j}\right)^{C_2} dxdt + \int_{Q_T} M_n dxdt < +\infty$$

The functions

$$\begin{aligned} \varphi_n &= N_n + C_1\left(w_{n_j} + z_{n_j}\right)^{C_2} \\ \psi_n &= M_n + C_1\left(w_{n_j} + z_{n_j}\right)^{C_2} \end{aligned}$$

are from $L^1(Q_T)$ and positive, moreover

$$\left|F\left(w_{n_j}, z_{n_j}\right)\right| \leq \varphi_n \quad \text{and} \quad \left|G\left(w_{n_j}, z_{n_j}\right)\right| \leq \psi_n \text{ a.e.}$$

We combine this result with (5.2) and by applying Lebesgue's dominated convergence Theorem, we obtain

$$F\left(w_{n_j}, z_{n_j}\right) \to F(w, z) \quad \text{and} \quad G\left(w_{n_j}, z_{n_j}\right) \to G(w, z) \quad \text{in } L^1(Q_T)$$

By passing to the limit of (5.1) when $j \to +\infty$ in $L^1(Q_T)$, we find

$$\begin{cases} w(t) = S_1(t)\, w_0 + \int_0^t S_1(t-s)\, F(w(s), z(s))\, ds \\ z(t) = S_2(t)\, z_0 + \int_0^t S_2(t-s)\, G(w(s), z(s))\, ds \end{cases}$$

which implies that (w, z) satisfies (2.15). Therefore (w, z) is a solution of (2.6)-(2.9). □

We conclude by (2.10) the existence in time of solutions of the reaction diffusion system (1.1)-(1.4).

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 375–385
DOI: 10.24193/subbmath.2023.2.12

Exponential growth of solutions with L_p-norm of a nonlinear viscoelastic wave equation with strong damping and source and delay terms

Abdelbaki Choucha and Djamel Ouchenane

Abstract. In this work, we are concerned with a problem for a viscoelastic wave equation with strong damping, nonlinear source and delay terms. We show the exponential growth of solutions with L_p-norm. i.e. $\lim_{t\to\infty} \|u\|_p^p \to \infty$.

Mathematics Subject Classification (2010): 35L05, 35L20, 58G16, 93D20.

Keywords: Strong damping, viscoelasticity, nonlinear source, exponential growth, delay.

1. Introduction

The well known "Growth" phenomenon is one of the most important phenomena of asymptotic behavior, where many researchers omit from its study especially when it comes from the evolution problems. It gives us very important information to know the behavior of the equation when time arrives at infinity, it differs from the Global existence and Blow up in both mathematically and in applications point of view. Although the interest of the scientific community for the study of delayed problems is fairly recent, multiple techniques have already been explored in depth. In this direction, we are concerned with the following problem

Received 18 May 2020; Accepted 06 July 2020.

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \displaystyle\int_0^t g(t-s)\Delta u(s)ds \\ \qquad +\mu_1 u_t + \mu_2 u_t(x,t-\tau) = b|u|^{p-2}.u, \quad x \in \Omega, t > 0, \\ u(x,t) = 0, \ x \in \partial\Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), \ \ (x,t) \in \Omega \times (0,\tau), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega, \end{cases} \tag{1.1}$$

where ω, b, μ_1 are positive constants, $p \geq 2$ and $\tau > 0$ is the time delay, and μ_2 is real number, and g is a differentiable function.

Viscous materials are the opposite of elastic materials that possess the ability to store and dissipate mechanical energy. As the mechanical properties of these viscous substances are of great importance when they appear in many applications of natural sciences. Many authors have given attention and attention to this problem since the beginning of the new millennium.

In the absence of the strong damping Δu_t, that is for $w = 0$, and in absence of the distributed delay term. Our problem (1.1) has been investigated by Berrimi and Messaoudi [2]. They established the local existence result by using the Galerkin method together with the contraction mapping theorem. Also, they showed that for a suitable initial data, then the local solution is global in time and in addition, they showed that the dissipation given by the viscoelastic integral term is strong enough to stabilize the oscillations of the solution with the same rate of decaying (exponential or polynomial) of the kernel g. Also their result has been obtained under weaker conditions than those used by Cavalcanti et al [4], in which a similar problem has been addressed. More precisely the authors in [5] looked into the following problem

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a(x)u_t + |u|^{\gamma}.u = 0 \tag{1.2}$$

the authors showed a decay result of an exponential rate. This later result has been improved by Berrimi and Messaoudi [2], in which they showed that the viscoelastic dissipation alone is strong enough to stabilize the problem even with an exponential rate. In many existing works on this field, under assumptions of the kernel g. For the problem (1.1) and with μ_1 concerning Cauchy problems, Kafini and Messaoudi [14] established a blow up result for the problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^{\infty} g(t-s)\Delta u(s)ds + u_t = |u|^{p-2}.u, \quad x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) \end{cases} \tag{1.3}$$

where g satisfies

$$\int_0^{\infty} g(s)ds < (2p-4)/(2p-3),$$

and the initial data were compactly supported with negative energy such that

$$\int u_0 u_1 dx > 0.$$

In the presence of the strong damping ($w > 0$). In [23], Song and Xue considered with the following viscoelastic equation with strong damping:

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^\infty g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{p-2}.u, \quad x \in \Omega, t > 0, \\ u(x,0) = u_0(x), \; u_t(x,0) = u_1(x) \end{cases} \tag{1.4}$$

They showed, under suitable conditions on g , that there were solutions of (1.4) with arbitrarily high initial energy that blow up in a finite time. For the same problem (1.4), in [24], Song and Zhong showed that there were solutions of (1.4) with positive initial energy that blew up in finite time. In [25], Zennir considered with the following viscoelastic equation with strong damping:

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \displaystyle\int_0^t g(t-s)\Delta u(s)ds \\ +a|u_t|^{m-2}.u_t = |u|^{p-2}.u, \quad x \in \Omega, t > 0, \\ u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \quad x \in \Omega \\ u(x,t) = 0, \quad x \in \partial\Omega. \end{cases} \tag{1.5}$$

they proved the exponential growth result under suitable assumptions.
In [17] the authors considered the following problem for a nonlinear viscoelastic wave equation with strong damping, nonlinear damping and source terms

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^\infty g(s)\Delta u(t-s)ds - \varepsilon_1 \Delta u_t + \varepsilon_2 u_t|u_t|^{m-2} = \varepsilon_3 u|u|^{p-2}, \\ u(x,t) = 0, \quad x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{1.6}$$

they proved a blow up result if $p > m$, and established the global existence.
In this article, we investigated problem (1.1), in which all the damping mechanism have been considered in the same time (i.e. $w > 0$; $g \neq 0$; and $\mu_1 > 0, \mu_2 \in L^\infty$), these assumptions make our problem different form those studied in the literature, specially the Exponential Growth of solutions. We will prove that if the initial energy $E(0)$ of our solutions is negative (this means that our initial data are large enough), then our local solutions in bounded and

$$\|u\|_p^p \to \infty \tag{1.7}$$

as t tends to $+\infty$ used idea in [25].
Our aim in the present work is to extend the existing Exponential Growth results to strong damping for a viscoelastic problem with delay under the following assumptions.
(**A1**) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable and decreasing function so that

$$g(t) \geq 0, \quad 1 - \int_0^\infty g(s)\,ds = l > 0. \tag{1.8}$$

(**A2**) There exists a constant $\xi > 0$ such that

$$g'(t) \leq -\xi g(t) \quad , \quad t \geq 0. \tag{1.9}$$

(**A3**) μ_2 is real number so that

$$|\mu_2| \leq \mu_1. \tag{1.10}$$

2. Main results

In this section, we prove the Exponential Growth result of solution of problem (1.1). First, as in [21], we introduce the new varible

$$z(x, \rho, t) = u_t(x, t - \tau\rho)$$

then we obtain

$$\begin{cases} \tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0 \\ z(x, 0, t) = u_t(x, t) \end{cases} \tag{2.1}$$

Let us denote by

$$gou = \int_\Omega \int_0^t g(t-s)|u(t) - u(s)|^2 ds. \tag{2.2}$$

Therefore, problem (1.1) takes the form:

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \displaystyle\int_0^t g(t-s)\Delta u(s) ds \\ +\mu_1 u_t + \mu_2 z(x, 1, t) = b|u|^{p-2}.u, \qquad x \in \Omega, t > 0, \\ \tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0 \end{cases} \tag{2.3}$$

with initial and boundary conditions

$$\begin{cases} u(x, t) = 0, \quad x \in \partial\Omega, \\ z(x, \rho, 0) = f_0\left(x, -\tau\rho\right), \\ u\left(x, 0\right) = u_0\left(x\right), \ u_t\left(x, 0\right) = u_1\left(x\right) \end{cases} \tag{2.4}$$

where

$$(x, \rho, t) \in \Omega \times (0, 1) \times (0, \infty)$$

We first state a local existence theorem that can be established by combining arguments of Georgiev and Todorova [10].

Theorem 2.1. *Assume (1.8),(1.9), and (1.10) holds. Let*

$$\begin{cases} 2 < p < \dfrac{2n-2}{n-2}, \qquad n \geq 3; \\ p \geq 2, \qquad\quad n = 1, 2 \end{cases} \tag{2.5}$$

Then for any initial data

$$(u_0, u_1, f_0) \in \mathcal{H} \ / \ \mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega \times (0, 1))$$

the problem (2.4) has a unique solution

$$u \in C([0, T]; \mathcal{H})$$

for some $T > 0$.

In the next theorem we give the global existence result, its proof based on the potential well depth method in which the concept of so-called stable set appears, where we show that if we restrict our initial data in the stable set, then our local solution obtained is global in time, that is to say, the norm

$$\|u_t\|_2 + \|\nabla u\|_2 \tag{2.6}$$

in the energy space $L^2(\Omega) \times H_0^1(\Omega)$ of our solution is bounded by a constant independent of the time t. We will make use of arguments in [22].

Theorem 2.2. *Suppose that (1.8),(1.9),(1.10), and (2.5) holds. If $u_0 \in W$, $u_1 \in H_0^1(\Omega)$ and*

$$\frac{bC_*^p}{l}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{2.7}$$

where C_* is the best Poincaré's constant. Then the local solution $u(t,x)$ is global in time.

The following lemma shows that the associated energy of the problem is nonincreasing under the condition (1.10), there exist ξ such that

$$\tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|), \quad |\mu_2| < \mu_1 \tag{2.8}$$

We introduce the energy functional

Lemma 2.3. *Assume (1.8),(1.9),(2.8) and (2.5) hold, let $u(t)$ be a solution of (2.3), then $E(t)$ is non-increasing, that is*

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(go\nabla u) \\ &\quad + \frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx - \frac{b}{p}\|u\|_p^p. \end{aligned} \tag{2.9}$$

satisfies

$$E(t) \le -c_1(\|u_t\|_2^2 + \int_\Omega z^2(x,1,t)dx) \tag{2.10}$$

Proof. By multiplying the equation $(2.3)_1$ by u_t and integrating over Ω, we get

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(go\nabla u) - \frac{b}{p}\|u\|_p^p\right\} \\ = &\ -\mu_1\|u_t\|_2^2 - \mu_2\int_\Omega u_t z(x,1,t)dx + \frac{1}{2}(g'o\nabla u) - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \omega\|\nabla u_t\|_2^2 \end{aligned} \tag{2.11}$$

and, multiplying $(2.3)_2$ by $\frac{\xi}{\tau}z$, we have

$$\begin{aligned} \frac{d}{dt}\frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx &= -\frac{\xi}{\tau}\int_\Omega\int_0^1 zz_\rho d\rho dx \\ &= -\frac{\xi}{2\tau}\int_\Omega[z^2(x,1,t) - z^2(x,0,t)]dx \\ &= \frac{\xi}{2\tau}\|u_t\|_2^2 - \frac{\xi}{2\tau}\int_\Omega z^2(x,1,t)dx. \end{aligned} \tag{2.12}$$

then, we get

$$
\begin{aligned}
\frac{d}{dt}E(t) \;&=\; -\mu_1\|u_t\|_2^2 - \mu_2\int_\Omega u_t z(x,1,t)dx + \frac{1}{2}(g'o\nabla u) \\
&\quad -\frac{1}{2}g(t)\|\nabla u\|_2^2 - \omega\|\nabla u_t\|_2^2 + \frac{\xi}{2\tau}\|u_t\|_2^2 \\
&\quad -\frac{\xi}{2\tau}\int_\Omega z^2(x,1,t)dx \qquad (2.13)
\end{aligned}
$$

By (2.11) and (2.12), we get (2.9).
And by using Young's inequality, (1.8),(1.9) in (2.13) , we get

$$
\frac{d}{dt}E(t) \le -\left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\|u_t\|_2^2 - \left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_\Omega z^2(x,1,t)dx \qquad (2.14)
$$

by (2.8) , we obtain (2.10). □

Now we are ready to state and prove our main result. For this purpose, we define

$$
\begin{aligned}
H(t) = -E(t) \;&=\; \frac{b}{p}\|u\|_p^p - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 \\
&\quad -\frac{1}{2}(go\nabla u) - \frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx.
\end{aligned} \qquad (2.15)
$$

Theorem 2.4. *Suppose that (1.8)-(1.10), and (2.5). Assume further that $E(0) < 0$ holds. Then the unique local solution of problem (2.3) grows exponentially.*

Proof. From (2.9), we have

$$
E(t) \le E(0) \le 0. \qquad (2.16)
$$

Hence

$$
\begin{aligned}
H'(t) = -E'(t) \;&\ge\; c_1\left(\|u_t\|_2^2 + \int_\Omega z^2(x,1,t)dx\right) \\
&\ge\; c_1\int_\Omega z^2(x,1,t)dx \ge 0. \qquad (2.17)
\end{aligned}
$$

and

$$
0 \le H(0) \le H(t) \le \frac{b}{p}\|u\|_p^p. \qquad (2.18)
$$

We set

$$
\mathcal{K}(t) = H(t) + \varepsilon\int_\Omega u u_t dx + \frac{\varepsilon\mu_1}{2}\int_\Omega u^2 dx + \frac{\varepsilon\omega}{2}\int_\Omega(\nabla u)^2 dx. \qquad (2.19)
$$

where $\varepsilon > 0$ to be specified later.
By multiplying $(2.3)_1$ by u and taking a derivative of (2.19), we obtain

$$
\begin{aligned}
\mathcal{K}'(t) \;&=\; H'(t) + \varepsilon\|u_t\|_2^2 + \varepsilon\int_\Omega \nabla u\int_0^t g(t-s)\nabla u(s)dsdx \\
&\quad -\varepsilon\|\nabla u\|_2^2 + \varepsilon b\int_\Omega |u|^p dx - \varepsilon\mu_2\int_\Omega u z(x,1,t)dx. \qquad (2.20)
\end{aligned}
$$

Using

$$\varepsilon\mu_2\int_\Omega uz(x,1,t)dx \le \varepsilon|\mu_2|\left\{\delta_1\|u\|_2^2+\frac{1}{4\delta_1}\int_\Omega z^2(x,1,t)dx\right\}. \tag{2.21}$$

and

$$\begin{aligned}\varepsilon\int_0^t g(t-s)ds\int_\Omega \nabla u.\nabla u(s)dxds &= \varepsilon\int_0^t g(t-s)ds\int_\Omega \nabla u.(\nabla u(s)-\nabla u(t))dxds\\ &\quad +\varepsilon\int_0^t g(s)ds\|\nabla u\|_2^2\\ &\ge \frac{\varepsilon}{2}\int_0^t g(s)ds\|\nabla u\|_2^2-\frac{\varepsilon}{2}(go\nabla u).\end{aligned} \tag{2.22}$$

we obtain, from (2.20),

$$\begin{aligned}\mathcal{K}'(t) &\ge H'(t)+\varepsilon\|u_t\|_2^2-\varepsilon\left(1-\frac{1}{2}\int_0^t g(s)ds\right)\|\nabla u\|_2^2+\varepsilon b\|u\|_p^p\\ &\quad -\varepsilon\delta_1|\mu_2|\|u\|_2^2-\frac{\varepsilon|\mu_2|}{4\delta_1}\int_\Omega z^2(x,1,t)dx\\ &\quad -\frac{\varepsilon}{2}(go\nabla u).\end{aligned} \tag{2.23}$$

Therefore, using (2.17) and by setting δ_1 so that

$$\frac{|\mu_2|}{4\delta_1 c_1}=\kappa,$$

substituting in (2.23), we get

$$\begin{aligned}\mathcal{K}'(t) &\ge [1-\varepsilon\kappa]H'(t)+\varepsilon\|u_t\|_2^2\\ &\quad -\varepsilon\left[\left(1-\frac{1}{2}\int_0^t g(s)ds\right)\right]\|\nabla u\|_2^2+\varepsilon b\|u\|_p^p\\ &\quad -\frac{\varepsilon|\mu_2|^2}{4c_1\kappa}\|u\|_2^2-\frac{\varepsilon}{2}(go\nabla u).\end{aligned} \tag{2.24}$$

For $0<a<1$, from (2.15),

$$\begin{aligned}\varepsilon b\|u\|_p^p &= \varepsilon p(1-a)H(t)+\frac{\varepsilon p(1-a)}{2}\|u_t\|_2^2+\varepsilon ba\|u\|_p^p\\ &\quad +\frac{\varepsilon p(1-a)}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2+\frac{\varepsilon}{2}p(1-a)(go\nabla u)\\ &\quad +\frac{\varepsilon p(1-a)\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx.\end{aligned} \tag{2.25}$$

substituting in (2.24), we get

$$\begin{aligned}\mathcal{K}'(t) \quad \geq \quad & [1-\varepsilon\kappa]H'(t)+\varepsilon\left[\frac{p(1-a)}{2}+1\right]\|u_t\|_2^2 \\ & +\varepsilon\left[\left(\frac{p(1-a)}{2}\right)\left(1-\int_0^t g(s)ds\right)-\left(1-\frac{1}{2}\int_0^t g(s)ds\right)\right]\|\nabla u\|_2^2 \\ & -\frac{\varepsilon|\mu_2|^2}{4c_1\kappa}\|u\|_2^2+\varepsilon p(1-a)H(t)+\varepsilon ba\|u\|_p^p \qquad (2.26) \\ & +\frac{\varepsilon p(1-a)\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx+\frac{\varepsilon}{2}[p(1-a)+1](go\nabla u)\end{aligned}$$

Using Poincaré's inequality, we obtain

$$\begin{aligned}\mathcal{K}'(t) \quad \geq \quad & [1-\varepsilon\kappa]H'(t)+\varepsilon\left[\frac{p(1-a)}{2}+1\right]\|u_t\|_2^2+\frac{\varepsilon}{2}(p(1-a)-1)(go\nabla u) \\ & +\varepsilon\left\{\left(\frac{p(1-a)}{2}-1\right)-\int_0^t g(s)ds(\frac{p(1-a)-1}{2})-\frac{c|\mu_2|^2}{4c_1\kappa}\right\}\|\nabla u\|_2^2 \\ & +\varepsilon ab\|u\|_p^p+\varepsilon p(1-a)H(t)+\frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx \qquad (2.27)\end{aligned}$$

At this point, we choose $a>0$ so small that

$$\alpha_1=\frac{p(1-a)}{2}-1>0$$

and assume

$$\int_0^\infty g(s)ds<\frac{\frac{p(1-a)}{2}-1}{\left(\frac{p(1-a)}{2}-\frac{1}{2}\right)}=\frac{2\alpha_1}{2\alpha_1+1} \qquad (2.28)$$

then we choose κ so large that

$$\alpha_2=(\frac{p(1-a)}{2}-1)-\int_0^t g(s)ds\left(\frac{p(1-a)-1}{2}\right)-\frac{c|\mu_2|^2}{4c_1\kappa}>0$$

Once κ and a are fixed, we pick ε so small enough so that

$$\alpha_4=1-\varepsilon\kappa>0$$

and

$$\mathcal{K}(t)\leq\frac{b}{p}\|u\|_p^p, \qquad (2.29)$$

Thus, for some $\beta>0$, estimate (2.27) becomes

$$\begin{aligned}\mathcal{K}'(t) \quad \geq \quad & \beta\{H(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+(go\nabla u)+\|u\|_p^p \\ & +\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\}. \qquad (2.30)\end{aligned}$$

and

$$\mathcal{K}(t)\geq\mathcal{K}(0)>0, \quad t>0. \qquad (2.31)$$

Next, using Young's and Poincaré's inequalities, from (2.19) we have

$$\begin{aligned}\mathcal{K}(t) &= \left(H+\varepsilon\int_\Omega uu_t dx+\frac{\varepsilon\mu_1}{2}\int_\Omega u^2dx+\frac{\varepsilon\omega}{2}\int_\Omega \nabla u^2 dx\right)\\ &\le c\left[H(t)+\left|\int_\Omega uu_t dx\right|+\|u\|_2^2+\|\nabla u\|_2^2\right]\\ &\le c[H(t)+\|\nabla u\|_2^2+\|u_t\|_2^2]. \end{aligned}\tag{2.32}$$

for some $c>0$: Since, $H(t)>0$, we have from (2.3)

$$\begin{aligned}&-\frac{1}{2}\|u_t\|_2^2-\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2-\frac{1}{2}(go\nabla u)\\ &-\frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx+\frac{b}{p}\|u\|_p^p>0. \end{aligned}\tag{2.33}$$

then

$$\begin{aligned}\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 &< \frac{b}{p}\|u\|_p^p<\frac{b}{p}\|u\|_p^p+(go\nabla u)\\ &\quad+\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx. \end{aligned}\tag{2.34}$$

In the other hand, using (1.8), to get

$$\begin{aligned}\frac{1}{2}(1-l)\|\nabla u\|_2^2 &< \frac{b}{p}\|u\|_p^p<\frac{b}{p}\|u\|_p^p+(go\nabla u)\\ &\quad+\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx. \end{aligned}\tag{2.35}$$

Consequently,

$$\begin{aligned}\|\nabla u\|_2^2 &< \frac{2b}{p}\|u\|_p^p+2(go\nabla u)+l\|\nabla u\|_2^2\\ &\quad+2\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx. \end{aligned}\tag{2.36}$$

Inserting (2.36) into (2.32), to see that there exists a positive constant k_1 such that

$$\begin{aligned}\mathcal{K}(t) &\le k_1[H(t)+\|\nabla u\|_2^2+\|u_t\|_2^2+\frac{b}{p}\|u\|_p^p+(go\nabla u)(t)\\ &\quad+\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx],\forall t>0. \end{aligned}\tag{2.37}$$

From inequalities (2.30) and (2.37) we obtain the differential inequality

$$\mathcal{K}'(t)\ge\lambda\mathcal{K}(t),\tag{2.38}$$

where $\lambda>0$, depending only on β and k_1.
a simple integration of (2.38), we obtain

$$\mathcal{K}(t)\ge\mathcal{K}(0)e^{(\lambda t)},\forall t>0\tag{2.39}$$

From (2.19) and (2.29), we have

$$\mathcal{K}(t) \leq \frac{b}{p}\|u\|_p^p. \tag{2.40}$$

By (2.39) and (2.40), we have

$$\|u\|_p^p \geq Ce^{(\lambda t)}, \forall t > 0$$

Therefore, we conclude that the solution in the L_p-norm growths exponentially. This completes the proof. □

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 387–398
DOI: 10.24193/subbmath.2023.2.13

A coupled system of fractional difference equations with anti-periodic boundary conditions

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Abstract. In this article, we give sufficient conditions for the existence, uniqueness and Ulam–Hyers stability of solutions for a coupled system of two-point nabla fractional difference boundary value problems subject to anti-periodic boundary conditions, using the vector approach of Precup [4, 14, 19, 21]. Some examples are included to illustrate the theory.

Mathematics Subject Classification (2010): 39A12, 39A70.

Keywords: Nabla fractional difference, boundary value problem, anti-periodic boundary conditions, existence, uniqueness, Ulam–Hyers stability.

1. Introduction

In [21], Precup described the advantage of vector-valued norms in the study of the semilinear operator system

$$\begin{cases} N_1(u_1, u_2) = u_1, \\ N_2(u_1, u_2) = u_2, \end{cases} \tag{1.1}$$

in a Banach space X with norm $|\cdot|$, by some methods of nonlinear analysis. Here N_1, $N_2 : X^2 \to X$ are given nonlinear operators. Obviously, this system can be viewed as a fixed point problem:

$$Nu = u, \tag{1.2}$$

Received 07 March 2020; Accepted 13 April 2020.

in the space X^2, where $u = (u_1, u_2)$ and $N = (N_1, N_2)$. Precup [21] proposed the applications of a few fixed point theorems to the system 1.1 in X^2, by using the vector-valued norm
$$\|u\| = \begin{pmatrix} |u_1| \\ |u_2| \end{pmatrix},$$
for $u = (u_1, u_2) \in X^2$. Also, Precup [21] demonstrated that the results obtained by using the vector-valued norm are better than those established by means of any scalar norm in X^2.

Theorem 1.1. [21] *Assume that*

(H1) *for each $i \in \{1, 2\}$, there exist nonnegative numbers a_i and b_i such that*
$$|N_i(u_1, u_2) - N_i(v_1, v_2)| \le a_i|u_1 - v_1| + b_i|u_2 - v_2|, \tag{1.3}$$
for all (u_1, u_2), $(v_1, v_2) \in X^2$;

(H2) *The spectral radius of $M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ is less than one.*

Then, (1.1) *has a unique solution $(u_1, u_2) \in X^2$.*

Theorem 1.2. [21] *Assume that*

(H3) *for each $i \in \{1, 2\}$, the operator N_i is completely continuous and, there exist nonnegative numbers a_i, b_i and c_i such that*
$$|N_i(u_1, u_2)| \le a_i|u_1| + b_i|u_2| + c_i, \tag{1.4}$$
for all $(u_1, u_2) \in X^2$.

In addition, assume that condition (H2) is satisfied. Then, (1.1) *has at least one solution $(u_1, u_2) \in X^2$ satisfying*
$$\begin{pmatrix} |u_1| \\ |u_2| \end{pmatrix} \le (I - M)^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{1.5}$$

Further, in [25], the author used the following theorem to establish Ulam–Hyers stability of solutions of (1.1):

Theorem 1.3. [25] *Assume that the hypothesis of Theorem 1.1 holds. Then, the system* (1.1) *is Ulam–Hyers stable.*

Motivated by these results, in this article, we consider the following coupled system of nabla fractional difference equations with anti-periodic boundary conditions
$$\begin{cases} \Big(\nabla_0^{\alpha_1 - 1}\big(\nabla u_1\big)\Big)(t) + f_1(u_1(t), u_2(t)) = 0, \quad t \in \mathbb{N}_2^T, \\ \Big(\nabla_0^{\alpha_2 - 1}\big(\nabla u_2\big)\Big)(t) + f_2(u_1(t), u_2(t)) = 0, \quad t \in \mathbb{N}_2^T, \\ u_1(0) + u_1(T) = 0, \quad \big(\nabla u_1\big)(1) + \big(\nabla u_1\big)(T) = 0, \\ u_2(0) + u_2(T) = 0, \quad \big(\nabla u_2\big)(1) + \big(\nabla u_2\big)(T) = 0, \end{cases} \tag{1.6}$$
and apply Theorems 1.1 - 1.3 to establish sufficient conditions on existence, uniqueness, and Ulam–Hyers stability [5, 6, 7, 17, 11, 13, 15, 22, 23, 24] of its solutions. For this purpose, we convert the system (1.6) in the form of (1.1). But the results may not be straightforward because the computation of nonnegative numbers in each theorem

for the system (1.6) is complicated due to the presence of nabla fractional difference operators in it.

Here $T \in \mathbb{N}_2$; $1 < \alpha_1, \alpha_2 < 2$; f_1, $f_2 : \mathbb{R}^2 \to \mathbb{R}$ are continuous, ∇_0^{ν} denotes the ν^{th}-th order Riemann–Liouville type backward (nabla) difference operator where $\nu \in \{\alpha_1 - 1, \alpha_2 - 1\}$ and ∇ denotes the first order nabla difference operator.

The present article is organized as follows: Section 2 contains preliminaries. In Section 3, we establish sufficient conditions on existence, uniqueness, and Ulam–Hyers stability of solutions of the system (1.6). We provide two examples in Section 4 to illustrate the applicability of established results.

2. Preliminaries

For our convenience, in this section, we present a few useful definitions and fundamental facts of nabla fractional calculus, which can be found in [1, 2, 3, 8, 9, 10, 16, 18, 20].

Denote by $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \ldots, b\}$ for any a, $b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$. The backward jump operator $\rho : \mathbb{N}_a \to \mathbb{N}_a$ is defined by $\rho(t) = \max\{a, t-1\}$, for all $t \in \mathbb{N}_a$. Define the μ^{th}-order nabla fractional Taylor monomial by

$$H_\mu(t,a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)} = \frac{\Gamma(t-a+\mu)}{\Gamma(t-a)\Gamma(\mu+1)}, \quad t \in \mathbb{N}_a, \quad \mu \in \mathbb{R} \setminus \{\ldots, -2, -1\}.$$

Here $\Gamma(\cdot)$ denotes the Euler gamma function. Observe that $H_\mu(a,a) = 0$ and $H_\mu(t,a) = 0$ for all $\mu \in \{\ldots, -2, -1\}$ and $t \in \mathbb{N}_a$. The first order backward (nabla) difference of $u : \mathbb{N}_a \to \mathbb{R}$ is defined by $\big(\nabla u\big)(t) = u(t) - u(t-1)$, for $t \in \mathbb{N}_{a+1}$.

Definition 2.1 (See [9]**).** Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u based at a is given by

$$\big(\nabla_a^{-\nu} u\big)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s)) u(s), \quad t \in \mathbb{N}_a,$$

where by convention $\big(\nabla_a^{-\nu} u\big)(a) = 0$.

Definition 2.2 (See [9]**).** Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $0 < \nu \leq 1$. The ν^{th}-order nabla difference of u is given by

$$\big(\nabla_a^{\nu} u\big)(t) = \Big(\nabla\big(\nabla_a^{-(1-\nu)} u\big)\Big)(t), \quad t \in \mathbb{N}_{a+1}.$$

Lemma 2.3 (See [9]**).** *We have the following properties of nabla fractional Taylor monomials.*

1. $\nabla H_\mu(t,a) = H_{\mu-1}(t,a)$, $t \in \mathbb{N}_a$.
2. $\sum_{s=a+1}^{t} H_\mu(s,a) = H_{\mu+1}(t,a)$, $t \in \mathbb{N}_a$.
3. $\sum_{s=a+1}^{t} H_\mu(t,\rho(s)) = H_{\mu+1}(t,a)$, $t \in \mathbb{N}_a$.

Proposition 2.4 (See [12]**).** *Let $s \in \mathbb{N}_a$ and $-1 < \mu$. The following properties hold:*

(a) $H_\mu(t,\rho(s)) \geq 0$ *for* $t \in \mathbb{N}_{\rho(s)}$ *and,* $H_\mu(t,\rho(s)) > 0$ *for* $t \in \mathbb{N}_s$.

(b) $H_\mu(t,\rho(s))$ *is a decreasing function with respect to* s *for* $t \in \mathbb{N}_{\rho(s)}$ *and* $\mu \in (0,\infty)$.
(c) *If* $t \in \mathbb{N}_s$ *and* $\mu \in (-1,0)$, *then* $H_\mu(t,\rho(s))$ *is an increasing function of* s.
(d) $H_\mu(t,\rho(s))$ *is a non-decreasing function with respect to* t *for* $t \in \mathbb{N}_{\rho(s)}$ *and* $\mu \in [0,\infty)$.
(e) *If* $t \in \mathbb{N}_s$ *and* $\mu \in (0,\infty)$, *then* $H_\mu(t,\rho(s))$ *is an increasing function of* t.
(f) $H_\mu(t,\rho(s))$ *is a decreasing function with respect to* t *for* $t \in \mathbb{N}_{s+1}$ *and* $\mu \in (-1,0)$.

Proposition 2.5 (See [12]**).** *Let* u *and* v *be two nonnegative real-valued functions defined on a set* S. *Further, assume* u *and* v *achieve their maximum values in* S. *Then,*

$$|u(t)-v(t)| \le \max\{u(t),v(t)\} \le \max\big\{\max_{t\in S} u(t), \max_{t\in S} v(t)\big\},$$

for every fixed t *in* S.

3. Green's function and its property

Assume $T \in \mathbb{N}_2$, $1<\alpha<2$, and $h : \mathbb{N}_2^T \to \mathbb{R}$. Consider the boundary value problem

$$\begin{cases} \Big(\nabla_0^{\alpha-1}\big(\nabla u\big)\Big)(t) + h(t) = 0, \quad t \in \mathbb{N}_2^T, \\ u(0)+u(T)=0, \quad \big(\nabla u\big)(1) + \big(\nabla u\big)(T) = 0. \end{cases} \tag{3.1}$$

First, we construct the Green's function, $G(t,s)$ corresponding to (3.1), and obtain an expression for its unique solution. Denote by

$$D_1 = \{(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T : t \ge s\}, \quad D_2 = \{(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T : t \le \rho(s)\},$$

and

$$\xi_\alpha = 2\left[1 + H_{\alpha-2}(T,0)\right]. \tag{3.2}$$

Theorem 3.1. *The unique solution of the nabla fractional boundary value problem* (3.1) *is given by*

$$u(t) = \sum_{s=2}^{T} G_\alpha(t,s)h(s), \quad t \in \mathbb{N}_0^T, \tag{3.3}$$

where

$$G_\alpha(t,s) = \begin{cases} K_\alpha(t,s) - H_{\alpha-1}(t,\rho(s)), & (t,s) \in D_1, \\ K_\alpha(t,s), & (t,s) \in D_2. \end{cases} \tag{3.4}$$

Here

$$K_\alpha(t,s) = \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(t,0)H_{\alpha-2}(T,\rho(s)) + H_{\alpha-1}(T,\rho(s))H_{\alpha-2}(T,0) - H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(s))\Big].$$

Proof. Denote by

$$\big(\nabla u\big)(t) = v(t), \quad t \in \mathbb{N}_1^T.$$

Subsequently, the difference equation in (3.1) takes the form

$$\big(\nabla_0^{\alpha-1} v\big)(t) + h(t) = 0, \quad t \in \mathbb{N}_2^T. \tag{3.5}$$

Let $v(1) = c_2$. Then, by Lemma 5.1 of [1], the unique solution of (3.5) is given by

$$v(t) = H_{\alpha-2}(t,0)c_2 - \big(\nabla_1^{-(\alpha-1)}h\big)(t), \quad t \in \mathbb{N}_1^T.$$

That is,

$$\big(\nabla u\big)(t) = H_{\alpha-2}(t,0)c_2 - \big(\nabla_1^{-(\alpha-1)}h\big)(t), \quad t \in \mathbb{N}_1^T. \tag{3.6}$$

Applying the first order nabla sum operator, ∇^{-1} on both sides of (3.6), we obtain

$$u(t) = c_1 + H_{\alpha-1}(t,0)c_2 - \big(\nabla_1^{-\alpha}h\big)(t), \quad t \in \mathbb{N}_0^T, \tag{3.7}$$

where $c_1 = u(0)$. We use the pair of anti-periodic boundary conditions considered in (3.1) to eliminate the constants c_1 and c_2 in (3.7). It follows from the first boundary condition $u(0) + u(T) = 0$ that

$$2c_1 + H_{\alpha-1}(T,0)c_2 = \big(\nabla_1^{-\alpha}h\big)(T). \tag{3.8}$$

The second boundary condition $\big(\nabla u\big)(1) + \big(\nabla u\big)(T) = 0$ yields

$$\left[1 + H_{\alpha-2}(T,0)\right] c_2 = \big(\nabla_1^{-(\alpha-1)}h\big)(T). \tag{3.9}$$

Solving (3.8) and (3.9) for c_1 and c_2, we obtain

$$c_1 = \frac{1}{2}\left[\sum_{s=2}^{T} H_{\alpha-1}(T,\rho(s))h(s) - \frac{2H_{\alpha-1}(T,0)}{\xi_\alpha}\sum_{s=2}^{T} H_{\alpha-2}(T,\rho(s))h(s)\right], \tag{3.10}$$

$$c_2 = \frac{2}{\xi_\alpha}\sum_{s=2}^{T} H_{\alpha-2}(T,\rho(s))h(s). \tag{3.11}$$

Substituting these expressions in (3.7), we achieve (3.4). □

Lemma 3.2. *Observe that*

$$|K_\alpha(t,s)| \le \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,1) + 2H_{\alpha-1}(T,0) + H_{\alpha-2}(T,0)H_{\alpha-1}(T,1)\Big], \tag{3.12}$$

for all $(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T$.

Proof. Denote by

$$K'_\alpha(t,s) = \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(t,0)H_{\alpha-2}(T,\rho(s)) + H_{\alpha-1}(T,\rho(s))H_{\alpha-2}(T,0)\Big], \tag{3.13}$$

and

$$K''_\alpha(t,s) = \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(s))\Big], \tag{3.14}$$

so that

$$K_\alpha(t,s) = K'_\alpha(t,s) - K''_\alpha(t,s), \quad (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T.$$

Clearly, from Proposition 2.4,

$$K'_\alpha(t,s) \ge 0, \quad K''_\alpha(t,s) > 0, \text{ for all } (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T.$$

From Proposition 2.5, it is obvious that

$$|K_\alpha(t,s)| \le \left\{ \max_{(t,s)\in\mathbb{N}_0^T\times\mathbb{N}_2^T} K'_\alpha(t,s), \max_{(t,s)\in\mathbb{N}_0^T\times\mathbb{N}_2^T} K''_\alpha(t,s) \right\}. \tag{3.15}$$

First, we evaluate the first backward difference of $K'_\alpha(t,s)$ with respect to t for a fixed s. Consider

$$\nabla K'_\alpha(t,s) = \frac{1}{\xi_\alpha}\Big[2H_{\alpha-2}(t,0)H_{\alpha-2}(T,\rho(s))\Big] > 0,$$

for all $(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T$, implying that $K'_\alpha(t,s)$ is an increasing function of t for a fixed s. Thus, we have

$$K'_\alpha(t,s) \le K'_\alpha(T,s), \quad (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T. \tag{3.16}$$

It follows from (3.13) - (3.16) that

$$\begin{aligned}
&|K_\alpha(t,s)| \\
&\le \left\{ \max_{(t,s)\in\mathbb{N}_0^T\times\mathbb{N}_2^T} K'_\alpha(t,s), \max_{(t,s)\in\mathbb{N}_0^T\times\mathbb{N}_2^T} K''_\alpha(t,s) \right\} \\
&\le \left\{ \max_{s\in\mathbb{N}_2^T} K'_\alpha(T,s), \max_{s\in\mathbb{N}_2^T} K''_\alpha(t,s) \right\} \\
&= \max_{s\in\mathbb{N}_2^T} K'_\alpha(T,s) \\
&= \frac{1}{\xi_\alpha} \max_{s\in\mathbb{N}_2^T} \Big[H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(s)) \\
&\quad + H_{\alpha-1}(T,\rho(s))H_{\alpha-2}(T,0)\Big] \\
&\le \frac{1}{\xi_\alpha}\Big[\max_{s\in\mathbb{N}_2^T} H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(T,0) \max_{s\in\mathbb{N}_2^T} H_{\alpha-2}(T,\rho(s)) \\
&\quad + H_{\alpha-2}(T,0) \max_{s\in\mathbb{N}_2^T} H_{\alpha-1}(T,\rho(s))\Big] \\
&= \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,\rho(2)) + 2H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(T)) + H_{\alpha-2}(T,0)H_{\alpha-1}(T,\rho(2))\Big] \\
&= \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,1) + 2H_{\alpha-1}(T,0) + H_{\alpha-2}(T,0)H_{\alpha-1}(T,1)\Big].
\end{aligned}$$

The proof is complete. □

4. Main results

Let $X = \mathbb{R}^{T+1}$ be the Banach space of all real $(T+1)$-tuples equipped with the maximum norm

$$|u| = \max_{t\in\mathbb{N}_0^T} |u(t)|.$$

Obviously, the product space X^2 is also a Banach space with the vector-norm

$$\|u\| = \begin{pmatrix} |u_1| \\ |u_2| \end{pmatrix},$$

for $u = (u_1, u_2) \in X^2$.
For our convenience, denote by

$$\Lambda_i = \frac{1}{\xi_{\alpha_i}} \Big[H_{\alpha_i - 1}(T, 1) + 2H_{\alpha_i - 1}(T, 0) + H_{\alpha_i - 2}(T, 0) H_{\alpha_i - 1}(T, 1) \Big], \tag{4.1}$$

$$a_i = l_i \left[\Lambda_i (T - 1) + H_{\alpha_i}(T, 1) \right], \tag{4.2}$$

$$b_i = m_i \left[\Lambda_i (T - 1) + H_{\alpha_i}(T, 1) \right], \tag{4.3}$$

$$c_i = n_i \left[\Lambda_i (T - 1) + H_{\alpha_i}(T, 1) \right], \tag{4.4}$$

for $i = 1, 2$.
Define the operator $T : X^2 \to X^2$ by

$$T(u_1, u_2)(t) = \begin{pmatrix} T_1(u_1, u_2)(t) \\ T_2(u_1, u_2)(t) \end{pmatrix}, \quad t \in \mathbb{N}_0^T, \tag{4.5}$$

where

$$\begin{aligned} & T_1(u_1, u_2)(t) \\ & = \sum_{s=2}^{T} G_{\alpha_1}(t, s) f_1(u_1(s), u_2(s)) \\ & = \sum_{s=2}^{T} K_{\alpha_1}(t, s) f_1(u_1(s), u_2(s)) - \sum_{s=2}^{t} H_{\alpha_1 - 1}(t, s) f_1(u_1(s), u_2(s)), \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} & T_2(u_1, u_2)(t) \\ & = \sum_{s=2}^{T} G_{\alpha_2}(t, s) f_2(u_1(s), u_2(s)) \\ & = \sum_{s=2}^{T} K_{\alpha_2}(t, s) f_2(u_1(s), u_2(s)) - \sum_{s=2}^{t} H_{\alpha_2 - 1}(t, s) f_2(u_1(s), u_2(s)). \end{aligned} \tag{4.7}$$

Theorem 4.1. *A couple* $(u_1, u_2) \in X^2$ *is a solution of* (1.6) *if, and only if,*

$$\begin{cases} T_1(u_1, u_2) = u_1, \\ T_2(u_1, u_2) = u_2. \end{cases} \tag{4.8}$$

In view of Theorem 4.1 it is enough to apply Theorems 1.1 - 1.3 to the system (4.8).

Theorem 4.2. *Assume that*

(I) *for each* $i \in \{1, 2\}$, *there exist nonnegative numbers* l_i *and* m_i *such that*

$$|f_i(u_1, u_2) - f_i(v_1, v_2)| \leq l_i |u_1 - v_1| + m_i |u_2 - v_2|, \tag{4.9}$$

for all (u_1, u_2), $(v_1, v_2) \in X^2$;

In addition, assume that condition (H2) is satisfied. Then, (4.8) *has a unique solution* $(u_1, u_2) \in X^2$.

Proof. For each $i \in \{1,2\}$ and for all (u_1,u_2), $(v_1,v_2) \in X^2$, consider

$$\begin{aligned}
&|T_i(u_1,u_2) - T_i(v_1,v_2)| \\
&\leq \sum_{s=2}^{T} |K_{\alpha_i}(t,s)|\,|f_i(u_1(s),u_2(s)) - f_i(v_1(s),v_2(s))| \\
&\quad + \sum_{s=2}^{t} H_{\alpha_i-1}(t,s)\,|f_i(u_1(s),u_2(s)) - f_i(v_1(s),v_2(s))| \\
&\leq [l_i|u_1-v_1| + m_i|u_2-v_2|] \left[\sum_{s=2}^{T} |K_{\alpha_i}(t,s)| + \sum_{s=2}^{t} H_{\alpha_i-1}(t,s)\right] \\
&\leq [l_i|u_1-v_1| + m_i|u_2-v_2|]\,[\Lambda_i(T-1) + H_{\alpha_i}(t,1)] \\
&\leq [l_i|u_1-v_1| + m_i|u_2-v_2|]\,[\Lambda_i(T-1) + H_{\alpha_i}(T,1)] \\
&\leq a_i|u_1-v_1| + b_i|u_2-v_2|,
\end{aligned}$$

implying that (H1) holds. Thus, by Theorem 1.1, the system (4.8) has a unique solution $(u_1,u_2) \in X^2$. □

Theorem 4.3. *Assume that*

(II) *for each $i \in \{1,2\}$, there exist nonnegative numbers a_i, b_i and c_i such that*

$$|f_i(u_1,u_2)| \leq l_i|u_1| + m_i|u_2| + n_i, \tag{4.10}$$

for all $(u_1,u_2) \in X^2$.

In addition, assume that condition (H2) is satisfied. Then, (4.8) *has at least one solution $(u_1,u_2) \in X^2$ satisfying* (1.5).

Proof. Since T_i, $i = 1,2$, is a summation operator on a discrete finite set, it is trivially completely continuous on X^2. For each $i \in \{1,2\}$ and for all $(u_1,u_2) \in X^2$, consider

$$\begin{aligned}
|T_i(u_1,u_2)| &\leq \sum_{s=2}^{T} |K_{\alpha_i}(t,s)|\,|f_i(u_1(s),u_2(s))| + \sum_{s=2}^{t} H_{\alpha_i-1}(t,s)\,|f_i(u_1(s),u_2(s))| \\
&\leq [l_i|u_1| + m_i|u_2| + n_i] \left[\sum_{s=2}^{T} |K_{\alpha_i}(t,s)| + \sum_{s=2}^{t} H_{\alpha_i-1}(t,s)\right] \\
&\leq [l_i|u_1| + m_i|u_2| + n_i]\,[\Lambda_i(T-1) + H_{\alpha_i}(t,1)] \\
&\leq [l_i|u_1| + m_i|u_2| + n_i]\,[\Lambda_i(T-1) + H_{\alpha_i}(T,1)] \\
&\leq a_i|u_1| + b_i|u_2| + c_i,
\end{aligned}$$

implying that (H3) holds. Thus, by Theorem 1.2, the system (4.8) has at least one solution $(u_1,u_2) \in X^2$ satisfying (1.5). □

Definition 4.4. [25] Let X be a Banach space and T_1, $T_2 : X \times X \to X$ be two operators. Then, the system (4.8) is said to be Ulam–Hyers stable if there exist C_1,

C_2, C_3, $C_4 > 0$ such that for each ε_1, $\varepsilon_2 > 0$ and each solution-pair $(u_1^*, u_2^*) \in X \times X$ of the in-equations:

$$\begin{cases} \|u_1 - T_1(u_1, u_2)\|_X \leq \varepsilon_1, \\ \|u_2 - T_2(u_1, u_2)\|_X \leq \varepsilon_2, \end{cases} \tag{4.11}$$

there exists a solution $(v_1^*, v_2^*) \in X \times X$ of (4.8) such that

$$\begin{cases} \|u_1^* - v_1^*\|_X \leq C_1\varepsilon_1 + C_2\varepsilon_2, \\ \|u_2^* - v_2^*\|_X \leq C_3\varepsilon_1 + C_4\varepsilon_2. \end{cases} \tag{4.12}$$

Theorem 4.5. *Assume that the hypothesis of Theorem 4.2 holds. Then, the system* (4.8) *is Ulam–Hyers stable.*

5. Examples

In this section, we provide two examples to illustrate the applicability of Theorem 4.2, Theorem 4.3, and Theorem 4.5.

Example 5.1. Consider the following boundary value problem for a coupled system of fractional difference equations

$$\begin{cases} \Big(\nabla_0^{0.5}\big(\nabla u_1\big)\Big)(t) + (0.001)\left[1 + \tan^{-1} u_1(t) + \tan^{-1} u_2(t)\right] = 0, \quad t \in \mathbb{N}_2^9, \\ \Big(\nabla_0^{0.5}\big(\nabla u_2\big)\Big)(t) + (0.002)\left[1 + \sin u_1(t) + \sin u_2(t)\right] = 0, \quad t \in \mathbb{N}_2^9, \\ u_1(0) + u_1(9) = 0, \quad \big(\nabla u_1\big)(1) + \big(\nabla u_1\big)(9) = 0, \\ u_2(0) + u_2(9) = 0, \quad \big(\nabla u_2\big)(1) + \big(\nabla u_2\big)(9) = 0. \end{cases} \tag{5.1}$$

Comparing (1.6) and (5.1), we have $T = 9$, $\alpha_1 = \alpha_2 = 1.5$,

$$f_1(u_1, u_2) = (0.001)\left[1 + \tan^{-1} u_1 + \tan^{-1} u_2\right],$$

and

$$f_2(u_1, u_2) = (0.002)\left[1 + \sin u_1 + \sin u_2\right],$$

for all $(u_1, u_2) \in \mathbb{R}^2$. Clearly, f_1 and f_2 are continuous on $\mathbb{R}^2$. Next, f_1 and f_2 satisfy assumption (I) with $l_1 = 0.001$, $m_1 = 0.001$, $l_2 = 0.002$ and $m_2 = 0.002$. We have,

$$\begin{aligned} a_1 &= l_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T,1)\right] = 0.0527, \\ a_2 &= l_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T,1)\right] = 0.1053, \\ b_1 &= m_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T,1)\right] = 0.0527, \\ b_2 &= m_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T,1)\right] = 0.1053. \end{aligned}$$

Further,

$$M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 0.0527 & 0.0527 \\ 0.1053 & 0.1053 \end{pmatrix}.$$

The spectral radius of M is 0.158, which is less than one, implying that M converges to zero. Hence, by Theorem 4.2, the system (5.1) has a unique solution $(u_1, u_2) \in X^2$. Also, by Theorem 4.5, the unique solution of (5.1) is Ulam–Hyers stable.

Example 5.2. Consider the following boundary value problem for a coupled system of fractional difference equations

$$\begin{cases} \left(\nabla_0^{0.5}\left(\nabla u_1\right)\right)(t) + (0.01)\left[1 + \frac{1}{\sqrt{1+u_1^2(t)}} + u_2(t)\right] = 0, & t \in \mathbb{N}_2^4, \\ \left(\nabla_0^{0.5}\left(\nabla u_2\right)\right)(t) + (0.02)\left[1 + u_1(t) + \frac{1}{\sqrt{1+u_2^2(t)}}\right] = 0, & t \in \mathbb{N}_2^4, \\ u_1(0) + u_1(4) = 0, \quad \left(\nabla u_1\right)(1) + \left(\nabla u_1\right)(4) = 0, \\ u_2(0) + u_2(4) = 0, \quad \left(\nabla u_2\right)(1) + \left(\nabla u_2\right)(4) = 0. \end{cases} \tag{5.2}$$

Comparing (1.6) and (5.2), we have $T = 4$, $\alpha_1 = \alpha_2 = 1.5$,

$$f_1(u_1, u_2) = (0.01)\left[1 + \frac{1}{\sqrt{1 + u_1^2(t)}} + u_2(t)\right],$$

and

$$f_2(u_1, u_2) = (0.02)\left[1 + u_1(t) + \frac{1}{\sqrt{1 + u_2^2(t)}}\right],$$

for all $(u_1, u_2) \in \mathbb{R}^2$. Clearly, f_1 and f_2 are continuous on $\mathbb{R}^2$. Next, f_1 and f_2 satisfy assumption (II) with $l_1 = 0.01$, $m_1 = 0.01$, $l_2 = 0.02$, $m_2 = 0.02$, $n_1 = 0.01$ and $n_2 = 0.02$. We have,

$$\begin{aligned} a_1 &= l_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T,1)\right] = 0.1219, \\ a_2 &= l_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T,1)\right] = 0.2438, \\ b_1 &= m_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T,1)\right] = 0.1219, \\ b_2 &= m_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T,1)\right] = 0.2438, \\ c_1 &= n_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T,1)\right] = 0.1219, \\ c_2 &= n_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T,1)\right] = 0.2438. \end{aligned}$$

Further,

$$M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 0.1219 & 0.1219 \\ 0.2438 & 0.2438 \end{pmatrix}.$$

The spectral radius of M is 0.3657, which is less than one, implying that M converges to zero. Hence, by Theorem 4.3, the system (5.2) has at least one solution $(u_1, u_2) \in X^2$ satisfying

$$\begin{pmatrix} |u_1| \\ |u_2| \end{pmatrix} \leq (I - M)^{-1}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0.1757 \\ 0.2658 \end{pmatrix}.$$

Conclusion

In this article, we obtained sufficient conditions on existence, uniqueness and Ulam–Hyers stability of solutions of the system (1.6) using the approaches of Precup and Urs. We also provided two examples to demonstrate the applicability of established results. Observe that Theorem 4.2 is not applicable to the system (5.1).

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 399–408
DOI: 10.24193/subbmath.2023.2.14

Statistical Korovkin-type theorem for monotone and sublinear operators

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Abstract. In this paper we generalize the result on statistical uniform convergence in the Korovkin theorem for positive and linear operators in $C([a,b])$, to the more general case of monotone and sublinear operators. Our result is illustrated by concrete examples.

Mathematics Subject Classification (2010): 41A35, 41A36, 41A63.

Keywords: Korovkin-type theorems, monotone and sublinear operator, nonlinear Choquet integral, statistical convergence.

1. Introduction

The celebrated theorem of Korovkin [29], [30] provides a very simple test of strong operator convergence to the identity for any sequence $(T_n)_n$ of positive linear operators that map $C\left([0,1]\right)$ into itself: the occurrence of this convergence for the functions $e_0(t)=1, e_1(t)=t$ and $e_2(t)=t^2$, $t\in[0,1]$. In other words, the fact that

$$\lim_{n\to\infty} T_n(f)=f \quad \text{uniformly on } [0,1],$$

for every $f\in C\left([0,1]\right)$ reduces to the status of the three aforementioned functions. Due to its simplicity and usefulness, this result has attracted a great deal of attention leading to numerous generalizations. Part of them are included in the authoritative monograph of Altomare-Campiti [7] and the excellent survey of Altomare [6].

Recently, the present authors have extended the Korovkin theorem to the framework of monotone and sublinear operators acting on function spaces endowed with the topology of uniform convergence on compact sets. See Gal-Niculescu [24], [25], [27].

Received 18 April 2022; Accepted 05 February 2023.

Let D be a subset of $\mathbb{N}$, the set of all natural numbers. The density of D is defined by

$$\delta(D) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_D(j),$$

whenever the limit exists, where χ_D is the characteristic function of D.

The sequence $(\alpha_k)_k$ is statistically convergent to the number L if, for every $\varepsilon > 0$, we have $\delta\{k \in \mathbb{N} : |\alpha_k - L| \geq \varepsilon\} = 0$, (see Conor [10]) or equivalently, there exists a subset $K \subset \mathbb{N}$ with $\delta(K) = 1$ and $n_0(\varepsilon)$ such that $k > n_0$ and $k \in K$ imply that $|\alpha_k - L| < \varepsilon$, see Šalát [34]. In this case we write $st - \ lim \quad \alpha_k = L$. It is known that any convergent sequence is statistically convergent, but not conversely. For example, the sequence defined by $\alpha_n = \sqrt{n}$ if n is square and $\alpha_n = 0$ otherwise, has the property that $st - \ lim \alpha_n = 0$.

Some basic properties of statistical convergence are exhibited in Connor [10], Salat [34], Schoenberg [35]. Over the years this concept has been examined in number theory Erdös - Jenenbaüm [16], trigonometric series Zygmund [37], probability theory Fridy-Khan [19], optimization Pehlivan-Mamedov [33], measure theory Miller [32] and summability theory Connor [10], Fridy [18], Fridy-Orhan [20].

Korovkin type theorems for statistical convergence of positive and linear operators were obtained by many authors, to make a selection see, e.g., Gadjev [22], [21], Agratini [1]-[4], Cárdenas-Morales - Garancho [8], Dirik [13], Duman-Khan-Orhan [14], Duman [15], Akdağ [5] and the references therein.

Since evidently that a positive linear operator is monotone and sublinear, it is the purpose of this paper to generalize the result on statistical uniform convergence in the Korovkin theorem for positive and linear operators, to monotone and sublinear operators.

2. Preliminaries on weakly nonlinear operators and on Choquet integral

In what follows we denote by X a metric measure space that is, a triple (X, d, m) consisting of a space X endowed with the metric d and the measure m defined on the sigma field of Borel subsets of X. Notice that every metric space can be seen as a metric measure considering on it any finite combination (with positive coefficients) of Dirac measures.

Attached to it is the vector lattice $\mathcal{F}(X)$ of all real-valued functions defined on X, endowed with the pointwise ordering. Some important vector sublattices of $\mathcal{F}(X)$ are

$$B(X) = \{f \in \mathcal{F}(X) : \ f \text{ bounded}\}.$$

$$C(X) = \{f \in \mathcal{F}(X) : \ f \text{ continuous and bounded}\}.$$

On $B(X)$ and $C(X)$ one considers the uniform norm $\|f\| = \sup\{|f(x)|; x \in [a,b]\}$.

Suppose that X and Y are two metric spaces and E and F are respectively ordered vector subspaces (or the positive cones) of $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ that contain the unity. An operator $T : E \to F$ is said to be a *weakly nonlinear operator* (respectively a *weakly nonlinear functional* when $F = \mathbb{R}$) if it satisfies the following three conditions:

(SL) (*Sublinearity*) T is subadditive and positively homogeneous, that is,

$$T(f+g) \leq T(f) + T(g) \quad \text{and} \quad T(af) = aT(f)$$

for all f, g in E and $a \geq 0$;

(M) (*Monotonicity*) $f \leq g$ in E implies $T(f) \leq T(g)$.

(TR) (*Translatability*) $T(f + \alpha \cdot 1) = T(f) + \alpha T(1)$ for all functions $f \in E$ and all numbers $a \geq 0$.

A stronger condition than translatability is that of *comonotonic additivity*,

(CA) $T(f+g) = T(f) + T(g)$ whenever the functions $f, g \in E$ are comonotone in the sense that

$$(f(s) - f(t)) \cdot (g(s) - g(t)) \geq 0 \quad \text{for all } s, t \in X.$$

The (CA) condition occurs naturally in the context of Choquet's integral (and thus in the case of Choquet type operators). See Gal-Niculescu [25], [26] and the references therein.

Suppose that E and F are respectively closed vector sublattices of the Banach lattices $C(X)$ and $C(Y)$.

Every monotone and subadditive operator (functional when $F = \mathbb{R}$) $T : E \to F$ verifies the inequality

$$|T(f) - T(g)| \leq T(|f-g|) \text{ for all } f, g. \tag{2.1}$$

Indeed, $f \leq g + |f-g|$ yields $T(f) \leq T(g) + T(|f-g|)$, that is,

$$T(f) - T(g) \leq T(|f-g|),$$

and interchanging the role of f and g we infer that

$$-(T(f) - T(g)) \leq T(|f-g|).$$

If T is linear, then the property of monotonicity is equivalent to that of *positivity*, that is, to the fact that

$$T(f) \geq 0 \quad \text{for all } f \geq 0.$$

If the operator (functional when $F = \mathbb{R}$) T is monotone and positively homogeneous, then necessarily

$$T(0) = 0.$$

The properties of weakly nonlinear operators were suggested by those of the nonlinear functional called Choquet integral. For this reason we shortly mention them below. Full details on this integral can be found in the books of D. Denneberg [12], M. Grabisch [28] and Z. Wang and G. J. Klir [36].

Let $(X, \mathcal{A})$ be an arbitrarily fixed measurable space, consisting of a nonempty abstract set X and a σ-algebra $\mathcal{A}$ of subsets of X.

Definition 2.1. (see, e.g., Denneberg [12] or Wang-Klir [36]) A set function $\mu : \mathcal{A} \to [0,1]$ is called a capacity if it verifies the following two conditions:

(a) $\mu(\emptyset) = 0$;

(b) $\mu(A) \leq \mu(B)$ for all $A, B \in \mathcal{A}$, with $A \subset B$ (monotonicity).

An important class of capacities is that of probability measures (that is, the capacities playing the property of σ-additivity). Probability distortions represents a major source of nonadditive capacities. Technically, one start with a probability measure $P : \mathcal{A} \rightarrow [0,1]$ and applies to it a distortion $u : [0,1] \rightarrow [0,1]$, that is, a nondecreasing and continuous function such that $u(0) = 0$ and $u(1) = 1$;for example, one may chose $u(t) = t^a$ with $\alpha > 0$. When the distortion u is concave (for example, when $u(t) = t^a$ with $0 < \alpha < 1$), then μ is also *submodular* in the sense that

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B) \quad \text{for all } A, B \in \mathcal{A}.$$

The Choquet concept of integrability with respect to a capacity refers to the whole class of random variables, that is, to all functions $f : X \to \mathbb{R}$ such that $f^{-1}(A) \in \mathcal{A}$ for every Borel subset A of $\mathbb{R}$.

Definition 2.2. (see, e.g., Denneberg [12] or Wang-Klir [36]) The Choquet integral of a random variable f with respect to the capacity μ is defined as the sum of two Riemann improper integrals,

$$(C)\int_X f d\mu = \int_0^{+\infty} \mu\left(\{x \in X : f(x) \geq t\}\right) dt + \int_{-\infty}^{0} \left[\mu\left(\{x \in X : f(x) \geq t\}\right) - 1\right] dt,$$

Accordingly, f is said to be Choquet integrable if both integrals above are finite.

If $f \geq 0$, then the last integral in the formula appearing in Definition 2.2 is 0.

The inequality sign $\geq$ in the above two integrands can be replaced by $>$; see [36], Theorem 11.1, p. 226.

The Choquet integral coincides with the Lebesgue integral when the underlying set function μ is a σ-additive measure.

As usually, a function f is said to be Choquet integrable on a set $A \in \mathcal{A}$ if $f\chi_A$ is integrable in the sense of Definition 2.2. We denote

$$(C)\int_A f d\mu = (C)\int_X f\chi_A d\mu.$$

The basic properties of the Choquet integral, seen as a functional are as follows: it is monotone, positive homogenous, comonotonic additive and subadditive (if μ is submodular).

Remark 2.3. Several extensions of Korovkin's theorem in the case of weakly nonlinear operators acting on a sublattice of a space $C(X)$ and the uniform convergence on compact sets can be found in the papers Gal-Niculescu [24], [25], [27].

In the next section we discuss an analogue in the space $C([a,b])$ and for statistical uniform convergence.

3. Main result, uniform convergence case

In this section we obtain an analogue result with the classical Korovkin theorem in $C([a,b])$ for the statistical uniform convergence of a sequence of monotone and sublinear operators.

Theorem 3.1. *If the sequence of monotone and sublinear operators* $A_n : C([a,b]) \to B([a,b])$ *satisfies the conditions*

$$st - \lim \|A_n(e_0) - e_0\| = 0; st - \lim \|A_n(e_1) - e_1\| = 0,$$

$$st - \lim \|A_n(e_2) - e_2\| = 0; st - \lim \|A_n(-e_1) + e_1\| = 0, \tag{3.1}$$

then for any nonnegative function $f \in C([a;b])$*, we have*

$$st - \lim \|A_n(f) - f\| = 0. \tag{3.2}$$

If, in addition, all A_n *are translatable, then the above convergence holds for all* $f \in C([a,b])$*. Here* $\|\cdot\|$ *denotes the uniform norm.*

Proof. Suppose firstly that $f \in C([a,b])$ is nonnegative on $[a,b]$. Since f is bounded, we can write

$$|f(t) - f(x)| \leq 2M, \text{ for all } t, x \in [a,b].$$

Also, since f is continuous on $[a,b]$, it follows that there exists a $\delta > 0$ (depending on ε) such that $|f(t) - f(x)| < \varepsilon$ for all $t, x \in [a,b]$ satisfying $|x-t| < \delta$, which implies that for all $t, x \in [a,b]$ we obtain

$$|f(t) - f(x)| \leq \varepsilon + \frac{2M}{\delta^2}(t-x)^2 = \varepsilon + \frac{2M}{\delta^2}(t^2 - 2xt + x^2). \tag{3.3}$$

We have two cases:
Case 1. $x \in [a,b],\ x \leq 0.$
Case 2. $x \in [a,b],\ x > 0.$

Applying A_n to (3.3), by the sublinearity of A_n and by the property (2.1), since $-2x \geq 0$, in Case 1 it follows

$$A_n(|f - f(x)|)(x) \leq \varepsilon A_n(e_0)(x) + \frac{2M}{\delta^2} A_n((e_1 - x)^2)(x) \leq \varepsilon A_n(e_0)(x)$$

$$+\frac{2M}{\delta^2}(A_n(e_2)(x) - x^2 + x^2 - 2xA_n(e_1)(x) + 2x^2 - 2x^2 + x^2 A_n(e_0)(x) - x^2 + x^2)$$

$$\leq \varepsilon \|A_n(e_0) - e_0\| + \varepsilon$$

$$+\frac{2M}{\delta^2}(\|A_n(e_2) - e_2\| + 2|x| \cdot \|A_n(e_1) - e_1\| + x^2 \|A_n(e_0) - e_0\|),$$

which by

$$A_n(f)(x) - f(x) = A_n(f)(x) - A_n(f(x))(x) + f(x)(A_n(e_0)(x) - e_0(x)),$$

immediately implies

$$|A_n(f)(x) - f(x)| \leq |A_n(f)(x) - A_n(f(x))(x)| + |f(x)| \cdot |A_n(e_0)(x) - e_0(x)|,$$

and therefore

$$\|A_n(f) - f\| \leq \|A_n(|f - f(x)|)\| + M \cdot \|A_n(e_0) - e_0\|$$

$$\leq (\varepsilon + M + \frac{2M\alpha^2}{\delta^2})\|A_n(e_0) - e_0\| + \frac{4M\alpha}{\delta^2}\|A_n(e_1) - e_1\| + \frac{2M}{\delta^2}\|A_n(e_2) - e_2\|$$

$$\leq C(\|A_n(e_0) - e_0\| + \|A_n(e_1) - e_1\| + \|A_n(e_2) - e_2\|), \tag{3.4}$$

where $C = \max\{\varepsilon + M + \frac{2M\alpha^2}{\delta^2}, \frac{4M\alpha}{\delta^2}\}$ and $\alpha = \max\{|a|, |b|\}$.

In the Case 2, since $-2x<0$ and applying the positive homogeneity of A_n too, it follows

$$A_n(|f-f(x)|)(x) \le \varepsilon A_n(e_0)(x) + \frac{2M}{\delta^2}A_n((e_1-x)^2)(x) \le \varepsilon A_n(e_0)(x)$$

$$+\frac{2M}{\delta^2}(A_n(e_2)(x)-x^2+x^2+2xA_n(-e_1)(x)+2x^2-2x^2+x^2A_n(e_0)(x)-x^2+x^2)$$

$$\le \varepsilon\|A_n(e_0)-e_0\|+\varepsilon$$

$$+\frac{2M}{\delta^2}(\|A_n(e_2)-e_2\|+2|x|\cdot\|A_n(-e_1)+e_1\|+x^2|A_n(e_0)-e_0\|),$$

which immediately implies

$$\|A_n(f)-f\| \le \|A_n(|f-f(x)|)\| + M\cdot\|A_n(e_0)-e_0\|$$

$$\le (\varepsilon+M+\frac{2M\alpha^2}{\delta^2})\|A_n(e_0)-e_0\| + \frac{4M\alpha}{\delta^2}\|A_n(-e_1)+e_1\| + \frac{2M}{\delta^2}\|A_n(e_2)-e_2\|$$

$$\le C(\|A_n(1,x)-1\|+\|A_n(-t,x)+x\|+\|A_n(t^2,x)-x^2\|), \tag{3.5}$$

where again $C=\max\{\varepsilon+M+\frac{2M\alpha^2}{\delta^2}, \frac{4M\alpha}{\delta^2}\}$ and $\alpha=\max\{|a|,|b|\}$.

Denoting

$$E=\left\{n:\|A_n(e_0)-e_0\|+\|A_n(e_1)-e_1\|+\|A_n(-e_1)+e_1\|+\|A_n(e_2)-e_2\|\ge\frac{\eta}{C}\right\},$$

$$E_1:=\left\{n:\|A_n(e_0)-e_0\|\ge\frac{\eta}{4C}\right\},$$

$$E_2:=\left\{n:\|A_n(e_1)-e_1\|\ge\frac{\eta}{4C}\right\},$$

$$E_3:=\left\{n:\|A_n(-e_1)+e_1\|\ge\frac{\eta}{4C}\right\},$$

$$E_4:=\left\{n:\|A_n(e_2)-e_2\|\ge\frac{\eta}{4C}\right\},$$

the inequalities (3.4) and (3.5) show that $E\subset E_1\cup E_2\cup E_3\cup E_4$, which implies

$$\chi_{E(j)} \le \chi_{E_1(j)}+\chi_{E_2(j)}+\chi_{E_3(j)}+\chi_{E_4(j)},\ \text{for all } j\in\mathbb{N}.$$

Therefore, denoting $D=\{n\in\mathbb{N};\|A_n(f)-f\|\ge\eta\}$, it is immediate that

$$\delta(D)\le\delta(E)\le\delta(E_1)+\delta(E_2)+\delta(E_3)+\delta(E_4)$$

and by using (3.1), we get (3.2) and therefore it follows that the proof is complete for nonnegative f.

Suppose now that f is of arbitrary sign. It follows that $f+\|f\|$ is nonnegative and therefore form the above conclusion we get

$$st-\lim\|A_n(f+\|f\|)-f-\|f\|\quad\|=0.$$

Since all A_n are translatable, it follows

$$A_n(f+\|f\|)(x)=A_n(f)(x)+\|f\|\cdot A_n(e_0)(x)$$

for all $n\in\mathbb{N}$ and therefore

$$\|A_n(f+\|f\|)-f-\|f\|\quad\|=\|A_n(f)-f+\|f\|\cdot(A_n(e_0)-e_0)\|,$$

which immediately leads to the desired conclusion. □

4. Concrete examples in Theorem 1

In this section we present three concrete examples illustrating Theorem 1.

Example 4.1. Firstly, let us consider the Bernstein-Kantorovich-Choquet polynomial operators for functions of one real variable defined by the formula

$$K_{n,\mu}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\mathrm{d}\mu(t)}{\mu([k/(n+1),(k+1)/(n+1)])},$$

with $\mu = \sqrt{m}$, m the Lebesgue measure and

$$p_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}, \quad \text{for } t \in [0,1] \text{ and } n \in \mathbb{N}.$$

According to the results in Section 3 in Gal-Niculescu [24], $K_{n,\mu}(e_k) \to e_k$, $k \in \{0,1,2\}$ and $K_{n,\mu}(-e_1) \to -e_1$, uniformly on $[0,1]$. Also, according to Section 5 in Gal-Niculescu [27], these operators are monotone, sublinear and translatable.

Define now the sequence

$$P_n(f)(x) = (1+\alpha_n)K_{n,\mu}(f)(x), x \in [0,1], n \in \mathbb{N},$$

where α_n is a sequence statistically convergent to zero but not convergent to zero in classical sense.

Therefore $P_n(f)(x)$ satisfies the conditions in Theorem 1 and consequently $P_n(f)$ converges statistically to f, for any $f \in C([a,b])$.

Example 4.2. We consider below an example which does not involve the Choquet integral, namely they are the so-called possibilistic Kantorovich operators introduced in Gal [23], defined by

$$T_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \sup\{f(x); x \in [k/(n+1),(k+1)/(n+1)]\}.$$

It is easy to see that each T_n is a monotone, sublinear and translatable operator and

$$T_n(e_0)(x) = e_0(x).$$

Also we have

$$\begin{aligned} T_n(e_1)(x) &= \sum_{k=0}^{n} p_{n,k}(x)\frac{k+1}{n+1} \\ &= \frac{n}{n+1} \cdot \sum_{k=0}^{n} p_{n,k}(x)\left[\frac{k}{n} + \frac{1}{n}\right] \\ &= \frac{n}{n+1}x + \frac{1}{n+1} \to e_1(x), \end{aligned}$$

$$\begin{aligned} T_n(-e_1)(x) &= \sum_{k=0}^{n} p_{n,k}(x)\left[-\frac{k}{n+1}\right] \\ &= \frac{n}{n+1}\sum_{k=0}^{n} p_{n,k}(x) - \frac{k}{n} \\ &= -\frac{n}{n+1}x \to -e_1(x), \end{aligned}$$

for $n \to \infty$, uniformly on $[0,1]$.

Moreover,

$$\begin{aligned} T_n(e_2)(x) &= \sum_{k=0}^{n} p_{n,k}(x)\cdot\left(\frac{k+1}{n+1}\right)^2 \\ &= \left(\frac{n}{n+1}\right)^2 \cdot \sum_{k=0}^{n} p_{n,k}(x)\frac{k^2+2k+1}{n^2} \\ &= \left(\frac{n}{n+1}\right)^2 x^2 + \frac{2n}{(n+1)^2}x + \frac{1}{(n+1)^2} \to e_2(x), \end{aligned}$$

for $n \to \infty$, uniformly on $[0,1]$.

Now, the sequence

$$Q_n(f)(x) = (1+\alpha_n)\cdot T_n(f)(x), x \in [a,b], n \in \mathbb{N},$$

where α_n is the sequence mentioned in Example 4.1 too, satisfies Theorem 1.

Example 4.3. Define now the sequence

$$Q_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x)\cdot\frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(\beta_n t)\mathrm{d}\mu(t)}{\mu([k/(n+1),(k+1)/(n+1)])},$$

where $\mu = \sqrt{m}$ and $0 \le \beta_n \le 1$, $n \in \mathbb{N}$, is statistically convergent to 1 but not convergent in the classical sense.

These operators are monotone, sublinear and translatable and we easily seen that we have

$$\begin{aligned} &Q_n(e_0)(x) = 1, \\ &Q_n(e_1)(x) = \beta_n K_{n,\mu}(e_1)(x), \\ &Q_n(-e_1)(x) = \beta_n K_{n,\mu}(-e_1)(x), \\ &Q_n(e_2)(x) = \beta_n^2 K_{n,\mu}(e_2)(x). \end{aligned}$$

Therefore, the hypothesis of Theorem 1 are satisfied.

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 409–422
DOI: 10.24193/subbmath.2023.2.15

A new class of Bernstein-type operators obtained by iteration

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Abstract. A new class of Bernstein-type operators are obtained by applying an iterative method of modifications starting from the Bernstein operators. These operators have good properties of approximation of functions and of their derivatives.

Mathematics Subject Classification (2010): 41A36, 41A10, 41A25.

Keywords: Modified Bernstein operators, degree of approximations, Voronovskaja theorem, higher order convexity, simultaneous approximation.

1. Introduction

Bernstein operators are defined by

$$B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right) \tag{1.1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \tag{1.2}$$

for $f : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$, $x \in [0,1]$.

They are the source of a vast literature with a multitude of modifications and generalizations. In this article we propose a new construction of a sequence of linear positive operators recursively obtained by applying a modification method starting from the Bernstein operators.

Received 23 February 2022; Accepted 22 September 2022.

For integers $0 \le r < n$ consider the operator

$$T_n^r(f)(x) = \sum_{i=0}^{n-r} p_{n-r,i}(x) F_{n,i}^r(f),\ f:[0,1] \to \mathbb{R},\ x \in [0,1], \tag{1.3}$$

where the functionals $F_{n,i}^r$ are defined recursively by $F_{n,i}^0(f) = f\left(\frac{i}{n}\right)$, $0 \le i \le n$ and, for $r \ge 1$:

$$F_{n,i}^r(f) = \left(1 - \frac{i}{n-r}\right) F_{n,i}^{r-1}(f) + \frac{i}{n-r} F_{n,i+1}^{r-1}(f),\ 0 \le i \le n-r. \tag{1.4}$$

Note that for $r = 0$, T_n^r coincides with the Bernstein operator, B_n. Also, the operator T_n^1 can be put in connection with operators $T_{n,\alpha}$, defined by

$$T_{n,\alpha} = \alpha B_n + (1-\alpha) T_n^1,\ \text{for } \alpha \in [0,1]$$

and introduced by Chen et alt. [1]. The Chlodovsky variant of operators $T_{n,\alpha}$ was studied in [7].

For operators T_n^r we study in this paper the explicit representation, the moments, estimates of the degree of approximation in terms of moduli of continuity, the Voronoskaja-type theorem, the preservation of the convexity of higher order and the simultaneous approximation. There exists a partial analogy between the operators T_n^r and the iteration by composition of Bernstein operators:

$$(B_n)^r := B_n \circ \cdot \circ B_n, \quad (r \text{ times}).$$

2. Basic identities

For $p \in \mathbb{N}$ define the monomial function $e_p(t) = t^p$, $t \in [0,1]$. Let $B[0,1]$ be the space of bounded functions defined on interval $[0,1]$, $C[0,1]$ be the space of continuous functions defined on interval $[0,1]$ and $C^k[0,1]$, $k \ge 1$ be the space of functions with k continuous derivatives.

Lemma 2.1. *For integers $0 \le r < n$, $0 \le i \le n-r$ there hold:*

i) $F_{n,i}^r(e_0) = 1$,

ii) $F_{n,i}^r(e_1) = \frac{i}{n-r}$.

Proof. The relations follows immediately by induction. □

Corollary 2.2. *For integers $0 \le r < n$, and $x \in [0,1]$, the following relation are true:*

i) $T_n^r(e_0)(x) = 1$,

ii) $T_n^r(e_1)(x) = x$.

Proof. Corollary 2.2 follows from Lemma 2.1 using the properties of Bernstein operators. □

For $a \in \mathbb{R}$, and $n \in \mathbb{N} \cup \{0\}$ denote by $(a)_n$ the Pochhammer symbol, i.e. $(a)_0 = 1$ and $(a)_n = a(a+1)\ldots(a+n-1)$, for $n \ge 1$.

For $n, r, i, k \in \mathbb{N} \cup \{0\}$, $0 \le r \le n$, $0 \le i \le n-r$, $0 \le k \le r$ define

$$c_{n,r,i,k} = \binom{r}{k} (n-i-r)_{r-k} (i)_k. \tag{2.1}$$

Lemma 2.3. *For $f \in C[0,1]$, $n \in \mathbb{N}$, $r \in \mathbb{N} \cup \{0\}$, $0 \le r < n$, $0 \le i \le n-r$, we have*

$$F^r_{n,i}(f) = \frac{1}{(n-r)_r} \sum_{k=0}^{r} c_{n,r,i,k} f\left(\frac{i+k}{n}\right). \tag{2.2}$$

Proof. We prove by mathematical induction with regards to r. For $r = 0$ equation (2.2) is clear. Suppose (2.2) true for $r < n-1$. Then, for $0 \le i \le n-r-1$, and $f : [0,1] \to \mathbb{R}$:

$$\begin{aligned}
F^{r+1}_{n,i}(f) &= \left(1 - \frac{i}{n-r-1}\right) F^r_{n,i}(f) + \frac{i}{n-r-1} F^r_{n,i+1}(f) \\
&= \frac{n-r-i-1}{n-r-1} \cdot \frac{1}{(n-r)_r} \sum_{k=0}^{r} \binom{r}{k} (n-r-i)_{r-k} (i)_k f\left(\frac{i+k}{n}\right) \\
&\quad + \frac{i}{n-r-1} \cdot \frac{1}{(n-r)_r} \sum_{k=0}^{r} \binom{r}{k} (n-r-i-1)_{r-k} (i+1)_k \left(\frac{i+1+k}{n}\right) \\
&= \frac{1}{(n-r-1)_{r+1}} \left\{ \sum_{k=0}^{r} \binom{r}{k} (n-r-i)_{r-k} (i)_k (n-r-i-1) f\left(\frac{i+k}{n}\right) \right. \\
&\quad \left. + \sum_{k=0}^{r} \binom{r}{k} (n-r-i-1)_{r-k} (i+1)_k i \left(\frac{i+1+k}{n}\right) \right\} \\
&= \frac{1}{(n-r-1)_{r+1}} \left\{ \sum_{k=0}^{r} \binom{r}{k} (n-r-i-1)_{r-k+1} (i)_k f\left(\frac{i+k}{n}\right) \right. \\
&\quad \left. + \sum_{k=0}^{r} \binom{r}{k} (n-r-i-1)_{r-k} (i)_{k+1} f\left(\frac{i+1+k}{n}\right) \right\}.
\end{aligned} \tag{2.3}$$

Since

$$\begin{aligned}
&\sum_{k=0}^{r} \binom{r}{k} (n-r-i-1)_{r-k} (i)_{k+1} f\left(\frac{i+1+k}{n}\right) \\
&= \sum_{k=1}^{r+1} \binom{r}{k-1} (n-r-i-1)_{r-k+1} (i)_k f\left(\frac{i+k}{n}\right)
\end{aligned}$$

and

$$\binom{r}{k} + \binom{r}{k-1} = \binom{r+1}{k},$$

by adding the last two sums in (2.3) one obtains

$$\begin{aligned}
F^{r+1}_{n,i}(f) &= \frac{1}{(n-r-1)_{r+1}} \sum_{k=0}^{r+1} \binom{r+1}{k} (n-r-i-1)_{r-k+1} (i)_k f\left(\frac{i+k}{n}\right) \\
&= \frac{1}{(n-r-1)_{r+1}} \sum_{k=0}^{r+1} c_{n,r+1,i,k} f\left(\frac{i+k}{n}\right).
\end{aligned}$$

□

Remark 2.4. From Lemma 2.3 it follows that

$$T_n^{n-1}(f)(x) = (1-x)f(0) + xf(1),\ f : [0,1] \to \mathbb{R},\ n \in \mathbb{N},\ x \in [0,1].$$

This relation, shows that the operators T_n^r make a link between the operators B_n and B_1, similarly with the link made by $(B_n)^r$, for $r = 1$ and the limit $r \to \infty$.

For any $n \in \mathbb{N}$ consider the operator

$$G_n(f)(t) = (1-t)f\left(\frac{n-1}{n}t\right) + tf\left(\frac{n-1}{n}t + \frac{1}{n}\right),\ f \in C[0,1],\ t \in [0,1]. \tag{2.4}$$

Lemma 2.5. *For $1 \le r < n$ and $f \in C[0,1]$ there holds*

$$T_n^r(f)(x) = (T_{n-1}^{r-1} \circ G_n)(f)(x),\ x \in [0,1]. \tag{2.5}$$

Proof. From relations (2.1) and (2.2) one has

$$F_{n,i}^r(f) = \frac{1}{(n-r)_r} \sum_{k=0}^{r} \binom{r}{k} (n-r-i)_{r-k} (i)_k f\left(\frac{i+k}{n}\right).$$

We decompose this sum in two sums denoted U_1 and U_2 using formula

$$\binom{r}{k} = \binom{r-1}{k-1} + \binom{r-1}{k}.$$

By changing the index one obtains

$$\begin{aligned}
U_1 &= \frac{1}{(n-r)_r} \sum_{k=0}^{r} \binom{r-1}{k-1} (n-r-i)_{r-k} (i)_k f\left(\frac{i+k}{n}\right) \\
&= \frac{1}{(n-r)_r} \sum_{k=1}^{r} \binom{r-1}{k-1} (n-r-i)_{r-k} (i)_{k-1} (i+k-1) f\left(\frac{i+k}{n}\right) \\
&= \frac{1}{(n-r)_r} \sum_{k=0}^{r-1} \binom{r-1}{k} (n-r-i)_{r-1-k} (i)_k (i+k) f\left(\frac{i+k+1}{n}\right) \\
&= \frac{1}{n-1} \cdot \frac{1}{(n-r)_{r-1}} \sum_{k=0}^{r-1} c_{n-1,r-1,i,k} (i+k) f\left(\frac{i+k+1}{n}\right).
\end{aligned}$$

Also, there holds

$$\begin{aligned}
U_2 &= \frac{1}{(n-r)_r} \sum_{k=0}^{r} \binom{r-1}{k} (n-r-i)_{r-k} (i)_k f\left(\frac{i+k}{n}\right) \\
&= \frac{1}{(n-r)_r} \sum_{k=0}^{r-1} \binom{r-1}{k} (n-r-i)_{r-1-k} (i)_k (n-k-i-1) f\left(\frac{i+k}{n}\right) \\
&= \frac{1}{n-1} \cdot \frac{1}{(n-r)_{r-1}} \sum_{k=0}^{r-1} c_{n-1,r-1,i,k} (n-k-i-1) f\left(\frac{i+k}{n}\right).
\end{aligned}$$

Then,

$$\begin{aligned} F^r_{n,i}(f) &= U_1 + U_2 \\ &= \frac{1}{(n-r)_{r-1}} \sum_{k=0}^{r-1} c_{n-1,r-1,i,k} \Big[\frac{i+k}{n-1} f\left(\frac{i+k+1}{n}\right) \\ &+ \frac{n-k-i-1}{n-1} f\left(\frac{i+k}{n}\right) \Big]. \end{aligned}$$

But:

$$\begin{aligned} & \frac{i+k}{n-1} f\left(\frac{i+k+1}{n}\right) + \frac{n-k-i-1}{n-1} f\left(\frac{i+k}{n}\right) \\ = \;& \frac{i+k}{n-1} f\left(\frac{n-1}{n}\frac{i+k}{n-1} + \frac{1}{n}\right) + \left(1 - \frac{k+i}{n-1}\right) f\left(\frac{n-1}{n} \cdot \frac{i+k}{n-1}\right) \\ = \;& G_n(f)\left(\frac{i+k}{n-1}\right). \end{aligned}$$

Then, for $0 \le i \le n-r$,

$$F^r_{n,i}(f) = \frac{1}{(n-r)_{r-1}} \sum_{k=0}^{r-1} c_{n-1,r-1,i,k} G_n(f)\left(\frac{i+k}{n-1}\right) = F^{r-1}_{n-1,i}(G_n(f)).$$

Finally,

$$\begin{aligned} T^r_n(f)(x) &= \sum_{i=0}^{n-r} p_{n-r,i}(x) F^r_{n,i}(f) = \sum_{i=0}^{n-r} p_{n-r,i}(x) F^{r-1}_{n-1,i}(G_n(f)) \\ &= T^{r-1}_{n-1}(G_n(f))(x). \end{aligned}$$

□

Corollary 2.6. *For integers $0 \le r < n$ there exists the representation*

$$T^r_n = B_{n-r} \circ G_{n-r+1} \circ G_{n-r+2} \circ \ldots \circ G_n. \tag{2.6}$$

3. The moments

Lemma 3.1. *For $n \in \mathbb{N}$, $p \in \mathbb{N}$ there holds*

$$G_n((e_1 - xe_0)^p)(t) = \sum_{j=0}^{p} (t-x)^j d_{n,p,j}(x), \; t, x \in [0,1], \tag{3.1}$$

where

$$\begin{aligned} d_{n,p,j}(x) &= \frac{1}{n^p}\binom{p}{j}(n-1)^j \Big[(1-x)(-x)^{p-j} + x(1-x)^{p-j}\Big] \\ &+ \frac{1}{n^p}\binom{p}{j-1}(n-1)^{j-1}\Big[x(-x)^{p-j} + (1-x)(1-x)^{p-j}\Big]. \end{aligned}$$

Proof. From the definition of G_n, grouping the terms with the same power of $t-x$ one obtains

$$
\begin{aligned}
& G_n((e_1 - xe_0)^p)(t) = (1-t)\left(\frac{n-1}{n}t - x\right)^p + t\left(\frac{n-1}{n}t + \frac{1}{n} - x\right)^p \\
= \ & (1-x+x-t)\left(\frac{n-1}{n}(t-x) - \frac{x}{n}\right)^p \\
& +(t-x+x)\left(\frac{n-1}{n}(t-x) + \frac{1}{n}(1-x)\right)^p \\
= \ & (1-x+x-t)\sum_{j=0}^{p}\binom{p}{j}\left(\frac{n-1}{n}\right)^j (t-x)^j\left(-\frac{x}{n}\right)^{p-j} \\
& +(t-x+x)\sum_{j=0}^{p}\binom{p}{j}\left(\frac{n-1}{n}\right)^j (t-x)^j\left(\frac{1-x}{n}\right)^{p-j} \\
= \ & \frac{1}{n^p}\sum_{j=0}^{p+1}(t-x)^j\left[\binom{p}{j}(1-x)(n-1)^j(-x)^{p-j}\right. \\
& \qquad \left. -\binom{p}{j-1}(n-1)^{j-1}(-x)^{p+1-j}\right] \\
& +\frac{1}{n^p}\sum_{j=0}^{p+1}(t-x)^j\left[\binom{p}{j}x(n-1)^j(1-x)^{p-j}\right. \\
& \qquad \left. +\binom{p}{j-1}(n-1)^{j-1}(1-x)^{p+1-j}\right].
\end{aligned}
$$

Finally, equation (3.1) follows, because the coefficient of $(t-x)^{p+1}$ is null. □

Define the moments of order p of operators T_n^r, by

$$M^p[T_n^r](x) = T_n^r((e_1 - xe_0)^p)(x),\ 0 \le r < n,\ p \ge 0,\ x \in [0,1]. \tag{3.2}$$

From Lemma 2.5 and Lemma 3.1 we have the following relation of recurrence

Corollary 3.2.

$$M^p[T_n^r](x) = \sum_{j=0}^{p} d_{n,p,j}(x) M^j[T_{n-1}^{r-1}](x),\ 1 \le r < n,\ p \ge 0,\ x \in [0,1]. \tag{3.3}$$

Lemma 3.3. *We have, for $x \in [0,1]$, $0 \le r < n$:*

$$M^0[T_n^r](x) = 1; \tag{3.4}$$

$$M^1[T_n^r](x) = 0; \tag{3.5}$$

$$M^2[T_n^r](x) = \frac{n+r+1}{n(n-r+1)}x(1-x); \tag{3.6}$$

$$M^3[T_n^r](x) = \frac{n^2+4nr+3n+r^2+3r+2}{n^2(n-r+1)(n-r+2)}x(1-x)(1-2x); \tag{3.7}$$

$$M^4[T_n^r](x) = x(1-x)a_{n,r}(x),\ \textit{with}\ |a_{n,r}(x)| \le C_r \cdot \frac{1}{n^2} \tag{3.8}$$

where C_r is independent on $n \in \mathbb{N}$, and $x \in [0,1]$.

Proof. Relations (3.4) and (3.5) can be obtained directly from Corollary 2.2. For the moment $M^2[T_n^r](x)$, first note that for $r = 0$ and $n \ge 1$, equality (3.6) becomes

$$M^2[T_n^0](x) = \frac{x(1-x)}{n},$$

which is known, from the property of Bernstein operators. For $r \ge 1$, from Corollary 3.2 and equations (3.4) and (3.5) one obtains

$$\begin{aligned} M^2[T_n^r](x) &= \frac{n^2-1}{n^2}M^2[T_{n-1}^{r-1}](x) + \frac{1-2x}{n^2}M^1[T_{n-1}^{r-1}](x) \\ &\quad + \frac{x(1-x)}{n^2}M^0[T_{n-1}^{r-1}](x) \\ &= \frac{n^2-1}{n^2}M^2[T_{n-1}^{r-1}](x) + \frac{x(1-x)}{n^2}. \end{aligned}$$

Then, equation (3.6) follows by induction since

$$\frac{n+r+1}{n(n-r+1)}x(1-x) = \frac{n^2-1}{n^2} \cdot \frac{n+r-1}{(n-1)(n-r+1)}x(1-x) + \frac{x(1-x)}{n^2}.$$

Equation (3.7) for $r = 0$, $n \in \mathbb{N}$ reads $M^3[T_n^0](x) = \frac{x(1-x)(1-2x)}{n^2}$, which coincides with the moment of order 3 of Bernstein operators. For $r \ge 1$, suppose that (3.7) is true for $r-1$ and $n-1$. From relations (3.3), (3.4), (3.5), (3.6) it follows after certain computations:

$$\begin{aligned} M^3[T_n^r](x) &= \frac{(n-1)^2(n+2)}{n^3}M^3[T_{n-1}^{r-1}](x) + 3\frac{n-1}{n^3}(1-2x)M^2[T_{n-1}^{r-1}](x) \\ &\quad + \frac{3nx(1-x)+1-6x+6x^2}{n^3}M^1[T_{n-1}^{r-1}](x) + \frac{x(1-x)(1-2x)}{n^3}M^0[T_{n-1}^{r-1}](x) \\ &= \frac{(n-1)^2(n+2)}{n^3} \cdot \frac{n^2+4nr+r^2-3n-3r+2}{(n-1)^2(n-r+1)(n-r+2)}x(1-x)(1-2x) \\ &\quad + 3\frac{n-1}{n^3}(1-2x)\frac{n+r-1}{(n-1)(n-r+1)}x(1-x) + \frac{1}{n^3}x(1-x)(1-2x) \\ &= \frac{n^2+4nr+3n+r^2+3r+2}{n^2(n-r+1)(n-r+2)}x(1-x)(1-2x). \end{aligned}$$

Finally, it is known that $B_n((e_1 - xe_0)^4)(x) = \mathrm{O}\left(\frac{1}{n^2}\right)$. Hence equation (3.8) is true for $r = 0$, $n \in \mathbb{N}$. For $1 \le r < n$ equation (3.3) yields

$$\begin{aligned} M^4[T_n^r](x) = {} & \frac{(n-1)^3(n+3)}{n^4} M^4[T_{n-1}^{r-1}](x) + 6\frac{(n-1)^2(1-2x)}{n^4} M^3[T_{n-1}^{r-1}](x) \\ & + \frac{(n-1)(6(n-3)x(1-x)+4)}{n^4} M^2[T_{n-1}^{r-1}](x) \\ & + \frac{4(n-1)x(1-x)(1-2x) - 4x^3 + 6x^2 - 4x + 1}{n^4} M^1[T_{n-1}^{r-1}](x) \\ & + \frac{x(1-x)(3x^2-3x+1)}{n^4} M^0[T_{n-1}^{r-1}](x). \end{aligned}$$

From this relation, from (3.4), (3.5), (3.6), (3.7) and supposing that

$$M^4[T_{n-1}^{r-1}](x) = x(1-x)\mathrm{O}\left(\frac{1}{n^2}\right)$$

it follows that

$$M^4[T_n^r](x) = x(1-x)\mathrm{O}\left(\frac{1}{n^2}\right).$$

So, relation (3.8) follows by induction. □

Lemma 3.4. *For integers n, r, p, with $n > r + p$ we have the representation*

$$T_n^r(e_p)(x) = \left(\frac{n-r}{n}\right)^p B_{n-r}(e_p)(x) + R_{n,p,r}(x), \tag{3.9}$$

where $R_{n,p}(x)$ is a polynomial with degree at most p having all the coefficients of type $\mathrm{O}\left(\frac{1}{n}\right)$, depending on p and r.

Proof. We have

$$G_n(e_p)(t) = (1-t)\left(\frac{n-1}{n}t\right)^p + t\left(\frac{n-1}{n}t + \frac{1}{n}\right)^p.$$

From this it follows that $G_n(e_p)(t) = \left(\frac{n-1}{n}t\right)^p + P_{n,p}(t)$, where $P_{n,p}(t)$ is a polynomial of degree at most p in variable t and all the coefficients of $P_{n,p}(t)$ are positive and of type $\mathrm{O}\left(\frac{1}{n}\right)$. Then, by induction we deduce that

$$(G_{n-r+1} \circ G_{n-r+2} \circ \cdots \circ G_n)(e_p) = \left(\frac{n-r}{n}\right)^p + \tilde{P}_{n,p,r}(t),$$

where $\tilde{P}_{n,p,r}(t)$ is a polynomial of degree at most p having all the coefficients of type $\mathrm{O}\left(\frac{1}{n}\right)$.
Using formula (2.6) we obtain

$$T_n^r(e_p) = \left(\frac{n-r}{n}\right)^p B_{n-r}(e_p) + B_{n-r}(\tilde{P}_{n,p,r}).$$

Denoting $R_{n,p,r}(x) = B_{n-r}(\tilde{P}_{n,p,r})(x)$ it follows that $R_{n,p}(x)$ satisfies the conditions from this lemma, because the Bernstein polynomials B_{n-r} preserve the degree of polynomials of degree up to $n - r$. □

4. Estimations of the degree of approximation by operators T_n^r.

In this section we deduce estimates of order of approximation using the first order modulus of continuity, the usual second order modulus of continuity and the second Ditzian-Totik modulus, which are given bellow, for a generic function $g \in B[0,1]$ and $h > 0$, respectively by

$$\begin{aligned}
\omega_1(g,h) &= \sup\{|g(u)-g(v)|,\ u,v \in [0,1],\ |u-v| \le h\};\\
\omega_2(g,h) &= \sup\{|g(x-\rho)-2g(x)+g(x+\rho)|,\ x \pm \rho \in [0,1],\ |\rho| \le h\};\\
\omega_2^{\varphi}(g,h) &= \sup\{|g(x-\rho)-2g(x)+g(x+\rho)|,\ x \pm \rho \in [0,1],\ |\rho| \le h\varphi(x)\},\\
&\quad \text{where } \varphi(x) = \sqrt{x(1-x)}.
\end{aligned}$$

Theorem 4.1. *For $f \in C[0,1]$, $x \in [0,1]$ and integers $0 \le r < n$ the following estimates are true:*

$$|T_n^r(f)(x) - f(x)| \le 2\omega_1\left(f, \mu_{n,r}(x)\right), \tag{4.1}$$

$$|T_n^r(f)(x) - f(x)| \le \frac{1}{2}\mu_{n,r}(x)\omega_1\left(f', 2\mu_{n,r}(x)\right), \tag{4.2}$$

$$|T_n^r(f)(x) - f(x)| \le \frac{3}{2}\omega_2\left(f, \mu_{n,r}(x)\right), \tag{4.3}$$

$$|T_n^r(f)(x) - f(x)| \le \frac{5}{2}\omega_2^{\varphi}\left(f, \sqrt{\frac{n+r+1}{n(n-r+1)}}\right), \tag{4.4}$$

where $\mu_{n,r}(x) = \sqrt{\frac{(n+r+1)x(1-x)}{n(n-r+1)}}$ and additionally, in inequality (4.2) we suppose that $f \in C^1[0,1]$, in inequality (4.3) we suppose that $\sqrt{\frac{(n+r+1)x(1-x)}{n(n-r+1)}} \le \frac{1}{2}$ and in inequality (4.4) we suppose that $\sqrt{\frac{n+r+1}{n(n-r+1)}} \le \frac{1}{2}$.

Proof. Inequality (4.1) follows from the general estimate of Mond [4]. For the rest of the estimates we can apply the estimates obtained in [5] for general operators in terms of the moments. So, inequality (4.2) follows from [5]- Cor. 2.3.2, inequality (4.3) follows from [5]- Cor. 2.2.1, and inequality (4.4) follows from [5]- Th. 2.5.1. □

Corollary 4.2. *For any $f \in C[0,1]$ and integer $r \ge 0$ there holds:*

$$\lim_{n\to\infty} \|T_n^r(f) - f\| = 0, \tag{4.5}$$

where $\|\cdot\|$ denotes the sup-norm.

We give now a quantitative version of the Voronovskaja theorem. For this we use the least concave majorant of the first modulus of continuity, given for a function $f \in B[a,b]$ and $h > 0$ by

$$\tilde{\omega}_1(f,h) = \begin{cases} \sup\limits_{\substack{0\le x\le h\le y\le b\\ x\ne y}} \frac{(h-x)\omega_1(f,y)+(y-h)\omega_1(f,x)}{y-x}, & 0 < h \le b-a \\ \omega_1(f,1), & h > b-a. \end{cases} \tag{4.6}$$

Theorem 4.3. *If $f \in C^2[0,1]$, $r \geq 0$ is an integer and $x \in [0,1]$, then we have*

$$\begin{aligned} &\left| T_n^r(f)(x) - f(x) - \frac{1}{2} \cdot \frac{(n+r+1)x(1-x)}{n(n-r+1)} \cdot f''(x) \right| \\ \leq \ & \tilde{C}_r \frac{x(1-x)}{n} \tilde{\omega}_1 \left(f'', \frac{1}{\sqrt{n}} \right), \end{aligned} \tag{4.7}$$

where $\tilde{C}_r > 0$ is a constant independent on f, n and x.

Proof. Using the estimate given in Gonska [2]-Th. 3.2 one obtains:

$$\begin{aligned} &\left| T_n^r(f)(x) - f(x) - \frac{1}{2} \cdot \frac{(n+r+1)x(1-x)}{n(n-r+1)} \cdot f''(x) \right| \\ \leq \ & \frac{1}{2} T_n^r((e_1 - xe_0)^2)(x) \tilde{\omega}_1 \left(f'', \frac{1}{3} \cdot \frac{T_n^r(|e_1 - xe_0|^3)(x)}{T_n^r((e_1 - xe_0)^2)(x)} \right). \end{aligned}$$

From the Cauchy-Schwartz inequality it results

$$\frac{T_n^r(|e_1 - xe_0|^3)(x)}{T_n^r((e_1 - xe_0)^2)(x)} \leq \sqrt{\frac{T_n^r((e_1 - xe_0)^4)(x)}{T_n^r((e_1 - xe_0)^2)(x)}}$$

Using Lemma 3.3 there is a constant C_r, independent on n and x such that

$$\frac{T_n^r((e_1 - xe_0)^4)(x)}{T_n^r((e_1 - xe_0)^2)(x)} \leq \frac{C_r \frac{x(1-x)}{n^2}}{\frac{(n+r+1)x(1-x)}{n(n-r+1)}} \leq \frac{C_r}{n}.$$

From the above relations it follows that there exists a constant $\tilde{C}_r$ such that relation (4.7) holds. □

5. Convexity of higher order. Simultaneous approximation

A function $f : I \to \mathbb{R}$, I interval, is named *convex of order* s, $s \geq -1$, or s-convex, in the sense of T. Popoviciu [6] if for any distinct points $x_0, x_1, \dots x_{s+1}$ in I the inequality $[f; x_0, x_1, \dots x_{s+1}] \geq 0$, holds, where $[f; x_0, x_1, \dots x_{s+1}] \geq 0$ is the divided difference of function f. In particular, if f is convex of order s, then $\Delta_h^{s+1} f(x) \geq 0$, for any $x \in I$, $h > 0$, such that $x + (s+1)h \in I$, where $\Delta_h^{s+1} f(x) = \sum_{i=0}^{r+1} (-1)^{s+1+i} \binom{s+1}{i} f(x + ih)$ is the finite difference of order $s+1$ of f. So that f is convex of order -1 iff it is positive, f is convex of order 0 iff f is increasing, f is convex of order 1, if it is usual convex and so on. Denote by D the derivative operator, and by $D^s := D \circ D \circ D \circ \cdots \circ D$, ($s$-times), the operator of derivative of order s. If $f \in C^{s+1}(I)$, then f is convex of order s if and only if $D^{s+1} f(x) \geq 0$, for all $x \in I$. An operator which transforms each s-convex function in a s-convex function is named convex operator of order s.

Lemma 5.1. *For $f \in C[0,1]$, and integers $0 \leq r < n$, $0 \leq s < n - r$ we have*

$$\begin{aligned} & D^s T_n^r(f)(x) \\ = & \frac{(n-r-s+1)_s}{(n-r)_r} \sum_{i=0}^{n-r-s} p_{n-r-s,i}(x) \sum_{k=0}^{r} c_{n+s,r,i+s,k} \Delta_{\frac{1}{n}}^s f\left(\frac{i+k}{n} \right). \end{aligned} \tag{5.1}$$

Proof. We prove by induction with regard to s. For $s=0$ it results from Lemma 2.3. Now suppose that (5.1) is true for s and prove it for $s+1$. We have

$$\begin{aligned}
& D^{s+1}T_n^r(f)(x) \\
= & \frac{(n-r-s+1)_s}{(n-r)_r}\sum_{i=0}^{n-r-s}\frac{\mathrm{d}}{\mathrm{d}x}p_{n-r-s,i}(x)\sum_{k=0}^{r}c_{n+s,r,i+s,k}\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+k}{n}\right) \\
= & \frac{(n-r-s+1)_s}{(n-r)_r}\sum_{i=0}^{n-r-s}(n-r-s)(p_{n-r-s-1,i-1}(x)-p_{n-r-s-1,i}(x))\times \\
& \qquad \times\sum_{k=0}^{r}c_{n+s,r,i+s,k}\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+k}{n}\right) \\
= & \frac{(n-r-s)_{s+1}}{(n-r)_r}\sum_{i=0}^{n-r-s-1}p_{n-r-s-1,i}(x)\times \\
& \times\Big[\sum_{k=0}^{r}c_{n+s,r,i+s+1,k}\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+k+1}{n}\right)-\sum_{k=0}^{r}c_{n+s,r,i+s,k}\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+k}{n}\right)\Big] \\
= & \frac{(n-r-s)_{s+1}}{(n-r)_r}\sum_{i=0}^{n-r-s-1}p_{n-r-s-1,i}(x)\sum_{j=0}^{r+1}\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+j}{n}\right)\times \\
& \qquad \times\Big[c_{n+s,r,i+s+1,j-1}-c_{n+s,r,i+s,j}\Big],
\end{aligned}$$

where $c_{n+s,r,i+s+1,-1}=0$ and $c_{n+s,r,i+s,r+1}=0$.
For n,r,i fixed, denote $\alpha_j=c_{n+s,r,i+s+1,j-1}-c_{n+s,r,i+s,j}$, $0\le j\le r+1$.
In order to prove the induction step it suffices to show for $0\le i\le n-r-s-1$:

$$\sum_{j=0}^{r+1}\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+j}{n}\right)\alpha_j=\sum_{k=0}^{r}c_{n+s+1,r,i+s+1,k}\Delta_{\frac{1}{n}}^{s+1}f\left(\frac{i+k}{n}\right). \tag{5.2}$$

Then

$$\begin{aligned}
& \sum_{j=0}^{r+1}\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+j}{n}\right)\alpha_j \\
= & \alpha_{r+1}\left[\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+r+1}{n}\right)-\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+r}{n}\right)\right] \\
& +(\alpha_r+\alpha_{r+1})\left[\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+r}{n}\right)-\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+r-1}{n}\right)\right]+\ldots \\
& +(\alpha_1+\alpha_2+\ldots+\alpha_{r+1})\left[\Delta_{\frac{1}{n}}^{s}f\left(\frac{i+1}{n}\right)-\Delta_{\frac{1}{n}}^{s}f\left(\frac{i}{n}\right)\right] \\
& +(\alpha_0+\alpha_1+\ldots+\alpha_{r+1})\Delta_{\frac{1}{n}}^{s}f\left(\frac{i}{n}\right).
\end{aligned}$$

Using Lemma 2.3 and then Lemma 2.1-i) we have

$$\begin{aligned}
\sum_{j=0}^{r+1}\alpha_j &= \sum_{j=0}^{r+1} c_{n+s,r,i+s+1,j-1} - \sum_{j=0}^{r+1} c_{n+s,r,i+s,j} \\
&= \sum_{j=0}^{r} c_{n+s,r,i+s+1,j} - \sum_{j=0}^{r} c_{n+s,r,i+s,j} \\
&= (n+s-r)_r F^r_{n+s,i+s+1}(e_0) - (n+s-r)_r F^r_{n+s,i+s}(e_0) \\
&= (n+s-r)_r - (n+s-r)_r = 0.
\end{aligned}$$

Therefore, it results

$$\sum_{j=0}^{r+1}\Delta^s_{\frac{1}{n}} f\left(\frac{i+j}{n}\right)\alpha_j = \sum_{k=0}^{r}\sum_{j=k+1}^{r+1}\alpha_j\cdot\Delta^{s+1}_{\frac{1}{n}} f\left(\frac{i+k}{n}\right). \tag{5.3}$$

In order to obtain relation (5.2) it suffices to prove for $0\le k\le r,\ 0\le i\le n-r-s-1$ that:

$$\sum_{j=k+1}^{r+1}\alpha_j = c_{n+s+1,r,i+s+1,k}. \tag{5.4}$$

Fix i. We prove relation (5.4) by descending induction with regard to k. For $k=r$ we have

$$\begin{aligned}
\sum_{j=r+1}^{r+1}\alpha_j &= \alpha_{r+1} = c_{n+s,r,i+s+1,r} = \binom{r}{r}(n-1-r)_0(i+s+1)_r \\
&= \binom{r}{r}(n-r)_0(i+s+1)_r = c_{n+s+1,r,i+s+1,r}.
\end{aligned}$$

Now, suppose that (5.4) is true for $k+1$, $0\le k\le r-1$ and prove it for k. One obtains

$$\begin{aligned}
&\sum_{j=k+1}^{r+1}\alpha_j \\
&= \alpha_{k+1} + \sum_{j=k+2}^{r+1}\alpha_j \\
&= \alpha_{k+1} + c_{n+s+1,r,i+s+1,k+1} \\
&= c_{n+s,r,i+s+1,k} - c_{n+s,r,i+s,k+1} + c_{n+s+1,r,i+s+1,k+1} \\
&= \binom{r}{k}(n-i-r-1)_{r-k}(i+s+1)_k - \binom{r}{k+1}(n-i-r)_{r-k-1}(i+s)_{k+1} \\
&\quad + \binom{r}{k+1}(n-i-r)_{r-k-1}(i+s+1)_{k+1}
\end{aligned}$$

$$\begin{aligned}
&= (i+s+1)_k(n-i-r)_{r-k-1}\Big[\binom{r}{k}(n-i-r-1)-\binom{r}{k+1}(i+s) \\
&\qquad +\binom{r}{k+1}(i+s+k+1)\Big] \\
&= (i+s+1)_k(n-i-r)_{r-k-1}\Big[\binom{r}{k}(n-i-r-1)+\binom{r}{k+1}(k+1)\Big] \\
&= \binom{r}{k}(i+s+1)_k(n-i-r)_{r-k-1}[(n-i-r-1)+(r-k)] \\
&= \binom{r}{k}(i+s+1)_k(n-i-r)_{r-k} \\
&= c_{n+s+1,r,i+s+1,k}.
\end{aligned}$$

Then equality (5.4) is true and consequently relation (5.2) is true. □

Theorem 5.2. *Let integers n, r be such that $n > r$. Then operator T_n^r is convex of order s for each integer $s \geq -1$ such that $n > r + s$.*

Proof. If f is s-convex, then $\Delta_{\frac{1}{n}}^{s+1} f\left(\frac{i+k}{n}\right) \geq 0$, for $0 \leq i \leq n-r-s-1$. From relation (5.1) with $s+1$, instead of s it follows that $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{s+1} T_n^r(f)(x) \geq 0$, i.e. $T_n^r(f)$ is s-convex. □

With the aid of this fact we can deduce the property of simultaneous approximation of operators T_n^r.

Theorem 5.3. *For any integers $0 \leq r < n$ and $0 \leq s < n-r$ we have*

$$\lim_{n\to\infty} \|(D^s \circ T_n^r)(f) - D^s f\| = 0 \tag{5.5}$$

Proof. It suffices to take $s \geq 1$. Let $n \in \mathbb{N}$ sufficiently large, such that $n > r + s$. Consider s-Kantorovich operator associated to the operator T_n^r, defined by

$$K_{n,r}^s = D^s \circ T_n^r \circ I_s,$$

where I_s is operator defined by

$$I_s(g)(x) = \int_0^x \frac{(x-t)^{s-1}}{(s-1)!} g(t)dt,\ x \in [0,1],\ g \in C[0,1],$$

for $s \geq 1$ and I_0 is the identical operator. Because operator T_n^r is convex of order $s-1$ if follows that $K_{n,r}^s$ is a linear positive operator. Note that

$$(D^s \circ T_n^r)(f) = K_{n,r}^s(D^s f).$$

So that, in order to prove relation (5.5) it is sufficient to prove that the sequence of operators $(K_{n,r}^s)_n$ satisfies the conditions in the theorem of Korovkin. In Knopp and Pottinger [3]- Korollar 2.2 it is shown that the necessary and sufficient condition for this is the following conditions

$$\lim_{n\to\infty} \|D^s e_{s+i} - (D^s \circ T_n^r)(e_{s+i})\| = 0,\ \text{for } i = 0,1,2. \tag{5.6}$$

hold. From Lemma 3.4 we obtain

$$(D^s \circ T_n^r)(e_{s+i}) = \left(\frac{n-r}{n}\right)^{s+i} D^s B_{n-r}(e_{s+i}) + D^s R_{n,s+i,r},\ i = 0,1,2$$

Because the sequence $(B_{n-r})_n$ has the property of simultaneous approximation, we infer

$$\lim_{n\to\infty} \left\| D^s e_{s+i} - \left(\frac{n-r}{n}\right)^{s+i} (D^s \circ B_{n-r})(e_{s+i}) \right\| = 0,\ i = 0,1,2.$$

Also, from properties of polynomials $R_{n,s+i,r}$ we obtain

$$\lim_{n\to\infty} \|D^s R_{n,s+i,r}\| = 0,\ i = 0,1,2. \qquad \square$$

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 423–438
DOI: 10.24193/subbmath.2023.2.16

An extension of Krasnoselskii's cone fixed point theorem for a sum of two operators and applications to nonlinear boundary value problems

Lyna Benzenati and Karima Mebarki

Abstract. The purpose of this work is to establish a new generalized form of the Krasnoselskii type compression-expansion fixed point theorem for a sum of an expansive operator and a completely continuous one. Applications to three nonlinear boundary value problems associated to second order differential equations of coincidence type are included to illustrate the main results.

Mathematics Subject Classification (2010): 47H10, 54H25, 34B18.

Keywords: Fixed point, Banach space, cone, expansive mapping, sum of operators, nonlinear boundary value problem, coincidence problems.

1. Introduction

One of the main results in fixed point theory is the cone expansion and compression theorem proved by Krasnoselskii in 1964 (see, e.g., [10, 11]). It represents a powerful existence tool in studying operator equations and showing existence of positive solutions to various boundary value problems. By this result, a solution is localized in a conical shell of a normed linear space. This theorem has been recently deeply improved in various directions; see [1, 2, 3, 6, 9, 12, 13, 14] and references therein. A vector version of Krasnoselskii's fixed point theorem in cones has been given in [4, 15, 16]. In practice, the vector version allows the nonlinear term of a system to have different behaviors both in components and in variables.

Received 21 March 2020; Accepted 10 April 2020.

In this paper, we first establish some user-friendly versions of Krasnoselskii type compression-expansion fixed point theorem for a sum of an expansive operator and a completely continuous one. A vector version of the main result is also given.
Next, using the main obtained fixed-point result, we study the existence of positive solutions for three nonlinear boundary value problems associated to second order differential equations and systems of coincidence type equations.

Let X be a normed linear space with norm $\|.\|$, and let $\mathcal{P} \subset X$ be a wedge, i.e., a closed convex subset of X, $\mathcal{P} \neq \{0\}$ with $\lambda\mathcal{P} \subset \mathcal{P} \neq \{0\}$ for every $\lambda \in \mathbb{R}_+$. If in addition $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$, then $\mathcal{P}$ is a cone, and we say that $x < y$ if and only if $y - x \in \mathcal{P} \setminus \{0\}$. For two numbers $0 < r < R$, we define the conical shell $\mathcal{P}_{r,R}$ by $\mathcal{P}_{r,R} := \{x \in \mathcal{P} : r \leq \|x\| \leq R\}$.

Let $N\colon D \subset X \to X$ be a continuous operator. The operator N is said to be *bounded* if it maps bounded sets into bounded sets, *completely continuous* if it maps bounded sets into relatively compact sets, and *compact* if the set $N(D)$ is relatively compact.

Consider the operator equation

$$Nx = x,$$

where N is a given nonlinear map acting in $\mathcal{P}$.

Theorem 1.1. (Krasnoselskii's compression-expansion fixed point theorem). *Let $\alpha, \beta > 0$, $\alpha \neq \beta$, $r := \min\{\alpha, \beta\}$ and $R := \max\{\alpha, \beta\}$. Assume that $N : \mathcal{P}_{r,R} \to \mathcal{P}$ is a compact map and there exists $p \in \mathcal{P} \setminus \{0\}$ such that the following conditions are satisfied:*

$$\begin{array}{lll} Nx \neq \lambda x & \text{for } \|x\| = \alpha & \text{and } \lambda > 1; \\ Nx + \mu p \neq x & \text{for } \|x\| = \beta & \text{and } \mu > 0. \end{array} \tag{1.1}$$

Then N has a fixed point x in $\mathcal{P}$ with $r \leq \|x\| \leq R$.

Remark 1.2. If $\beta < \alpha$, then the conditions (1.1) represents a compression property of N upon the conical shell $\mathcal{P}_{r,R}$, while if $\beta > \alpha$, then the conditions (1.1) expresses an expansion property of N upon $\mathcal{P}_{r,R}$.

Consider a system of two operator equations

$$\begin{cases} N_1(x_1, x_2) = x_1 \\ N_2(x_1, x_2) = x_2, \end{cases}$$

where N_1, N_2 act from $\mathcal{P} \times \mathcal{P}$ to $\mathcal{P}$.

Theorem 1.3. ([16, Theorem 2.1]). *Let $(X, \|.\|)$ be a normed linear space; $\mathcal{P}_1$, $\mathcal{P}_2 \subset X$ two wedges; $\mathcal{P} := \mathcal{P}_1 \times \mathcal{P}_2$; $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$ for $i = 1, 2$ and let $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$ for $i = 1, 2$. Assume that $N : \mathcal{P}_{r,R} = (\mathcal{P}_1)_{r_1,R_1} \times (\mathcal{P}_2)_{r_2,R_2} \to \mathcal{P}$, $N = (N_1, N_2)$, is a compact map and there exist $p_i \in \mathcal{P}_i \setminus \{0\}$, $i = 1, 2$ such that for each $i \in \{1, 2\}$ the following conditions are satisfied in $\mathcal{P}_{r,R}$:*

$$\begin{array}{lll} N_i x \neq \lambda x_i & \text{for } \|x_i\| = \alpha_i & \text{and } \lambda > 1; \\ N_i x + \mu p_i \neq x_i & \text{for } \|x_i\| = \beta_i & \text{and } \mu > 0. \end{array} \tag{1.2}$$

Then N has a fixed point $x = (x_1, x_2)$ in $\mathcal{P}$ such that $r_i \leq \|x_i\| \leq R_i$ for $i = 1, 2$.

A mapping $T : D \subset Y \to Y$, where (Y, d) is a metric space, is said to be *expansive* if there exists a constant $h > 1$ such that

$$d(Tx, Ty) \geq h\, d(x, y) \text{ for all } x, y \in D.$$

To establish our results, we need the following technical lemma concerning expansive mappings.

Lemma 1.4. *Let $(X, \|.\|)$ be a linear normed space and $D \subset X$. Assume that the mapping $T : D \to X$ is expansive with constant $h > 1$. Then the mapping $T : D \to T(D)$ is invertible and*

$$\|T^{-1}x - T^{-1}y\| \leq \frac{1}{h}\|x - y\|, \;\; \forall\, x, y \in T(D).$$

2. Main results

Theorem 2.1. *Let K be a subset of a Banach space X and $\mathcal{P} \subset X$ a wedge. Assume that $T : K \to X$ is an expansive mapping with constant $h > 1$ and $F : K \to X$ is a mapping such that $I - F : K \to \mathcal{P}$ is completely continuous one with $\mathcal{P} \subset T(K)$. Let $\alpha, \beta > 0$, $\alpha \neq \beta$, $p \in \mathcal{P} \setminus \{0\}$, $r := \min\{\alpha, \beta\}$ and $R := \max\{\alpha, \beta\}$.*
Suppose that the following conditions are satisfied:

$$x \neq \lambda Tx + Fx \quad \text{for } x \in T^{-1}(\mathcal{P}),\ \|Tx\| = \alpha \ \text{ and } \lambda > 1. \tag{2.1}$$

$$x \neq Tx + Fx - \mu p \quad \text{for } x \in T^{-1}(\mathcal{P}),\ \|Tx\| = \beta \ \text{ and } \mu > 0. \tag{2.2}$$

Then $T + F$ has a fixed point x in $T^{-1}(\mathcal{P})$ such that $r \leq \|Tx\| \leq R$.

Proof. By Lemma 1.4, the operator $T^{-1} : T(K) \to K$ is a $\frac{1}{h}$-contraction. Then the operator N defined by

$$\begin{array}{rcl} N : \mathcal{P} & \to & \mathcal{P} \\ y & \mapsto & Ny = T^{-1}y - FT^{-1}y \end{array}$$

is well defined and it is completely continuous.
Claim 1. We show that Condition (2.1) implies that

$$Ny \neq \lambda y \text{ for } \|y\| = \alpha \text{ and } \lambda > 1.$$

On the contrary, assume the existence of $\lambda_0 > 1$ and $y_1 \in \mathcal{P}$ with $\|y_1\| = \alpha$ such that

$$Ny_1 = \lambda_0 y_1.$$

Let $x_1 := T^{-1}y_1$. Then

$$x_1 - Fx_1 = \lambda_0 Tx_1.$$

The hypotheses $y_1 \in \mathcal{P}$, $\|y_1\| = \alpha$ imply that $x_1 \in T^{-1}(\mathcal{P})$ and $\|Tx_1\| = \alpha$. Which lead to a contradiction with Condition (2.1).
Claim 2. We show that Condition (2.2) implies that

$$Ny + \mu p \neq y \text{ for } \|y\| = \beta \text{ and } \mu > 0.$$

On the contrary, assume the existence of $\mu_0 > 1$ and $y_2 \in \mathcal{P}$ with $\|y_2\| = \beta$ such that

$$y_2 - Ny_2 = \mu_0 p.$$

Let $x_2 := T^{-1}y_2$. Then

$$x_2 = Tx_2 + Fx_2 - \mu_0 p.$$

The hypotheses $y_2 \in \mathcal{P}$, $\|y_2\| = \beta$ imply that $x_2 \in T^{-1}(\mathcal{P})$ and $\|Tx_2\| = \beta$. Which lead to a contradiction with Condition (2.2).

Consequently, by Theorem 1.1, the operator N has a fixed point $y \in \mathcal{P}$ such that $r \leq \|y\| \leq R$. That is

$$T^{-1}y - FT^{-1}y = y.$$

Let $x := T^{-1}y$. Then $x \in T^{-1}(\mathcal{P})$, it is a fixed point of $T + F$, and

$$r \leq \|Tx\| \leq R. \qquad \square$$

If in addition $\mathcal{P}$ is a cone, as a consequence of Theorem 2.1, we derive the following cone compression and expansion fixed point theorems, the first in terms of the partial order relation induced by $\mathcal{P}$ and the second of norm type.

Corollary 2.2. *Let K be a subset of a Banach space X and $\mathcal{P} \subset X$ a cone. Assume that $T : K \to X$ is an expansive mapping with constant $h > 1$ and $F : K \to X$ is a mapping such that $I - F : K \to \mathcal{P}$ is completely continuous one with $\mathcal{P} \subset T(K)$. Let $\alpha, \beta > 0$, $\alpha \neq \beta$, $r := \min\{\alpha, \beta\}$ and $R := \max\{\alpha, \beta\}$.*
Suppose that the following conditions are satisfied:

$$x \not> Tx + Fx \quad \text{for } x \in T^{-1}(\mathcal{P}) \text{ with } \|Tx\| = \alpha. \tag{2.3}$$

$$x \not< Tx + Fx \quad \text{for } x \in T^{-1}(\mathcal{P}) \text{ with } \|Tx\| = \beta. \tag{2.4}$$

Then $T + F$ has a fixed point x in $T^{-1}(\mathcal{P})$ such that $r \leq \|Tx\| \leq R$.

Proof. The conditions (2.1) and (2.2) of Theorem 2.1 are satisfied. Indeed, assume the contrary of Condition (2.1). Then there exist $\lambda_0 > 1$ and $x_0 \in T^{-1}(\mathcal{P})$ with $\|Tx_0\| = \alpha$ such that

$$x_0 = \lambda_0 Tx_0 + Fx_0.$$

Thus, $Tx_0 = \frac{1}{\lambda_0}(x_0 - Fx_0) < x_0 - Fx_0$, that is $x_0 > Tx_0 + Fx_0$, which contradicts (2.3).
Assume the contrary of Condition (2.2). Then there exist $p \in \mathcal{P} \setminus \{0\}$, $\mu_0 > 0$ and $x_1 \in T^{-1}(\mathcal{P})$ with $\|Tx_1\| = \beta$ such that

$$x_1 = Tx_1 + Fx_1 - \mu_0 p.$$

Since $\mu_0 p \in \mathcal{P} \setminus \{0\}$, we obtain

$$x_1 < Tx_1 + Fx_1,$$

which contradicts (2.4). $\square$

Corollary 2.3. *Let K be a subset of a Banach space X and $\mathcal{P} \subset X$ a cone. Assume that $T : K \to X$ is an expansive mapping with constant $h > 1$ and $F : K \to X$ is a mapping such that $I - F : K \to \mathcal{P}$ is completely continuous one with $\mathcal{P} \subset T(K)$. Let $\alpha, \beta > 0$, $\alpha \neq \beta$, $r := \min\{\alpha, \beta\}$ and $R := \max\{\alpha, \beta\}$.*
Suppose that the following conditions are satisfied:

$$\|x - Fx\| \leq \|Tx\| \quad \text{for } x \in T^{-1}(\mathcal{P}) \text{ with } \|Tx\| = \alpha. \tag{2.5}$$

$$\|x - Fx\| \geq \|Tx\| \quad \text{for } x \in T^{-1}(\mathcal{P}) \text{ with } \|Tx\| = \beta. \tag{2.6}$$

Then $T+F$ has a fixed point x in $T^{-1}(\mathcal{P})$ such that $r \leq \|Tx\| \leq R$.

Proof. The conditions (2.1) and (2.2) of Theorem 2.1 are satisfied. Indeed, assume the contrary of Condition (2.1). Then there exist $\lambda_0 > 1$ and $x_0 \in T^{-1}(\mathcal{P})$ with $\|Tx_0\| = \alpha$ such that

$$x_0 = \lambda_0 Tx_0 + Fx_0.$$

Then $x_0 - Fx_0 = \lambda_0 Tx_0$, that is

$$\|x_0 - Fx_0\| = \lambda_0 \|Tx_0\| > \|Tx_0\|,$$

which contradicts (2.5).
Assume the contrary of Condition (2.2). Then there exist $p \in \mathcal{P} \setminus \{0\}$, $\mu_0 > 0$ and $x_1 \in T^{-1}(\mathcal{P})$ with $\|Tx_1\| = \beta$ such that

$$x_1 = Tx_1 + Fx_1 - \mu_0 p.$$

$x_1 - Fx_1 = Tx_1 - \mu_0 p$ that is

$$\|x_1 - Fx_1\| < \|Tx_1\|,$$

which contradicts (2.6). □

The vector version of Theorem 2.1 is presented in the following theorem. In what follows, we shall consider two Banach spaces $(X_1, \|.\|_1), (X_2, \|.\|_2)$; two wedges $\mathcal{P}_1 \subset X_1, \mathcal{P}_2 \subset X_2$, the product space $X := X_1 \times X_2$, the corresponding wedge $\mathcal{P} := \mathcal{P}_1 \times \mathcal{P}_2$ of X. For $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$, let $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$ for $i = 1, 2$, and $r = (r_1, r_2)$, $R = (R_1, R_2)$.

Theorem 2.4. *Let $K := K_1 \times K_2$ be a subset of X.
Assume that $T_i : K_i \subset X_i \to X_i$ be an expansive mapping with constant $h_i > 1$ and $F_i : K \to X_i$ is a mapping such that $I_i - F_i : K \to X_i$ be a completely continuous one with $\mathcal{P}_i \subset T(K_i)$, $i = 1, 2$ and $x_i - F_i(x_1, x_2) \in \mathcal{P}_i$ for $x_i \in K_i$, $i = 1, 2$.
Suppose that there exist $p_i \in \mathcal{P}_i \setminus \{0\}$, $i = 1, 2$ such that for each $i \in \{1, 2\}$ the following conditions are satisfied:*

$$x_i \neq \lambda T_i x_i + F_i x \;\; \textit{for} \;\; x_i \in T_i^{-1}(\mathcal{P}_i),\; \|T_i x_i\| = \alpha_i \;\textit{and}\; \lambda > 1. \tag{2.7}$$

$$x_i \neq T_i x_i + F_i x - \mu p_i \;\; \textit{for} \;\; x_i \in T_i^{-1}(\mathcal{P}_i),\; \|T_i x_i\| = \beta_i \;\textit{and}\; \mu > 0. \tag{2.8}$$

Then $T + F = (T_1 + F_1, T_2 + F_2)$ has a fixed point $x = (x_1, x_2)$ in $T_1^{-1}(\mathcal{P}_1) \times T_2^{-1}(\mathcal{P}_2)$ such that

$$r_i \leq \|T_i x_i\| \leq R_i \;\; \textit{for} \;\; i = 1, 2.$$

Proof. By Lemma 1.4, for $i \in \{1, 2\}$ the operator $T_i^{-1} : T(K_i) \to K_i$ is an $\frac{1}{h_i}$-contraction. Then the operator N defined by

$$\begin{aligned} N : \mathcal{P} \;&\to\; \mathcal{P} \\ y \;&\mapsto\; N(y_1, y_2) = (N_1(y_1, y_2), N_2(y_1, y_2)) \end{aligned}$$

where

$$\begin{cases} N_1(y_1, y_2) = T_1^{-1} y_1 - F_1(T_1^{-1} y_1, T_2^{-1} y_2) \\ \\ N_2(y_1, y_2) = T_2^{-1} y_2 - F_2(T_1^{-1} y_1, T_2^{-1} y_2) \end{cases}$$

is well defined and it is completely continuous.

Claim 1. We show that Condition (2.7) implies that

$$N_i y \neq \lambda y_i \text{ for } \|y_i\| = \alpha_i \text{ and } \lambda > 1 \text{ for } i = 1, 2.$$

On the contrary, assume the existence of $\lambda_0 > 1$ and , $y^0 = (y_1^0,\ y_2^0) \in \mathcal{P}$ with $\|y_i^0\| = \alpha_i$ such that

$$N_1 y^0 = \lambda_0 y_1^0 \text{ or } N_2 y^0 = \lambda_0 y_2^0.$$

Let $x_i^0 := T_i^{-1} y_i^0$ for $i = 1, 2$. Then, we obtain

$$x_1^0 - F_1(x_1^0, x_2^0) = \lambda_0 T_1 x_1^0$$

or

$$x_2^0 - F_1(x_1^0, x_2^0) = \lambda_0 T_2 x_2^0.$$

The hypotheses $y^0 \in \mathcal{P}$, $\|y_i^0\| = \alpha_i$ imply that $x_i^0 \in T_i^{-1}(\mathcal{P}_i)$ for $i = 1, 2$ with $\|T_i x_i^0\| = \alpha_i$, which lead to a contradiction with Condition (2.7).

Claim 2. We show that condition (2.8) implies that

$$N_i y + \mu p_i \neq y_i \text{ for } \|y_i\| = \beta_i \text{ and } \mu > 0 \text{ for } i = 1, 2.$$

On the contrary, assume the existence of $\mu_0 > 0$ and $z^0 = (z_1^0, z_2^0) \in \mathcal{P}$ with $\|z_i^0\| = \beta_i$ such that

$$z_1^0 - N_1 z^0 = \mu_0 p_1 \text{ or } z_2^0 - N_2 z^0 = \mu_0 p_2.$$

Let $t_i^0 := T_i^{-1} z_i^0$ for $i = 1, 2$. Then, we obtain

$$t_1^0 = T_1 t_1^0 + F_1(t_1^0, t_2^0) - \mu_0 p_1$$

or

$$t_2^0 = T_2 t_2^0 + F_2(t_1^0, t_2^0) - \mu_0 p_2.$$

The hypotheses $z^0 \in \mathcal{P}$, $\|z_i^0\| = \beta_i$ imply that $t_i^0 \in T_i^{-1}(\mathcal{P}_i)$ for $i = 1, 2$ with $\|T_i t_i^0\| = \beta_i$, which lead to a contradiction with condition (2.8). Our result then follows from Theorem 1.3. $\square$

Remark 2.5. Since the compact operator N in Theorems 1.1 and 1.3 may be generalized to a strict-set contraction, the conclusion of Theorems 2.1 (and its Corollaries) and Theorems 2.4 can be extended to the case of a ℓ-set contraction mapping $I - F$ $(0 < \ell < h)$ with respect to some measure of noncompactness (see [5]).

3. Applications

3.1. Example 1

Consider the following nonlinear boundary value problem

$$\begin{cases} -\frac{d^2}{dt^2} f(t, x(t)) = g(t) h(x(t)),\ 0 < t < 1 \\ \\ x(0) = x(1) = 0, \end{cases} \tag{3.1}$$

where $f : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous function defined by:

$$f(t, u) = u^3 + a(t)u,\ a \in \mathcal{C}^2([0, 1], \mathbb{R}_+),\ \text{with} \min_{t \in [0,1]} a(t) > 1,$$

$g \in \mathcal{C}([0, 1], \mathbb{R}_+)$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous increasing function.

Problem (3.1) is equivalent to the integral equation

$$f(t,x(t)) = \int_0^1 G(t,s)g(s)h(x(s))ds,\ t \in [0,1], \tag{3.2}$$

where G is the corresponding Green's function defined in $[0,1]\times[0,1]$ by:

$$G(t,s) = \begin{cases} t(1-s), & if \quad 0 \le t \le s \le 1, \\ s(1-t), & if \quad 0 \le s \le t \le 1. \end{cases} \tag{3.3}$$

The Green function satisfies the following properties:

$$\begin{array}{rcl} 0 \le G(t,s) & \le & G(s,s),\ \forall\,(t,s) \in [0,1]\times[0,1], \\ G(t,s) & \ge & \frac{1}{4}G(s,s),\ \forall (t,s) \in [\frac{1}{4},\frac{3}{4}]\times[0,1]. \\ \int_0^1 G(t,s)\,ds & \le & \frac{1}{8},\ \forall\, t \in [0,1]. \\ \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)\,ds & \ge & \frac{1}{16},\ \forall\, t \in [\frac{1}{4},\frac{3}{4}]. \end{array}$$

We will set

$$A := \max_{t\in[0,1]} \int_0^1 G(t,s)g(s)\,ds,$$

$$B := \frac{1}{4}\int_{\frac{1}{4}}^{\frac{3}{4}} G(t_0,s)g(s)\,ds, \quad \text{for some } t_0 \in [0,1].$$

We let

($\mathcal{C}_0$) $1 < a_0 := \min_{t\in[0,1]} a(t) \le a^0 := \max_{t\in[0,1]} a(t).$

Assume that the following assumptions hold for some positive reals α, β with $\alpha \ne \beta$:

($\mathcal{C}_1$) $Ah\left(\frac{1}{a_0}\alpha\right) \le \alpha,$

($\mathcal{C}_2$) $Bh\left(\frac{1}{4}\beta_0\right) \ge \beta$, where $\beta_0 = \beta_0(\beta) > 0$ such that $\beta_0^3 + a^0\beta_0 = \beta$.

Remark 3.1. From the properties of Green's function, we get

$$\max_{t\in[0,1]} \int_0^1 G(t,s)g(s)\,ds \le \frac{1}{8}\max_{t\in[0,1]} g(t)$$

and

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)g(s)\,ds \ge \frac{1}{16}\min_{t\in[\frac{1}{4},\frac{3}{4}]} g(t).$$

Then, for the conditions ($\mathcal{C}_1$) and ($\mathcal{C}_2$) to be satisfied it is enough that constants α and β satisfy

$$\frac{1}{8}\max_{t\in[0,1]} g(t)h\left(\frac{1}{a_0}\alpha\right) \le \alpha \ \text{ and } \ \frac{1}{16}\min_{t\in\in[\frac{1}{4},\frac{3}{4}]} g(t)h\left(\frac{1}{4}\beta_0\right) \ge \beta.$$

Now we state our main result

Theorem 3.2. *Let Assumptions ($\mathcal{C}_0$)-($\mathcal{C}_2$) be satisfied. Then the nonlinear boundary value problem has a solution x which belongs to $\mathcal{C}([0,1],\mathbb{R}_+)$.*

Proof. Consider the Banach space $X = \mathcal{C}([0,1])$ normed by $\|x\| = \max\limits_{t\in[0,1]} |x(t)|$, the set

$$K = \{x \in X \mid x(t) \geqslant 0, \forall\, t \in [0,1]\}$$

and the positive cone $\mathcal{P}$

$$\mathcal{P} = \left\{ x \in X : x \geq 0 \text{ on } [0,1] \text{ and } x(t) \geq \frac{1}{4}\|x\| \text{ for } \frac{1}{4} \leq t \leq \frac{3}{4} \right\}.$$

Define the operators $T : K \to K$ and $F : K \to X$ by

$$Tx(t) = x(t)^3 + a(t)x(t)$$

$$Fx(t) = x(t) - \int_0^1 G(t,s)g(s)h(x(s))\, ds,$$

respectively, for $t \in [0,1]$. Then the integral equation (3.2) is equivalent to the operational equation $x = Tx + Fx$. We check that all assumptions of Theorem 2.1 are satisfied.

(a) The operator $T : K \to K$ is surjective and it is expansive with constant $a_0 > 1$.

(b) Using the Arzela-Ascoli compactness criteria, we can show that $I - F$ maps bounded sets of K into relatively compact sets. In view of the sup-norm and the continuity of functions G, g and h, it is easily checked that $I - F$ is continuous. Therefore, the operator $I - F : K \to \mathcal{P}$ is completely continuous.

(c) Assume the existence of $x_0 \in T^{-1}(\mathcal{P})$ with $\|Tx_0\| = \alpha$ and $\lambda_0 > 1$ such that

$$x_0 = \lambda_0 T x_0 + F x_0,$$

Then, $\lambda_0 T x_0 = x_0 - F x_0 = \int_0^1 G(.,s)g(s)h(x_0(s))\, ds$ on $[0,1]$.

So

$$\alpha < \lambda_0 \|Tx_0\| = \max_{t\in[0,1]} \int_0^1 G(t,s)g(s)h(x_0(s))\, ds. \tag{3.4}$$

On the other hand, we have

$$\|x_0\| = \|T^{-1}Tx_0\| \leq \frac{1}{a_0}\, \|Tx_0\| = \frac{1}{a_0}\alpha,$$

where $\frac{1}{a_0} < 1$ is the Liptchiz constant of T^{-1}, which implies that

$$0 \leq x_0(t) \leq \frac{1}{a_0}\alpha \ \text{ for } \ t \in [0,1].$$

Since the function h is increasing, we get

$$0 \leq h(x_0(t)) \leq h\left(\frac{1}{a_0}\alpha\right) \ \text{ for } \ t \in [0,1].$$

Thus, for all $t \in [0,1]$, we obtain

$$\begin{aligned}\int_0^1 G(t,s)g(s)h(x_0(s))\,ds &\le h\left(\tfrac{1}{a_0}\alpha\right)\int_0^1 G(t,s)g(s)\,ds\\ &\le \|\textstyle\int_0^1 G(.,s)g(s)\,ds\|\, h\left(\tfrac{1}{a_0}\alpha\right)\\ &\le Ah\left(\tfrac{1}{a_0}\alpha\right) \le \alpha.\end{aligned}$$

By passage to the maximum, we obtain

$$\max_{t\in[0,1]}\int_0^1 G(t,s)g(s)h(x_0(s))\,ds \le \alpha,$$

which leads to a contradiction with (3.4).

(d) Assume the existence of $x_1 \in T^{-1}(\mathcal{P})$ with $\|Tx_1\| = \beta$ and $\mu_0 > 0$ such that

$$x_1 = Tx_1 + Fx_1 - \mu_0 y_0,$$

where $y_0 \in \mathcal{P}$ with $y_0(t) > 0$ on $[0,1]$. Then

$$\int_0^1 G(.,s)g(s)h(x_1(s))\,ds = x_1 - Fx_1 = Tx_1 - \mu_0 y_0 < Tx_1 \text{ on } [0,1].$$

Since for all $t \in [0,1]$, $(Tx_1)(t) \le \|Tx_1\| = \beta$, we get

$$\int_0^1 G(t,s)g(s)h(x_1(s))\,ds < (Tx_1)(t) \le \beta,\ \forall\, t \in [0,1]. \tag{3.5}$$

On the other hand, from the property of Green's function G, for all $t \in [\frac{1}{4},\frac{3}{4}]$, we have

$$\begin{aligned}\int_0^1 G(t,s)g(s)h(x_1(s))\,ds &\ge \tfrac{1}{4}\int_{\frac14}^{\frac34} G(s,s)g(s)h(x_1(s))\,ds\\ &\ge \tfrac{1}{4}\int_{\frac14}^{\frac34} G(t_0,s)g(s)h(x_1(s))\,ds.\end{aligned}$$

Since $\|Tx_1\| = \beta$ there exists $t_1 \in [0,1]$ such that $(Tx_1)(t_1) = \beta$. That is

$$(x_1(t_1))^3 + a(t_1)x_1(t_1) = \beta \le (x_1(t_1))^3 + a^0 x_1(t_1),$$

where $a^0 = \max\limits_{t\in[0,1]} a(t)$. Let $\beta_0 = \beta_0(\beta) > 0$ such that $\beta_0^3 + a^0\beta_0 = \beta$. So $x_1(t_1) \ge \beta_0$, which implies that $\|x_1\| \ge \beta_0$. Hence $x_1(s) \ge \frac{1}{4}\beta_0$, $\forall s \in [\frac{1}{4},\frac{3}{4}]$, which gives

$$h(x_1(s)) \ge h\left(\frac{1}{4}\beta_0\right).$$

Thus

$$\int_0^1 G(t,s)g(s)h(x_1(s))\,ds \geq \frac{1}{4}\,h\left(\frac{1}{4}\beta_0\right)\int_{\frac{1}{4}}^{\frac{3}{4}} G(t_0,s)g(s)\,ds = Bh\left(\frac{1}{4}\beta_0\right) \geq \beta,$$

which leads to a contradiction with (3.5). Therefor Theorem 2.1 applies and assure that Problem (3.1) has at least one positive solution $x \in \mathcal{C}([0,1])$ such that

$$r \leq \|Tx\| \leq R,$$

where $r = \min(\alpha,\beta)$ and $R = \max(\alpha,\beta)$. □

3.2. Example 2

Consider the following second-order nonlinear boundary value problem posed on the positive half-line

$$\begin{cases} -\frac{d^2}{dt^2}f(t,x(t)) + k^2 f(t,x(t)) = g(t)h(t,x(t)),\ t \in (0,+\infty). \\ x(0) = 0, \quad \lim\limits_{t\to+\infty} x(t) = 0, \end{cases} \tag{3.6}$$

where k is a positive real parameter and $f : [0,+\infty) \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function defined by:

$$f(t,u) = u^3 + a(t)u,\ a \in \mathcal{C}^2([0,+\infty),\mathbb{R}_+).$$

The functions $g : [0,+\infty) \to \mathbb{R}_+$ and $h : [0,+\infty) \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous.
Problem (3.6) is equivalent to the integral equation

$$f(t,x(t)) = \int_0^{+\infty} G(t,s)g(s)h(s,x(s))ds, \tag{3.7}$$

where G is the corresponding Green's function defined by:

$$G(t,s) = \frac{1}{2k}\begin{cases} e^{-ks}(e^{kt} - e^{-kt}), & if \quad 0 < t \leq s < \infty, \\ e^{-kt}(e^{ks} - e^{-ks}), & if \quad 0 < s \leq t < \infty. \end{cases}$$

The Green function G satisfies the following useful estimates:

$$G(t,s) \leq G(s,s) \leq \tfrac{1}{2k},\ \ \forall\, t,s \in [0,+\infty).$$

$$G(t,s)e^{-\mu t} \leq G(s,s)e^{-ks},\ \forall\, t,s \in [0,+\infty),\ \forall\, \mu \geq k.$$

$$G(t,s) \geq \Lambda G(s,s)e^{-ks},\ \forall\,(0 < \gamma < \delta),\ \ \forall\, t \in [\gamma,\delta]\,,\ \forall\, s \in [0,+\infty),$$

where

$$0 < \Lambda = \min(e^{-k\delta}, e^{k\gamma} - e^{-k\gamma}) < 1.$$

Assume that the following conditions are satisfied

$$(\mathcal{H}_0)\ \ 1 < a_0 \colon = \inf_{t\in[0,+\infty)} a(t) \leq a^0 \colon = \sup_{t\in[0,+\infty)} a(t).$$

$(\mathcal{H}_1)$ $h : [0, +\infty) \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and satisfies the polynomial growth condition:

$$\exists\, d > 0 : d \neq 1,\ 0 \leq h(t,x) \leq b(t) + c(t)x^d,\ \forall (t,x) \in [0,+\infty) \times \mathbb{R}_+,$$

where the functions $b, c \in \mathcal{C}([0,+\infty), \mathbb{R}_+)$.

$(\mathcal{H}_2)$ Assume the integrals

$$\begin{cases} M_1 & := & \int\limits_0^\infty e^{-ks} b(s) G(s,s) g(s) ds \\ M_2 & := & \int\limits_0^\infty e^{(d\theta - k)s} c(s) G(s,s) g(s) ds \end{cases}$$

are convergent and satisfy

$$\exists\, R > 0,\ M_1 + M_2 \frac{1}{a_0^d} R^d \leq R.$$

$(\mathcal{H}_3)$ There exists r with $0 < r < R$ such that

$$\Lambda \int\limits_\gamma^\delta e^{-ks} G(s,s) g(s) h(s,u)\, ds \geq r e^{\theta \delta} \quad \text{for all} \quad u \geq \Lambda r_0,$$

where $r_0 = r_0(r) > 0$ such that $r_0^3 + a^0 r_0 = r$.

Now we state our main result.

Theorem 3.3. *Let Assumptions $(\mathcal{H}_0)$-$(\mathcal{H}_3)$ be satisfied. Then the nonlinear boundary value problem (3.6) has at least one positive solution.*

Proof. Given a real parameter $\theta \geq k$ and consider the weighted Banach space

$$X = \left\{ x \in \mathcal{C}([0,+\infty), \mathbb{R}) : \sup_{t \in [0,+\infty)} \{ e^{-\theta t} |x(t)| \} < \infty \right\}$$

normed by

$$\|x\|_\theta = \sup_{t \in [0,+\infty)} \{ e^{-\theta t} |x(t)| \}.$$

Consider the set

$$K = \{ x \in X \mid x(t) \geqslant 0, \forall\, t \in [0,+\infty) \}.$$

For arbitrary positive real numbers $0 < \gamma < \delta$, let $\mathcal{P}$ the positive cone defined in X by

$$\mathcal{P} = \left\{ x \in X : \ x \geq 0 \text{ on } [0,+\infty) \text{ and } \min_{t \in [\gamma,\delta]} x(t) \geq \Lambda \|x\|_\theta \right\}.$$

Define the operators $T : K \to K$ and $F : K \to X$ by:

$$Tx(t) = x(t)^3 + a(t)x(t)$$

$$Fx(t) = x(t) - \int\limits_0^{+\infty} G(t,s) g(s) h(s, x(s))\, ds,$$

respectively, for $t \in [0,+\infty)$.Then the integral equation (3.7) is equivalent to the operational equation $x = Tx + Fx$. We check that all assumptions of Theorem 2.1 are satisfied:

(a) The operator $T : K \to K$ is surjective and it is expansive with constant $a_0 > 1$.

(b) Using the properties of Green function G and appealing to the Zima compactness criteria (see [17, 18]), we can show that the operator $I - F : K \to \mathcal{P}$ is completely continuous (see [7, 8]).

(c) Assume the existence of $x_0 \in T^{-1}(\mathcal{P})$ with $\|Tx_0\|_\theta = R$ and $\lambda_0 > 1$ such that

$$x_0 = \lambda_0 T x_0 + F x_0,$$

Then, $\lambda_0 T x_0 = x_0 - F x_0 = \int\limits_0^{+\infty} G(.,s)g(s)h(s,x_0(s))\,ds$ on $[0,+\infty)$.

So

$$R < \lambda_0\|Tx_0\|_\theta = \|(I-F)x_0\|_\theta. \tag{3.8}$$

On the other hand, we have

$$\|x_0\|_\theta = \|T^{-1}Tx_0\|_\theta \leq \frac{1}{a_0}\,\|Tx_0\|_\theta = \frac{1}{a_0}\,R,$$

where $\frac{1}{a_0} < 1$ is the Liptchiz constant of T^{-1}. Thus, by Assumptions $(\mathcal{H}_1), (\mathcal{H}_2)$ and the properties of function G, for all $t \in [0,+\infty)$, we obtain

$$\begin{aligned}
|(I-F)x_0(t)|e^{-\theta t} &= \int\limits_0^{+\infty} e^{-\theta t} G(t,s)g(s)h(s,x_0(s))\,ds \\
&\leq \int\limits_0^{+\infty} e^{-ks} G(s,s)g(s)[b(s) + c(s)|x_0(s)|^d]\,ds \\
&\leq \int\limits_0^{+\infty} e^{-ks} G(s,s)g(s)b(s)\,ds \\
&\quad + \|x_0\|_\theta^d \int\limits_0^{+\infty} e^{(d\theta-k)s} G(s,s)g(s)c(s)\,ds \\
&\leq M_1 + M_2\|x_0\|_\theta^d \\
&\leq M_1 + \tfrac{1}{a_0^d}\,R^d \leq R.
\end{aligned}$$

By passage to the supremum over t, we get

$$\sup_{t\in[0,+\infty)} \{|(I-F)x_0(t)|e^{-\theta t}\} \leq M_1 + M_2\|x_0\|_\theta^d \leq R,$$

which leads to a contradiction with (3.8).

(d) Assume the existence of $x_1 \in T^{-1}(\mathcal{P})$ with $\|Tx_1\|_\theta = r$ and $\mu_0 > 0$ such that

$$x_1 = Tx_1 + Fx_1 - \mu_0 y_0,$$

where $y_0 \in \mathcal{P}$ with $y_0(t) > 0$ on $[0,+\infty)$. Then

$$\int\limits_0^{+\infty} G(t,s)g(s)h(s,x_1(s))\,ds = x_1 - Fx_1 = Tx_1 - \mu_0 y_0 < Tx_1.$$

Since for all $t \in [0,+\infty)$, $|(Tx_1)(t)|e^{-\theta t} \le \|Tx_1\|_\theta = r$, we get

$$\int_0^{+\infty} G(t,s)g(s)h(s,x_1(s))\,ds < (Tx_1)(t) \le re^{\theta\delta},\ \forall\, t \in [\gamma,\delta]. \tag{3.9}$$

On the other hand, $\|Tx_1\|_\theta = r$ implies one of the following cases:
Case 1. There exists $t_1 \in [0,+\infty)$ such that $|(Tx_1)(t_1)|e^{-\theta t_1} = r$. That is

$$(e^{-\theta t_1}x_1(t_1))^3 + a(t_1)e^{-\theta t_1}x_1(t_1) = r \le (e^{-\theta t_1}x_1(t_1))^3 + a^0 e^{-\theta t_1}x_1(t_1),$$

where $a^0 = \sup\limits_{t\in[0,+\infty)} a(t)$. Let $r_0 = r_0(r) > 0$ such that $r_0^3 + a^0 r_0 = r$.
Thus, $e^{-\theta t_1}x_1(t_1) \ge r_0$, which implies that $\|x_1\|_\theta \ge r_0$. Hence $x_1(s) \ge \Lambda\, r_0$, $\forall s \in [\gamma,\delta]$.
Case 2. $\lim\limits_{t\to+\infty} |(Tx_1)(t)|e^{-\theta t} = r$. That is

$$\begin{aligned} &\lim_{t\to+\infty}(e^{-\theta t}x_1(t))^3 + \lim_{t\to+\infty} a(t) \lim_{t\to+\infty} e^{-\theta t}x_1(t) = r \\ \le\ &\lim_{t\to+\infty}(e^{-\theta t}x_1(t))^3 + a^0 \lim_{t\to+\infty} e^{-\theta t}x_1(t). \end{aligned}$$

Thus, there exists $r_0 = r_0(r) > 0$ such that

$$\lim_{t\to+\infty} e^{-\theta t}x_1(t) \ge r_0,$$

which gives $\|x_1\|_\theta \ge r_0$.
Consequently, from Assumption $(\mathcal{H}_2)$ and the properties of Green function G, for all $t \in [\gamma,\delta]$, we have

$$\begin{aligned} \int_0^{+\infty} G(t,s)g(s)h(s,x_1(s))\,ds &\ge \Lambda \int_0^{+\infty} e^{-ks}G(s,s)g(s)h(s,x_1(s))\,ds \\ &\ge \Lambda \int_\gamma^\delta e^{-ks}G(s,s)g(s)h(s,x_1(s))\,ds \\ &\ge re^{\theta\delta}, \end{aligned}$$

which leads to a contradiction with (3.9). Then Theorem 2.1 applies. Therefore, Problem (3.6) has at least one solution $x \in K$ such that

$$r \le \|Tx\| \le R. \qquad \square$$

3.3. Example 3

In the following example, we will use the Theorem 2.4 to study the existence of positive solutions to a boundary value problem for a system of differential equations of the second order. A study that allows the nonlinear term of our system to have different behaviors both in components and in variables, and it gives a kind of localization of each component of a solution.

Consider the following nonlinear boundary value problem for system of two differential equations with Dirichlet condition

$$\begin{cases} -\frac{d^2}{dt^2} f_1(t, x_1(t)) = g_1(t) h_1(x_1(t), x_2(t)), \ 0 < t < 1 \\ -\frac{d^2}{dt^2} f_2(t, x_2(t)) = g_2(t) h_2(x_1(t), x_2(t)), \ 0 < t < 1 \\ x_1(0) = x_1(1) = 0, \\ x_2(0) = x_2(1) = 0, \end{cases} \tag{3.10}$$

where for $i \in \{1,2\}$, $f_i : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions defined by:

$$f_i(t, u) = u^3 + a_i(t) u, \ a_i \in \mathcal{C}^2([0,1], \mathbb{R}_+).$$

$g_i \in \mathcal{C}([0,1], \mathbb{R}_+)$ and $h_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous increasing functions with respect to its two variables.

The system (3.10) is equivalent to the integral system

$$\begin{cases} f_1(t, x_1(t)) = \int\limits_0^1 G(t,s) g_1(s) h_1(x(s)) ds, \ t \in [0,1] \\ f_2(t, x_2(t)) = \int\limits_0^1 G(t,s) g_2(s) h_2(x(s)) ds, \ t \in [0,1], \end{cases} \tag{3.11}$$

where $x = (x_1, x_2)$ and G is the corresponding Green's function given in (3.3). We will set

$$\begin{aligned} A_i &:= \max_{t \in [0,1]} \int_0^1 G(t,s) g_i(s)\, ds, \\ B_i &:= \frac{1}{4} \int\limits_{\frac{1}{4}}^{\frac{3}{4}} G(t_i^0, s) g_i(s)\, ds, \quad \text{for some } t_i^0 \in [0,1]. \end{aligned}$$

In what follows we consider $i \in \{1,2\}$ and let

($\mathbf{C}_0$) $1 < a_i^0 := \min_{t \in [0,1]} a_i(t) \le b_i^0 := \max_{t \in [0,1]} a_i(t)$.

Assume that the following assumptions hold for some α_i, β_i with $\alpha_i \neq \beta_i$:

($\mathbf{C}_1$) $A_i h_i(\frac{1}{a_1^0}\alpha_1, \frac{1}{a_2^0}\alpha_2) \le \alpha_i$,

($\mathbf{C}_2$) $B_i h_i(\frac{1}{4}\beta_1^0, \frac{1}{4}\beta_2^0) \ge \beta_i$, where $\beta_i^0 = \beta_i^0(\beta_i) > 0$ such that $(\beta_i^0)^3 + b_i^0 \beta_i^0 = \beta_i$.

Our main existence result on system (3.10) is

Theorem 3.4. *Let Assumptions ($\mathbf{C}_0$)-($\mathbf{C}_2$) be satisfied. Then the system (3.10) has a solution $x = (x_1, x_2)$ which belongs to $C([0,1], \mathbb{R}_+) \times C([0,1], \mathbb{R}_+)$.*

Proof. We apply Theorem 2.4. Here $X_1 = X_2 = C[0,1]$ with norm

$$\|u\|_\infty = \max_{t \in [0,1]} |u(t)|,$$

and

$$K_1 = K_2 = \{u \in C[0,1] : \ u(t) \ge 0 \text{ for all } t \in [0,1]\};$$

$$\mathcal{P}_1 = \mathcal{P}_2 = \left\{ u \in u \in C[0,1] : u \ge 0 \text{ on } [0,1] \text{ and } u(t) \ge \frac{1}{4}\|u\| \ \text{ for } \ \frac{1}{4} \le t \le \frac{3}{4} \right\}.$$

Define the operators $T_i : K_i \to K_i$ and $F_i : K_1 \times K_2 \to X_i$, for $i = 1, 2$, by:

$$T_i x_i(t) = x_i(t)^3 + a_i(t) x_i(t)$$

$$F_ix(t) = x_i(t) - \int_0^1 G(t,s)g_i(s)h_i(x(s))\,ds,$$

respectively, for $t \in [0,1]$.

Then, the integral system (3.11) is equivalent to the operator equation

$$(x_1, x_2) = (T_1x_1 + F_1(x_1, x_2), T_2x_2 + F_2(x_1, x_2)),$$

According to Theorem 2.4 and in a way similar to the one used to show Theorem 3.2, we can easily show that the system (3.10) has at least one positive solution $x = (x_1, x_2)$ which belongs to $C[0,1] \times C[0,1]$ such that

$$r_i \leq \|T_ix_i\| \leq R_i,$$

where $r_i = \min(\alpha_i, \beta_i)$ and $R_i = \max(\alpha_i, \beta_i)$ for $i = 1, 2$. □

Acknowledgments. The authors are grateful to Professor Radu Precup, Babeş–Bolyai University, Romania, for his valuable remarks and suggestions. The authors acknowledge support of "Direction Générale de la Recherche Scientifique et du Développement Technologique (DGRSDT)", MESRS, Algeria.

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 439–448
DOI: 10.24193/subbmath.2023.2.17

Four-dimensional Riemannian product manifolds with circulant structures

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Abstract. A 4-dimensional Riemannian manifold equipped with an additional tensor structure, whose fourth power is the identity, is considered. This structure has a circulant matrix with respect to some basis, i.e. the structure is circulant, and it acts as an isometry with respect to the metric. The Riemannian product manifold associated with the considered manifold is studied. Conditions for the metric, which imply that the Riemannian product manifold belongs to each of the basic classes of Staikova-Gribachev's classification, are obtained. Examples of such manifolds are given.

Mathematics Subject Classification (2010): 53B20, 53C15, 15B05.

Keywords: Riemannian metric, almost product structure, circulant matrix.

1. Introduction

The study of Riemannian manifolds M with a metric g and an almost product structure P is initiated by K. Yano in [15]. The classification of the almost product manifolds (M, g, P) with respect to the covariant derivative of P is made by A. M. Naveira in [11]. The manifolds (M, g, P) with zero trace of P are classified with respect to the covariant derivative of P by M. Staikova and K. Gribachev in [14]. The basic classes of this classification are $\mathcal{W}_1, \mathcal{W}_2$ and $\mathcal{W}_3$. Their intersection is the class $\mathcal{W}_0$ of Riemannian P-manifolds. The class of the Riemannian product manifolds is $\mathcal{W}_1 \oplus \mathcal{W}_2$. It is formed by manifolds with an integrable structure P. The class $\mathcal{W}_1$ consists of the conformal Riemannian P-manifolds. Some of the recent studies of the Riemannian almost product manifolds are made in [4], [5], [7], [8] and [12].

Problems of differential geometry of a 4-dimensional Riemannian manifold M with a tensor structure Q of type $(1, 1)$, which satisfies $Q^4 = \mathrm{id}$, $Q^2 \neq \pm\mathrm{id}$, are

Received 16 May 2020; Accepted 14 July 2021.

considered in [1], [2], [3], [10] and [13]. The matrix of Q in some basis is circulant and Q is compatible with the metric g, so that an isometry is induced in any tangent space on M. Such a manifold (M, g, Q) is associated with a Riemannian almost product manifold (M, g, P), where $P = Q^2$ and $\mathrm{tr}P = 0$.

In the present work we continue studying the manifold (M, g, Q) and the associated manifold (M, g, P). Our purpose is to determine their position, among the well-known manifolds, by using the classifications in [11] and [14]. In Sect. 2, we recall some necessary facts about (M, g, Q) and about (M, g, P). In Sect. 3, we obtain the components of the fundamental tensor F, determined by the metric g and by the covariant derivative of P. We establish that (M, g, P) is a Riemannian product manifold, i.e. (M, g, P) belongs to the class $\mathcal{W}_1 \oplus \mathcal{W}_2$. We find necessary and sufficient conditions under which (M, g, P) belongs to each of the classes $\mathcal{W}_0$, $\mathcal{W}_1$ and $\mathcal{W}_2$. In Sect. 4, we give some examples of the considered Riemannian product manifolds.

2. Preliminaries

Let M be a 4-dimensional Riemannian manifold equipped with a tensor structure Q of type $(1, 1)$. The structure Q has a circulant matrix, with respect to some basis, as follows:

$$(Q_j^k) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.1}$$

Then Q has the properties

$$Q^4 = \mathrm{id}, \quad Q^2 \neq \pm\mathrm{id}.$$

Let the metric g and the structure Q satisfy

$$g(Qx, Qy) = g(x, y), \quad x, y \in \mathfrak{X}(M).$$

The above condition and (2.1) imply that the matrix of g has the form:

$$(g_{ij}) = \begin{pmatrix} A & B & C & B \\ B & A & B & C \\ C & B & A & B \\ B & C & B & A \end{pmatrix}. \tag{2.2}$$

Here A, B and C are smooth functions of an arbitrary point $p(x^1, x^2, x^3, x^4)$ on M. It is supposed that $A > C > B > 0$ in order g to be positive definite. The manifold (M, g, Q) is introduced in [13].

Anywhere in this work, x, y, z will stand for arbitrary elements of the algebra $\mathfrak{X}(M)$ of the smooth vector fields on M. The Einstein summation convention is used, the range of the summation indices being always $\{1, 2, 3, 4\}$.

In [1], it is noted that the manifold (M, g, P), where $P = Q^2$, is a Riemannian manifold with an almost product structure P, because $P^2 = \mathrm{id}$, $P \neq \pm\mathrm{id}$ and $g(Px, Py) = g(x, y)$. Moreover $\mathrm{tr}P = 0$. For such manifolds Staikova-Gribachev's classification is valid [14]. This classification was made with respect to the tensor F of type $(0,3)$ and the associated 1-form θ, which are defined by

$$F(x,y,z)=g((\nabla_x P)y,z), \quad \theta(x)=g^{ij}F(e_i,e_j,x). \tag{2.3}$$

Here ∇ is the Levi-Civita connection of g, and g^{ij} are the components of the inverse matrix of (g_{ij}) with respect to an arbitrary basis $\{e_i\}$. The tensor F has the following properties

$$F(x,z,y)=F(x,y,z), \quad F(x,Py,Pz)=-F(x,y,z). \tag{2.4}$$

The manifolds (M,g,P) with an integrable structure P are called Riemannian product manifolds and they form the class $\mathcal{W}_1\oplus\mathcal{W}_2$. The characteristic conditions for the classes $\mathcal{W}_0$, $\mathcal{W}_1,\mathcal{W}_2$ and $\mathcal{W}_1\oplus\mathcal{W}_2$ are the following:

i) $\mathcal{W}_0$:

$$F(x,y,z)=0, \tag{2.5}$$

ii) $\mathcal{W}_1$:

$$\begin{aligned} F(x,y,z)=\frac{1}{4}\big((g(x,y)\theta(z)+g(x,z)\theta(y)\\ -g(x,Py)\theta(Pz)-g(x,Pz)\theta(Py)\big), \end{aligned} \tag{2.6}$$

iii) $\mathcal{W}_2$:

$$F(x,y,Pz)+F(y,z,Px)+F(z,x,Py)=0, \quad \theta(z)=0, \tag{2.7}$$

iv) $\mathcal{W}_1\oplus\mathcal{W}_2$:

$$F(x,y,Pz)+F(y,z,Px)+F(z,x,Py)=0. \tag{2.8}$$

In the next section, we obtain conditions under which (M,g,P) belongs to each of the classes i) – iv) and to some of their subclasses. Thus the following statements are useful for the completeness of our research.

Theorem 2.1. [10] *If the structure Q of (M,g,Q) satisfies $\nabla Q=0$, then $\nabla P=0$, i.e. (M,g,P) belongs to the class $\mathcal{W}_0$.*

Theorem 2.2. [13] *The structure Q of (M,g,Q) satisfies $\nabla Q=0$ if and only if the following equalities are valid:*

$$\begin{aligned} &A_3=C_1, \quad A_1=C_3, \quad B_3=B_1, \quad 2B_1=C_4+C_2,\\ &B_4=B_2, \quad A_2=C_4, \quad A_4=C_2, \quad 2B_2=C_1+C_3, \end{aligned} \tag{2.9}$$

where $A_i=\frac{\partial A}{\partial x^i}$, $B_i=\frac{\partial B}{\partial x^i}$, $C_i=\frac{\partial C}{\partial x^i}$.

3. The fundamental tensor F on (M,g,P)

The components of Nijenhuis tensor N of the almost product structure P are determined by the equalities

$$N_{ij}^k=P_i^a(\partial_a P_j^k-\partial_j P_a^k)-P_j^a(\partial_a P_i^k-\partial_i P_a^k).$$

It is known from [14] that the vanishing of the Nijenhuis tensor N is equivalent to the condition (2.8).

Using (2.1), we get that the components of the almost product structure $P = Q^2$ on (M, g, P) are given by the matrix

$$(P_j^k) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

Evidently the Nijenhuis tensor N of P vanishes, so we have the following

Lemma 3.1. *The manifold (M, g, P) belongs to the class $\mathcal{W}_1 \oplus \mathcal{W}_2$.*

Further, we will consider each of the cases when the fundamental tensor F on (M, g, P) satisfies the identities (2.5), (2.6) or (2.7), since F satisfies (2.8). First we calculate the components of F.

Lemma 3.2. *The nonzero components $F_{ijk} = F(e_i, e_j, e_k)$ of the fundamental tensor F on the manifold (M, g, P) are given by*

$$\begin{aligned}
&F_{111} = -F_{133} = A_3 - C_1, && 2F_{112} = -2F_{134} = A_4 - B_1 - C_2 + B_3, \\
&F_{122} = -F_{144} = B_4 - B_2, && 2F_{114} = -2F_{123} = A_2 - B_1 - C_4 + B_3, \\
&F_{211} = -F_{233} = B_3 - B_1, && 2F_{212} = -2F_{234} = A_3 - B_2 - C_1 + B_4, \\
&F_{222} = -F_{244} = A_4 - C_2, && 2F_{223} = -2F_{214} = A_1 + B_4 - C_3 - B_2, \\
&F_{311} = -F_{333} = C_3 - A_1, && 2F_{334} = -2F_{312} = A_2 + B_1 - C_4 - B_3, \\
&F_{322} = -F_{344} = B_4 - B_2, && 2F_{323} = -2F_{314} = A_4 + B_1 - C_2 - B_3, \\
&F_{411} = -F_{433} = B_3 - B_1, && 2F_{434} = -2F_{412} = A_1 + B_2 - C_3 - B_4, \\
&F_{422} = -F_{444} = C_4 - A_2, && 2F_{414} = -2F_{423} = A_3 - B_4 - C_1 + B_2.
\end{aligned} \tag{3.2}$$

Proof. The inverse matrix of (g_{ij}) has the form:

$$(g^{ik}) = \frac{1}{D} \begin{pmatrix} \overline{A} & \overline{B} & \overline{C} & \overline{B} \\ \overline{B} & \overline{A} & \overline{B} & \overline{C} \\ \overline{C} & \overline{B} & \overline{A} & \overline{B} \\ \overline{B} & \overline{C} & \overline{B} & \overline{A} \end{pmatrix}, \tag{3.3}$$

where

$$\begin{aligned}
\overline{A} =& A(A + C) - 2B^2, \ \overline{B} = B(C - A), \ \overline{C} = 2B^2 - C(A + C), \\
D =& (A - C)\big((A + C)^2 - 4B^2\big).
\end{aligned} \tag{3.4}$$

The next formula for the Christoffel symbols Γ of ∇ is well known:

$$2\Gamma_{ij}^k = g^{ak}(\partial_i g_{aj} + \partial_j g_{ai} - \partial_a g_{ij}).$$

Then, using (2.2), (3.3) and (3.4), we calculate these coefficients. They are given below:

$$
\begin{aligned}
\Gamma^i_{ii} &= \frac{1}{2D}\big(B(C-A)(4B_i - A_j - A_s) + 2B^2(2C_i - A_i - A_k) \\
&\qquad + A(A+C)A_i - C(A+C)(2C_i - A_k)\big), \\
\Gamma^j_{ii} &= \frac{1}{2D}\big(B(C-A)(A_i + 2C_i - A_k) + 2B^2(A_j - A_s) \\
&\qquad + A(A+C)(2B_i - A_j) - C(A+C)(2B_i - A_s)\big), \\
\Gamma^k_{ii} &= \frac{1}{2D}\big(B(C-A)(4B_i - A_j - A_s) + 2B^2(A_i + A_k - 2C_i) \\
&\qquad + A(A+C)(2C_i - A_k) - C(A+C)A_i\big), \\
\Gamma^i_{ij} &= \frac{1}{2D}\big(B(C-A)(A_i + C_i + B_j - B_s) + 2B^2(B_i + C_j - B_k - A_j) \\
&\qquad + A(A+C)A_j - C(A+C)(B_i + C_j - B_k)\big), \\
\Gamma^k_{ij} &= \frac{1}{2D}\big(B(C-A)(A_i + C_i + B_j - B_s) + 2B^2(B_k - C_j - B_i + A_j) \\
&\qquad + A(A+C)(B_i + C_j - B_k) - C(A+C)A_j\big), \\
\Gamma^j_{ik} &= \frac{1}{2D}\big(B(C-A)(A_i + A_k) + 2B^2(C_j - C_s) \\
&\qquad + A(A+C)(B_i + B_k - C_j) - C(A+C)(B_i + B_k - C_s)\big), \\
\Gamma^i_{ik} &= \frac{1}{2D}\big(B(C-A)(2B_i + 2B_k - C_j - C_s) + 2B^2(A_i - A_k) \\
&\qquad + A(A+C)A_k - C(A+C)A_i\big).
\end{aligned} \tag{3.5}
$$

In the equalities (3.5) it is assumed that $i \neq j \neq k \neq s$ and the numbers in the pair (i,k) (resp. in (j,s)) are simultaneously even or odd.
The matrix of the associated metric $\tilde{g}$, determined by $\tilde{g}(x,y) = g(x,Py)$, is of the type:

$$
(\tilde{g}_{ij}) = \begin{pmatrix} C & B & A & B \\ B & C & B & A \\ A & B & C & B \\ B & A & B & C \end{pmatrix}. \tag{3.6}
$$

Due to (2.3) the components of F are $F_{ijk} = \nabla_i \tilde{g}_{jk}$. The well-known are the following identities for a Riemannian metric:

$$
\nabla_i \tilde{g}_{jk} = \partial_i \tilde{g}_{jk} - \Gamma^a_{ij}\tilde{g}_{ak} - \Gamma^a_{ik}\tilde{g}_{aj}. \tag{3.7}
$$

Applying (3.5) and (3.6) into (3.7), and bearing in mind (2.3) and (2.4), we find the nonzero components of F, given in (3.2). $\square$

Immediately, we have the following

Corollary 3.3. *The components $\theta_k = g^{ij}F(e_i, e_j, e_k)$ of the 1-form θ on the manifold (M, g, P) are expressed by the equalities*

$$\begin{aligned}
\theta_1 &= \frac{1}{D}\big(\overline{C}(2C_3 - 2A_1) + \overline{B}(4B_3 - 4B_1) + \overline{A}(2A_3 - 2C_1)\big),\\
\theta_2 &= \frac{1}{D}\big(\overline{A}(2A_4 - 2C_2) + \overline{B}(4B_4 - 4B_2) + \overline{C}(2C_4 - 2A_2)\big),\\
\theta_3 &= \frac{1}{D}\big(\overline{C}(2C_1 - 2A_3) + \overline{B}(4B_1 - 4B_3) + \overline{A}(2A_1 - 2C_3)\big),\\
\theta_4 &= \frac{1}{D}\big(\overline{A}(2A_2 - 2C_4) + \overline{B}(4B_2 - 4B_4) + \overline{C}(2C_2 - 2A_4)\big).
\end{aligned} \tag{3.8}$$

Proof. The equalities (3.8) follow by direct computations from (3.2), (3.3) and (3.4). □

Having in mind Lemma 3.1, Lemma 3.2 and Corollary 3.3 we obtain the next statements.

Theorem 3.4. *The manifold (M, g, P) belongs to the class $\mathcal{W}_0$ if and only if the following equalities are valid:*

$$A_3 = C_1, \quad A_1 = C_3, \quad B_3 = B_1, \quad B_4 = B_2, \quad A_2 = C_4, \quad A_4 = C_2. \tag{3.9}$$

Proof. Due to (3.2) we get that (2.5) is satisfied if and only if (3.9) holds true. □

Theorem 3.5. *The manifold (M, g, P) belongs to the class $\mathcal{W}_1$ if and only if the following equalities are valid:*

$$\begin{aligned}
(A + C)(B_4 - B_2) &= B(A_4 - C_2 + C_4 - A_2),\\
(A + C)(B_3 - B_1) &= B(A_3 - C_1 + C_3 - A_1).
\end{aligned} \tag{3.10}$$

Proof. Using (2.2), (3.1), (3.2), (3.6), (3.8) and (3.10) we obtain

$$F_{kij} = \frac{1}{4}\big(g_{kj}\theta_i + g_{ki}\theta_j - \tilde{g}_{kj}\tilde{\theta}_i - \tilde{g}_{ki}\tilde{\theta}_j\big), \quad \tilde{\theta}_i = P_i^a\theta_a, \tag{3.11}$$

which is equivalent to (2.6).

Vice versa, if (3.11) holds true, then (2.2), (3.1), (3.2), (3.6) and (3.8) imply (3.10). □

In [14], it is proved that $\mathcal{W}_1 = \overline{\mathcal{W}}_3 \oplus \overline{\mathcal{W}}_6$, where $\overline{\mathcal{W}}_3$ and $\overline{\mathcal{W}}_6$ are two basic classes of Naveira's classification. These classes have the following characteristic conditions ([6], [9]):

$$\begin{aligned}
\overline{\mathcal{W}}_3 : \ F(x, y, z) = \frac{1}{4}\Big(&\big(g(x, y) + g(x, Py)\big)\theta(z)\\
&+ \big(g(x, z) + g(x, Pz)\big)\theta(y)\Big), \ \theta(Pz) = -\theta(z).
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
\overline{\mathcal{W}}_6 : \ F(x, y, z) = \frac{1}{4}\Big(&\big(g(x, y) - g(x, Py)\big)\theta(z)\\
&+ \big(g(x, z) - g(x, Pz)\big)\theta(y)\Big), \ \theta(Pz) = \theta(z).
\end{aligned} \tag{3.13}$$

Corollary 3.6. *The manifold* (M, g, P) *belongs to the class* $\overline{\mathcal{W}}_3$ *if and only if the following equalities are valid:*

$$\begin{array}{ll} A_4 - C_2 = C_4 - A_2, & (A + C)(B_4 - B_2) = 2B(A_4 - C_2), \\ A_3 - C_1 = C_3 - A_1, & (A + C)(B_3 - B_1) = 2B(A_3 - C_1). \end{array} \tag{3.14}$$

Proof. The local form of (3.12) is

$$F_{kij} = \frac{1}{4}\big((g_{ki} + \tilde{g}_{ki})\theta_j + (g_{kj} + \tilde{g}_{kj})\theta_i\big), \quad \tilde{\theta}_i = -\theta_i. \tag{3.15}$$

Taking into account (3.8) and (3.10), we get that (3.15) is satisfied if and only if (3.14) holds true. □

Corollary 3.7. *The manifold* (M, g, P) *belongs to the class* $\overline{\mathcal{W}}_6$ *if and only if the following equalities are valid:*

$$A_4 - C_2 = A_2 - C_4, \quad B_4 = B_2, \quad A_3 - C_1 = A_1 - C_3, \quad B_3 = B_1. \tag{3.16}$$

Proof. The local form of (3.13) is

$$F_{kij} = \frac{1}{4}\big((g_{ki} - \tilde{g}_{ki})\theta_j + (g_{kj} - \tilde{g}_{kj})\theta_i\big), \quad \tilde{\theta}_i = \theta_i. \tag{3.17}$$

Having in mind (3.8) and (3.10), we get that (3.17) is satisfied if and only if (3.16) holds true. □

Theorem 3.8. *The manifold* (M, g, P) *belongs to the class* $\mathcal{W}_2$ *if and only if the following equalities are valid:*

$$\begin{array}{ll} (A + C)(C_3 - A_1) = 2B(B_3 - B_1), & C_3 - A_1 = A_3 - C_1, \\ (A + C)(C_4 - A_2) = 2B(B_4 - B_2), & C_4 - A_2 = A_4 - C_2. \end{array} \tag{3.18}$$

Proof. From (2.7) and (3.8) we have

$$\begin{aligned} &\overline{C}(2C_3 - 2A_1) + \overline{B}(4B_3 - 4B_1) + \overline{A}(2A_3 - 2C_1) = 0, \\ &\overline{A}(2A_4 - 2C_2) + \overline{B}(4B_4 - 4B_2) + \overline{C}(2C_4 - 2A_2) = 0, \\ &\overline{C}(2C_1 - 2A_3) + \overline{B}(4B_1 - 4B_3) + \overline{A}(2A_1 - 2C_3) = 0, \\ &\overline{A}(2A_2 - 2C_4) + \overline{B}(4B_2 - 4B_4) + \overline{C}(2C_2 - 2A_4) = 0. \end{aligned} \tag{3.19}$$

The equalities (3.4) and (3.19) imply (3.18).

Vice versa. We apply (3.18) into (3.8) and we get that (2.7) holds true. □

4. Examples of manifolds (M, g, P)

In this section we give a solution of each system of differential equations (3.9), (3.10), (3.14), (3.16) and (3.18), in order to get examples of (M, g, P) of the classes considered.

4.1. An example in $\mathcal{W}_0$

Let (M,g,P) be a manifold with

$$\begin{aligned} &A=(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2,\ B=x^1+x^2+x^3+x^4,\\ &C=2x^1x^3+2x^2x^4, \end{aligned} \tag{4.1}$$

where $x^i>1$.

Evidently $A>C>B>0$ are valid. We check directly that the functions (4.1) and their derivatives satisfy the equalities (3.9).

Thus we have the following

Proposition 4.1. *The manifold* (M,g,P) *with* (4.1) *belongs to* $\mathcal{W}_0$.

Remark 4.2. We note that the functions (4.1) do not satisfy (2.9). Then we have $\nabla Q\neq 0$ for (M,g,P), where $P=Q^2$. An example of a manifold (M,g,Q) with $\nabla Q=0$ is given in [13].

4.2. An example in $\overline{\mathcal{W}}_3$

Let (M,g,P) be a manifold with

$$\begin{aligned} &A=a(x^1+x^2-x^3-x^4),\quad B=b(x^1+x^2-x^3-x^4),\\ &C=c(x^1+x^2-x^3-x^4), \end{aligned} \tag{4.2}$$

where $a,c,b\in\mathbb{R},\ a>c>b>0,\ x^1+x^2-x^3-x^4>0$.

The inequalities $A>C>B>0$ hold true. The functions (4.2) and their derivatives satisfy the equalities (3.14) and do not satisfy the conditions (3.9).

Therefore, we establish the following

Proposition 4.3. *The manifold* (M,g,P) *with* (4.2) *belongs to* $\overline{\mathcal{W}}_3$ *but does not belong to* $\mathcal{W}_0$.

4.3. An example in $\overline{\mathcal{W}}_6$

Let (M,g,P) be a manifold with

$$\begin{aligned} &A=a(x^1+x^2+x^3+x^4),\quad B=b(x^1+x^2+x^3+x^4),\\ &C=c(x^1+x^2+x^3+x^4), \end{aligned} \tag{4.3}$$

where $a,c,b\in\mathbb{R},\ a>c>b>0,\ x^1+x^2+x^3+x^4>0$.

The inequalities $A>C>B>0$ are satisfied. The functions (4.3) and their derivatives satisfy the equalities (3.16) and do not satisfy the conditions (3.9).

Immediately, we state the following

Proposition 4.4. *The manifold* (M,g,P) *with* (4.3) *belongs to* $\overline{\mathcal{W}}_6$ *but does not belong to* $\mathcal{W}_0$.

4.4. An example in $\mathcal{W}_1$

Let (M, g, P) be a manifold with

$$\begin{aligned} A =& a\exp(x^1 - x^2),\ C = c\exp(x^4 - x^3), \\ B =& a\exp(x^1 - x^2) - c\exp(x^4 - x^3), \end{aligned} \tag{4.4}$$

where $a, c \in \mathbb{R}^+$, $\ln\dfrac{c}{a} < x^1 - x^2 + x^3 - x^4 < \ln\dfrac{2c}{a}$.

Then $A > C > B > 0$ are valid. The functions (4.4) satisfy the equalities (3.10) but do not satisfy any of the conditions (3.14) and (3.16).

Therefore, we establish the following

Proposition 4.5. *The manifold* (M, g, P) *with* (4.4) *belongs to* $\mathcal{W}_1$ *but does not belong to* $\overline{\mathcal{W}}_3$ *or to* $\overline{\mathcal{W}}_6$.

4.5. An example in $\mathcal{W}_2$

Let (M, g, P) be a manifold with

$$\begin{aligned} &A = \exp\left(x^1 + x^2 - x^3 - x^4\right), \quad B = \sinh(x^1 + x^2 - x^3 - x^4), \\ &C = \exp\left(x^3 + x^4 - x^1 - x^2\right), \end{aligned} \tag{4.5}$$

where $0 < x^1 + x^2 - x^3 - x^4 < \ln\sqrt{3}$.

The inequalities $A > C > B > 0$ are satisfied. The functions from (4.5) give a solution to (3.18) but do not give a solution to (3.9).

Consequently we have the following

Proposition 4.6. *The manifold* (M, g, P) *with* (4.5) *belongs to* $\mathcal{W}_2$ *but does not belong to* $\mathcal{W}_0$.

5. Conclusion

In this paper we classify the 4-dimensional Riemannian product manifolds (M, g, P), associated with the Riemannian manifolds with circulant structures (M, g, Q), using the well-known classifications in [11] and [14].

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Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 449–463
DOI: 10.24193/subbmath.2023.2.18

Around metric coincidence point theory

Ioan A. Rus

Dedicated to Prof. Adrian Petruşel on the occasion of his 60^{th} anniversary

Abstract. Let (X,d) be a complete metric space, (Y,ρ) be a metric space and $f,g : X \to Y$ be two mappings. The problem is to give metric conditions which imply that, $C(f,g) := \{x \in X \mid f(x) = g(x)\} \neq \emptyset$. In this paper we give an abstract coincidence point result with respect to which some results such as of Peetre-Rus (I.A. Rus, *Teoria punctului fix în analiza funcţională*, Babeş-Bolyai Univ., Cluj-Napoca, 1973), A. Buică (A. Buică, *Principii de coincidenţă şi aplicaţii*, Presa Univ. Clujeană, Cluj-Napoca, 2001) and A.V. Arutyunov (A.V. Arutyunov, *Covering mappings in metric spaces and fixed points*, Dokl. Math., 76(2007), no.2, 665-668) appear as corollaries. In the case of multivalued mappings our result generalizes some results given by A.V. Arutyunov and by A. Petruşel (A. Petruşel, *A generalization of Peetre-Rus theorem*, Studia Univ. Babeş-Bolyai Math., 35(1990), 81-85). The impact on metric fixed point theory is also studied.

Mathematics Subject Classification (2010): 54H25, 47H10, 47H04, 54C60, 47H09.

Keywords: Metric space, singlevalued and multivalued mapping, coincidence point metric condition, fixed point metric condition, covering mapping, coincident point displacement, fixed point displacement, iterative approximation of coincidence point, iterative approximation of fixed point, weakly Picard mapping, pre-weakly Picard mapping, Ulam-Hyers stability, well-posedness of coincidence point problem.

1. Introduction

Let (X,d) be a complete metric space, (Y,ρ) be a metric space and $f,g : X \to Y$ be continuous mappings. The following results are well known:

Received 04 January 2023; Accepted 01 March 2023.

Peetre-Rus' Theorem ([42]). *We suppose that there exist two mappings* $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $M > 0$ *such that:*

(1) *there exists* $x \in X$ *such that,* $\rho(f(x), g(x)) \leq M$*;*
(2) *for each* $x \in X$ *with* $\rho(f(x), g(x)) \leq M$*, there exists* $x_1 \in X$ *such that,*
$$\rho(f(x_1), g(x_1)) \leq \varphi(\rho(f(x), g(x)))$$
and
$$d(x, x_1) \leq \psi(\rho(f(x), g(x)));$$
(3) φ *and* ψ *are increasing,* $\varphi(M) \leq M$*,* $\varphi^n(M) \to 0$ *as* $n \to \infty$ *and*
$$\sum_{i=0}^{\infty} \psi(\varphi^i(M)) < +\infty.$$

In these conditions, $C(f, g) := \{x \in X \mid f(x) = g(x)\} \neq \emptyset$*.*

Buică's Theorem ([15]; see also [2]). *We suppose that there exists,* $0 < l < 1$*,* $k > 0$ *and* $h : X \to X$ *such that:*
$$\rho(f(h(x)), g(h(x))) \leq l\rho(f(x), g(x)), \ \forall\ x \in X,$$
and
$$d(x, h(x)) \leq k\rho(f(x), g(x)), \ \forall\ x \in X.$$
Then we have that:

(*i*) $C(f, g) \neq \emptyset$*;*
(*ii*) *for each* $x_0 \in X$*,* $h^n(x_0) \to x^*(x_0)$ *as* $n \to \infty$ *and* $x^*(x_0) \in C(f, g)$*;*
(*iii*) $d(x_0, x^*(x_0)) \leq \frac{k}{1-l}\rho(f(x_0), g(x_0))$*,* $\forall\ x_0 \in X$*.*

Arutyunov's Theorem ([5]). *We suppose that:*

(1) f *is* α*-covering with* $\alpha > 0$*, i.e.,* $B_Y(f(x), \alpha r) \subset f(B_X(x, r))$*,* $\forall\ x \in X$*,* $\forall\ r > 0$*;*
(2) g *is* L*-Lipschitz with* $L < \alpha$*.*

Then for any $x_0 \in X$*, there exists* $x^*(x_0) \in X$ *such that:*

(*i*) $f(x^*(x_0)) = g(x^*(x_0))$*;*
(*ii*) $d(x_0, x^*(x_0)) \leq (\alpha - L)^{-1}\rho(f(x_0), g(x_0))$*,* $\forall\ x_0 \in X$*.*

In this paper we give an abstract result with respect to which the above results appear as corollaries. In the last section we present a similar result in the case of multivalued mappings, result which generalizes some results given by A.V. Arutyunov ([5]) and by A. Petruşel ([35]).

The impact of our results on metric fixed point theory is also analyzed.

The paper has the following structure:

2. Preliminaries
   2.1. Comparison functions
   2.2. Pre-weakly Picard mappings
   2.3. Covering mappings
   2.4. Conditions, on a functional on metric space, weaker then continuity
3. Basic coincidence point results in metric spaces
4. Ulam-Hyers stability of a coincidence point equation

5. Well-posedness of the coincidence point problem
6. The case of multivalued mappings

2. Preliminaries

2.1. Comparison functions

For $M \in]0, +\infty]$, a function $\varphi : [0, M[\to [0, M[$ is called:

(a) a *comparison function on* $[0, M[$ if φ is increasing and $\varphi^n(t) \to 0$ as $n \to \infty$, $\forall\, t \in [0, M[$;

(b) a *strong comparison function on* $[0, M[$ if φ is a comparison function on $[0, M[$ and

$$\sum_{i=0}^{\infty} \varphi^i(t) < +\infty, \ \forall\, t \in [0, M[.$$

We remark that if φ is a comparison function on $[0, M[$ then, $\varphi(t) < t$, $\forall\, t \in]0, M[$ and $\varphi(0) = 0$, i.e., φ is a Picard function.

Now, let $\varphi : [0, M[\to [0, M[$ and $\psi : [0, M[\to \mathbb{R}_+$ be two functions. By definition, the pair (φ, ψ) is a *comparison pair on* $[0, M[$ if:

(1) φ is a comparison function on $[0, M[$;

(2) ψ is increasing, $\psi(0) = 0$ and ψ is continuous in 0;

(3) $\sum_{i=0}^{\infty} \psi(\varphi^i(t)) < +\infty$, $\forall\, t \in [0, M[$.

Example 2.1. For each $M \in]0, +\infty]$, if $\varphi(t) := lt$, where $0 < l < 1$ and $\psi(t) := kt$, where $k > 0$ and $t \in [0, M[$, then the pair (φ, ψ) is a comparison pair on $[0, M[$. In this case, $\sum_{i=0}^{\infty} \psi(\varphi^i(t)) = \dfrac{kt}{1-l}$, $\forall\, t \in [0, M[$.

Example 2.2. If $\varphi : [0, M[\to [0, M[$ is a strong comparison function on $[0, M[$ and $\psi(t) := kt$, $\forall\, t \in [0, M[$, with $k > 0$, then the pair (φ, ψ) is a comparison pair on $[0, M[$.

2.2. Pre-weakly Picard mappings

Let (X, d) be a metric space. By definition, a mapping $f : X \to X$ is a *pre-weakly Picard mapping* (*pre-WPM*) if the sequence $(f^n(x))_{n \in \mathbb{N}}$ is a convergent sequence for all $x \in X$.

If $f : X \to X$ is pre-WPM, then we consider the mapping $f^\infty : X \to X$, defined by $f^\infty(x) := \lim_{n\to\infty} f^n(x)$.

By definition, if $f : X \to X$ is pre-WPM with

$$f^\infty(x) \in F_f := \{x \in X \mid f(x) = x\}, \ \forall\, x \in X,$$

then f is a *weakly Picard mapping* (*WPM*).

Example 2.3. If (X, d) is a complete metric space and $f : X \to X$ is a *graphic contraction* (i.e., $d(f^2(x), f(x)) \leq ld(x, f(x))$, $\forall\, x \in X$ with $0 < l < 1$) then f is a pre-WPM (see [13] and [47]).

Example 2.4. If (X, d) is a complete metric space and $f : X \to X$ is a *Caristi mapping* (i.e., $d(x, f(x)) \leq \theta(x) - \theta(f(x))$, $\forall\, x \in X$, with the functional $\theta : X \to \mathbb{R}_+$) then f is a pre-WPM (see [13] and [47]).

Example 2.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) := \begin{cases} \frac{1}{2}x, \text{ for } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{2}(x+1), \text{ for } x \in \mathbb{Q} \end{cases} .$$

In this case:

(a) f is pre-WPM;

(b) $f^\infty(x) = \begin{cases} 0, \text{ for } x \in \mathbb{R} \setminus \mathbb{Q} \\ 1, \text{ for } x \in \mathbb{Q} \end{cases}$;

(c) $F_f = \{1\}$;

(d) $f^\infty(0) = 1$ and $f^\infty(1) = 1$.

In the sequel of our paper we need the following result.

Invariant partition lemma. *Let (X, d) be a metric space. If $f : X \to X$ is pre-WPM, then there exists a partition of X,*

$$X = \bigcup_{u \in f^\infty(X)} X_u$$

such that $f(X_u) \subset X_u$.

Proof. For $u \in f^\infty(X)$, we take $X_u := \{x \in X \mid f^n(x) \to u \text{ as } n \to \infty\}$. □

2.3. Covering mappings

If (X, d) is a metric space, then we denote by

$B_X(x, r) := \{u \in X \mid d(x, u) < r\}$ - the open ball of radius $r \in \mathbb{R}_+^*$, centered at $x \in X$;

$\tilde{B}_X(x, r) := \{u \in X \mid d(x, u) \leq r\}$ - the closed ball of radius $r \in \mathbb{R}_+$, centered at $x \in X$.

Let (X, d) and (Y, ρ) be two metric spaces, $f : X \to Y$ be a mapping and $\alpha \in \mathbb{R}_+^*$. By definition, f is an *α-covering mapping* if,

$$\tilde{B}_Y(f(x), \alpha r) \subset f(\tilde{B}_X(x, r)),\ \forall\, x \in X,\ \forall\, r \in \mathbb{R}_+. \tag{CV}$$

The condition (CV) is equivalent with each of the following ones:

(CV_1) For all $r \in \mathbb{R}_+$ the following implication holds,
$x \in X$, $y \in Y$ and $\rho(f(x), y) \leq \alpha r \Rightarrow$ there exists $x_1 \in X$ such that, $f(x_1) = y$ and $d(x, x_1) \leq r$;

(CV_2) For all $r \in \mathbb{R}_+$, the following implication holds,
$x \in X$, $y \in Y$ and $\rho(f(x), y) \leq r \Rightarrow$ there exists $x_1 \in X$ such that, $f(x_1) = y$ and $d(x, x_1) \leq \frac{r}{\alpha}$.

It is clear that each covering mapping is surjective.

For more considerations on covering mappings (also named *open with linear rate*) and its relations with metric regularity see [2]-[8], [17], [54], [55], [14], [25], [18], [16].

2.4. Conditions, on a functional on metric space, weaker than continuity

Let (X, d) be a metric space and $F : X \to \mathbb{R}$ be a functional. By definition (Angrisani [3], Kirk-Saliga [26], Aamri-Chaira [1]), the functional F is called a *regular-global-inf (r.g.i.)* if for each $x \in X$, $F(x) > \inf\limits_X F := \inf\{F(u) \mid u \in X\}$ implies that there exist $\varepsilon > 0$ such that, $\varepsilon < F(x) - \inf\limits_X F$, and a neighborhood $V(x)$ of x, such that, $F(y) > F(x) - \varepsilon$, for each $y \in V(x)$.

From this definition it follows that:

(1) ([26]) The functional F is an r.g.i. on X if and only if for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$, we have the following implication:
$$x_n \to x^* \text{ and } F(x_n) \to \inf_X F \Rightarrow F(x^*) = \inf_X F;$$

(2) If $F : X \to \mathbb{R}_+$ is an r.g.i. on X, $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x^*$ and $F(x_n) \to 0$, then $F(x^*) = 0$.

We also have:

(3) If $F : X \to \mathbb{R}_+$ is a lower semicontinuous (l.s.c.) functional and $(x_n)_{n\in\mathbb{N}} \subset X$, then the following implication holds:
$$x_n \to x^* \text{ and } F(x_n) \to 0 \Rightarrow F(x^*) = 0.$$

3. Basic coincidence point results in metric spaces

Let (X, d) and (Y, ρ) be two metric spaces and $f, g : X \to Y$ be two mappings. For $M \in]0, +\infty]$ we denote by $X_M := \{x \in X \mid \rho(f(x), g(x)) < M\}$. We remark that:
$$X_\infty = X \text{ and } C(f, g) \subset X_M,\ \forall\ M \in]0, \infty].$$

More general, if $\lambda : Y \times Y \to \mathbb{R}_+$ is a functional, we denote by
$$X_M := \{x \in X \mid \lambda(f(x), g(x)) < M\}.$$

For some M, X_M may be $\emptyset$. For example, for $f, g : \mathbb{R} \to \mathbb{R}$,
$$f(x) = x,\ g(x) = x + 1,\ X_M = \emptyset,$$
for $M \le 1$ and $X_M = \mathbb{R}$, for $M \in]1, +\infty]$.

Our basic result, in the case of singlevalued mappings, is the following one:

Theorem 3.1. *Let (X, d) be a complete metric space, (Y, ρ) be a metric space, $f, g : X \to Y$ be two mappings, $M \in]0, +\infty]$ and $\lambda : Y \times Y \to \mathbb{R}_+$ be a functional. We suppose that:*

(1) *$X_M := \{x \in X \mid \lambda(f(x), g(x)) < M\} \neq \emptyset$;*
(2) *The coincidence point λ-displacement, $\lambda_{f,g} : X_M \to \mathbb{R}_+$, $\lambda_{f,g}(x) := \lambda(f(x), g(x))$ is l.s.c.;*
(3) *There exists a comparison pair, (φ, ψ), on $[0, M[$ with respect to which, for each, $x \in X_M$ there exists $x_1 \in X_M$ such that:*
   (a) *$\lambda(f(x_1), g(x_1)) \le \varphi(\lambda(f(x), g(x)))$;*
   (b) *$d(x, x_1) \le \psi(\lambda(f(x), g(x)))$.*

Then there exists a pre-WPM, $h : X_M \to X_M$ such that:

(*i*) $\lambda(f(h^\infty(x)), g(h^\infty(x))) = 0$, $\forall\, x \in X_M$;

(*ii*) $d(x, h^\infty(x)) \leq \sum_{i=0}^{\infty} \psi(\varphi^i(\lambda(f(x), g(x))))$, $\forall\, x \in X_M$;

(*iii*) *If in addition,*

$$u, v \in Y,\ \lambda(u, v) = 0 \ \Rightarrow\ u = v,$$

then, $h^\infty(x) \in C(f, g)$, $\forall\, x \in X_M$, *i.e.,* $C(f, g) \neq \emptyset$.

Proof. From (a) and (b) there exists $h : X_M \to X_M$ such that,

$$\lambda(f(h(x)), g(h(x))) \leq \varphi(\lambda(f(x), g(x))),\ \forall\, x \in X_M,$$

and

$$d(x, h(x)) \leq \psi(\lambda(f(x), g(x))),\ \forall\, x \in X_M.$$

These imply that,

$$\lambda(f(h^n(x)), h^{n+1}(x)) \leq \varphi^n(\lambda(f(x), g(x))) \to 0 \text{ as } n \to \infty,\ \forall\, x \in X_M$$

and

$$d(h^n(x), h^{n+1}(x)) \leq \psi(\varphi^n(\lambda(f(x), g(x)))),\ \forall\, x \in X_M.$$

Since (X, d) is a complete metric space and (φ, ψ) is a comparison pair on $[0, M[$, it follows that h is pre-WPM.

On the other hand, from (2) we have,

$$0 \leq \lambda(f(h^\infty(x)), g(h^\infty(x))) \leq \underline{\lim}_{n\to\infty} \lambda(f(h^n(x)), g(h^n(x))) = 0.$$

It is clear that, from the above considerations, we have (i), (ii) and (iii). □

Remark 3.2. If f and g are continuous, $M < +\infty$, $\lambda := \rho$, then from Theorem 3.1 we have Peetre-Rus' theorem.

Remark 3.3. If f and g are continuous, $M = +\infty$, $\lambda := \rho$, $\varphi(t) := lt$, where $0 < l < 1$ and $\psi(t) := kt$, with $k > 0$, $\forall\, t \in [0, M[$, then from Theorem 3.1 we have Buică's theorem.

Remark 3.4. Let f and g be as in Arutyunov's theorem. Since f is α-covering, for $r := \frac{t}{\alpha}$ we have that, if $x \in X$, $y \in Y$ with $\rho(f(x), y) \leq t$, there exists $x_1 \in X$ such that, $f(x_1) = y$ and $d(x_1, x) \leq \frac{t}{\alpha}$. So, $\rho(f(x), f(x_1)) \leq t$.

If we take, $t := \rho(f(x), g(x))$ and $y := g(x) = f(x_1)$, we have

$$\rho(f(x_1), g(x_1)) \leq \frac{L}{\alpha}\rho(f(x), g(x)) \text{ and } d(x, x_1) \leq \frac{1}{\alpha}\rho(f(x), g(x)).$$

If we take in Theorem 3.1, f continuous and α-covering, g L-Lipschitz with $L < \alpha$, $M := +\infty$, $\lambda := \rho$, $\varphi(t) := \frac{L}{\alpha}t$ and $\psi(t) := \frac{t}{\alpha}$, we have Arutyunov's theorem. Moreover, from the above proof, we have:

Theorem 3.5. *Let (X, d) be a complete metric space, (Y, ρ) be a metric space, $f : X \to Y$ be continuous, $g : X \to Y$ be L-Lipschitz and $\alpha > 0$ with $L < \alpha$. We suppose that the following implication holds:*

$x \in X$, $y \in Y$, $\rho(f(x), y) \leq \rho(f(x), g(x)) \Rightarrow$ there exists $x_1 \in X$ such that, $f(x_1) = y$ and $d(x, x_1) \leq \frac{1}{\alpha}\rho(f(x), g(x))$.

Then there exists a pre-WPM, $h : X \to X$ such that:

(*i*) $h^\infty(x) \in C(f,g)$, $\forall\ x \in X$;
(*ii*) $d(x, h^\infty(x)) \le (\alpha - L)^{-1}\rho(f(x), g(x))$.

Remark 3.6. Let us consider in Theorem 3.1, $M := +\infty$, $\lambda := \rho$, $\varphi(t) := lt$, where $0 < l < 1$, $\psi(t) := kt$, with $k > 0$, $\forall\ t \in [0, M[$. In this case Theorem 3.1 takes the following form:

Theorem 3.7. *Let (X,d) be a complete metric space, (Y,ρ) be a metric space, $f,g : X \to Y$ be two mappings. We suppose that:*

$(2')$ *The coincidence point displacement, $\rho_{f,g} : X \to \mathbb{R}_+$, $x \mapsto \rho(f(x), g(x))$ is l.s.c.;*
$(3')$ *There exist $0 < l < 1$ and $k > 0$ w.r.t. which for each $x \in X$ there exists $x_1 \in X$ such that:*
  (a') $\rho(f(x_1), g(x_1)) \le l\rho(f(x), g(x))$;
  (b') $d(x, x_1) \le k\rho(f(x), g(x))$.

Then there exists a pre-WPM, $h : X \to X$, such that:

(i') $h^\infty(x) \in C(f,g)$, $\forall\ x \in X$, *i.e.*, $C(f,g) \neq \emptyset$;
(ii') $d(x, h^\infty(x)) \le \frac{k}{1-l}\rho(f(x), g(x))$, $\forall\ x \in X$.

Remark 3.8. If in Theorem 3.7 we take, $Y := X$ and $g := 1_X$ we have the following result:

Theorem 3.9. *Let (X,d) be a complete metric space, ρ be a metric on X and $f : X \to X$ be a mapping. We suppose that:*

$(2'')$ *The fixed point displacement, $\rho_f : (X,d) \to \mathbb{R}_+$, $\rho_f(x) := \rho(x, f(x))$, is l.s.c.;*
$(3'')$ *There exist $0 < l < 1$ and $k > 0$ w.r.t. which for each $x \in X$ there exists $x_1 \in X$ such that:*
  (a'') $\rho(x_1, f(x_1)) \le l\rho(x, f(x))$;
  (b'') $d(x, x_1) \le k\rho(x, f(x))$.

Then there exists a pre-WPM, $h : (X,d) \to (X,d)$ such that:

(i'') $h^\infty(x) \in F_f$, $\forall\ x \in X$, *i.e.*, $F_f \neq \emptyset$;
(ii'') $d(x, h^\infty(x)) \le \frac{k}{1-l}\rho(x, f(x))$, $\forall\ x \in X$.

Remark 3.10. If in Theorem 3.9 we take, $\rho := d$ and f and l-graphic contraction, then we have:

Theorem 3.11. *Let (X,d) be a complete metric space and $f : X \to X$ be an l-graphic contraction. If the fixed point displacement $d_f : X \to \mathbb{R}_+$, $x \mapsto d(x, f(x))$ is l.s.c., then f is a WPM and*

$$d(x, f^\infty(x)) \le \frac{1}{1-l} d(x, f(x)),\ \forall\ x \in X.$$

Proof. We take, $h(x) := f(x)$. □

Remark 3.12. If in Theorem 3.9 we take, $\rho := d$ and f an l-contraction, then we have the following variant of contraction principle:

Theorem 3.13. *Let (X,d) be a complete metric space and $f : X \to X$ be an l-contraction. Then we have that:*

(*i*) $F_f = F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}^*$;
(*ii*) $f^n(x) \to x^*$ *as* $n \to \infty$, $\forall\ x \in X$;
(*iii*) $d(x, x^*) \leq \frac{1}{1-l} d(x, f(x))$, $\forall\ x \in X$.

The above consideration give rise to the following questions:

Problem 3.14. To translate Arutyunov's theorem in terms of metric regularity.

References: [17], [14], [25], [18], [16].

Problem 3.15. Which metric conditions on $f : X \to X$ imply that:
(*i*) f is a graphic contraction ?
(*ii*) $d_f : X \to \mathbb{R}_+$, $d_f(x) := d(x, f(x))$ is l.s.c. ?

References: [47], [49], [27], [32], [28], [1], [21], [26], [12], [44].

Problem 3.16. Let (X, d) be a metric space and $f : X \to X$ be a mapping. The problem is to compare the following conditions on f:
(1) the graphic of f is closed;
(2) f is orbitally continuous;
(3) the fixed point displacement of f, $d_f : X \to X$, $d_f(x) := d(x, f(x))$ is l.s.c.

References: [1], [47], [27], [32], [21], [26], [3].

4. Ulam-Hyers stability of coincidence point equations

Let (X, d) and (Y, ρ) be two metric spaces, $f, g : X \to Y$ be two mappings and $\lambda : Y \times Y \to \mathbb{R}_+$ be such that the following implications hold,

$$u, v \in Y,\ \lambda(u, v) = 0 \ \Leftrightarrow\ u = v.$$

Let us consider the coincidence point equation,

$$f(x) = g(x) \tag{0}$$

and for each $\varepsilon > 0$, the ε-coincidence point inequation, with respect to λ,

$$\lambda(f(x), g(x)) \leq \varepsilon \tag{ε}$$

We denote by, $C_{\varepsilon,\lambda}(f, g) := \{x \in X \mid \lambda(f(x), g(x)) \leq \varepsilon\}$, the solution set of (ε).

By definition, the equation (0) is Ulam-Hyers stable, with respect to the functional λ, if there exists $c > 0$ such that for each $\varepsilon > 0$ we have that: for each $u^* \in C_{\varepsilon,\lambda}(f, g)$ there exists $x^* \in C(f, g)$ with, $d(u^*, x^*) \leq c\varepsilon$.

Our result is the following.

Theorem 4.1. *Let $f, g : X \to Y$ be as in Theorem 3.1. If in addition, $M := +\infty$ and the comparison pair, (φ, ψ) is such that there exists $c > 0$ for which, $\sum_{i=0}^{\infty} \psi(\varphi^i(t)) \leq ct$, for all $t \geq 0$, then the equation (0) is Ulam-Hyers stable with respect to the functional λ.*

Proof. For $u^* \in C_{\varepsilon,\lambda}(f, g)$ we take $x^* := h^{\infty}(u^*)$. $\square$

Remark 4.2. If we take $\lambda := \rho$ then we have the Ulam-Hyers stability with respect to ρ, in the conditions of Buică's theorem and in the conditions of Arutyunov's theorem.

Remark 4.3. For more considerations on Ulam-Hyers stability of fixed point equations and of coincidence point equations see: [45], [30] and the references therein.

5. Well-posedness of the coincidence point problem

Let (X, d) and (Y, ρ) be two metric spaces and $f, g : X \to Y$ with $C(f, g) \neq \emptyset$ and $r : X \to C(f, g)$ be a set retraction. By definition, the coincidence point problem for the pair (f, g) is well-posed with respect to r and to the functional $\lambda : Y \times Y \to \mathbb{R}_+$ if for each $x^* \in C(f, g)$ and each $(x_n)_{n\in\mathbb{N}} \subset r^{-1}(x^*)$ the following implication holds:

$$\lambda(f(x_n), g(x_n)) \to 0 \text{ as } n \to \infty \;\Rightarrow\; x_n \to x^* \text{ as } n \to \infty.$$

We have the following result:

Theorem 5.1. *Let $f, g : (X, d) \to (Y, \rho)$, $\lambda : Y \times Y \to \mathbb{R}_+$ and $h : X \to X$ as in Theorem 3.1. If in addition, $M := +\infty$ and $\lambda(u, v) = 0 \Leftrightarrow u = v$, then the coincidence point problem for the pair (f, g) is well-posed with respect to h^∞ and to λ, if the pair (φ, ψ) is such that $\sum_{i=0}^{\infty} \psi(\varphi^i(t)) \to 0$ as $t \to 0$.*

Proof. First, we remark that $h^\infty : X \to C(f, g)$ is a set retraction. Let $(x_n)_{n\in\mathbb{N}} \subset (h^\infty)^{-1}(x^*)$ with $x^* \in C(f, g)$. Then,

$$d(x_n, x^*) = d(x_n, h^\infty(x_n)) \leq \sum_{i=0}^{\infty} \psi(\varphi^i(\lambda(f(x_n), g(x_n)))) \to 0 \text{ as } n \to \infty. \qquad \square$$

For more consideration on well-posedness of fixed point problem and of coincidence point problem, see: [47], [22], [46], [32].

6. The case of multivalued mappings

Throughout this section we follow the notations and terminology in [39]. See also: [37], [38], [49], [9], [29], [11], [16].

The basic result of this section is the following:

Theorem 6.1. *Let (X, d) be a complete metric space, (Y, ρ) be a metric space, $T, S : X \to P_{cl}(Y)$ be two multivalued mappings, $M \in]0, +\infty]$, (φ, ψ) be a comparison pair on $[0, M[$ and $\Lambda : P_{cl}(Y) \times P_{cl}(Y) \to \mathbb{R}_+ \cup \{+\infty\}$ be a functional. We suppose that:*

(1) *$X_M := \{x \in X \mid \Lambda(T(x), S(x)) < M\} \neq \emptyset$;*
(2) *The Λ-coincidence point displacement functional, $\Lambda_{T,S} : X_M \to \mathbb{R}_+$,*
$$\Lambda_{T,S}(x) := \Lambda(T(x), S(x))$$
is l.s.c.;
(3) *For each $x \in X_M$ there exists $x_1 \in X_M$ such that:*
    (a) *$\Lambda(T(x_1), S(x_1)) \leq \varphi(\Lambda(T(x), S(x)))$;*

(b) $d(x,x_1) \le \psi(\Lambda(T(x),S(x)))$.

Then there exists a pre-WPM, $h: X_M \to X_M$ *such that:*

(i) $\Lambda(T(h^\infty(x)),S(h^\infty(x))) = 0,\ \forall\ x \in X_M$;

(ii) $d(x,h^\infty(x)) \le \sum_{i=0}^{\infty} \psi(\varphi^i(\Lambda(T(x),S(x)))),\ \forall\ x \in X_M$;

(iii) If in addition, for $A,B \in P_{cl}(Y)$, $\Lambda(A,B) = 0$ *implies that:*

(iii₁) $A \cap B \neq \emptyset$,
then, $C(T,S) := \{x \in X \mid T(x) \cap S(x) \neq \emptyset\} \neq \emptyset$;

(iii₂) $A = B$,
then, $C(T,S) \neq \emptyset$ *and* $T(h^\infty(x)) = S(h^\infty(x)),\ \forall\ x \in X_M$;

(iii₃) $A = B = \{y^*\}$,
then $C(T,S) \neq \emptyset$ *and* $T(h^\infty(x)) = S(h^\infty(x)) = \{y^*_x\}$.

Proof. If we take, $h(x) := x_1$, then we have that:

$$\Lambda(T(h(x)),S(h(x))) \le \varphi(\Lambda(T(x),S(x))),\ \forall\ x \in X_M,$$

and

$$d(x,h(x)) \le \psi(\Lambda(T(x),S(x))),\ \forall\ x \in X_M.$$

These imply that,

$$\Lambda(T(h^n(x)),S(h^n(x))) \to 0 \text{ as } n \to \infty,$$

and h is a pre-WPM, and

$$d(x,h^\infty(x)) \le \sum_{i=0}^{\infty} \psi(\varphi^i(\Lambda(T(x),S(x)))),\ \forall\ x \in X_M.$$

Since, $\Lambda_{T,S}$ is l.s.c., it follows that,

$$\begin{aligned} 0 \le \Lambda(T(h^\infty(x)),S(h^\infty(x))) &\le \underline{\lim}_{n\to\infty} \Lambda(T(h^n(x)),S(h^n(x))) \\ &= \lim_{n\to\infty} \Lambda(T(h^n(x)),S(h^n(x))) = 0. \end{aligned}$$

So, we have the conclusions *(i)*, *(ii)* and *(iii)*. □

Remark 6.2. If we take in Theorem 6.1, $\Lambda := H_\rho$, the Pompeiu-Hausdorff metric on $P_{cl}(Y)$, then we have:

Theorem 6.3. *Let* $\Lambda := H_\rho$ *in Theorem 6.1. Then we have the following conclusions:*
There exists a pre-WPM, $h: X_M \to X_M$ *such that:*

(i′) $T(h^\infty(x)) = S(h^\infty(x)),\ \forall\ x \in X_M$;

(ii′) $d(x,h^\infty(x)) \le \sum_{i=0}^{\infty} \psi(\varphi^i(H_\rho(T(x),S(x)))),\ \forall\ x \in X_M$.

Remark 6.4. Let $\Lambda := e$, in Theorem 6.1., the excess functional. In this case, $A,B \in P_{cl}(Y)$, $e(A,B) := \sup\{\rho(a,B) \mid a \in A\} = 0 \Rightarrow A \subset B$. So, we have the following result:

Theorem 6.5. *If in Theorem 6.1., we take* $\Lambda := e$*, then the conclusions of this theorem take the following form:*

There exists a pre-WPM, $h: X_M \to X_M$ *such that:*

(i'') $T(h^\infty(x)) \subset S(h^\infty(x))$, *i.e.*, $C(T,S) \neq \emptyset$;

(ii'') $d(x,h^\infty(x)) \leq \sum_{i=0}^{\infty} \psi(\varphi^i(e(T(x),S(x))))$, $\forall\, x \in X_M$.

Remark 6.6. If we take $\Lambda := D$, where for $A, B \in P_{cl}(Y)$,

$$D(A,B) := \inf\{\rho(a,b) \mid a \in A,\ b \in B\},$$

then we have a theorem given by A. Petruşel in [35] and A.V. Arutyunov in [5].

Remark 6.7. If we take in Theorem 6.1, $Y := X$, $S := d$, $S(x) := \{x\}$, $\forall\, x \in X$, then Theorem 6.1 takes the following form:

Theorem 6.8. *Let (X,d) be a complete metric space, $T : X \to P_{cl}(X)$, $M \in]0,+\infty]$, and (φ,ψ) be a comparison pair on $[0,M[$, $\Lambda : X \times P_{cl}(X) \to \mathbb{R}_+ \cup \{+\infty\}$ be a functional. We suppose that:*

(1) $X_M := \{x \in X \mid \Lambda(x,T(x)) < M\} \neq \emptyset$;
(2) $\Lambda_T : X_M \to \mathbb{R}_+$, $\Lambda_T(x) = \Lambda(x,T(x))$, *the Λ-fixed point displacement is l.s.c.;*
(3) *For each $x \in X_M$, there exists $x_1 \in X_M$ such that:*
   (a) $\Lambda(x_1,T(x_1)) \leq \varphi(\Lambda(x,T(x)))$;
   (b) $d(x,x_1) \leq \psi(\Lambda(x,T(x)))$.

Then there exists a pre-WPM, $h : X_M \to X_M$ such that:

(i) $\Lambda(h^\infty(x),T(h^\infty(x))) = 0$, $\forall\, x \in X_M$;

(ii) $d(x,h^\infty(x)) \leq \sum_{i=0}^{\infty} \psi(\varphi^i(\Lambda(x,T(x))))$, $\forall\, x \in X_M$.

Remark 6.9. It is clear that, if in Theorem 6.8 we take:

- $\Lambda := H_d$, then $T(h^\infty(x)) = \{h^\infty(x)\}$, i.e., $h^\infty(x)$ is a strict fixed point of T, $\forall\, x \in X_M$;
- $\Lambda := D$, then $h^\infty(x) \in T(h^\infty(x))$, $\forall\, x \in X_M$, i.e., $h^\infty(x)$ is a fixed point of T, $\forall\, x \in X_M$.

Remark 6.10. In [19] Y. Feng and S. Liu, have given the following fixed point result:

Let (X,d) be a complete metric space, $T : X \to P_{cl}(X)$ be a multivalued mapping. For a positive constant $b \in]0,1[$, set

$$I_b^x := \{y \in T(x) \mid bd(x,y) \leq d(x,T(x))\}.$$

If there exists a constant $c \in]0,1[$ such that for any $x \in X$, there is $y \in I_b^x$ satisfying

$$d(y,T(y)) \leq cd(x,y),$$

then T has a fixed point in X provided $c < b$ and the fixed point displacement, d_T is l.s.c. We remark that we are in the conditions of Theorem 6.8, with, $M := +\infty$, $\varphi(t) := \frac{c}{b}t$, $\psi(t) := \frac{t}{b}$ and $\Lambda := D$.

From the considerations presented in this section, the following questions follow:

Problem 6.11. Let (X, d) and (Y, ρ) be two metric spaces and $T, S : X \to P_{cl}(Y)$ be two multivalued mapping. Let $P : P_{cl}(Y) \times P_{cl}(Y) \to \mathbb{R}_+ \cup \{+\infty\}$ be a metric $(H_\rho, H_\rho^+, \ldots)$. In which conditions the $P_{T,S}$-coincidence displacement, $P_{T,S} : X \to \mathbb{R}_+$, $P_{T,S}(x) := P(T(x), S(x))$, is l.s.c. ?

References: [37], [39], [27], [9], [29], [4].

Problem 6.12. To use Theorem 6.1 in studying the Ulam-Hyers stability of a coincidence equation.

References: [45], [49], [30], [50].

Problem 6.13. To use Theorem 6.1 to study the well-posedness of coincidence point problem.

References: [39], [40], [49], [9], [29], [11], [53], [50].

Problem 6.14. Which metric fixed point theorems appear as consequences of Theorem 6.8 ?

References: [17], [22], [37], [48], [49], [10], [41], [51], [11], [52], [36], [12], [44].

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