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CONTENTS

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GÁBOR KASSAY – IN MEMORIAM

Petra Renáta Rigó and Ferenc Szenkovits

Abstract. The scientific activity of Professor Gábor Kassay (1956-2021), one of the most prolific mathematician in Cluj-Napoca (Romania) is presented, through the memories of some co-authors, with whom he collaborated throughout the four decades of his scientific activity.

Mathematics Subject Classification (2010): 01A85, 90C33, 90C47, 46N10.

Keywords: Gábor Kassay, equilibrium problems, minimax theorems, convex analysis, monotone operators.

1. Life and scientific activity of GÁBOR KASSAY (1956–2021)

Gábor Kassay was born on December 24, 1956 in Odorheiu Secuiesc (Székelyudvarhely). He studied elementary and high school in his hometown and mathematics at Babeş-Bolyai University in Cluj-Napoca (1976–1980). In 1994 he obtained his scientific degree in mathematics at the same university, with a thesis summarizing his researches on minimax problems, under the supervision of Professor József Kolumbán.

He started his teaching career in secondary schools in Cluj-Napoca (1980–1987) and continued at Babeş-Bolyai University as teaching assistant (1987–1990), assistant professor (1990–1995), associate professor (1995–2002, 2004–2005), professor (2005–2021). In the period 2002–2004 he was a visiting professor at Eastern Mediterranean University in Famagusta, Northern Cyprus.

His university lectures covered the following topics: mathematical analysis, optimization theory, functional analysis, operations research, convex analysis, game theory.

The list of publications of Gábor Kassay totals 87 scientific articles published in prestigious international journals such as: Mathematical Methods of Operations Research, SIAM Journal on Optimization, Journal of Optimization Theory and Applications, Nonlinear Analysis, Journal of Global Optimization; four books, five book

chapters and a conference proccedings volume edited by him. His recognition is also indicated by the fact that he worked together with more than thirty-five coauthors from different countries. His works total over 2000 citations, including several articles with over 100 independent citations. The complete list of his publications can be found at [36].

He was leader of successful group research programs, co-organizer of scientific conferences, leader of scientific seminars on analysis and optimization. He presented his results at several international conferences around the world.

In this article we try to present this special scientific personality through the testimonies of some of his collaborators.

2. Memories from coauthors

Gábor Kassay was a great master of scientific collaboration. He successfully established and maintained contacts with specialists involved in his fields of interest, publishing joint results with over thirty-five co-authors. The confessions presented below give us a real picture about his ability to establish scientific relationships, about his work style, as well as about the special man and friend who Gabi Kassay was for many. For more details see [36].

József Kolumbán, Babeş-Bolyai University, Cluj-Napoca, Romania:

"I noticed Gábor Kassay from the first year of his studies as one of the most diligent and passionate about mathematics. In particular, his work capacity, intuitive mindset and task-solving skills were extraordinary. Already at that time excelled in finding examples and counterexamples. Although Gábor Kassay graduated from the university with excellent results and could be very useful and necessary in our faculty, he could not be appointed to the university in the circumstances of that time.

His interest in mathematical research kept him in Cluj even after graduating. He chose a high school in this university center, in order to be able to continue actively participating in the activities within the Tiberiu Popoviciu Scientific Seminar. This seminar was very helpful in Gabi's scientific activity throughout his career, he even published some of his first papers in the volumes of this seminar [1, 2]. It was at this time that we wrote our first collaborations [3].

Gábor Kassay's 1994 doctoral (PhD) dissertation was titled "New results in minimax theory applied to variational inequalities and optimization tasks". Throughout his career, the theory of equilibrium, which includes these types of tasks, has been the focus of his attention. It includes, among other topics, optimization, minimax problems, Nash-equilibrium, complementarity, fixed point tasks, variational inequalities, and many other problems in applied mathematics. Gábor Kassay has been publishing articles on this topic since the early 1990s [4, 5, 6], when the synthesizing name "equilibrium theory" had not yet been born. Since then, this theory has evolved enormously. His practitioners have appeared all over the world, who publish hundreds of papers on this topic every year. Gabi has exploited this professional environment very cleverly. He had a working relationship with the best of the profession, from whom he learned a lot, and returned home and shared his experiences with his colleagues.

His curiosity, polite action, reliability and dear manners helped him greatly in this regard.

By leaving, Gabi left a great void in my soul. He gave me one of the most beautiful gifts of my life by being a close colleague and friend for over 40 years. Two years before his death, he presented me with a copy of the monograph on the latest results of the theory of equilibrium, including some of his own, written with Vicențiu Rădulescu [31], with the following dedication: "To my mentor, József Kolumbán, without whom this book (among many others) would not have been written. With friendly love, Gabi, Cluj, 2019 March 7." In the Acknowledgements section of the book, the following sentence is included: "Gábor Kassay is indebted to Joseph Kolumbán, his former teacher and supervisor: their joint pepers and interesting discussions on equilibrium problems opened the author's interest toward this topic." These words are also evidence of Gabi's spiritual richness."

Zsolt Páles, University of Debrecen, Hungary:

"After the political changes in Hungary, in 1989, my first visit to Cluj-Napoca became possible in 1992 with a small group of mathematicians from Debrecen. In Cluj, we received a very warm welcome and immediately made friendship with many Hungarian and Romanian mathematicians. Being one of our hosts, Gábor Kassay spent a lot of time with us and we both realized that we had many fields of common interest. In particular, the theory of convexity, nonsmooth analysis and variational inequalities were in the focus of research for both of us. After this visit to Cluj, starting from the year 1995, I became a regular participant of the conferences organized by the Babeş-Bolyai University, I visited Cluj almost every year and Gábor also visited Debrecen several times to deliver seminar and conference lectures. I still have a vivid memory of our participation at the first Joint Conference of Mathematics and Computer Science in Illyefalva in 1995, where also József Kolumbán joined our discussions and the snooker games in the local pub of the village. Due to this active cooperation, we published our first paper with Gábor in 1999 [9], and then two further papers jointly also with József Kolumbán [8, 11]. These works still receive many citations, they are the most important papers for all of us.

The events that we shared keeps Gábor's memory in us. We still cannot understand and accept how and why all this happened to him. Nothing can compensate his loss."

Monica Bianchi, Catholic University of the Sacred Heart and
Rita Pini, University of Milano-Bicocca, Milan:

"Gábor has been not only a great coworker, but especially a very dear friend during the last eighteen years. We met him the first time in 2003, at the 18th International Symposium on Mathematical Programming in Copenhagen. After attending our lecture, he came to us and gave us a card with his e-mail address, since he was interested in the topic and, why not?, to begin a collaboration. We wrote the first paper [12] about the existence of equilibria via Ekeland's principle working at distance, via e-mail essentially. But since then almost every year we succeeded in getting together for one week or more, in Milan, in general, and also by attending the same conferences. We also visited a few times Cluj, where he was always a thoughtful host, pleased to

show us what he liked most in the nearby. Our studies about well-posedness [18, 20], stability of equilibria and generalized equations [15, 23, 24, 25, 26, 27, 28], regularization of variational inequalities and equilibrium problems [33], that have been finalized in twelve publications, usually took the start when we could discuss face-to-face, and went on by exchanging several e-mails. Only during the pandemia we got used to meet via web, and our last work was done completely in this way [35]. Many years passed by, but we keep vivid memories of several moments with him. We will never forget his rigor, his eye for details, his intellectual honesty, but also his consideration for others, his good manners and his extreme courtesy. We will miss him a lot."

Hans Frenk, Sabanci University, Istanbul, Turkey:

"My scientific collaboration with Gábor Kassay lasted from 1998 until 2008. During that period I visited Gábor almost every year in Kolozsvár and later for one time in Cyprus while Gábor visited me several times in Rotterdam at the Erasmus University. Our collaboration started due to our mutual acquaintance Tibor Illés from Eötvös University in Budapest. We shared a common interest in generalisations of convexity and related minmax theorems [7]. Gábor had a lot of experience in this field due to his work on generalisations of so-called K-convex functions and I was interested in extending the classical theory of minmax theorems and convexity. Also around that time I completed my work with my former Ph.D students J.Gromicho and A.I de Barros on the ellipsiod method and fractional programming involving quasiconvex functions. Since immediately we liked each other personally and felt together that our knowledge was complementary we started our collaboration. This collaboration would last for almost 10 years starting with our first paper appearing in Journal of Optimization Theory and Applications in 1999 [7] and ending with the last paper in the same journal in 2007 [16]. In total we wrote 7 joint published papers (also sometimes with other coauthors) and two book chapters of which the last one appeared in 2008 [14, 17]. During that time we also visited several conferences on generalisations of convexity presenting our work. After the publication of the last chapter in 2008 our scientific cooperation ended since we both felt that our work was finished and we continued separately with other research topics. Gábor with his work on variational inequalities and me on applications of stochastic processes and optimization in Operations Research. This was also partly caused by my transition to Sabanci University in Istanbul. Although we irregularly stayed in contact and even planned a kind of reunion to visit each other in either Istanbul or Kolozsvár, this never happened due to our busy schedules. I regret now we never did this. I will remember Gábor not only as a dedicated and talented researcher but also on a personal basis as somebody who was very enthusiastic and curious about everything in life and his love for mountain climbing. A nice, friendly and curious person and a scientific friend."

Qamrul Hasan Ansari, Aligarh Muslim University, India:

"Gábor Kassay visited Aligarh Muslim University, India in November 2017 and was a guest of honour in an open ceremony of an international conference on anlaysis and its applications. He also visited several times the Department of Mathematics & Statistics, King Fahd University of Petroleum and Minerals, Saudi Arabia. Prof. Kassay worked as a consultant in KFUPM funded research project at King Fahd

University of Petroleum and Minerals, Saudi Arabia with Prof. S. Al-Homidan. It is our honour to work with Prof. Kassay and we published jointly several research papers, namely [29, 30, 32, 34]."

Radu Ioan Bot, University of Vienna, Austria:

"Gábor Kassay was a good friend and a great companion from the very early days of my academic career [13]. I have great memories with him from his visits in Chemnitz, and also from the various optimization conferences we jointly attended."

Cornel Pintea, Babeş-Bolyai University, Cluj-Napoca, Romania:

"I first met Gabi Kassay in 1985 as a freshman student at Faculty of Mathematics, Babeş-Bolyai University, as he taught me and my group of colleagues a tutorial of Mathematical Analysis. Gabi Kassay was a teacher and researcher of high order. I certainly appreciated, during my first academic year, the rigorous and meticulous way in which he prepared and delivered his topics such as the Cantor sets, the Cantor intersection theorem, the structure of the open subsets of the real line, a Whitney type decomposition theorem, integrals and so on. At that time I also noticed his ability to enter the world of the students he taught as he considered himself and used to be considered by most of his students as part of their own world. His teaching activity has obviously reached higher and higher levels, due to its own dynamic along the last decades, and its outcome consists in several realized and well established former students. Such an accommodation with the students he taught was only possible through extraordinary communication skills. Therefore, I am also sure that he was widely appreciated by his students along the almost four decades of his teaching activity and most of them still remember his lectures.

The research component of his professional activity is also very reach and highly appreciated within the Mathematical Analysis community, with emphasis on Optimization, Variational Analysis and Equilibrium Problems, as his published scientific papers have great impact in this community. Indeed, Gabi has extensively published in national and especially in international journals with good standards and was the author of several books and book chapters among which we just mention here the monographs *The Equilibrium Problem and Related Topics* [10] and *Equilibrium Problems and Applications* [31]. The outcome of his research activity was significantly influenced, in my opinion, by his communication skills as he used to have direct contacts with his collaborators on a regular basis. In this respect he traveled a lot and used these opportunities, not only for mathematical production, but also to understand the local culture and the history of the communities he visited. I had several opportunities to observe this face of his cultural interests when we both traveled for common scientific events such as those in Isfahan (Iran) for a conference on Nonlinear Analysis and Optimization in 2009, in Pisa (Italy) for a workshop on Variational Analysis, Equilibria and Optimization organized, in May 2017, in the honor of his 60th birthday or in Granada (Spain) for a conference on Minimax Inequalities and Equilibrium Problems in May 2019. In fact Gabi was one of the greatest fruitful travelers, in professional purposes, in our department. Indeed the outcome of his research activity does not only reduces to his publications but is also visible through the PhD students he supervised who are currently occupying important positions both

in Romania and abroad. Gabi has had an extensive coordination activity. Indeed, he coordinated 3 exploratory and research projects (IDEAS) obtained by competition at the national level, all with significant scientific output e.g. [21, 22, 19]. Gabi was also the coordinator of the Analysis and Optimization Research Group within our Faculty of Mathematics and Computer Science, a group with important scientific production. Last, but not least, Gabi had an extensive editorial activity, being a member of the editorial board of 6 international journals."

Szilárd Csaba László, Technical University of Cluj-Napoca, Romania:

"Professor Gábor Kassay was my PhD supervisor, mentor and, last but not least, my good friend. He was full of zest and enthusiasm for living, he was driven by curiosity about new things. In his mathematical proofs he was characterized by strict logic and consistency, but at the same time he was able to pass on even the newly acquired knowledge to his students or colleagues.

During my doctoral studies, I had the opportunity to observe his attitude towards science and mathematics. I always listened to his scientific lectures and refined explanations with great interest. He taught that not all mathematical results are worth publishing and that we should distinguish between really valuable and negligible mathematical results. He also showed me the importance of examples and counterexamples in a mathematical study. He shared the open questions and obstacles that arose during his research with his colleagues and friends. He was happy when someone could give a counterexample or an explanation. In such cases, he gladly involved the given person in his current research, he made no difference whether he was a student or a professor.

Personally, I can thank Gábor a lot. He introduced me into the world of research and taught me how to write a scientific article [22]. Later, he was also my mentor in a postdoctoral project. He kept track of my scientific work, and I often held presentations at the research seminar he led. The loss of Gábor left a huge space behind, but his memory continues to live for us, those who knew him and respected his consciousness, helpfulness and optimism."

3. Concluding remarks

Gábor Kassay was driven by a desire to learn and discover new things. He also reached several places on each continent of the world and he shared many stories and experiences with his friends and colleagues. The presented memories show that Gábor Kassay was an excellent researcher, instructor, a good colleague and a great friend, whose loss leaves a hole in our hearts.

We would like to express our thanks to all who contributed to the realization of this article through memories and useful recommendations.

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Porosity-based methods for solving stochastic feasibility problems

Kay Barshad, Simeon Reich and Alexander J. Zaslavski

Dedicated to the memory of Professor Gábor Kassay.

Abstract. The notion of porosity is well known in Optimization and Nonlinear Analysis. Its importance is brought out by the fact that the complement of a σ-porous subset of a complete pseudo-metric space is a residual set, while the existence of the latter is essential in many problems which apply the generic approach. Thus, under certain circumstances, some refinements of known results can be achieved by looking for porous sets. In 2001 Gabour, Reich and Zaslavski developed certain generic methods for solving stochastic feasibility problems. This topic was further investigated in 2021 by Barshad, Reich and Zaslavski, who provided more general results in the case of unbounded sets. In the present paper we introduce and examine new generic methods that deal with the aforesaid problems, in which, in contrast with previous studies, we consider sigma-porous sets instead of meager ones.

Mathematics Subject Classification (2010): 37B25, 46N10, 47J25, 54E50, 54E52, 90C30, 90C48.

Keywords: Baire category, Banach space, common fixed point problem, generic convergence, porous set, residual set, stochastic feasibility problem.

1. Introduction and background

We consider (generalized) stochastic feasibility problems from the point of view of the generic approach (for more applications of this approach, see, for example, [7]). These are the problems of finding almost common fixed points of measurable (with respect to a probability measure) families of mappings. Namely, we provide generic methods for finding almost common fixed points by using the notion of porosity. Our results are applicable to both the consistent case (that is, the case where the aforesaid

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almost common fixed points exist) and the inconsistent case (that is, the case where there are no common fixed points at all).

We begin by recalling the definitions of porosity and local convexity.

Given a pseudo-metric space (Y, ρ), we denote by $B_\rho(y, r)$, for each $y \in Y$ and $r > 0$, the open ball in (Y, ρ) of center y and radius r. Recall that a subset E of a complete pseudo-metric space (Y, ρ) is called a *porous* subset of Y if there exist $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$, there exists a point $z \in Y$ for which

$$B_\rho(z, \alpha r) \subset B_\rho(y, r) \setminus E.$$

A subset of Y is called a *σ-porous* subset of Y if it is a countable union of porous subsets of Y. Note that since a porous set is nowhere dense, any σ-porous set is of the first category and hence its complement is residual in (Y, ρ), that is, it contains a countable intersection of open and dense subsets of (Y, ρ). For this reason, there is a considerable interest in σ-porous sets while searching for generic solutions to optimization problems. More information concerning the notion of porosity and its applications can be found, for example, in [3], [6], [7] and [8].

Recall that a topological vector space V with the topology T is said to be a *locally convex space* if there exists a family $\mathscr{P}$ of pseudo-norms on V such that the family of open balls $\{B_\rho(x_0, \varepsilon) : x_0 \in V, \varepsilon > 0, \rho \in \mathscr{P}\}$ is a subbasis for T and $\cap_{\rho \in \mathscr{P}} Z_\rho = \{0\}$, where $Z_\rho = \{x \in V : \rho(x) = 0\}$ for each $\rho \in \mathscr{P}$. Clearly, every normed space (as a topological vector space with respect to its norm) is a locally convex space. In the sequel we use the following result (see Theorem 3.9 in [2]).

Theorem 1.1. *Let V be a real locally convex topological vector space, and let A and B be two disjoint closed and convex subsets of V. If either A or B is compact, then A and B are strictly separated, that is, there is $\alpha \in \mathbb{R}$ and a continuous linear functional $\phi : V \to \mathbb{R}$ such that $\phi(a) > \alpha$ for each $a \in A$ and $\phi(b) < \alpha$ for each $b \in B$.*

Now we introduce the spaces for which we investigate the stochastic feasibility problem. Other spaces which can be considered regarding this problem, can be found, for example, in [1] and [5].

Suppose that $(X, \|\cdot\|)$ is a normed vector space with norm $\|\cdot\|$, F is a nonempty, closed, convex and bounded subset of X, $(\Omega, \mathcal{A}, \mu)$ is a probability measure space (more information on measure spaces and measurable mappings can be found, for example, in [3]) and K is a subset of X which contains F. Denote by $\mathcal{N}$ the set of all nonexpansive mappings $A : K \to F$, that is, all mappings $A : K \to F$ such that $\|Ax - Ay\| \le \|x - y\|$ for each $x, y \in K$. For the set $\mathcal{N}$, define a metric $\rho_{\mathcal{N}} : \mathcal{N} \times \mathcal{N} \to \mathbb{R}$ by

$$\rho_{\mathcal{N}}(A, B) := \sup\{\|Ax - Bx\| : x \in K\}, \ A, B \in \mathcal{N}.$$

Clearly, the metric space $(\mathcal{N}, \rho_{\mathcal{N}})$ is complete if $(X, \|\cdot\|)$ is a Banach space.

Denote by $\mathcal{N}_\Omega$ the set of all mappings $T : \Omega \to \mathcal{N}$ such that for each $x \in K$, the mapping $T'_x : \Omega \to F$, defined, for each $\omega \in \Omega$, by $T'_x(\omega) := T(\omega)(x)$, is measurable. It is not difficult to see that if $T \in \mathcal{N}_\Omega$, then T'_x is integrable on Ω. For each $T \in \mathcal{N}_\Omega$, define an operator $\widetilde{T} : K \to F$ by $\widetilde{T}x = \int_\Omega T'_x(\omega)\, d\mu(\omega)$ for each $x \in K$. By Theorem 1.1, this is indeed a mapping the image of which is contained in F. Note that the

mapping defined on $\mathcal{N}_\Omega$ by $T \mapsto \widetilde{T}$ is onto $\mathcal{N}$. Clearly, for each $T \in \mathcal{N}_\Omega$, we have $\widetilde{T} \in \mathcal{N}$. Thus we consider the topology defined by the following pseudo-metric on $\mathcal{N}_\Omega$:

$$\rho_{\mathcal{N}_\Omega}(T,S) := \rho_{\mathcal{N}}\left(\widetilde{T},\widetilde{S}\right),\ T,S \in \mathcal{N}_\Omega.$$

It is not difficult to see that the pseudo-metric space $(\mathcal{N}_\Omega, \rho_{\mathcal{N}_\Omega})$ is complete if $(X, \|\cdot\|)$ is a Banach space.

Denote by $\mathcal{M}_\Omega$ the set of all sequences $\{T_n\}_{n=1}^\infty \subset \mathcal{N}_\Omega$. We define a pseudo-metric $\rho_{\mathcal{M}_\Omega} : \mathcal{M}_\Omega \times \mathcal{M}_\Omega \to \mathbb{R}$ on $\mathcal{N}_\Omega$ in the following way:

$$\rho_{\mathcal{M}_\Omega}\left(\{T_n\}_{n=1}^\infty, \{S_n\}_{n=1}^\infty\right) := \sup\left\{\rho_{\mathcal{N}_\Omega}(T_n,S_n) :\ n = 1,2\dots\right\},$$
$$\{T_n\}_{n=1}^\infty, \{S_n\}_{n=1}^\infty \in \mathcal{M}_\Omega.$$

Obviously, this space is complete if $(X, \|\cdot\|)$ is a Banach space.

The rest of the paper is organized as follows. In Section 2 we state our main results. Two auxiliary assertions are presented in Section 3. In Section 4 we provide the proofs of our main results.

In all our results we also assume that $(X, \|\cdot\|)$ is a Banach space.

2. Statements of the main results

In this section we state our main results. We establish them in Section 4 below.

Recall that for each $T \in \mathcal{N}_\Omega$, a point $x \in K$ is an *almost common fixed point* of the family $\{T(\omega)\}_{\omega\in\Omega}$ if $T(\omega)x = x$ for almost all $\omega \in \Omega$. Similarly, for each sequence $\{T_n\}_{n=1}^\infty \in \mathcal{M}_\Omega$, a point $x \in K$ is an *almost common fixed point* of the family $\{T_n(\omega)\}_{\omega\in\Omega,\, n=1,2\dots}$ if $T_n(\omega)x = x$ for all $n = 1,2,\dots$ and almost all $\omega \in \Omega$.

Theorem 2.1. *There exists a set $\mathcal{F} \subset \mathcal{M}_\Omega$ such that $\mathcal{M}_\Omega \backslash \mathcal{F}$ is a σ-porous subset of $\mathcal{M}_\Omega$ and for each $\{T_n\}_{n=1}^\infty \in \mathcal{F}$, the following assertion holds true:*

For each $\varepsilon > 0$, there is a positive integer N such that for each integer $n \geq N$ and each mapping $s : \{1,2,\dots\} \to \{1,2,\dots\}$, we have

$$\left\|\widetilde{T_{s(n)}}\dots\widetilde{T_{s(1)}}x - \widetilde{T_{s(n)}}\dots\widetilde{T_{s(1)}}y\right\| < \varepsilon$$

for each $x, y \in K$. Consequently, if there is an almost common fixed point of the family $\{T_n(\omega)\}_{\omega\in\Omega,\, n=1,2\dots}$, then it is unique and for each $x \in K$, the sequence $\left\{\widetilde{T_{s(n)}}\dots\widetilde{T_{s(1)}}x\right\}_{n=1}^\infty$ converges to it as $n \to \infty$, uniformly on K, for each mapping $s : \{1,2,\dots\} \to \{1,2,\dots\}$.

Theorem 2.2. *There exists a set $\mathcal{F} \subset \mathcal{N}_\Omega$ such that the set $\mathcal{G} := \mathcal{N}_\Omega \backslash \mathcal{F}$ a σ-porous subset of $\mathcal{N}_\Omega$, and for each $T \in \mathcal{F}$, the following assertion holds true:*

There exists $x_T \in K$ which is the unique fixed point of the operator $\widetilde{T}$ such that for each $x \in K$, the sequence $\left\{\widetilde{T}^n x\right\}_{n=1}^\infty$ converges to x_T as $n \to \infty$, uniformly on K. Moreover, the set $\mathfrak{F}$ of all almost common fixed points of the family $\{T(\omega)\}_{\omega\in\Omega}$ is contained in $\{x_T\}$. As a result, if $\mathfrak{F} \neq \emptyset$, then x_T is the unique almost common fixed point of the family $\{T(\omega)\}_{\omega\in\Omega}$.

3. Auxiliary results

In this section we present two lemmata which will be used in the proofs of our main results. We start by defining three sequences which we use in the proofs of these lemmata.

Choose $z_0 \in F$ and set $r_0 = 1$. We first define the sequence $\{\alpha_k\}_{k=1}^{\infty}$ of positive numbers by

$$\alpha_k = 2^{-1}\left(1 + 2k\left(2\sup_{z\in F}\|z\| + 1\right)\right)^{-1} \in (0,1)\,. \tag{3.1}$$

Clearly, for each positive integer k,

$$(1-\alpha_k)\left(2\sup_{z\in F}\|z\| + 1\right)^{-1} \in (0,1) \tag{3.2}$$

and

$$(1-\alpha_k)\left(2\sup_{z\in F}\|z\| + 1\right)^{-1} - 2\alpha_k k = 2^{-1}\left(2\sup_{z\in F}\|z\| + 1\right)^{-1} > 0. \tag{3.3}$$

Using (3.3), for each $r \in (0, r_0]$, we choose sequences $\{\gamma_k^r\}_{k=1}^{\infty}$ and $\{N_k^r\}_{k=1}^{\infty}$ of positive numbers such that

$$\gamma_k^r \in \left(2\alpha_k k r, (1-\alpha_k)\, r\left(2\sup_{z\in F}\|z\| + 1\right)^{-1}\right) \tag{3.4}$$

and

$$N_k^r > 2\left(\gamma_k^r k^{-1} - 2\alpha_k r\right)^{-1}\sup_{z\in F}\|z\| + 1 \tag{3.5}$$

for each positive integer k. Evidently, by(3.1), (3.2) and (3.4), $\gamma_k^r \in (0,1)$.

Lemma 3.1. *Assume that k is a positive integer and let $\mathcal{F}_k$ be the set of all sequences $\{T_n\}_{n=1}^{\infty} \in \mathcal{M}_\Omega$ for which there exists a positive integer N such that for each mapping $s : \{1,2,\dots\} \to \{1,2,\dots\}$, we have*

$$\left\|\widetilde{T_{s(N)}}\dots\widetilde{T_{s(1)}}x - \widetilde{T_{s(N)}}\dots\widetilde{T_{s(1)}}y\right\| < k^{-1}$$

for each $x, y \in K$. Then the set $\mathcal{G}_k := \mathcal{M}_\Omega \backslash \mathcal{F}_k$ is a porous subset of $\mathcal{M}_\Omega$.

Proof. Assume that $\{T_n\}_{n=1}^{\infty} \in \mathcal{M}_\Omega$ and $r \in (0, r_0]$. Define a sequence of mappings $\left\{T_n^{\gamma_k^r}\right\}_{n=1}^{\infty}$, $T_n^{\gamma_k^r} : \Omega \to \mathcal{N}$, by

$$T_n^{\gamma_k^r}(\omega)\, x := (1-\gamma_k^r)\, T_n(\omega)\, x + \gamma_k^r z_0,\ n = 1,2,\dots$$

for each $\omega \in \Omega$ and each $x \in K$. Clearly, $\left\{T_n^{\gamma_k^r}\right\}_{n=1}^{\infty} \in \mathcal{M}_\Omega$ and for each $n = 1,2,\dots,$

$$\begin{aligned}\widetilde{T_n^{\gamma_k^r}}x &= \int_\Omega \left((1-\gamma_k^r)\, T_n(\omega)\, x + \gamma_k^r z_0\right) d\mu(\omega) = \gamma_k^r z_0 + (1-\gamma_k^r)\int_\Omega T_n(\omega)\, x d\mu(\omega)\\ &= (1-\gamma_k^r)\,\widetilde{T_n}x + \gamma_k^r z_0\end{aligned}$$

for each $x \in K$. We have

$$\rho_{\mathcal{M}_\Omega}\left(\left\{T_n^{\gamma_k^r}\right\}_{n=1}^{\infty}, \{T_n\}_{n=1}^{\infty}\right) \le 2\gamma_k^r \sup_{z\in F}\|z\|\,, \tag{3.6}$$

as well as, for each positive integer n,

$$\left\|\widetilde{T_n^{\gamma_k^r}}x - \widetilde{T_n^{\gamma_k^r}}y\right\| \leq (1-\gamma_k^r)\left\|x-y\right\| \tag{3.7}$$

for each $x, y \in K$.
Let $\{S_n\}_{n=1}^{\infty} \in \mathcal{M}_\Omega$ satisfy

$$\rho_{\mathcal{M}_\Omega}\left(\left\{T_n^{\gamma_k^r}\right\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty}\right) < \alpha_k r. \tag{3.8}$$

Assume that $s : \{1, 2, \dots\} \to \{1, 2, \dots\}$ is an arbitrary mapping. We claim that

$$\left\|\widetilde{S_{s\left(N_k^r\right)}} \dots \widetilde{S_{s(1)}}x - \widetilde{S_{s\left(N_k^r\right)}} \dots \widetilde{S_{s(1)}}y\right\| < k^{-1} \tag{3.9}$$

for each $x, y \in K$. Suppose to the contrary that this does not hold. Then there exist points $x_0, y_0 \in K$ such that for each $i = 0 \dots N_k^r$, we have

$$\left\|\widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}}x_0 - \widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}}y_0\right\| \geq k^{-1}. \tag{3.10}$$

Using the triangle inequality, (3.8), (3.7) and (3.10), we obtain that for each $i = 1 \dots N_k^r$,

$$\begin{aligned}
&\left\|\widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}}x_0 - \widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}}y_0\right\| \\
&\leq \left\|\widetilde{S_{s(i)}}\widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}}x_0 - \widetilde{T_{s(i)}^{\gamma_k^r}}\widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1}}x_0\right\| \\
&+ \left\|\widetilde{T_{s(i)}^{\gamma_k^r}}\widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}}x_0 - \widetilde{T_{s(i)}^{\gamma_k^r}}\widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}}y_0\right\| \\
&+ \left\|\widetilde{T_{s(i)}^{\gamma_k^r}}\widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}}y_0 - \widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}}y_0\right\| \\
&< 2\alpha_k r + (1-\gamma_k^r)\left\|\widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}}x_0 - \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}}y_0\right\| \\
&\leq \left\|\widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}}x_0 - \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}}y_0\right\| + 2\alpha_k r - \gamma_k^r k^{-1}.
\end{aligned}$$

Hence by (3.4), we have

$$\begin{aligned}
&\left\|\widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}}x_0 - \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}}y_0\right\| - \left\|\widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}}x_0 - \widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}}y_0\right\| \\
&> \gamma_k^r k^{-1} - 2\alpha_k r > 0
\end{aligned}$$

for each $i = 1 \dots N_k^r$. Therefore

$$\begin{aligned}
2\sup_{z\in F}\|z\| &\geq \left\|\widetilde{S_{s(1)}}x_0 - \widetilde{S_{s(1)}}y_0\right\| - \left\|\widetilde{S_{s\left(N_k^r\right)}} \dots \widetilde{S_{s(1)}}x_0 - \widetilde{S_{s\left(N_k^r\right)}} \dots \widetilde{S_{s(1)}}y_0\right\| \\
&= \Sigma_{i=2}^{N_k^r}\left(\left\|\widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}}x_0 - \widetilde{S_{s(i-1)}} \dots \widetilde{S_{s(1)}}y_0\right\|\right. \\
&\left. - \left\|\widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}}x_0 - \widetilde{S_{s(i)}} \dots \widetilde{S_{s(1)}}y_0\right\|\right) > \left(N_k^r - 1\right)\left(\gamma_k^r k^{-1} - 2\alpha_k r\right).
\end{aligned}$$

As a result,

$$N_k^r < 2\left(\gamma_k^r k^{-1} - 2\alpha_k r\right)^{-1}\sup_{z\in F}\|z\| + 1.$$

This, however, contradicts (3.5). Thus (3.9) does hold. Next, using the triangle inequality, we see by (3.6), (3.8) and (3.4) that

$$\begin{aligned}\rho_{\mathcal{M}_\Omega}\left(\{T_n\}_{n=1}^{\infty},\{S_n\}_{n=1}^{\infty}\right) &\leq \rho_{\mathcal{M}_\Omega}\left(\{T_n\}_{n=1}^{\infty},\left\{T_n^{\gamma_k^r}\right\}_{n=1}^{\infty}\right)\\ &+\rho_{\mathcal{M}_\Omega}\left(\left\{T_n^{\gamma_k^r}\right\}_{n=1}^{\infty},\{S_n\}_{n=1}^{\infty}\right)\\ &< 2\gamma_k^r\sup_{z\in F}\|z\|+\alpha_k r<(1-\alpha_k)\,r+\alpha_k r=r. \end{aligned}\tag{3.11}$$

From (3.9) and (3.11) it now follows that

$$B_{\rho_{\mathcal{M}_\Omega}}\left(\left\{T_n^{\gamma_k^r}\right\}_{n=1}^{\infty},\alpha_k r\right)\subset B_{\rho_{\mathcal{M}_\Omega}}\left(\{T_n\}_{n=1}^{\infty},r\right)\cap\mathcal{F}_k=B_{\rho_{\mathcal{M}_\Omega}}\left(\{T_n\}_{n=1}^{\infty},r\right)\backslash\mathcal{G}_k.$$

Hence $\mathcal{G}_k$ is indeed a porous subset of $\mathcal{M}_\Omega$, as asserted. □

Lemma 3.2. *Assume that k is a positive integer and let $\mathcal{F}_k$ be the set of all mappings $T\in\mathcal{N}_\Omega$ for which there exists a positive integer N such that*

$$\left\|\widetilde{T}^N x-\widetilde{T}^N y\right\|<k^{-1}$$

for each $x,y\in K$. Then the set $\mathcal{G}_k:=\mathcal{N}_\Omega\backslash\mathcal{F}_k$ is a porous subset of $\mathcal{N}_\Omega$.

Proof. Assume that $T\in\mathcal{N}_\Omega$ and $r\in(0,r_0]$. Define a mapping $T_{\gamma_k^r}$, $T_{\gamma_k^r}:\Omega\to\mathcal{N}$, by

$$T_{\gamma_k^r}(\omega)\,x:=(1-\gamma_k^r)\,T(\omega)\,x+\gamma_k^r z_0$$

for each $\omega\in\Omega$ and each $x\in K$. Clearly, $T_{\gamma_k^r}\in\mathcal{N}_\Omega$ and

$$\begin{aligned}\widetilde{T_{\gamma_k^r}}x&=\int_\Omega\left((1-\gamma_k^r)\,T(\omega)\,x+\gamma_k^r z_0\right)d\mu(\omega)=\gamma_k^r z_0+(1-\gamma_k^r)\int_\Omega T(\omega)\,x\,d\mu(\omega)\\ &=(1-\gamma_k^r)\,\widetilde{T}x+\gamma_k^r z_0\end{aligned}$$

for each $x\in K$. We have

$$\rho_{\mathcal{N}_\Omega}\left(T_{\gamma_k^r},T\right)\leq 2\gamma_k^r\sup_{z\in F}\|z\|,\tag{3.12}$$

as well as

$$\left\|\widetilde{T_{\gamma_k^r}}x-\widetilde{T_{\gamma_k^r}}y\right\|\leq(1-\gamma_k^r)\,\|x-y\|\tag{3.13}$$

for each $x,y\in K$.
Let $S\in\mathcal{N}_\Omega$ satisfy

$$\rho_{\mathcal{N}_\Omega}\left(T_{\gamma_k^r},S\right)<\alpha_k r.\tag{3.14}$$

We claim that

$$\left\|\widetilde{S}^{N_k^r}x-\widetilde{S}^{N_k^r}y\right\|<k^{-1}\tag{3.15}$$

for each $x,y\in K$. Suppose to the contrary that this does not hold. Then there exist points $x_0,y_0\in K$ such that for each $i=0\ldots N_k^r$, we have

$$\left\|\widetilde{S}^i x_0-\widetilde{S}^i y_0\right\|\geq k^{-1}.\tag{3.16}$$

Using the triangle inequality, (3.14), (3.13) and (3.16), we see that for each $i = 1 \dots N_k^r$,

$$\begin{aligned}\left\|\widetilde{S}^i x_0 - \widetilde{S}^i y_0\right\| &\leq \left\|\widetilde{S}\widetilde{S}^{i-1} x_0 - \widetilde{T_{\gamma_k^r}}\widetilde{S}^{i-1} x_0\right\| \\ &+ \left\|\widetilde{T_{\gamma_k^r}}\widetilde{S}^{i-1} x_0 - \widetilde{T_{\gamma_k^r}}\widetilde{S}^{i-1} y_0\right\| + \left\|\widetilde{T_{\gamma_k^r}}\widetilde{S}^{i-1} y_0 - \widetilde{S}\widetilde{S}^{i-1} y_0\right\| \\ &< 2\alpha_k r + (1-\gamma_k^r)\left\|\widetilde{S}^{i-1} x_0 - \widetilde{S}^{i-1} y_0\right\| \\ &\leq \left\|\widetilde{S}^{i-1} x_0 - \widetilde{S}^{i-1} y_0\right\| + 2\alpha_k r - \gamma_k^r k^{-1}.\end{aligned}$$

Hence by (3.4),

$$\left\|\widetilde{S}^{i-1} x_0 - \widetilde{S}^{i-1} y_0\right\| - \left\|\widetilde{S}^i x_0 - \widetilde{S}^i y_0\right\| > \gamma_k^r k^{-1} - 2\alpha_k r > 0,$$

for each $i = 1 \dots N_k^r$. Therefore

$$\begin{aligned}2\sup_{z\in F}\|z\| &\geq \left\|\widetilde{S} x_0 - \widetilde{S} y_0\right\| - \left\|\widetilde{S}^{N_k^r} x_0 - \widetilde{S}^{N_k^r} y_0\right\| \\ &= \Sigma_{i=2}^{N_k^r}\left(\left\|\widetilde{S}^{i-1} x_0 - \widetilde{S}^{i-1} y_0\right\| - \left\|\widetilde{S}^i x_0 - \widetilde{S}^i y_0\right\|\right) \\ &> \left(N_k^r - 1\right)\left(\gamma_k^r k^{-1} - 2\alpha_k r\right).\end{aligned}$$

As a result,

$$N_k^r < 2\left(\gamma_k^r k^{-1} - 2\alpha_k r\right)^{-1}\sup_{z\in F}\|z\| + 1.$$

This, however, contradicts (3.5). Thus (3.15) does hold. Next, using the triangle inequality, we see by (3.12), (3.14) and (3.4) that

$$\begin{aligned}\rho_{\mathcal{N}_\Omega}(T,S) &\leq \rho_{\mathcal{N}_\Omega}\left(T, T_{\gamma_k^r}\right) + \rho_{\mathcal{N}_\Omega}\left(T_{\gamma_k^r}, S\right) \\ &< 2\gamma_k^r \sup_{z\in F}\|z\| + \alpha_k r < (1-\alpha_k) r + \alpha_k r = r.\end{aligned} \tag{3.17}$$

From (3.15) and (3.17) it now follows that

$$B_{\rho_{\mathcal{N}_\Omega}}\left(T_{\gamma_k^r}, \alpha_k r\right) \subset B_{\rho_{\mathcal{N}_\Omega}}(T, r) \cap \mathcal{F}_k = B_{\rho_{\mathcal{N}_\Omega}}(T, r) \backslash \mathcal{G}_k.$$

Hence $\mathcal{G}_k$ is indeed a porous subset of $\mathcal{N}_\Omega$, as asserted. □

4. Proofs of the main results

Proof of Theorem 2.1. By Lemma 3.1, there is a sequence of subsets $\{\mathcal{F}_n\}_{n=1}^\infty$ of $\mathcal{M}_\Omega$ such that for each positive integer n, the set $\mathcal{G}_n := \mathcal{M}_\Omega \backslash \mathcal{F}_n$ is a porous subset of $\mathcal{M}_\Omega$ and $\mathcal{F}_n$ is the set of all sequences $\{T_n\}_{n=1}^\infty \in \mathcal{M}_\Omega$ for which there exists a positive integer N such that for each mapping $s : \{1, 2, \dots\} \to \{1, 2, \dots\}$, we have

$$\left\|\widetilde{T_{s(N)}} \dots \widetilde{T_{s(1)}} x - \widetilde{T_{s(N)}} \dots \widetilde{T_{s(1)}} y\right\| < n^{-1} \tag{4.1}$$

for each $x, y \in K$. Set $\mathcal{F} := \cap_{n=1}^\infty \mathcal{F}_n$. Then $\mathcal{M}_\Omega \backslash \mathcal{F} = \cup_{n=1}^\infty \mathcal{G}_n$ is a σ-porous subset of $\mathcal{M}_\Omega$.

Let $\{T_n\}_{n=1}^{\infty} \in \mathcal{F}$ and let $\varepsilon > 0$. Choose a positive integer n_0 such that $n_0^{-1} < \varepsilon$. Since $\{T_n\}_{n=1}^{\infty} \in \mathcal{F}_{n_0}$, we infer from (4.1) that there exists a positive integer N such that for each integer $n \geq N$ and each mapping $s : \{1, 2, \dots\} \to \{1, 2, \dots\}$,

$$\left\|\widetilde{T_{s(n)}} \dots \widetilde{T_{s(1)}} x - \widetilde{T_{s(n)}} \dots \widetilde{T_{s(1)}} y\right\| \leq \left\|\widetilde{T_{s(N)}} \dots \widetilde{T_{s(1)}} x - \widetilde{T_{s(N)}} \dots \widetilde{T_{s(1)}} y\right\| < n_0^{-1} < \varepsilon \tag{4.2}$$

for each $x, y \in K$. This completes the proof. □

Proof of Theorem 2.2. By Lemma 3.2, there is a sequence of subsets $\{\mathcal{F}_n\}_{n=1}^{\infty}$ of $\mathcal{N}_\Omega$ such that for each positive integer n, the set $\mathcal{G}_n := \mathcal{N}_\Omega \backslash \mathcal{F}_n$ is a porous subset of $\mathcal{N}_\Omega$ and $\mathcal{F}_n$ is the set of all mappings $T \in \mathcal{N}_\Omega$ for which there exists a positive integer N satisfying

$$\left\|\widetilde{T}^N x - \widetilde{T}^N y\right\| < n^{-1} \tag{4.3}$$

for each $x, y \in K$. Set $\mathcal{F} := \cap_{n=1}^{\infty} \mathcal{F}_n$. Then $\mathcal{N}_\Omega \backslash \mathcal{F} = \cup_{n=1}^{\infty} \mathcal{G}_n$ is a σ-porous subset of $\mathcal{N}_\Omega$.

Let $T \in \mathcal{F}$ and let $\varepsilon > 0$ be arbitrary. Choose a positive integer n_0 such that $n_0^{-1} < \varepsilon$. Since $T \in \mathcal{F}_{n_0}$, we infer from (4.3) that there exists a positive integer N such that for each integer $n \geq N$,

$$\left\|\widetilde{T}^n x - \widetilde{T}^n y\right\| < \left\|\widetilde{T}^N x - \widetilde{T}^N y\right\| < n_0^{-1} < \varepsilon \tag{4.4}$$

for each $x, y \in K$. Clearly, for all integers $n, m \geq N$, we have

$$\left\|\widetilde{T}^n x - \widetilde{T}^m x\right\| < \varepsilon \tag{4.5}$$

for each $x \in K$. Since ε is an arbitrary positive number, inequality (4.5) and the completeness of the subspace F of $(X, \|\cdot\|)$ imply that the sequence $\left\{\widetilde{T}^n\right\}_{n=1}^{\infty}$ converges to an operator $P : K \to F$, uniformly on K. By taking the limit in (4.4), we see that P is constant on K, that is, there exists a point $x_T \in K$ such that the sequence $\left\{\widetilde{T}^n x\right\}_{n=1}^{\infty} \to x_T$ as $n \to \infty$, uniformly on K. Pick an arbitrary point $x_0 \in K$. Since the operator $\widetilde{T}$ is continuous, it follows that

$$\widetilde{T} x_T = \widetilde{T} \lim_{n\to\infty} \widetilde{T}^k x_0 = \lim_{k\to\infty} \widetilde{T}^{k+1} x_0 = x_T.$$

Hence $x_T \in K$ is the unique fixed point of the operator $\widetilde{T}$, as asserted. □

Remark 4.1. We take this opportunity to correct two misprints in [1].

- Page 332, second paragraph: The sentence "Note that this mapping is onto K." should be replaced by the sentence "Note that the mapping defined on $\mathcal{N}_\Omega$ by $T \mapsto \widetilde{T}$ is onto $\mathcal{N}$."
- Page 347: The formula

$$\widetilde{R_n} x_R = \widetilde{R_n} \lim_{k\to\infty} \widetilde{R_n}^k x = \lim_{k\to\infty} \widetilde{R_n}^{k+1} x_R = x_R$$

should be replaced by the formula

$$\widetilde{R_n} x_R = \widetilde{R_n} \lim_{k\to\infty} \widetilde{R_n}^k x = \lim_{k\to\infty} \widetilde{R_n}^{k+1} x = x_R.$$

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A maximum theorem for generalized convex functions

Zsolt Páles

Dedicated to the memory of Professors Gábor Kassay and Csaba Varga.

Abstract. Motivated by the Maximum Theorem for convex functions (in the setting of linear spaces) and for subadditive functions (in the setting of Abelian semigroups), we establish a Maximum Theorem for the class of generalized convex functions, i.e., for functions $f : X \to \mathbb{R}$ that satisfy the inequality $f(x \circ y) \leq pf(x) + qf(y)$, where $\circ$ is a binary operation on X and p, q are positive constants. As an application, we also obtain an extension of the Karush–Kuhn–Tucker theorem for this class of functions.

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Keywords: Maximum theorem, generalized convex function.

1. Introduction

In what follows, a linear space X always means a vector space over the field of real numbers. If X is a topological linear space, then its (topological) dual space is denoted by X^*. The Maximum Theorem for convex functions, which is due to Dubovitskii and Milyutin (cf. [9]), can be stated as follows.

Theorem 1.1. *Let X be a linear space, let $D \subseteq X$ be a convex set and let $f_1, \dots, f_n : D \to \mathbb{R}$ be convex functions such that*

$$0 \leq \max(f_1(x), \dots, f_n(x)) \qquad (x \in D).$$

Then there exist $\lambda_1, \dots, \lambda_n \geq 0$ with $\lambda_1 + \cdots + \lambda_n = 1$ such that

$$0 \leq \lambda_1 f_1(x) + \cdots + \lambda_n f_n(x) \qquad (x \in D).$$

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A standard application of the Maximum Theorem is to prove the subdifferential formula for the pointwise maximum of convex functions, which was established by Dubovitskii and Milyutin (see [9]). For the standard terminologies and notations, we refer to the list of monographs in the list of references, where the reader can find many more details and applications.

Theorem 1.2. *Let X be a topological vector space, $D \subseteq X$ be an open convex set, $p \in D$ and $f_1, \dots, f_n : D \to \mathbb{R}$ be continuous convex functions with $f_1(p) = \cdots = f_n(p)$ and define $f := \max(f_1, \dots, f_n)$. Then*

$$\partial f(p) = \operatorname{conv}\big(\partial f_1(p) \cup \cdots \cup \partial f_n(p)\big).$$

Proof. Using that $f(p) = f_1(p) = \cdots = f_n(p)$, for all $h \in X$, we obtain

$$\begin{aligned}
f'(p,h) &:= \lim_{t\to 0^+} \frac{f(p+th)-f(p)}{t} \\
&= \lim_{t\to 0^+} \frac{\max(f_1(p+th),\dots,f_n(p+th)) - f(p)}{t} \\
&= \lim_{t\to 0^+} \max\left(\frac{f_1(p+th)-f(p)}{t},\dots,\frac{f_n(p+th)-f(p)}{t}\right) \\
&= \lim_{t\to 0^+} \max\left(\frac{f_1(p+th)-f_1(p)}{t},\dots,\frac{f_n(p+th)-f_n(p)}{t}\right) \\
&= \max\left(\lim_{t\to 0^+} \frac{f_1(p+th)-f_1(p)}{t},\dots,\lim_{t\to 0^+} \frac{f_n(p+th)-f_n(p)}{t}\right) \\
&= \max(f_1'(p,h),\dots,f_n'(p,h)).
\end{aligned}$$

First assume that a continuous linear functional $\varphi \in X^*$ belongs to $\partial f(p)$. Then, in view of the above formula for directional derivatives, we get

$$\varphi(h) \le f'(p,h) = \max(f_1'(p,h),\dots,f_n'(p,h)) \qquad (h \in X).$$

This relation implies that

$$0 \le \max(f_1'(p,h)-\varphi(h),\dots,f_n'(p,h)-\varphi(h)) \qquad (h \in X).$$

This inequality states that the maximum of the convex functions $h \mapsto f_i'(p,h) - \varphi(h)$ is nonnegative. Thus, by the Maximum Theorem, there exist $\lambda_1, \dots, \lambda_n \ge 0$ with $\lambda_1 + \cdots + \lambda_n = 1$ such that

$$0 \le \lambda_1(f_1'(p,h)-\varphi(h)) + \cdots + \lambda_n(f_n'(p,h)-\varphi(h)) \qquad (h \in X),$$

equivalently,

$$\varphi(h) \le \lambda_1 f_1'(p,h) + \cdots \lambda_n f_n'(p,h) = (\lambda_1 f_1 + \cdots \lambda_n f_n)'(p,h) \qquad (h \in X).$$

Using the so-called Sum Rule, we get

$$\begin{aligned}
\varphi \in \partial(\lambda_1 f_1 + \cdots \lambda_n f_n)(p) &= \lambda_1 \partial f_1(p) + \cdots + \lambda_n \partial f_n(p) \\
&\subseteq \operatorname{conv}\big(\partial f_1(p) \cup \cdots \cup \partial f_n(p)\big).
\end{aligned}$$

The proof of the reversed inclusion is simpler, thus it is left to the reader. □

Another motivation for this paper comes from the theory of subadditive functions defined on Abelian semigroups. The following result was stated in the monograph [7] of Fuchssteiner and Lusky.

Theorem 1.3. *Let $(X,+)$ be an Abelian semigroup and let $f_1,\dots,f_n : X \to \mathbb{R}$ be subadditive functions such that*

$$0 \le \max(f_1(x),\dots,f_n(x)) \qquad (x \in X).$$

Then there exist $\lambda_1,\dots,\lambda_n \ge 0$ with $\lambda_1+\dots+\lambda_n = 1$ such that

$$0 \le \lambda_1 f_1(x) + \dots + \lambda_n f_n(x) \qquad (x \in X).$$

This result has beautiful applications in the book [7], for instance, the Phragmen–Lindelöf Principle and the Hadamard Three Circle Theorem (both results belong to the theory of complex functions) can elegantly be verified in terms of them.

2. The general maximum problem

The two Maximum Theorems described in the Introduction motivate the following definition.

Definition 2.1. Let X be a nonempty set. A family $\mathcal{F} \subseteq \{f : X \to \mathbb{R}\}$ is said to have the *discrete maximum property* if

$$f_1,\dots,f_n \in \mathcal{F}, \qquad 0 \le \max(f_1(x),\dots,f_n(x)) \qquad (x \in X)$$

implies that there exist $(\lambda_1,\dots,\lambda_n) \in S_n$ such that

$$0 \le \lambda_1 f_1(x) + \dots + \lambda_n f_n(x) \qquad (x \in X).$$

Here, for convenience, S_n denotes the $(n-1)$-dimensional simplex

$$\{(\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n \mid \lambda_1,\dots,\lambda_n \ge 0,\ \lambda_1+\dots+\lambda_n = 1\}.$$

If X has at least two elements, then the set of all functions $\mathcal{F} := \{f : X \to \mathbb{R}\}$ does not have the discrete maximum property. Indeed, Let $\{A_1, A_2\}$ be a partition of X and $f_i(x) := 0$ if $x \in A_i$, $f_i(x) := -1$ if $x \notin A_i$. Then $\max(f_1,f_2) = 0$, but $\lambda f_1 + (1-\lambda) f_2 < 0$ for all $\lambda \in [0,1]$. This example shows that, in order to possess the discrete maximum property, the family $\mathcal{F} \subseteq \{f : X \to \mathbb{R}\}$ must satisfy some additional nontrivial conditions.

In the next result we characterize the situation when a finite family of given functions possess a nonnegative convex combination.

Theorem 2.2. *Let X be nonempty and $f_1,\dots,f_n : X \to \mathbb{R}$. Then there exists $(\lambda_1,\dots,\lambda_n) \in S^n$ such that*

$$0 \le \lambda_1 f_1(x) + \dots + \lambda_n f_n(x) \qquad (x \in X) \tag{2.1}$$

if and only if

$$0 \le \max_{i \in \{1,\dots,n\}} \big(t_1 f_i(x_1) + \dots + t_n f_i(x_n)\big) \quad (x_1,\dots,x_n \in X,\ (t_1,\dots,t_n) \in S_n). \tag{2.2}$$

Proof. Assume first that (2.1) holds for some $\lambda \in S_n$. To verify the necessity of (2.2), let $x_1, \dots, x_n \in X$ and $(t_1, \dots, t_n) \in S_n$ be arbitrary. Then, using (2.1) for $x \in \{x_1, \dots, x_n\}$, we get

$$\begin{aligned} 0 &\le \sum_{j=1}^{n} t_j\big(\lambda_1 f_1(x_j) + \cdots + \lambda_n f_n(x_j)\big) \\ &= \sum_{i=1}^{n} \lambda_i\big(t_1 f_i(x_1) + \cdots + t_n f_i(x_n)\big) \\ &\le \max_{i\in\{1,\dots,n\}} \big(t_1 f_i(x_1) + \cdots + t_n f_i(x_n)\big). \end{aligned}$$

This shows the necessity of condition (2.2).
Now assume that (2.2) holds and, for $x \in X$, define the set $\Lambda_x \subseteq S_n$ by

$$\Lambda_x := \big\{(\lambda_1, \dots, \lambda_n) \in S_n \mid 0 \le \lambda_1 f_1(x) + \cdots + \lambda_n f_n(x)\big\}. \tag{2.3}$$

The inequality (2.1) is now equivalent to the condition

$$\bigcap_{x\in X} \Lambda_x \ne \emptyset, \tag{2.4}$$

because every element λ of the above intersection will satisfy (2.1). It easily follows from the definition that Λ_x is a compact convex subset of the $(n-1)$-dimensional affine space

$$\{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \lambda_1 + \cdots + \lambda_n = 1\}.$$

Therefore, according to Helly's Theorem, the condition (2.4) is satisfied if and only every n-member subfamily of $\{\Lambda_x \mid x \in X\}$ has a nonempty intersection. To verify this, let $x_1, \dots .x_n \in X$ be fixed arbitrarily. According to inequality (2.2), the pointwise maximum of the convex functions

$$S_n \ni (t_1, \dots, t_n) \mapsto t_1 f_i(x_1) + \cdots + t_n f_i(x_n)$$

is nonnegative over S_n.
Therefore, in view of Theorem 1.1, there exists $(\lambda_1, \dots, \lambda_n) \in S_n$ such that

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} \lambda_i\big(t_1 f_i(x_1) + \cdots + t_n f_i(x_n)\big) \\ &= \sum_{j=1}^{n} t_j\big(\lambda_1 f_1(x_j) + \cdots + \lambda_n f_n(x_j)\big) \qquad ((t_1, \dots, t_n) \in S_n). \end{aligned}$$

If $i \in \{1, \dots, n\}$, then substituting $(t_1, \dots, t_n) := (\delta_{i,j})_{j=1}^n$ into the above inequality, we get that

$$\lambda_1 f_1(x_i) + \cdots + \lambda_n f_n(x_i) \qquad (i \in \{1, \dots, n\}).$$

This shows that $\lambda \in \Lambda_{x_1} \cap \cdots \cap \Lambda_{x_n}$, proving that this intersection is nonempty, as it was desired. □

In the case $n = 2$, the above theorem immediately implies the following statement.

Corollary 2.3. *Let X be a nonempty set and $f, g : X \to \mathbb{R}$. Then there exists $\lambda \in [0,1]$ such that*

$$0 \le \lambda f(x) + (1-\lambda) g(x) \qquad (x \in X) \tag{2.5}$$

if and only if

$$0 \le \max\big(tf(x) + (1-t)f(y), tg(x) + (1-t)g(y)\big) \qquad (x, y \in X,\ t \in [0,1]). \tag{2.6}$$

3. Generalized convexity

The general convexity property that we introduce below is going to play an important role in the sequel.

Definition 3.1. Let X be a nonempty set, $\circ : X \times X \to X$ be a binary operation, $p, q > 0$ be constants. A function $f : X \to \mathbb{R}$ is called $(\circ, p, q)$-*convex* if

$$f(x \circ y) \le pf(x) + qf(y) \qquad (x, y \in X).$$

Trivially, if X is a convex subset of a linear space, $p = q = \frac{1}{2}$, and $x \circ y = \frac{x+y}{2}$, then f is $(\circ, p, q)$-convex if and only if f is Jensen convex. On the other hand, if X is an Abelian semigroup, $p = q = 1$, and $x \circ y = x + y$, then f is $(\circ, p, q)$-convex if and only if f is subadditive.

The proof of the following assertion is elementary, therefore it is omitted.

Theorem 3.2. *The family of $(\circ, p, q)$-convex functions is closed with respect to addition, multiplication by positive scalars and pointwise maximum.*

The main result of this paper is stated in the following theorem.

Theorem 3.3. *Let X be a nonempty set, $\circ : X \times X \to X$ be a binary operation, and $p, q > 0$ be constants. Let $f_1, \dots, f_n : X \to \mathbb{R}$ be $(\circ, p, q)$-convex functions such that*

$$0 \le \max(f_1(x), \dots, f_n(x)) \qquad (x \in X).$$

Then there exist $\lambda_1, \dots, \lambda_n \ge 0$ with $\lambda_1 + \dots + \lambda_n = 1$ such that

$$0 \le \lambda_1 f_1(x) + \dots + \lambda_n f_n(x) \qquad (x \in X).$$

The following auxiliary result establishes the key tool for the proof of Theorem 3.3.

Lemma 3.4. *Let X be a nonempty set, $\circ : X \times X \to X$ be a binary operation, and $p, q > 0$ be constants. Let*

$$S := \left\{ \frac{a}{a+b} \;\middle|\; \begin{array}{l} \textit{There is an operation } * : X \times X \to X \textit{ such that} \\ \textit{every } (\circ, p, q)\textit{-convex function is } (*, a, b)\textit{-convex.} \end{array} \right\}$$

Then $1 - S \subseteq S$ and S is dense multiplicative subsemigroup of $[0,1]$.

Proof. If $s \in S$, then there exists an operation $* : X \times X \to X$ and $a, b > 0$ such that $s = \frac{a}{a+b}$ and f is $(*, a, b)$-convex, i.e.,

$$f(x * y) \le af(x) + bf(y) \qquad (x, y \in X).$$

Thus, interchanging the roles of x and y, we get

$$f(y * x) \le bf(x) + af(y) \qquad (x, y \in X),$$

which means that f is $(*', b, a)$-convex, where $x *' y := y * x$. Therefore $1 - s = \frac{b}{a+b} \in S$, which shows that $1 - S \subseteq S$.
Additionally, let $t \in S$ be arbitrary.
Then there exists a binary operation $\cdot : X \times X \to X$ and $c, d > 0$ such that $t = \frac{c}{c+d}$ and f is also $(\cdot, c, d)$-convex, i.e.,

$$f(x \cdot y) \leq cf(x) + df(y) \qquad (x, y \in X).$$

Using the $(\cdot, c, d)$- and the $(*, a, b)$-convexity of f (twice), for all $x, y \in X$, we obtain

$$\begin{aligned} f((x * y) \cdot (y * y)) &\leq cf(x * y) + df(y * y) \\ &\leq c(af(x) + bf(y)) + d(af(y) + bf(y)) \\ &= acf(x) + (bc + ad + bd)f(y). \end{aligned}$$

This implies that f is $(\diamond, ac, bc + ad + bd)$-convex, where $x \diamond y := (x * y) \cdot (y * y)$. Therefore,

$$st = \frac{ac}{ac + bc + ad + bd} \in S,$$

which proves that S is closed with respect to multiplication.
By induction, it follows that

$$s^n \in S \qquad (s \in S,\, n \in \mathbb{N}). \tag{3.7}$$

The assumption that f is $(\circ, p, q)$-convex implies that $S \cap]0, 1[\neq \emptyset$. Therefore, (3.7) yields that $\inf S = 0$. Using the inclusion $1 - S \subseteq S$, we can see that $\sup S = 1$.
Finally, to prove the density of S in $[0, 1]$, let $0 < a < b < 1$ be arbitrary. By $\sup S = 1$, we can choose $s \in S$ so that $\dfrac{a}{b} < s < 1$. Then, for some $n \in \mathbb{N}$, (in particular, with $n := \lfloor \frac{\log(a)}{\log(s)} \rfloor$), we have $s^n \in [a, b]$, which implies that $S \cap [a, b]$ is nonempty. □

In the next result, we verify the Maximum Theorem for two functions.

Theorem 3.5. *Let X be a nonempty set, $\circ : X \times X \to X$ be a binary operation, and $p, q > 0$ be constants. If $f, g : X \to \mathbb{R}$ are $(\circ, p, q)$-convex functions satisfying*

$$0 \leq \max(f(x), g(x)) \qquad (x \in X), \tag{3.8}$$

then there exists $\lambda \in [0, 1]$ such that (2.5) *holds true.*

Proof. First we show that f and g satisfy the inequality (2.6). To verify this, let $x, y \in X$ and let $s \in S$ (where the set S was defined in Lemma 3.4.) Then there exist a binary operation $* : X \times X \to X$ and constans $a, b > 0$ such that the $(\circ, p, q)$-convexity of f and g implies the $(*, a, b)$-convexity of them. Thus, by the maximum inequality (3.8) at $x * y$, we get

$$0 \leq \max(f(x * y), g(x * y)) \leq \max(af(x) + bf(y), ag(x) + bg(y)).$$

Therefore

$$0 \leq \max\Big(\frac{a}{a+b} f(x) + \frac{b}{a+b} f(y), \frac{a}{a+b} g(x) + \frac{b}{a+b} g(y)\Big),$$

and hence

$$0 \leq \max\big(sf(x) + (1 - s)f(y), sg(x) + (1 - s)g(y)\big).$$

Because $s \in S$ was arbitrary and S is dense in $[0,1]$ (according to Lemma 3.4), we can conclude that (2.6) is satisfied for all $t \in [0,1]$.
Having proved that (2.6) is valid, in view of Corollary 2.3, it follows that there exists $\lambda \in [0,1]$ such that (2.5) holds. □

Proof of the discrete Maximum Theorem. The statement is trivial for $n = 1$ and it has been proved for $n = 2$. Assume its validity for some $n \geq 2$. Let $f_0, f_1, \dots, f_n$ be $(\circ, p, q)$-convex functions such that

$$0 \leq \max(f_0(x), f_1(x), \dots, f_n(x)) \qquad (x \in X).$$

Let $g(x) := \max(f_1(x), \dots, f_n(x))$. Then, by Theorem 3.2, we have that g is $(\circ, p, q)$-convex and

$$0 \leq \max(f_0(x), g(x)) \qquad (x \in X).$$

Using now Theorem 3.5, we obtain the existence of $\lambda \in [0,1]$ such that

$$\begin{aligned} 0 &\leq \lambda f_0(x) + (1-\lambda) g(x) \\ &= \max\big(\lambda f_0(x) + (1-\lambda) f_1(x), \dots, \lambda f_0(x) + (1-\lambda) f_n(x)\big) \qquad (x \in X). \end{aligned}$$

By the inductive assumption, there exists $(\lambda_1, \dots, \lambda_n) \in S_n$ such that

$$\begin{aligned} 0 &\leq \lambda_1\big(\lambda f_0(x) + (1-\lambda) f_1(x)\big) + \dots + \lambda_n\big(\lambda f_0(x) + (1-\lambda) f_n(x)\big) \\ &= \lambda f_0(x) + \lambda_1(1-\lambda) f_1(x) + \dots + \lambda_n(1-\lambda) f_n(x) \qquad (x \in X), \end{aligned}$$

which proves the statement for $(n+1)$ functions. □

4. An application

In the subsequent result we establish an extension of the Karush–Kuhn–Tucker Theorem.

Theorem 4.1. *Let X be a nonempty set, $\circ : X \times X \to X$ be a binary operation, and $p, q > 0$ be constants. Let $f_0, f_1, \dots, f_n : X \to \mathbb{R}$ be $(\circ, p, q)$-convex functions and assume that $f_0(x_0) = 0$ and $x_0 \in X$ is a solution of the constrained optimization problem*

$$\textit{Minimize} \quad f_0(x) \quad \textit{subject to} \quad f_1(x), \dots, f_n(x) \leq 0. \tag{4.9}$$

Then there exist $(\lambda_0, \lambda_1, \dots, \lambda_n) \in S_{n+1}$ such that

$$\lambda_1 f_1(x_0) = \dots = \lambda_1 f_1(x_0) = 0 \tag{4.10}$$

and

$$0 \leq \lambda_0 f_0(x) + \lambda_1 f_1(x) + \dots + \lambda_n f_n(x) \qquad (x \in X). \tag{4.11}$$

Conversely, if conditions (4.10) *and* (4.11) *hold for some $(\lambda_0, \lambda_1, \dots, \lambda_n) \in S_{n+1}$ with $\lambda_0 > 0$, then x_0 is a solution of the optimization problem* (4.9).

Proof. If x_0 is a solution of the optimization problem then, for all $x \in X$, the inequalities

$$f_0(x) < f_0(x_0) = 0 \qquad \text{and} \qquad f_1(x), \dots, f_n(x) \leq 0$$

cannot hold simultaneously. Hence

$$0 \leq \max(f_0(x), f_1(x), \dots, f_n(x)) \qquad (x \in X).$$

Therefore, in view of Theorem 3.3, there exist $(\lambda_0, \lambda_1, \dots, \lambda_n) \in S_{n+1}$ such that (4.11) holds.
Being a solution to (4.9), x_0 is admissible for the optimization problem, that is, we have that $f_1(x_0), \dots, f_n(x_0) \le 0$. Hence

$$0 \le \lambda_0 f_0(x_0) + \lambda_1 f_1(x_0) + \cdots + \lambda_n f_n(x_0) = \lambda_1 f_1(x_0) + \cdots + \lambda_n f_n(x_0) \le 0.$$

The terms in the last sum are nonpositive, therefore, the only way this sum can be zero is that it is zero termwise. Hence the transversality condition (4.10) is also true.
To prove the reversed statement, assume that (4.10) and (4.11) hold for some $(\lambda_0, \lambda_1, \dots, \lambda_n) \in S_{n+1}$ with $\lambda_0 > 0$. Let $x \in X$ be an admissible point with respect to problem (4.9), i.e., assume that $f_1(x_0), \dots, f_n(x_0) \le 0$. Then, by (4.10) and (4.11), we get

$$\begin{aligned}\lambda_0 f_0(x_0) &= \lambda_0 f_0(x_0) + \lambda_1 f_1(x_0) + \cdots + \lambda_n f_n(x_0)\\ &= 0 \le \lambda_0 f_0(x) + \lambda_1 f_1(x) + \cdots + \lambda_n f_n(x) \le \lambda_0 f_0(x),\end{aligned}$$

which, using that $\lambda_0 > 0$, implies $f_0(x_0) \le f_0(x)$, and proves the minimality of x_0. □

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Continuity and maximal quasimonotonicity of normal cone operators

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Dedicated to the memory of Professor Gábor Kassay.

Abstract. In this paper we study some properties of the adjusted normal cone operator of quasiconvex functions. In particular, we introduce a new notion of maximal quasimotonicity for set-valued maps different from similar ones recently appeared in the literature, and we show that it is enjoyed by this operator. Moreover, we prove the $s \times w^*$ cone upper semicontinuity of the normal cone operator in the domain of f in case the set of global minima has non empty interior.

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Keywords: Quasimonotone operator, maximal quasimonotone operator, cone upper semicontinuity, upper sign continuity, quasiconvex function.

1. Introduction

The notion of maximal monotone operator dates back to the sixties and, since then, it has been extensively studied in literature (see, for instance, [9] and the references therein). One of the main interests for maximal monotone operators is the strong relationship between convexity of a function and maximal monotonicity of its associated subdifferential operator.

In recent years different generalizations of monotonicity have been proposed, both in the scalar (see [16]) and in the set-valued case, in finite and infinite dimensional spaces. Among them the most studied are, without a doubt, pseudomonotonicity and quasimonotonicity. Many nice properties of these classes of operators have been proved, but little effort has been devoted to the study of a suitable notion of maximality. To fill this gap, Hadjisavvas in [14] introduced and studied maximal pseudomonotone operators $T : X \rightrightarrows X^*$, where X is a Banach space and X^* denotes its dual, while the notion of maximality for quasimonotone operators has been addressed in the recent works by Aussel and Eberhard [6], and by Bueno and Cotrina

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[11]. In particular, in [11] the authors extend the notion of polarity introduced by Martínez-Legaz and Svaiter in 2005 ([17]), by defining the quasimonotone polar of a set-valued operator in order to characterize maximal quasimonotone operators via graph inclusion.

In this work we define a new notion of maximality for a quasimonotone operator defined on a Banach space, that is based both on the notion of quasimonotone polar of an operator T and on its behaviour at the points in the interior of the effective domain of T. This property is enjoyed, in particular, by the Clarke subdifferential $\partial^o f$, where f is quasiconvex and locally Lipschitz, under suitable restrictions on ∂^o, as well as by the adjusted normal cone operator to the sublevel sets of a quasiconvex, lower semicontinuous and solid function, provided suitable assumptions on the minima are satisfied. The interest in studying the properties of the adjusted normal cone operator is due to the crucial role it plays in characterizing quasiconvexity (see [7]).

The paper is organized as follows: In Section 2 we present some preliminary notions and results. In Section 3 the new definition of maximal quasimonotonicity for operators is introduced; some properties of maximal quasimonotone operators are established, together with a sufficient condition that can be compared with a similar one for maximal monotone operators. Section 4 is devoted to the investigation of the properties of the adjusted normal cone operator of a lower semicontinuous and quasiconvex function in terms of maximal quasimonotonicity and cone upper semicontinuity. In particular, the cone upper semicontinuity is proved in the domain of f in case the set of global minima has non empty interior, thereby extending a result in [7].

2. Preliminaries

Let X be a real Banach space, X^* its topological dual, and $\langle \cdot, \cdot \rangle$ the duality mapping. In the following, $\{x_\alpha\}$ and $\{x^*_\alpha\}$, with $\alpha \in \Gamma$ will denote nets in X and X^*, respectively.

For $x \in X$ and $r > 0$, $B(x,r)$, $\overline{B}(x,r)$ and $S(x,r)$ will denote the open ball, the closed ball and the sphere centered at x with radius r, respectively. Also, given a nonempty set $A \subseteq X$, let $B(A,\epsilon) = \{x \in X : \mathrm{dist}(x,A) < \epsilon\}$ and $\overline{B}(A,\epsilon) = \{x \in X : \mathrm{dist}(x,A) \leq \epsilon\}$, where $\mathrm{dist}(x,A) = \inf_{y\in A}\|x-y\|$ is the distance of x from A. A set L in a topological vector space is said to be a cone if it is closed under multiplication by nonnegative scalars; a set L is said to be an open cone if it is an open set, closed under multiplication by positive scalars. A convex set B is called a base of a cone L if and only if $0 \notin \overline{B}$ and $L = \cup_{t\geq 0} tB$.

The domain and the graph of a set valued map $T : X \rightrightarrows X^*$ will be denoted by $\mathrm{dom}(T)$ and $\mathrm{Gr}(T)$, while the *effective domain* of T is given by

$$\mathrm{edom}(T) = \{x \in \mathrm{dom}(T) : T(x) \neq \{0\}\}.$$

For any $x^* \in X^*$, let $\mathbb{R}_+ x^* = \{tx^* \in X^* : t \geq 0\}$ and for any $B \subseteq X^*$ let $\mathbb{R}_+ B = \cup_{x^*\in B}\mathbb{R}_+ x^*$. The operator $(\mathbb{R}_+ T) : X \rightrightarrows X^*$ is given by

$$(\mathbb{R}_+ T)(x) = \mathbb{R}_+(T(x)) = \cup_{x^*\in T(x)}\mathbb{R}_+ x^*.$$

Given $(x, x^*), (y, y^*) \in X \times X^*$, (x, x^*) is said to be *quasimonotonically related* to (y, y^*), denoted by $(x, x^*) \uparrow (y, y^*)$, if

$$\min\{\langle x^*, y - x\rangle, \langle y^*, x - y\rangle\} \leq 0$$

(see for instance [11] and the references therein). Note that $(x, 0)$ is quasimonotonically related to any $(y, y^*) \in X \times X^*$. Relation $\uparrow$ is a tolerance relation, i.e., it is reflexive and symmetric but in general not transitive.

The *quasimonotone polar* $T^\nu : X \rightrightarrows X^*$ of T is given by

$$\begin{aligned} T^\nu(x) &= \{x^* \in X^* : (x, x^*) \uparrow (y, y^*)\ \forall y^* \in T(y), y \in \mathrm{dom}(T)\} \\ &= \{x^* \in X^* : (x, x^*) \uparrow (y, y^*)\ \forall y^* \in T(y), y \in \mathrm{edom}(T)\} \end{aligned}$$

Note that $0 \in T^\nu(x)$ and that $T^\nu(x)$ is a cone for all $x \in X$. Moreover, $T^\nu(x)$ is a convex and w^*-closed set (see Corollary 3.8 in [11]), that can be not pointed (see, for instance, the next Example 3.2).

Moreover, the following proposition, related to Lemma 1 in [6] and to Proposition 3.5 in [14] holds :

Proposition 2.1. *Let $T : X \rightrightarrows X^*$ be an operator. If $(x_\alpha, x^*_\alpha) \in \mathrm{Gr}(T^\nu)$, $(x_\alpha, x^*_\alpha) \to (x, x^*)$ in the $w \times w^*$ topology, and $\limsup_\alpha \langle x^*_\alpha, x_\alpha\rangle \leq \langle x^*, x\rangle$, then $x^* \in T^\nu(x)$. In particular, $\mathrm{Gr}(T^\nu)$ is sequentially closed in the $s \times w^*$ topology and in the $w \times s$ topology.*

Proof. Take any $(y, y^*) \in \mathrm{Gr}(T)$. Since $(x_\alpha, x^*_\alpha) \uparrow (y, y^*)$, we have

$$\min\left\{\langle x^*_\alpha, y - x_\alpha\rangle, \langle y^*, x_\alpha - y\rangle\right\} \leq 0.$$

By our assumptions,

$$\liminf_\alpha \langle x^*_\alpha, y - x_\alpha\rangle = \langle x^*, y\rangle - \limsup_\alpha \langle x^*_\alpha, x_\alpha\rangle \geq \langle x^*, y - x\rangle .$$

Thus

$$\min\left\{\langle x^*, y - x\rangle, \langle y^*, x - y\rangle\right\} \leq 0$$

which says that $(x, x^*) \in \mathrm{Gr}(T^\nu)$.

In particular, $\mathrm{Gr}(T^\nu)$ is sequentially closed in the $s \times w^*$ and in the $w \times s$ topologies, because, in these cases, we have $\lim \langle x^*_n, x_n\rangle = \langle x^*, x\rangle$. □

In the sequel we will introduce the notions of quasimonotonicity, cone upper semicontinuity, upper sign continuity for an operator T. The reader can easily convince himself that all the definitions hold for T if and only if they hold for $\mathbb{R}_+T$.

A map $T : X \rightrightarrows X^*$ is said to be

(i) *quasimonotone* if $T(x) \subseteq T^\nu(x)$, for all $x \in X$; equivalently, for every $x, y \in X$, $x^* \in T(x)$, $y^* \in T(y)$,

$$\min\{\langle x^*, y - x\rangle, \langle y^*, x - y\rangle\} \leq 0;$$

(ii) *$s \times w^*$ cone upper semicontinuous (cone usc)* at $x \in \mathrm{edom}(T)$ if for every w^*-open cone K such that $T(x) \subseteq K \cup \{0\}$, there exists a neighborhood U of x such that $T(y) \subseteq K \cup \{0\}$ for all $y \in U$ (see Definition 5 in [6]);

(iii) *upper sign continuous* at x if for every $v \in X$,

$$\begin{aligned} &\exists \delta > 0 : \forall t \in]0, \delta[\,, \exists x^* \in T(x+tv)\backslash\{0\} : \langle x^*, v\rangle \geq 0 \\ &\quad \Rightarrow \exists x^* \in T(x)\backslash\{0\} : \langle x^*, v\rangle \geq 0 \end{aligned} \tag{2.1}$$

In particular, the second definition fits well with operators $T : X \rightrightarrows X^*$ whose values are unbounded convex cones. In this case, if $T(x)$ has a base for every $x \in \mathrm{edom}(T)$, the notion is equivalent to Definition 2.1 in [7]. Moreover, our definition of upper sign continuity is slightly different from Definition 9 in [6].

It is easy to verify that the definition (ii) is stronger than (iii). Indeed, the following result holds:

Proposition 2.2. *If $T : X \rightrightarrows X^*$ is cone upper semicontinuous at $x \in \mathrm{edom}(T)$, then T is upper sign continuous at x.*

Proof. Suppose first that for every $v \in X$, the l.h.s. in (2.1) is never satisfied; in this case, there is nothing to prove. Otherwise, suppose that there exists $v \in X$ such that the l.h.s. holds, but $\langle x^*, v\rangle < 0$ for every $x^* \in T(x) \setminus \{0\}$. The set

$$K_v = \{x^* \in X^* : \langle x^*, v\rangle < 0\}$$

is a w^*-open cone with $T(x) \subseteq K_v \cup \{0\}$. From the cone upper semicontinuity at x, for t small enough, $T(x+tv) \subseteq K_v \cup \{0\}$, a contradiction. □

The cone upper semicontinuity of a conic valued operator, under mild conditions, implies also the closedness of the graph of the operator in the $s \times w^*$ topology as shown in the following result:

Proposition 2.3. *Let $T : X \rightrightarrows X^*$ be such that for all $x \in X$, $T(x)$ is a convex, w^*-closed cone with a w^*-compact base. If* $\mathrm{dom}(T)$ *is closed and T is cone usc, then* $\mathrm{Gr}(T)$ *is closed in the $s \times w^*$ topology.*

Proof. Let (x_α, x^*_α), $\alpha \in \mathcal{A}$ be a net in $\mathrm{Gr}(T)$, converging to (x, x^*) in the $s \times w^*$ topology. Since $\mathrm{dom}(T)$ is closed, $x \in \mathrm{dom}(T)$. We have to show that $x^* \in T(x)$. If $x^* = 0$ this is trivial, so we suppose that $x^* \neq 0$ and $x^* \notin T(x)$. Let $B(x)$ be a w^*-compact base of $T(x)$. Then $B(x) \cap \mathbb{R}_+ x^* = \emptyset$.

By Lemma 3.3 of [14], there exists $b \in X$ such that $\langle x^*, b\rangle > 0 > \langle y^*, b\rangle$ for all $y^* \in B(x)$, so $\langle x^*, b\rangle > 0 > \langle y^*, b\rangle$ for all $y^* \in T(x)\backslash\{0\}$. The set $V := \{y^* \in X^* : \langle y^*, b\rangle < 0\}$ is an open cone and $T(x) \subseteq V \cup \{0\}$. By cone upper semicontinuity, there exists $\alpha_0 \in \mathcal{A}$ such that $T(x_\alpha) \subseteq V \cup \{0\}$ for $\alpha \succcurlyeq \alpha_0$. Thus, $\langle x^*_\alpha, b\rangle \leq 0$ for $\alpha \succcurlyeq \alpha_0$. This contradicts $\langle x^*, b\rangle > 0$ and $x^*_\alpha \xrightarrow{w^*} x^*$. □

Remark 2.4. In the Euclidean setting, a conic-valued map with closed graph is always cone usc. Indeed, one can consider the operator $T'(x) = T(x) \cap S(0,1)$; T' has compact range and closed graph, and therefore it is upper semicontinuous. This is equivalent to say that T is cone usc (see for instance [1], [8]). This is no longer true in infinite dimensional settings, as the following example shows. Let $X = X^* = \ell^2$, $\{x_n\}_n \subset \ell^2$ be a sequence of points different from 0 and strongly convergent to 0, and consider the set-valued map $T : \ell^2 \rightrightarrows \ell^2$ with domain $\{x_n\}_n \cup \{0\}$ and defined as follows:

$$T(0) = \{0\}, \quad T(x_n) = \mathbb{R}_+ e_n,$$

where e_n denotes the sequence $\{e_n^i\}_i$ such that $e_n^i = 1$ if $i = n$, and $e_n^i = 0$, otherwise. This operator is not cone usc at $x = 0$; indeed, taking $V = \emptyset$, $T(0) \subset V \cup \{0\}$, but $T(x_n) \notin V \cup \{0\}$, for any n. Let us show that $\mathrm{Gr}(T)$ is in fact $s \times w^*$ closed. Suppose that $(x_n, x_n^*) \in \mathrm{Gr}(T)$, and $x_n^* \xrightarrow{w^*} x^*$. From the definition of T, $x_n^* = t_n e_n$, for some $t_n \geq 0$. In addition, the sequence $\{t_n e_n\}$ is bounded. This implies that, for every $x \in \ell^2$, $\langle t_n e_n, x\rangle \to \langle x^*, x\rangle$ if and only if $x^* = 0$, thereby showing the closedness of $\mathrm{Gr}(T)$.

In order to define the notion of the operator we are interested in, i.e., the adjusted normal cone operator, we need first to recall some necessary preliminary definitions.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function and $\mathrm{dom}\, f = \{x \in X : f(x) < +\infty\}$ its domain, which is always assumed nonempty.

For every $\lambda \in \mathbb{R}$ define the sublevel set $S_{f,\lambda} = \{x \in X : f(x) \leq \lambda\}$ and the strict sublevel set $S_{f,\lambda}^{<} = \{x \in X : f(x) < \lambda\}$. In particular, in order to simplify the notation, for every $x \in \mathrm{dom}\, f$, we set

$$S_f(x) = S_{f,f(x)}, \qquad S_f^{<}(x) = S_{f,f(x)}^{<}.$$

The function f is said to be *lower semicontinuous (lsc)* if $S_{f,\lambda}$ is a closed set for every $\lambda \in \mathbb{R}$, and *solid* if $\mathrm{int}\, S_{f,\lambda} \neq \emptyset$ for every $\lambda > \inf_X f$.

Moreover, let $\rho_x^f = \mathrm{dist}(x, S_f^{<}(x))$ and for any $x \in \mathrm{dom}\, f$ define the adjusted sublevel set $S_f^a(x)$ by

$$S_f^a(x) = \begin{cases} S_f(x) \cap \overline{B}(S_f^{<}(x), \rho_x^f) & x \in \mathrm{dom}\, f \setminus \mathrm{argmin} f \\ S_f(x) & x \in \mathrm{argmin} f. \end{cases}$$

In particular, $S_f^a(x) = S_f(x)$ for every $x \in \mathrm{dom}\, f$ whenever every minimum of f is global.

In general, $S_f^{<}(x) \subset S_f^a(x) \subseteq S_f(x)$ for any $x \in \mathrm{dom}\, f$.

The function f is said to be *quasiconvex* if for every $x, y \in \mathrm{dom}\, f$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

It is well known that the convexity of the sublevel sets $S_f(x)$, of the strict sublevel sets $S_f^{<}(x)$ as well as of the adjusted sublevel sets $S_f^a(x)$ for every $x \in X$, characterizes the quasiconvexity of the function f (see [7]).

Let us recall that a map $T : X \rightrightarrows X$ is said to be *lower semicontinuous* at x if for every $x_n \xrightarrow{s} x$ with $x \in \mathrm{dom}(T)$, and for every $y \in T(x)$, there exists $y_n \in T(x_n)$ such that $y_n \xrightarrow{s} y$ (see for instance [3], p. 39-40).

The following result, whose proof is very similar to the proof in the finite dimensional case in [1, Th. 3.1], holds:

Theorem 2.5. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be quasiconvex. If $S_f(x)$ is closed for all $x \in \mathrm{dom} f$, then the map $x \rightrightarrows S_f^a(x)$ is lower semicontinuous on* $\mathrm{dom} f$.

For any function f let us define the normal cone operator $N_f : X \rightrightarrows X^*$ and the adjusted normal cone operator $N_f^a : X \rightrightarrows X^*$ as follows: if $x \in \operatorname{dom} f$,

$$N_f(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0,\ \forall y \in S_f(x)\}$$
$$N_f^a(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0,\ \forall y \in S_f^a(x)\};$$

if $x \notin \operatorname{dom} f$, we set $N_f(x) = N_f^a(x) = \emptyset$. Obviously, $N_f(x) \subseteq N_f^a(x)$.

These operators are always quasimonotone, indeed they satisfy a stronger property known as cyclic quasimonotonicity (see [7] and the references therein).

3. A new notion of maximal quasimonotone map

The study of a suitable definition of maximal quasimonotone set-valued map was recently addressed by Aussel and Eberhard [6] and also by Bueno and Cotrina [11]. The new notion of maximal quasimonotonicity we introduce in this section is enjoyed, in particular, by the Clarke subdifferential operator of a locally Lipschitz and quasiconvex function, and by the adjusted normal cone operator of a quasiconvex function.

Definition 3.1. Let $T : X \rightrightarrows X^*$ be a quasimonotone operator with $\operatorname{int}\operatorname{edom}(T) \neq \emptyset$. T is maximal quasimonotone if for every $x^* \in T^\nu(x)$ with $x \in \operatorname{int}\operatorname{edom}(T)$, we have $x^* \in \mathbb{R}_+T(x)$, i.e. $T^\nu(x) = \mathbb{R}_+T(x)$ for every $x \in \operatorname{int}\operatorname{edom}(T)$.

As a consequence of [11, Th. 4.7(4)], our notion of maximal quasimonotone operator is weaker than the notion introduced in [6].

The following trivial example exhibits a maximal quasimonotone map according to Definition 3.1 which is not maximal quasimonotone neither according to [6] or [11].

Example 3.2. Define $T : \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$T(x) = \begin{cases} 0, & \text{if } x < 0 \\ [0, +\infty) & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$$

Then $\operatorname{edom}(T) = [0, +\infty)$. It is straightforward to verify that T is maximal quasimonotone according to Definition 3.1. Indeed, for $x \in (0, +\infty)$, $(x, x^*) \uparrow (y, y^*)$ for every $y^* \in T(y)$ if and only if $x^* \in \mathbb{R}_+T(x)$.

On the other hand, a quasimonotone extension of T on $[0, +\infty)$ can be provided by setting $T(0) = (-\infty, +\infty)$. Thus T is not maximal quasimonotone either according to Definition 1 in [6] or according to the definition in [11] given in terms of inclusion. In addition, note that T is not even pre-maximal quasimonotone as defined in [11] since

$$T^\nu(x) = \begin{cases} (-\infty, +\infty) & \text{if } x \leq 0 \\ [0, +\infty) & \text{if } x > 0 \end{cases}$$

is not quasimonotone.

The following example shows a quasimonotone operator which is not maximal quasimonotone according to Definition 3.1.

Example 3.3. Define $T : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ by

$$T(x,y) = \begin{cases} \mathbb{R}_+ (1,1) & \text{if } x \geq 0, y \geq 0,\ (x,y) \neq (0,0) \\ \mathbb{R}_+ (1,-1) & \text{if } x > 0, y < 0 \\ \mathbb{R}_+ (-1,1) & \text{if } x < 0, y > 0 \\ \mathbb{R}_+ (-1,-1) & \text{if } x \leq 0, y \leq 0 \end{cases}$$

It is straighforward to verify that this operator is quasimonotone with $\operatorname{edom} T = \mathbb{R}^2$ but it is not maximal quasimonotone; indeed, $T^\nu(0,0) = \mathbb{R}^2$, but $T(0,0) = \mathbb{R}_+ (-1,-1)$.

In the next proposition some properties of maximal quasimonotone operators are summarized. Some of them extend to maximal quasimonotone operators results similar to those involving maximal monotone ones (see, for instance, [15], Ch. 3).

Proposition 3.4. *Let $T : X \rightrightarrows X^*$ be a maximal quasimonotone operator. Then,*

i) *$\mathbb{R}_+ T : X \rightrightarrows X^*$ is maximal quasimonotone.*
ii) *$\mathbb{R}_+ T(x)$ is convex for all $x \in \operatorname{int}\operatorname{edom}(T)$.*
iii) *If $x \in \operatorname{int}\operatorname{edom}(T)$, $x_n \xrightarrow{s} x$, $x_n^* \xrightarrow{w^*} x^*$ with $x_n^* \in T(x_n)$, then $x^* \in \mathbb{R}_+ T(x)$. In particular, $\mathbb{R}_+ T(x)$ is sequentially w^*-closed, for every $x \in \operatorname{int}\operatorname{edom}(T)$.*
iv) *If $x \in \operatorname{int}\operatorname{edom}(T)$, $x_n \xrightarrow{w} x$, $x_n^* \xrightarrow{s} x^*$ with $x_n^* \in T(x_n)$, then $x^* \in \mathbb{R}_+ T(x)$.*

Proof. Recall that by definition of maximal quasimonotone operators,

$$T^\nu(x) = \mathbb{R}_+ T(x) \text{ for all } x \in \operatorname{int}\operatorname{dom}(T).$$

i) Trivial, noting that $(\mathbb{R}_+ T)^\nu(x) = T^\nu(x) = \mathbb{R}_+ T(x)$ for all

$$x \in \operatorname{int}\operatorname{edom}(\mathbb{R}_+ T) = \operatorname{int}\operatorname{edom}(T).$$

ii) follows from Corollary 3.8 in [11].

iii) and iv) follows from Proposition 2.1 observing that for quasimonotone operators $x_n^* \in T(x_n) \subseteq T^\nu(x_n)$.

□

Remark 3.5. Note that $\mathbb{R}_+ T(x)$ is not necessarily convex or w^*-closed at the boundary of $\operatorname{edom}(T)$. For example, take $X = \mathbb{R}^2$ and define T by

$$T(x) = \begin{cases} \mathbb{R}_+ \times \{0\} & \text{if } x > 0, y \geq 0 \\ \mathbb{R}_- \times \{0\} & \text{if } x < 0, y \geq 0 \\ \mathbb{R} \times \mathbb{R}_+ & \text{if } x = 0, y > 0 \\ \{(x,y) : -2\,|x| < y < -\,|x|\} & \text{if } x = y = 0 \\ \emptyset & \text{if } y < 0 \end{cases}$$

Then T is maximal quasimonotone according to Definition 3.1, but $\mathbb{R}_+ T(0,0)$ is neither closed, nor convex.

The next two results try to adapt known properties of maximal monotone operators to the case of maximal quasimonotone ones.

It is well known that any maximal monotone operator is upper semicontinuous in the interior of its domain (see Theorem 1.28, Section 3 in [15]). In case of maximal quasimonotone operators a similar result holds in a finite dimensional setting.

Proposition 3.6. *If $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal quasimonotone, then T is cone usc at every $x \in \operatorname{int}\operatorname{edom}(T)$.*

Proof. Without loss of generality we will suppose that $T = \mathbb{R}_+T$. Let $x \in \operatorname{int}\operatorname{edom}(T)$ be a point where T is not cone usc. Then, there exist an open cone K, and sequences $\{x_n\}$ and $\{x_n^*\}$ such that: $T(x) \subset K \cup \{0\}$, $x_n \to x$ and $x_n^* \in T(x_n) \setminus (K \cup \{0\})$. Without loss of generality, suppose that $\|x_n^*\| = 1$ and $x_n^* \to x^*$, with $\|x^*\| = 1$. From Proposition 3.4-iii), $x^* \in T(x)$. On the other hand, $x_n^* \in (K \cup \{0\})^c \subset K^c$, which is a closed set, so $x^* \in K^c$, $x^* \neq 0$, a contradiction. □

The next result provides a sufficient condition for maximal quasimonotonicity, that can be compared with a similar one for maximal monotone operators (see Theorem 1.33, Section 3 in [15]; see also Lemma 9.i-ii. in [6]):

Proposition 3.7. *Let $T : X \rightrightarrows X^*$ be upper sign-continuous, with convex, w^*-compact values. If $\operatorname{int}\operatorname{edom}(T) \neq \emptyset$ and $0 \notin T(x)$ for every $x \in \operatorname{int}\operatorname{edom}(T)$, then $T^\nu(x) \subseteq \mathbb{R}_+T(x)$, for every $x \in \operatorname{int}\operatorname{edom}(T)$. In particular, if T is quasimonotone, then it is maximal quasimonotone.*

Proof. Let us assume that there exists $x \in \operatorname{int}\operatorname{edom}(T)$ and $x_0^* \neq 0$, such that $x_0^* \in T^\nu(x) \setminus \mathbb{R}_+T(x)$. From the assumption $0 \notin T(x)$, and thus $\mathbb{R}_+x_0^* \cap T(x) = \emptyset$. Therefore, we can apply Lemma 3.3. in [14] and find $b \in X$ such that

$$\langle x_0^*, b\rangle > 0 > \langle x^*, b\rangle, \quad \forall x^* \in T(x). \tag{3.1}$$

Set $x_t = x + tb \in \operatorname{int}\operatorname{edom}(T)$ for $t > 0$ sufficiently small. Since $\langle x_0^*, x_t - x\rangle > 0$, from the definition of quasimonotone polar it follows that $\langle x^*, b\rangle \geq 0$ for all $x^* \in T(x_t)$. By upper sign-continuity, there exists $x^* \in T(x) \setminus \{0\}$ such that $\langle x^*, b\rangle \geq 0$, contradicting (3.1).

In case T is quasimonotone, from the inclusion $T^\nu(x) \supseteq \mathbb{R}_+T(x)$ the maximal quasimonotonicity easily follows. □

The example below shows that the assumption $0 \notin T(x)$ cannot be dropped, even in case we strengthen the continuity of T by imposing its cone upper semicontinuity:

Example 3.8. Define $T : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ by

$$T(x,y) = \begin{cases} \mathbb{R} \times (-\infty, 0] & \text{if } x = 0, y = 0 \\ [0, +\infty) \times \{0\} & \text{if } x > 0, y \geq 0 \\ (-\infty, 0] \times \{0\} & \text{if } x < 0, y \geq 0 \\ \mathbb{R} \times \{0\} & \text{if } x = 0, y > 0 \\ \mathbb{R}_+(x,y) & \text{if } x \in \mathbb{R}, y < 0 \end{cases}$$

It is straighforward to verify that $\operatorname{edom}(T) = \mathbb{R}^2$, T is quasimonotone, cone usc with closed, conic and convex values, but it is not maximal quasimonotone. As a matter of fact, $T^\nu(0,0) = \mathbb{R}^2$, while $T(0,0) = \mathbb{R} \times (-\infty, 0]$.

In the last result of this section we apply Proposition 3.7 to show the maximal quasimonotonicity of the Clarke subdifferential.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a locally Lipschitz function and denote by $\partial^o f : X \rightrightarrows X^*$ its Clarke subdifferential. It is well known that $\operatorname{dom}(\partial^o f) = \operatorname{dom} f$, $\partial^o f(x)$

is w^*-compact and convex for all $x \in \operatorname{dom}(\partial^o f)$, and $\partial^o f$ is upper semicontinuous in the $s \times w^*$ topology (see [12], and [18] Prop. 7.3.8). Thus, Proposition 3.7 and Theorem 4.1 in [5] imply

Corollary 3.9. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a locally Lipschitz quasiconvex function. Assume that $0 \notin \partial^o f(x)$ for all $x \in \operatorname{int}\operatorname{edom}(\partial^o f)$. Then, $\partial^o f$ is maximal quasimonotone.*

Note that a function satisfying the assumptions of the corollary above is necessarily pseudoconvex (see [4], Theorem 4.1). This means that $\partial^o f$ is D-maximal pseudomonotone (see [14], Corollary 3.2). However, this does not automatically imply maximal quasimonotonicity, as shown by the next example. The example also shows that the assumption $0 \notin \partial^o f(x)$, $\forall x \in \operatorname{dom} f$, cannot be omitted from Corollary 3.9.

Example 3.10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x_1, x_2) = \frac{1}{2}x_1^2 + |x_2|$. Then f is convex, thus quasiconvex. Its subdifferential $\partial f = \partial^o f$ is given by

$$\partial f\,(x_1, x_2) = \begin{cases} \{(x_1, 1)\} & \text{if } x_2 > 0 \\ \{(x_1, -1)\} & \text{if } x_2 < 0 \\ \{x_1\} \times [-1, 1] & \text{if } x_2 = 0 \end{cases}$$

Note that $\partial^o f$ is usc with compact convex values and $\operatorname{edom}(\partial^o f) = \mathbb{R}^2$. The operator $\partial^o f$ is maximal monotone and D-maximal pseudomonotone. It is not maximal quasimonotone, because $(1, 0) \in (\partial^o f)^\nu\,(0, 0)$, but $(1, 0) \notin \mathbb{R}_+ \partial^o f(0, 0)$.

Finally, note that the function $f(x) = |x|$ does not satisfy the assumptions of Corollary 3.9, but $\partial^o f$ is maximal quasimonotone.

4. Maximal quasimonotonicity and continuity properties of the adjusted normal cone operator

We start by proving the maximal quasimonotonicity of the normal operator N_f^a. To this purpose, it is necessary to describe the interior of the effective domain of this operator.

Let us first introduce some preliminary useful notions. Given a convex set $K \subseteq X$, a point $x_0 \in K$ is called a *support point* of K if there exists $x^* \in X^* \backslash \{0\}$ such that

$$\langle x^*, x_0 \rangle = \sup_{x \in K} \langle x^*, x \rangle\,,$$

or equivalently, if $x_0 \in \operatorname{edom}(N_K)$, where $N_K : K \rightrightarrows X^*$ is defined as follows

$$N_K(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0,\ \forall y \in K\}.$$

The set of support points of K is denoted by $\operatorname{supp}(K)$; this definition is consistent with the one in [2], Ch. 7, but is different from the one in [10], that corresponds in fact to the notion of proper support points given in [2]. The set of nonsupport points (or quasi-interior points, see [13] Prop. 2.2) is the set

$$\operatorname{nsupp}(K) := K \backslash \operatorname{supp}(K).$$

Note that, if K is a nonempty, convex and closed set with nonempty interior, then every boundary point x of K is a support point for K (see Lemma 7.7 in [2]). Therefore, $\operatorname{nsupp}(K) = \operatorname{int} K$. In infinite dimensional spaces we may have nonsupport points even if $\operatorname{int} K$ is empty (see Example 7.8 in [2]).

If $\operatorname{nsupp}(K) \neq \emptyset$, then $\operatorname{nsupp}(K)$ is dense in K. In fact, we have the easy property:

Proposition 4.1. *Let $K \subseteq X$ be convex. If $x_1 \in K$ and $x_2 \in \operatorname{nsupp}(K)$, then*

$$]x_1, x_2] \subseteq \operatorname{nsupp}(K).$$

In particular, $\operatorname{nsupp}(K)$ *is dense in K.*

Proof. Assume that there exists $x_3 = tx_1 + (1-t)x_2$, $t \in]0,1[$, such that

$$x_3 \notin \operatorname{nsupp}(K).$$

Then there exists $x^* \in X^* \backslash \{0\}$ such that

$$\langle x^*, tx_1 + (1-t)x_2 \rangle = \sup_{x \in K} \langle x^*, x \rangle \geq \langle x^*, x_1 \rangle \tag{4.1}$$

$$\langle x^*, tx_1 + (1-t)x_2 \rangle = \sup_{x \in K} \langle x^*, x \rangle > \langle x^*, x_2 \rangle \tag{4.2}$$

The strict inequality in (4.2) is due to the fact that

$$\langle x^*, tx_1 + (1-t)x_2 \rangle = \langle x^*, x_2 \rangle$$

would imply that $x_2 \in \operatorname{supp}(K)$, contrary to our assumption.

Combining (4.1) and (4.2) we get a contradiction. Hence, $x_3 \in \operatorname{nsupp}(K)$. □

Let now $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc, solid and quasiconvex function and set

$$C = \operatorname{argmin} f.$$

Under the assumptions on f, C is closed and convex, and $\operatorname{int} \operatorname{dom} f \neq \emptyset$.

Proposition 4.2. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a quasiconvex, lsc and solid function. Then*

$$\operatorname{int} \operatorname{edom}(N_f^a) = \begin{cases} \operatorname{int} \operatorname{dom} f & \text{if } \operatorname{nsupp}(C) = \emptyset \\ (\operatorname{int} \operatorname{dom} f) \setminus C & \text{if } \operatorname{nsupp}(C) \neq \emptyset \end{cases}$$

Proof. By Proposition 3.4 in [7] we have $\operatorname{dom} f \setminus C \subseteq \operatorname{edom}(N_f^a)$, so

$$(\operatorname{int} \operatorname{dom} f) \setminus C \subseteq \operatorname{edom}(N_f^a).$$

Since $(\operatorname{int} \operatorname{dom} f) \setminus C$ is open, we obtain

$$(\operatorname{int} \operatorname{dom} f) \setminus C \subseteq \operatorname{int} \operatorname{edom}(N_f^a). \tag{4.3}$$

We consider two cases:

(i) Let $\operatorname{nsupp}(C) = \emptyset$. Then $C = \operatorname{supp}(C) = \operatorname{edom}(N_C) \subseteq \operatorname{edom}(N_f^a)$. Combining with $(\operatorname{int} \operatorname{dom} f) \setminus C \subseteq \operatorname{edom}(N_f^a)$ we obtain $\operatorname{int} \operatorname{dom} f \subseteq \operatorname{edom}(N_f^a)$. Hence $\operatorname{int} \operatorname{dom} f \subseteq \operatorname{int} \operatorname{edom}(N_f^a)$. The reverse implication is obvious, since $\operatorname{edom}(N_f^a) \subseteq \operatorname{dom} f$, so $\operatorname{int} \operatorname{edom}(N_f^a) = \operatorname{int} \operatorname{dom} f$.

(ii) Let $\operatorname{nsupp}(C) \neq \emptyset$. Take $x_0 \in \operatorname{int} \operatorname{edom}(N_f^a)$. There exists $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq \operatorname{int} \operatorname{edom}(N_f^a)$. Then $B(x_0, \varepsilon) \subseteq \operatorname{int} \operatorname{dom} f$. If we had $B(x_0, \varepsilon) \cap C \neq \emptyset$, then

we would also have $B(x_0,\varepsilon)\cap \operatorname{nsupp} C\neq\emptyset$, due to Proposition 4.1. But then there would exist a point $y\in B(x_0,\varepsilon)\subseteq \operatorname{edom}(N_f^a)$ such that $y\in \operatorname{nsupp} C$. This is clearly impossible. Hence, $B(x_0,\varepsilon)\subseteq(\operatorname{int}\operatorname{dom} f)\setminus C$, which shows that

$$\operatorname{int}\operatorname{edom}(N_f^a)\subseteq(\operatorname{int}\operatorname{dom} f)\setminus C.$$

The reverse implication was already shown in (4.3). □

An immediate consequence of Proposition 4.2 is the following:

Corollary 4.3. *Let $f:X\to\mathbb{R}\cup\{+\infty\}$ be a quasiconvex, lsc function. Assume that* $\operatorname{int} C\neq\emptyset$. *Then*

$$\operatorname{int}\operatorname{edom}(N_f^a)=(\operatorname{int}\operatorname{dom} f)\setminus C.$$

Proof. If $\operatorname{int} C\neq\emptyset$, then f is solid, and $\operatorname{nsupp}(C)=\operatorname{int} C\neq\emptyset$. Proposition 4.2 yields the result. □

We are now in position to prove maximality of the quasimonotone operator N_f^a. To this aim, it is necessary to provide a description for $(N_f^a)^\nu$.

Theorem 4.4. *Let $f:X\to\mathbb{R}\cup\{+\infty\}$ be a quasiconvex, lsc and solid function. Then*

$$(N_f^a)^\nu(x)=\begin{cases} N_f^a(x), & \text{if } x\in\operatorname{dom} f\backslash C\\ X^*, & \text{if } x\in C\end{cases}$$

Proof. Let $x\in C$. Take any $(y,y^*)\in \operatorname{Gr}(N_f^a)$. If $y\in C$, then $x\in S_f(y)=S_f^a(y)$. If $y\notin C$, then $x\in S_f^<(y)\subseteq S_f^a(y)$. In both cases, $x\in S_f^a(y)$ so $\langle y^*,x-y\rangle\leq 0$. It follows that for every $x^*\in X^*$,

$$\min\{\langle x^*,y-x\rangle,\langle y^*,x-y\rangle\}\leq 0.$$

Thus, $(x,x^*)\uparrow(y,y^*)$ so $(N_f^a)^\nu(x)=X^*$.
Now let $x\in\operatorname{dom} f\backslash C$. Since N_f^a is quasimonotone, we always have $N_f^a(x)\subseteq(N_f^a)^\nu(x)$, so we have to show that

$$(N_f^a)^\nu(x)\subseteq N_f^a(x). \tag{4.4}$$

Suppose by contradiction that there exists $x_0^*\in(N_f^a)^\nu(x)\setminus N_f^a(x)$. It follows that $\langle x_0^*,y'-x\rangle>0$ for some $y'\in S_f^a(x)$.
Since f is solid, $\operatorname{int} S_f^a(x)\neq\emptyset$ and $S_f^a(x)=\overline{\operatorname{int} S_f^a(x)}$. Thus, there exists some $\overline{y}$ such that

$$\langle x_0^*,\overline{y}-x\rangle>0,\quad \overline{y}\in\operatorname{int} S_f^a(x). \tag{4.5}$$

Set $y_t=x+t(\overline{y}-x)$, $t\in(0,1]$. Then (4.5) implies that for all $t\in(0,1]$,

$$\langle x_0^*,y_t-x\rangle>0,\quad y_t\in\operatorname{int} S_f^a(x).$$

Combining with $x_0^*\in(N_f^a)^\nu(x)$ and $\langle y^*,y_t-x\rangle=t\langle y^*,\overline{y}-x\rangle$, we deduce

$$\langle y^*,\overline{y}-x\rangle\geq 0,\quad \forall y^*\in N_f^a(y_t),\ t\in(0,1]. \tag{4.6}$$

By Proposition 3.4 (ii) in [7], for every quasiconvex, lsc and solid function f and $x\in\operatorname{dom} f\backslash C$, we have $N_f^a(x)\backslash\{0\}\neq\emptyset$. Thus, $x\in\operatorname{edom}(N_f^a)$. Take any $x^*\in N_f^a(x)\backslash\{0\}$. Then

$$\overline{y}\in\operatorname{int} S_f^a(x)\subseteq\operatorname{int}\{y\in X:\langle x^*,y-x\rangle\leq 0\}=\{y\in X:\langle x^*,y-x\rangle<0\}.$$

This means that $\langle x^*, \overline{y} - x\rangle < 0$. Hence,

$$N_f^a(x) \subset \{z^* \in X^* : \langle z^*, \overline{y} - x\rangle < 0\} \cup \{0\}. \tag{4.7}$$

Set $K = \{z^* \in X^* : \langle z^*, \overline{y} - x\rangle < 0\}$. This is a w^*-open cone, and $N_f^a(x) \subseteq K \cup \{0\}$. Taking into account the cone upper semicontinuity of the map N_f^a at $x \in \operatorname{dom} f \backslash C$ implied by Proposition 3.5 in [7], we obtain $N_f^a(y_t) \subseteq K \cup \{0\}$ for all $t > 0$ small enough. From (4.6), we get $N_f^a(y_t) = \{0\}$. But for $t > 0$ small enough, we have that $y_t \notin C$ so $y_t \in \operatorname{edom}(N_f^a)$, a contradiction. □

Theorem 4.5. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a quasiconvex, lsc and solid function. In addition, if $\sharp C \geq 2$, we assume that $\operatorname{int} C \neq \emptyset$. If $\operatorname{int}\operatorname{edom}(N_f^a) \neq \emptyset$, then N_f^a is maximal quasimonotone.*

Proof. Let $x \in \operatorname{int}\operatorname{edom}(N_f^a)$. In the special case $\sharp C = 1$ and $C = \{x\}$, we have $N_f^a(x) = X^* = (N_f^a)^\nu(x)$ by Theorem 4.4. According to Corollary 4.3, in all other the cases we have $x \notin C$. Applying again Theorem 4.4 we obtain $N_f^a(x) = (N_f^a)^\nu(x)$, so N_f^a is maximal quasimonotone. □

Remark 4.6. The assumption about the set C in the theorem above cannot be relaxed. Take, for instance, the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = |x_1|$.
The set $C = \{0\} \times \mathbb{R}$ has empty interior, $(N_f^a)^\nu(0,0) = \mathbb{R}^2$ from Theorem 4.4, but $(0,1) \notin \mathbb{R}_+ N_f^a(0,0) = N_f^a(0,0)$.

Note that N_f^a can be maximal quasimonotone also in case the function f is not quasiconvex. Take for instance, $f(x) = xe^{-x}$. Indeed, it is easy to verify that

$$N_f^a(x) = \begin{cases} [0, +\infty) & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases}$$

is maximal quasimonotone, despite f being trivially not quasiconvex.

In this last part we will investigate some continuity properties of the map N_f^a. Let us first state the following result:

Proposition 4.7. *Let $A : X \rightrightarrows X$ be a map which is lsc on its domain. Define $M : \operatorname{dom}(A) \rightrightarrows X^*$ by $M(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0, \forall y \in A(x)\}$. Then the graph of M is $s \times w^*$ sequentially closed on $\operatorname{dom}(A) \times X^*$.*

Proof. Assume that $x_n \xrightarrow{s} x \in \operatorname{dom}(A)$, $x_n^* \in M(x_n)$ and $x_n^* \xrightarrow{w^*} x^*$. Since A is a lsc map, for every $y \in A(x)$ there exists a subnet x_{n_i} of x_n and $y_{n_i} \in A(x_{n_i})$ s.t. $y_{n_i} \xrightarrow{s} y$. Let β be a bound of the sequence $\{x_n^*\}$. Then

$$\begin{aligned} \left|\langle x^*, y - x\rangle - \left\langle x_{n_i}^*, y_{n_i} - x_{n_i}\right\rangle\right| &\leq \left|\left\langle x^* - x_{n_i}^*, y - x\right\rangle\right| \\ &\quad + \left|\left\langle x_{n_i}^*, (y - x) - (y_{n_i} - x_{n_i})\right\rangle\right| \\ &\leq \left|\left\langle x^* - x_{n_i}^*, y - x\right\rangle\right| + \beta \left\|(y - x) - (y_{n_i} - x_{n_i})\right\| \to 0. \end{aligned}$$

We find

$$\langle x^*, y - x\rangle = \lim \left\langle x_{n_i}^*, y_{n_i} - x_{n_i}\right\rangle \leq 0.$$

Hence, $x^* \in M(x)$. □

As an immediate consequence of Theorem 2.5 and Proposition 4.7 we find the following:

Corollary 4.8. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be quasiconvex. If $S_f(x)$ is closed for all $x \in \operatorname{dom} f$, then the graph of the map $x \rightrightarrows N_f^a(x)$ is sequentially closed on $\operatorname{dom} f \times X^*$ in the $s \times w^*$ topology.*

In finite dimensions, the above corollary entails that N_f^a is cone usc (see Corollaries 3.1 and 3.2 in [1]).

In infinite dimensions, by assuming that f is solid, we can show, via the $s \times w^*$ closedness of the graph, the cone upper semicontinuity of the normal cone operator N_f^a in $\operatorname{dom} f$ under a suitable assumption on C. In particular, we recover Proposition 3.5 in [7].

Theorem 4.9. *Let f be quasiconvex, lsc and solid. Then N_f^a is $s \times w^*$ cone upper semicontinuous in $\operatorname{dom} f \setminus C$. If in addition $\#C \leq 1$, or $\#C \geq 2$ and $\operatorname{int} C \neq \emptyset$, then N_f^a is $s \times w^*$ cone upper semicontinuous in $\operatorname{dom} f$.*

Proof. First of all note that if C is a singleton, then N_f^a is $s \times w^*$ cone upper semicontinuous at that point. In the following we will assume that C is not a singleton. Let $x \in \operatorname{dom} f$.

Suppose by contradiction that there exist a w^*-open cone M and a sequence $x_n \in \operatorname{dom} f$, $x_n \xrightarrow{s} x$, such that $N_f^a(x) \subseteq M \cup \{0\}$, but

$$N_f^a(x_n) \nsubseteq M \cup \{0\}. \tag{4.8}$$

Thus, there exists $z_n^* \neq 0$, with $z_n^* \in N_f^a(x_n) \setminus M$. We will show that there exist $n_0 \in \mathbb{N}$, $\varepsilon > 0$ and $y_0 \in X$ such that for all $n \geq n_0$ and $v \in \overline{B}(0,1)$, we have $y_0 + \varepsilon v \in S_f^a(x_n)$. To see this, we consider two cases:

(i) If $x \notin C$, then take λ such that $\inf f < \lambda < f(x)$. Since f is solid, $\operatorname{int} S_{f,\lambda} \neq \emptyset$. By lower semicontinuity of f, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $f(x_n) > \lambda$. Now take $y_0 \in X$ and $\varepsilon > 0$ such that $\overline{B}(y_0, \varepsilon) \subseteq S_{f,\lambda}$. Then for every $v \in \overline{B}(0,1)$ and $n \geq n_0$, we have $y_0 + \varepsilon v \in S_{f,\lambda} \subseteq S_f^<(x_n) \subseteq S_f^a(x_n)$.

(ii) If $x \in C$, then by assumption $\operatorname{int} C \neq 0$; take $y_0 \in \operatorname{int} C$ and $\varepsilon > 0$ such that $\overline{B}(y_0, \varepsilon) \subseteq C$. Then we obtain $y_0 + \varepsilon v \in C \subseteq S_f^a(x_n)$ for all $n \in \mathbb{N}$ and $v \in \overline{B}(0,1)$.

In both cases, $z_n^* \in N_f^a(x_n)$ implies that for $n \geq n_0$,

$$\varepsilon \langle z_n^*, v \rangle \leq \langle z_n^*, x_n - y_0 \rangle \quad \forall v \in \overline{B}(0,1),$$

so

$$\varepsilon \|z_n^*\| \leq \langle z_n^*, x_n - y_0 \rangle.$$

Consequently, taking $n_1 \geq n_0$ such that $\|x_n - x\| \leq \frac{\varepsilon}{2}$ for $n \geq n_1$, we find

$$\varepsilon \|z_n^*\| \leq \langle z_n^*, x_n - x \rangle + \langle z_n^*, x - y_0 \rangle \leq \frac{\varepsilon}{2} \|z_n^*\| + \langle z_n^*, x - y_0 \rangle, \quad n \geq n_1.$$

Thus,

$$0 < \frac{\varepsilon}{2} \|z_n^*\| \leq \langle z_n^*, x - y_0 \rangle, \quad n \geq n_1. \tag{4.9}$$

Since $\langle z_n^*, x - y_0\rangle > 0$, we can choose $t_n > 0$ such that $\langle t_n z_n^*, x - y_0\rangle = 1$. From (4.9) we deduce $\|t_n z_n^*\| \leq \frac{2}{\varepsilon}$, $n \geq n_1$. Thus there exists $z^* \in X^*$ and a subsequence $t_{n_k} z_{n_k}^* \xrightarrow{w^*} z^*$. From the $s \times w^*$ sequential closedness of $\mathrm{Gr}(N_f^a)$, it follows that $z^* \in N_f^a(x) \subseteq M \cup \{0\}$. But from (4.8) we obtain that $t_n z_n^*$ belongs to the w^*-closed set $X^* \backslash M$ for all n, so $z^* \notin M$. It follows that $z^* = 0$. Therefore $\langle t_n z_n^*, x - y_0\rangle \to 0$, a contradiction. □

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On some qualitative properties of Ćirić's fixed point theorem

Mădălina Moga

Dedicated to the memory of Professor Gábor Kassay.

Abstract. It is well known that of all the extensions of the Banach-Caccioppoli Contraction Principle, the most general result was established by Ćirić in 1974. In this paper, we will present some results related to Ćirić type operator in complete metric spaces. Existence and uniqueness are re-called and several stability properties (data dependence and Ostrowski stability property) are proved. Using the retraction-displacement condition, we will establish the well-posedness and the Ulam-Hyers stability property of the fixed point equation $x = f(x)$.

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1. Introduction and preliminaries

Let (X, d) be a metric space and $f : X \to X$ be an operator. For $A \subset X$, let $\delta(A) := sup\{d(a, b) : a, b \in A\}$ the diameter of the set A. For each $x \in X$, we denote:

$$O(x, n) = \{x, f(x), ..., f^n(x)\}, n = 1, 2, ...,$$

$$O(x, \infty) = \{x, f(x), ..., f^n(x), ...\}.$$

Definition 1.1. (Ćirić [2]) Let (X, d) be a metric space and $f : X \to X$ be an operator. Then X is said to be f-orbitally complete if every Cauchy sequence which is contained in $O(x, \infty)$, for some $x \in X$ converges in X.

The following classes of operators in a metric space (X, d) are important for our approach.

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Definition 1.2. Let (X, d) be a metric space and $f : X \to X$ be an operator. Then f is said to be an α-contraction if there exists $\alpha \in [0, 1)$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y), \text{ for all } x, y \in X. \tag{1.1}$$

Definition 1.3. (Rus [6]) Let (X, d) be a metric space and $f : X \to X$ be an operator. Then f is said to be a graphic α-contraction if there exists $\alpha \in [0, 1)$ such that

$$d(f^2(x), f(x)) \leq \alpha d(x, f(x)), \text{ for all } x \in X. \tag{1.2}$$

Through this paper we denote $\mathbb{N} := \{0, 1, 2, \cdots\}$ the set of all natural numbers and by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

We recall that, $Fix(f) = \{x \in X | x = f(x)\}$ is the fixed point set of f and we denote by $(f^n(x))_{n\in\mathbb{N}}$ the sequence of Picard iterates for f starting from $x_0 \in X$, where $f^n = f \circ f \circ \cdots \circ f$ for n-times. Notice that the sequence of Picard iterates for f starting from $x_0 \in X$ can be recursively defined by the formula $x_{n+1} = f(x_n)$, for $n \in \mathbb{N}$, where $x_n := f^n(x_0), n \in \mathbb{N}$.

Definition 1.4. (Ćirić [2]) An operator $f : X \to X$ is said to be a generalized contraction if and only if for every $x, y \in X$ there exists nonnegative numbers q, r, s and t, which may depend on both x and y, such that $sup\{q + r + s + 2t : x, y \in X\} < 1$ and

$$\begin{aligned} d(f(x), f(y)) \leq q \cdot d(x, y) + r \cdot d(x, f(x)) + \\ + s \cdot d(y, f(y)) + t \cdot [d(x, f(y)) + d(y, f(x))]. \end{aligned}$$

Definition 1.5. (Ćirić [2]) Let (X, d) be a metric space and $f : X \to X$ be an operator. Then X is said to be a Ćirić type operator (named a quasi-contraction in the original paper [2]) if there exists a number $q \in (0, 1)$, such that

$$d(f(x), f(y)) \leq q \cdot max\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}, \tag{1.3}$$

for all $x, y \in X$.

It is well known (see [4]) that of all the extensions of Banach-Caccioppoli Contraction Principle, the most general result was established by Ćirić in 1974 for the above class of operators.

In the following example we present a Ćirić type operator which is not a generalized contraction.

Example 1.6. Let

$$X_1 = \left\{\frac{m}{n} : m = 0, 1, 2, 4, 6, ...; n = 1, 3, 7, ..., 2k + 1, ...\right\},$$

$$X_2 = \left\{\frac{n}{n} : m = 1, 2, 4, 6, 8, ...; n = 2, 5, 8, ..., 3k + 2, ...\right\},$$

where $k \in \mathbb{N}$ and let $X = X_1 \cup X_2$. Let us define $f : X \to X$ by

$$f(x) = \begin{cases} \frac{2}{3}x & , x \in X_1, \\ \\ \frac{1}{5}x & , x \in X_2. \end{cases}$$

The mapping f is a Ćirić type operator with $q = \frac{2}{3}$. If both x and y are in X_1 or in X_2, then

$$d(f(x), f(y)) \leq \frac{2}{3} d(x, y)$$

If we take $x \in X_1$ and $y \in X_2$, then we have that

$$x \geq \frac{3}{10} y \text{ implies } d(f(x), f(y)) = \frac{2}{3}\left(x - \frac{3}{10} y\right) \leq \frac{2}{3}\left(x - \frac{1}{5} y\right) = \frac{2}{3} d(x, f(y))$$

$$x < \frac{3}{10} y \text{ implies } d(f(x), f(y)) = \frac{2}{3}\left(\frac{3}{10} y - x\right) \leq \frac{2}{3}(y - x) = \frac{2}{3} d(x, y)$$

Thus, we have that f satisfies the following condition:

$$d(f(x), f(y)) \leq \frac{2}{3} max\left\{d(x, y), d(x, f(y)), d(y, f(x))\right\}$$

and, hence, it is Ćirić type operator.

In the following step we show that f is not a generalized contraction on X. Let $x = 1$ and $y = \frac{1}{2}$. Then we have that

$$\begin{aligned} q \cdot d(x, y) + r \cdot d(x, f(x)) + s \cdot d(y, f(y)) + t \cdot [d(x, f(y)) + d(y, f(x))] \\ = \frac{1}{2} q + \frac{1}{3} r + \frac{4}{10} s + \frac{32}{30} t < (q + r + s + 2t) \frac{32}{60} \\ < \frac{32}{60} < \frac{17}{30} = d(f(x), f(y)), \end{aligned}$$

as $q + r + s + 2t < 1$, we can see that f is not a generalized contraction.

In this paper, we will present some results related to Ćirić type operator in complete metric spaces. Existence and uniqueness are re-called and several stability properties (data dependence and Ostrowski stability property) are proved. Using the retraction-displacement condition, we will establish the well-posedness and the Ulam-Hyers stability property of the fixed point equation $x = f(x)$.

Our results generalize and complement some theorems given in [1], [2], [3], [5], [6], [7], [8].

2. Main results

In this section we will consider a metric space (X, d) and $f : X \to X$ a Ćirić type operator. Besides the usual properties which are proved by Ćirić in [2], we will prove some other stability properties. More precisely, we will establish the continuous data dependence property of the fixed point and the Ostrowski stability property for the operator f. Moreover, using the retraction-displacement condition and we also prove that the fixed point equation $x = f(x)$ is well-posed and Ulam-Hyers stable.

Theorem 2.1. *(Ćirić [2]) Let (X, d) be a metric space and $f : X \to X$ be a Ćirić type operator. Suppose that X is f-orbitally complete. Then:*

1. *f has a unique fixed point x^* in X and $\lim_{n\to\infty} f^n(x) = x^*$, i.e., f is a Picard operator;*

2. $d(f^n(x), x^*) \leq \frac{q^n}{1-q} d(x, f(x))$, *for every* $x \in X$ *and every* $n \in \mathbb{N}^*$;

The idea of the proof is based on the following two relations:

(i) if $n \in \mathbb{N}^*$, then for each $x \in X$ we have that $d(f^i(x), f^j(x)) \leq q\delta(O(x,n))$, for every $i, j \in \mathbb{N}^*$;

(ii) for each $x \in X$ we have that $\delta(O(x,\infty)) \leq \frac{1}{1-q} d(x, f(x))$.

A second result in [2] shows that if there exists $p \in \mathbb{N}$ with $p \geq 2$ such that f^p is a Ćirić type operator, then f is a Picard operator.

Remark 2.2. If $f : X \to X$ satisfies all the assumptions in Theorem 2.1, then we have the following additional conclusion:

3. f satisfies the retraction-displacement condition

$$d(x, x^*) \leq \frac{1}{1-q} d(x, f(x)), \text{ for all } x \in X. \tag{2.1}$$

Remark 2.3. The conclusion 3. follows by 2. in the following way. Take $n = 1$ in 2. Then, we have

$$d(f(x), x^*) \leq \frac{q}{1-q} d(x, f(x)), \text{ for all } x \in X.$$

Hence

$$d(x, x^*) \leq d(x, f(x)) + d(f(x), x^*) \leq \frac{1}{1-q} d(x, f(x)), \text{ for all } x \in X.$$

Lemma 2.4 (Cauchy-Toeplitz Lemma). *Let* $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ *be two sequences of positive numbers such that* $\sum_{n\geq 0} a_n < \infty$ *and* $\lim_{n\to\infty} b_n = 0$. *Then*

$$\lim_{n\to\infty} \left(\sum_{k=0}^{n} a_{n-k} b_k \right) = 0$$

The following notion is essential in our approach.

Definition 2.5. (Rus [8]) Let (X, d) be a metric space and $f : X \to X$ be an operator such that $Fix(f) \neq \emptyset$. We say that f satisfies the retraction-displacement condition if there exists $c > 0$ and a set retraction $\rho : X \to Fix(f)$ such that

$$d(x, \rho(x)) \leq c d(x, f(x)), \text{ for all } x \in X. \tag{2.2}$$

If $Fix(f) = \{x^*\}$ then we have

$$d(x, x^*) \leq c d(x, f(x)), \text{ for all } x \in X.$$

For example, if $f : X \to X$ is an α-contraction and (X, d) is a complete metric space then f satisfies the following retraction-displacement condition

$$d(x, x^*) \leq \frac{1}{1-\alpha} d(x, f(x)), \text{ for all } x \in X.$$

On the same lines, if $f : X \to X$ is a graphic α-contraction then it satisfies the retraction-displacement condition

$$d(x, \rho(x)) \leq \frac{1}{1-\alpha} d(x, f(x)), \text{ for all } x \in X,$$

where $\rho : X \to Fix(f)$ is defined by

$$\rho(x) = \lim_{n\to\infty} f^n(x).$$

The following theorem is the main result of the paper.

Theorem 2.6. *Let (X, d) be a metric space, $f : X \to X$ be a Ćirić type operator and suppose that X is f-orbitally complete. Denote by $x^* \in X$ the unique fixed point of f. Then the following conclusions hold:*

1. *the fixed point $x = f(x)$ equation has the data dependence property, i.e., for any operator $g : X \to X$ such that $Fix(g) \neq \emptyset$ and $d(f(x), g(x)) \leq \eta$, for all $x \in X$ and some $\eta > 0$, we have*
$$d(x^*, u^*) \leq \frac{1+q}{1-q}\eta,$$
for all $u^* \in Fix(g)$.
2. *the fixed point equation is well-posed, i.e., for every sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that*
$$d(u_n, f(u_n)) \to 0,$$
as $n \to \infty$, we have that $u_n \to x^*$, as $n \to \infty$;
3. *the fixed point equation is Ulam-Hyers stable, i.e., there exists $c > 0$ such that for any $\varepsilon > 0$ and any $u^* \in X$ an ε-solution of the fixed point equation (in the sense that $d(u^*, f(u^*)) \leq \varepsilon$), we have*
$$d(u^*, x^*) \leq c \cdot \varepsilon.$$
4. *if $q < \frac{1}{2}$, then the fixed point equation has the Ostrowski stability property, i.e., for any sequence $(u_n)_{n\in\mathbb{N}} \subset X$ with $d(u_{n+1}, f(u_n)) \to 0$ as $n \to \infty$, we have that $u_n \to x^*$;*
5. *if $q < \frac{1}{2}$, then f is a graphic $\frac{q}{1-q}$-contraction;*
6. *if $q < \frac{1}{3}$, then the operator f is a quasi-contraction, in the sense that there exists $\beta := \frac{q}{1-2q} < 1$ such that*
$$d(f(x), x^*) \leq \beta d(x, x^*), \text{ for every } x \in X.$$

Proof.

1. To prove data dependence we will take $u^* \in Fix(g)$ such that $d(f(x), g(x)) \leq \eta$. Then, we will prove that $d(x^*, u^*) \leq \dfrac{1+q}{1-q}\eta$.
$$\begin{aligned}
d(x^*, u^*) = d(f(x^*), g(u^*)) &\leq d(f(x^*), f(u^*)) + d(f(u^*), g(u^*)) \\
&\leq q \cdot max\,\{d(x^*, u^*), d(x^*, f(x^*)), d(u^*, f(u^*)), d(x^*, f(u^*)), \\
&\quad d(u^*, f(x^*))\} + d(f(u^*), g(u^*)) \\
&\leq q \cdot max\,\{d(x^*, u^*), d(u^*, g(u^*)) + d(g(u^*), f(u^*)), \\
&\quad d(x^*, g(u^*)) + d(g(u^*), f(u^*)), d(x^*, u^*)\} + \eta \\
&\leq q \cdot max\,\{d(x^*, u^*), \eta, d(x^*, u^*) + \eta, d(x^*, u^*)\} + \eta \\
&\leq q(d(x^*, u^*) + \eta) + \eta
\end{aligned}$$

Hence, we get that

$$d(x^*, u^*) \leq \frac{1+q}{1-q}\eta$$

2. We will prove that the fixed point equation is well-posed. Let us estimate the distance between u_n and x^*, where $(u_n)_n \in \mathbb{N}$ is a sequence in X such that $d(u_n, f(u_n)) \to 0$ as $n \to \infty$.
   In order to prove this, we will use the retraction-displacement condition (2.1). We have:

$$d(u_n, x^*) \leq \frac{1}{1-q} d(u_n, f(u_n)) \to 0 \text{ as } n \to \infty$$

3. Let $\varepsilon > 0$ and $u^* \in X$ be an ε-solution of the fixed point equation $x = f(x)$, i.e., $d(u^*, f(u^*)) \leq \varepsilon$. Using the retraction-displacement condition (2.1) we will estimate the distance between x^* and u^*:

$$d(x^*, u^*) = d(u^*, x^*) \leq \frac{1}{1-q} d(u^*, f(u^*)) \leq \frac{1}{1-q}\varepsilon$$

   There exists $c > 0$ such that $c := \frac{1}{1-q}$. Then it follows that

$$d(x^*, u^*) \leq c \cdot \varepsilon$$

   which proves that the fixed point equation $x = f(x)$ is Ulam-Hyers stable.
4. We will show that the operator $f : X \to X$ has the Ostrowski property. We observe that:

$$d(u_{n+1}, x^*) \leq d(u_{n+1}, f(u_n)) + d(f(u_n), x^*) \tag{2.3}$$

   We take separately $d(f(u_n), x^*)$ from the above inequality and we have:

$$\begin{aligned} d(f(u_n), x^*) &= d(f(u_n), f(x^*)) \\ &\leq q \cdot max\,\{d(u_n, x^*), d(u_n, f(u_n)), d(x^*, f(x^*), d(u_n, f(x^*)), \\ &\quad d(x^*, f(u_n))\} \\ &\leq q \cdot max\,\{d(u_n, x^*), d(u_n, x^*) + d(x^*, f(u_n)), d(u_n, x^*), \\ &\quad d(x^*, f(u_n))\} \\ &\leq q(d(u_n, x^*) + d(x^*, f(u_n))). \end{aligned}$$

   Thus

$$d(f(u_n), x^*) \leq \frac{q}{1-q} d(u_n, x^*) \tag{2.4}$$

   We replace in (2.3) the relation obtained in inequality (2.4):

$$d(u_{n+1}, x^*) \leq d(u_{n+1}, f(u_n)) + \frac{q}{1-q} d(u_n, x^*) \tag{2.5}$$

We denote: $\alpha := \dfrac{q}{1-q} < 1$.

We will use Cauchy-Toeplitz Lemma and we obtain:

$$\begin{aligned} d(u_{n+1}, x^*) &\leq d(u_{n+1}, f(y_n)) + \alpha d(u_n, x^*) \\ &\leq d(u_{n+1}, f(u_n)) + \alpha[d(u_n, f(u_{n-1})) + \alpha d(u_{n-1}, x^*)] \\ &\leq d(u_{n+1}, f(u_n)) + \alpha d(u_n, f(u_{n-1})) + \alpha^2 d(u_{n-1}, x^*) \leq ... \leq \\ &\leq d(u_{n+1}, f(u_n)) + \alpha d(u_n, f(u_{n-1})) + \alpha^2 d(u_{n-1}, f(u_{n-2})) \\ &\quad + ... + \alpha^n d(u_1, f(u_0)) + \alpha^{n+1} d(u_0, x^*) \to 0 \text{ as } n \to \infty. \end{aligned}$$

5. If we put $y := f(x)$ in the Ćirić type operator condition, we get

$$\begin{aligned} d(f(x), f^2(x)) &\leq q \max\{d(x, f(x)), d(f(x), f^2(x)), d(x, f^2(x))\} \\ &\leq q\left(d(x, f(x)) + d(f(x), f^2(x))\right). \end{aligned}$$

   Thus, we get that $d(f(x), f^2(x)) \leq \frac{q}{1-q} d(x, f(x))$, for every $x \in X$.

6. We will show now that f is a quasi-contraction, in the sense that

$$d(f(x), x^*) \leq \beta d(x, x^*), \text{ for every } x \in X,$$

   where $\beta := \frac{q}{1-2q} < 1$. Indeed, by the second conclusion of Theorem 2.1 for $n = 1$, we have $d(f(x), x^*) \leq \frac{q}{1-q} d(x, f(x))$, for every $x \in X$. Then, we can write successively:

$$d(f(x), x^*) \leq \frac{q}{1-q} d(x, f(x)) \leq \frac{q}{1-q}\left(d(x, x^*) + d(f(x), x^*)\right).$$

   As a consequence,

$$d(f(x), x^*) \leq \frac{q}{1-2q} d(x, x^*), \text{ for each } x \in X. \qquad \square$$

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Relative and mutual monotonicity

Cornel Pintea

Dedicated to the memory of Professor Gábor Kassay.

Abstract. In this work we first consider a certain monotonicity relative to some given one-to-one operator and prove the counterparts, adjusted to this new context, of most results obtained before in the joint work with G. Kassay [10]. For two operators with the same status relative to injectivity, such as two local injective operators, we define what we call *mutual h-monotonicity* and prove that every two mutual h-monotone local diffeomorphisms can be obtained from each other via a composition with a h-monotone diffeomorphism.

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1. Introduction

The importance of the Minty-Browder monotonicity stands in its applications to the theory of partial differential equations (see for example [2, 3, 4],[13, 14]) and in its connection with convex analysis, due to the characterization ofconvexity, within the class of semicontinuous functions, through the Minty-Browder monotonicity of the subdifferential operator (see for instance [6]). In the previous joint work with G. Kassay ([10]) we extended the class of Minty-Browder monotone operators to the class of h-monotone operators. While the inverse images of maximal Minty-Browder monotone operators are well-known to be convex sets [16, p. 105], we only proved in [10] that the inverse images of such operators, with finite dimensional source space, are indivisible by closed connected hypersurfaces. In a joint work with G. Kassay and F. Szenkowitz [11] we provided an elementary proof for the convexity of inverse images of Minty-Browder monotone operators. The lack of divisibility of the inverse images of the h-monotone operators through closed connected hypersurfaces allowed us to establish some global injectivity results.

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In this work we first consider a certain monotonicity concept relative to some given one-to-one operator. Note that the Minty-Browder monotonicity as well as the h-monotonicity are particular notions of this relative monotonicity. Indeed, the role of the given operator in the definitions of Minty-Browder monotonicity and in that of h-monotone operators is played by the identity operator. We also prove the counterparts, adjusted to this new context, of most results obtained before in [10]. For two operators with the same status relative to injectivity, such as two local injective operators, we define what we call mutual h-monotonicity and prove that every two mutual h-monotone local diffeomorphisms can be obtained from each other via a composition with a h-monotone diffeomorphism. As a consequence we observe that two mutual h-monotone local diffeomorphisms have the same valence and provide some examples of h-monotone operators relative to the gradient operator of some strictly convex functions.

2. h-monotonicity relative to an injective operator

In this section we first emphasize some geometrical properties of the Minty-Browder monotone operators which suggest an interesting enlargement of this class.

Let $S^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere and $d_{S^n} : S^n \times S^n \longrightarrow \mathbb{R}_+$ be the metric associated to the Riemann structure of S^n, i.e., $d_{S^n}(x, y) = \arccos\langle x, y\rangle$, $x, y \in S^n$ is the measure of the angle between the vectors x and y. Note that $0 \leq d_{S^n} \leq \pi$. Denote by pr_{S^n} the radial projection

$$\mathbb{R}^{n+1} \setminus \{0\} \longrightarrow S^n, z \longmapsto \frac{z}{||z||}.$$

The next important geometric characterizations of Minty-Browder monotonicity allow us to enlarge this class.

Let D be a subset of $\mathbb{R}^{n+1}$. The following statements hold:

1. $T : D \longrightarrow \mathbb{R}^{n+1}$ is a Minty-Browder increasing operator if and only if
$$d_{S^n}(pr_{S^n}(x - y), pr_{S^n}(Tx - Ty)) \leq \frac{\pi}{2}$$
for all $x, y \in D$, $Tx \neq Ty$.
2. $T : D \longrightarrow \mathbb{R}^{n+1}$is a Minty-Browder decreasing operator if and only if
$$d_{S^n}(pr_{S^n}(x - y), pr_{S^n}(Tx - Ty)) \geq \frac{\pi}{2}$$
for all $(x, y) \in (D \times D) \setminus \ker T$.

Indeed, the stated facts follow from the following obvious relation

$$\frac{\langle x - y, Tx - Ty\rangle}{||x - y|| \cdot ||Tx - Ty||} = \cos[d_{S^n}(pr_{S^n}(x - y), pr_{S^n}(Tx - Ty))].$$

Taking into account the characterizations above, a natural extension of monotonicity occurs. Recall that $0 \leq d_{S^n} \leq \pi$ for any $x, y \in S^n$.

Definition 2.1. Let $T : D \longrightarrow \mathbb{R}^{n+1}$ be a given operator, where D is a subset of $\mathbb{R}^{n+1}$.

1. T is said to be *h-increasing* if $d_{S^n}(pr_{S^n}(x-y), pr_{S^n}(Tx-Ty)) < \pi$ for all $(x,y) \in (D \times D) \setminus \ker T$.
2. T is said to be *h-decreasing* if $d_{S^n}(pr_{S^n}(x-y), pr_{S^n}(Tx-Ty)) > 0$ for all $(x,y) \in (D \times D) \setminus \ker T$.
3. T is said to be *h-monotone* if T is either h-increasing or T is h-decreasing.

Remark 2.2. Let $T : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ be a linear isometry.

1. T is h-increasing if and only if $-1 \notin \mathrm{Spec}(A)$.
2. T is h-decreasing if and only if $1 \notin \mathrm{Spec}(A)$.

Indeed, if T is not h-increasing, then $d_{S^n}(pr_{S^n}(x-y), pr_{S^n}(Tx-Ty)) = \pi$ for some $(x,y) \in (D \times D) \setminus \ker T$, i.e.

$$\frac{Tx-Ty}{\|Tx-Ty\|} = -\frac{x-y}{\|x-y\|} \Leftrightarrow T(x-y) = -(x-y) \Rightarrow -1 \in \mathrm{Spec}(T).$$

Conversely, if $-1 \in \mathrm{Spec}(T)$, then $Tx = -x$ for some $x \neq 0$, i.e.

$$\frac{Tx}{\|x\|} = -\frac{x}{\|x\|} \Leftrightarrow \frac{Tx-To}{\|Tx-T0\|} = -\frac{x-0}{\|x-0\|} \Leftrightarrow d_{S^n}(pr_{S^n}(x-0), pr_{S^n}(Tx-T0)) = \pi$$

which shows that T is not h-increasing. The statement (1) can be similarly proved.

Remark 2.3. The vector-valued function $T : D \longrightarrow \mathbb{R}^{n+1}$ is h-monotone but not Minty-Browder monotone whenever $-1 < i_T < 0$, where i_T stands for

$$\inf\left\{\frac{\langle Tx-Ty, x-y\rangle}{\|Tx-Ty\| \cdot \|x-y\|} \mid (x,y) \in D \times D \setminus \ker T\right\}.$$

Several estimates of some parameters of monotonicity of type i_T are provided in [12].

Definition 2.4. Let $T, A : D \longrightarrow \mathbb{R}^{n+1}$ be given operators with A injective, where D is a subset of $\mathbb{R}^{n+1}$.

1. T is said to be *h-increasing relative to A* or simply *A-increasing* if
$$d_{S^n}(pr_{S^n}(Ax-Ay), pr_{S^n}(Tx-Ty)) < \pi, \forall (x,y) \in (D \times D) \setminus \ker T.$$
2. T is said to be *h-decreasing relative to A* or simply *A-decreasing* if
$$d_{S^n}(pr_{S^n}(Ax-Ay), pr_{S^n}(Tx-Ty)) > 0, \forall (x,y) \in (D \times D) \setminus \ker T.$$
3. T is said to be *h-monotone relative to A* or simply *A-monotone* if T is either A-increasing or T is A-decreasing.

Remark 2.5. Analyzing Definition 2.1, the next (geometric) interpretations become obvious.

1. T is A-increasing if and only if
$$\frac{Tx-Ty}{||Tx-Ty||} \neq \frac{Ay-Ax}{||Ay-Ax||},$$
for all $(x,y) \in (D \times D) \setminus \ker T$.

2. T is A-decreasing if and only if
$$\frac{Tx - Ty}{||Tx - Ty||} \neq \frac{Ax - Ay}{||Ay - Ax||},$$
for all $(x, y) \in (D \times D) \setminus \ker T$. In other words, for A-increasing operators, the action represented by Figure 1(a) is not allowed, while for A-decreasing operators, the action represented by Figure 1(b) is not allowed.

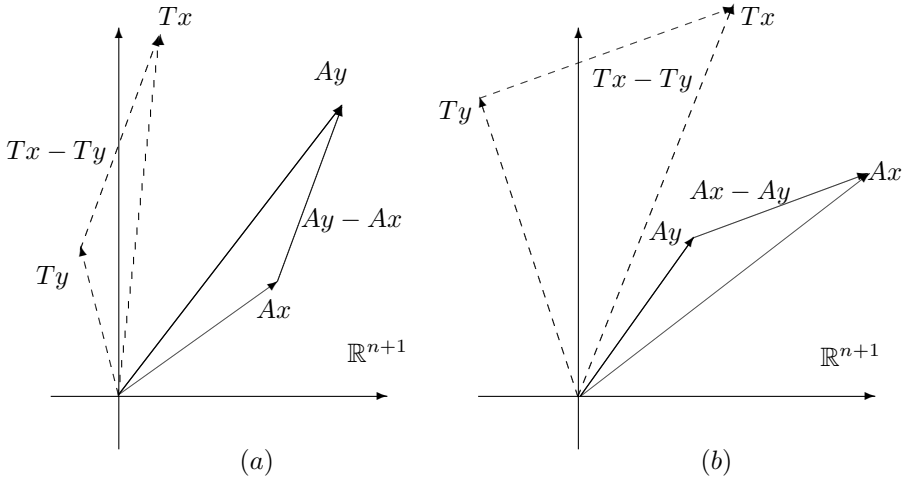

FIGURE 1. Actions not allowed for A-increasing/decreasing operators

3. T is A-increasing if and only if
$$\langle Tx - Ty, Ax - Ay\rangle > -\|Tx - Ty\| \cdot \|Ax - Ay\|, \forall (x, y) \in (D \times D) \setminus \ker T.$$

4. T is A-decreasing if and only if
$$\langle Tx - Ty, Ax - Ay\rangle < \|Tx - Ty\| \cdot \|Ax - Ay\|, \forall (x, y) \in (D \times D) \setminus \ker T.$$

5. T is A-increasing and A-decreasing if and only if
$$|\langle Tx - Ty, Ax - Ay\rangle| < \|Tx - Ty\| \cdot \|Ax - Ay\|, \forall x, y \in (D \times D) \setminus \ker(T).$$
6. If T is h-increasing/decreasing, then $T \circ A$ is A-increasing/decreasing.
7. Let $A : D \longrightarrow \mathbb{R}^{n+1}$ be an injective local homeomorphism/diffeomorphism, i.e. the range of A is open as well as the restriction and the corestriction
$$D \longrightarrow \mathrm{Im}(A),\ x \mapsto Ax$$
is a homeomorphism/diffeomorphism still denoted by A. Then T is A-increasing/decreasing if and only if the composition $T \circ A^{-1} : A(D) \longrightarrow \mathbb{R}^{n+1}$ is h-increasing/decreasing.

8. The h-increasing/decreasing monotonicity coincides with the i_D-increasing/decreasing monotonicity, where $i_D : D \hookrightarrow \mathbb{R}^{n+1}$ stands for the inclusion operator.
9. If $D \subseteq \mathbb{R}^{n+1}$ is a convex open set and $f : D \longrightarrow \mathbb{R}$ is a strictly convex function whose convexity is ensured by the everywhere positive definiteness of its Hessian matrix, then its gradient is an injective local diffeomorphism.

Remark 2.6. In the Definition 2.4 and in Remark 2.5, the role of the injective operator A can be taken over by a (possibly non-injective) local diffeomorphism which we could still denote by A. Thus, we obtain the definition and equivalent forms of monotonicity with respect to (possibly non-injective) local diffeomorphisms.

The increasing A-monotonicity allows the angles between the vectors $Tx-Ty$ and $Ax-Ay$ to exceed $\pi/2$ and approach π arbitrarily close for $(x,y) \in (D \times D) \setminus \ker(T)$, although the upper bound π is never reached by these angles in the case of increasing A-monotone operators. The classes of (A,η)-increasing and (A,η)-decreasing operators $\eta \in (-1,1)$ can still be defined by means of these angles which are not allowed to exceed the upper bound $\arccos\eta$, i.e.

$$\langle Tx - Ty, Ax - Ay\rangle \geq \eta\|Tx - Ty\| \cdot \|Ax - Ay\|, \ \forall x, y \in D$$

for the increasing option and they are not allowed to decrease under the lower bound $\arccos\eta$ for the decreasing option, i.e.

$$\langle Tx - Ty, Ax - Ay\rangle \leq \eta\|Tx - Ty\| \cdot \|Ax - Ay\|, \ \forall x, y \in D.$$

For $\eta = 0$ we call the first type of operators A-Minty-Browder increasing (or shortly $A-M-B$-increasing operators) and the second type A-Minty-Browder decreasing operators (or shortly $A-M-B$-decreasing operators). These angles are therefore allowed to exceed $\pi/2$ when $\eta \in (-1,0)$, for the increasing option, but not to approach π arbitrarily close. This ensure, for the class of (A,η)-increasing operators when $\eta \in (-1,0)$, the status of intermediate class between the class of A-Minty-Browder increasing operators and the class of h-increasing operators. If, on the contrary $\eta \in [0,1)$, then the class of η-monotone operators is contained in the class of A-Minty-Browder operators. A similar discussion can be done for decreasing operators. The η-increasing/decreasing monotonicity corresponds to the (i_D,η)-increasing/decreasing monotonicity, where $i_D : D \hookrightarrow \mathbb{R}^{n+1}$ stands for the inclusion. (see [12]).

Remark 2.7. Another direction in which the A-monotonicity can be extended, due to the Remarks 2.5[(1)-(4)], is for operators $T : D \longrightarrow H$, where $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space, $D \subseteq H$ is an open set and $A : D \longrightarrow H$ is injective. This is also the case for (A,η)-monotonicity.

Remark 2.8. Let $T : D \longrightarrow H$ be a given operator. If $A : H \longrightarrow H$ is a linear isomorphism, then the A-Minty-Browder increasing/decreasing monotonicity of T is equivalent with the Minty-Browder increasing/decreasing monotonicity of $A^* \circ T$.

Remark 2.9. 1. Let $A : H \longrightarrow H$ be a linear unitary automorphism, i.e. A is an isometric linear automorphism. Then T is $\left(A\big|_D, \eta\right)$-increasing/decreasing if and only if $A^* \circ T$ is η-increasing/decreasing.

2. If $A : H \longrightarrow H$ is a linear unitary automorphism, then T is $A\big|_D$-increasing/decreasing if and only if $A^* \circ T$ is h-increasing/decreasing.
3. If $A : D \longrightarrow H$ be an injective operator, then T is $\left(A\big|_D, \eta\right)$-increasing/decreasing if and only if $T \circ A^{-1}$ is η-increasing/decreasing.

Proposition 2.10. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $D \subseteq H$ be an open set. Let also $T : D \longrightarrow H$ be a given operator and let $A : H \longrightarrow H$ be a bounded linear isomorphism such that $b_{A^*} := \inf\{\|A^* z\| \ : \ \|z\| = 1\} > 0$. If $-b_{A^*} < \eta a_A \leq 0$ and T is (A, η)-increasing, then $A^* \circ T$ is $(\eta a_A)/b_{A^*}$-increasing, where a_A stands for $\|A\|$. If $0 \leq \eta a_A < b_{A^*}$ and T is (A, η)-decreasing, then $A^* \circ T$ is $(\eta a_A)/b_{A^*}$-decreasing.*

Proof. Assume that T is (A, η)-increasing for $\eta \in (-1, 0)$, i.e. we have

$$\langle Tx - Ty, Ax - Ay \rangle \geq \eta \|Tx - Ty\| \cdot \|Ax - Ay\| \Longleftrightarrow$$
$$\langle (A^* \circ T)x - (A^* \circ T)y, x - y \rangle \geq \eta \|Tx - Ty\| \cdot \|Ax - Ay\|, \ \forall x, y \in D.$$

Therefore, for $x, y \in D$, $x \neq y$, we have

$$\begin{aligned}
&(A^* \circ T)x - (A^* \circ T)y, x - y \rangle \geq \eta \|Tx - Ty\| \cdot \|x - y\| \frac{\|Ax - Ay\|}{\|x - y\|} \\
&= \eta \|A^*(Tx) - A^*(Ty)\| \frac{1}{\left\| A^* \left(\dfrac{Tx - Ty}{\|Tx - Ty\|} \right) \right\|} \cdot \|x - y\| \cdot \left\| A \left(\frac{x - y}{\|x - y\|} \right) \right\| \\
&\geq \eta \left(\frac{\sup\limits_{\|z\|=1} \|Az\|}{\inf\limits_{\|z\|=1} \|A^* z\|} \right) \|(A^* \circ T)x - (A^* \circ T)y\| \cdot \|x - y\| \\
&= \frac{\eta a_A}{b_{A^*}} \cdot \|(A^* \circ T)x - (A^* \circ T)y\| \cdot \|x - y\|
\end{aligned}$$

and the proof of the first statement is now complete. The second statement can be similarly proved. □

Corollary 2.11. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set, let $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a linear isomorphism. If $-b_{A^*} < \eta a_A \leq 0$ and T is (A, η)-increasing, then $A^* \circ T$ is $(\eta a_A)/b_{A^*}$-increasing. If $0 \leq \eta a_A < b_{A^*}$ and T is (A, η)-decreasing, then $A^* \circ T$ is $(\eta a_A)/b_{A^*}$-decreasing.*

Remark 2.12. The gradients of strictly convex functions whose strict convexity is ensured by the everywhere positive definiteness of the Hessian matrix are good candidates to play the role of the injective operator A. Indeed, the gradient of such a function defined on a convex open subset D of $\mathbb{R}^n$ is injective, as the everywhere positive definiteness of the Hessian matrix is equivalent with the everywhere positive definiteness the Fréchet differentials of the gradient. In fact the Jacobian matrix of the gradient of such a C^2-smooth function is precisely the Hessian matrix of that function.

In fact the h-monotonicity of a certain operator coincides with its ∇f-monotonicity, where $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is the strictly convex function given by

$$f(x) = \frac{1}{2}\|x\|^2.$$

Proposition 2.13. *Let $T, A : D \longrightarrow \mathbb{R}^{n+1}$ be given operators with A injective, where D is a subset of $\mathbb{R}^{n+1}$. The operator $T + A$ is A-increasing and A-decreasing if and only if T is A-increasing and A-decreasing.*

Proof. Indeed,

$$\langle Tx + Ax - Ty - Ay, Ax - Ay\rangle^2 = \big(\|Ax - Ay\|^2 + \langle Tx - Ty, Ax - Ay\rangle\big)^2$$
$$= \|Ax - Ay\|^4 + 2\|Ax - Ay\|^2\langle Tx - Ty, Ax - Ay\rangle + \langle Tx - Ty, Ax - Ay\rangle^2.$$

and

$$\|Tx + Ax - Ty - Ay\|^2 \cdot \|Ax - Ay\|^2 = \|Tx - Ty + Ax - Ay\|^2 \cdot \|Ax - Ay\|^2$$
$$= \big(\|Ax - Ay\|^2 + 2\langle Tx - Ty, Ax - Ay\rangle + \|Tx - Ty\|^2\big) \cdot \|Ax - Ay\|^2$$
$$= \|Ax - Ay\|^4 + 2\|Ax - Ay\|^2\langle Tx - Ty, Ax - Ay\rangle + \|Tx - Ty\|^2\|Ax - Ay\|^2.$$

The statement follows easily by using Remark 2.5(5). □

Corollary 2.14. *Let $D \subseteq \mathbb{R}^{n+1}$ be a convex open set and $f : D \longrightarrow \mathbb{R}$ be a C^2-smooth strictly convex function whose convexity is ensured by the everywhere positive definiteness of its Hessian matrix. The operator $T + \nabla f$ is ∇f-increasing and ∇f-decreasing if and only if $T : D \longrightarrow \mathbb{R}^{n+1}$ is ∇f-increasing and ∇f-decreasing.*

3. On the degree of some spherical projections

Since in our study on h-monotone operators the *degree* of differentiable maps plays an important role, in this section we discuss some of its properties. In this respect we first recall the notions of *critical/regular points* and *critical/regular values*.

Let M, N be differential manifolds and $f : M \to N$ be a differentiable mapping. We first define the *rank* of f at a point $p \in M$ as $\mathrm{rank}_p f := \mathrm{rank}(df)_p = \dim[\mathrm{Im}(df)_p$, where $(df)_p : T_p(M) \longrightarrow T_{f(p)}(N)$ is the differential (or tangent map) of f at p, and observe that $\mathrm{rank}_p f \leq \min\{m, n\}$, where $m = \dim(M)$ and $n = \dim(N)$. The point $p \in M$ is called a *regular* point of f if $\mathrm{rank}_p f = \min\{m, n\}$ and it is called *critical point* of f if $\mathrm{rank}_p f < \min\{m, n\}$. One can immediately observe that the set $\mathcal{R}(f)$ of all regular points of f is open while the set $\mathcal{C}(f) := M\backslash\mathcal{R}(f)$ of all critical points of f is closed. A value $y \in f(\mathcal{C}(f)) =: \mathcal{B}(f)$ is called *critical value* of f, and a point $q \in N\backslash\mathcal{B}(f)$ is called *regular value* of f.

If $m = n$, then a point $x \in M$ is a regular point of $f : M \longrightarrow N$ if and only if f is a local diffeomorphism at x. Consequently the preimage $f^{-1}(y)$ of a regular value y of f is discrete. If f is additionally proper (i.e. the inverse images of compact sets are compact), then the preimage $f^{-1}(y)$ of such a regular value is finite.

If $\mathcal{H} \subset \mathbb{R}^{n+1}$ is a hypersurface, i.e. an n-dimensional submanifold, and $p \in \mathcal{H}$, then denote by $\mathcal{T}_p(\mathcal{H})$ the collection of all tangent vectors $\gamma'(0)$, where

$\gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathcal{H}$ is a parameterized differentiable curve such that $\gamma(0) = p$ and recall that $\mathcal{T}_p(\mathcal{H})$ is an n-dimensional vector subspace of $\mathbb{R}^{n+1}$. Denote by $i_{\mathcal{H}} : \mathcal{H} \hookrightarrow \mathbb{R}^{n+1}$ the inclusion mapping and recall that two hyperplanes of $\mathbb{R}^{n+1}$ are orthogonal if their normal vectors are orthogonal. A compact hypersurface of $\mathbb{R}^{n+1}$ without boundary (in the sense of manifold theory) will be called *closed hypersurface.*
If M, N are compact connected oriented n-dimensional manifolds, $f : M \longrightarrow N$ is a differentiable map and $y \in N$ is a regular value of f, then

$$\deg_y(f) := \begin{cases} \displaystyle\sum_{x \in f^{-1}(y)} \varepsilon_x & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset, \end{cases}$$

where

$$\varepsilon_x := \begin{cases} 1 & \text{if } (df)_x \text{ preserves the orientation} \\ -1 & \text{if } (df)_x \text{ reverses the orientation.} \end{cases}$$

In fact $\deg_y(f)$ does not depend on y and is called the *degree* of f, being simply denoted by $\deg(f)$ (see [1], pp. 253]). If f is not onto, observe that $\deg(f) = 0$, since every $y \in N \setminus \mathrm{Im}(f)$ is a regular value of f. On the other hand, one can show that deg is invariant on differential homotopy classes of maps from M to N. Since every continuous homotopy class of maps from M to N contains a differentiable map, the notion of *degree* can be extended to the class of all continuous maps and its invariance on continuous homotopy classes is part of the extension procedure. For more details the reader could consult [8], pp. 165, 166, 21-221]. A different approach of degree theory for continuous maps appears in [7], pp. 62-65, 266-271].

Proposition 3.1. *If X is a topological space and $f, g : X \longrightarrow S^n, n \geq 1$ are continuous maps such that $d_{S^n}(f(x), g(x)) < \pi$ for all $x \in X$, then $f \simeq g$, i.e. f and g are homotopic.*

Proof. Indeed, the following homotopy

$$H : X \times [0,1] \longrightarrow S^n, \ H(x,t) := \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$$

is well defined and $H(\cdot, 0) = f, H(\cdot, 1) = g$. □

Remark 3.2. If X is a topological space and $f, g : X \longrightarrow S^n, n \geq 1$ are continuous maps such that $d_{S^n}(f(x), g(x)) > 0$ for all $x \in X$, then $f \simeq -g$, i.e. f and $-g$ are homotopic.

For a given function $f : X \longrightarrow Y$ we define its kernel as the equivalence relation on X whose graph is $\ker(f) := \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$. The next statements reveal some important homotopy properties of the A-monotone operators.

Corollary 3.3. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set and let $A : D \longrightarrow \mathbb{R}^{n+1}$ be an injective local diffeomorphism. If $T : D \to \mathbb{R}^{n+1}$ is an A-monotone operator, then the map*

$$D \times D \setminus \ker(T) \to S^n, (x, y) \longmapsto pr_{S^n}(Tx - Ty)$$

is homotopic to one of the maps

$$D \times D \setminus \ker(T) \to S^n, (x,y) \longmapsto pr_{S^n}(Ax - Ay)$$

or

$$D \times D \setminus \ker(T) \to S^n, (x,y) \longmapsto pr_{S^n}(Ay - Ax).$$

Corollary 3.4. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set and let $A : D \longrightarrow \mathbb{R}^{n+1}$ be an injective local diffeomorphism. If $T : D \to \mathbb{R}^{n+1}$ is a differentiable A-monotone operator and $\mathcal{H} \subset D$ is a closed connected hypersurface, then the degree of $\mathrm{pr}_{S^n} \circ (A\big|_{\mathcal{H}} - Az)$ is invariant over every connected component of $D \setminus T^{-1}(T(\mathcal{H}))$.*

Proof. We assume that T is A-increasing, as the decreasing option can be similarly treated. Let us consider a continuous path $\gamma : [0,1] \longrightarrow D \setminus T^{-1}(T(\mathcal{H}))$. Therefore $\gamma(0) = z_0$ and $\gamma(1) = z_1$ belong to the same connected component of $T^{-1}(T(\mathcal{H}))$. By using Corollary 3.3 one can deduce that

$$\mathrm{pr}_{S^n} \circ (A\big|_{\mathcal{H}} - Az_0) \simeq \mathrm{pr}_{S^n} \circ (T\big|_{\mathcal{H}} - Tz_0)$$

and

$$\mathrm{pr}_{S^n} \circ (A\big|_{\mathcal{H}} - Az_1) \simeq \mathrm{pr}_{S^n} \circ (T\big|_{\mathcal{H}} - Tz_1)$$

along with

$$\deg\ \mathrm{pr}_{S^n} \circ (A\big|_{\mathcal{H}} - Az_0) = \deg\ \mathrm{pr}_{S^n} \circ (T\big|_{\mathcal{H}} - Tz_0) \tag{3.1}$$

and

$$\deg\ \mathrm{pr}_{S^n} \circ (A\big|_{\mathcal{H}} - Az_1) = \deg\ \mathrm{pr}_{S^n} \circ (T\big|_{\mathcal{H}} - Tz_1). \tag{3.2}$$

On the other hand

$$H : \mathcal{H} \times [0,1] \longrightarrow T^{-1}(T(\mathcal{H})),\ H(x,t) = \frac{Tx - T(\gamma(t))}{\|Tx - T(\gamma(t))\|}$$

realizes a homotopy between $\mathrm{pr}_{S^n} \circ (T\big|_{\mathcal{H}} - Tz_0)$ and $\mathrm{pr}_{S^n} \circ (T\big|_{\mathcal{H}} - Tz_1)$. Therefore

$$\deg\ \mathrm{pr}_{S^n} \circ (T\big|_{\mathcal{H}} - Tz_0) = \deg\ \mathrm{pr}_{S^n} \circ (T\big|_{\mathcal{H}} - Tz_1)$$

which combined with (3.1) and (3.2) leads us to the equality

$$\deg\ \mathrm{pr}_{S^n} \circ (A\big|_{\mathcal{H}} - Az_0) = \deg\ \mathrm{pr}_{S^n} \circ (A\big|_{\mathcal{H}} - Az_1).$$

□

Remark 3.5. Let X be a compact differential n-dimensional manifold and

$$f, g : X \longrightarrow S^n,\ n \geq 1$$

be continuous maps such that

1. If $d_{S^n}(f(x), g(x)) < \pi$ for all $x \in X$, then $\deg(g) = \deg(f)$. Indeed, f and g are, according to Proposition 3.1, homotopic to each other.

2. If $d_{S^n}(f(x), g(x)) > 0$ for all $x \in X$, then $\deg(g) = (-1)^{n+1}\deg(f)$. Indeed,
$$d_{S^n}(f(x), g(x)) > 0,\ \forall\, x \in X \Leftrightarrow d_{S^n}(-f(x), g(x)) < \pi,\ \forall\, x \in X,$$
which shows, according to Proposition 3.1, that
$$\deg(g) = \deg(-f) = \deg(A \circ f) = \deg(A)\deg(f) = (-1)^{n+1}\deg(f),$$
where $A : S^n \longrightarrow S^n, Ax = -x$ is the antipodal map.
Consequently, if $\deg(g) \neq (-1)^{n+1}\deg(f)$, then the *coincidence set*
$$C(f, g) := \{x \in X : f(x) = g(x)\}$$
is not empty.

If $\mathcal{H} \subset \mathbb{R}^{n+1}$ is a closed connected hypersurface, then, according to [9], Theorem 4.6] and the related results therein, $\mathcal{H}$ separates $\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+1} \setminus \mathcal{H}$ has precisely two connected components, one of which is bounded and denoted by $int(\mathcal{H})$ and another one which is unbounded and denoted by $ext(\mathcal{H})$.
On the other hand $\partial\big[int(\mathcal{H})\big] = \mathcal{H} = \partial\big[ext(\mathcal{H})\big]$, where ∂S stands for the topological frontier of S.

Proposition 3.6. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set and let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth injective local diffeomorphism. If $\mathcal{H} \subset D \subseteq \mathbb{R}^{n+1}$ is a closed connected hypersurface, then*
$$\deg[pr_{S^n} \circ (A\big|_{\mathcal{H}} - Az)] = 0,\ \forall z \in A^{-1}(ext(A(\mathcal{H}))),$$
and either
$$\deg[pr_{S^n} \circ (A\big|_{\mathcal{H}} - Az)] = 1, \forall z \in A^{-1}(int(A(\mathcal{H})))$$
or
$$\deg[pr_{S^n} \circ (A\big|_{\mathcal{H}} - Az)] = -1, \forall z \in A^{-1}(int(A(\mathcal{H})).$$

Proof. Since $A : D \longrightarrow \mathbb{R}^{n+1}$ is an injective local diffeomorphism, the image $A(\mathcal{H})$ of $\mathcal{H}$ through A is a closed connected hypersurface and according to [10, Proposition 3.7] we conclude that
$$\deg[pr_{S^n} \circ (i_{A(\mathcal{H})} - Az)] = 0 \text{ for all } z \in A^{-1}(ext(\mathcal{A}(\mathcal{H}))),$$
as well as either
$$\deg[pr_{S^n} \circ (i_{\mathcal{H}} - Az)] = 1 \text{ for all } z \in A^{-1}(int(A(\mathcal{H})))$$
or
$$\deg[pr_{S^n} \circ (i_{A(\mathcal{H})} - Az)] = -1 \text{ for all } z \in A^{-1}(int(A(\mathcal{H}))).$$
Note that
$$pr_{S^n} \circ (A\big|_{\mathcal{H}} - Az)] = [pr_{S^n} \circ (i_{A(\mathcal{H})} - Az)] \circ r,$$
where r stands for the restriction and corestriction $\mathcal{H} \longrightarrow A(\mathcal{H})$, $x \mapsto Ax$, which is a diffeomorphism. The multiplicative property of the degree combined with the obvious fact that either $\deg r \equiv 1$ or $\deg r \equiv -1$, concludes the proof. $\square$

4. Properties of the inverse images of A-monotone operators

In this section we provide some examples of closed subsets of the Euclidean space $\mathbb{R}^{n+1}$ which can be separated by closed connected hypersurfaces. We close this section by proving that the inverse images of continuous A-monotone operators cannot be separated by closed smooth hypersurfaces.

Definition 4.1. A subset X of $\mathbb{R}^{n+1}$ is separated by a closed connected hypersurface $\mathcal{H}$ of $\mathbb{R}^{n+1}$ if $\mathcal{H} \subseteq \mathbb{R}^{n+1} \setminus X$ and each $int(\mathcal{H})$, $ext(\mathcal{H})$ contains a connected component of X at least. We say that X is *divisible by closed connected hypersurfaces* if X is separated by one closed connected hypersurface, at least. Otherwise we say that X is *indivisible by closed connected hypersurfaces.*

Theorem 4.2. ([10]) *If the closed set $C \subset \mathbb{R}^{n+1}$ has a compact connected component K such that $C \setminus K$ is nonempty and closed, then C is divisible by closed connected hypersurfaces.*

Theorem 4.3. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set, let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth injective local diffeomorphism and $T : D \longrightarrow \mathbb{R}^{n+1}$ be a continuous A-monotone operator. If $y \in \mathrm{Im}(T)$, then $T^{-1}(y)$ is indivisible by closed connected hypersurfaces.*

Proof. Assume that $T^{-1}(y)$ is divisible by closed connected hypersurfaces, for some $y \in \mathrm{Im}(T)$ and consider a closed connected hypersurface $\mathcal{H} \subset \mathbb{R}^{n+1}$ with the property that one component of $T^{-1}(y)$, say C, is contained in $int(\mathcal{H})$ and another component of $T^{-1}(y)$, say K, is contained in $ext(\mathcal{H})$.
If $z_0 \in C \subseteq A^{-1}(int(A(\mathcal{H}))$ and $z_1 \in K \subseteq A^{-1}(ext(A(\mathcal{H}))$, then, according to Proposition 3.6 and Corollary 3.4, one gets

$$\pm 1 = \deg[pr_{S^n} \circ (A\big|_{\mathcal{H}} - Az_0)] = \pm \deg[pr_{S^n} \circ (A\big|_{\mathcal{H}} - Az_1)] = 0,$$

which is absurd. □

Corollary 4.4. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set, let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth injective local diffeomorphism and $T : D \longrightarrow \mathbb{R}^{n+1}$ be a continuous A-monotone operator. If $y \in \mathrm{Im}(T)$, then either $T^{-1}(y)$ is connected or the set $T^{-1}(y) \setminus K$ is not closed for every compact connected component K of $T^{-1}(y)$.*

Proof. Assume that $T^{-1}(y)$ is not connected and $T^{-1}(y) \setminus K$ is closed for some compact connected component K of $T^{-1}(y)$. Then $T^{-1}(y) \setminus K$ is nonempty and the statement follows by using Theorems 4.2, 4.3. □

Theorem 4.5. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set, let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth injective local diffeomorphism and $T : D \longrightarrow \mathbb{R}^{n+1}$ be a continuous A-monotone operator. If $q \in \mathrm{Im}(T)$, then either $T^{-1}(q)$ is a singleton or* $\dim\left(T^{-1}(q)\right) \geq 1$.

Proof. Recall that, according to Remark 2.5(7), the operator $T : D \longrightarrow \mathbb{R}^{n+1}$ is A-monotone if and only if $T \circ A^{-1} : A(D) \longrightarrow \mathbb{R}^{n+1}$ is h-monotone. By using [10, Theorem 4.8] one gets that either $(T \circ A^{-1})^{-1}(q) = A(T^{-1}(q))$ is a singleton or

$$\dim(T \circ A^{-1})^{-1}(q) = \dim(A(T^{-1}(q)) \geq 1,$$

i.e., $T^{-1}(q)$ is a singleton or $\dim\left(T^{-1}(q)\right) \geq 1$. □

The next properties of the inverse images of A-monotone operators are immediate consequences of Theorem 4.5.

Corollary 4.6. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set, let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth injective local diffeomorphism and $T : D \longrightarrow \mathbb{R}^{n+1}$ be a continuous A-monotone operator. Then either T is injective or* $\dim\left(T^{-1}(y)\right) \geq 1$ *for some $y \in \mathrm{Im}(T)$.*

Proof. If T is not injective, then $\mathrm{card}\left(T^{-1}(y)\right) \geq 2$ for some $y \in \mathrm{Im}(T)$. According to Theorem 4.5, $\dim\left(T^{-1}(y)\right) \geq 1$. □

Definition 4.7. ([5]) A continuous map $f : X \to Y$ is said to be *light* if

$$\dim\left(f^{-1}(y)\right) \leq 0 \text{ for every } y \in Y.$$

Observe that locally injective operators are light, as their inverse images are discrete sets.

Corollary 4.8. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set and let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth injective local diffeomorphism. If $T : D \longrightarrow \mathbb{R}^{n+1}$ a continuous A-monotone light operator, then T is injective.*

Proof. We need to prove that $\mathrm{card}\left[A^{-1}(q)\right] = 1$ for each $q \in \mathrm{Im}(A)$. Indeed, if $\mathrm{card}\left[A^{-1}(q)\right]$ were at least 2 for some $q \in \mathrm{Im}(A)$, then, according to Theorem 4.5, we would get $\dim\left(A^{-1}(q)\right) \geq 1$. □

Corollary 4.9. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set and let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth injective local diffeomorphism. If $T : D \longrightarrow \mathbb{R}^{n+1}$ a continuous A-monotone local homeomorphism, then T is injective.*

Corollary 4.10. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set and let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth local diffeomorphism. If $T : D \longrightarrow \mathbb{R}^{n+1}$ is a C^1-smooth A-monotone operator, then T is locally injective. Indeed, according to Corollary 4.9, the restriction $T\big|_U$, where $U \subseteq D$ is an open set, is injective whenever the restriction $A\big|_U$ is injective. The later type of restrictions are one-to-one for suitable choices of the open set $U \subseteq D$, as A is a local diffeomorphism. If T is additionally open, then one can conclude that T is a local homeomorphism.*

Remark 4.11. Observe that Corollary 4.9 can be also obtained from Theorem 4.3. Indeed the inverse images of local diffeomorphisms, as discrete sets, are divisible by closed connected hypersurfaces provided their cardinality is at least two.

5. Pairs of mutual monotone local homeomorphisms

In this section we deal with pairs operators having *a priori* the same status relative to injectivity, i.e. they are local homeomorphisms.

Definition 5.1. Let $D \subseteq \mathbb{R}^n$ be an open set and let $T, Q : D \longrightarrow \mathbb{R}^n$ be two local homeomorphisms. The two local homeomorphisms are said to be

1. *mutual h*-increasing if

$$d_{S^n}\left(pr_{S^n}(Tx - Ty), pr_{S^n}(Qx - Qy)\right) < \pi, \ \forall (x, y) \in (D \times D) \setminus (\ker T \cup \ker Q).$$

2. *mutual* h-decreasing if

$$d_{S^n}(pr_{S^n}(Tx-Ty), pr_{S^n}(Qx-Qy)) > 0, \forall (x,y) \in (D\times D)\setminus(\ker S \cup \ker Q).$$

3. *mutual* h-monotone if they are either *mutual* h-increasing or *mutual* h-decreasing.

Remark 5.2. Let $D \subseteq \mathbb{R}^{n+1}$ be an open set and let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth local diffeomorphism. If $T : D \longrightarrow \mathbb{R}^{n+1}$ is an open C^1-smooth A-monotone operator, then, according to Corollary 4.10, the operators T and A have *a posteriori* some rather close status with respect to injectivity, i.e. A is a local diffeomorphism and T is a local homeomorphism and the two operartors T and A are mutual h-monotone local homeomorphisms.

Remark 5.3. Let $D \subseteq \mathbb{R}^n$ be an open set and let $T, Q : D \longrightarrow \mathbb{R}^n$ be two local homeomorphisms.

1. The two local homeomorphisms are *mutual* h-increasing if and only if

$$\langle Tx-Ty, Qx-Qy\rangle > -\|Tx-Ty\|\cdot\|Qx-Qy\|, \forall (x,y)\in(D\times D)\setminus(\ker T\cup\ker Q);$$

2. The two local homeomorphisms are *mutual* h-decreasing if and only if

$$\langle Tx-Ty, Qx-Qy\rangle < \|Tx-Ty\|\cdot\|Qx-Qy\|, \forall (x,y)\in(D\times D)\setminus(\ker S\cup\ker Q).$$

3. The relation of being *mutual* h-increasing/decreasing is symmetric.

Theorem 5.4. *If $D \subseteq \mathbb{R}^n$ is an open set and $T, Q : D \longrightarrow \mathbb{R}^n$ are two mutual h-monotone local diffeomorphisms, then* $\ker S = \ker T$.

Proof. By using Corollary 4.9 it follows that the two local diffeomorphisms are simultaneously injective or non-injective. Their injectivity is equivalent with

$$\ker T = \ker Q = \Delta_D := \{(x,x) \ : \ x \in D\}.$$

We now assume that none of them is injective as well as $\ker T \setminus \ker Q \neq \emptyset$ and consider $(u,v) \in \ker T \setminus \ker Q$, i.e. $Tu = Tv$ and $Qu \neq Qv$. Let $r, \varepsilon > 0$ be such that $T(B(u, r+\varepsilon))$, $Q(B(u, r+\varepsilon))$ are open and the restrictions

$$\bar{B}(u,r) \longrightarrow T(\bar{B}(u,r)),\ x \mapsto Tx$$
$$\bar{B}(u,r) \longrightarrow Q(\bar{B}(u,r)),\ x \mapsto Qx$$

are diffeomorphisms and $Qv \notin Q(\bar{B}(u,r))$. In particular the sphere $S(p,r)$ is mapped by T onto a closed hypersurface $T(S(p,r))$. Since the local diffeomorphisms T, Q are mutual h-monotone, it follows that either

$$d_{S^n}(pr_{S^n}\circ(T\big|_{S(u,r)} - Tv), pr_{S^n}\circ(T\big|_{S(u,r)} - Tv)) < \pi$$

or

$$d_{S^n}(pr_{S^n}\circ(T\big|_{S(u,r)} - Tv), pr_{S^n}\circ(Q\big|_{S(u,r)} - Qv)) > 0.$$

In both cases we get, via Remark 3.5, that

$$\deg pr_{S^n}\circ(Q\big|_{S(u,r)} - Qv) = \pm\deg pr_{S^n}\circ(T\big|_{S(u,r)} - Tv) = \pm\deg pr_{S^n}\circ(T\big|_{S(u,r)} - Tu).$$

On the other hand, by using Proposition 3.6

$$\deg pr_{S^n}\circ(Q\big|_{S(u,r)} - Qv) = 0,$$

as $Qv \in ext\ (S(u,r))$ and

$$\deg pr_{S^n} \circ (T\big|_{S(u,r)} - Tu) = \pm 1,$$

as $Tu \in int\ T(S(u,r))$, which is absurd.
Therefore $\ker T \backslash \ker Q = \emptyset \Longleftrightarrow \ker T \subseteq \ker Q$. The opposite inclusion can be similarly done by interchanging the roles of T and Q. □

Theorem 5.5. *Let $D \subseteq \mathbb{R}^n$ be an open set and let $T, Q : D \longrightarrow \mathbb{R}^n$ be two local diffeomorphisms. Then T, Q are mutual h-monotone if and only if there exists a h-monotone diffeomorphism*

$$\Phi : \mathrm{Im}(Q) \longrightarrow \mathrm{Im}(T)$$

such that $T = \Phi \circ Q$.

Proof. If Φ is h-increasing and $T = \Phi \circ Q$, then

$$\begin{aligned}\langle Tx - Ty, Qx - Qy\rangle &= \langle \Phi(Qx) - \Phi(Qy), Qx - Qy\rangle \\ &> -\|\Phi(Qx) - \Phi(Qy)\| \cdot \|Qx - Qy\| \\ &= -\|Tx - Ty\| \cdot \|Qx - Qy\|,\end{aligned}$$

i.e. $T = \Phi \circ Q$ is h-increasing. If Φ is h-decreasing and $T = \Phi \circ Q$, then

$$\begin{aligned}\langle Tx - Ty, Qx - Qy\rangle &= \langle \Phi(Qx) - \Phi(Qy), Qx - Qy\rangle < \|\Phi(Qx) - \Phi(Qy)\| \cdot \|Qx - Qy\| \\ &= \|Tx - Ty\| \cdot \|Qx - Qy\|\end{aligned}$$

i.e. $T = \Phi \circ Q$ is h-decreasing. Conversely, if T, Q are mutual h-monotone, then $\ker T = \ker Q$, due to Theorem 5.4. The functions

$$\alpha : D/\ker T \longrightarrow \mathrm{Im}\ T,\ \alpha(d + \ker T) = T(d)$$
$$\beta : D/\ker Q \longrightarrow \mathrm{Im}\ Q,\ \beta(d + \ker Q) = Q(d)$$

are well-defined, bijective and $T = \alpha \circ \pi_{\ker T}$ and $Q = \beta \circ \pi_{\ker Q}$, where

$$\pi_{\ker T} : D \longrightarrow D/\ker T \text{ and } \pi_{\ker Q} : D \longrightarrow D/\ker Q$$

are the canonical projections. The bijections α and β are also unique with their corresponding properties. Since $\ker T = \ker Q$, it follows that $D/\ker T = D/\ker Q$ and

$$\mathrm{Im}\ Q \xleftarrow{\beta} D/\ker Q = D/\ker T \xrightarrow{\alpha} \mathrm{Im}\ T$$

are bijections. Therefore $\Phi := \alpha \circ \beta^{-1} : \mathrm{Im}(Q) \longrightarrow \mathrm{Im}(T)$ is a bijection and

$$\Phi \circ Q = \alpha \circ \beta^{-1} \circ Q = \alpha \circ \pi_{\ker Q} = \alpha \circ \pi_{\ker T} = T.$$

Since T and Q are local diffeomorphisms it follows that Φ is differentiable and a diffeomorphism therefore. Finally

$$\begin{aligned}\langle \Phi(Qx) - \Phi(Qy), Qx - Qy\rangle &= \langle Tx - Ty, Qx - Qy\rangle > -\|Tx - Ty\| \cdot \|Qx - Qy\| \\ &= -\|\Phi(Qx) - \Phi(Qy)\| \cdot \|Qx - Qy\|\end{aligned}$$

if T, Q are mutual h-increasing and

$$\begin{aligned}\langle \Phi(Qx) - \Phi(Qy), Qx - Qy\rangle &= \langle Tx - Ty, Qx - Qy\rangle < \|Tx - Ty\| \cdot \|Qx - Qy\| \\ &= \|\Phi(Qx) - \Phi(Qy)\| \cdot \|Qx - Qy\|\end{aligned}$$

if T, Q are mutual h-decreasing. In other words, Φ is h-increasing/decreasing if T, Q are mutual h-increasing/decreasing. □

Corollary 5.6. *Let $D \subseteq \mathbb{R}^{n+1}$ be an open set and let $A : D \longrightarrow \mathbb{R}^{n+1}$ be a C^1-smooth local diffeomorphism. If $T : D \longrightarrow \mathbb{R}^{n+1}$ is an open C^1-smooth A-monotone operator, then there exists a h-monotone homeomorphism*

$$\Phi : \mathrm{Im}(Q) \longrightarrow \mathrm{Im}(T)$$

such that $T = \Phi \circ Q$.

Proof. According to Remark 5.2, T is a local homeomorphism and T, A are obviously mutual h-monotone local homeomorphisms. From now on the proof works along the same lines with the proof of Theorem 5.5. □

Corollary 5.7. *Let $D \subseteq \mathbb{R}^n$ be an open set and let $T, Q : D \longrightarrow \mathbb{R}^n$ be two local diffeomorphisms. If T, Q are mutual h-monotone, then*

$$\mathrm{Val}(T) = \mathrm{Val}(Q),$$

where

$$\mathrm{Val}(F) := \sup\{\mathrm{card} F^{-1}(y) : y \in \mathbb{R}^n\}$$

stands for the valence of $F : D \longrightarrow \mathbb{R}^n$, as defined in [15].

Proof. Indeed, according to Theorem 5.5 there exists an h-monotone diffeomorphism

$$\Phi : \mathrm{Im}(Q) \longrightarrow \mathrm{Im}(T)$$

such that $T = \Phi \circ Q$. Thus $T^{-1}(y) = (\Phi \circ Q)^{-1}(y) = Q^{-1}(\Phi^{-1}(y))$ for every $y \in \mathrm{Im}(T)$, which implies that

$$\mathrm{card}\ T^{-1}(y) = \mathrm{card}\ Q^{-1}(\Phi^{-1}(y)), \forall y \in \mathrm{Im}(T)$$

and shows that

$$\begin{aligned}\mathrm{Val}(T) &= \sup\{\mathrm{card}\ T^{-1}(y) : y \in \mathrm{Im}(T)\}\\ &= \sup\{\mathrm{card}\ Q^{-1}(\Phi^{-1}(y)) : y \in \mathrm{Im}(T)\}\\ &= \sup\{\mathrm{card}\ Q^{-1}(z) : z \in \mathrm{Im}(Q)\} = \mathrm{Val}(Q).\end{aligned}$$

□

Remark 5.8. In the proof of Corollary 5.7, we only used the quality of Φ to be globally injective, not its quality to be differentiable with differentiable inverse.

6. Final comments and remarks

Throughout the section we make use of the notation described below (see [18]). Let D be a nonempty open convex subset of $\mathbb{R}^n$, and let $f : D \to \mathbb{R}$ be a C^2-smooth convex function. The Hessian matrix of f at an arbitrary point $x \in D$ will be denoted by $H_x(f)$. Recall that $H_x(f)$ is a symmetric matrix and it defines a symmetric bilinear functional

$$\mathcal{H}_x(f) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R},\ \mathcal{H}_x(f)(u, v) := u \cdot H_x(f) \cdot v^T.$$

The following region

$$\mathrm{Hess}^{+}(f) := \{x \in D | H_x(f) \text{ is positive definite}\}$$

associated to some C^2-smooth regular function $f : D \longrightarrow \mathbb{R}$ was described in [17] for the particular polynomial function

$$f_a : \mathbb{R}^2 \longrightarrow \mathbb{R},\ f_a(x,y) = (x^2+y^2)^2 - 2a^2(x^2-y^2).$$

Denote by $h_x(f) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the linear transformation defined by the following equality $\mathcal{H}_x(f)(u,v) := \langle h_x(f)u, v\rangle$, $\forall u, v \in \mathbb{R}^n$. and set

$$\sigma_f := \sup_{z\in D} \|h_f(z)\|.$$

Let further $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator, and let $T : D \to \mathbb{R}^n$ be the vector-valued function defined by $Tx := \nabla f(x) + Ax$. We shall denote by $[A]$ the matrix representation of A with respect to the standard basis of $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere (i.e., centered at the origin) in $\mathbb{R}^n$, and let

$$W(A) := \{\langle Ax, x\rangle \mid x \in S^{n-1}\}$$

be the *numerical range* of A. It is well known that $W(A) = [\lambda_A, \mu_A]$, where λ_A and μ_A denote the smallest and the greatest eigenvalue, respectively, of the symmetric operator $(A + A^*)/2$. Let also λ_A^* and μ_A^* denote the smallest and the greatest eigenvalue, respectively, of the symmetric positive semidefinite operator A^*A. It is well known that

$$\|A\| := \max_{x\in S^{n-1}} \|Ax\| = \sqrt{\mu_A^*} \qquad \text{and} \qquad \min_{x\in S^{n-1}} \|Ax\| = b_A. \tag{6.1}$$

Sometimes we set, for brevity, $a_A := \|A\| = \sqrt{\mu_A^*}$ and $b_A := 1/\|A^{-1}\|$ if A is invertible. Since $\|A^{-1}\|$ equals the square root of the greatest eigenvalue of

$$(A^{-1})^* A^{-1} = (AA^*)^{-1},$$

it follows that $b_A = \sqrt{\lambda_A^*}$.

Theorem 6.1. ([18]) *Let $D \subseteq \mathbb{R}^n$ be a convex open set, let $f : D \longrightarrow \mathbb{R}$ be a C^2-smooth convex function and let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear automorphism. If the following inequalities are satisfied*

$$\sigma_f < b_A + \lambda_A \text{ and } \inf_{z\in D} \|h_f(z)\| < -\mu_A,$$

then $-1 < i_{\nabla f + A} < 0$, namely $\nabla f + A$ is h-monotone but not monotone.

Theorem 6.2. ([18]) *Let $D \subseteq \mathbb{R}^n$ be a convex open set, let $f : D \longrightarrow \mathbb{R}$ be a C^2-smooth convex function and let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear automorphism. If the following inequality is satisfied*

$$\sigma_f < \min\{b_A + \lambda_A, -\mu_A\}, \tag{6.2}$$

then $T := \nabla f + A$ is injective.

Remark 6.3. Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be C^2-smooth functions such that $\mathrm{Hess}^+(f) \neq \emptyset$ and $\mathrm{Hess}^+(g) = \mathbb{R}^n$. Then $\nabla f\big|_{\mathrm{Hess}^+(f)} \circ \nabla g\big|_D$ and $(\nabla f\big|_{\mathrm{Hess}^+(f)} + A) \circ \nabla g\big|_D$ are $\nabla g\big|_D$-increasing for every convex open subset D of $\mathbb{R}^n$ such that the range of $\nabla g\big|_D$ is contained in $\mathrm{Hess}^+(f)$, where $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear automorphism related with f through the inequality (6.2). Thus $\nabla f\big|_{\mathrm{Hess}^+(f)} \circ \nabla g\big|_D$ and $(\nabla f\big|_{\mathrm{Hess}^+(f)} + A) \circ \nabla g\big|_D$ are one-to-one as ∇g is a Minty-Browder monotone global diffeomorphism.

Example 6.4. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a C^2-smooth function such that $\mathrm{Hess}^+(f) \neq \emptyset$ and the smooth function

$$g : \mathbb{R}^n \longrightarrow \mathbb{R},\ g(x) = \frac{1}{2} e^{\|x\|^2}.$$

Then its gradient $(\nabla g)_x = e^{\|x\|^2} \cdot x$ is a Minty-Browder monotone global diffeomorphism, as the Hessian matrix $H_x(g) = e^{\|x\|^2}(I_n + 2x \cdot x^T)$ of g, which is actually the Jacobian matrix of ∇g, is positive definite. Indeed, the diagonal determinants $\Delta_k = 1 + 2(x_1^2 + \cdots + x_k^2)$ of $I_n + 2x \cdot x^T$ are all positive and the positive definiteness of $I_n + 2x \cdot x^T$ follows via the Sylvester criterion. Therefore $\nabla f\big|_{\mathrm{Hess}^+(f)} \circ \nabla g\big|_D$ along with $(\nabla f\big|_{\mathrm{Hess}^+(f)} + A) \circ \nabla g\big|_D$ are $\nabla g\big|_D$-increasing for every convex open subset D of $\mathbb{R}^n$ such that the range of $\nabla g\big|_D$ is contained in $\mathrm{Hess}^+(f)$, where $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear automorphism related with f through the inequality (6.2). Thus $\nabla f\big|_{\mathrm{Hess}^+(f)} \circ \nabla g\big|_D$ and $(\nabla f\big|_{\mathrm{Hess}^+(f)} + A) \circ \nabla g\big|_D$ are one-to-one as ∇g is a Minty-Browder monotone global diffeomorphism.

Remark 6.5. For the global injectivity of ∇g alone, in Example 6.4, we need neither the Minty-Browder monotonicity of ∇g nor the positive definiteness of $H(g)$, as the injectivities of its restrictions to the spheres centered at the origin and to the half lines starting from the origin are rather obvious. For example the injectivity of the restriction of ∇g to the half line $\{\lambda x |\ \lambda > 0\}$ generated by $x \neq 0$ reduces to the injectivity of the function

$$\varphi : (0, \infty) \longrightarrow \mathbb{R},\ \varphi(\lambda) = \frac{\|(\nabla g)_{\lambda x}\|}{\|x\|^2} = \lambda e^{\lambda^2 \|x\|^2}.$$

Note however that the outcome of the Minty-Browder monotonicity of ∇g along with the positive definiteness of $H(g)$, in Example 6.4, does not reduce to the global injectivity of ∇g alone, but also ensure the differentiability of its inverse.

Remark 6.6. Let $D \subseteq \mathbb{R}^n$ be a convex open set and $f, g : D \longrightarrow \mathbb{R}$ be C^2-smooth functions such that $\mathrm{Hess}^+(f) \neq \emptyset$. Then $(\nabla f + A) \circ \nabla f\big|_{\mathrm{Hess}^+(f)}$ is $\nabla f\big|_{\mathrm{Hess}^+(f)}$-increasing, where $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear automorphism related with f through the inequality (6.2). Indeed, $\nabla f + A$ is, according to Theorems 6.1 and 6.2, an h-monotone global injective operator. Therefore, according to Corollary 5.7 and the Remark 5.8 afterwards, we obtain:

$$\mathrm{Val}\,(\nabla f + A) \circ \nabla f\big|_{\mathrm{Hess}^+(f)} = \mathrm{Val}\left(\nabla f\big|_{\mathrm{Hess}^+(f)}\right).$$

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A relaxed version of the gradient projection method for variational inequalities with applications

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Dedicated to the memory of Professor Gábor Kassay.

Abstract. In this paper, we propose a relaxed version of the gradient projection method for strongly monotone variational inequalities defined on a level set of a (possibly non-differentiable) convex function. Our algorithm can be implemented easily since it computes on every iteration one projection onto some half-space containing the feasible set and only one value of the underlying mapping. Under mild and standard conditions we establish the strong convergence of the proposed algorithm. Numerical results and comparisons for the image deblurring problem show that our method can outperform related algorithms in the literature.

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Keywords: Variational inequality, gradient projection method, strong convergence, LASSO problem, image deblurring problem.

1. Introduction

The variational inequality problem (VIP) is to find a point $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0 \quad \forall y \in C, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ with the inner product $\langle ., . \rangle$ and its induced norm $\|.\|$, and $A : \mathcal{H} \to \mathcal{H}$ is a single-valued mapping. Let us denote by $\mathrm{Sol}(C, A)$ the solution set of the problem (1.1), i.e.,

$$\mathrm{Sol}(C, A) = \{x \in C : \langle Ax, y - x \rangle \geq 0 \quad \forall y \in C\}.$$

The variational inequality problem (VIP) has received much attention in the past several decades due to its applications in a large variety of problems arising in economics, optimization, transportation research, game theory, signal and image

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processing, data science, etc., see [1, 4, 8, 14, 15, 18, 19, 22, 20] and the references therein. There are many iterative methods for solving variational inequalities, most of which are based on projection methods. The simplest form is the gradient projection method [5] as follows:

$$\begin{cases} x^0 \in C, \\ x^{k+1} = P_C(x^k - \lambda A x^k), \ k \geq 0, \end{cases}$$

where P_C denotes the metric projection of $\mathcal{H}$ onto the set C, λ is a positive real number. The convergence of this method can be proved under a strong condition that the mapping A is strongly monotone and Lipschitz continuous. In order to relax the strong monotonicity assumption, Korpelevich [15] proposed the extragradient method which requires an additional projection at each iteration. Under the conditions that A is monotone and Lipschitz continuous, this method is shown to be weakly convergent in the setting of Hilbert spaces. Many researchers proposed improvements of the extragradient method, see, e.g., Censor et al. [4], He [6], Iusem-Svaiter [11], Khobotov [13], Malitsky and Semenov [22], Popov [23], Solodov and Svaiter [24], Tinti [25], Tseng [26], Malitsky [20], Maingé [18], Maingé and Gobinddass [19], Malitsky [21] and the references therein. In many real world applications, the feasible set is given in the form of $C = \{x \in \mathcal{H} : c(x) \leq 0\}$, where c is a convex function but not necessarily differentiable. For example, in LASSO problem, the function $c(x) = \|x\|_1 - \tau, \ \tau > 0$ satisfies the above requirement. Very recently, the authors in [2, 7, 9] used the subgradient extragradient method [4] and projection and contraction method [6] to propose relaxed projection algorithms for the variational inequality (1.1). However, the convergence of algorithms in [2, 9, 7] requires that c is a continuously differentiable convex function such that $c'(x)$ is Lipschitz continuous. This makes the real applications of their method very restrictive.

Our concern now is the following: *Can we design a new relaxed projection method to solve the variational inequality (*1.1*) efficiently without demanding differentiability of the convex function c?*

In this paper, we give a positive answer to this question. Motivated by the algorithms in [2, 7, 8, 9], we will introduce an efficient new algorithm for solving the VIP (1.1). The main feature of our method is that it requires only one value of the underlying mapping per iteration with no need for projections onto the feasible set. Theoretical analysis and experimental results show that our algorithm is more efficient than the previous ones for variational inequality problems.

The rest of the paper is organized as follows. After collecting some definitions and basic results in Section 2, we prove in Section 3 the strong convergence of the proposed algorithm. Finally, in Section 4 we provide some numerical results to illustrate the convergence of our algorithm and compare it with the previous algorithms.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space endowed with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$. When $\{x^k\}$ is a sequence in $\mathcal{H}$, we denote strong convergence of $\{x^k\}$ to $x \in \mathcal{H}$ by $x^k \to x$ and weak convergence by $x^k \rightharpoonup x$. For a given sequence $\{x^k\} \subset \mathcal{H}$, $\omega_w(x^k)$ denotes the weak ω-limit set of $\{x^k\}$, i.e.,

$$\omega_w(x^k) := \{x \in \mathcal{H} : x^{k_j} \rightharpoonup x \ \text{ for some subsequence } \{k_j\} \text{ of } \{k\}\}.$$

A useful and simple norm equality is the following

$$\begin{aligned} \|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 \\ - \gamma\beta\|y-z\|^2 - \alpha\gamma\|x-z\|^2, \end{aligned} \tag{2.1}$$

for all $x, y, z \in \mathcal{H}$ and $\alpha, \beta, \gamma \in [0,1]$ satisfying $\alpha + \beta + \gamma = 1$. Let C be a nonempty closed convex subset of $\mathcal{H}$. For every element $x \in \mathcal{H}$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$P_C x := \operatorname*{argmin}_{y \in C} ||x - y||.$$

P_C is called the metric projection of $\mathcal{H}$ onto C.

Lemma 2.1. *The metric projection P_C has the following basic properties:*

(1) $\langle x - P_C x, y - P_C x \rangle \leq 0$ *for all* $x \in \mathcal{H}$ *and* $y \in C$*;*

(2) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2$ *for all* $x \in \mathcal{H}$, $y \in C$*;*

(3) $\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle$ *for every* $x, y \in \mathcal{H}$*;*

(4) $\|P_C(x) - P_C(y)\| \leq \|x - y\|$ *for all* $x, y \in \mathcal{H}$*.*

We will focus on solving the problem (1.1) governed by Lipschitz continuous and strongly monotone A, i.e., there exist two positive constants L and η such that

$$\|Ax - Ay\| \leq L\|x - y\| \quad \forall x, y \in \mathcal{H},$$

and

$$\langle Ax - Ay, x - y \rangle \geq \eta\|x - y\|^2 \quad \forall x, y \in \mathcal{H},$$

respectively. In this case, we also say that A is L-Lipschitz continuous and η-strongly monotone.

Let $g : \mathcal{H} \to (-\infty, \infty]$, $\mathrm{dom} g := \{x \in \mathcal{H} : g(x) < +\infty\}$. We recall that the subdifferential of g at $x \in \mathcal{H}$ is defined as the set of all subgradients of g at x:

$$\partial g(x) := \{w \in \mathcal{H} : g(y) - g(x) \geq \langle w, y - x \rangle \quad \forall y \in \mathcal{H}\}. \tag{2.2}$$

g is strongly convex with constant $m > 0$ if and only if $g(x) - \frac{m}{2}\|x\|^2$ is convex. We already know that if g is lower semicontinuous convex at $x \in \mathrm{int}(\mathrm{dom} g)$, then $\partial g(x)$ is nonempty and bounded. The next lemmas are essential for our analysis in the sequel.

Lemma 2.2. (Cegielski and Zalas [3], Theorem 5) *Assume that A is a L-Lipschtz continuous and η-strongly monotone operator and μ is a constant such that $\mu \in \left(0, \dfrac{2\eta}{L^2}\right)$. Let $T^\mu = P_C(I - \mu A)$ (or $I - \mu A$), where I is the identity operator on $\mathcal{H}$. Then T^μ is a strict contraction with coefficient $1 - \tau$, where $\tau = \frac{1}{2}\mu(2\eta - \mu L^2)$.*

Lemma 2.3. (Maingé [16], Lemma 3.1; Xu [27], Lemma 2.5) *Let* $\{a_k\}, \{b_k\}$ *and* $\{c_k\}$ *be sequences of nonnegative real numbers such that*

$$a_{k+1} \leq (1-\delta_k)a_k + b_k + c_k, \quad k \geq 1,$$

where $\{\delta_k\}$ *is a sequence in* $(0,1)$ *and* $\{b_k\}$ *is a real sequence. Assume that*

$$\sum_{k=1}^{\infty} c_k < \infty.$$

Then the following results hold:

(1) *If* $b_k \leq \delta_k M$ *for some* $M \geq 0$ *and for all* $k \geq 1$ *then* $\{a_k\}$ *is a bounded sequence.*

(2) *If* $\sum_{k=1}^{\infty} \delta_k = \infty$ *and* $\limsup_{k\to\infty} b_k/\delta_k \leq 0$, *then* $\lim_{k\to\infty} a_k = 0$.

Lemma 2.4. (Maingé [17], Lemma 3.1) *Let* $\{\Gamma_n\}$ *be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence* $\{\Gamma_{n_j}\}$ *of* $\{\Gamma_n\}$ *such that* $\Gamma_{n_j} < \Gamma_{n_j+1}$ *for all* $j \geq 0$. *Also consider the sequence of integers* $\{\tau(n)\}_{n\geq n_0}$ *defined by*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ *is a nondecreasing sequence verifying* $\lim_{n\to\infty} \tau(n) = \infty$ *and, for all* $n \geq n_0$,

$$\max\{\Gamma_{\tau(n)}, \Gamma_n\} \leq \Gamma_{\tau(n)+1}.$$

3. A relaxed gradient projection algorithm

In this section, we consider VIP (1.1) in which C is given by

$$C = \{x \in \mathcal{H} : c(x) \leq 0\}.$$

where $c : \mathcal{H} \to \mathbb{R}$ is a convex and lower semicontinuous function.

We need the following basic assumptions for VIP (1.1):

(C1) $\mathrm{Sol}(C, A) \neq \emptyset$;

(C2) The mapping A is strongly monotone and L-Lipschitz continuous;

(C3) ∂c is a bounded operator (i.e., bounded on bounded sets).

3.1. The algorithm

The algorithm is designed as follows.

Algorithm 3.1 (Relaxed gradient projection algorithm)

Step 0 (Initialization): Select initial $x^0, x^1 \in C$, $\theta \in [0,1)$ and two positive real number sequences $\{\beta_k\}$, $\{\epsilon_k\}$ satisfying

$$\lim_{k\to\infty} \beta_k = 0, \quad \sum_{k=0}^{\infty} \beta_k = +\infty, \quad \epsilon_k = o(\beta_k), \tag{3.1}$$

where $\epsilon_k = o(\beta_k)$ means that the sequence $\{\epsilon_k\}$ is an infinitesimal of higher order than $\{\beta_k\}$. Set $k := 1$.

Step 1: Given x^{k-1} and x^k $(k \geq 1)$, choose α_k such that

$$\alpha_k = \begin{cases} \min\left\{\theta, \dfrac{\epsilon_k}{\|x^k - x^{k-1}\|}\right\} & \text{if } x^k \neq x^{k-1}, \\ \theta & \text{otherwise.} \end{cases} \tag{3.2}$$

Compute $w^k = x^k + \alpha_k(x^k - x^{k-1})$ and take $\xi^k \in \partial c(w^k)$. Construct the half-space

$$C_k = \{x \in \mathcal{H} : c(w^k) + \langle \xi^k, x - w^k \rangle \leq 0\},$$

and calculate

$$x^{k+1} = P_{C_k}(w^k - \beta_k A w^k). \tag{3.3}$$

Step 2: If $x^{k+1} = w^k$ then stop. Otherwise set $k := k+1$ and return to **Step 1**.

Remark 3.1. We have $C \subseteq C_k$ for every $k \geq 0$. Indeed, we obtain by (2.2) and $\xi^k \in \partial c(w^k)$ that

$$c(x) - c(w^k) \geq \langle \xi^k, x - w^k \rangle \quad \forall x \in \mathcal{H}.$$

If $x \in C$ then we get $c(w^k) + \langle \xi^k, x - w^k \rangle \leq 0$, i.e., $x \in C_k$. Hence, the statement is true.

3.2. Convergence analysis

We first show that the stopping criterion Algorithm 3.1 is valid.

Lemma 3.2. *If $w^k = x^{k+1}$ then $w^k \in \mathrm{Sol}(C, A)$.*

Proof. If $w^k = x^{k+1}$ then by (3.3) and Lemma 2.1 (1), we have

$$\langle w^k - \lambda_k A w^k - w^k, y - w^k \rangle \leq 0 \quad \forall y \in C_k,$$

or equivalently,

$$\langle A w^k, y - w^k \rangle \geq 0 \quad \forall y \in C_k.$$

Therefore, we get

$$\langle A w^k, y - w^k \rangle \geq 0 \quad \forall y \in C.$$

Hence $w^k \in \mathrm{Sol}(C, A)$. □

A key lemma for our convergence theorem is presented next.

Lemma 3.3. *Assume that the conditions (C1)-(C3) hold. Then the sequence $\{x^k\}$ generated by Algorithm* 3.1 *is bounded.*

Proof. We have

$$\begin{aligned}\|x^{k+1}-z\| &= \|P_{C_k}(w^k-\beta_k Aw^k)-z\| \\ &\le \|(I-\beta_k A)w^k-(I-\beta_k A)z-\beta_k Az\| \\ &= (1-\gamma_k)\|w^k-z\|+\beta_k\|Az\|. \end{aligned} \tag{3.4}$$

Moreover, we have

$$\begin{aligned}\|w^k-z\| &= \|x^k-z+\alpha_k(x^k-x^{k-1})\| \\ &\le \|x^k-z\|+\alpha_k\|x^k-x^{k-1}\|. \end{aligned} \tag{3.5}$$

Combining (3.5) and (3.4), we immediately get

$$\|x^{k+1}-z\| \le (1-\gamma_k)\|x^k-z\|+(1-\gamma_k)\alpha_k\|x^k-x^{k-1}\|+\beta_k\|Az\|.$$

By (3.1) and (3.2), we see that

$$\begin{aligned}\lim_{k\to\infty}\frac{b_k}{\gamma_k} &= \lim_{k\to\infty}\frac{(1-\gamma_k)\alpha_k\|x^k-x^{k-1}\|+\beta_k\|Az\|}{\gamma_k}, \\ &= \lim_{k\to\infty}\left[\frac{2(1-\gamma_k)}{2\eta-\beta_k L^2}\frac{\alpha_k}{\beta_k}\|x^k-x^{k-1}\|+\frac{2}{2\eta-\beta_k L^2}\|Az\|\right] = \frac{\|Az\|}{\eta},\end{aligned}$$

where $b_k=(1-\gamma_k)\alpha_k\|x^k-x^{k-1}\|+\beta_k\|Az\|$.
This implies that the sequence $\left\{\frac{b_k}{\gamma_k}\right\}$ is bounded. Using Lemma 2.3 (1), we conclude that the sequence $\{\|x^k-z\|\}$ is bounded. This shows that the sequence $\{x^k\}$ is bounded and so is $\{w^k\}$. □

Lemma 3.4. *Assume that the conditions (C1)-(C3) hold and let $\{x^k\}$ be the sequence generated by Algorithm* 3.1. *Then, for each $z\in C$, we have*

$$\begin{aligned}\|x^{k+1}-z\|^2 \le (1-\gamma_k)(\|x^k-z\|^2+2\alpha_k\|x^k-x^{k-1}\|\|x^k-z\|+\alpha_k^2\|x^k-x^{k-1}\|^2) \\ +\gamma_k\left[\frac{-4}{2\eta-\beta_k L^2}\langle Az,w^k-z\rangle+\frac{4\beta_k}{2\eta-\beta_k L^2}\|Az\|\|Aw^k\|\right].\end{aligned}$$

Proof. Let $\gamma_k=\frac{1}{2}\beta_k(2\eta-\beta_k L^2)$. Since $\beta_k\to 0$ as $k\to\infty$, there exists some positive integer k_0 such that

$$0<\beta_k<\frac{\eta}{L^2} \tag{3.6}$$

for all $k\ge k_0$. In view of Lemma 2.6, we obtain from (3.6) that $P_{C_k}(I-\beta_k A)$ (so is $I-\beta_k A$) is a strict contraction with coefficient $1-\gamma_k$ for all $k\ge k_0$. For each $z\in C$,

we have

$$\begin{aligned}
\|x^{k+1}-z\|^2 &= \|P_{C_k}(w^k-\beta_k Aw^k)-z\|^2\\
&\le \|(I-\beta_k A)w^k-(I-\beta_k A)z-\beta_k Az\|^2\\
&= (1-\gamma_k)\|w^k-z\|^2-2\beta_k\langle Az, w^k-z-\beta_k Aw^k\rangle\\
&\le (1-\gamma_k)\|w^k-z\|^2-2\beta_k\langle Az, w^k-z\rangle+2\beta_k^2\|Az\|\|Aw^k\|\\
&= (1-\gamma_k)\|w^k-z\|^2+\gamma_k\left[\frac{-4}{2\eta-\beta_k L^2}\langle Az, w^k-z\rangle\right.\\
&\qquad\left.+\frac{4\beta_k}{2\eta-\beta_k L^2}\|Az\|\|Aw^k\|\right].
\end{aligned}$$

Using (3.5) we arrive at

$$\begin{aligned}
\|x^{k+1}-z\|^2 &\le (1-\gamma_k)(\|x^k-z\|+\alpha_k\|x^k-x^{k-1}\|)^2\\
&\qquad+\gamma_k\left[\frac{-4}{2\eta-\beta_k L^2}\langle Az, w^k-z\rangle+\frac{4\beta_k}{2\eta-\beta_k L^2}\|Az\|\|Aw^k\|\right]\\
&= (1-\gamma_k)(\|x^k-z\|^2+2\alpha_k\|x^k-x^{k-1}\|\|x^k-z\|+\alpha_k^2\|x^k-x^{k-1}\|^2)\\
&\qquad+\gamma_k\left[\frac{-4}{2\eta-\beta_k L^2}\langle Az, w^k-z\rangle+\frac{4\beta_k}{2\eta-\beta_k L^2}\|Az\|\|Aw^k\|\right].
\end{aligned}$$

Therefore, the proof is complete. □

We are now in a position to establish the strong convergence theorem of Algorithm 3.1.

Theorem 3.5. *Assume that the conditions (C1)-(C3) hold. Then any sequence* $\{x^k\}$ *generated by Algorithm* 3.1 *converges strongly to the unique solution of the variational inequality problem* (1.1)*.*

Proof. For each $z \in C$, using the nonexpansive property of projection operators, we have

$$\begin{aligned}
\|x^{k+1}-z\|^2 &= \|P_{C_k}(w^k-\beta_k Aw^k)-P_{C_k}w^k+P_{C_k}w^k-P_{C_k}z\|^2\\
&= \|P_{C_k}w^k-P_{C_k}z\|^2+2\beta_k\|w^k-z\|\|Aw^k\|+\beta_k^2\|Aw^k\|^2\\
&\le \|w^k-z\|^2-\|w^k-P_{C_k}w^k\|^2+2\beta_k\|w^k-z\|\|Aw^k\|+\beta_k^2\|Aw^k\|^2\\
&= \|w^k-z\|^2-\|w^k-P_{C_k}w^k\|^2+\beta_k M, \qquad (3.7)
\end{aligned}$$

where $M \ge \sup_k\{2\|w^k-z\|\|Aw^k\|+\beta_k\|Aw^k\|^2\}$.
On the other hand, by applying (2.1) we get

$$\begin{aligned}
\|w^k-z\|^2 &= \|(1+\alpha_k)(x^k-z)-\alpha_k(x^{k-1}-z)\|^2\\
&= (1+\alpha_k)\|x^k-z\|^2-\alpha_k\|x^{k-1}-z\|^2+\alpha_k(1+\alpha_k)\|x^k-x^{k-1}\|^2\\
&\le (1+\alpha_k)\|x^k-z\|^2-\alpha_k\|x^{k-1}-z\|^2+2\alpha_k\|x^k-x^{k-1}\|^2. \qquad (3.8)
\end{aligned}$$

Combining (3.7) and (3.8) we have

$$\|x^{k+1}-z\|^2 \le (1+\alpha_k)\|x^k-z\|^2 - \alpha_k\|x^{k-1}-z\|^2 + 2\alpha_k\|x^k-x^{k-1}\|^2 - \|w^k - P_{C_k}w^k\|^2 + \beta_k M.$$

Putting $\Gamma_k := \|x^k - z\|^2$ for all $k \in \mathbb{N}$ we have

$$\|w^k - P_{C_k}w^k\|^2 \le \Gamma_k - \Gamma_{k+1} + \alpha_k(\Gamma_k - \Gamma_{k-1}) + 2\alpha_k\|x^k - x^{k-1}\|^2 + \beta_k M. \quad (3.9)$$

Now, we consider two possible cases:

Case 1. Assume that there exists $k_0 \ge 0$ such that for each $k \ge k_0$, $\Gamma_{k+1} \le \Gamma_k$. In this case, $\lim_{k\to\infty} \Gamma_k$ exists and $\lim_{k\to\infty}(\Gamma_k - \Gamma_{k+1}) = 0$.
Since $\lim_{k\to\infty} \beta_k = 0$ and $\lim_{k\to\infty} \alpha_k\|x^k - x^{k-1}\|^2 = 0$, it follows from (3.9) that

$$\lim_{k\to\infty} \|w^k - P_{C_k}w^k\|^2 = 0. \quad (3.10)$$

We now show that $\omega_w(x^k) \subset C$. Let $\bar{x} \in \omega_w(x^k)$ be an arbitrary element. Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_l}\}$ that converges weakly to $\bar{x} \in C_k$. Note that

$$\lim_{k\to\infty} \|w^k - x^k\| = \lim_{k\to\infty} \alpha_k\|x^k - x^{k-1}\| = 0. \quad (3.11)$$

It follows from (3.11) that $\{w^{k_l}\}$ also converges weakly to $\bar{x}$. Next we verify that $\bar{x} \in C$.
Due to $P_{C_{k_l}}w^{k_l} \in C_{k_l}$, it follows from the definition of C_{k_l} that

$$c(w^{k_l}) + \langle \xi^{k_l}, P_{C_{k_l}}w^{k_l} - w^{k_l}\rangle \le 0,$$

where $\xi^{k_l} \in \partial c(w^{k_l})$. The use of the Cauchy-Schwart inquality implies that

$$c(w^{k_l}) \le \|\xi^{k_l}\|\big\|P_{C_{k_l}}w^{k_l} - w^{k_l}\big\|. \quad (3.12)$$

From the boundedness assumption of ξ^{k_l} and (3.10), (3.12), we have

$$c(w^{k_l}) \le \|\xi^{k_l}\|\big\|P_{C_{k_l}}w^{k_l} - w^{k_l}\big\| \to 0. \quad (3.13)$$

From the weak lower-semicontinuity of the convex function $c(x)$ and since $w^{k_l} \rightharpoonup \bar{x}$, it follows from (3.13) that

$$c(\bar{x}) \le \liminf_{l\to\infty} c(w^{k_l}) \le 0,$$

which means that $\bar{x} \in C$.
Using Lemma 3.4 we have

$$\|x^{k+1}-z\|^2 \le (1-\gamma_k)(\|x^k-z\|^2 + 2\alpha_k\|x^k-x^{k-1}\|\|x^k-z\| + \alpha_k^2\|x^k-x^{k-1}\|^2) + \gamma_k\left[\frac{-4}{2\eta-\beta_k L^2}\langle Az, w^k - z\rangle + \frac{4\beta_k}{2\eta-\beta_k L^2}\|Az\|\|Aw^k\|\right]. \quad (3.14)$$

Besides, we obtain

$$\begin{aligned}\|x^k - z\|^2 + 2\alpha_k\|x^k - z\|\left\|x^k - x^{k-1}\right\| + \alpha_k^2\|x^k - x^{k-1}\|^2 \\ \leq \|x^k - z\|^2 + 2\alpha_k\|x^k - z\|\left\|x^k - x^{k-1}\right\| + \alpha_k\|x^k - x^{k-1}\|^2 \\ \leq \|x^k - z\|^2 + 3M_1\alpha_k\|x^k - x^{k-1}\|, \end{aligned} \tag{3.15}$$

where $M_1 = \sup_{k\in\mathbb{N}}\{\|x^k - z\|, \|x^k - x^{k-1}\|\}$.
Combining (3.14) and (3.15) we get

$$\begin{aligned}\|x^{k+1} - z\|^2 &\leq (1-\gamma_k)\|x^k - z\|^2 + 3M_1(1-\gamma_k)\alpha_k\|x^k - x^{k-1}\| \\ &\quad + \gamma_k\left[\frac{-4}{2\eta - \beta_k L^2}\langle Az, w^k - z\rangle + \frac{4\beta_k}{2\eta - \beta_k L^2}\|Az\|\|Aw^k\|\right] \\ &\leq (1-\gamma_k)\|x^k - z\|^2 + \gamma_k\left[3M_1(1-\gamma_k)\frac{\alpha_k}{\gamma_k}\|x^k - x^{k-1}\|\right. \\ &\quad \left. + \frac{-4}{2\eta - \beta_k L^2}\langle Az, w^k - z\rangle + \frac{4\beta_k}{2\eta - \beta_k L^2}\|Az\|\|Aw^k\|\right]. \end{aligned} \tag{3.16}$$

It is easy to see that

$$\lim_{k\to\infty}\left[(1-\gamma_k)\frac{\alpha_k}{\gamma_k}\|x^k - x^{k-1}\| + \frac{4\beta_k}{2\eta - \beta_k L^2}\|Az\|\|Aw^k\|\right] = 0. \tag{3.17}$$

To apply Lemma 2.3, it remains to show that $\limsup_{k\to\infty}\langle Az, w^k - z\rangle \geq 0$. Indeed, since $z \in \mathrm{Sol}(C, A)$, we get that

$$\limsup_{k\to\infty}\langle Az, w^k - z\rangle = \max_{\hat{z}\in\omega_w(\{w^k\})}\langle Az, \hat{z} - z\rangle \geq 0.$$

By applying Lemma 2.3 to (3.16) with the data

$$\begin{aligned} a_k &:= \|x^k - z\|^2, \quad \delta_k := \gamma_k, \quad c_k := 0, \\ b_k &:= 3M_1(1-\gamma_k)\frac{\alpha_k}{\gamma_k}\|x^k - x^{k-1}\| + \frac{-4}{2\eta - \beta_k L^2}\langle Az, w^k - z\rangle \\ &\qquad + \frac{4\beta_k}{2\eta - \beta_k L^2}\|Az\|\|Aw^k\| \end{aligned}$$

we immediately deduce that the sequence $\{x^k\}$ converges strongly to $z \in \mathrm{Sol}(C, A)$.

Case 2. Assume that there exists a subsequence $\{\Gamma_{k_m}\} \subset \{\Gamma_k\}$ such that $\Gamma_{k_m} \leq \Gamma_{k_m+1}$ for all $m \in \mathbb{N}$. In this case, we can define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(k) = \max\{n \leq k : \Gamma_n < \Gamma_{n+1}\}.$$

Then we have from Lemma 2.4 that $\tau(k) \to \infty$ as $k \to \infty$ and $\Gamma_{\tau(k)} < \Gamma_{\tau(k)+1}$. So, we have from (3.9) that

$$\begin{aligned}\|w^{\tau(k)} - P_{C_{\tau(k)}} w^{\tau(k)}\|^2 &\le \Gamma_{\tau(k)} - \Gamma_{\tau(k)+1} + \alpha_{\tau(k)}(\Gamma_{\tau(k)} - \Gamma_{\tau(k)-1}) \\ &\qquad + 2\alpha_{\tau(k)}\|x^{\tau(k)} - x^{\tau(k)-1}\|^2 + \beta_{\tau(k)}M \\ &\le \alpha_{\tau(k)}\|x^{\tau(k)} - x^{\tau(k)-1}\|\left(\sqrt{\Gamma_{\tau(k)}} + \sqrt{\Gamma_{\tau(k)-1}}\right) \\ &\qquad + 2\alpha_{\tau(k)}\|x^{\tau(k)} - x^{\tau(k)-1}\|^2 + \beta_{\tau(k)}M \\ &\to 0. \end{aligned} \tag{3.18}$$

Following the same lines as in the proof of Case 1, we get from (3.18) that

$$\begin{aligned}&\lim_{k\to\infty} \|w^{\tau(k)} - P_{C_{\tau(k)}} w^{\tau(k)}\|^2 = 0, \\ &\limsup_{k\to\infty} \langle Az, w^{\tau(k)} - z\rangle = \max_{\hat{z}\in\omega_w(\{w^{\tau(k)}\})} \langle Az, \hat{z} - z\rangle \ge 0 \end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\|x^{\tau(k)+1} - z\|^2 &\le (1-\gamma_{\tau(k)})\|x^{\tau(k)} - z\|^2 \\ &\quad + \gamma_{\tau(k)}\Bigg[3M_1(1-\gamma_{\tau(k)})\frac{\alpha_{\tau(k)}}{\gamma_{\tau(k)}}\|x^{\tau(k)} - x^{\tau(k)-1}\| \\ &\quad + \frac{-4}{2\eta - \beta_{\tau(k)}L^2}\langle Az, w^{\tau(k)} - z\rangle + \frac{4\beta_{\tau(k)}}{2\eta - \beta_{\tau(k)}L^2}\|Az\|\|Aw^{\tau(k)}\|\Bigg]. \end{aligned} \tag{3.20}$$

Since $\Gamma_{\tau(k)} < \Gamma_{\tau(k)+1}$, we have from (3.20) that

$$\begin{aligned}\|x^{\tau(k)} - z\|^2 &\le 3M_1(1-\gamma_{\tau(k)})\frac{\alpha_{\tau(k)}}{\gamma_{\tau(k)}}\|x^{\tau(k)} - x^{\tau(k)-1}\| \\ &\quad + \frac{-4}{2\eta - \beta_{\tau(k)}L^2}\langle Az, w^{\tau(k)} - z\rangle + \frac{4\beta_{\tau(k)}}{2\eta - \beta_{\tau(k)}L^2}\|Az\|\|Aw^{\tau(k)}\|. \end{aligned} \tag{3.21}$$

Combining (3.17), (3.19) and (3.21) yields

$$\limsup_{k\to\infty} \|x^{\tau(k)} - z\|^2 \le 0,$$

and hence

$$\lim_{k\to\infty} \|x^{\tau(k)} - z\|^2 = 0.$$

From (3.20), we have

$$\limsup_{k\to\infty} \|x^{\tau(k)+1} - z\|^2 \le \limsup_{k\to\infty} \|x^{\tau(k)} - z\|^2.$$

Thus

$$\lim_{k\to\infty} \|x^{\tau(k)+1} - z\|^2 = 0.$$

Therefore, by Lemma 2.4, we obtain

$$0 \le \|x^k - z\| \le \max\{\|x^{\tau(k)} - z\|, \|x^k - z\|\} \le \|x^{\tau(k)+1} - z\| \to 0.$$

Consequently, $\{x^k\}$ converges strongly to $z \in \mathrm{Sol}(C, A)$. □

4. Numerical results

Example 4.1. Image restoration problems can be formulated as an inverse problem as follows:

$$y = Ax + v, \tag{4.1}$$

where $A \in \mathbb{R}^{m\times n}$ represents a known blurring operator (which is called the point spread function: PSF), $y \in \mathbb{R}^{m\times 1}$ represents the blurred image, and $v \in \mathbb{R}^{m\times 1}$ stands for the additive noises or perturbation signals, $x \in \mathbb{R}^{n\times 1}$ is the unknown original image whose size is assumed to be the same as that of y (that is, $m = n$). In most cases, this problem is ill-posed, hence directly inverting A would lead to bad and possibly multiple solutions. To overcome this difficulty, a popular strategy is to use a regularization based method, which provides the prior knowledge of images that one wants to reconstruct. In this paper, the problem (4.1) is approximately solved by the following optimization model:

$$\begin{aligned} \min_{x\in\mathbb{R}^{n^2}} f(x) :=& \frac{1}{2}\|Ax - y\|^2 + \frac{1}{2}\alpha\|x\|^2, \\ \text{s.t. } & \|x\|_1 \le t, \end{aligned} \tag{4.2}$$

where α is a positive parameter, and $\|.\|_1$ is the ℓ_1-norm, which is to make small component of x to become zero. The objective function of the problem (4.2) is strongly convex. Note that, the objective f is strongly convex and differentiable with the gradient given by

$$\nabla f(x) = A^*(Ax - y) + \alpha x,$$

where A^* is the adjoint of A.
We observe that the gradient ∇f is $(\|A\|^2 + \alpha)$-Lipschitz continuous and α-strongly monotone. We already know that x^* solves (4.2) if and only if x^* solves the variational inequality problem of finding $x \in C$ such that

$$\langle \nabla f(x), y - x\rangle \ge 0 \quad \forall y \in C,$$

where $C := \{x \in \mathbb{R}^{n^2} : \|x\|_1 \le t\}$.

The quality of the restoration is measured by the peak signal-to-noise ratio (PSNR) in decibel (dB):

$$\mathrm{PSNR}(x) = 20\log_{10}\frac{x_{\max}}{\sqrt{\mathrm{Var}(x,\bar{x})}},$$

where

$$\mathrm{Var}(x,\bar{x}) = \frac{\sum_{j=1}^{n^2}[\bar{x}(j) - x(j)]^2}{n^2},$$

and $\bar{x}$ is the true image and $x_{\max}$ is the maximum possible pixel value of the image.

FIGURE 1. Cameraman original and blurred and noisy images on top; Lena original and blurred and noisy images below.

All the codes were written in Matlab (R2016a) and run on PC with Intel(R) Core(TM) i3-370M Processor 2.40 GHz. In the numerical results reported in the following tables, 'Iter.' and 'Sec.' stand for the number of iterations and the CPU time in seconds, respectively. We now apply our proposed algorithm - Algorithm 3.1 (IGPM) and the strongly convergent algorithms in the literature including Algorithm 1 of Hieu and Thong [10] (VPRGM), Algorithm 3.1 of Khanh and Vuong [12] (GPM), and the golden ratio algorithm of Malitsky [21] (GRA) with diminishing step sizes to recover the blurred Lena and Cameraman images. The size of the image is $m = n = 256$. The original and the blurred images are shown in Figure 1. For all tested algorithms, we use the same starting points $x^0 = x^1 = \mathbf{0}$ ($\mathbf{0}$ is a vector in $\mathbb{R}^{n^2}$ in which all components are zero) and limit the number of iterations by 2500 for all algorithms as well. Moreover, we set $A = RW$, where R is the blur matrix and W denotes the inverse wavelet transform. The blur kernel is taken to be $h_{ij} = \frac{1}{1+i^2+j^2}$, for $i, j = -4, ..., 4$. An additive zero-mean white Gaussian noise with standard deviation 10^{-3} was added to the images.

Moreover, for Algorithm 3.1 (IGPM), we take $\epsilon_k = \frac{1}{k^{1.1}}$, $\theta = 0.6$; α_k is computed by (3.2).

We take the same stepsizes $\lambda_k = \frac{1}{k^{0.3}}$, the regularization parameter $\alpha = 2e{-}5$ for all algorithms. Besides, we choose $\theta_k = 1$ for VPRGM of [10]. The comparison of four algorithms with Cameraman and Lena images are reported in Table 1 and Table 2, respectively. The reconstructed images are presented in Figures 2, 4. The convergence behaviour of algorithms is given in Figures 3, 5. In these figures, the value of PSNR for all algorithms is represented by the y-axis, the running time is represented by the x-axis.

	Sec.	Iter	PSNR
GRA	112.9531	2500	28.4390
VPRGM	113.5313	2500	31.8681
GPM	113.6406	2500	31.8692
Our algorithm (IGPM)	**83.6**	2500	**37.0024**

TABLE 1. Comparison of four algorithms for reconstructing the blurred Cameraman image.

	Sec.	Iter	PSNR
GRA	112.9531	2500	31.9244
VPRGM	117.6094	2500	35.2395
GPM	117.0469	2500	35.7691
Our algorithm (IGPM)	**94.2031**	2500	**44.5633**

TABLE 2. Comparison of four algorithms for reconstructing the blurred Lena image.

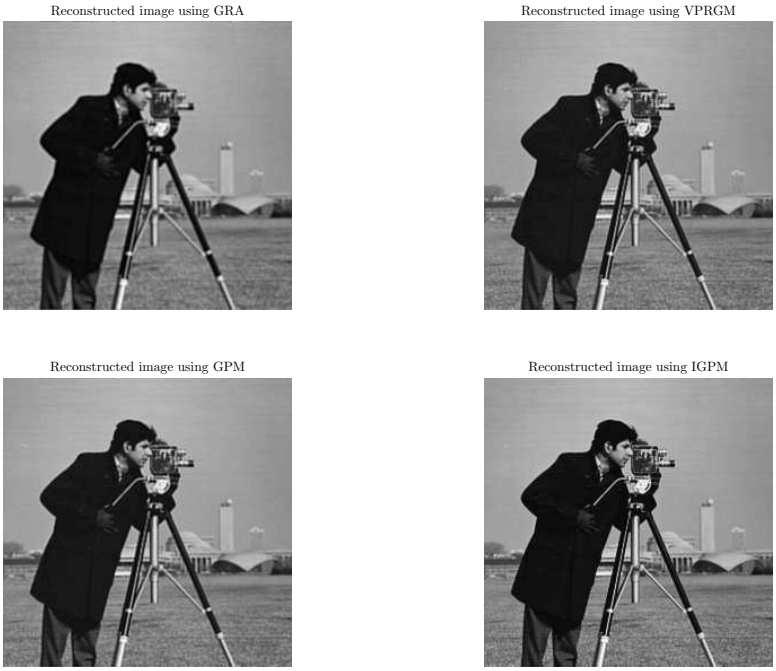

FIGURE 2. The reconstructed images with the Cameraman image

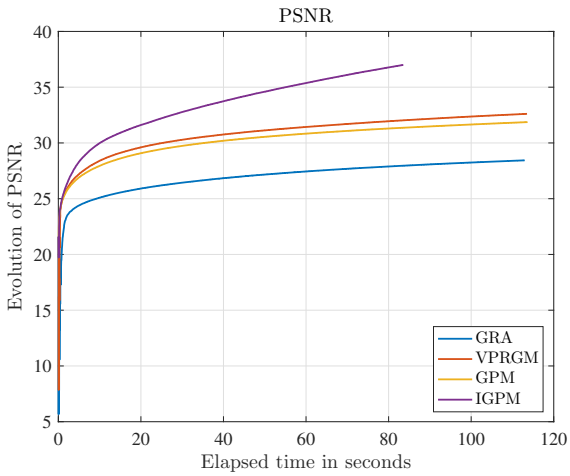

FIGURE 3. Evolution of PSNR with the Cameraman image

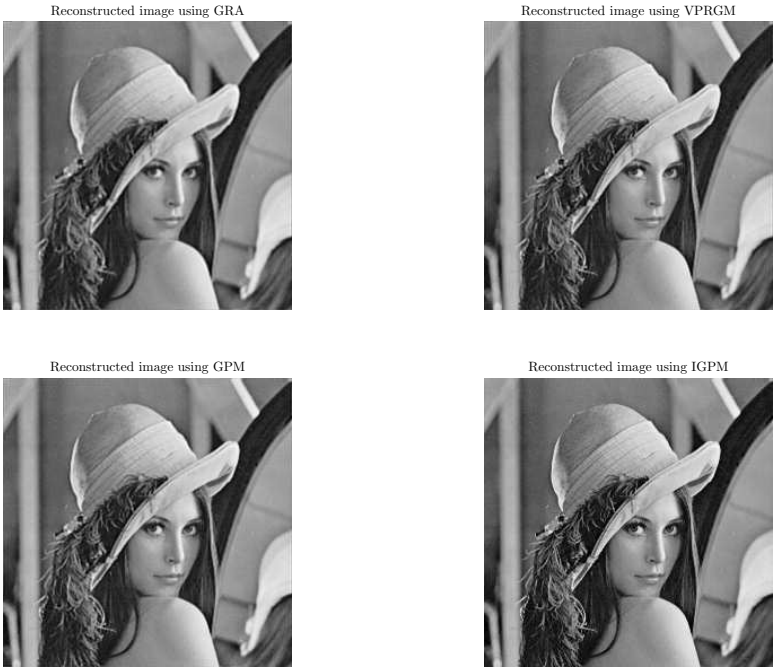

FIGURE 4. The reconstructed images with the Lena image

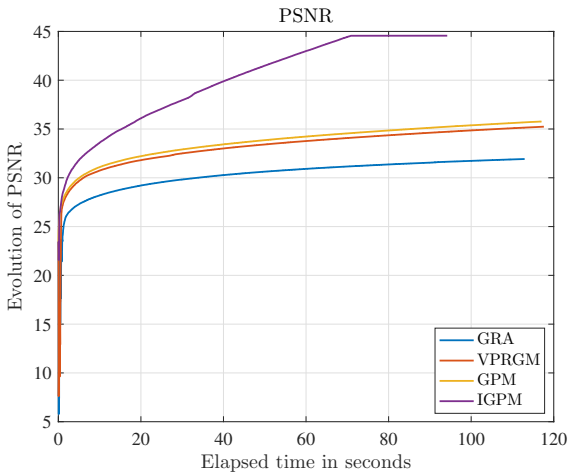

FIGURE 5. Evolution of PSNR with the Lena image

Figures 3, 5 clearly demonstrate that IGPM gives lower running time compared to others. Clearly, our method provides clearer images and improved PSNR values. We emphasize here that these numerical results are very preliminary.

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Well-posedness for set-valued equilibrium problems

Mihaela Miholca

Dedicated to the memory of Professor Gábor Kassay.

Abstract. In this paper we extend a concept of well-posedness for vector equilibrium problems to the more general framework of set-valued equilibrium problems in topological vector spaces using an appropriate reformulation of the concept of minimality for sets. Sufficient conditions for well-posedness are given in the generalized convex settings and we are able to single out classes of well-posed set-valued equilibrium problems.
On the other hand, in order to relax some conditions, we introduce a concept of minimizing sequences for a set-valued problem, in the set criterion sense, and further we will have a concept of well-posedness for the set-valued equilibrium problem we are interested in. Sufficient results are also given for this well-posedness concept.

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Keywords: Set-valued equilibrium problems, well-posedness, maximizing sequences, minimizing sequences.

1. Introduction

In the last few years, set-valued optimization problems have received much attention by many authors due to their extensive applications in many fields such as optimal control, economics, game theory, multiobjective optimization and so on(see, e.g., [1], [2], [8] and the references therein). For some motivating examples one may refer also to the book by Khan et al.[11].
Approaches in set-valued optimization can be made using two types of criteria of solutions: the vector criterion and the set optimization criterion. The first criterion is equivalent to finding efficient solutions of the image set but this criterion is not always suitable for all types of set-valued optimization problems.

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Kuroiwa [14] introduced an alternative criterion of solutions for set-valued optimization problems, called the set optimization criterion, which is based on a comparison among the values of the objective set-valued map.
On the other hand, well-posedness plays a crucial role in the stability theory for optimization problems. The classical notion of well-posednesss for a scalar optimization problem was first introduced by Tykhonov [19] and is known as Tykhonov well-posedness. In the literature, various notions of well-posedness for vector optimization problems have been introduced and studied(see, e.g., [4], [5], [9], [12], [16] and the references therein).
Apart from its theoretical interest, important problems arising from economics, mechanics, electricity, chemistry and other practical sciences motivate the study of equilibrium problems. Recently, equilibrium problems for vector mappings have been considered by many authors. For a nice survey, we refer to the research monograph devoted to the analysis of equilibrium problems in pure and applied nonlinear analysis and mathematical economics by Kassay et al.[10].
Some concepts of well-posedness for the strong vector equilibrium problem in topological vector spaces were introduced and studied by Bianchi et al.[3]. Also, they gave sufficient conditions, in concave settings, in order to guarantee the well-posedness.
Inspired by the work of Bianchi et al.[3], in this paper we study the well-posedness of a set-valued equilibrium problem in topological vector spaces. We consider and study two notions of well-posedness; the first one generalizes the concept of well-posedness of strong vector equilibrium problem introduced by Bianchi et al.[3] and the second one is linked to the behaviour of a suitable set-valued problem.
The first concept of well-posedness for our set-valued equilibrium problem is also named M-well-posedness like in vectorial case and we are able to give sufficient conditions for M-well-posedness in generalized convex settings assuming alternative conditions only on a suitable set-valued map.
In order to drop some assumptions, we consider a concept of well-posedness for a suitable set-valued map with respect to a quasi-order relation, strongly related to a concept of well-posedness of our set-valued equilibrium problem. Some sufficient conditions concerning this kind of well-posedness are also established.
The paper in four sections is organized as follows. Section 2 presents the preliminaries required throughout the paper. Section 3 generalizes the concept of well-posedness of the strong vector equilibrium problem to a set-valued equilibrium problem and establishes some sufficient conditions for well-posedness in finite and infinite dimensional settings pointing out classes of well-posed set-valued equilibrium problems. Section 4 introduces a new concept of well-posedness for our set-valued equilibrium problem under weaker assumptions than those in Section 3. Some sufficient results for well-posedness are also obtained in infinite dimensional settings. For a clear understanding of the concepts and to illustrate our results, we give also some examples.

2. Preliminaries

Let X and Y be topological vector spaces with countable local bases. Let $\mathcal{P}(Y)$ be the collection of all nonempty subsets of Y and K be a proper nonempty closed convex

pointed cone in the real topological vector space Y. For $A \in \mathcal{P}(Y)$ we denote the topological interior, the topological closure, the topological boundary and complement of A by $intA$, clA, ∂A and A^c, respectively.
We consider also a preference relation on $\mathcal{P}(Y)$ introduced by Kuroiwa [14]: the lower set less quasi-order relation induced by the cone K. Also, we denote by $K_0 = K \setminus \{0\}$. For $A, B \in \mathcal{P}(Y)$

$$A \preceq_K B \Leftrightarrow B \subseteq A + K.$$

We now consider S a nonempty proper subset of Y. A preference relation based on the solution concept equipped with the set S was proposed by Flores-Bazán et al.[6]. For $a, b \in S$,

$$a \preceq_S b \Longleftrightarrow a - b \in S.$$

Khushboo et al.[13] reformulate a notion of minimality for a set $A \in \mathcal{P}(Y)$ considered for vector optimization problems by Flores-Bazan et al.[6]. An element $\overline{a} \in A$ is said to be an S-minimal point of A if

$$a \not\preceq_S \overline{a}, \text{ for all } a \in A \setminus \{\overline{a}\},$$

or, equivalently,

$$A \setminus \{\overline{a}\} \subseteq \overline{a} + S^c.$$

We denote the set of S-minimal points of A by $E_S(A)$.
It is obvious that if $0 \in S^c$ then

$$\overline{a} \in E_S(A) \Longleftrightarrow \overline{a} \in A \text{ and } A \subseteq \overline{a} + S^c. \tag{2.1}$$

It is well-known that vector equilibrium problems are natural extensions of several problems of practical interest like vector optimization and vector variational inequality problems. In the literature, there are some kinds of extensions of scalar equilibrium problems to the vector equilibrium problems. Further, vector equilibrium problems are extended to set-valued equilibrium problems in several manners.
In this paper we consider the set-valued equilibrium problem (SEP) which consists in finding $\overline{x} \in D$ such that

$$f(\overline{x}, y) \subseteq (-K_0)^c \text{ for all } y \in D,$$

where $D \subseteq X, f : D \times D \rightrightarrows Y$. This problem generalizes, in a certain sense, the strong vector equilibrium problem considered by Bianchi et al.[3]
We denote by S_0 the solution set of the problem (SEP) and we will suppose in the sequel that S_0 is nonempty.
Our purpose is to try to assign reasonable definitions of well-posedness for (SEP) that recover some previous existing concepts in vector criterion, see Bianchi et al.[3]. In order to start our approach, we introduce the set-valued map $\varphi : D \rightrightarrows Y$ given by

$$\varphi(x) = E_{-K_0}(f(x, D)).$$

The map φ generalizes the definition of the function ϕ in Bianchi et al.[3]; indeed, taking into account (2.1) we have that

$$z \in \varphi(x) \Leftrightarrow z \in f(x, D) \text{ and } f(x, D) \subseteq z + (-K_0)^c$$
$$\Leftrightarrow z \in f(x, D) \text{ and } (f(x, D) - z) \cap (-K) = \{0\}.$$

Throughout the paper is assumed that $\varphi(x) \neq \emptyset$ for every $x \in D$. The domain of φ, denoted by domφ, is defined as dom$\varphi := \{x \in D : \varphi(x) \neq \emptyset\}$ and therefore dom$\varphi = D$.
In the sequel, we shall denote by $\mathcal{V}_X(x_0)$ a neighbourhood base of x_0 in the topological space X. The same notation will be used for other spaces.
We now recall some notions of continuity for set-valued maps. Let $\varphi : D \rightrightarrows Y$ be a set-valued map.

Definition 2.1. [11] The map φ is said to be

(i) upper semicontinuous at $x_0 \in D$ if for every $W \subseteq Y, W$ open, $\varphi(x_0) \subseteq W$, there exists a neighbourhood $U \in \mathcal{V}_X(x_0)$ such that $\varphi(x) \subseteq W$ for every $x \in U \cap D$.
(ii) lower semicontinuous at $x_0 \in D$ if for every $W \subseteq Y, W$ open, $\varphi(x_0) \cap W \neq \emptyset$, there exists a neighbourhood $U \in \mathcal{V}_X(x_0)$ such that $\varphi(x) \cap W \neq \emptyset$ for every $x \in U \cap D$.

Definition 2.2. [7] The map φ is said to be upper Hausdorff continuous at $x_0 \in D$ if for every $W \in \mathcal{V}_Y(0)$, there exists a neighbourhood $U \in \mathcal{V}_X(x_0)$ such that $\varphi(x) \subseteq \varphi(x_0) + W$ for every $x \in U \cap D$.

The graph of φ, denoted by graph φ, is defined as graph$\varphi := \{(x, y) \in D \times Y : y \in \varphi(x)\}$.

Definition 2.3. [7] The map φ is said to be compact at $x_0 \in D$ if for every sequence $((x_n, y_n))_{n\in\mathbb{N}} \subseteq$ graphφ with $x_n \to x_0$ there exists a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that $y_{n_k} \to y_0 \in \varphi(x_0)$. Also φ is said to be compact on D if φ is compact at every $x_0 \in D$.

In metric spaces, Crespi et al.[4] pointed out, the following results obtained by Göpfert et al.[7] regarding the compactness of a set-valued map. These results also hold when we deal with topological vector spaces with countable local bases.

Theorem 2.4. [7],[4] *The following statements are equivalent*

(i) *φ is compact at $x_0 \in D$;*
(ii) *φ is upper semicontinuous at x_0 and $\varphi(x_0)$ is compact;*
(iii) *φ is upper Hausdorff continuous at x_0 and $\varphi(x_0)$ is compact.*

In order to obtain our main results we need the following characterization of upper and lower semicontinuity for set-valued maps.

Theorem 2.5. *Let $\varphi : D \rightrightarrows Y$ be a set-valued map.*

(i) *If $x_0 \in D$ and $\varphi(x_0)$ is compact, then φ is upper semicontinuous at x_0 if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ with $x_n \to x_0$ and for any $y_n \in \varphi(x_n)$, $n \in \mathbb{N}$, there exist $y_0 \in \varphi(x_0)$ and a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that $y_{n_k} \to y_0$(see [7]).*
(ii) *φ is lower semicontinuous at $x_0 \in D$ if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ with $x_n \to x_0$ and for any $y_0 \in \varphi(x_0)$, there exists $y_n \in \varphi(x_n)$, $n \in \mathbb{N}$, such that $y_n \to y_0$(see [1]).*

In particular, we focus on l-type K-convex set-valued maps, a concept of generalized convexity introduced by Kuroiwa [14], see also Seto et al.[18].

Definition 2.6. Let $D \subseteq X$ be a nonempty convex subset of X. A set-valued map $\varphi : D \rightrightarrows Y$ is said to be l-type K-convex if for any $x_0, x_1 \in \mathrm{dom}\varphi$ and $\lambda \in (0,1)$,

$$\varphi((1-\lambda)x_0 + \lambda x_1) \preceq_K (1-\lambda)\varphi(x_0) + \lambda\varphi(x_1).$$

3. M-well-posed set-valued equilibrium problems

In this section we keep the assumption that $0 \in f(x, D)$ for all $x \in D$ (see Bianchi et al.[3]) and investigate the properties of the set-valued map φ. Also, the concept of maximizing sequence for the set-valued map φ and a concept of well-posedness for the problem (SEP) are provided, similarly with those considered in Bianchi et al.[3](see also [15]). Further, sufficient conditions for the problem (SEP) to be well-posed are given and we discuss the role of l-type K-convexity of the set-valued map φ in order to single out classes of well-posed set-valued equilibrium problems in finite and infinite dimensional spaces.

Proposition 3.1. *For the map φ the following assertions hold:*

(i) $\varphi(x) \cap K_0 = \emptyset$ *for all* $x \in D$;
(ii) $\overline{x} \in S_0 \Longleftrightarrow 0 \in \varphi(\overline{x})$;
(iii) $\overline{x} \in S_0 \Longleftrightarrow \varphi(\overline{x}) \cap K \neq \emptyset$.

Proof. (i) Assume that for some $x_0 \in D, \varphi(x_0) \cap K_0 \neq \emptyset$. Therefore, there exists $z \in K_0, z \neq 0$, such that $z \in E_{-K_0}(f(x_0, D))$. Hence $z \in f(x_0, D)$ and $f(x_0, D) \subseteq z + (-K_0)^c$. Since $0 \in f(x, D)$ for all $x \in D$, we obtain that $0 \in z + (-K_0)^c$, i.e., $-z \in (-K_0)^c$ which contradicts the fact that $z \in K_0$.
(ii) Since $0 \in f(x, D)$ for each $x \in D$,

$$\overline{x} \in S_0 \Longleftrightarrow f(\overline{x}, y) \subseteq (-K_0)^c \text{ for every } y \in D \Longleftrightarrow$$
$$\Longleftrightarrow f(\overline{x}, D) \subseteq (-K_0)^c \Longleftrightarrow 0 \in f(\overline{x}, D), f(\overline{x}, D) \subseteq 0 + (-K_0)^c,$$

i.e., $0 \in E_{-K_0}(f(\overline{x}, D)) = \varphi(\overline{x})$.
(iii) Trivial, by (i) and (ii). □

Let us recall the following notion of upper Hausdorff convergence of a sequence of points to a set (see, e.g., Miglierina et al.[17]).

Definition 3.2. The sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is said to be upper Hausdorff convergent to the set $A \subseteq X$ ($x_n \rightharpoonup A$) if for every neighbourhood $W \in \mathcal{V}_X(0)$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in A + W$, for every $n \geq n_0$.

It is well-known that the well-posedness concepts are formulate in terms of convergence of suitable minimizing sequences. Bianchi et al.[3] introduced the following concept for a sequence and proved that is related to some concept for sequences introduced by Miglierina et al.[17].

Definition 3.3. [3] A sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ is said to be a maximizing sequence for φ if for every $V \in \mathcal{V}_Y(0)$ there exists $n_0 \in \mathbb{N}$ such that

$$\varphi(x_n) \cap V \neq \emptyset, \ \forall n \geq n_0.$$

Clearly, every sequence $(x_n)_{n\in\mathbb{N}} \subseteq S_0$ is a maximizing sequence.

The following definition reproduces, in set-valued settings, the classical notion of Tykhonov well-posedness given in metric spaces, see also [3].

Definition 3.4. [3] We say that the set-valued equilibrium problem (SEP) is M-well-posed if every maximizing sequence is upper Hausdorff convergent to S_0.

Next theorem gives sufficient conditions for the set-valued equilibrium problem (SEP) to be M-well-posed. It is given in finite dimensional spaces and is a variant of Theorem 1 in Bianchi et al.[3] where the hypotheses are given with respect to the maps φ and f. In our version, the hypotheses are imposed only on the map φ which makes our result to be much easier to verify.
Similarly with Bianchi et al.[3], we suppose that the topological vector space Y is regular, i.e., every nonempty closed set and every singleton disjoint from it can be separated by open sets.

Theorem 3.5. *Let X be a finite dimensional vector space and $D \subseteq X$ be a closed convex set such that:*

(i) *$S_0 \subseteq D$ is bounded;*
(ii) *φ compact on $D \setminus S_0$;*
(iii) *φ is l-type $(-K)$-convex on D.*

Then the problem (SEP) is M-well-posed.

Proof. Suppose by contradiction that there exists a maximizing sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ which is not upper Hausdorff convergent to the set S_0. Therefore, there exists a neighbourhood $V \in \mathcal{V}_X(0)$ such that

$$x_n \notin S_0 + V, \text{ for infinitely many } n. \tag{3.1}$$

Since S_0 is bounded, the set $S_0 + V$ is bounded, $V \in \mathcal{V}_X(0)$, and therefore the set $cl(S_0+V)$ is compact. Consider the compact set $bd(S_0+V) = cl(S_0+V)\setminus int(S_0+V)$. Fix an arbitrary $\overline{x} \in S_0$. We can always find $\lambda_n \in (0,1)$ such that

$$\overline{x}_n = \lambda_n\overline{x} + (1-\lambda_n)x_n \in bd(S_0+V).$$

The set $bd(S_0+V)$ being compact, we can extract from the sequence

$$(\lambda_n\overline{x} + (1-\lambda_n)x_n)_{n\in\mathbb{N}}$$

a subsequence $(\lambda_{n_k}\overline{x}+(1-\lambda_{n_k})x_{n_k})_{k\in\mathbb{N}}$ converging to $x^* \in bd(S_0+V)$. By the l-type $(-K)$-convexity of φ, we have for every $k \in \mathbb{N}$,

$$\lambda_{n_k}\varphi(\overline{x}) + (1-\lambda_{n_k})\varphi(x_{n_k}) \subseteq \varphi(\lambda_{n_k}\overline{x} + (1-\lambda_{n_k})x_{n_k}) - K.$$

Therefore, since $0 \in \varphi(\overline{x})$ and $(x_{n_k})_{k\in\mathbb{N}} \subseteq D$ is a maximizing sequence, there exists $u_{n_k} \in \varphi(x_{n_k})$ such that $u_{n_k} \to 0$; hence we have

$$\lambda_{n_k}0 + (1-\lambda_{n_k})u_{n_k} \in \varphi(\lambda_{n_k}\overline{x} + (1-\lambda_{n_k})x_{n_k}) - K.$$

Thus, there exists $v_{n_k} \in \varphi(\lambda_{n_k}\overline{x} + (1-\lambda_{n_k})x_{n_k})$ such that

$$\lambda_{n_k}0 + (1-\lambda_{n_k})u_{n_k} - v_{n_k} \in -K.$$

Since φ is compact at $x^* \in bd(S_0 + V)$, from Theorem 2.4(ii) and Theorem 2.5(i), it follows that there exist a subsequence $(v_{n_{k_l}})_{l\in\mathbb{N}}$ of $(v_{n_k})_{k\in\mathbb{N}}$ and $v^* \in \varphi(x^*)$ such that $v_{n_{k_l}} \to v^*$. Hence, we have

$$\lambda_{n_{k_l}} 0 + (1 - \lambda_{n_{k_l}}) u_{n_{k_l}} - v_{n_{k_l}} \in -K. \tag{3.2}$$

Since $\lambda_{n_{k_l}} \in (0, 1)$, there exist a subsequence of $\lambda_{n_{k_l}}$ (denoted also $\lambda_{n_{k_l}}$) and $\lambda_0 \in [0, 1]$ such that $\lambda_{n_{k_l}} \to \lambda_0$. Now taking in (3.2) the limit as $l \to \infty$, from the closedness of K, we obtain that $0 \in v^* - K$ and then $0 \in \varphi(x^*) - K$, a contradiction.
Indeed, if not, $0 \in \varphi(x^*) - K$. Hence there exist $\overline{z} \in \varphi(x^*), \overline{z} \neq 0$ and $k \in K$ such that $\overline{z} - k = 0$. Therefore $\overline{z} = k$ and it follows that $\varphi(x^*) \cap K \neq \emptyset$, a contradiction since $x^* \notin S_0$.
The proof is complete. □

We now provide sufficient conditions for M-well-posedness in infinite dimensional settings assuming that the set $D \setminus S_0$ is compact.

Theorem 3.6. *If the following conditions hold:*

(i) *the set D is convex and $D \setminus S_0$ is compact;*
(ii) *φ compact on $D \setminus S_0$;*
(iii) *φ is l-type $(-K)$-convex on D,*

then the problem (SEP) is M-well-posed.

Proof. Suppose by contradiction that there exist a maximizing sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ and a neighbourhood $V \in \mathcal{V}_X(0)$ such that

$$x_n \notin S_0 + V, \text{ for infinitely many } n.$$

Since $(x_n)_{n\in\mathbb{N}}$ is a maximizing sequence, one can choose a sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \in \varphi(x_n)$ such that $u_n \to 0$. Let now $\overline{x} \in S_0$. Therefore, there exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \subseteq (0, 1)$ such that $\lambda_n \overline{x} + (1 - \lambda_n) x_n \in D \setminus S_0$. Since $D \setminus S_0$ is compact there exists a subsequence $(\lambda_{n_k} \overline{x} + (1 - \lambda_{n_k}) x_{n_k})_{k\in\mathbb{N}} \subseteq D \setminus S_0$ such that $\lambda_{n_k} \overline{x} + (1 - \lambda_{n_k}) x_{n_k} \to x^* \in D \setminus S_0$ when $k \to \infty$.
From the l-type $(-K)$-convexity of φ on D we obtain that

$$\lambda_{n_k} \varphi(\overline{x}) + (1 - \lambda_{n_k}) \varphi(x_{n_k}) \subseteq \varphi(\lambda_{n_k} \overline{x} + (1 - \lambda_{n_k}) x_{n_k}) - K. \tag{3.3}$$

Since $0 \in \varphi(\overline{x})$ and $u_{n_k} \in \varphi(x_{n_k})$, $u_{n_k} \to 0$, from (3.3) we have

$$\lambda_{n_k} 0 + (1 - \lambda_{n_k}) u_{n_k} \in \lambda_{n_k} \varphi(\overline{x}) + (1 - \lambda_{n_k}) \varphi(x_{n_k}) \subseteq \varphi(\lambda_{n_k} \overline{x} + (1 - \lambda_{n_k}) x_{n_k}) - K.$$

Therefore, there exists $v_{n_k} \in \varphi(\lambda_{n_k} \overline{x} + (1 - \lambda_{n_k}) x_{n_k})$, $k \in \mathbb{N}$, such that

$$\lambda_{n_k} 0 + (1 - \lambda_{n_k}) u_{n_k} - v_{n_k} \in -K. \tag{3.4}$$

The map φ is compact at $x^* \in D \setminus S_0$; taking into account Theorem 2.4(ii) and Theorem2.5(i), it follows that there exist a subsequence $(v_{n_{k_l}})_{l\in\mathbb{N}}$ of $(v_{n_k})_{k\in\mathbb{N}}$ and $v^* \in \varphi(x^*)$ such that $v_{n_{k_l}} \to v^*$. By (3.4) we have that

$$\lambda_{n_{k_l}} 0 + (1 - \lambda_{n_{k_l}}) u_{n_{k_l}} - v_{n_{k_l}} \in -K.$$

Similarly with Theorem 3.5 we obtain that $0 \in \varphi(x^*) - K$, which is a contradiction because $x^* \notin S_0$.
The proof is complete. □

4. Well-posedness with respect to set criterion

In the previous section, the assumption that $0 \in f(x, D)$ for every $x \in D$, gave us the possibility to characterize the solutions of the problem (SEP) via the set-valued map φ.
Since there exist set-valued equilibrium problems which are well-posed without fulfilling the condition above, in this section we want to drop the assumption that $0 \in f(x, D)$ for every $x \in D$. To this point, we consider the following set-valued problem (SP) which consists in finding $\overline{x} \in D$ such that

$$\varphi(\overline{x}) \subseteq (-K_0)^c,$$

where $D \subseteq X$ and $\varphi : D \rightrightarrows Y$, $\varphi(x) = E_{-K_0}(f(x, D))$. We will denote by $\overline{S}_0$ the solution set of this problem. From the definition of the map φ it follows that $S_0 \subseteq \overline{S}_0$. Since we supposed that $S_0 \neq \emptyset$ we have also that $\overline{S}_0 \neq \emptyset$.
Further we will introduce a well-posedness concept for the set-valued problem (SP) which will lead to some concept of well-posedness for (SEP). While dealing with set-valued problems it is more relevant to consider solution concepts based on comparison among the sets corresponding to each value of the objective map.
For $0 \neq e \in Y$ and $A \in \mathcal{P}(Y)$, Khushboo et al.[13] considered the scalarization function $\phi_{e,A} : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined as $\phi_{e,A}(y) = \inf\{t \in \mathbb{R} : y \in te + A - S\}$, where S is a nonempty proper subset of Y. When $S = -K$ we obtain that $\phi_{e,A}(y) = \inf\{t \in \mathbb{R} : y \in te + A + K\}$.
We now consider the following generalized Gerstewitz function introduced by Khushboo et al.[13].

Definition 4.1. Let $H_e : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$ be defined as

$$H_e(A, B) = sup_{b \in B} \phi_{e,A}(b). \tag{4.1}$$

Further, under the assumptions that $S = -K$ and $e \in -K$, since the set $-K$ is closed and $\mathrm{cl}(-K) + \mathbb{R}_{++}e \subseteq -K$ holds, the following two lemmas are particular cases of Theorem 4.1(ii) and Lemma 4.2 in Khushboo et al.[13], respectively.

Lemma 4.2. *If $r \in \mathbb{R}$ and $A \in \mathcal{P}(Y)$ then*

$$\{y \in Y : \phi_{e,A}(y) \leq r\} = re + A + K.$$

Lemma 4.3. *If $r \in \mathbb{R}$ and $A, B \in \mathcal{P}(Y)$ then*

$$H_e(A, B) \leq r \Longleftrightarrow B \subseteq A + re + K. \tag{4.2}$$

In the sequel, we suppose that $S = -K$ and $e \in -K$. Inspired by the two lemmas above, we introduce the following notion for a sequence $(x_n)_{n \in \mathbb{N}} \subseteq D$ to be a minimizing sequence for the set-valued problem φ.

Definition 4.4. A sequence $(x_n)_{n \in \mathbb{N}} \subseteq D$ is said to be a minimizing sequence for φ if there exist a sequence $(y_n)_{n \in \mathbb{N}} \subseteq \overline{S}_0$ and a sequence $(\varepsilon_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}, \varepsilon_n > 0, \varepsilon_n \to 0$ such that

$$H_e(\varphi(y_n), \varphi(x_n)) \leq \varepsilon_n.$$

Remark 4.5. In our settings, it is obvious that a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ is a minimizing sequence for φ if and only if there exist a sequence $(y_n)_{n\in\mathbb{N}} \subseteq \overline{S}_0$ and a sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}, \varepsilon_n > 0, \varepsilon_n \to 0$ such that

$$\varphi(x_n) \subseteq \varphi(y_n) + e\varepsilon_n + K.$$

We observe that every sequence from $\overline{S}_0$ is a minimizing sequence.

The following example shows that the maximizing sequence and the minimizing sequence concepts introduced before for φ, are different.

Example 4.6. Let $e = (-1, 0)$ and $f : D \times D \rightrightarrows Y$ where $D = [-1, 1]$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, be defined as

$$f(x, y) = \{(x, - \mid y \mid)\}.$$

Is is easy to check that $\varphi : D \rightrightarrows Y, \varphi(x) = E_{-K_0}(f(x, D))$, is defined by

$$\varphi(x) = \{(x, -1)\},$$

and $S_0 = \overline{S}_0 = (0, 1]$. Let $x_n = (-\frac{1}{n})_{n\in\mathbb{N}^*}$. Since there exist the sequence $(y_n)_{n\in\mathbb{N}} \subseteq \overline{S}_0, y_n = \frac{1}{n}, n \in \mathbb{N}^*$, and the sequence $(\varepsilon_n)_{n\in\mathbb{N}}, \varepsilon_n = \frac{2}{n}, n \in \mathbb{N}^*$, such that

$$\varphi(x_n) \subseteq \varphi(y_n) + e\varepsilon_n + K,$$

it follows that $(x_n)_{n\in\mathbb{N}^*}$ is a minimizing sequence for the map φ. We can notice that for each $V \in \mathcal{V}_Y(0)$, $\varphi(x_n) \cap V = \emptyset$; therefore $(x_n)_{n\in\mathbb{N}}$ is not a maximizing sequence. Also, $0 \in f(x, D)$ does not hold for every $x \in D$.

Definition 4.7. We say that the set-valued problem (SP) is M_1-well-posed if every minimizing sequence is upper Hausdorff convergent to the set $\overline{S}_0$.

Now we provide sufficient conditions for the problem (SP) to be M_1-well-posed in infinite dimensional spaces.

Theorem 4.8. *If D is compact, $\overline{S}_0$ is closed, φ is lower semicontinuous on $D \setminus \overline{S}_0$ and compact on $\overline{S}_0$, then the problem (SP) is M_1-well-posed.*

Proof. Suppose by contradiction that there exist a minimizing sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ and $V \in \mathcal{V}_X(0)$ such that

$$x_n \notin \overline{S}_0 + V, \tag{4.3}$$

for infinitely many n. Since the set D is compact it follows that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} \to x_0$, $x_0 \in D$. From (4.3), obviously $x_0 \notin \overline{S}_0$.
On the other hand, $(x_n)_{n\in\mathbb{N}} \subseteq D$ is a minimizing sequence, therefore there exist a sequence $(y_n)_{n\in\mathbb{N}} \subseteq \overline{S}_0$ and a sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}, \varepsilon_n > 0, \varepsilon_n \to 0$, such that

$$H_e(\varphi(y_n), \varphi(x_n)) \leq \varepsilon_n. \tag{4.4}$$

By Lemma 4.3, we have that

$$\varphi(x_n) \subseteq \varphi(y_n) + \varepsilon_n e + K, \ \varepsilon_n \to 0.$$

The set $\overline{S}_0 \subseteq D$ is closed, D is compact and therefore $\overline{S}_0$ is compact. Thus, there exists a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that $y_{n_k} \to y_0$ for some $y_0 \in \overline{S}_0$.
Let $v_0 \in \varphi(x_0)$. The map φ is lower semicontinuous on $D \setminus \overline{S}_0$ and therefore at

$x_0 \in D \setminus \overline{S}_0$. From Theorem 2.5$(ii)$, it follows that there exists $v_{n_k} \in \varphi(x_{n_k})$ such that $v_{n_k} \to v_0$.
The inclusion

$$\varphi(x_{n_k}) \subseteq \varphi(y_{n_k}) + \varepsilon_{n_k} e + K, \ \varepsilon_{n_k} \to 0,$$

implies that there exists $u_{n_k} \in \varphi(y_{n_k})$ such that

$$v_{n_k} \in u_{n_k} + \varepsilon_{n_k} e + K, \text{ for every } k \in \mathbb{N}.$$

By Theorem 2.4, the map φ is upper semicontinuous at $y_0 \in \overline{S}_0$ and $\varphi(y_0)$ is compact; thus, taking into account Theorem 2.5(ii), there exist a subsequence $(u_{n_{k_l}})_{l\in\mathbb{N}}$ of $(u_{n_k})_{k\in\mathbb{N}}$ and $u_0 \in \varphi(y_0)$ such that $u_{n_{k_l}} \to u_0$. We have that

$$v_{n_{k_l}} \in u_{n_{k_l}} + \varepsilon_{n_{k_l}} e + K, \text{ for every } l \in \mathbb{N}.$$

When $l \to \infty$, it follows from the closedness of K that $v_0 \in u_0 + K \subseteq \varphi(y_0) + K$ and therefore $\varphi(x_0) \subseteq \varphi(y_0) + K$, which is a contradiction since $\varphi(y_0) \subseteq (-K_0)^c$ and $x_0 \notin \overline{S}_0$. □

In the next example all the assumptions of the theorem above are fulfilled and the problem (SP) is well-posed. Also, $0 \in f(x, D)$ does not hold for every $x \in D$.

Example 4.9. Let $f : D \times D \rightrightarrows Y$ where $D = [-1, 1]$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, be defined as

$$f(x, y) = \{(x, \mid y \mid)\}, x \in [-1, 1], \ y \in [-1, 1].$$

The map $\varphi : D \rightrightarrows Y$ is defined by

$$\varphi(x) = \{(x, 0)\}, x \in [-1, 1].$$

The solution set for (SP) is $\overline{S}_0 = S_0 = [0, 1]$.

Remark 4.10. Obviously, if $(x_n)_{n\in\mathbb{N}}$ is a minimizing sequence of φ we have that there exist $(y_n)_{n\in\mathbb{N}} \subseteq \overline{S}_0$ and $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}, \varepsilon_n > 0$, $\varepsilon_n \to 0$, such that

$$\varphi(x_n) \subseteq \varphi(y_n) + \varepsilon_n e + K \subseteq f(y_n, D) + \varepsilon_n e + K.$$

Now we introduce the concept of M_1-well-posedness for the problem (SEP), strongly related to the concept of M_1-well-posedness of (SP).

Definition 4.11. The set-valued equilibrium problem (SEP) is said to be M_1-well-posed if every minimizing sequence $(x_n)_{n\in\mathbb{N}}$ is upper Hausdorff convergent to the set S_0.

The following theorem makes the connection between the special set-valued problem (SP) and the set-valued equilibrium problem (SEP) we are interested in. Also, it provides sufficient conditions for M_1-well-posedness of the set-valued equilibrium problem (SEP).

Theorem 4.12. *If the problem (SP) is M_1-well-posed and for every $V \in \mathcal{V}_X(0)$, $\overline{S}_0 \subseteq S_0 + V$, then the set-valued equilibrium problem (SEP) is M_1-well-posed.*

Proof. Let $V \in \mathcal{V}_X(0)$ be a neighbourhood of 0. Hence there exists $W \in \mathcal{V}_X(0)$ such that $W + W \subseteq V$.
Let $(x_n)_{n\in\mathbb{N}} \subseteq D$ be a minimizing sequence for φ. Since the problem (SP) is M_1-well-posed it follows that for W there exists $n_0 \in \mathbb{N}$ such that $x_n \in \overline{S}_0 + W$ for every $n \geq n_0$. Also, from the hypothesis $\overline{S}_0 \subseteq S_0 + W$. Therefore,

$$x_n \in \overline{S}_0 + W \subseteq S_0 + W + W \subseteq S_0 + V,$$

for every $n \geq n_0$.
Hence for every $V \in \mathcal{V}_X(0)$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in S_0 + V$ for every $n \geq n_0$.
The proof is complete. □

The following example illustrates Theorem 4.12.

Example 4.13. Let $e = (0, -1)$ and $f : D \times D \rightrightarrows Y$, where

$$D = [-1, 1],\ Y = \mathbb{R}^2,\ K = \mathbb{R}^2_+,$$

be defined as

$$f(x,y) = \begin{cases} \{((1-\mid y\mid)x, \mid y\mid x)\}, x \in [0,1], x \neq \frac{1}{n}, n \geq 2; y \in [-1,1]; \\ \{((1-\mid y\mid)(-x), (1-\mid y\mid)(\pm x))\}, x = \frac{1}{n}, n \geq 2, y \neq 0; \\ \{(-x,x)\}, x = \frac{1}{n}, n \geq 2, y = 0; \\ \{(x, \mid y\mid)\}, x \in [-1,0), y \in [-1,1]. \end{cases}$$

The map $\varphi : D \rightrightarrows Y$ is defined by

$$\varphi(x) = \begin{cases} [(0,x);(x,0)], x \in [0,1], x \neq \frac{1}{n}, n \geq 2; \\ \{(-x,x)\}, x = \frac{1}{n}, n \geq 2; \\ \{(x,0)\}, x \in [-1,0). \end{cases}$$

The solution set for (SP) is $\overline{S}_0 = [0,1]$ and the solution set of the set-valued equilibrium problem (SEP) is $S_0 = [0,1] \setminus \{\frac{1}{n}, n \geq 2\}$. It is easy to observe that $\overline{S}_0 \subseteq S_0 + V$ for every $V \in \mathcal{V}_X(0)$. Also, every minimizing sequence of the set-valued problem (SP) is upper Hausdorff convergent to the set $\overline{S_0}$ and therefore the problem (SP) is M_1-well-posed. Finally, we observe that the problem (SEP) is also M_1-well-posed.

5. Conclusions

In this paper, we introduce some concepts of well-posedness for a set-valued equilibrium problem; the first of them generalizes a concept of well-posedness of the strong vector equilibrium problem studied by Bianchi et al.[3] in topological vector spaces. First, we focus on several properties of a suitable set-valued map φ and we obtain some sufficient results for well-posedness for our set-valued equilibrium problem in the presence of l-type K-convexity of the set-valued map φ in finite and infinite settings. The quasi-order relation induced by the nonempty closed convex pointed cone K in the topological vector space Y and the nice properties of the Gerstewitz map considered by Khushboo et al.[13], conducted us to another well-posedness concept for

the set-valued equilibrium problem we are interested in. Some sufficient conditions for this well-posedness concept have been obtained via an appropriate set-valued problem.

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Quasiconvex functions: how to separate, if you must!

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Dedicated to the memory of Professor Gábor Kassay.

Abstract. Since quasiconvex functions have convex lower level sets it is possible to minimize them by means of separating hyperplanes. An example of such a procedure, well-known for convex functions, is the subgradient method. However, to find the normal vector of a separating hyperplane is in general not easy for the quasiconvex case. This paper attempts to gain some insight into the *computational* aspects of determining such a normal vector and the geometry of lower level sets of quasiconvex functions. In order to do so, the directional differentiability of quasiconvex functions is thoroughly studied. As a consequence of that study, it is shown that an important subset of quasiconvex functions belongs to the class of quasidifferentiable functions. The main emphasis is, however, on computing actual separators. Some important examples are worked out for illustration.

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1. Introduction

The backbone of every successful procedure to minimize a general nonsmooth convex function is *separation*. For example, so-called *subgradient methods* as discussed in [18], refinements of such methods with space dilation yielding the *ellipsoid algorithm*, [9, 22], use the important property of a finite-valued convex function that every nonoptimal point in its domain can be properly separated by an affine functional, or hyperplane, from the nonempty set of points with lower functional value, the so-called *lower level set*. Also, the important class of *bundle methods*, [13], is based on the construction of hyperplanes supporting the epigraph and so these methods can be seen as refinements of the *cutting plane* idea of Kelley, [14]. Since the epigraph of

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a convex function and its lower level sets are convex it is possible to separate both the epigraph and a lower level set from points outside their relative interiors and use the corresponding separating hyperplanes to minimize the function. Moreover, for finite-valued convex functions the normal vectors of both types of hyperplane are determined by elements of the nonempty subgradient set at the corresponding point. To extend the above results to a larger class of functions it is natural to consider quasiconvex functions. These functions, by definition, have convex lower level sets. We first observe that for an important subset of the quasiconvex functions, the so-called *lower subdifferentiable functions*, one can define the concept of a *lower subgradient*, [17]. This lower subgradient at some point satisfies the subgradient inequality on the corresponding lower level set (therefore its name!) and this enables us to apply the cutting plane approach of Kelley, [17]. Since this lower subgradient can be identified by means of a hyperplane separating the point from its convex lower level set it is important to be able to compute such a separating hyperplane. A similar observation holds for all quasiconvex functions and this paper addresses the question how to compute the normal vector of a hyperplane separating the lower level set of a quasiconvex function from any given nonminimal point on its domain. We try to keep the class of quasiconvex functions as general as possible by not assuming lower subdifferentiability. Unfortunately some results are only valid under some additional assumptions. These assumptions cease to hold for quasiconvex functions which are constant in some neighborhood of a nonminimal point. If this happens it seems impossible to compute a normal vector of a separating hyperplane using only local information. However, this does not imply that every algorithm based on the construction of separating hyperplanes will get trapped in such a "bad" point. In a pair of subsequent papers, [6, 8], an adaptation of the ellipsoid method is considered which keeps track of a hyperrectangle containing a minimal point. This hyperrectangle is in general much smaller than the current ellipsoid and can be constructed without increasing the complexity order of the algorithm. This gives the opportunity, in case the center of the current ellipsoid is such a "bad" point, to search this "easy" hyperrectangle in order to either prove optimality of the present point or find another point from where it is possible to proceed. In this paper we also show that every quasiconvex function with a Lipschitz continuous directional derivative is quasidifferentiable, [5]. This result relates these two function classes.

2. Quasiconvex functions

We recall that a function $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is called *proper* if the *domain* of f, given by $\mathrm{dom}(f) := \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) < \infty\}$, is nonempty and if $f(\boldsymbol{x}) > -\infty$ for every $\boldsymbol{x} \in \mathbb{R}^n$. Among the set of proper functions we will now concentrate on the so-called evenly quasiconvex functions defined below.

Definition 2.1. A function $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is called quasiconvex if the lower level sets $\mathcal{L}_f^{\leq}(\alpha) := \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) \leq \alpha\}$ are convex for every $\alpha \in \mathbb{R}$. The function is called evenly quasiconvex if its lower level sets are all evenly convex. Observe a set is called *evenly convex* if it can be represented by the intersection of open halfspaces.

Observe that every lower semicontinuous quasiconvex function is evenly quasiconvex, since it has closed convex (hence evenly convex) lower level sets. Moreover, it can also be shown (see [15]) that every upper semicontinuous quasiconvex function is evenly quasiconvex Clearly, for f quasiconvex, it is well known that $\mathrm{dom}(f) = \bigcup_{\alpha\in\mathbf{R}} \mathcal{L}_f^{\leq}(\alpha)$ is convex due to $\mathcal{L}_f^{\leq}(\alpha) \subseteq \mathcal{L}_f^{\leq}(\beta)$ for every $\alpha \leq \beta$. The following result lists some well-known equivalent characterizations of quasiconvexity, see [19].

Lemma 2.2. *The following conditions are equivalent.*

1. *The function* $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ *is quasiconvex.*
2. *The sets* $\mathcal{L}_f^{<}(\alpha) := \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) < \alpha\}$ *are convex for each* $\alpha \in \mathbb{R}$.
3. $f(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leq \max\{f(\boldsymbol{x}), f(\boldsymbol{y})\}$ *for every* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *and* $0 < \lambda < 1$.

In the next section we will consider the special class of proper positively homogeneous evenly quasiconvex functions.

3. On properties of proper positively homogeneous evenly quasiconvex functions

This section mainly derives similar results as those obtained by Crouzeix in [2, 3, 4]. However, while Crouzeix considers proper, positively homogeneous, lower semicontinuous quasiconvex functions we replace lower semicontinuity and quasiconvexity by evenly quasiconvexity. Despite this weaker assumption it is possible to derive similar results by means of easier proofs. Since the main results in this section are a consequence of duality results for quasiconvex functions these simple proofs are possible using a more natural generalization, [16, 7], of the well-known biconjugate or Fenchel-Moreau theorem for convex functions, [12, 21]. It turns out that proper evenly quasiconvex functions originate a more symmetrical representation in the dual space than proper lower semicontinuous quasiconvex functions, [16, 7], and using this more suitable representation one can give simpler proofs. Moreover, since the definition of an evenly quasiconvex function already "includes" a separation result for convex sets it is also possible to give a very simple and easy proof for this dual representation of proper evenly quasiconvex functions. For a proof of the next result the reader should consult Theorem 1.16 and 1.18 of [6]. Observe that $\langle .,. \rangle$ denotes the well known innerproduct.

Lemma 3.1. *Let* $\varphi : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ *be a proper positively homogeneous evenly quasiconvex function satisfying* $\varphi(\mathbf{0}) = 0$. *For every* $\boldsymbol{x} \in \mathbb{R}^n$ *it follows*

$$\varphi(\boldsymbol{x}) = \sup\left\{\psi\Big(\boldsymbol{x}^\star, \langle \boldsymbol{x}^\star, \boldsymbol{x}\rangle\Big) : \boldsymbol{x}^\star \in \mathbb{R}^n\right\} \tag{3.1}$$

with

$$\psi(\boldsymbol{x}^\star, r) := \inf\{\varphi(\boldsymbol{y}) : \langle \boldsymbol{x}^\star, \boldsymbol{y}\rangle \geq r,\ \boldsymbol{y} \in \mathbb{R}^n\}. \tag{3.2}$$

Moreover, for every $\boldsymbol{x}^\star \in \mathbb{R}^n$ *the function* $r \longmapsto \psi(\boldsymbol{x}^\star, r)$ *is a nondecreasing positively homogeneous function.*

The above lemma is the alluded dual representation. Using it, the next results provide slight improvements over related results in [4, 2]. Recall that a convex positively homogeneous function is also called *sublinear*, see [12].

Lemma 3.2. *If $\varphi : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is a proper positively homogeneous evenly quasiconvex nonnegative function satisfying $\varphi(\mathbf{0}) = 0$ then φ is a lower semicontinuous sublinear function with its subgradient set $\partial\varphi(\mathbf{0})$ at $\mathbf{0}$ nonempty.*

Proof. Since the function φ is assumed to be positively homogeneous it remains to prove that it is lower semicontinuous and convex. By (3.1) it is sufficient to prove that the function $r \longmapsto \psi(\boldsymbol{x}^\star, r)$, and hence the function $\boldsymbol{x} \longmapsto \psi\Big(\boldsymbol{x}^\star, \langle \boldsymbol{x}^\star, \boldsymbol{x}\rangle\Big)$, is lower semicontinuous and convex for every $\boldsymbol{x}^\star \in \mathbb{R}^n$ with ψ defined by (3.2) in Lemma 3.1. Clearly it follows by the nonnegativity of the function φ that $0 \leq \psi(\boldsymbol{x}^\star, r)$ for every $(\boldsymbol{x}^\star, r) \in \mathbb{R}^{n+1}$. Also, $\psi(\boldsymbol{x}^\star, 0) = \inf\{\varphi(\boldsymbol{y}) : \langle \boldsymbol{x}^\star, \boldsymbol{y}\rangle \geq 0,\ \boldsymbol{y} \in \mathbb{R}^n\} \leq \varphi(\mathbf{0}) = 0$ and so $\psi(\boldsymbol{x}^\star, 0) = 0$. Hence by the nonnegativity of ψ and the function $r \longmapsto \psi(\boldsymbol{x}^\star, r)$ is nondecreasing we conclude that $\psi(\boldsymbol{x}^\star, r) = 0$ for every $r \leq 0$. Again using φ is a positively homogeneous function and hence the function $r \longmapsto \psi(\boldsymbol{x}^\star, r)$ is also positively homogeneous it follows for $r > 0$ that $\psi(\boldsymbol{x}^\star, r) = r\psi(\boldsymbol{x}^\star, 1)$ with $\psi(\boldsymbol{x}^\star, 1) \geq 0$. This shows that the convexity and lower semicontinuity of the function $r \longmapsto \psi(\boldsymbol{x}^\star, r)$ is established whether $\psi(\boldsymbol{x}^\star, 1)$ is finite or not. To prove the last part we observe, since the function φ is proper lower semicontinuous and sublinear, that by Theorem V.3.1.1 of [12] the function φ is the support function of the closed nonempty convex set $\mathcal{C} := \{\boldsymbol{x}^\star \in \mathbb{R}^n : \langle \boldsymbol{x}^\star, \boldsymbol{x}\rangle \leq \varphi(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathbb{R}^n\}$. Since $\varphi(\mathbf{0}) = 0$ we have $\mathcal{C} = \partial\varphi(\mathbf{0})$ and the proof is finished. □

Another consequence of Lemma 3.1 is given by the following result. Remember K° denotes the well known polar of the cone K given by $K^\circ = \{\boldsymbol{x}^* \in \mathbb{R}^n : \langle \boldsymbol{x}^*, \boldsymbol{x}\rangle \leq 0$ for every $\boldsymbol{x} \in K\}$.

Lemma 3.3. *If $\varphi : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is a proper positively homogeneous evenly quasiconvex function satisfying $\varphi(\mathbf{0}) = 0$ and $\mathrm{dom}(\varphi) \subseteq \mathrm{cl}\left(\mathcal{L}_\varphi^<(0)\right)$ then φ is a lower semicontinuous sublinear function with its subgradient set $\partial\varphi(\mathbf{0})$ at $\mathbf{0}$ nonempty.*

Proof. To ensure the first part of the result only the convexity and lower semicontinuity of the function φ require a proof. This will once again be based on analyzing the function $r \longmapsto \psi(\boldsymbol{x}^\star, r)$ for each $\boldsymbol{x}^\star \in \mathbb{R}^n$. We discuss the following mutually exclusive cases for $\boldsymbol{x}^\star$.

1. Let $\boldsymbol{x}^\star$ not belong to $\Big(\mathrm{cl}\left(\mathcal{L}_\varphi^<(0)\right)\Big)^\circ$. If this holds we can find some $\boldsymbol{x}_0$ satisfying $\varphi(\boldsymbol{x}_0) < 0$ and $\langle \boldsymbol{x}^\star, \boldsymbol{x}_0\rangle > 0$. Since the function φ is proper the value $\varphi(\boldsymbol{x}_0)$ must be finite. Hence, for every $r > 0$ we obtain by Lemma 3.1 that

$$\begin{aligned}\psi(\boldsymbol{x}^\star, r) &= \langle \boldsymbol{x}^\star, \boldsymbol{x}_0\rangle^{-1}\psi\Big(\boldsymbol{x}^\star, r\langle \boldsymbol{x}^\star, \boldsymbol{x}_0\rangle\Big)\\ &\leq \langle \boldsymbol{x}^\star, \boldsymbol{x}_0\rangle^{-1}\varphi(r\boldsymbol{x}_0) = r\langle \boldsymbol{x}^\star, \boldsymbol{x}_0\rangle^{-1}\varphi(\boldsymbol{x}_0) < 0\end{aligned}$$

and so $\lim_{r\uparrow\infty}\psi(\boldsymbol{x}^\star, r) = -\infty$. This yields using $r \longmapsto \psi(\boldsymbol{x}^\star, r)$ is nondecreasing that $\psi(\boldsymbol{x}^\star, r) = -\infty$ for each $r \in \mathbb{R}$ and we obtain by relation (3.1) and the

function φ is proper that

$$\varphi(\boldsymbol{x}) = \sup\{\psi(\boldsymbol{x}, \langle \boldsymbol{x}^*, \boldsymbol{x}\rangle) : \boldsymbol{x}^* \in \mathrm{cl}\left(\mathcal{L}_\varphi^<(0)\right)\Big)^\circ\}.$$

2. Let $\boldsymbol{x}^\star$ belong $\left(\mathrm{cl}\left(\mathcal{L}_\varphi^<(0)\right)\right)^\circ$ and consider $\psi(\boldsymbol{x}^\star, r)$ for $r > 0$. If the vector $\boldsymbol{y} \in \mathbb{R}^n$ satisfies $\langle \boldsymbol{x}^\star, \boldsymbol{y}\rangle \geq r > 0$ then $\boldsymbol{y}$ does not belong to $\mathrm{cl}\left(\mathcal{L}_\varphi^<(0)\right)$ and so $\boldsymbol{y}$ is not an element of $\mathrm{dom}(\varphi)$. This implies $\varphi(\boldsymbol{y}) = +\infty$ for each $\boldsymbol{y} \in \mathbb{R}^n$ satisfying $\langle \boldsymbol{x}^\star, \boldsymbol{y}\rangle \geq r > 0$ and by relation (3.2) we obtain $\psi(\boldsymbol{x}^\star, r) = +\infty$ for $r > 0$. To analyze $\psi(\boldsymbol{x}^\star, r)$ for $r \leq 0$ we consider the following two mutually exclusive cases.
   (a) There exists an $\boldsymbol{x}_0$ belonging to $\mathcal{L}_\varphi^<(0)$ such that $\langle \boldsymbol{x}^\star, \boldsymbol{x}_0\rangle = 0$. If this holds we obtain by relation (3.2) for every $\alpha > 0$ that

   $$\psi(\boldsymbol{x}^\star, 0) = \psi\Big(\boldsymbol{x}^\star, \langle \boldsymbol{x}^\star, \boldsymbol{x}_0\rangle\Big) \leq \varphi(\alpha \boldsymbol{x}_0) = \alpha\varphi(\boldsymbol{x}_0) < 0$$

   and as in part 1 we obtain $\psi(\boldsymbol{x}^\star, 0) = -\infty$.This shows $\psi(\boldsymbol{x}^\star, r) = -\infty$ for every $r \leq 0$.
   (b) For every $\boldsymbol{x}$ belonging to $\mathcal{L}_\varphi^<(0)$ it follows that $\langle \boldsymbol{x}^\star, \boldsymbol{x}\rangle < 0$. To compute $\psi(\boldsymbol{x}^\star, 0)$ we first observe for each $\boldsymbol{y}$ satisfying $\langle \boldsymbol{x}^\star, \boldsymbol{y}\rangle \geq 0$ that by our assumption the vector $\boldsymbol{y}$ does not belong to $\mathcal{L}_\varphi^<(0)$ and so $\varphi(\boldsymbol{y}) \geq 0$. Since $\mathbf{0}$ is one of those elements $\boldsymbol{y}$ and $\varphi(\mathbf{0}) = 0$ it follows from (3.2) that $\psi(\boldsymbol{x}^\star, 0) = 0$. Clearly, Lemma 3.1 yields for $r < 0$ that

   $$\psi(\boldsymbol{x}^\star, r) = -r\psi(\boldsymbol{x}^\star, -1)$$

   with $-\infty \leq \psi(\boldsymbol{x}^\star, -1) < 0$.

To finish the proof it follows by the above analysis that we must only concentrate on part 2b and verify that there exists some $\boldsymbol{x}^\star$ belonging to $\left(\mathrm{cl}\left(\mathcal{L}_\varphi^<(0)\right)\right)^\circ$ satisfying $\psi(\boldsymbol{x}^\star, -1) > -\infty$. If such an $\boldsymbol{x}^\star$ does not exists then $\varphi(\boldsymbol{x}) = -\infty$ for every $\boldsymbol{x} \in \mathcal{L}_\varphi^<(0)$ and this contradicts that the function φ is proper. Hence, to represent the function φ as in relation (3.1) it is enough to consider elements of the set

$$\mathcal{S} := \left\{\boldsymbol{x}^\star \in \left(\mathrm{cl}\left(\mathcal{L}_\varphi^<(0)\right)\right)^\circ : -\infty < \psi(\boldsymbol{x}^\star, -1) < +\infty\right\}$$

and we have verified that

$$\varphi(\boldsymbol{x}) = \sup\left\{\psi\Big(\boldsymbol{x}^\star, \langle \boldsymbol{x}^\star, \boldsymbol{x}\rangle\Big) : \boldsymbol{x}^\star \in \mathcal{S}\right\}. \tag{3.3}$$

Since for every $\boldsymbol{x}^\star \in \mathcal{S}$ the function $\boldsymbol{x} \longmapsto \psi\Big(\boldsymbol{x}^\star, \langle \boldsymbol{x}^\star, \boldsymbol{x}\rangle\Big)$ is convex and lower semicontinuous this shows by relation (3.3) that the function φ is lower semicontinuous and convex. The last part follows from similar arguments as used in the proof of Lemma 3.2. □

Observe for φ convex with $\varphi(\mathbf{0}) = 0$ that $\varphi(\boldsymbol{x}) \geq 0$ for every $\boldsymbol{x} \in \mathbb{R}^n$ is equivalent to $\mathbf{0} \in \partial\varphi(\mathbf{0})$. So, for φ satisfying the conditions of Lemma 3.2 we have $\mathbf{0} \in \partial\varphi(\mathbf{0})$ while for φ satisfying the conditions of Lemma 3.3 we have $\mathbf{0} \notin \partial\varphi(\mathbf{0})$.

An immediate consequence of the previous two lemmas is the following theorem, which improves a related result in [4, 2]. Before discussing this theorem we introduce for any function φ the related functions φ_- and φ_+ given by

$$\varphi_-(\boldsymbol{x}) := \begin{cases} \varphi(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \operatorname{cl}\left(\mathcal{L}_\varphi^<(0)\right) \\ +\infty & \text{otherwise} \end{cases} \tag{3.4}$$

and

$$\varphi_+(\boldsymbol{x}) := \begin{cases} 0 & \text{if } \boldsymbol{x} \in \operatorname{cl}\left(\mathcal{L}_\varphi^<(0)\right) \\ \varphi(\boldsymbol{x}) & \text{otherwise.} \end{cases} \tag{3.5}$$

Theorem 3.4. *Every proper, positively homogeneous evenly quasiconvex function φ satisfying $\varphi(\mathbf{0}) = 0$ is lower semicontinuous and is the minimum of two lower semicontinuous sublinear functions φ_- and φ_+.*

Proof. If $\varphi(\boldsymbol{x}) \geq 0$ for every $\boldsymbol{x}$ or equivalently $\mathcal{L}_\varphi^<(0)$ is empty we obtain by relation (3.4) and (3.5) that $\varphi_-(\boldsymbol{x}) = +\infty$ and $\varphi_+(\boldsymbol{x}) = \varphi(\boldsymbol{x})$ for every $\boldsymbol{x}$ and the desired result follows by Lemma 3.2. If $\mathcal{L}_\varphi^<(0)$ is nonempty it follows using φ is a proper positively homogeneous evenly quasiconvex function and $\operatorname{cl}\left(\mathcal{L}_\varphi^<(0)\right)$ a nonempty closed convex cone (hence evenly convex) that φ_+ satisfies the conditions of Lemma 3.2 and φ_- the conditions of Lemma 3.3. Hence the functions φ_- and φ_+ are lower semicontinuous and sublinear. This also implies by relation (3.4) that $\varphi_-(\boldsymbol{x}) \leq 0$ for every $\boldsymbol{x} \in \operatorname{cl}\left(\mathcal{L}_\varphi^<(0)\right)$ and by relations (3.4) and (3.5) we obtain $\varphi(\boldsymbol{x}) = \min\{\varphi_-(\boldsymbol{x}), \varphi_+(\boldsymbol{x})\}$ showing the desired result. □

By Theorem 3.4 every proper evenly quasiconvex positively homogeneous function which is finite at $\mathbf{0}$ must be lower semicontinuous. This is a rather remarkable result which does not hold in general for evenly quasiconvex functions. As an example we mention the evenly quasiconvex function

$$\operatorname{sign}(x) := \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

which is neither lower nor upper semicontinuous at 0.

If φ is a finite positively homogeneous evenly quasiconvex function one can show, under some additional condition, that φ is continuous on $\mathbb{R}^n$. To establish this result we need the following lemma.

Lemma 3.5. *If the function φ is proper positively homogeneous and evenly quasiconvex and its lower level set $\mathcal{L}_\varphi^<(0)$ is nonempty then the following conditions are equivalent.*

1. $\operatorname{rbd}(\mathcal{L}_\varphi^<(0)) \subseteq \mathcal{L}_\varphi^=(0)$.
2. *$\mathcal{L}_\varphi^<(0)$ is relatively open.*

Proof. To verify $1 \Rightarrow 2$ it is sufficient to prove that $\mathcal{L}_\varphi^<(0) \subseteq \operatorname{ri}(\mathcal{L}_\varphi^<(0))$. Let $\boldsymbol{d}_0 \in \mathcal{L}_\varphi^<(0) \subseteq \operatorname{cl}(\mathcal{L}_\varphi^<(0))$ and suppose that $\boldsymbol{d}_0$ does not belong to $\operatorname{ri}(\mathcal{L}_\varphi^<(0))$. Then $\boldsymbol{d}_0 \in \operatorname{rbd}(\mathcal{L}_\varphi^<(0))$ and hence by 1 we obtain that $\varphi(\boldsymbol{d}_0) = 0$. This contradicts $\boldsymbol{d}_0 \in \mathcal{L}_\varphi^<(0)$ and we have shown that $\mathcal{L}_\varphi^<(0) \subseteq \operatorname{ri}(\mathcal{L}_\varphi^<(0))$. To prove $2 \Rightarrow 1$ we observe for $\boldsymbol{d} \in \operatorname{rbd}(\mathcal{L}_\varphi^<(0))$

that $\boldsymbol{d}$ does not belong to $\mathrm{ri}(\mathcal{L}_\varphi^<(0)) = \mathcal{L}_\varphi^<(0)$. Hence, $\varphi(\boldsymbol{d}) \geq 0$ and since $\varphi_+(\boldsymbol{d}) = 0$ for every $\boldsymbol{d} \in \mathrm{cl}(\mathcal{L}_\varphi^<(0))$ with φ_+ defined as in relation (3.5), it follows by Theorem 3.4 that $0 \leq \varphi(\boldsymbol{d}) = \min\{\varphi_-(\boldsymbol{d}), \varphi_+(\boldsymbol{d})\} \leq 0$ or equivalently $\varphi(\boldsymbol{d}) = 0$. □

In the next result we show that under some additional condition a finite positively homogeneous evenly quasiconvex functions is actually continuous.

Lemma 3.6. *If the function φ is a finite positively homogeneous evenly quasiconvex function and the set $\mathcal{L}_\varphi^<(0)$ is relatively open then the function φ is continuous on $\mathbb{R}^n$.*

Proof. If $\mathcal{L}_\varphi^<(0)$ is empty then by Lemma 3.2 we obtain that $\varphi(\boldsymbol{d}) = \varphi_+(\boldsymbol{d})$ for every $\boldsymbol{d} \in \mathbb{R}^n$. Since $\mathrm{dom}(\varphi_+) = \mathbb{R}^n$ and by Lemma 3.2 the function φ_+ is convex it follows by Corollary 10.1.1 of [21] that φ is continuous. If $\mathcal{L}_\varphi^<(0)$ is nonempty then by Theorem 10.1 of [21], Lemma 3.3, Lemma 3.2 and Theorem 3.4 it is sufficient to prove that φ is upper semicontinuous on $\mathrm{rbd}(\mathcal{L}_\varphi^<(0))$. Since by assumption the set $\mathcal{L}_\varphi^<(0)$ is relatively open we obtain by Lemma 3.5 that $\varphi(\boldsymbol{d}) = 0$ for every $\boldsymbol{d} \in \mathrm{rbd}(\mathcal{L}_\varphi^<(0))$. Suppose now by contradiction that $\limsup_{\boldsymbol{d}\to\boldsymbol{d}_0} \varphi(\boldsymbol{d}) > \varphi(\boldsymbol{d}_0) = 0$ for some $\boldsymbol{d}_0 \in \mathrm{rbd}(\mathcal{L}_\varphi^<(0))$. Hence, there exists a sequence $\{\boldsymbol{d}_k : k \geq 1\}$ with $\lim_{k\uparrow\infty} \boldsymbol{d}_k = \boldsymbol{d}_0$ such that $\lim_{k\uparrow\infty} \varphi(\boldsymbol{d}_k) > 0$. Since by Lemma 3.4 the function φ_+ is sublinear and finite it follows as in the first part of this proof that φ_+ is continuous and we obtain by Theorem 3.4 that $\varphi_+(\boldsymbol{d}_0) = \lim_{k\uparrow\infty} \varphi_+(\boldsymbol{d}_k) \geq \lim_{k\uparrow\infty} \varphi(\boldsymbol{d}_k) > 0$. This implies using the definition of φ_+ in relation (3.5) that $\boldsymbol{d}_0 \notin \mathrm{cl}(\mathcal{L}_\varphi^<(0))$ contradicting $\boldsymbol{d}_0 \in \mathrm{rbd}(\mathcal{L}_\varphi^<(0))$. Therefore the function φ must be upper semicontinuous for every $\boldsymbol{d} \in \mathrm{rbd}(\mathcal{L}_\varphi^<(0))$ and this proves the desired result. □

It is now immediately clear for φ continuous on $\mathbb{R}^n$ that the set $\mathcal{L}_\varphi^<(0)$, if not empty, has full dimension n and so $\mathrm{ri}(\mathcal{L}_\varphi^<(0)) = \mathrm{int}(\mathcal{L}_\varphi^<(0)) = \mathcal{L}_\varphi^<(0)$.

The properties of the above special class of positively homogeneous evenly quasiconvex functions will be useful to study the local properties of more general quasiconvex functions. A way to do this is to look at directional derivatives of quasiconvex functions as functions of the direction. This will be discussed in the next section.

4. Directional derivatives of quasiconvex functions

Unlike convex functions, quasiconvex functions do not always have directional derivatives. An important generalization of directional derivatives is given by the *Dini upper derivative* of f at $\boldsymbol{x}_0$ in the direction $\boldsymbol{d}$. This generalization coincides with the definition of Dini upper derivative used within the theory of quasidifferentiable functions if f is locally Lipschitz around $\boldsymbol{x}_0$, see [5].

Definition 4.1. If $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is some function with $f(\boldsymbol{x}_0)$ finite the Dini upper derivative of f at $\boldsymbol{x}_0$ in the direction $\boldsymbol{d}$ is given by

$$f'_+(\boldsymbol{x}_0; \boldsymbol{d}) := \limsup_{t\downarrow 0} \frac{f(\boldsymbol{x}_0 + t\boldsymbol{d}) - f(\boldsymbol{x}_0)}{t}.$$

We observe by the definition of lim sup that $f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ always exists, i.e. $-\infty \leq f'_+(\boldsymbol{x}_0;\boldsymbol{d}) \leq +\infty$, for any function $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ with $f(\boldsymbol{x}_0)$ finite. Moreover, it is easy to verify that $\boldsymbol{d} \longmapsto f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ is positively homogeneous and that $f'_+(\boldsymbol{x}_0;\boldsymbol{0}) = \boldsymbol{0}$. If we know additionally that f is quasiconvex then the next result is easy to prove using Lemma 2.2, see [2].

Lemma 4.2. *Let $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ be a quasiconvex function with $f(\boldsymbol{x}_0)$ finite. Then the function $\boldsymbol{d} \longmapsto f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ is positively homogeneous, quasiconvex and $f'_+(\boldsymbol{x}_0;\boldsymbol{0}) = \boldsymbol{0}$.*

In the remainder of this paper we will always assume that $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is a quasiconvex function with $f(\boldsymbol{x}_0)$ finite and $\boldsymbol{d} \longmapsto f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ is a proper evenly quasiconvex function. Introducing the function $\varphi(\boldsymbol{d}) := f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ we observe by Lemma 4.1 that this function satisfies the properties of the functions studied in Section 3. Although this function depends on $\boldsymbol{x}_0$, whenever no risk of confusion exists we do not refer to it for the sake of notation convenience.

Lemma 4.3. *If the function f is quasiconvex and finite at $\boldsymbol{x}_0$ and the function φ given by $\varphi(\boldsymbol{d}) := f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ is a finite evenly quasiconvex function then the function φ is continuous on $\mathbb{R}^n$.*

Proof. The function φ is positively homogeneous and satisfies $\varphi(\boldsymbol{0}) = 0$. Applying now Lemma 3.6 it is sufficient to verify that $\mathcal{L}^<_\varphi(0)$ is relatively open. By definition this holds for $\mathcal{L}^<_\varphi(0)$ empty. Hence assume that $\mathcal{L}^<_\varphi(0)$ nonempty and let $\boldsymbol{d} \in \mathcal{L}^<_\varphi(0)$. This implies that there exists some $t_0 > 0$ satisfying $\boldsymbol{x}_0 + t\boldsymbol{d} \in \operatorname{ri}(\mathcal{L}^<_f(f(\boldsymbol{x}_0))$. Hence by Theorem 6.8.2 of [21] we obtain that $\boldsymbol{d} \in t_0^{-1}(\operatorname{ri}(\mathcal{L}^<_f(f(\boldsymbol{x}_0))) - \boldsymbol{x}_0) \subseteq \operatorname{ri}(\operatorname{cone}(\mathcal{L}^<_f(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0))$ and we obtain by Lemma 4.4 that $\boldsymbol{d} \in \operatorname{ri}(\mathcal{L}^<_\varphi(0))$. This shows the result. □

Notice that Crouzeix in [2, 3, 4] observed that φ might not be lower semicontinuous even if φ is a finite, positively homogeneous and quasiconvex function. Finally, if f is quasiconvex and additionally locally Lipschitz around $\boldsymbol{x}_0$, (see [1] fot the definition of locally Lipschitz), it is easy to show by a direct proof that φ is Lipschitz continuous (and hence continuous) on $\mathbb{R}^n$.

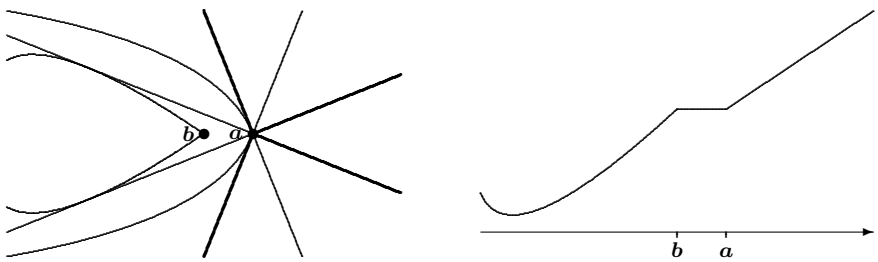

FIGURE 1. Interpretation of the partial description

As already pointed out in the introduction, it is crucial for many optimization methods to be able to compute an element of the normal cone of $\mathcal{L}^<_f(f(\boldsymbol{x}_0))$ at $\boldsymbol{x}_0$.

It is essential to consider the strict lower level set $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ since, unlike for convex functions, a nonminimal point may be in the interior of its lower level set. In order to see that take any point in the segment connecting $\boldsymbol{a}$ and $\boldsymbol{b}$ in Figure 1. The first picture is drawn in the domain and shows two lower level sets. The one with a dashed boundary is $\mathcal{L}_f^<(f(\boldsymbol{b}))$ and the one with a full boundary is $\mathcal{L}_f^\leq(f(\boldsymbol{a})) = \mathcal{L}_f^\leq(f(\boldsymbol{b}))$. The second picture is drawn in the epigraph space and corresponds to slicing the graph of the function along the line going through $\boldsymbol{a}$ and $\boldsymbol{b}$.

Clearly, in order to seek a vector normal to $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ we must know that the set $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ is nonempty. A sufficient condition to ensure that this strict lower level set is nonempty is the nonemptiness of the set of *strict descent directions* at $\boldsymbol{x}_0$ defined as $\mathcal{L}_\varphi^<(0) := \{\boldsymbol{d} \in \mathbb{R}^n : f'_+(\boldsymbol{x}_0; \boldsymbol{d}) < 0\}$. Unfortunately, in the case of quasiconvex functions, contrary to convex functions (see [12]), the nonemptiness of the set of strict descent directions is not necessary as shown by $f(x) = x^3$ at 0.

This function is differentiable at 0 and its derivative at this point equals 0. Therefore $f'(0; d) = 0$ for every $d \in \mathbb{R}$, while $\mathcal{L}_f^<(0) = (-\infty, 0)$ is clearly nonempty.

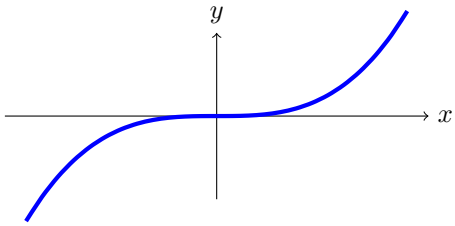

FIGURE 2. A simple but "nasty" quasiconvex function: x^3

For quasiconvex functions a necessary condition is given by the nonemptiness of the set $\mathcal{L}_\varphi^\leq(0) \setminus \{\boldsymbol{0}\}$ with $\mathcal{L}_\varphi^\leq(0) := \{\boldsymbol{d} \in \mathbb{R}^n : f'_+(\boldsymbol{x}_0; \boldsymbol{d}) \leq 0\}$ the set of *descent directions*. It turns out, see Section 4.2 ahead, that the function φ completely characterizes the normal cone of $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ at $\boldsymbol{x}_0$ if $\mathcal{L}_\varphi^<(0)$ is nonempty. For $\mathcal{L}_\varphi^<(0)$ empty we also need global information to find out whether $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ is nonempty or not and so the local information given by φ is insufficient even to decide whether $\boldsymbol{x}_0$ minimizes f or not.

To discuss the case with $\mathcal{L}_\varphi^<(0)$ nonempty we first observe that $\mathcal{L}_\varphi^<(0)$ is a convex subset of the nonempty convex cone $\operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0)$. For $\mathcal{L}_\varphi^<(0)$ nonempty it is shown in the next result that $\operatorname{ri}(\operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0))$ equals $\operatorname{ri}(\mathcal{L}_\varphi^<(0))$. The same result is proven by Crouzeix in [3] but for completeness we list a more detailed proof.

Lemma 4.4. *If $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is a quasiconvex function with $f(\boldsymbol{x}_0)$ finite and $\mathcal{L}_\varphi^<(0)$ nonempty then* $\operatorname{ri}(\operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0))$ *equals* $\operatorname{ri}(\mathcal{L}_\varphi^<(0))$.

Proof. Since $\mathcal{L}_\varphi^<(0) \subseteq \operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0)$ and $\mathcal{L}_\varphi^<(0)$ is nonempty it is sufficient by Theorem 6.3.1 of [21] to verify that $\operatorname{ri}(\operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0)) \subseteq \mathcal{L}_\varphi^<(0)$. Consider now some $\boldsymbol{d}_0 \in \operatorname{ri}(\operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0))$ and let

$$\boldsymbol{d} \in \mathcal{L}_\varphi^<(0) \subseteq \operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0).$$

By Theorem 6.4 of [21] there exists some $\mu < 0$ such that

$$\boldsymbol{d}_\mu := \boldsymbol{d}_0 + \mu(\boldsymbol{d} - \boldsymbol{d}_0) \in \mathrm{ri}(\mathrm{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0))$$

and so $\boldsymbol{d}_0 = \frac{1}{1-\mu}\boldsymbol{d}_\mu - \frac{\mu}{1-\mu}\boldsymbol{d}$. Moreover, since $\boldsymbol{d}_\mu \in \mathrm{ri}(\mathrm{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0))$ and by Theorem 6.8.1 of [21] it follows that $\mathbf{0}$ does not belong to $\mathrm{ri}(\mathrm{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0))$ we can find some $t_0 > 0$ satisfying $f(\boldsymbol{x}_0 + t_0\boldsymbol{d}_\mu) < f(\boldsymbol{x}_0)$. Construct now for each $t > 0$ the line $\mathcal{L}_t$ going through $\boldsymbol{x}_0 + t_0\boldsymbol{d}_\mu$ and $\boldsymbol{m}_t := \boldsymbol{x}_0 + t\boldsymbol{d}_0$ and crossing $\boldsymbol{x}_0 + \alpha\boldsymbol{d}$ in $\boldsymbol{n}_t := \boldsymbol{x}_0 + \xi_t\boldsymbol{d}$, see Figure 3. By the quasiconvexity of f it follows that

$$f(\boldsymbol{m}_t) = f(\boldsymbol{x}_0 + t\boldsymbol{d}_0) \le \max\{f(\boldsymbol{x}_0 + t_0\boldsymbol{d}_\mu), f(\boldsymbol{x}_0 + \xi_t\boldsymbol{d})\}. \tag{4.1}$$

To compute ξ_t we intersect the line $\mathcal{L}_t$ with the line $\boldsymbol{x}_0 + \alpha\boldsymbol{d}$. After some computations we obtain

$$\xi_t = \frac{-\mu t t_0}{(1-\mu)t_0 - t}.$$

Substituting this into (4.1) yields

$$\frac{f(\boldsymbol{x}_0 + t\boldsymbol{d}_0) - f(\boldsymbol{x}_0)}{t} \le \max\left\{\frac{f(\boldsymbol{x}_0 + t_0\boldsymbol{d}_\mu) - f(\boldsymbol{x}_0)}{t}, \frac{f\left(\boldsymbol{x}_0 + \frac{-\mu t t_0}{(1-\mu)t_0 - t}\boldsymbol{d}\right) - f(\boldsymbol{x}_0)}{t}\right\}$$

and due to $\boldsymbol{x}_0 + t_0\boldsymbol{d}_\mu \in \mathcal{L}_f^<(f(\boldsymbol{x}_0))$ and $\lim_{t\downarrow 0}\frac{\xi_t}{t} = \frac{-\mu}{1-\mu} > 0$ we obtain

$$f'_+(\boldsymbol{x}_0; \boldsymbol{d}_0) \le \frac{-\mu}{1-\mu} f'_+(\boldsymbol{x}_0; \boldsymbol{d}) < 0.$$

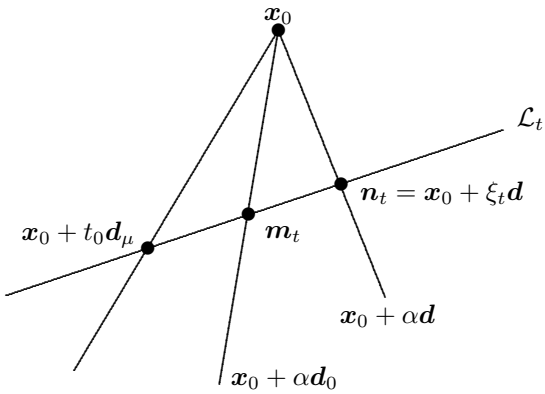

FIGURE 3. Construction of $\mathcal{L}_t$

Hence, $\boldsymbol{d}_0 \in \mathcal{L}_\varphi^<(0)$ and the desired result is proven. □

The previous result will play an important role in the sequel. Its main importance is to show that if the set $\mathcal{L}_\varphi^<(0)$ is nonempty then this set is indistinguishable by polarity from the set $\mathrm{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0)$.

4.1. Quasidifferentiability of quasiconvex functions

This section shows the important result that a quasiconvex function with a Lipschitz continuous directional derivative at $\boldsymbol{x}_0$ is quasidifferentiable at $\boldsymbol{x}_0$.

Definition 4.5 ([5])**.** If $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is some function with $f(\boldsymbol{x}_0)$ finite the directional derivative of f at $\boldsymbol{x}_0$ in the direction $\boldsymbol{d}$ is given by

$$f'(\boldsymbol{x}_0; \boldsymbol{d}) := \lim_{t \downarrow 0} \frac{f(\boldsymbol{x}_0 + t\boldsymbol{d}) - f(\boldsymbol{x}_0)}{t}.$$

Moreover, f is said to be quasidifferentiable at $\boldsymbol{x}_0$ if $\boldsymbol{d} \longmapsto f'(\boldsymbol{x}_0; \boldsymbol{d})$ exists for every $\boldsymbol{d} \in \mathbb{R}^n$ and

$$f'(\boldsymbol{x}_0; \boldsymbol{d}) = \max_{\boldsymbol{y} \in \underline{\partial} f(\boldsymbol{x}_0)} \langle \boldsymbol{d}, \boldsymbol{y} \rangle + \min_{\boldsymbol{y} \in \overline{\partial} f(\boldsymbol{x}_0)} \langle \boldsymbol{d}, \boldsymbol{y} \rangle$$

with $\underline{\partial} f(\boldsymbol{x}_0)$, resp. $\overline{\partial} f(\boldsymbol{x}_0)$, compact convex subsets of $\mathbb{R}^n$. The sets $\underline{\partial} f(\boldsymbol{x}_0)$ and $\overline{\partial} f(\boldsymbol{x}_0)$ are called respectively the subdifferential and the superdifferential of f at $\boldsymbol{x}_0$ being $\mathbb{D} f(\boldsymbol{x}) := [\underline{\partial} f(\boldsymbol{x}_0), \overline{\partial} f(\boldsymbol{x}_0)] \subseteq \mathbb{R}^{2n}$ the quasidifferential of f at $\boldsymbol{x}_0$.

Observe that whenever $f'(\boldsymbol{x}_0; \boldsymbol{d})$ exists it equals $f'_+(\boldsymbol{x}_0; \boldsymbol{d})$. It is well-know that every finite convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is quasidifferentiable at every $\boldsymbol{x} \in \mathbb{R}^n$ with $\mathbb{D} f(\boldsymbol{x}) := [\partial f(\boldsymbol{x}), \{\boldsymbol{0}\}]$ and $\partial f(\boldsymbol{x})$ the nonempty subgradient set of f at $\boldsymbol{x}$. Moreover, it can be easily shown, [5], that the function $\boldsymbol{x} \longmapsto \min\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\}$ is quasidifferentiable at $\boldsymbol{x}_0$ if f_i, $i = 1, 2$, is quasidifferentiable at $\boldsymbol{x}_0$. In general, the set of quasidifferentiable functions at $\boldsymbol{x}_0$ is a linear space closed with respect to all algebraic operations and, more importantly, to the operations of taking maxima and minima. Also for f quasidifferentiable at $\boldsymbol{x}_0$ it is easy to verify that $\boldsymbol{d} \longmapsto f'(\boldsymbol{x}_0; \boldsymbol{d})$ is Lipschitz continuous. To relate the previous results for quasiconvex functions to the above class of functions we observe for $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ quasiconvex, $\mathcal{L}^<_\varphi(0)$ empty with $\varphi(\boldsymbol{d}) := f'(\boldsymbol{x}_0; \boldsymbol{d})$ a finite continuous function that by Theorem (3.4) and Lemma 3.2 the function f is quasidifferentiable at $\boldsymbol{x}_0$ with $\mathbb{D} f(\boldsymbol{x}_0) = [\partial \varphi_+(\boldsymbol{0}), \{\boldsymbol{0}\}]$. If this holds the Lipschitz continuity of φ follows by the finiteness of φ_+. However, if $\mathcal{L}^<_\varphi(0)$ is nonempty we have to assume that φ is Lipschitz continuous (with Lipschitz constant $L > 0$) and by Theorem 3.4 this implies $\varphi(\boldsymbol{d}) = \min\{\varphi_-(\boldsymbol{d}), \varphi_+(\boldsymbol{d})\}$. Applying Lemma 3.3 and Lemma 3.2 we know that φ_+ is a finite positively homogeneous convex function and φ_- a proper lower semicontinuous positively homogeneous convex function. Hence to prove that f is quasidifferentiable at $\boldsymbol{x}_0$ it is sufficient to replace φ_- by a finite positively homogeneous convex function without destroying Theorem 3.4. Clearly for φ Lipschitz continuous it follows that $\mathcal{L}^<_\varphi(0)$ is open and hence $\operatorname{int}(\operatorname{dom}(\varphi_-)) = \mathcal{L}^<_\varphi(0)$. This implies by Theorem 23.4 of [21] that $\partial \varphi_-(\boldsymbol{d})$ is a nonempty compact convex set for every $\boldsymbol{d} \in \mathcal{D}^<_f(\boldsymbol{x}_0)$ and since φ is Lipschitz continuous with Lipschitz constant L it is easy to show by the subgradient inequality applied to φ_- that $\partial \varphi_-(\boldsymbol{d}) \subseteq L\mathcal{B}$ for every $\boldsymbol{d} \in \mathcal{L}^<_\varphi(0)$ with $\mathcal{B} := \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \leq 1\}$ the closed unit Euclidean ball. On the other hand, since φ_- is positively homogeneous it follows that $\partial \varphi_-(\lambda \boldsymbol{d}) = \partial \varphi_-(\boldsymbol{d})$ for every $\lambda > 0$ and $\boldsymbol{d} \in \mathcal{L}^<_\varphi(0)$ and so by the previous observations one can pick for every $\boldsymbol{d}_0 \in \mathcal{L}^<_\varphi(0)$ and corresponding ray $\{\lambda \boldsymbol{d}_0 : \lambda > 0\} \subseteq \mathcal{L}^<_\varphi(0)$ a subgradient $\boldsymbol{\xi}(\boldsymbol{d}_0) \in \partial \varphi_-(\lambda \boldsymbol{d}_0)$, $\lambda > 0$, satisfying

$\|\boldsymbol{\xi}(\boldsymbol{d}_0)\| \leq L$. Consider now the function $\hat{\varphi}_- : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ given by

$$\hat{\varphi}_-(\boldsymbol{d}) := \sup_{\boldsymbol{d}_0 \in \mathcal{L}_\varphi^<(0)} \varphi_-(\boldsymbol{d}_0) + \langle \boldsymbol{\xi}(\boldsymbol{d}_0), \boldsymbol{d} - \boldsymbol{d}_0 \rangle. \tag{4.2}$$

For this function the following result holds.

Lemma 4.6. *If $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is a quasiconvex function with $f(\boldsymbol{x}_0)$ finite, φ is Lipschitz continuous with $\varphi(\boldsymbol{d}) := f'(\boldsymbol{x}_0; \boldsymbol{d})$ and $\mathcal{L}_\varphi^<(0)$ is nonempty then the function $\hat{\varphi}_-$ given by (4.2) is a finite, positively homogeneous and convex function. Moreover, $\hat{\varphi}_-(\boldsymbol{d})$ equals $\varphi(\boldsymbol{d})$ for every $\boldsymbol{d} \in \mathrm{cl}(\mathcal{L}_\varphi^<(0))$ and $\hat{\varphi}_-(\boldsymbol{d}) > 0$ for every $\boldsymbol{d} \notin \mathrm{cl}(\mathcal{L}_\varphi^<(0))$.*

Proof. Clearly $\hat{\varphi}_-$ is a convex function. Since $\|\boldsymbol{\xi}(\boldsymbol{d})\| \leq L$ and $\boldsymbol{\xi}(\boldsymbol{d}) \in \partial\varphi_-(\boldsymbol{d})$ for every $\boldsymbol{d} \in \mathcal{L}_\varphi^<(0)$ we obtain for $\boldsymbol{d}_0 \in \mathcal{L}_\varphi^<(0)$ fixed and any $\boldsymbol{d} \in \mathbb{R}^n$ that

$$\varphi_-(\boldsymbol{d}_0) + \langle \boldsymbol{\xi}(\boldsymbol{d}_0), \boldsymbol{d} - \boldsymbol{d}_0 \rangle \leq \varphi_-(\boldsymbol{0}) + \langle \boldsymbol{\xi}(\boldsymbol{d}_0), \boldsymbol{d} \rangle \leq L\|\boldsymbol{d}\|$$

and so by (4.2) the function $\hat{\varphi}_-$ is finite. To prove that $\hat{\varphi}_-$ is positively homogeneous we first observe using $\boldsymbol{\xi}(\boldsymbol{d}_0) \in \partial\varphi_-(\lambda\boldsymbol{d}_0)$, $\lambda > 0$, for every $\boldsymbol{d}_0 \in \mathcal{L}_\varphi^<(0)$ that

$$\begin{aligned} \varphi_-(\boldsymbol{d}_0) + \langle \boldsymbol{\xi}(\boldsymbol{d}_0), \lambda\boldsymbol{d} - \boldsymbol{d}_0 \rangle &= \lambda\varphi_-\left(\lambda^{-1}\boldsymbol{d}_0\right) + \lambda\left\langle \boldsymbol{\xi}(\boldsymbol{d}_0), \boldsymbol{d} - \lambda^{-1}\boldsymbol{d}_0 \right\rangle \\ &= \lambda\varphi_-\left(\lambda^{-1}\boldsymbol{d}_0\right) + \lambda\left\langle \boldsymbol{\xi}\left(\lambda^{-1}\boldsymbol{d}_0\right), \boldsymbol{d} - \lambda^{-1}\boldsymbol{d}_0 \right\rangle \\ &\leq \lambda\hat{\varphi}_-(\boldsymbol{d}) \end{aligned}$$

for every $\boldsymbol{d} \in \mathbb{R}^n$. This yields by the definition of $\hat{\varphi}_-$ that $\hat{\varphi}_-(\lambda\boldsymbol{d}) \leq \lambda\hat{\varphi}_-(\boldsymbol{d})$ for every $\lambda > 0$ and hence $\hat{\varphi}_-(\lambda\boldsymbol{d}) \leq \lambda\hat{\varphi}_-(\boldsymbol{d}) = \lambda\hat{\varphi}_-\left(\lambda^{-1}\lambda\boldsymbol{d}\right) \leq \hat{\varphi}_-(\lambda\boldsymbol{d})$ implying $\hat{\varphi}_-$ is positively homogeneous. Also for every $\boldsymbol{d}_0 \in \mathcal{L}_\varphi^<(0)$ it follows that $\varphi_-(\boldsymbol{d}) \geq \varphi_-(\boldsymbol{d}_0) + \langle \boldsymbol{\xi}(\boldsymbol{d}_0), \boldsymbol{d} - \boldsymbol{d}_0 \rangle$ and so $\varphi_-(\boldsymbol{d}) \geq \hat{\varphi}_-(\boldsymbol{d})$. If $\boldsymbol{d} \in \mathcal{L}_\varphi^<(0)$ we obtain by (4.2) that $\hat{\varphi}_-(\boldsymbol{d}) \geq \varphi_-(\boldsymbol{d})$ and this yields that $\hat{\varphi}_-$ equals φ_- on $\mathcal{L}_\varphi^<(0)$. By the lower semicontinuity of φ_- and the continuity of $\hat{\varphi}_-$ the functions are equal on $\mathrm{cl}(\mathcal{L}_\varphi^<(0))$. Finally, assume that $\hat{\varphi}_-(\boldsymbol{d}_1) \leq 0$ for some $\boldsymbol{d}_1 \notin \mathrm{cl}(\mathcal{L}_\varphi^<(0))$ and consider a fixed $\boldsymbol{d}_0 \in \mathcal{L}_\varphi^<(0)$. Since $\mathcal{L}_\varphi^<(0)$ is open there exists some $0 < \mu < 1$ such that $\boldsymbol{d}_\mu := \mu\boldsymbol{d}_0 + (1-\mu)\boldsymbol{d}_1 \in \mathrm{rbd}(\mathcal{L}_\varphi^<(0))$. This implies by the convexity of $\hat{\varphi}_-$ that $\hat{\varphi}_-(\boldsymbol{d}_\mu) \leq \mu\hat{\varphi}_-(\boldsymbol{d}_0) + (1-\mu)\hat{\varphi}_-(\boldsymbol{d}_1) < 0$, and so $\varphi_-(\boldsymbol{d}_\mu) = \hat{\varphi}_-(\boldsymbol{d}_\mu) < 0$ contradicting Lemma 3.5. This yields $\hat{\varphi}_-(\boldsymbol{d}) > 0$ for every $\boldsymbol{d} \notin \mathrm{cl}(\mathcal{L}_\varphi^<(0))$ and the proof of the result is finished. □

Introduce now the function $\tilde{\varphi}_- : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$\tilde{\varphi}_-(\boldsymbol{d}) := \varphi_+(\boldsymbol{d}) + \hat{\varphi}_-(\boldsymbol{d}). \tag{4.3}$$

Using this function one can show the following result.

Theorem 4.7. *If $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is a quasiconvex function with $f(\boldsymbol{x}_0)$ finite and φ Lipschitz continuous then f is quasidifferentiable at $\boldsymbol{x}_0$.*

Proof. The result is already verified for $\mathcal{L}_\varphi^<(0)$ empty. Assume therefore that $\mathcal{L}_\varphi^<(0)$ is nonempty. If this holds, it follows by Lemma 4.6 and Lemma 3.2 that $\tilde{\varphi}_-$ given by relation (4.3) and φ_+ are quasidifferentiable. Moreover, it is easy to verify by Theorem 3.4 and again relation (4.3) using $\hat{\varphi}_-(\boldsymbol{d}) > 0$ for every $\boldsymbol{d} \notin \mathrm{cl}(\mathcal{L}_\varphi^<(0))$ that

$$f'(\boldsymbol{x}_0; \boldsymbol{d}) = \min\left\{\tilde{\varphi}_-(\boldsymbol{d}), \varphi_+(\boldsymbol{d})\right\}$$

for every $\boldsymbol{d} \in \mathbb{R}^n$ and hence the desired result is proved. □

4.2. Where are the separators?

This subsection, based on Lemma 4.4 and on the properties of the Dini upper derivative, characterizes the elements of the normal cone of the set $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ at $\boldsymbol{x}_0$. Introduce now the set

$$\Gamma_f := \{\boldsymbol{x} \in \mathbb{R}^n : f'_+(\boldsymbol{x}; \boldsymbol{d}) \geq 0 \text{ for every } \boldsymbol{d} \in \mathbb{R}^n\} \tag{4.4}$$

which is sometimes called the set of stationary points. For reasons to be soon clarified we call this the set of *"bad" points*. Before deriving the announced characterization, we observe by Theorem 11.3 of [21] that the normal cone

$$\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0) := \left\{\boldsymbol{x}^\star \in \mathbb{R}^n : \langle \boldsymbol{x}^\star, \boldsymbol{x} - \boldsymbol{x}_0\rangle \leq 0 \text{ for every } \boldsymbol{x} \in \mathcal{L}_f^<(f(\boldsymbol{x}_0))\right\} \tag{4.5}$$

of $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ at $\boldsymbol{x}_0$ is a proper nonempty convex cone of $\mathbb{R}^n$ if $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ is nonempty. In the next lemma a partial description of $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$ is given by means of the function φ if the set $\mathcal{L}_\varphi^<(0)$ is empty. Introducing the nonempty sets

$$\mathcal{L}_\varphi^=(0) := \{\boldsymbol{d} \in \mathbb{R}^n : f'_+(\boldsymbol{x}_0; \boldsymbol{d}) = 0\}$$

and

$$\mathcal{L}_\varphi^\leq(0) := \{\boldsymbol{d} \in \mathbb{R}^n : f'_+(\boldsymbol{x}_0; \boldsymbol{d}) \leq 0\}$$

it follows that $\mathcal{L}_\varphi^<(0)$ is empty if and only if $\mathcal{L}_\varphi^\leq(0) = \mathcal{L}_\varphi^=(0)$. A sufficient condition for $\mathcal{L}_\varphi^<(0)$ to be empty is given by $\boldsymbol{x}_0 \in \operatorname{int}(\mathcal{L}_f^\leq(f(\boldsymbol{x}_0)))$. The example $f(x) = x^3$ at $x = 0$, shows that this condition is not necessary. Although trivial the next result seems to be new.

Lemma 4.8. *If the function $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is a quasiconvex function satisfying $f(\boldsymbol{x}_0)$ finite and the set $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ is nonempty and the set $\mathcal{L}_\varphi^<(0)$ is empty or equivalently $\boldsymbol{x}_0 \in \Gamma_f$, then*

$$(\mathcal{L}_\varphi^=(0))^\circ \subseteq \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$$

with $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$ the normal cone of $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ at $\boldsymbol{x}_0$ defined in relation (4.5).

Proof. Since $\mathcal{L}_\varphi^<(0)$ is empty and f is quasiconvex it must follow by Lemma 4.2 that $\mathcal{L}_\varphi^=(0)$ is a nonempty convex cone. Moreover, the nonemptyness of $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ and the emptyness of $\mathcal{L}_\varphi^<(0)$ enable us to verify that $\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0 \subseteq \mathcal{L}_\varphi^=(0)$ and so $\operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0) \subseteq \mathcal{L}_\varphi^=(0)$. This implies that

$$(\mathcal{L}_\varphi^=(0))^\circ \subseteq (\operatorname{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0))^\circ = \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$$

and hence the desired result is proven. □

Clearly, the above result reduces to a useless observation if $\mathcal{L}_\varphi^=(0)$ equals $\mathbb{R}^n$. In this case we obtain $(\mathcal{L}_\varphi^=(0))^\circ = \{\boldsymbol{0}\}$ and this happens for $f(x) = x^3$ at $x_0 = 0$.

Figure 1 provides an interpretation of Lemma 4.8. The first picture is drawn in the domain and shows two lower level sets. The one with a dashed boundary is $\mathcal{L}_f^<(f(\boldsymbol{b}))$ and the one with a full boundary is $\mathcal{L}_f^\leq(f(\boldsymbol{a})) = \mathcal{L}_f^\leq(f(\boldsymbol{b}))$. The second picture is drawn in the epigraph space and corresponds to slicing the graph of the function along the line going through $\boldsymbol{a}$ and $\boldsymbol{b}$. Observe first that if $\boldsymbol{x}_0 \in (\boldsymbol{a}, \boldsymbol{b})$ then

$\mathcal{L}_\varphi^=(0) = \mathbb{R}^n$ and so no useful information is provided. On the other hand, if $\boldsymbol{x}_0 = \boldsymbol{a}$ then $\mathcal{L}_\varphi^=(0) \neq \mathbb{R}^n$ and so $\left(\mathcal{L}_\varphi^=(0)\right)^\circ$ also contains nonzero elements.
Applying Lemma 4.4 and Theorem 6.3 of [21] it follows for $\mathcal{L}_\varphi^<(0)$ nonempty that

$$\mathrm{cl}(\mathcal{L}_\varphi^<(0)) = \mathrm{cl}(\mathrm{cone}(\mathcal{L}_f^<(f(\boldsymbol{x}_0)) - \boldsymbol{x}_0)).$$

Similar as for convex functions (see [12, 21]), this yields

$$\begin{aligned} \mathcal{N}_{\mathcal{L}_f^<}(\mathbf{x}_0) &= \{\mathbf{x}^* \in \mathbb{R}^n : \langle \mathbf{x}^*, \mathbf{x} - \mathbf{x}_0\rangle \leq 0 \text{ for every } \mathbf{x} \in \mathcal{L}_f^<(f(\mathbf{x}_0)\} \\ &= \Big(\mathrm{cone}(\mathcal{L}_f^<(f(\mathbf{x}_0)) - \mathbf{x}_0)\Big)^\circ \\ &= \Big(cl(\mathrm{cone}(\mathcal{L}_f^<(f(\mathbf{x}_0)) - \mathbf{x}_0))\Big)^\circ \\ &= (cl(\mathcal{L}_\varphi(0)))^\circ \end{aligned} \tag{4.6}$$

with $\mathcal{K}^\circ$ denoting the polar cone of $\mathcal{K}$. Hence, to give an alternative description of the set $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$, it is sufficient by relation (4.6) or Lemma 4.8 to show that the set $\mathrm{cl}(\mathcal{L}_\varphi^<(0))$ or $\mathrm{cl}(\mathcal{L}_\varphi^=(0))$ is the polar cone of some other closed cone K and then apply the bipolar theorem.

Lemma 4.9. *If the function $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is a quasiconvex function satisfying $f(\boldsymbol{x}_0)$ finite and the function φ given by $\varphi(\boldsymbol{d}) := f'_+(\boldsymbol{x}_0; \boldsymbol{d})$ is a proper evenly quasiconvex function satisfying $\mathcal{L}_\varphi^<(0)$ is nonempty then*

$$\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0) = \mathrm{cl}(\mathrm{cone}(\partial\varphi_-(\mathbf{0})))$$

with $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$ the normal cone of $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ at $\boldsymbol{x}_0$ defined in relation (4.5).

Proof. By Theorem 3.4 and Proposition VI.1.3.3 of [12] we obtain that

$$\begin{aligned} \mathrm{cl}(\mathcal{L}_\varphi^<(0)) &= \{\boldsymbol{d} \in \mathbb{R}^n : \varphi_-(\boldsymbol{d}) \leq 0\} \\ &= \{\boldsymbol{d} \in \mathbb{R}^n : \langle \boldsymbol{x}^\star, \boldsymbol{d}\rangle \leq 0 \text{ for every } \boldsymbol{x}^\star \in \partial\varphi_-(\mathbf{0})\} \\ &= (\mathrm{cone}(\partial\varphi_-(\mathbf{0})))^\circ. \end{aligned}$$

Hence by (4.6) and Proposition III.4.2.7 of [12] it follows that

$$\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0) = (\mathrm{cl}(\mathcal{L}_\varphi^<(0))^\circ = (\mathrm{cone}(\partial\varphi_-(\mathbf{0})))^{\circ\circ} = \mathrm{cl}(\mathrm{cone}(\partial\varphi_-(\mathbf{0})))$$

and this shows the desired result. □

An interpretation of Lemma 4.9 is provided by Figure 4.

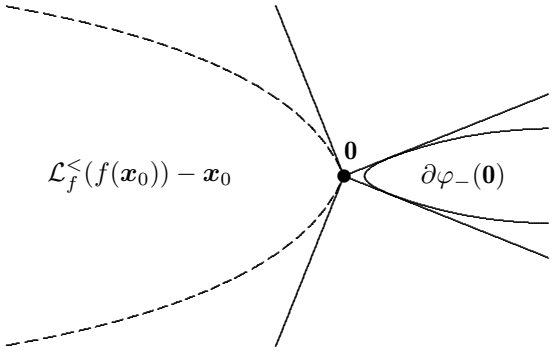

FIGURE 4. The normal cone to the strict lower level set in the favorable case

Compare now this figure with Figure 1, The situation described in Figure 4 corresponds to taking $\boldsymbol{x}_0 = \boldsymbol{b}$ in Figure 1.

Since φ_- is sublinear it follows that $\partial\varphi_-(\boldsymbol{d}) \subseteq \partial\varphi_-(\mathbf{0}) \subseteq \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$ for every $\boldsymbol{d} \in \mathrm{dom}(\varphi_-) = \mathrm{cl}(\mathcal{L}_\varphi^<(0))$. If additionally, $\mathcal{L}_\varphi^<(0)$ is a convex cone of dimension n this implies by Theorem IV.4.2.3 of [12] that φ_- is differentiable on a dense subset of $\mathrm{int}(\mathcal{L}_\varphi^<(0))$, and so we can conclude for a point belonging to this dense subset that $\nabla\varphi_-(\boldsymbol{d}) \in \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$.

An immediate consequence of Lemma 4.8 and Lemma 3.2 is given by the following result. Although this result is not difficult to prove it appears to be new and improves Lemma 4.8 for φ evenly quasiconvex.

Lemma 4.10. *If the function $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is a quasiconvex function satisfying $f(\boldsymbol{x}_0)$ finite and the set $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ nonempty and the function φ given by $\varphi(\boldsymbol{d}) = f'_+(\boldsymbol{x}_0; \boldsymbol{d})$ is proper and evenly quasiconvex and the set $\mathcal{L}_\varphi^<(0)$ empty then*

$$\mathrm{cl}(\mathrm{cone}(\partial\varphi_+(\mathbf{0}))) \subseteq \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$$

with $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$ the normal cone of $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ at $\boldsymbol{x}_0$ defined in relation (4.5).

Proof. By our assumptions it follows that $\mathcal{L}_\varphi^=(0)$ is nonempty and $\mathcal{L}_\varphi^\leq(0) = \mathcal{L}_\varphi^=(0)$. This implies by Lemma 3.2 and φ_+ being the support function of $\partial\varphi_+(\mathbf{0})$ that

$$\begin{aligned}
\mathcal{L}_\varphi^=(0) &= \{\boldsymbol{d} \in \mathbb{R}^n : \varphi_+(\boldsymbol{d}) = 0\} \\
&= \{\boldsymbol{d} \in \mathbb{R}^n : \langle \boldsymbol{x}^\star, \boldsymbol{d}\rangle \leq 0 \text{ for every } \boldsymbol{d} \in \partial\varphi_+(\mathbf{0})\} \\
&= (\mathrm{cone}(\partial\varphi_+(\mathbf{0})))^\circ.
\end{aligned}$$

Applying now Lemma 4.8 and Proposition III.4.2.7 of [12] yields

$$\mathrm{cl}(\mathrm{cone}(\partial\varphi_+(\mathbf{0}))) = (\mathrm{cone}(\partial\varphi_+(\mathbf{0})))^{\circ\circ} = (\mathcal{L}_\varphi^=(0))^\circ \subseteq \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$$

and the desired result is proven. □

As already observed, if $\boldsymbol{d} \longmapsto f'_+(\boldsymbol{x}_0; \boldsymbol{d})$ is the zero functional or equivalently $\partial\varphi_+(\mathbf{0}) = \{\mathbf{0}\}$, the above result does not provide any useful information.

5. How to separate, if you must!

In this section we analyze the problem of computing an element of the normal cone $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$ if $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ is nonempty. As already observed, a sufficient condition for $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ to be nonempty is given by the nonemptiness of the set $\mathcal{L}_\varphi^<(0)$ with $\varphi(\boldsymbol{d}) := f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ and so it is natural to consider the optimization problem

$$\vartheta(S) = \inf\{\varphi(\mathbf{d}) : \mathbf{d} \in \mathcal{C}\} \tag{S}$$

with $\mathcal{C}$ a compact convex set satisfying $\mathbf{0} \in \operatorname{int}(\mathcal{C})$. Notice, since $\dim(\mathcal{C}) = n$ that the boundary $\operatorname{bd}(\mathcal{C})$ of $\mathcal{C}$ is given by $\mathcal{C} \setminus \operatorname{int}(\mathcal{C})$. In order to guarantee that the optimization problem (S) is solvable, i.e. there exists some $\boldsymbol{d}_0 \in \mathcal{C}$ satisfying $\varphi(\boldsymbol{d}_0) = \vartheta(S)$, it is sufficient by Theorem 3.4 to assume that φ is a proper evenly quasiconvex function. In the remainder of this section we always assume that $\boldsymbol{d} \longmapsto f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ satisfies this property and so the set $\mathcal{S}$ of optimal solutions of optimization problem (S) is always nonempty. Clearly, one should choose the compact convex set $\mathcal{C}$ with $\mathbf{0} \in \operatorname{int}(\mathcal{C})$ in such a way that optimization problem (S) is "easy" solvable. Since φ is a proper, positively homogeneous and evenly quasiconvex function the following result is easy to verify and so its proof is omitted.

Lemma 5.1. *It follows $\vartheta(S) < 0$ if and only if $\mathcal{L}_\varphi^<(0)$ is nonempty. If this holds then $\mathcal{S} \subseteq \operatorname{bd}(\mathcal{C})$. Moreover, if $\vartheta(S) = 0$, i.e. $\boldsymbol{x}_0 \in \Gamma_f$, then either $\mathbf{0}$ is the unique solution of (S) or $\mathcal{S} \cap \operatorname{bd}(\mathcal{C})$ is nonempty.*

Clearly, if $\mathbf{0}$ is the unique solution of optimization problem (S) then due to $\mathbf{0} \in \operatorname{int}(\mathcal{C})$, $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ quasiconvex and $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ proper and positively homogeneous, it follows that $\mathcal{L}_f^<(f(\boldsymbol{x}_0))$ is empty. Also, if $\vartheta(S) < 0$ we obtain by Lemma 4.9 that an optimal solution $\boldsymbol{d}_0$ of optimization problem (S) is also an optimal solution of the optimization problem

$$\vartheta(S') = \inf\{\varphi_-(\mathbf{d}) : \gamma_C(\mathbf{d}) \le 1, \mathbf{d} \in cl(\mathcal{L}_\varphi^<(0)\} \tag{S'}$$

with $\gamma_{\mathcal{C}}(\boldsymbol{d}) := \inf\{t > 0 : \boldsymbol{d} \in t\mathcal{C}\}$ the *gauge* of $\mathcal{C}$. Since by Lemma 3.3 the function φ_- is proper and convex with $\operatorname{dom}(\varphi_-) = \operatorname{cl}(\mathcal{L}_\varphi^<(0))$ and the function $\gamma_{\mathcal{C}}$ is finite and convex due to $\mathcal{C}$ compact, convex and $\mathbf{0} \in \operatorname{int}(\mathcal{C})$, the optimization problem (S) satisfies the properties of a convex program given in Section 28 of [21].

Using now so-called *primal-dual* information given by the *Karush-Kuhn-Tucker* conditions it is possible to prove the next result.

Lemma 5.2. *If $\boldsymbol{d}_0$ is an optimal solution of (S) with $\vartheta(S) < 0$ then the set $\partial\varphi_-(\boldsymbol{d}_0) - \vartheta(S)\partial\gamma_{\mathcal{C}}(\boldsymbol{d}_0)$ contains $\mathbf{0}$.*

Proof. If $\boldsymbol{d}_0$ is an optimal solution of (S) with $\varphi(\boldsymbol{d}_0) = \vartheta(S) < 0$ then $\lambda\boldsymbol{d}_0 \in \mathcal{L}_\varphi^<(0)$ for every $\lambda > 0$. Also, by Lemma 5.1 we obtain that $\boldsymbol{d}_0 \in \operatorname{bd}(\mathcal{C})$ implying that $\gamma_{\mathcal{C}}(\boldsymbol{d}_0) = 1$ and so $\gamma_{\mathcal{C}}(\lambda\boldsymbol{d}_0) = \lambda\gamma_{\mathcal{C}}(\boldsymbol{d}_0) < 1$ for every $0 < \lambda < 1$. Hence, by Corollary 28.2.1 of [21] a Karush-Kuhn-Tucker vector λ_1 of (S) exists and this yields by Theorem 28.3 of [21] that $\mathbf{0} \in \partial\varphi_-(\boldsymbol{d}_0) + \lambda_1\partial\gamma_{\mathcal{C}}(\boldsymbol{d}_0)$, $\lambda_1(\gamma_{\mathcal{C}}(\boldsymbol{d}_0) - 1) = 0$ and $\lambda_1 \ge 0$. If $\lambda_1 = 0$ it follows that $\mathbf{0} \in \partial\varphi_-(\boldsymbol{d}_0)$ and this yields $\varphi_-(\boldsymbol{d}) \ge \varphi_-(\boldsymbol{d}_0)$ for every $\boldsymbol{d} \in \mathbb{R}^n$. However, since φ_- is positively homogeneous and $\varphi_-(\boldsymbol{d}_0) < 0$ it follows that $\varphi_-(\lambda\boldsymbol{d}_0) = \lambda\varphi_-(\boldsymbol{d}_0) < \varphi_-(\boldsymbol{d}_0)$

for every $\lambda > 1$ contradicting $\mathbf{0} \in \partial\varphi_-(\boldsymbol{d}_0)$. Hence, $\lambda_1 > 0$ and to compute λ_1 we observe the following. It is well-known, [12, 21], that

$$\partial\gamma_{\mathcal{C}}(\boldsymbol{d}_0) = \{\boldsymbol{d}_0^\star \in \mathcal{C}^\circ : \langle \boldsymbol{d}_0^\star, \boldsymbol{d}_0\rangle = \gamma_{\mathcal{C}}(\boldsymbol{d}_0)\}$$

with $\mathcal{C}^\circ$ the polar of $\mathcal{C}$ and so by the Karush-Kuhn-Tucker conditions and Lemma 5.1 there exists some $\boldsymbol{d}_0^\star \in \mathbb{R}^n$ with $-\boldsymbol{d}_0^\star \in \partial\varphi_-(\boldsymbol{d}_0)$, $\langle \boldsymbol{d}_0^\star, \boldsymbol{d}_0\rangle = \lambda_1$ and $\langle \boldsymbol{d}_0^\star, \boldsymbol{d}\rangle \leq \lambda_1$ for every $\boldsymbol{d} \in \mathcal{C}$. Since $-\boldsymbol{d}_0^\star \in \partial\varphi_-(\boldsymbol{d}_0)$ it follows by Theorem 23.5 of [21] that

$$\varphi_-(\boldsymbol{d}_0) + \varphi_-^*(-\boldsymbol{d}_0^\star) = -\langle \boldsymbol{d}_0^\star, \boldsymbol{d}_0\rangle$$

with φ_-^* the conjugate function of φ_-. Since φ_- is positively homogeneous and thus φ_-^* is either 0 or $+\infty$ we obtain by the above equality that

$$\varphi_-(\boldsymbol{d}_0) = -\langle \boldsymbol{d}_0^\star, \boldsymbol{d}_0\rangle = -\lambda_1$$

and so the result is proven. □

The following result is an immediate consequence of the previous lemma.

Corollary 5.3. *If $\gamma_{\mathcal{C}}$ is differentiable in $\boldsymbol{d}_0$ then $-\nabla\gamma_{\mathcal{C}}(\boldsymbol{d}_0) \in \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$.*

Proof. The previous result shows for $\vartheta(S) < 0$ and $\boldsymbol{d}_0$ an optimal solution of (S) that the sets $\partial\varphi_-(\boldsymbol{d}_0)$ and $\vartheta(S)\partial\gamma_{\mathcal{C}}(\boldsymbol{d}_0)$ intersect. Hence, if $\gamma_{\mathcal{C}}$ is differentiable in $\boldsymbol{d}_0$ with gradient $\nabla\gamma_{\mathcal{C}}(\boldsymbol{d}_0)$ then

$$\vartheta(S)\partial\gamma_{\mathcal{C}}(\boldsymbol{d}_0) = \{\vartheta(S)\nabla\gamma_{\mathcal{C}}(\boldsymbol{d}_0)\}$$

and so $\vartheta(S)\nabla\gamma_{\mathcal{C}}(\boldsymbol{d}_0) \in \partial\varphi_-(\boldsymbol{d}_0) \subseteq \partial\varphi_-(\mathbf{0})$. Now, by Lemma 3.3 it follows that $\partial\varphi_-(\mathbf{0}) \subseteq \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$ and since $\vartheta(S) < 0$ and $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$ is a cone this leads to the stated result. □

In the next example we discuss the well know Lp norm.

Example 5.4. *Take $\mathcal{C} := \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_p \leq 1\}$ with $\|\cdot\|_p$, $1 < p < \infty$, the ℓ_p-norm. Clearly, $\gamma_{\mathcal{C}}(\boldsymbol{x}) = \|\boldsymbol{x}\|_p$ and $\gamma_{\mathcal{C}}$ is differentiable everywhere except at $\mathbf{0}$. Moreover, for every $\boldsymbol{x} \neq \mathbf{0}$ it is easy to verify that*

$$\nabla\gamma_{\mathcal{C}}(\boldsymbol{x}) = \|\boldsymbol{x}\|_p^{1-p} \begin{bmatrix} \operatorname{sign}(x_1)|x_1|^{p-1} \\ \vdots \\ \operatorname{sign}(x_s)|x_s|^{p-1} \end{bmatrix}$$

with x_i the ith component of $\boldsymbol{x}$ and $\operatorname{sign}(x)$ the sign function.

If $\vartheta(S) < 0$, $\boldsymbol{d}_0$ solves optimization problem (S) and $\gamma_{\mathcal{C}}$ is not differentiable in $\boldsymbol{d}_0$ while $\varphi(\boldsymbol{d})$ is differentiable in $\boldsymbol{d}_0$ then it is easy to show, due to $\boldsymbol{d}_0 \in \operatorname{int}(\mathcal{L}_\varphi^<(0))$ and the definition of φ_-, that $\nabla\varphi(\boldsymbol{d}_0) = \nabla\varphi_-(\boldsymbol{d}_0) \in \partial\varphi_-(\mathbf{0})$ with $\varphi(\boldsymbol{d}) := f_+'(\boldsymbol{x}_0; \boldsymbol{d})$. In this case, the optimization problem (S) is only used to identify an interior element of $\mathcal{L}_\varphi^<(0)$. This also shows that selecting some $\boldsymbol{d}_1 \in \operatorname{int}(\mathcal{L}_\varphi^<(0))$ with φ differentiable in $\boldsymbol{d}_1$ already yields an element of the normal cone $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$.

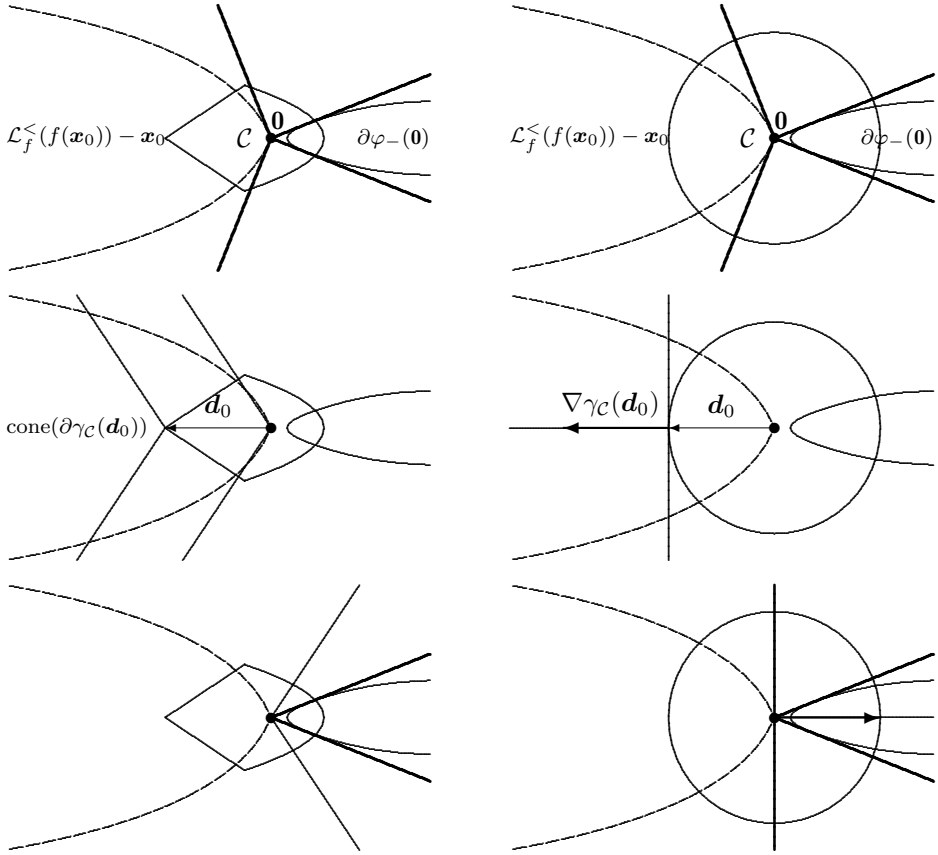

FIGURE 5. Geometric interpretation of the separation oracle

Finally, we provide in Figure 5 a geometrical interpretation of Lemma 5.2. The first picture shows a set $\mathcal{C}$ with a kink at $\boldsymbol{d}_0$ and for which $\mathrm{cone}(\partial\gamma_{\mathcal{C}}(\boldsymbol{d}_0))$ is a cone (shifted in the picture to the vertex of $\mathcal{C}$ for clarity) whose symmetric cone intersects $\partial\varphi_-(\boldsymbol{d}_0)$ (by Lemma 5.2) but includes elements which do not belong to $\mathrm{cone}(\partial\varphi_-(\boldsymbol{d}_0)) = \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$. On the other hand, the second picture corresponds to a smooth $\mathcal{C}$. Hence $\partial\gamma_{\mathcal{C}}(\boldsymbol{d}_0)$ is a singleton and so the symmetric of its conical hull (a half line) intersects (by Lemma 5.2 again) $\mathrm{cone}(\partial\varphi_-(\boldsymbol{d}_0))$ and it must be included in $\mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$.

We triplicate each picture for clarity. The top picture shows the conical hull of the strict lower level set, the corresponding normal cone and the compact convex set $\mathcal{C}$ corresponding to the feasible region of optimization problem (S). The middle picture shows the solution of problem (S), direction $\boldsymbol{d}_0$, and the conical hull of $\partial\gamma_{\mathcal{C}}(\boldsymbol{d}_0)$. Finally, the bottom picture shows the intersection of the symmetric of this conical hull and the normal cone.

We consider in the next section several quasiconvex functions for which we do not have to solve the optimization problem (S).

6. Examples

This section illustrates classes of functions for which optimization problem (S) can be replaced by easier membership problems.

6.1. Regular functions

In this subsection we discuss a separation oracle for the following subclass of quasiconvex functions.

Definition 6.1. Let $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ be a quasiconvex function with $f(\boldsymbol{x}_0)$ finite. Then f is called regular at $\boldsymbol{x}_0$ if φ is a lower semicontinuous sublinear function with $\varphi(\boldsymbol{d}) := f'_+(\boldsymbol{x}_0; \boldsymbol{d})$.

Following Pshenichnyi in [20] these functions are sometimes called *quasidifferentiable*. However, we prefer to follow Clarke, [1], and call them *regular* since the term quasidifferentiable has nowadays a broader meaning, see [5].

As the next lemma shows the above class of functions is closed under the finite max operator.

Lemma 6.2. *Let $f_i : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$, $i = 1, \ldots, n$, be quasiconvex functions with $f_i(\boldsymbol{x}_0)$ finite for every $1 \le i \le n$. If each function f_i is regular at $\boldsymbol{x}_0$ then the function $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ given by $f(\boldsymbol{x}) := \max_{1 \le i \le n} f_i(\boldsymbol{x})$ is also regular at $\boldsymbol{x}_0$.*

Proof. It is easy to verify that f is quasiconvex and $f(\boldsymbol{x}_0)$ is finite. Moreover, by Lemma 2.5.3 of [11] we obtain that

$$f'_+(\boldsymbol{x}_0; \boldsymbol{d}) = \max_{i \in I(\boldsymbol{x}_0)} f'_{i_+}(\boldsymbol{x}_0; \boldsymbol{d}) \tag{6.1}$$

with $I(\boldsymbol{x}_0) := \{1 \le i \le n : f(\boldsymbol{x}_0) = f_i(\boldsymbol{x}_0)\}$ the set of active indices of f at $\boldsymbol{x}_0$. Since by assumption it follows that $\boldsymbol{d} \longmapsto f'_{i_+}(\boldsymbol{x}_0; \boldsymbol{d})$ is a lower semicontinuous sublinear function for every $1 \le i \le n$ the desired result follows by (6.1). □

An important class of regular functions is given by the next lemma. These functions are extremely important in location analysis, see [11].

Lemma 6.3. *Let $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ be a finite nondecreasing quasiconvex function and $\boldsymbol{v} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ a finite-valued convex vector function, i.e. $\boldsymbol{v}(\boldsymbol{x}) := (v_1(\boldsymbol{x}), \ldots, v_m(\boldsymbol{x}))$ with $v_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, \ldots, m$, finite-valued convex functions. If the function $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is given by $f(\boldsymbol{x}) = g(\boldsymbol{v}(\boldsymbol{x}))$ and g is regular and locally Lipschitz at $\boldsymbol{v}(\boldsymbol{x}_0)$ then f is a quasiconvex function regular at $\boldsymbol{x}_0$. Moreover, $f'_+(\boldsymbol{x}_0; \boldsymbol{d}) = g'_+(\boldsymbol{v}(\boldsymbol{x}_0); \boldsymbol{v}'(\boldsymbol{x}_0; \boldsymbol{d}))$ with $\boldsymbol{v}'(\boldsymbol{x}_0; \boldsymbol{d}) = (v'_1(\boldsymbol{x}_0; \boldsymbol{d}), \ldots, v'_m(\boldsymbol{x}_0; \boldsymbol{d}))$ and the function $\boldsymbol{d} \longmapsto f'_+(\boldsymbol{x}_0; \boldsymbol{d})$ is Lipschitz continuous.*

Proof. Since g is a nondecreasing quasiconvex function and $\boldsymbol{v}$ a convex vector function it is easy to verify that f is quasiconvex. Also, by Lemma 2.5.2 of [11] it follows that

$f'_+(\boldsymbol{x}_0;\boldsymbol{d}) = g'_+(\boldsymbol{v}(\boldsymbol{x}_0);\boldsymbol{v}'(\boldsymbol{x}_0;\boldsymbol{d}))$. Moreover, since g is regular at $\boldsymbol{v}(\boldsymbol{x}_0)$ and nondecreasing we obtain for every $0<\lambda<1$ and $\boldsymbol{d}_1,\boldsymbol{d}_2\in\mathbb{R}^n$ that

$$\begin{aligned}
&f'_+(\boldsymbol{x}_0;\lambda\boldsymbol{d}_1+(1-\lambda)\boldsymbol{d}_2)\\
&\quad= g'_+(\boldsymbol{v}(\boldsymbol{x}_0);\boldsymbol{v}'(\boldsymbol{x}_0;\lambda\boldsymbol{d}_1+(1-\lambda)\boldsymbol{d}_2))\\
&\quad\le g'_+(\boldsymbol{v}(\boldsymbol{x}_0);\lambda\boldsymbol{v}'(\boldsymbol{x}_0;\boldsymbol{d}_1)+(1-\lambda)\boldsymbol{v}'(\boldsymbol{x}_0;\boldsymbol{d}_2))\\
&\quad\le \lambda g'_+(\boldsymbol{v}(\boldsymbol{x}_0);\boldsymbol{v}'(\boldsymbol{x}_0;\boldsymbol{d}_1))+(1-\lambda)g'_+(\boldsymbol{v}(\boldsymbol{x}_0);\boldsymbol{v}'(\boldsymbol{x}_0;\boldsymbol{d}_2))\\
&\quad= \lambda f'_+(\boldsymbol{x}_0;\boldsymbol{d}_1)+(1-\lambda)f'_+(\boldsymbol{x}_0;\boldsymbol{d}_2)
\end{aligned}$$

and by the Lipschitz continuity of $\boldsymbol{d}\longmapsto\boldsymbol{v}'(\boldsymbol{x}_0;\boldsymbol{d})$ and $\boldsymbol{d}\longmapsto g'_+(\boldsymbol{v}(\boldsymbol{x}_0);\boldsymbol{d})$, the Lipschitz continuity of $\boldsymbol{d}\longmapsto f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ follows. □

For the class of functions given in Definition 6.1 it is now easy, using only classical results of convex analysis, to prove the next result.

Lemma 6.4. *If $f:\mathbb{R}^n\longrightarrow[-\infty,+\infty]$ is a quasiconvex function regular at $\boldsymbol{x}_0$ and $\mathcal{L}^<_\varphi(0)$ is nonempty then*

$$\mathrm{cl}(\mathrm{cone}(\partial\varphi(\mathbf{0})))=\mathcal{N}_{\mathcal{L}^<_f}(\boldsymbol{x}_0)$$

with $\partial\varphi(\mathbf{0})$ the subgradient set of the convex function $\varphi(\boldsymbol{d}):=f'_+(\boldsymbol{x}_0;\boldsymbol{d})$ at $\mathbf{0}$. Moreover, if $\mathcal{L}^<_f(f(\boldsymbol{x}_0))$ is nonempty and $\mathcal{L}^<_\varphi(0)$ is empty then

$$\mathrm{cl}(\mathrm{cone}(\partial\varphi(\mathbf{0})))\subseteq\mathcal{N}_{\mathcal{L}^<_f}(\boldsymbol{x}_0).$$

Proof. To prove the first result we observe by relation (4.6) that

$$\mathcal{N}_{\mathcal{L}^<_f}(\boldsymbol{x}_0)=(\mathrm{cl}(\mathcal{L}^<_\varphi(0)))^\circ.$$

Since f is a quasiconvex function regular at $\boldsymbol{x}_0$ it follows by Proposition VI.1.3.3 of [12] that $\mathrm{cl}(\mathcal{L}^<_\varphi(0))=\mathcal{L}^\le_\varphi(0)$. Moreover, by Theorem V.3.1.1 of [12] we obtain that $\mathcal{L}^\le_\varphi(0)$ equals $\{\boldsymbol{d}:\langle\boldsymbol{x}^\star,\boldsymbol{d}\rangle\le 0$ for every $\boldsymbol{x}^\star\in\partial\varphi(\mathbf{0})\}$. Clearly this set also equals $(\mathrm{cone}(\partial\varphi(\mathbf{0})))^\circ$ and hence by (4.6) and Proposition III.4.2.7 of [12] we obtain that

$$\mathcal{N}_{\mathcal{L}^<_f}(\boldsymbol{x}_0)=(\mathrm{cone}(\partial\varphi(\mathbf{0})))^{\circ\circ}=\mathrm{cl}(\mathrm{cone}(\partial\varphi(\mathbf{0}))).$$

The second result can be proved in a similar way and this completes the proof. □

Finally we can show the main result of this subsection. Recall that the set of "bad" points Γ_f is defined in (4.4).

Lemma 6.5. *Let $g_i:\mathbb{R}^m\longrightarrow\mathbb{R}$, $i=1,\dots,n$, be quasiconvex and continuously differentiable functions and suppose $v_i:\mathbb{R}^n\longrightarrow\mathbb{R}$, $1\le i\le m$, are finite-valued convex functions. Then it follows for $f(\boldsymbol{x}):=\max_{1\le i\le n}f_i(\boldsymbol{x})$ with $f_i(\boldsymbol{x}):=g_i(\boldsymbol{v}(\boldsymbol{x}))$ that $\boldsymbol{x}_0$ belongs to Γ_f if and only if*

$$\mathbf{0}\in\mathrm{conv}\left(\bigcup_{i\in I(\boldsymbol{x}_0)}\sum_{j=1}^m\tfrac{\partial g_i}{\partial z_j}(\boldsymbol{v}(\boldsymbol{x}_0))\partial v_j(\boldsymbol{x}_0)\right)$$

where $I(\boldsymbol{x}_0) := \{1 \le i \le n : f(\boldsymbol{x}_0) = f_i(\boldsymbol{x}_0)\}$. *Moreover, if* $\mathcal{L}_\varphi^<(0)$ *is nonempty, i.e.* $\boldsymbol{x}_0 \notin \Gamma_f$, *then*

$$\operatorname{cone}\left(\operatorname{conv}\left(\bigcup_{i\in I(\boldsymbol{x}_0)} \sum_{j=1}^m \tfrac{\partial g_i}{\partial z_j}(\boldsymbol{v}(\boldsymbol{x}_0))\partial v_j(\boldsymbol{x}_0)\right)\right) = \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0)$$

Proof. Clearly every function $g_i : \mathbb{R}^m \longrightarrow \mathbb{R}$ is regular and locally Lipschitz at $\boldsymbol{x}_0$. Applying now Lemma 6.3 yields

$$f_i'(\boldsymbol{x}_0;\boldsymbol{d}) = \sum_{j=1}^m \tfrac{\partial g_i}{\partial z_j}(\boldsymbol{v}(\boldsymbol{x}_0))v_j'(\boldsymbol{x}_0;\boldsymbol{d})$$

and so by Theorem V.3.1.1 of [12] we obtain for $1 \le i \le n$ that

$$f_i'(\boldsymbol{x}_0;\boldsymbol{d}) = \max\{\langle \boldsymbol{d}, \boldsymbol{x}^\star\rangle : \boldsymbol{x}^\star \in \partial\varphi_i(\mathbf{0})\}$$

with

$$\partial\varphi_i(\mathbf{0}) := \sum_{j=1}^m \frac{\partial g_i}{\partial z_j}(\boldsymbol{v}(\boldsymbol{x}_0))\partial v_j(\boldsymbol{x}_0).$$

By Lemma 2.5.3 of [11] this implies

$$\begin{aligned} f'(\boldsymbol{x}_0;\boldsymbol{d}) &= \max_{i\in I(\boldsymbol{x}_0)} f_i'(\boldsymbol{x}_0;\boldsymbol{d}) \\ &= \max_{i\in I(\boldsymbol{x}_0)} \max\{\langle \boldsymbol{d}, \boldsymbol{x}^\star\rangle : \boldsymbol{x}^\star \in \partial\varphi_i(\mathbf{0})\} \\ &= \max\left\{\langle \boldsymbol{d}, \boldsymbol{x}^\star\rangle : \boldsymbol{x}^\star \in \operatorname{conv}\left(\bigcup_{i\in I(\boldsymbol{x}_0)} \partial\varphi_i(\mathbf{0})\right)\right\}. \end{aligned}$$

Using the above relation it follows that $f'(\boldsymbol{x}_0;\boldsymbol{d}) \ge 0$ for every $\boldsymbol{d} \in \mathbb{R}^n$ if and only if $\mathbf{0}$ belongs to $\operatorname{conv}\left(\bigcup_{i\in I(\boldsymbol{x}_0)} \partial\varphi_i(\mathbf{0})\right)$. This proves the first part. To prove the second part we observe that $\operatorname{conv}\left(\bigcup_{i\in I(\boldsymbol{x}_0)} \partial\varphi_i(\mathbf{0})\right)$ is the subgradient set of the finite valued convex function φ at $\mathbf{0}$ and since $\partial\varphi_i(\mathbf{0})$ is compact for each i and $\mathbf{0}$ does not belong to $\operatorname{conv}\left(\bigcup_{i\in I(\boldsymbol{x}_0)} \partial\varphi_i(\mathbf{0})\right)$ the second result follows by Lemma 6.4 together with Lemma III.1.4.7 and Theorem III.1.4.3 of [12]. □

In the next subsection we consider another class of quasiconvex functions for which the separation problem is easy.

6.2. Another class of easy functions

Let $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuously differentiable and convex function and $\gamma_i \in \mathbb{R}$, $1 \le i \le m$, and introduce the functions $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$f_i(\boldsymbol{x}) := \min\{g_i(\boldsymbol{x}), \gamma_i\}.$$

Clearly, the functions f_i are quasiconvex and so is the function

$$f(\boldsymbol{x}) := \max_{1\le i\le m} f_i(\boldsymbol{x}).$$

A representation of such a function for the case of affine g_i is given in Figure 6.

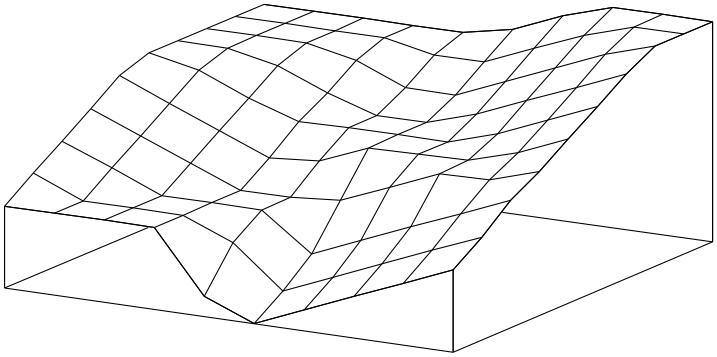

FIGURE 6. A bivariate quasiconvex function with horizontal regions

Moreover, if $I(\boldsymbol{x}_0) := \{1 \le i \le m : f(\boldsymbol{x}_0) = f_i(\boldsymbol{x}_0)\}$ it follows that

$$f'(\boldsymbol{x}_0; \boldsymbol{d}) = \max_{i \in I(\boldsymbol{x}_0)} f_i'(\boldsymbol{x}_0; \boldsymbol{d}) \tag{6.2}$$

and

$$f_i'(\boldsymbol{x}_0; \boldsymbol{d}) = \begin{cases} \min\{\langle \nabla g_i(\boldsymbol{x}_0), \boldsymbol{d}\rangle, 0\} & \text{if } g_i(\boldsymbol{x}_0) = \gamma_i \\ \langle \nabla g_i(\boldsymbol{x}_0), \boldsymbol{d}\rangle & \text{if } g_i(\boldsymbol{x}_0) < \gamma_i \\ 0 & \text{if } g_i(\boldsymbol{x}_0) > \gamma_i \end{cases}. \tag{6.3}$$

By (6.2) and (6.3) it is clear that $f'(\boldsymbol{x}_0; \boldsymbol{d}) \ge 0$ for every $\boldsymbol{d} \in \mathbb{R}^n$ if there exists some $i \in I(\boldsymbol{x}_0)$ satisfying $g_i(\boldsymbol{x}_0) > \gamma_i$ and this implies that $\vartheta(S) = \min_{\boldsymbol{d} \in \mathcal{C}} f'(\boldsymbol{x}_0; \boldsymbol{d}) \ge 0$. Therefore assume for every $i \in I(\boldsymbol{x}_0)$ that $g_i(\boldsymbol{x}_0) \le \gamma_i$. If this holds the following result is easy to prove.

Lemma 6.6. *If for every $i \in I(\boldsymbol{x}_0)$ it follows that $g_i(\boldsymbol{x}_0) \le \gamma_i$ then $\vartheta(S) < 0$ if and only if $\mathbf{0} \notin \operatorname{conv}(\{\nabla g_i(\boldsymbol{x}_0), i \in I(\boldsymbol{x}_0)\})$. Moreover, if this holds then*

$$\operatorname{conv}(\{\nabla g_i(\boldsymbol{x}_0), i \in I(\boldsymbol{x}_0)\}) \subseteq \mathcal{N}_{\mathcal{L}_f^<}(\boldsymbol{x}_0).$$

Proof. Clearly, by the assumption $g_i(\boldsymbol{x}_0) \le \gamma_i$ for every $i \in I(\boldsymbol{x}_0)$, (6.2) and (6.3) we obtain that $\min_{\boldsymbol{d} \in \mathcal{C}} f'(\boldsymbol{x}; \boldsymbol{d})$ is equivalent to the optimization problem

$$\begin{array}{lllll} \min & t & & & \\ \text{st}: & t & \ge & \min\{\langle \nabla g_i(\boldsymbol{x}_0), \boldsymbol{d}\rangle, 0\} & \text{for every } i \in J(\boldsymbol{x}_0) \\ & t & \ge & \langle \nabla g_i(\boldsymbol{x}_0), \boldsymbol{d}\rangle & \text{for every } \ i \in I(\boldsymbol{x}_0) \setminus J(\boldsymbol{x}_0) \\ & \boldsymbol{d} & \in & \mathcal{C} & \end{array}$$

with $J(\boldsymbol{x}_0) := \{i \in I(\boldsymbol{x}_0) : g_i(\boldsymbol{x}_0) = \gamma_i\}$. This implies that $\vartheta(S) < 0$ if and only if the optimization problem

$$\begin{array}{llll} \min & t & & \\ \text{st}: & t & \ge & \langle \nabla g_i(\boldsymbol{x}_0), \boldsymbol{d}\rangle \quad \text{for every } \ i \in I(\boldsymbol{x}_0) \\ & \boldsymbol{d} & \in & \mathcal{C} \end{array}$$

has a negative objective value. This problem in turn is equivalent to

$$\min_{\boldsymbol{d} \in \mathcal{C}} \max_{i \in I(\boldsymbol{x}_0)} \langle \nabla g_i(\boldsymbol{x}_0), \boldsymbol{d}\rangle = \min_{\boldsymbol{d} \in \mathcal{C}} \varphi(\boldsymbol{d})$$

with

$$\varphi(\boldsymbol{d}) := \max\{\langle \boldsymbol{d}, \boldsymbol{y}\rangle : \boldsymbol{y} \in \operatorname{conv}(\{\nabla g_i(\boldsymbol{x}_0), i \in I(\boldsymbol{x}_0)\})\}.$$

We finally obtain that $\vartheta(S) < 0$ if and only if there exists some $\boldsymbol{d} \in \mathcal{C}$ with $\varphi(\boldsymbol{d}) < 0$ or equivalently $\boldsymbol{0} \notin \operatorname{conv}(\{\nabla g_i(\boldsymbol{x}_0), i \in I(\boldsymbol{x}_0)\})$. Observe that for $s = 2$ this decision can be carried out by means of the linear time algorithm presented in [10]. By the definition of φ_- and the representation of $f'(\boldsymbol{x}_0; \boldsymbol{d})$ it follows that

$$\varphi_-(\boldsymbol{d}) = \begin{cases} \varphi(\boldsymbol{d}) & \text{if } \boldsymbol{d} \in \operatorname{cl}(\mathcal{L}_\varphi^<(0)) \\ +\infty & \text{otherwise} \end{cases}$$

and so any $\nabla g_i(\boldsymbol{x}_0)$, $i \in I(\boldsymbol{x}_0)$, belongs to $\partial\varphi_-(\boldsymbol{0})$. This implies

$$\operatorname{conv}(\{\nabla g_i(\boldsymbol{x}_0), i \in I(\boldsymbol{x}_0)\}) \subseteq \partial\varphi_-(\boldsymbol{0})$$

and by Lemma 3.3 the desired result follows. □

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A new splitting algorithm for equilibrium problems and applications

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Dedicated to the memory of Professor Gábor Kassay.

Abstract. In this paper, we discuss a new splitting algorithm for solving equilibrium problems arising from Nash-Cournot oligopolistic equilibrium problems in electricity markets with non-convex cost functions. Under the strong pseudomonotonicity of the original bifunction and suitable conditions of the component bifunctions, we prove the strong convergence of the proposed algorithm. Our results improve and develop previously discussed extragradient-like splitting algorithms and general extragradient algorithms. We also present some numerical experiments and compare our algorithm with the existing ones.

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Keywords: Equilibrium problem, splitting algorithm, strong pseudomonotonicity, extragradient algorithm.

1. Introduction

In recent years, equilibrium problems (EP) have been investigated by many researchers. It is well known that various classes of optimization, variational inequality, Kakutani fixed point, Nash equilibrium in noncooperative game theory and minimax problems can be formulated as an equilibrium problem [5].

An equilibrium problem can be formulated by means of Ky Fan's inequality [5]:

$$\text{find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \text{ for all } y \in C, \qquad EP(f, C)$$

where C is a nonempty closed convex subset in a Hilbert space H and $f : C \times C \to \mathbb{R}$ is a bifunction such that $f(x, x) = 0$ for all $x \in C$. The set of solutions of $EP(f, C)$ is denoted by $Sol(f, C)$.

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Projection-type methods are very popular for solving equilibrium problems because the iterations can be performed cheaply. At each iteration of these algorithms, we have to solve the strongly convex problem

$$\min\{\lambda_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\}, \tag{1.1}$$

where $\lambda_k > 0$ is the step size and x^k is the current approximation of the solution. Note that, in the variational inequalities case, when $f(x, y) := \langle F(x), y - x\rangle$, where $F : C \to C$ is a mapping, problem (1.1) becomes

$$\text{find} \quad P_C\left(x^k - \lambda_k F(x^k)\right), \tag{1.2}$$

where P_C is the projection onto C.

The computational cost of solving problems (1.1) is the main factor influencing performance of projection-type methods. One effective way to reduce the computational cost is to decompose f into two or more component bifunctions. Then, instead of solving (1.1), we have to solve only the simpler subproblems for these component bifunctions [4, 6, 11, 12]. Since 1950s, operator splitting techniques have been successfully used in PDE, large-scale optimization problems and signal processing to reduce complex problems into a series of simpler subproblems [7]. In the past decade, this technique has been received much attention due to its vast applications [2, 6, 4, 12]. Recently, in [1], the authors have introduced splitting algorithms for equilibrium problems when $f = f_1 + f_2$. Under the strong pseudomonotonicity of the bifunction f and suitable conditions of f_1 and f_2, the algorithm proposed in [1] is strongly convergent. However, it may happen that the bifunction f is decomposed into three components, i.e., $f = f_1 + f_2 + f_3$ (see Example 4.1 in Sect. 4). Then, the two-component splitting algorithm in [1] is not suitable. In this paper, inspired by work in [1, 9], we propose a new splitting algorithm for solving this class of equilibrium problems.

The rest of this article is divided into three sections. Section 2 recalls some mathematical preliminaries needed in the sequel. Section 3 presents a three-component splitting algorithm for equilibrium problems and provides the convergence analysis of the proposed algorithm. Some preliminary computational results are presented in the last section. Also in this section, we introduce a new Nash-Cournot equilibrium model for electricity markets. In contrast to the existing ones, the new model contains nonconvex cost functions, and hence, the bifunction f of the corresponding equilibrium problem is decomposed into three components. We then apply the proposed algorithm to solve this problem.

2. Preliminaries

In this section, we present some basic concepts, properties, and notations which will be useful in the sequel. Let H be a real Hilbert space, equipped with the Euclidean inner product $\langle ., .\rangle$ and the associated norm $\|.\|$, C be a nonempty closed convex subset in H. Let $f : C \times C \to \mathbb{R}$ be a bifunction on C, satisfying $f(x, x) = 0$ for all $x \in C$.

Definition 2.1. [15] A bifunction $f : C \times C \to \mathbb{R}$ is said to be

1. γ-strongly monotone on C if there exists a constant $\gamma > 0$ such that for all $x, y \in C$,
$$f(x, y) + f(y, x) \leq -\gamma\|x - y\|^2;$$
2. monotone on C if for all $x, y \in C$,
$$f(x, y) + f(y, x) \leq 0;$$
3. γ-strongly pseudomonotone on C if there exists a constant $\gamma > 0$ such that for all $x, y \in C$,
$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq -\gamma\|x - y\|^2;$$
4. pseudomonotone on C if for all $x, y \in C$,
$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0.$$

Definition 2.2. [1] A bifunction $f : C \times C \to \mathbb{R}$ is said to be Lipschitz-type continuous if there exists a constant $Q > 0$ such that for all $x, y, z \in C$,
$$|f(x, y) + f(y, z) - f(x, z)| \leq Q\|x - y\|\|y - z\|. \tag{2.1}$$

Note that, if we choose $f(x, y) := \langle Fx, y - x\rangle$, where $F : C \to C$ is a Lipschitz continuous mapping, then the corresponding bifunction f is Lipschitz-type continuous.

Definition 2.3. [1] A bifunction $f : C \times C \to \mathbb{R}$ is said to be partially τ-Hölder continuous on C if there exist a constant $L > 0$ and $\tau \in (0, 1]$ such that for all $x, y, z \in C$, at least one of the following conditions is satisfied:

(i) $|f(x, y) - f(z, y)| \leq L\|x - z\|^\tau$;
(ii) $|f(x, y) - f(x, z)| \leq L\|y - z\|^\tau$.

It is easy seen that, if an equilibrium bifunction f is τ-Hölder continuous on C then
$$|f(x, y)| \leq L\|x - y\|^\tau \quad \forall x, y \in C. \tag{2.2}$$

Definition 2.4. The subdifferential of a function $u : H \to \mathbb{R}$ at x is the set:
$$\partial u(x) := \{w \in H : u(y) - u(x) \geq \langle w, y - x\rangle \ \ \forall y \in H\}.$$
The normal cone of C at $x \in C$ is defined by
$$N_C(x) := \{q \in H : \langle q, y - x\rangle \leq 0 \ \forall y \in C\}.$$

In order to prove our main results, we need the following lemmas.

Lemma 2.5. [16] *Let $f : C \to \mathbb{R}$ be convex and subdifferentiable on C. Then, x^* is a solution of the problem*
$$\min\{f(x) : x \in C\}$$
if and only if $0 \in \partial f(x^*) + N_C(x^*)$.

Lemma 2.6. (Lemma 2.5 [18]) *Let* $\{\alpha_k\}$, $\{\beta_k\}$, $\{\lambda_k\}$ *be sequences of nonnegative numbers satisfying*

$$\alpha_{k+1} \leq (1-\lambda_k)\alpha_k + \lambda_k\gamma_k + \beta_k \ \forall k \geq 1.$$

If $\lambda_k \in (0,1)$ $\forall k \geq 1$, $\sum_{k=1}^{\infty} \lambda_k = \infty$, $\limsup_{k\to\infty} \gamma_k \leq 0$ *and* $\sum_{k=1}^{\infty} \beta_k < \infty$ *then* $\lim_{k\to\infty} \alpha_k = 0$.

3. Three-component splitting algorithm

Let C be a nonempty, closed, convex subset in a Hilbert space H and $f : C\times C :\to \mathbb{R}$ be a bifunction on C. We are interested in the equilibrium problem

$$\text{find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \text{ for all } y \in C, \tag{3.1}$$

where f can be decomposed into three components: $f = f_1 + f_2 + f_3$, f_i $(i = 1, 2, 3)$ are equilibrium bifunctions on C, i.e., $f_i(x, x) = 0$ for all $x \in C$.

Assumption 1. In this paper, we assume that

A.1 For each $x \in C$, the function $f_i(x, .)$ $(i = 1, 2, 3)$ is lower semicontinuous, convex and for each $y \in C$, the function $f(., y)$ is hemicontinuous on C, i.e.

$$\lim_{t\to 0} f(tz + (1-t)x, y) = f(x, y), \quad \forall x, y, z \in C.$$

A.2 The bifunction f is γ-strongly pseudomonotone.

Note that under assumptions **A.1** and **A.2**, problem $EP(f, C)$ has a unique solution [13]. To find this solution, we propose the following three-component splitting algorithm.

Algorithm 3.1. (Three-component splitting algorithm - 3-CSA))
Step 0. Choose $x^0 \in C$, $\lambda_k \subset (0, +\infty)$. Set $k = 0$.

Step 1. Given x^k, compute x^{k+1} as

$$\bar{x}^k = \operatorname{argmin}\Big\{\lambda_k f_1(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\Big\},$$
$$\tilde{x}^k = \operatorname{argmin}\Big\{\lambda_k f_2(\bar{x}^k, y) + \frac{1}{2}\|y - \bar{x}^k\|^2 : y \in C\Big\},$$
$$x^{k+1} = \operatorname{argmin}\Big\{\lambda_k f_3(\tilde{x}^k, y) + \frac{1}{2}\|y - \tilde{x}^k\|^2 : y \in C\Big\}.$$

Step 2. Update $k := k + 1$ and go to **Step 1**.

Theorem 3.2. *Assume that conditions* **A.1**, **A.2** *hold,* f_1 *is* Q*-Lipschitz-type continuous,* f_i *is partially* τ_i*-Holder continuous,* $i = 2, 3$. *Moreover, suppose that*

(B.1) $\sum_{k=1}^{+\infty} \lambda_k = +\infty$,

(B.3) $\sum_{k=1}^{+\infty} (\lambda_k)^{\frac{2}{2-\tau}} < +\infty$,

where $\tau = \min\{\tau_2, \tau_3\}$. *Then, the sequence* $\{x^k\}$ *generated by Algorithm* 3.1 *strongly converges to the unique solution* x^* *of* $EP(f, C)$.

Proof. Since $\bar{x}^k$ is the unique solution of the problem

$$\min\Big\{\lambda_k f_1(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\Big\},$$

there exist $\omega^k \in \partial f_1(x^k, .)(\bar{x}^k)$ and $q^k \in N_C(\bar{x}^k)$ such that

$$0 = \lambda_k \omega^k + \bar{x}^k - x^k + q^k.$$

From the definition of $N_C(.)$, we have

$$\big\langle x^k - \bar{x}^k - \lambda_k \omega^k, y - \bar{x}^k \big\rangle \leq 0 \ \ \forall y \in C. \tag{3.2}$$

Hence,

$$\lambda_k \big\langle \omega^k, y - \bar{x}^k \big\rangle \leq \lambda_k (f_1(x^k, y) - f_1(x^k, \bar{x}^k)) \ \ \forall y \in C. \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\big\langle x^k - \bar{x}^k, y - x^k \big\rangle \leq \lambda_k (f_1(x^k, y) - f_1(x^k, \bar{x}^k)) - \|x^k - \bar{x}^k\|^2 \ \ \forall y \in C. \tag{3.4}$$

Analogously, since $\tilde{x}^k$ and x^{k+1} are the solutions of the problems

$$\min\Big\{\lambda_k f_2(\bar{x}^k, y) + \frac{1}{2}\|y - \bar{x}^k\|^2 : y \in C\Big\},$$

$$\min\Big\{\lambda_k f_3(\tilde{x}^k, y) + \frac{1}{2}\|y - \tilde{x}^k\|^2 : y \in C\Big\},$$

it follows that

$$\big\langle \bar{x}^k - \tilde{x}^k, y - \bar{x}^k \big\rangle \leq \lambda_k (f_2(\bar{x}^k, y) - f_2(\bar{x}^k, \tilde{x}^k)) - \|\bar{x}^k - \tilde{x}^k\|^2 \ \ \forall y \in C. \tag{3.5}$$

and

$$\big\langle \tilde{x}^k - x^{k+1}, y - \tilde{x}^k \big\rangle \leq \lambda_k (f_3(\tilde{x}^k, y) - f_3(\tilde{x}^k, x^{k+1})) - \|\tilde{x}^k - x^{k+1}\|^2 \ \ \forall y \in C. \tag{3.6}$$

In (3.6), taking $y = \tilde{x}^k$, we get

$$\|x^{k+1} - \tilde{x}^k\|^2 \leq -\lambda_k f_3(\tilde{x}^k, x^{k+1}) \leq \lambda_k |f_3(\tilde{x}^k, x^{k+1})|. \tag{3.7}$$

Since f_3 is partially τ_3-Holder continuous and $f_3(x, x) = 0$ for all $x \in C$, there exists a constant $L_3 > 0$ such that for all $k \geq 1$, it holds that

$$|f_3(\tilde{x}^k, x^{k+1})| \leq L_3 \|x^{k+1} - \tilde{x}^k\|^{\tau_3}. \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$\|x^{k+1} - \tilde{x}^k\| \leq (L_3 \lambda_k)^{\frac{1}{2-\tau_3}} \tag{3.9}$$

and

$$\lambda_k |f_3(\tilde{x}^k, x^{k+1})| \leq (L_3 \lambda_k)^{\frac{2}{2-\tau_3}}. \tag{3.10}$$

In (3.5), taking $y = \bar{x}^k$ and using the partial Holder continuity of f_2, we get

$$\|\tilde{x}^k - \bar{x}^k\| \leq (L_2 \lambda_k)^{\frac{1}{2-\tau_2}} \tag{3.11}$$

and

$$\lambda_k |f_2(\bar{x}^k, \tilde{x}^k)| \leq (L_2 \lambda_k)^{\frac{2}{2-\tau_2}}. \tag{3.12}$$

From (3.4) and the Q-Lipschitz continuity of f_1, we arrive at

$$\begin{aligned}\|\bar{x}^k - y\|^2 &= \|\bar{x}^k - x^k\|^2 + \|x^k - y\|^2 + \langle \bar{x}^k - x^k, x^k - y\rangle \\ &\le \|x^k - y\|^2 - \|\bar{x}^k - x^k\|^2 + 2\lambda_k(f_1(x^k, y) - f_1(x^k, \bar{x}^k)) \\ &\le \|x^k - y\|^2 - \|\bar{x}^k - x^k\|^2 + 2\lambda_k(f_1(\bar{x}^k, y) + Q\|\bar{x}^k - x^k\|.\|\bar{x}^k - y\|) \\ &\le \|x^k - y\|^2 - \|\bar{x}^k - x^k\|^2 + 2\lambda_k f_1(\bar{x}^k, y) \\ &+ (Q\lambda_k)^2\|\bar{x}^k - y\|^2 + \|\bar{x}^k - x^k\|^2 \\ &= \|x^k - y\|^2 + 2\lambda_k f_1(\bar{x}^k, y) + (Q\lambda_k)^2\|\bar{x}^k - y\|^2. \end{aligned} \tag{3.13}$$

Analogously to (3.5), (3.6) we get

$$\begin{aligned}\|\tilde{x}^k - y\|^2 &\le \|\bar{x}^k - y\|^2 - \|\tilde{x}^k - \bar{x}^k\|^2 + 2\lambda_k(f_2(\bar{x}^k, y) - f_2(\bar{x}^k, \tilde{x}^k)) \\ &\le \|\bar{x}^k - y\|^2 + 2\lambda_k(f_2(\bar{x}^k, y) - f_2(\bar{x}^k, \tilde{x}^k)) \\ &\le \|\bar{x}^k - y\|^2 + 2\lambda_k f_2(\bar{x}^k, y) + 2\lambda_k|f_2(\bar{x}^k, \tilde{x}^k)|. \end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\|x^{k+1} - y\|^2 &\le \|\tilde{x}^k - y\|^2 - \|x^{k+1} - \tilde{x}^k\|^2 + 2\lambda_k(f_3(\tilde{x}^k, y) - f_3(\tilde{x}^k, x^{k+1})) \\ &\le \|\tilde{x}^k - y\|^2 + 2\lambda_k(f_3(\tilde{x}^k, y) - f_3(\tilde{x}^k, x^{k+1})) \\ &= \|\tilde{x}^k - y\|^2 + 2\lambda_k f_3(\bar{x}^k, y) - 2\lambda_k f_3(\tilde{x}^k, x^{k+1}) \\ &+ 2\lambda_k(f_3(\tilde{x}^k, y) - f_3(\bar{x}^k, y)) \\ &\le \|\tilde{x}^k - y\|^2 + 2\lambda_k f_3(\bar{x}^k, y) + 2\lambda_k|f_3(\tilde{x}^k, x^{k+1})| \\ &+ 2\lambda_k|f_3(\tilde{x}^k, y) - f_3(\bar{x}^k, y)|. \end{aligned} \tag{3.15}$$

From (3.9)-(3.15) and the partial τ_3-Holder continuity of f_3, we have

$$\begin{aligned}\|x^{k+1} - y\|^2 &\le \|x^k - y\|^2 + 2\lambda_k f(\bar{x}^k, y) + (Q\lambda_k)^2\|\bar{x}^k - y\|^2 \\ &+ 2\lambda_k|f_2(\bar{x}^k, \tilde{x}^k)| + 2\lambda_k|f_3(\tilde{x}^k, x^{k+1})| \\ &+ 2\lambda_k|f_3(\tilde{x}^k, y) - f_3(\bar{x}^k, y)| \\ &\le \|x^k - y\|^2 + 2\lambda_k f(\bar{x}^k, y) + (Q\lambda_k)^2\|\bar{x}^k - y\|^2 \\ &+ 2(L_2\lambda_k)^{\frac{2}{2-\tau_2}} + 2(L_3\lambda_k)^{\frac{2}{2-\tau_3}} + 2\lambda_k(L_3\lambda_k)^{\frac{\tau_3}{2-\tau_2}}. \end{aligned} \tag{3.16}$$

In (3.16), taking $y = x^*$ and using the γ-strong pseudomonotonicity of f, we obtain

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\le \|x^k - x^*\|^2 - \lambda_k(2\gamma - Q^2\lambda_k)\|\bar{x}^k - x^*\|^2 + \\ &+ 2(L_2\lambda_k)^{\frac{2}{2-\tau_2}} + 2(L_3\lambda_k)^{\frac{2}{2-\tau_3}} + 2\lambda_k(L_3\lambda_k)^{\frac{\tau_3}{2-\tau_2}}. \end{aligned} \tag{3.17}$$

Using the inequality $\|a + b\| \ge |\|a\| - \|b\||$ for all $a, b \in H$, we infer that

$$\begin{aligned}\lambda_k\|\bar{x}^k - x^*\|^2 &\ge \lambda_k(\|\bar{x}^k - x^{k+1}\| - \|x^{k+1} - x^*\|)^2 \\ &\ge (\lambda_k - 1)\|\bar{x}^k - x^{k+1}\|^2 + \lambda_k(1 - \lambda_k)\|x^{k+1} - x^*\|^2. \end{aligned} \tag{3.18}$$

Since $\lim_{k\to+\infty} \lambda_k = 0$, without loss of generality we can assume that $1-\lambda_k > 0$ and $2\gamma - Q^2\lambda_k > 0$ for all $k \geq 1$. Combining (3.17) and (3.18), we arrive at

$$\begin{aligned}\|x^{k+1}-x^*\|^2 \leq \|x^k - x^*\|^2 &+ (2\gamma - Q^2\lambda_k)(1-\lambda_k)\|\bar{x}^k - x^{k+1}\|^2 \\ &+ \lambda_k(1-\lambda_k)(2\gamma - Q^2\lambda_k)\|x^{k+1}-x^*\|^2 \\ &+ 2(L_2\lambda_k)^{\frac{2}{2-\tau_2}} + 2(L_3\lambda_k)^{\frac{2}{2-\tau_3}} + 2\lambda_k(L_3\lambda_k)^{\frac{\tau_3}{2-\tau_2}}. \end{aligned} \tag{3.19}$$

On the other hand, it holds that

$$\begin{aligned}\|\bar{x}^k - x^{k+1}\|^2 &= \|\bar{x}^k - \tilde{x}^k\|^2 + \|\tilde{x}^k - x^{k+1}\|^2 + 2\langle \tilde{x}^k - x^{k+1}, \bar{x}^k - \tilde{x}^k\rangle \\ &\leq (L_2\lambda_k)^{\frac{2}{2-\tau_2}} + (L_3\lambda_k)^{\frac{2}{2-\tau_3}} + 2\langle \tilde{x}^k - x^{k+1}, \bar{x}^k - \tilde{x}^k\rangle. \end{aligned} \tag{3.20}$$

In (3.6), taking $y = \bar{x}^k$, we get

$$\begin{aligned}2\langle \tilde{x}^k - x^{k+1}, \bar{x}^k - \tilde{x}^k\rangle &\leq 2\lambda_k(f_3(\tilde{x}^k, \bar{x}^k) - f_3(\tilde{x}^k, x^{k+1})) \\ &\leq 2\lambda_k|f_3(\tilde{x}^k, \bar{x}^k)| + 2\lambda_k|f_3(\tilde{x}^k, x^{k+1})| \\ &\leq 2\lambda_k L_3\|\tilde{x}^k - \bar{x}^k\|^{\tau_3} + 2(L_3\lambda_k)^{\frac{2}{2-\tau_3}} \\ &\leq 2\lambda_k L_3(L_2\lambda_k)^{\frac{\tau_3}{2-\tau_2}} + 2(L_3\lambda_k)^{\frac{2}{2-\tau_3}}. \end{aligned} \tag{3.21}$$

Combining (3.19)-(3.21), we have

$$\begin{aligned}[1+\lambda_k(1-\lambda_k)(2\gamma - Q^2\lambda_k)]\|x^{k+1}-x^*\|^2 \leq \|x^k - x^*\|^2 &+ 2(\gamma+1)(L_2\lambda_k)^{\frac{2}{2-\tau_2}} \\ &+ 2(2\gamma+1)(L_3\lambda_k)^{\frac{2}{2-\tau_3}} \\ &+ 2\Big(L_3L_2^{\frac{\tau_3}{2-\tau_2}} + L_3^{\frac{\tau_3}{2-\tau_2}}\Big)\lambda_k^{\frac{2+\tau_3-\tau_2}{2-\tau_2}}, \end{aligned}$$

or

$$\|x^{k+1}-x^*\|^2 \leq (1-A_k)\|x^k - x^*\|^2 + B_k + C_k + D_k,$$

where

$$\begin{aligned} A_k &= \frac{\lambda_k(1-\lambda_k)(2\gamma - Q^2\lambda_k)}{1+\lambda_k(1-\lambda_k)(2\gamma - Q^2\lambda_k)}, \\ B_k &= \frac{2(\gamma+1)(L_2\lambda_k)^{\frac{2}{2-\tau_2}}}{1+\lambda_k(1-\lambda_k)(2\gamma - Q^2\lambda_k)}, \\ C_k &= \frac{2(2\gamma+1)(L_3\lambda_k)^{\frac{2}{2-\tau_3}}}{1+\lambda_k(1-\lambda_k)(2\gamma - Q^2\lambda_k)}, \\ D_k &= \frac{2\Big(L_3L_2^{\frac{\tau_3}{2-\tau_2}} + L_3^{\frac{\tau_3}{2-\tau_2}}\Big)\lambda_k^{\frac{2+\tau_3-\tau_2}{2-\tau_2}}}{1+\lambda_k(1-\lambda_k)(2\gamma - Q^2\lambda_k)}. \end{aligned}$$

We have

$$\begin{aligned}&\lim_{k\to+\infty}\Big(\frac{\lambda_k(1-\lambda_k)(2\gamma - Q^2\lambda_k)}{1+\lambda_k(1-\lambda_k)(2\gamma - Q^2\lambda_k)}.\frac{1}{\lambda_k}\Big) \\ &= \lim_{k\to+\infty}\Big(\frac{(1-\lambda_k)(2\gamma - Q^2\lambda_k)}{1+\lambda_k(1-\lambda_k)(2\gamma - Q^2\lambda_k)}\Big) = 2\gamma, \end{aligned}$$

moreover, since $\sum\limits_{k=1}^{+\infty} \lambda_k = +\infty$, it follows that

$$\sum_{k=1}^{+\infty} A_k = \sum_{k=1}^{+\infty} \Big(\frac{\lambda_k(1-\lambda_k)(2\gamma - Q^2\lambda_k)}{1+\lambda_k(1-\lambda_k)(2\gamma - Q^2\lambda_k)} \Big) = +\infty.$$

On the other hand,

$$\lim_{k\to+\infty} \big(B_k.\frac{1}{\lambda_k^{\frac{2}{2-\tau}}}\big) = 0, \ \lim_{k\to+\infty} \big(C_k.\frac{1}{\lambda_k^{\frac{2}{2-\tau}}}\big) = 0, \ \lim_{k\to+\infty} \big(D_k.\frac{1}{\lambda_k^{\frac{2}{2-\tau}}}\big) = 0$$

it implies that

$$\sum_{k=1}^{+\infty} (B_k + C_k + D_k) < +\infty.$$

Applying Lemma 2.6, we get $\lim\limits_{k\to+\infty} \|x^k - x^*\|^2 = 0$ or $\lim\limits_{k\to+\infty} x^k = x^*$. □

Remark 3.3. (a) Since $\tau \in (0,1]$, $\frac{2}{2-\tau} \in (1,2]$, we can choose a sequence $\{\lambda_k\}$ satisfying conditions (B.1)-(B.3), for example, $\lambda_k = \frac{1}{k^\alpha}$, where $\alpha \in (\frac{2-\tau}{2}, 1)$.

(b) In Algorithm 3.1, we need not to know the Lipschitz constant Q of f_1.

(c) Algorithm 3.1 reminds the so-called General Extragradient Algorithm in [9], in the sense that the both algorithms require three subproblems at each iteration. However, our algorithm has a clear advantage: at each iteration, we have to solve only subproblems for the component bifunctions f_i, instead of solving subproblems for the whole bifunction f. Hence, our algorithm may have a low computational cost when the function f has a complicated structure, while the component bifunctions f_i are simpler.

(d) If $f_3 = 0$, i.e. $f = f_1 + f_2$, then the new algorithm collapses to the existing one in [1].

4. Numerical examples

In this section, we provide an application of the proposed algorithm to electricity markets. We also compare our algorithm with some existing ones. All the programmings are implemented in MATLAB R2010b running on a PC with Intel®Core2™ Quad Processor Q9400 2.66Ghz 4GB Ram.

Example 4.1. (*Nash-Cournot oligopolistic equilibrium model for electricity markets with non-convex cost functions*)
We introduce a Nash-Cournot oligopolistic equilibrium model for electricity markets. In contrast to the existing ones considered in [14, 1], the new model contains non-convex cost functions. In this situation, the three-component splitting algorithm seem to be the most suitable one for solving the corresponding equilibrium problem.

Consider an electricity market with N companies. Suppose that x_j is the power generation level of company j, $(j = 1, \ldots, N)$. Then, the total power generation of

the market is

$$\sigma := \sum_{k=1}^{N} x_k.$$

Obviously, the more electricity companies produce, the lower electricity price is. Hence, we assume that the electricity price p is inversely proportional to σ and is defined by

$$p(x) = 200 - 2\sum_{k=1}^{N} x_k.$$

To produce electricity, companies have to pay two costs: production cost and environmental cost. The cost of production per unit of electricity decreases as the production level increases. Hence, we assume that the production cost h_j^{prod} of company j is a concave function of x_j:

$$h_j^{\text{prod}}(x_j) = a_j\sqrt{x_j} + b_j.$$

Meanwhile, the environmental charge per unit of electricity increases as the product level increases. Hence, the environmental cost h_j^{env} of company j is a convex function of x_j:

$$h_j^{\text{env}}(x^j) = c_j {x_j}^2 + d_j.$$

And so, the total cost h_j of company j is:

$$h_j(x_j) = h_j^{\text{prod}}(x_j) + h_j^{\text{env}}(x^j) = a_j\sqrt{x_j} + b_j + c_j {x_j}^2 + d_j.$$

Let $N = 6$. The parameters a_j, b_j, c_j, d_j are given in Table 1.

j	a_j	b_j	c_j	d_j
1	1.0	2.0	0.05	2.2
2	0.7	2.1	0.06	2.1
3	0.8	1.9	0.03	1.9
4	0.9	1.8	0.02	1.8
5	0.8	2.2	0.01	2.3
6	0.6	2.3	0.04	1.8

TABLE 1. The parameters of the cost function

The profit ξ_j of a company j is

$$\xi_j(x) := p(x)x_j - h_j(x_j) = \left(200 - 2\sum_{k=1}^{N} x_k\right) x_j - h_j(x_j),$$

where $x = (x_1, \ldots, x_N)^T \in C := \left\{x \in \mathbb{R}^N : \alpha_j \le x_j \le \beta_j\right\}$, α_j, β_j are given in Table 2.

j	1	2	3	4	5	6
α_j	10	10	10	10	10	10
β_j	90	70	100	60	110	50

TABLE 2. The lower and upper bounds for power generation levels x_j

We are interested in a such point $x^* = (x_1^*, \dots, x_N^*) \in C$ satisfying

$$\xi_j(x_1^*, \dots, x_{j-1}^*, y_j, x_{j+1}^*, \dots, x_N^*) \le \xi_j(x_1^*, \dots, x_N^*).$$

for all $y = (y_1, \dots, y_N) \in C, j = 1, \dots, N$. The point x^* is called the Nash equilibrium. Let

$$\zeta(x, y) := \varphi(x, x) - \varphi(x, y),$$

where

$$\begin{aligned}\varphi(x, y) &= \sum_{j=1}^{N} \xi_j(x_1, \dots, x_{j-1}, y_j, x_{j+1}, \dots, x_N) \\ &= \sum_{j=1}^{N} \left[200 - 2\left(\sum_{k \ne j} x_k + y_j\right)\right] y_j - \sum_{j=1}^{N} h_j(y_j).\end{aligned}$$

Then x^* a Nash equilibrium point of this model if and only if it is a solution of the equilibrium problem (see [10]):

$$\text{find } x^* \in C \text{ such that } \zeta(x^*, y) \ge 0 \ \forall y \in C. \tag{4.1}$$

We have

$$\zeta(x, y) = \langle (A + B)x + By + q, y - x\rangle + h(y) - h(x), \tag{4.2}$$

where

$$A := \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 2 & 0 \end{pmatrix}, \ B := \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

and $q = -(100, 100, 100, 100, 100, 100)^T$. However, the bifunction ζ given by (4.2) is not strongly pseudomonotone (even not pseudomonotone). This bifunction can be rewritten as

$$\zeta(x, y) = f(x, y) + 0.6\langle B(y - x), y - x\rangle,$$

where $f(x, y) = \langle (A + 1.6B)x + 0.4By + q, y - x\rangle + h(y) - h(x)$. It is easy seen that the matrix B is positive definite, hence x^* is a solution of the equilibrium problem (4.1) if and only if it is a solution of the problem (see [14]):

$$\text{find } x^* \in C \text{ such that } f(x^*, y) \ge 0 \ \forall y \in C. \tag{4.3}$$

Let us prove the bifunction f is strongly pseudomonotone. Indeed, for all $x, y \in C$, we have

$$f(x,y) + f(y,x) = -\langle (A + 1.2B)(x-y), (x-y)\rangle .$$

Since $A+1.2B$ is a positive definite matrix, it implies that the bifunction f is strongly monotone, and hence, is strongly pseudomonotone. Let

$$f_1(x,y) = \langle (A+1.6B)x + 0.4By + q, y - x\rangle,$$

$$f_2(x,y) = \sum_{j=0}^{N} \left(h_j^{\text{env}}(y_j) - h_j^{\text{env}}(x_j)\right).$$

and

$$f_3(x,y) = \sum_{j=0}^{N} \left(h_j^{\text{prod}}(y_j) - h_j^{\text{prod}}(x_j)\right),$$

It is easy seen that f_1, f_2 and f_3 satisfy all conditions of the proposed algorithm. Now we will apply this algorithms to solve problem (4.3). Note that in this example, subproblems of f_1 and f_2 have quadratic forms and are much easier to solve than general convex problems. Moreover, although the cost functions h_j^{prod} are concave, the subproblems of f_3 is convex if $\lambda_k \leq \frac{1}{6}$.
We implement the algorithm with the starting point $x^0 = (0,0,0,0,0,0)^T$, $\lambda_k = \frac{1}{k+6}$ and the stopping criteria $\|x^{k+1} - x^k\| \leq 10^{-4}$. The test results are reported in Table 3. The algorithm finds the approximation of the solution after 105 iterations.

Iter(k)	x_k^1	x_k^2	x_k^3	x_k^4	x_k^5	x_k^6	$\|x_{k-1} - x^k\|$
0	0	0	0	0	0	0	
1	22.9133	22.8534	23.0463	23.1103	23.1777	22.9841	31.4327
2	10.0597	10.0000	10.2182	10.2922	10.3731	10.1480	12.9558
3	15.3184	15.2412	15.5167	15.6095	15.7111	15.4289	3.7680
4	13.7630	13.6767	13.9837	14.0868	14.2002	13.8865	0.7174
5	14.0422	13.9487	14.2802	14.3913	14.5139	14.1756	0.0755
6	14.0034	13.9046	14.2542	14.3713	14.5007	14.1441	0.0200
7	13.9975	13.8947	14.2579	14.3796	14.5143	14.1437	0.0152
...	...	...	...	...	...	...	...
105	13.9815	13.8658	14.2731	14.4099	14.5630	14.1455	9.903810^{-5}

TABLE 3. Iterations of the proposed algorithm with starting point $x_0 = (0,0,0,0,0,0)^T$

Example 4.2. We compare our algorithm with the Armijo Line Search Algorithm (ALS) (Algorithm 1 in [8]), the General Extragradient Algorithm (GEA) in [9], the Splitting Sequential Algorithm (SAL) (Algorithm 1 in [1]) and the Subgradient Algorithm (SGA) given by Santos in [17]. Consider the equilibrium problem

$$\text{find } x \in C \text{ such that } \langle Ax + P(x), y - x\rangle + \varphi(y) - \varphi(x) \geq 0 \ \forall y \in C,$$

where the feasible set $C \subset \mathbb{R}^5$ is given by

$$C := \{x \in \mathbb{R}^5 : -5 \leq x_i \leq 5 \ \forall i = 1, \ldots, 5\},$$

$$\varphi : \mathbb{R}^5 \to \mathbb{R},\ \varphi(x) = \|x\|^2,$$
$$F : \mathbb{R}^5 \to \mathbb{R}^5,\ F(x) = Ax + P(x),$$

with

$$A := \begin{pmatrix} 3 & 1 & 0 & 1 & 2 \\ 1 & 5 & -1 & 0 & 1 \\ 0 & -1 & 4 & 2 & -2 \\ 1 & 0 & 2 & 6 & -1 \\ 2 & 1 & -2 & -1 & 5 \end{pmatrix},$$

and $P : \mathbb{R}^5 \to \mathbb{R}^5$ is the proximal mapping of the function

$$h(x) := \frac{\|x\|^4}{4},$$

i.e.,

$$P(x) := \operatorname{argmin}\left\{ \frac{\|y\|^4}{4} + \frac{1}{2}\|y - x\|^2 : y \in \mathbb{R}^5 \right\}.$$

Note that, since we do not have a closed form of $P(x)$, to compute the value of this mapping, we have to solve a strongly convex problem. In our algorithm, let

$$f_1(x, y) := \langle Ax, y - x\rangle,$$
$$f_2(x, y) := \langle P(x), y - x\rangle,$$
$$f_3(x, y) := \varphi(y) - \varphi(x)$$

and $f := f_1 + f_2 + f_3$.

In Algorithm SAL, let

$$f_1(x, y) := \langle Ax + P(x), y - x\rangle,$$
$$f_2(x, y) := \varphi(y) - \varphi(x)$$

and $f := f_1 + f_2$. It is easy seen that all the conditions of the four algorithms are satisfied. Moreover, the mapping P is nonexpansive and the Lipschitz constants of f (defined in [8]) are

$$c_1 = c_2 = \frac{1}{2}(\|A\| + 1).$$

We apply the four algorithms for solving $EP(f, C)$ with the parameters:

- In Algorithm GEA: $\alpha_k = \beta_k = \frac{1}{7c_1}$.
- In Algorithm ALS: $G(x) := \|x\|^2$, $\eta = 0.5$; $\rho = 1$.
- In Algorithm SAL $\lambda_k = \frac{1}{k}$.
- In Algorithm SGA $\beta_k = \frac{1}{k}$, $\rho_k = 1$, $\epsilon_k = 0$, $\xi_k = 0$.
- In our algorithm: $\lambda_k = \frac{1}{k}$.

All the algorithms use the same starting points and the same stopping rule:

$$\|x^k - x^*\| \le 3.10^{-4},$$

where $x^* = (0, 0, 0, 0, 0)^T$ is the unique solution of the $EP(f, C)$. The results are presented in Table 4.

	$x^0 = (5,5,5,5,5)^T$		$x^0 = (1,1,1,1,1)^T$		$x^0 = (1,2,3,4,5)^T$		$x^0 = (-3,-5,2,-4,4)^T$	
	CPU times	Iter.	CPU times	Iter.	CPU times	Iter.	CPU times	Iter.
Alg. GEA	-	-	-	-	-	-	-	-
Alg. SGA	-	-	0.6004	14	-	-	-	-
Alg. ALS	11.1323	27	8.6821	21	11.3570	29	11.3248	26
Alg. SAL	0.4357	11	0.4470	10	0.5019	12	0.3509	10
Alg. 3-CSA	0.4604	12	0.3839	9	0.4860	11	0.5082	13

TABLE 4. Comparision of the algorithms. (-) means the algorithm does not obtain the required accuracy after 100s.

From this table we can see that, if the initial approximation x^0 is close enough to the exact solution x^*, say, $||x^0 - x^*|| \leq 7.4$, then the performance of 3-CSA is the best among four above mentioned algorithms.

Example 4.3. Consider problem $EP(f, C)$ with

$$C := \left\{x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : 2x_1^2 + x_2^2 + \ldots + x_m^2 \leq 1\right\}$$

and $f : C \times C \to \mathbb{R}$, defined by

$$f(x, y) := \langle Ax, y - x\rangle + y^2 - x^2 + \langle y, y - x\rangle \ \ \forall x, y \in C,$$

where $A = (a_{ij})$ is a $m \times m$ matrix and

$$a_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1.1 & \text{if } i = j. \end{cases}$$

It is easy seen that $f(x, y) + f(y, x) = -0.1\|x - y\|^2$ for all $x, y \in C$, and hence f is strongly monotone on C. All conditions of the three-component splitting algorithm (3-CSA) and the splitting sequential algorithm (SAL) (Algorithm 1 in[1]) are satisfied. We will use this problem to compare them. For 3-CSA, let

$$f_1(x, y) := \langle Ax, y - x\rangle\,,\ f_2(x, y) := y^2 - x^2,\ f_3(x, y) := \langle y, y - x\rangle \ \ \forall x, y \in C.$$

For SAL, define

$$\tilde{f}_1(x, y) := \langle Ax, y - x\rangle\,,\ \tilde{f}_2(x, y) := y^2 - x^2 + \langle y, y - x\rangle \ \ \forall x, y \in C.$$

Note that the problem has a unique solution $x^* = (0, 0, \ldots, 0)^T$. In the both algorithms, we use the same step-size $\lambda_k = \frac{1}{k}$ for all $k \geq 1$, the same stopping criteria $\|x^k - x^*\| \leq \epsilon$ and the same starting point x^0, which is randomly generated. 3-CSA now becomes

$$\begin{cases} x^0 \in C, \\ \bar{x}^k = \operatorname{argmin}\left\{\lambda_k \left\langle Ax^k, y - x\right\rangle + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}, \\ \tilde{x}^k = \operatorname{argmin}\left\{\lambda_k(\|y\|^2 - \|\bar{x}^k\|^2) + \frac{1}{2}\|y - \bar{x}^k\|^2 : y \in C\right\}, \\ x^{k+1} = \operatorname{argmin}\left\{\lambda_k \left\langle y, y - \tilde{x}^k\right\rangle + \frac{1}{2}\|y - \tilde{x}^k\|^2 : y \in C\right\}. \end{cases}$$

From the definition of $\bar{x}^k$, it implies that

$$\lambda_k Ax^k + \bar{x}^k - x^k + q = 0,$$

where q is a normal vector of C at $\bar{x}^k$. Hence,

$$\left\langle x^k - \lambda_k A x^k - \bar{x}^k, z - \bar{x}^k \right\rangle \leq 0 \ \forall z \in C.$$

It follows that $\bar{x}^k = P_C(x^k - \lambda_k A x^k)$. Similarly, we have

$$\tilde{x}^k = P_C\left(\frac{1}{1+2\lambda_k}\bar{x}^k\right).$$

Since 0 and $\bar{x}^k$ belong to C, it implies that $\frac{1}{1+2\lambda_k}\bar{x}^k \in C$, and hence, $\tilde{x}^k = \frac{1}{1+2\lambda_k}\bar{x}^k$. Analogously, we have $x^{k+1} = \frac{1+\lambda_k}{1+2\lambda_k}\tilde{x}^k$, and hence, 3-CSA has the following closed form:

$$\begin{cases} x^0 \in C, \\ x^{k+1} = \frac{1+\lambda_k}{(1+2\lambda_k)^2} P_C(x^k - \lambda_k A x^k). \end{cases}$$

Similarly, in this problem, SAL can be rewritten as

$$\begin{cases} y^0 \in C, \\ y^{k+1} = \frac{1+\lambda_k}{1+4\lambda_k} P_C(y^k - \lambda_k A y^k). \end{cases}$$

We have $\frac{1+\lambda_k}{(1+2\lambda_k)^2} < \frac{1+\lambda_k}{1+4\lambda_k}$. Hence, by induction, it is easy seen that $\|x^k - x^*\| < \|y^k - y^*\|$ for all $k \geq 1$. This means that in this problem, 3-CSA requires less iterations than SAL does. For more specific comparisons, we test these two algorithms in the problem with different m and ϵ. The results are presented in Table 5. From this table, we can see that the results of 3-CSA are better than those of SAL in terms of iterations and computational time.

		3-CSA		SAL	
m	ϵ	CPU times	Iter.	CPU times	Iter.
50	10^{-3}	0.0012	5	0.0024	8
	10^{-4}	0.0018	10	0.0032	15
	10^{-5}	0.0021	18	0.0041	27
100	10^{-3}	0.0018	6	0.0027	9
	10^{-4}	0.0022	11	0.0039	16
	10^{-5}	0.0036	20	0.0057	30
500	10^{-3}	0.3019	7	0.4019	10
	10^{-4}	0.5169	12	0.7019	19
	10^{-5}	0.9674	22	1.3214	34
2000	10^{-3}	0.5436	7	0.7503	11
	10^{-4}	0.9746	13	1.3976	20
	10^{-5}	1.8324	24	2.5864	36

TABLE 5. Comparision of 3-CSA and SAL

Next, we compare 3-CSA with the Subgradient Algorithm (SGA), the Armijo Line Search Algorithm (ALS), the Ergodic Algorithm (EDA) [3]. In EDA, we choose $\lambda_k = \frac{1}{k}$ for all $k \geq 1$. The parameters for the remaining algorithms are selected as in Example 4.2. The comparisons results are presented in Figure 1. As we can see, 3-CSA shows a better behavior in terms of the computational time.

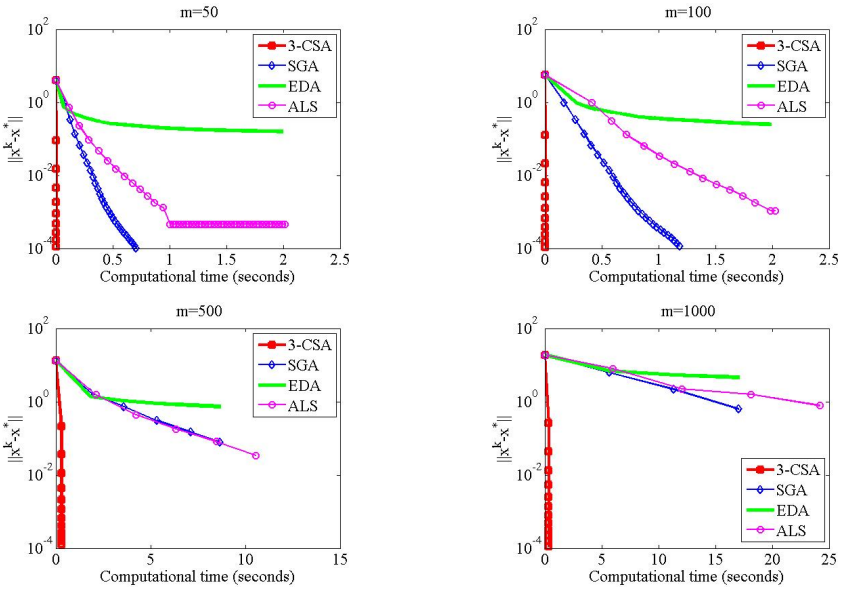

FIGURE 1. Comparisions of 3-SCA with some existing algorithms in Example 4.3

5. Conclusion

In this paper, we have proposed a three-component splitting algorithm for solving equilibrium problems in Hilbert spaces. Under the assumptions that the involving bifunction is strongly pseudomonotone and the component bifunctions satisfy suitable conditions, we have proved that the proposed algorithm strongly converges to the unique solution of the problem. Our algorithm is particularly effective when applied to equilibrium problems with complicated bifunctions, given as the sum of three components. The effectiveness of the proposed algorithm has been tested by some numerical experiments and comparisons. Also, the new algorithm has been applied to the Nash-Cournot oligopolistic equilibrium model for electricity markets with a non-convex cost function.

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Characterizations of hilbertian norms involving the areas of triangles in a smooth space

Teodor Precupanu

Dedicated to the memory of Professor Gábor Kassay.

Abstract. In the previous paper, we have defined together with I. Ionică the heights of a nontrivial triangle with respect to Birkhoff orthogonality in a real smooth space X, $\dim X \geq 2$. In the present paper, we remark that, generally, the area of a nontrivial triangle in X has not the same value for different heights of the triangle. The purpose of this paper is to characterize the norm of X if this space has the property that the area of any triangle is well defined (independent of considered height). In this line we give five equivalent properties using the directional derivative of the norm. If X is strictly convex and $\dim X \geq 3$, then each of these five properties characterizes the hilbertian norms (generated by inner products).

Mathematics Subject Classification (2010): 54AXX.

Keywords: Smooth space, strictly convex space, norm derivative, Birkhoff orthogonality, height of a triangle, hilbertian norm.

1. Introduction

Let X be a real normed space and let X^* be its dual space. We recall that two elements $x, y \in X$ are *Birkhoff orthogonal*, $x \perp y$, if

$$\|x\| \leq \|x + ty\|, \quad \text{for all } t \in \mathbb{R},$$

where we denote by $\mathbb{R}$ the set of real numbers.

If the norm of X is generated by an inner product then this norm is called *hilbertian*. Also, we recall that the space X is smooth if there exists

$$\lim_{t \to 0} \frac{\|x + ty\|^2 - \|x\|^2}{2t} = n'(x; y), \text{for all } x, y \in X.$$

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Since the function $t \mapsto \|x+ty\|, t \in \mathbb{R}$, is convex, it follows that

$$x \perp y \text{ if and only if } n'(x;y) = 0.$$

In a real smooth space X the basic properties of the norm derivative are the following:

(P_1) $n'(x;x) = \|x\|^2$, for all $x \in X$.

(P_2) $n'(x; a_1y_1 + a_2y_2) = a_1n'(x;y_1) + a_2n'(x;y_2)$, for all $a_1, a_2 \in \mathbb{R}$ and $x, y_1, y_2 \in X$.

(P_3) For every $x \in X$, the map $y \mapsto n'(x;y), y \in X$, is a linear continuous functional, that is $n'(x;\cdot) \in X^*$.

(P_4) $n'(x;y) \leq \|x\|\|y\|$, for all $x, y \in X$.

(P_5) $n'(ax; x+by) = a\|x\|^2 + abn'(x;y)$, for all $a, b \in \mathbb{R}, x, y \in X$.

(P_6) The mapping $x \mapsto n'(x;\cdot)$, $x \in X$, is continuous from X, endowed with norm topology, into X^* with the w^*-topology.

Particularly, we have the following homogeneous property:

$$n'(ax;by) = abn'(x;y), \text{ for all } a, b \in \mathbb{R}, x, y \in X.$$

Also, if Y is a finite dimensional subspace of X, then $x \mapsto n'(x;y)$ is continuous on Y for any fixed $y \in X$. (For details concerning these properties see for instance [5-8].)

A simple and useful characterization of the hilbertian norm in a smooth space using norm derivative was established by Joichi [11], Leduc [12], Tapia [17], namely

$$n'(x;y) = n'(y;x), \text{ for all } x, y \in X.$$

Moreover, Leduc [12] proved that if $\dim X \geq 3$, then it is sufficient to have the following weaker property:

$$n'(x;y) = 0 \text{ whenever } n'(y;x) = 0. \tag{1.1}$$

This property was extended using right norm derivatives by Papini [13]. There exists many different characterizations of hilbertian norms using norm derivatives or norm directional derivatives if smoothness is not request (see, for example [1-4,9-11,15-16] and the monography of Amir [5]). Generally, these characterizations are obtained using some known properties of remarkable lines of a triangle.

We recall a characterization of strictly convex spaces established by Tapia [17], namely, a linear normed space X is strictly convex if and only if the equality $|n'(x;y)| = \|x\|\|y\|$ holds only if x, y are linear dependent elements.

In this paper we consider the heights of a triangle defined in [14], using Birkhoff orthogonality. We establish five properties such that the areas of a triangle corresponding to each height of the triangle have the same value, that is the area is well defined. If X is also strictly convex and $\dim X \geq 3$, then every of these five properties characterizes the hilbertian norms.

2. Main result

Let x, y, z be three distinct elements in a real smooth space X. In [14] there are defined the *heights of the triangle*, having the vertices x, y, z, with respect to Birkhoff orthogonality. This concept is different to other concepts considered in [1,2,4]. In our

paper, we use the concept of height defined in [14]. Thus, the height corresponding to vertex z is as follows:

$$h_{z;x,y} = \left\{ z + t\left(x - z + \frac{n'(x-y;x-z)}{\|x-y\|^2}(y-x) \right) \mid t \in \mathbb{R} \right\}.$$

Indeed, this straight line has the following property: $x - y \perp h_{z;x,y}$. Consequently, the area of triangle corresponding to this height is

$$A_{z;x,y} = \frac{1}{2}\|x-y\| \left\| x - z + \frac{n'(x-y;x-z)}{\|x-y\|^2}(y-x) \right\|. \tag{2.1}$$

Similarly, we get the areas $A_{x;z,y}$ and $A_{y;z,x}$. We remark that

$$A_{z;x,y} = A_{z;y,x} = A_{z+u,x+u,y+u}, \text{ for any } x, y, z, u \in X.$$

Since the area of a triangle is conserved by translation we can consider only triangles having a vertex in origin. We say that *the area of nontrivial triangle is well defined* if

$$A_{x;y,z} = A_{y;z,x} = A_{z;x,y}. \tag{2.2}$$

Obviously, we can suppose in the paper that $\dim X \geq 2$.

Theorem 2.1. *Let X be a real smooth space with $\dim X \geq 2$. The areas of nontrivial triangles in X are well defined if and only if one of the following equivalent properties is true:*

(i) The area of any triangle having a vertex in the origin is well defined;

(ii) $\|x-y\| \cdot \|x - z + \frac{n'(x-y;x-z)}{\|x-y\|^2}(y-x)\| = \|y-z\| \cdot \|y - x + \frac{n'(y-z;y-x)}{\|y-z\|^2}(z-y)\|$ $= \|z-x\| \cdot \|z - y + \frac{n'(z-x;z-y)}{\|z-x\|^2}(x-z)\|$, for all distinct elements $x, y, z \in X$;

(iii) $\|x-y\| \cdot \|x - z + \frac{n'(x-y;x-z)}{\|x-y\|^2}(y-x)\| = \|y-z\| \cdot \|y - x + \frac{n'(y-z;y-x)}{\|y-z\|^2}(z-y)\|$, for all distinct elements $x, y, z \in X$;

(iv) $\|x\| \cdot \|\|y\|^2 x - n'(y;x)y\| = \|y\| \cdot \|\|x\|^2 y - n'(x;y)x\|$, for all $x, y \in X$;

(v) for any two-dimensional subspace $Y \subset X$ there exists a constant $K > 0$ such that, for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in Y, the following equality holds:

$$\|\|x\|^2 y - n'(x;y)x\| = K\|x\||x_1 y_2 - x_2 y_1|.$$

If X is strictly convex and $\dim X \geq 3$, then every of these properties is true if and only if the norm of X is hilbertian.

Proof. According to the equalities (2.1) and (2.2) the area of every triangle with vertices x, y, z is well defined if and only if the equalities (ii) hold. Since we can consider only triangles with a vertex in origin, we have the equivalence $(i)\Leftrightarrow(ii)$. Also, (ii) is equivalent with the equalities obtained if $z = 0$, that is

$$\begin{aligned} \|x-y\| \left\| x + \frac{n'(x-y;x)}{\|x-y\|^2}(y-x) \right\| &= \|y\| \left\| y - x - \frac{n'(y;y-x)}{\|y\|^2} y \right\| \\ &= \|x\| \left\| y - \frac{n'(x;y)}{\|x\|^2} x \right\|, \end{aligned}$$

which proves that equality (iv) holds for all non zero distinct elements $x, y \in X$. Therefore, $(ii)\Leftrightarrow(iv)$. Conversely, if we change x, y with $x - y$ and $z - y$ respectively,

we get $(iv)\Rightarrow(iii)$. The implications $(ii)\Rightarrow(iii)$ and $(v)\Rightarrow(iv)$ are obvious. Next we observe that the equality (iii) applied to the elements y, z, x proves that $(iii)\Rightarrow(ii)$.

Now, we prove the equivalence $(iv)\Leftrightarrow(v)$. If Y is the linear subspace generated by the linear independent elements x, y, then there exist two functions $A, B : \mathbb{R}^2 \to \mathbb{R}$ such that

$$n'(u; v) = A(u_1, u_2)v_1 + B(u_1, u_2)v_2, \text{ for all } u = (u_1, u_2), v = (v_1, v_2),$$

where A, B are homogeneous. Thus, we have

$$\begin{aligned}
&\|x\|^2 y - n'(x; y)x = ((A(x_1, x_2)x_1 + B(x_1, x_2)x_2)y_1 \\
&- (A(x_1, x_2)y_1 + B(x_1, x_2)y_2)x_1, \\
&(A(x_1, x_2)x_1 + B(x_1, x_2)x_2)y_2 - (A(x_1, x_2)y_1 + B(x_1, x_2)y_2)x_2) \\
&= (B(x_1, x_2)(x_2y_1 - x_1y_2), -A(x_1, x_2)(x_2y_1 - x_1x_2)) \\
&= (x_2y_1 - x_1y_2)(B(x_1, x_2), -A(x_1, x_2)).
\end{aligned}$$

Therefore, the equality (iv) becomes

$$\begin{aligned}
&\|y\| \cdot |x_1y_2 - x_2y_2| \cdot \|(B(x_1, x_2), -A(x_1, x_2))\| \\
&= \|x\| \cdot |x_1y_2 - x_2y_1| \cdot \|(B(y_1, y_2), -A(y_1, y_2)\|,
\end{aligned}$$

for all $x, y \in Y$, that is the function

$$x \mapsto \|x\|^{-1}\|(B(x_1, x_2), -A(x_1, x_2)\|, \ x = (x_1, x_2) \in Y\backslash\{0\},$$

is a constant function. Since

$$|x_1y_2 - x_2y_1| \cdot \|(B(x_1, x_2), -A(x_1, x_2))\| = \|\|x\|^2 y - n'(x; y\|x\|)\|,$$

it follows that (iv) is equivalent with (v).

Assume that X is strictly convex. If $n'(x; y) = 0$, it follows by (iv) that

$$\|\|y\|^2 x - n'(y; x)y\| = \|x\|\|y\|^2.$$

Since $n'(x; \|y\|^2 x - n'(y; x)y) = \|x\|^2\|y\|^2$, we obtain that

$$n'(x; \|y\|^2 x - n'(y; x)y) = \|x\|\|\|y\|^2 x - n'(y; x)y\|.$$

By Tapia's characterization of strictly convex spaces [17], we get that the elements x and $\|y\|^2 x - n'(y; x)y$ are linear dependent, that is $n'(y; x) = 0$. But $\dim X \geq 3$, and so, by Leduc's result [12], it follows that the norm is necessary hilbertian.

Remark 2.2. The characterization of hilbertian norms by property (iv) was given in [10].

Remark 2.3. By the homogeneity property of the norm it follows that we can consider only elements having equal norms. Consequently, we have the following equivalent properties:

(i′) the area of any isosceles triangle is well defined;

(iv′) $\|y - n'(x; y)x\| = \|x - n'(y; x)y\|$, for all $x, y \in X$, with $\|x\| = \|y\| = 1$.

Also, in equality (v) we can consider only x, y having the same norm.

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Hermite–Hadamard type inequalities for F–convex functions involving generalized fractional integrals

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Abstract. In this paper, we firstly summarize some properties of the family $\mathcal{F}$ and F–convex functions which are defined by B. Samet. Utilizing generalized fractional integrals new Hermite–Hadamard type inequalities for F–convex functions have been provided. Some results given earlier works are also as special cases of our results.

Mathematics Subject Classification (2010): 26A51, 26A33, 26D07, 26D10, 26D15.

Keywords: Hermite–Hadamard inequality, F–convex, general fractional integral.

1. Introduction

Let $f : I \subseteq R \to R$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. If f is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality holds [17]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f . Both inequalities in (1.1) hold in the reversed direction if f is concave.

It is well known that the Hermite–Hadamard inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, see [7, 6, 12, 16], [24]-[23] and the references therein. Also, many type of convexity have been defined, such as quasi-convex in [5], pseudo-convex in [13], strongly convex in [19], ε-convex in [10], s-convex in [9], h-convex

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in [28], etc. Recently, Samet in [20], have defined a new concept of convexity that depends on a certain function satisfying some axioms, that generalizes different types of convexity, including ε-convex functions, α-convex functions, h-convex functions, and many others.
Recall the family $\mathcal{F}$ of mappings $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ satisfying the following axioms:
(A1) If $u_i \in L^1(0,1)$, $i = 1,2,3$, then for every $\lambda \in [0,1]$, we have

$$\int_0^1 F(e_1(t), e_2(t), e_3(t), \lambda)dt = F\left(\int_0^1 e_1(t)dt, \int_0^1 e_2(t)dt, \int_0^1 e_3(t)dt, \lambda\right).$$

(A2) For every $u \in L^1(0,1)$, $w \in L^\infty(0,1)$ and $(z_1, z_2) \in \mathbb{R}^2$, we have

$$\int_0^1 F(w(t)u(t), w(t)z_1, w(t)z_2, t)dt = T_{F,w}\left(\int_0^1 w(t)u(t)dt, z_1, z_2\right),$$

where $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function that depends on (F, w), and it is nondecreasing with respect to the first variable.
(A3) For any $(w, e_1, e_2, e_3) \in \mathbb{R}^4$, $e_4 \in [0,1]$, we have

$$wF(e_1, e_2, e_3, e_4) = F(we_1, we_2, we_3, e_4) + L_w,$$

where $L_w \in \mathbb{R}$ is a constant that depends only on w.

Definition 1.1. Let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be a given function. We say that f is a convex function with respect to some $F \in \mathcal{F}$ (or F-convex function), iff

$$F(f(tx + (1-t)y), f(x), f(y), t) \leq 0, \quad (x,y,t) \in [a,b] \times [a,b] \times [0,1].$$

Remark 1.2. 1) Let $\varepsilon \geq 0$, and let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be an ε-convex function, see [10], that is

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon, \ \ (x,y,t) \in [a,b] \times [a,b] \times [0,1].$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$F(e_1, e_2, e_3, e_4) = e_1 - e_4 e_2 - (1 - e_4)e_3 - \varepsilon \tag{1.2}$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{F,w}(e_1, e_2, e_3) = e_1 - \left(\int_0^1 tw(t)dt\right) e_2 - \left(\int_0^1 (1-t)w(t)dt\right) e_3 - \varepsilon. \tag{1.3}$$

For

$$L_w = (1-w)\varepsilon, \tag{1.4}$$

it is clear that $F \in \mathcal{F}$ and

$$F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - tf(x) - (1-t)f(y) - \varepsilon \leq 0,$$

that is f is an $F-$convex function. Particularly, taking $\varepsilon = 0$, we show that if f is a convex function then f is an $F-$convex function with respect to F defined above.

2) Let $f : [a, b] \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an α-convex function, $\alpha \in (0, 1]$, that is

$$f(tx + (1-t)y) \leq t^\alpha f(x) + (1-t^\alpha)f(y), \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1].$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ by

$$F(e_1, e_2, e_3, e_4) = e_1 - e_4^\alpha e_2 - (1 - e_4^\alpha)e_3 \tag{1.5}$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{F,w}(e_1, e_2, e_3) = e_1 - \left(\int_0^1 t^\alpha w(t)dt\right) e_2 - \left(\int_0^1 (1-t^\alpha)w(t)dt\right) e_3. \tag{1.6}$$

For $L_w = 0$, it is clear that $F \in \mathcal{F}$ and

$$F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - t^\alpha f(x) - (1-t^\alpha)f(y) \leq 0,$$

that is, f is an F-convex function.
3) Let $h : J \to [0, +\infty)$ be a given function which is not identical to 0, where J is an interval in $\mathbb{R}$ such that $(0, 1) \subseteq J$. Let $f : [a, b] \to [0, +\infty)$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an h-convex function, see [28], that is

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y), \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1].$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ by

$$F(e_1, e_2, e_3, e_4) = e_1 - h(e_4)e_2 - h(1 - e_4)e_3 \tag{1.7}$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{F,w}(e_1, e_2, e_3) = e_1 - \left(\int_0^1 h(t)w(t)dt\right) e_2 - \left(\int_0^1 h(1-t)w(t)dt\right) e_3. \tag{1.8}$$

For $L_w = 0$, it is clear that $F \in \mathcal{F}$ and

$$F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - h(t)f(x) - h(1-t)f(y) \leq 0,$$

that is, f is an F-convex function.

In [20], author established the following Hermite-Hadamard type inequalities using the new convexity concept:

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an F-convex function, for some $F \in \mathcal{F}$. Suppose that $f \in L^1[a, b]$. Then*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{1}{b-a}\int_a^b f(x)dx, \frac{1}{b-a}\int_a^b f(x)dx, \frac{1}{2}\right) \leq 0,$$

$$T_{F,1}\left(\frac{1}{b-a}\int_a^b f(x)dx, f(a), f(b)\right) \leq 0.$$

Theorem 1.4. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a,b) \in I^\circ \times I^\circ$, $a < b$. Suppose that:*
(i) $|f'|$ is F-convex on $[a,b]$ for some $F \in \mathcal{F}$;
(ii) the function $t \in (0,1) \to L_{w(t)}$ belongs to $L^1(0,1)$, where $w(t) = |1-2t|$. Then

$$T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|, |f'(a)|, |f'(b)|\right) + \int_0^1 L_{w(t)}dt \leq 0.$$

Theorem 1.5. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a,b) \in I^\circ \times I^\circ$, $a < b$ and let $p > 1$. Suppose that $|f'|^{p/(p-1)}$ is F-convex on $[a,b]$, for some $F \in \mathcal{F}$ and $|f'| \in L^{p/(p-1)}(a,b)$. Then*

$$T_{F,1}\left(A(p,f), |f'(a)|^{p/(p-1)}, |f'(b)|^{p/(p-1)}\right) \leq 0,$$

where

$$A(p,f) = \sqrt[p-1]{p+1}\left(\frac{2}{b-a}\right)^{\frac{p}{p-1}}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|^{\frac{p}{p-1}}.$$

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. For more details, see [8, 11, 14, 18].

Definition 1.6. Let $f \in L^1[a,b]$. The Riemann-Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

It is remarkable that Sarikaya et al. in [25], first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.7. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L^1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\right] \leq \frac{f(a)+f(b)}{2}, \tag{1.9}$$

with $\alpha > 0$.

Meanwhile, Sarikaya et al. in [25], presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha > 0$.

Lemma 1.8. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right]$$
$$= \frac{b-a}{2}\int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(ta+(1-t)b)\,dt. \tag{1.10}$$

Budak et al. in [3], prove the following Hermite-Hadamard type inequalities for F-convex functions via fractional integrals:

Theorem 1.9. *Let $I \subseteq R$ be an interval, $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping on I°, $a, b \in I^{\circ}$, $a < b$. If f is F-convex on $[a,b]$ for some $F \in \mathcal{F}$, then we have*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{a^+}^{\alpha}f(b), \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b^-}^{\alpha}f(a), \frac{1}{2}\right)$$
$$+\int_0^1 L_{w(t)}dt \leq 0 \tag{1.11}$$

and

$$T_{F,w}\left(\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b) + J_{b^-}^{\alpha}f(a)\right], f(a)+f(b), f(a)+f(b)\right)$$
$$+\int_0^1 L_{w(t)}dt \leq 0, \tag{1.12}$$

where $w(t) = \alpha t^{\alpha-1}$.

Theorem 1.10. *Let $I \subseteq R$ be an interval, $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$, $a < b$. Suppose that $|f'|$ is F-convex on $[a,b]$ for some $F \in \mathcal{F}$ and the function $t \in [0,1] \to L_{w(t)}$ belongs to $L^1[0,1]$, where $w(t) = |(1-t)^{\alpha} - t^{\alpha}|$. Then*

$$T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b) + J_{b^-}^{\alpha}f(a)\right]\right|, |f'(a)|, |f'(b)|, t\right)$$
$$+\int_0^1 L_{w(t)}dt \leq 0. \tag{1.13}$$

For the other papers on inequalities for F-convex functions, see [2, 4, 15, 26, 27].
Now we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [22].
Let's define a function $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t}dt < +\infty, \tag{1.14}$$

$$\frac{1}{A} \leq \frac{\varphi(s)}{\phi(r)} \leq A \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2, \tag{1.15}$$

$$\frac{\varphi(r)}{r^2} \leq B\frac{\varphi(s)}{s^2} \text{ for } s \leq r, \tag{1.16}$$

$$\left|\frac{\varphi(r)}{r^2}-\frac{\varphi(s)}{s^2}\right|\leq C|r-s|\frac{\varphi(r)}{r^2} \text{ for } \frac{1}{2}\leq\frac{s}{r}\leq 2, \tag{1.17}$$

where $A, B, C > 0$ are independent of $r, s > 0$. If $\varphi(r)r^{\alpha}$ is increasing for some $\alpha \geq 0$ and $\frac{\varphi(r)}{r^{\beta}}$ is decreasing for some $\beta \geq 0$, then ϕ satisfies the conditions (1.14)–(1.17).
The following left-sided and right-sided generalized fractional integral operators are defined respectively, as follows:

$$_{a^+}I_{\varphi}f(x)=\int_a^x\frac{\varphi\left(x-t\right)}{x-t}f(t)dt, \quad x>a, \tag{1.18}$$

$$_{b^-}I_{\varphi}f(x)=\int_x^b\frac{\varphi\left(t-x\right)}{t-x}f(t)dt, \quad x<b. \tag{1.19}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc.
Sarikaya and Ertuğral in [22], establish the following Hermite-Hadamard inequality and lemmas for the generalized fractional integral operators:

Theorem 1.11. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ with $a<b$, then the following inequalities for fractional integral operators hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2\Lambda(1)}\left[_{a+}I_{\varphi}f(b)+_{b-}I_{\varphi}f(a)\right]\leq\frac{f(a)+f(b)}{2}, \tag{1.20}$$

where the mapping $\Lambda:[0,1]\to\mathbb{R}$ is defined by

$$\Lambda(x)=\int_0^x\frac{\varphi\left(\left(b-a\right)t\right)}{t}dt. \tag{1.21}$$

Lemma 1.12. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L\left[a,b\right]$, then the following equality for generalized fractional integrals hold:*

$$\begin{aligned}&\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{2\Lambda(1)}\left[_{a^+}I_{\varphi}f(b)+\ _{b^-}I_{\varphi}f(a)\right]\\ =\ &\frac{(b-a)}{2\Lambda(1)}\int_0^1\left[\Lambda(1-t)-\Lambda(t)\right]f'\left(ta+(1-t)b\right)dt.\end{aligned} \tag{1.22}$$

Motivated by the above literatures, the main objective of this article is to establish some new Hermite–Hadamard type inequalities for F–convex functions via generalized fractional integrals. Some special cases will be obtain from main results. At the end, a briefly conclusion will be given as well.

2. Hermite–Hadamard type inequality via generalized fractional integrals

In this section, we establish some inequalities of Hermite–Hadamard type including generalized fractional integrals via F–convex functions.

Theorem 2.1. *Let $I \subseteq R$ be an interval, $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping on I°, $a, b \in I^\circ$, $a < b$. If f is F-convex on $[a, b]$ for some $F \in \mathcal{F}$, then we have*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{1}{\Lambda(1)}\ {}_{a+}I_\varphi f(b), \frac{1}{\Lambda(1)}\ {}_{b-}I_\varphi f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0 \tag{2.1}$$

and

$$T_{F,w}\left(\frac{1}{\Lambda(1)}\left[{}_{a+}I_\varphi f(b) + {}_{b-}I_\varphi f(a)\right], f(a)+f(b), f(a)+f(b)\right)$$

$$+\int_0^1 L_{w(t)} dt \leq 0, \tag{2.2}$$

where $w(t) = \frac{\varphi((b-a)t)}{t\Lambda(1)}$.

Proof. Since f is F–convex, we have

$$F\left(f\left(\frac{x+y}{2}\right), f(x), f(y), \frac{1}{2}\right) \leq 0,\ \forall\, x, y \in [a, b].$$

For

$$x = ta + (1-t)b \text{ and } y = tb + (1-t)a,$$

we have

$$F\left(f\left(\frac{a+b}{2}\right), f(ta+(1-t)b), f(tb+(1-t)a), \frac{1}{2}\right) \leq 0,\ \forall\, t \in [0, 1].$$

Multiplying this inequality by $w(t) = \frac{\varphi((b-a)t)}{t\Lambda(1)}$ and using axiom (A3), we get

$$F\left(\frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)} f\left(\frac{a+b}{2}\right), \frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)} f(ta+(1-t)b),\right.$$

$$\left.\frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)} f(tb+(1-t)a), \frac{1}{2}\right) + L_{w(t)} \leq 0,$$

for all $t \in [0, 1]$. Integrating over $[0, 1]$ with respect to the variable t and using axiom (A1), we obtain

$$F\left(\frac{f\left(\frac{a+b}{2}\right)}{\Lambda(1)} \int_0^1 \frac{\varphi\left((b-a)\,t\right)}{t} dt, \frac{1}{\Lambda(1)} \int_0^1 \frac{\varphi\left((b-a)\,t\right)}{t} f(ta+(1-t)b) dt,\right.$$

$$\left.\frac{1}{\Lambda(1)} \int_0^1 \frac{\varphi\left((b-a)\,t\right)}{t} f(tb+(1-t)a) dt, \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0.$$

Using the facts that

$$\int_0^1 \frac{\varphi\left((b-a)\,t\right)}{t} f(ta+(1-t)b)dt = \int_a^b \frac{\varphi\left(b-x\right)}{b-x} f(x)dx = {}_{a+}I_{\varphi} f(b)$$

and

$$\int_0^1 \frac{\varphi\left((b-a)\,t\right)}{t} f(tb+(1-t)a)dt = \int_a^b \frac{\varphi\left(x-a\right)}{x-a} f(x)dx = {}_{b-}I_{\varphi} f(a),$$

we obtain

$$F\left(f\left(\frac{a+b}{2}\right), \frac{1}{\Lambda(1)}\ {}_{a+}I_{\varphi} f(b), \frac{1}{\Lambda(1)}\ {}_{b-}I_{\varphi} f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0,$$

which gives (2.1).
On the other hand, since f is F-convex, we have

$$F\left(f\left(ta+(1-t)b\right), f(a), f(b), t\right) \leq 0,\ \forall\, t \in [0,1]$$

and

$$F\left(f\left(tb+(1-t)a\right), f(b), f(a), t\right) \leq 0,\ \forall\, t \in [0,1]\,.$$

Using the linearity of F, we get

$$F\left(f\left(ta+(1-t)b\right) + f\left(tb+(1-t)a\right), f(a)+f(b), f(a)+f(b), t\right) \leq 0,$$

$\forall\, t \in [0,1]$. Applying the axiom (A3) for $w(t) = \frac{\varphi((b-a)t)}{t\Lambda(1)}$, we obtain

$$F\Bigg(\frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)}\left[f\left(ta+(1-t)b\right) + f\left(tb+(1-t)a\right)\right],$$

$$\frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)}\left[f(a)+f(b)\right], \frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)}\left[f(a)+f(b)\right], t\Bigg) + L_{w(t)} \leq 0,$$

for all $t \in [0,1]$. Integrating over $[0,1]$ and using axiom (A2), we have

$$T_{F,w}\Bigg(\int_0^1 \frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)}\left[f\left(ta+(1-t)b\right) + f\left(tb+(1-t)a\right)\right] dt,$$

$$f(a)+f(b), f(a)+f(b)\Bigg) + \int_0^1 L_{w(t)} dt \leq 0,$$

that is

$$T_{F,w}\left(\frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi} f(b) + {}_{b-}I_{\varphi} f(a)\right], f(a)+f(b), f(a)+f(b)\right)$$

$$+ \int_0^1 L_{w(t)} dt \leq 0.$$

The proof of Theorem 2.1 is completed. □

Remark 2.2. If we take $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$ in Theorem 2.1, then the inequalities (2.1) and (2.2) reduce to the inequalities (1.11) and (1.12).

Corollary 2.3. *If we take* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *in Theorem* 2.1*, then we have the following inequalities for k-Riemann-Liouville fractional integrals*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}} I^{\alpha}_{a+,\ k} f(b), \frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}} I^{\alpha}_{b-,\ k} f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0$$

and

$$T_{F,w}\left(\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k} f(b) + I^{\alpha}_{b-,\ k} f(a)\right], f(a)+f(b), f(a)+f(b)\right) + \int_0^1 L_{w(t)} dt \leq 0.$$

where $w(t) = \frac{\alpha}{k} t^{\frac{\alpha}{k}-1}$.

Corollary 2.4. *If we choose* $F(e_1, e_2, e_3, e_4) = e_1 - e_4 e_2 - (1-e_4)e_3 - \varepsilon$ *in Theorem* 2.1*, then the function* f *is* ε*-convex on* $[a,b]$*, where* $\varepsilon \geq 0$ *and we have the following new double inequality:*

$$f\left(\frac{a+b}{2}\right) + \varepsilon \leq \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi} f(b) + {}_{b-}I_{\varphi} f(a)\right] \leq \frac{f(a)+f(b)}{2} + \frac{\varepsilon}{2}. \tag{2.3}$$

Proof. Using (1.4) with $w(t) = \frac{\varphi((b-a)t)}{t\Lambda(1)}$, we have

$$\int_0^1 L_{w(t)} dt = \varepsilon \int_0^1 \left(1 - \frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)}\right) dt = 0. \tag{2.4}$$

Using (1.2), (2.1) and (2.4), we get

$$F\left(f\left(\frac{a+b}{2}\right), \frac{1}{\Lambda(1)}\ {}_{a+}I_{\varphi} f(b), \frac{1}{\Lambda(1)}\ {}_{b-}I_{\varphi} f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0.$$

So

$$f\left(\frac{a+b}{2}\right) - \frac{1}{2\Lambda(1)}\left[\ {}_{a+}I_{\varphi} f(b) + {}_{b-}\ I_{\varphi} f(a)\right] - \varepsilon \leq 0,$$

that is

$$f\left(\frac{a+b}{2}\right) + \varepsilon \leq \frac{1}{2\Lambda(1)}\left[\ {}_{a+}I_{\varphi} f(b) + {}_{b-}\ I_{\varphi} f(a)\right].$$

On the other hand, using (1.3) with $w(t) = \frac{\varphi((b-a)t)}{t\Lambda(1)}$, we have

$$T_{F,w}(e_1, e_2, e_3) = e_1 - \left(\int_0^1 t \frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)} dt\right) e_2 - \left(\int_0^1 (1-t) \frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)} dt\right) e_3 - \varepsilon \tag{2.5}$$

for $e_1, e_2, e_3 \in \mathbb{R}$. Hence, from (2.2) and (2.5), we obtain

$$\begin{aligned}
0 &\geq T_{F,w}\left(\frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right], f(a)+f(b), f(a)+f(b)\right) \\
&\quad + \int_0^1 L_{w(t)}dt \\
&= \frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right] - \left(\int_0^1 t\frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)}dt\right)\left[f(a)+f(b)\right] \\
&\quad - \left(\int_0^1 (1-t)\frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)}dt\right)\left[f(a)+f(b)\right] - \varepsilon \\
&= \frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right] - \left[f(a)+f(b)\right] - \varepsilon.
\end{aligned}$$

This implies that

$$\frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right] \leq f(a)+f(b)+\varepsilon$$

and thus the proof is completed. □

Remark 2.5. If we take $\varepsilon = 0$ in Corollary 2.4, then f is convex and we have the inequality (1.20).

Corollary 2.6. *If we choose $F(e_1, e_2, e_3, e_4) = e_1 - h(e_4)e_2 - h(1-e_4)e_3$ in Theorem 2.1, then the function f is h-convex on $[a, b]$ and we have the following new double inequality:*

$$\begin{aligned}
\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right] \\
&\leq \frac{\left[f(a)+f(b)\right]}{2\Lambda(1)} \\
&\times \int_0^1 \frac{\varphi((b-a)t)}{t}[h(t)+h(1-t)]dt. \qquad (2.6)
\end{aligned}$$

Proof. Using (1.7) and (2.1) with $L_{w(t)} = 0$, we have

$$\begin{aligned}
0 &\geq F\left(f\left(\frac{a+b}{2}\right), \frac{1}{\Lambda(1)}\,{}_{a+}I_{\varphi}f(b), \frac{1}{\Lambda(1)}\,{}_{b-}I_{\varphi}f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)}dt \\
&= f\left(\frac{a+b}{2}\right) - h\left(\frac{1}{2}\right)\frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right],
\end{aligned}$$

that is

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right].$$

On the other hand, using (1.8) and (2.2) with $w(t)=\frac{\varphi((b-a)t)}{t\Lambda(1)}$, we obtain

$$\begin{aligned}
0 &\geq T_{F,w}\left(\frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right],f(a)+f(b),f(a)+f(b)\right)\\
&\quad+\int_0^1 L_{w(t)}dt\\
&= \frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\\
&\quad-\left[\int_0^1 h(t)\frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)}dt+\int_0^1 h(1-t)\frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)}dt\right]\left[f(a)+f(b)\right]\\
&= \frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\\
&\quad-\frac{1}{\Lambda(1)}\left(\int_0^1\left[h(t)+h(1-t)\right]\frac{\varphi\left((b-a)\,t\right)}{t}dt\right)\left[f(a)+f(b)\right],
\end{aligned}$$

that is

$$\begin{aligned}
&\frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\\
&\leq\frac{\left[f(a)+f(b)\right]}{\Lambda(1)}\left(\int_0^1\left[h(t)+h(1-t)\right]\frac{\varphi\left((b-a)\,t\right)}{t}dt\right)
\end{aligned}$$

and thus the proof is completed. □

Theorem 2.7. *Let $I\subseteq R$ be an interval, $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$, $a<b$. Suppose that $|f'|$ is F-convex on $[a,b]$, for some $F\in\mathcal{F}$ and the function $t\in[0,1]\to L_{w(t)}$ belongs to $L^1[0,1]$, where $w(t)=\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}$. Then, we have the following inequality:*

$$\begin{aligned}
&T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right|,\right.\\
&\left.\left|f'(a)\right|,\left|f'(b)\right|,t\right)+\int_0^1 L_{w(t)}dt\leq 0. \qquad (2.7)
\end{aligned}$$

Proof. Since $|f'|$ is F-convex, we have

$$F\left(\left|f'(ta+(1-t)b)\right|,\left|f'(a)\right|,\left|f'(b)\right|,t\right)\leq 0,\ \forall\, t\in[0,1].$$

Using axiom (A3) with $w(t)=\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}$, we get

$$F\left(w(t)\left|f'(ta+(1-t)b)\right|,w(t)\left|f'(a)\right|,w(t)\left|f'(b)\right|,t\right)+L_{w(t)}\leq 0,\ \forall\, t\in[0,1].$$

Integrating over $[0,1]$ and using axiom (A2), we obtain

$$T_{F,w}\left(\int_0^1 w(t)\left|f'(ta+(1-t)b)\right|dt,\left|f'(a)\right|,\left|f'(b)\right|,t\right)+\int_0^1 L_{w(t)}dt\leq 0,$$

$\forall\, t \in [0,1]$. From Lemma 1.8, we have

$$\begin{aligned} & \frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b)+{}_{b^-}I_{\varphi}f(a)\right]\right| \\ \leq & \int_0^1 w(t)\left|f'(ta+(1-t)b)\right|dt. \end{aligned}$$

Since $T_{F,w}$ is nondecreasing with respect to the first variable, we establish

$$T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b)+{}_{b^-}I_{\varphi}f(a)\right]\right|,\right.$$

$$\left.\left|f'(a)\right|,\left|f'(b)\right|,t\right)+\int_0^1 L_{w(t)}dt\leq 0.$$

The proof of Theorem 2.7 is completed. □

Corollary 2.8. *Under assumptions of Theorem* 2.7*, if we choose*

$$F(e_1,e_2,e_3,e_4)=e_1-e_4e_2-(1-e_4)e_3-\varepsilon,$$

then the function $|f'|$ *is* ε*-convex on* $[a,b]$, $\varepsilon\geq 0$ *and we have the following new inequality:*

$$\begin{aligned} & \left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b)+{}_{b^-}I_{\varphi}f(a)\right]\right| \\ \leq & \frac{(b-a)}{2\Lambda(1)}\left(\int_0^1 t\left|\Lambda(1-t)-\Lambda(t)\right|dt\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right] \qquad (2.8) \\ & +\varepsilon\frac{(b-a)}{2\Lambda(1)}\int_0^1\left|\Lambda(1-t)-\Lambda(t)\right|dt. \end{aligned}$$

Proof. From (1.4) with $w(t)=\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}$, we have

$$\begin{aligned} \int_0^1 L_{w(t)}dt &= \varepsilon\int_0^1\left(1-\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}\right)dt \\ &= \varepsilon\left(1-\int_0^1\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}dt\right). \end{aligned}$$

Using (1.3) with $w(t) = |\Lambda(1-t) - \Lambda(t)|$, we get

$$\begin{aligned} T_{F,w}(e_1, e_2, e_3) &= e_1 - \left(\int_0^1 t \frac{|\Lambda(1-t) - \Lambda(t)|}{\Lambda(1)} dt \right) e_2 \\ &\quad - \left(\int_0^1 (1-t) \frac{|\Lambda(1-t) - \Lambda(t)|}{\Lambda(1)} dt \right) e_3 - \varepsilon \\ &= e_1 - \left(\int_0^1 t \frac{|\Lambda(1-t) - \Lambda(t)|}{\Lambda(1)} dt \right) (e_2 + e_3) - \varepsilon \end{aligned}$$

for $e_1, e_2, e_3 \in \mathbb{R}$. Then, by Theorem 2.7, we have

$$\begin{aligned} 0 &\geq T_{F,w} \left(\frac{2}{b-a} \left| \frac{f(a) + f(b)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{a^+}I_\varphi f(b) + {}_{b^-}I_\varphi f(a) \right] \right|, \right. \\ &\left. |f'(a)|, |f'(b)|, t \right) + \int_0^1 L_{w(t)} dt \\ &= \frac{2}{b-a} \left| \frac{f(a) + f(b)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{a^+}I_\varphi f(b) + {}_{b^-}I_\varphi f(a) \right] \right| \\ &- \left(\int_0^1 t \frac{|\Lambda(1-t) - \Lambda(t)|}{\Lambda(1)} dt \right) \left[|f'(a)| + |f'(b)| \right] \\ &+ \varepsilon \left(1 - \int_0^1 \frac{(|\Lambda(1-t) - \Lambda(t)|}{\Lambda(1)} dt \right) - \varepsilon. \end{aligned}$$

This completes the proof. □

Remark 2.9. If we choose $\varepsilon = 0$ in Corollary 2.8, then $|f'|$ is convex and we have the inequality:

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{a^+}I_\varphi f(b) + {}_{b^-}I_\varphi f(a) \right] \right| \\ &\leq \frac{(b-a)}{2\Lambda(1)} \left(\int_0^1 t \left| \Lambda(1-t) - \Lambda(t) \right| dt \right) \left[|f'(a)| + |f'(b)| \right], \end{aligned} \tag{2.9}$$

which is given by Sarikaya and Ertuğral in [22].

Corollary 2.10. *Under assumption of Theorem* 2.7*, if we choose*

$$F(e_1, e_2, e_3, e_4) = e_1 - h(e_4)e_2 - h(1-e_4)e_3,$$

then the function $|f'|$ is h-convex on $[a,b]$ and we have the inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b)+{}_{b^-}I_{\varphi}f(a)\right]\right|$$
$$\leq\frac{(b-a)}{\Lambda(1)}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]\left(\int_0^1 h(t)\left|\Lambda(1-t)-\Lambda(t)\right|dt\right), \tag{2.10}$$

which is given by Ali et al. in [1].

Proof. From (1.8) with $w(t)=|\Lambda(1-t)-\Lambda(t)|$, we have

$$T_{F,w}(e_1,e_2,e_3)=e_1-\left(\int_0^1 h(t)\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}dt\right)e_2$$
$$-\left(\int_0^1 h(1-t)\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}dt\right)e_3$$
$$=e_1-\left(\int_0^1 h(t)\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}dt\right)e_2-\left(\int_0^1 h(t)\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}dt\right)e_3$$
$$=e_1-\left(\int_0^1 h(t)\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}dt\right)(e_2+e_3)$$

for $e_1,e_2,e_3\in\mathbb{R}$. Then, by Theorem 2.7, we have

$$T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b)+{}_{b^-}I_{\varphi}f(a)\right]\right|,|f'(a)|,|f'(b)|,t\right)$$
$$=\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b)+{}_{b^-}I_{\varphi}f(a)\right]\right|$$
$$-\left(\int_0^1 h(t)\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}dt\right)\left[|f'(a)|+|f'(b)|\right]\leq 0.$$

This completes the proof. □

Remark 2.11. If we take $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ in Theorem 2.7, then the inequality (2.7) reduces to the inequality (1.13).

Corollary 2.12. *If we take $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.7, then we have the following inequalities for k-Riemann-Liouville fractional integrals*

$$T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b)+I^{\alpha}_{b-,\ k}f(a)\right]\right|,|f'(a)|,|f'(b)|,t\right)$$
$$+\int_0^1 L_{w(t)}dt\leq 0,\ \textit{where } w(t)=\left|(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right|.$$

3. Conclusion

In the development of this work, using the definition of F–convex functions some new Hermite–Hadamard type inequalities via generalized fractional integrals have been deduced. Also, this class of functions can be applied to obtain several results in convex analysis, related optimization theory, etc. The authors hope that these results will serve as a motivation for future work in this fascinating area.

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On a pure traction problem for the nonlinear elasticity system in Sobolev spaces with variable exponents

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Abstract. The paper deals with a nonlinear elasticity system with nonconstant coefficients. The existence and uniqueness of the solution of Neumann's problem is proved using Galerkin techniques and monotone operator theory, in Sobolev spaces with variable exponents.

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1. Introduction

The study of PDE problems with variable exponents is a novel and quite interesting topic. It comes from the theory of nonlinear elasticity, elastic mechanics, fluid dynamics, electrorheological fluids, and image processing, etc. (see [1], [15], [16]). First, we introduce the notations needed in this article. Let Ω an connected open bounded domain of $\mathbb{R}^{\mathbb{N}}$ ($\mathbb{N} = 3$) with Lipschitz boundary Γ. To a given field of displacement u, we associate a nonlinear deformation tensor E defined by

$$E\left(\nabla u(x)\right) = \frac{1}{2}\left(\nabla u^T + \nabla u + \nabla u^T \nabla u\right),$$

whose components are:

$$E_{ij}\left(\nabla u(x)\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{m=1}^{3} \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right),\ 1 \leq i, j \leq 3. \tag{1.1}$$

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The corresponding nonlinear constraints tensor $\sigma(u) = (\sigma_{ij}(u(x)))_{1\leq i,j\leq 3}$ is then given by:

$$\sigma_{ij}(u(x)) = \sum_{k,h=1}^{3} a_{ijkh}(x)\ E_{kh}(\nabla u(x)),\ 1 \leq i,j \leq 3, \tag{1.2}$$

which describes a nonlinear relation between the stress tensor $(\sigma_{ij})_{i,j=1,2,3}$ and the deformation tensor $(E_{ij})_{i,j=1,2,3}$. The coefficients of elasticity a_{ijkh} satisfy the following symmetry properties:

$$a_{ijkh} = a_{jikh} = a_{ijhk},\ \text{for all } 1 \leq i,j,k,h \leq 3. \tag{1.3}$$

The aim of this paper is to prove the existence and uniqueness of weak solutions for the following nonlinear elliptic problem, encountered in the theory of nonlinear elasticity:

$$\left\{\begin{array}{c} -\sum\limits_{j=1}^{3} \dfrac{\partial}{\partial x_j}\sigma_{ij}(u(x)) = f_i(x,u(x)) \text{ in } \Omega,\ 1 \leq i \leq 3, \\ \sigma_{ij}(u(x)) = \sum\limits_{k,h=1}^{3} a_{ijkh}(x)\ E_{kh}(\nabla u(x)) \text{ in } \Omega,\ 1 \leq i,j \leq 3, \\ E_{ij}\left(\nabla u(x)\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum\limits_{m=1}^{3} \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right) \text{ in } \Omega,\ 1 \leq i,j \leq 3, \\ \sum\limits_{j=1}^{3} \sigma_{ij}\left(u\left(x\right)\right)\eta_j = 0 \text{ on } \Gamma,\ 1 \leq i \leq 3. \end{array}\right. \tag{P}$$

Problem (P) models the behavior of a heterogeneous material with Neumann's condition on the boundary. The consideration of this general material is in no way restrictive. Indeed, we can applied this study to the most particular elastic materials, but this particular case makes it easy, to describe the different stages of this work. The tensor of the constraints considered here is nonlinear and grouped, as special cases, some models used in Ciarlet [2], Dautry-Lions [4] and Lions [10]. Let us cite by way of example (see [2] , [8]):

1. The problem of displacement for a homogeneous or heterogeneous material of St Vennan-Kirchhoff where:
   - the applied volumetric forces f are dead (does not depend on u),
   - the tensor of stress is in the form (material of StVennan-Kirchhoff):

$$\left\{\begin{array}{c} \sigma_{ij}(u(x)) = \lambda(trE_{ij}(\nabla u(x))) + 2\mu E_{ij}(\nabla u(x)), \\ 1 \leq i,j \leq 3,\ \lambda > 0,\ \mu > 0, \end{array}\right.$$

2. The coefficients of elasticity have the form:

$$a_{ijpq} = \lambda\delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}), 1 \leq i,\ j,\ p,\ q \leq 3$$

   with, λ and μ depend on x or not,
3. The applied volumetric forces f have the form $f(\xi) = |\xi|^{p(x)-1}\,\xi$,
4. Some models called "LES" (Large Eddy Simulations) used in fluid mechanics. These problems are:

$$-\operatorname{div}(\psi(x)a(\nabla u(x))) = f(x).$$

For $\psi \equiv 1$ and $a(\xi) = |\xi|^{p(x)-2}\,\xi$, the above equation may be described by:

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\,\nabla u) = f.$$

The operator $\Delta_{p(x)} : u \longrightarrow \Delta_{p(x)}(u) = \operatorname{div}(|\nabla u|^{p(x)-2}\,\nabla u)$ is called the $p\,(x)$-Laplacian.

Several authors studied the system of elasticity with laws of particular behavior and using various techniques in constant exposants Sobolev spaces for example in [2] Ciarlet used the implicit function theorem to show the existence and uniqueness of a solution, in [4] Dautry-Lions studied the linear problem in a regular boundary domain, in [11], [12], [13] Merouani studied the Lamé (elasticity) system in a polygonal boundary domain.

The bibliography quoted here does not claim to be exhaustive and the deficiencies it certainly entails must be attributed to the author's ignorance and not to the author's ill will.

To solve our problem, we will consider an operator: $u \to A(u) = -\sum\limits_{j=1}^{3} \dfrac{\partial}{\partial x_j}\sigma_{ij}(u(x))$ as operator of Leray-Lions [9], with Neumann's condition on Γ, and we prove a theorem of existence and uniqueness of solution using Galerkin techniques and monotone operator theory.

This paper is organized as follows:

- Notations and properties of variable exponent Lebesgue-Sobolev spaces,
- Hypotheses and main result,
- Proof of theorem,
- Conclusion and bibliography.

2. Properties of variable exponent Lebesgue-Sobolev spaces

In this section, we recall some definitions and basic properties of the generalized Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, when Ω is a bounded open set of $\mathbb{R}^{\mathbb{N}}(\mathbb{N} \geq 1)$ with a smooth boundary.

Let $p : \overline{\Omega} \to [1, +\infty)$ be a continuous, real-valued function.

Denote by $p_- = \min\limits_{x\in\overline{\Omega}} p(x)$ and $p_+ = \max\limits_{x\in\overline{\Omega}} p(x)$.

We introduce the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R}; u \text{ is measurable with } \int_\Omega |u\,(x)|^{p(x)}\,dx < \infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u\,(x)}{\lambda}\right|^{p(x)} dx \leq 1 \right\}.$$

The following inequality will be used later

$$\min\left\{ \|u\|^{p_-}_{L^{p(x)}(\Omega)}, \|u\|^{p_+}_{L^{p(x)}(\Omega)} \right\} \leq \int_\Omega |u\,(x)|^{p(x)}\,dx \leq \max\left\{ \|u\|^{p_-}_{L^{p(x)}(\Omega)}, \|u\|^{p_+}_{L^{p(x)}(\Omega)} \right\}$$

for any $u \in L^{p(x)}(\Omega)$.

Lemma 2.1. [3], [5], [6], [7]

- *The space* $\left(L^{p(x)}(\Omega), \|.\|_{L^{p(x)}(\Omega)}\right)$ *is a Banach space.*
- *If* $p_- > 1$, *then* $L^{p(x)}(\Omega)$ *is reflexive and its conjugate space can be identified with* $L^{p'(x)}(\Omega)$ *where,* $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. *Moreover, for any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{p'(x)}(\Omega)$, *we have the Hölder inequality*

$$\int_\Omega |uv|\, dx \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)} \leq 2 \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)}.$$

- *If* $p_+ < +\infty$, *then* $L^{p(x)}(\Omega)$ *is separable.*
- *Some embedding stay true, for example, if* $0 < |\Omega| < \infty$ *and* p_1, p_2 *are variable exponent so that* $p_1(x) \leq p_2(x)$ *almost everywhere in* Ω, *then we have* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Now, we define also the variable Sobolev space by

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega);\ |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

endowed with the following norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

Definition 2.2. The variable exponent $p : \overline{\Omega} \to [1, +\infty)$ is said to satisfy the log-Hölder continuous condition if

$$\forall x, y \in \overline{\Omega},\ \ |x - y| < 1,\ \ |p(x) - p(y)| < w(|x - y|),$$

where $w : (0, \infty) \to \mathbb{R}$ is a nondecreasing function with $\lim\limits_{\alpha \to 0} \sup w(\alpha) \ln\left(\frac{1}{\alpha}\right) < \infty$.

Lemma 2.3. [3], [5], [6], [7]

- *If* $1 < p_- \leq p_+ < \infty$, *then the space* $\left(W^{1,p(x)}(\Omega), \|.\|_{W^{1,p(x)}(\Omega)}\right)$ *is a separable and reflexive Banach space.*
- *If* $p(x)$ *satisfies the log-Hölder continuous condition, then* $C^\infty(\Omega)$ *is dense in* $W^{1,p(x)}(\Omega)$. *Moreover, we can define the Sobolev space with zero boundary values,* $W_0^{1,p(x)}(\Omega)$ *as the completion of* $C_0^\infty(\Omega)$ *with respect to the norm* $\|.\|_{W^{1,p(x)}(\Omega)}$.
- *For all* $u \in W_0^{1,p(x)}(\Omega)$, *the Poincaré inequality*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C \|\nabla u\|_{L^{p(x)}(\Omega)},$$

holds. Moreover, $\|u\|_{W_0^{1,p(x)}(\Omega)} = \|\nabla u\|_{L^{p(x)}(\Omega)}$ *is a norm in* $W_0^{1,p(x)}(\Omega)$.

Throughout this paper, we shall assume that the variable exponent $p(x)$ satisfy the log-Hölder condition, and $\mathbb{N} < p_- \leq p_+ < \infty$ because if $p(x) > \mathbb{N}$ then $W^{1,p(x)}(\Omega) \subset C(\Omega)$ for every $x \in \Omega$.

3. Hypotheses and main result

We consider the following problem:

$$\begin{cases} -\sum_{j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(u(x)) = f_i(x, u(x)) \text{ in } \Omega,\ 1 \leq i \leq 3, \\ \sigma_{ij}(u(x)) = \sum_{k,h=1}^{3} a_{ijkh}(x)\ E_{kh}(\nabla u(x)) \text{ in } \Omega,\ 1 \leq i,j \leq 3, \\ E_{ij}\left(\nabla u(x)\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{m=1}^{3} \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right) \text{ in } \Omega,\ 1 \leq i,j \leq 3, \\ \sum_{j=1}^{3} \sigma_{ij}\left(u\left(x\right)\right)\eta_j = 0 \text{ on } \Gamma,\ 1 \leq i \leq 3. \end{cases} \tag{3.1}$$

This problem being that of Neumann, we must impose the necessary conditions of existence namely the condition of compatibility:

$$\int_{\Omega} f dx = 0.$$

This is the hypotheses which concern E_{kh} and f :

$$\begin{cases} \forall i,j,k,h = 1 \text{ to } 3: \\ 1)\ E_{kh} \text{ is a continuous function,} \\ 2)\ \text{(Coercivity) } \exists \alpha > 0;\ \text{such that } E_{kh}\left(\xi\right)\xi_{ij} \geq \alpha\left|\xi\right|^{p(x)}, \\ \quad \forall \xi \in \mathbb{R}^{3\times 3} \text{ and, } \xi_{ij} \in \mathbb{R}, \\ 3)\ \text{(Increase) } \exists C \in \mathbb{R};\ \left|E_{kh}\left(\xi\right)\right| \leq C\left(1 + \left|\xi\right|^{p(x)-1}\right), \\ 4)\ \left(E_{kh}\left(\xi\right) - E_{kh}\left(\eta\right)\right)\left(\xi_{ij} - \eta_{ij}\right) \geq 0, \forall \xi, \eta \in \mathbb{R}^{3\times 3}, \text{ and} \\ \quad \xi_{ij}, \eta_{ij} \in \mathbb{R}, \\ 5)\ a_{ijkh} \in L^{\infty}\left(\Omega\right);\ \exists \alpha_0 > 0;\ a_{ijkh} \geq \alpha_0 \text{ a.e. in } \Omega, \\ 6)\ f = \left(f_1, f_2, f_3\right) \text{ is a Caratheodory function and,} \\ \quad f \in \left(L^{\frac{p(x)}{p(x)-1}}\left(\Omega\right)\right)^3. \end{cases} \tag{3.2}$$

Let us look for an adequate weak form of (3.1). Note that if $w \in \left(L^{p(x)}(\Omega)\right)^9$, then the growth condition on E_{kh} gives

$$\begin{aligned} \left|E_{kh}\left(w\right)\right| &\leq C\left(1 + \left|w\right|^{p(x)-1}\right) \\ &\leq \left(C + C\left|w\right|^{p(x)-1}\right) \in L^{\frac{p(x)}{p(x)-1}}\left(\Omega\right),\ 1 \leq k,h \leq 3. \end{aligned}$$

So, if $u \in H$, we have $E_{kh}\left(\nabla u\right) \in L^{p'(x)}\left(\Omega\right)$. Or

$$H = \left\{u \in \left(W^{1,p(x)}\left(\Omega\right)\right)^3,\ \frac{1}{mes\left(\Omega\right)}\int_{\Omega} u\left(x\right)dx = 0\right\},$$

is a closed vector subspace of $\left(W^{1,p(x)}\left(\Omega\right)\right)^3$, provided with the norm

$$\left\|u\right\|_H = \left\|\nabla u\right\|_{L^{p(x)}(\Omega)},$$

which is equivalent to the norm of $\left(W^{1,p(x)}(\Omega)\right)^3$. We note that:

$$\left(W^{1,p(x)}(\Omega)\right)^3 = H \oplus F,$$

where F is the space of constants. Let's take then $v \in H$, we have $\nabla v \in \left(L^{p(x)}(\Omega)\right)^9$. So we obtain from the inequality of Hölder:

$$E_{kh}(\nabla u)\frac{\partial v_i}{\partial x_j} \in L^1(\Omega), \forall\, i,j,k,h = 1 \text{ to } 3.$$

It is therefore natural to look $u \in H$ and take the test functions in H. We also recall that if $f(.,s) \in \left(L^{p'(x)}(\Omega)\right)^3$, the mapping $v \to \int_\Omega f(x,u(x))v(x)\,dx$ acting from H to $\mathbb{R}$, is an element of H'. We denote by f this element, that is to say for $f \in \left(L^{p'(x)}(\Omega)\right)^3$, we have

$$\langle f, v\rangle_{H',H} = \int_\Omega f(x,u(x))\,v(x)\,dx,\ \forall v \in H.$$

The weak form of (3.1) is thus:

$$\begin{cases} u \in H, \\ \displaystyle\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u(x))\frac{\partial v_i}{\partial x_j}dx = \langle f, v\rangle_{H',H},\ \forall v \in H. \end{cases} \tag{3.3}$$

Theorem 3.1. *Under the hypotheses* (3.2)*, there exist* $u \in H$ *solution of* (3.3)*. If, moreover,* $(E_{kh}(\xi) - E_{kh}(\eta))(\xi_{ij} - \eta_{ij}) > 0$*, for all* $\xi, \eta \in \mathbb{R}^{3\times 3}, \xi_{ij}, \eta_{ij} \in \mathbb{R}, \xi_{ij} \neq \eta_{ij}$ *then there exist a unique solution* u *of* (3.3)*.*

For the proof of this theorem, we will need the following (classical) integration lemmas:

Lemma 3.2. *Let* $p : \Omega \to]1, +\infty[$*. If* $f_n \to f$ *in* $L^{p(x)}(\Omega)$ *and* $g_n \to g$ *weakly in* $L^{p'(x)}(\Omega)$*. So*

$$\int_\Omega f_n g_n dx \to \int_\Omega fg dx \text{ when } n \to \infty.$$

Demonstration of lemma (3.2). We have:

$$\begin{aligned}\left|\int_\Omega (f_n\ g_n - f\ g)\,dx\right| &= \left|\int_\Omega (f_n\ g_n - f\ g - f\ g_n + f\ g_n)\,dx\right| \\ &= \left|\int_\Omega [(f_n - f)\ g_n + f\ \ (g_n - g)]\,dx\right| \\ &\le \int_\Omega |f_n - f|\ \ |g_n|\,dx + \left|\int_\Omega f\ (g_n - g)dx\right| \\ &\le 2.\,\|f_n - f\|_{L^{p(x)}(\Omega)}\,\|g_n\|_{L^{p'(x)}(\Omega)} + |\langle g_n - g, f\rangle| \to 0.\end{aligned}$$

Lemma 3.3. *If* $E_{kh} \in C\left(\mathbb{R}^{3\times 3}, \mathbb{R}\right)$, $|E_{kh}(\xi)| \le C\left(1 + |\xi|^{p(x)-1}\right)$, $k,h = 1$ *to* 3*, for all* $\xi \in \mathbb{R}^{3\times 3}$ *and if* $u_n \to u$ *in* $\left(W^{1,p(x)}(\Omega)\right)^3$ *then* $E_{kh}(\nabla u_n) \to E_{kh}(\nabla u)$, $k,h = 1$ *to* 3*, in* $L^{p'(x)}(\Omega)$*.*

The lemma (3.3) is proved by Lebesgue's dominated convergence theorem.

Remark 3.4. [14] Let $p \in L_+^\infty(\Omega) = \{p \in L^\infty(\Omega), p_- \geq 1\}, (u_n) \subset L^{p(x)}(\Omega)$ and $u \in L^{p(x)}(\Omega)$. If $\lim_{n\to\infty} \|u_n - u\|_{L^{p(x)}(\Omega)} = 0$. Then there exist a subsequence $(u_{nj}) \subset (u_n)$ and a function $g \in L^{p(x)}(\Omega)$ such that:

(i) $u_{nj} \to u$ a.e. in Ω,

(ii) $|u_{nj}| \leq g(x)$ a.e. in Ω.

Demonstration of lemma (3.3). $u_n \to u$ in $\left(W^{1,p(x)}(\Omega)\right)^3$ involves: $u_n \to u$ in $\left(L^{p(x)}(\Omega)\right)^3$ and $\nabla u_n \to \nabla u$ in $\left(L^{p(x)}(\Omega)\right)^9$.
$\nabla u_n \to \nabla u$ in $\left(L^{p(x)}(\Omega)\right)^9$ involves $\nabla u_n \to \nabla u$ a.e. in Ω, and as E_{kh} is continuous then:

$$E_{kh}(\nabla u_n) \to E_{kh}(\nabla u) \text{ a.e., } k, h = 1 \text{ to } 3$$

we have also

$$|E_{kh}(\nabla u_n)| \leq (C + C|\nabla u_n|^{p(x)-1}) \in L^{\frac{p(x)}{p(x)-1}}(\Omega), \ k, h = 1 \text{ to } 3.$$

So we deduce that

$$E_{kh}(\nabla u_n) \to E_{kh}(\nabla u) \text{ in } L^{\frac{p(x)}{p(x)-1}}(\Omega).$$

We will also need for the proof the following lemma:

Lemma 3.5. *(Finite-dimensional coercive operator) Let V be a finite-dimensional space, and $T : V \to V'$ continuous. We suppose that T is coercive, namely:*

$$\frac{\langle T(v).v\rangle_{V',V}}{\|v\|_V} \to +\infty \text{ when } \|v\|_V \to +\infty.$$

Then, for every $b \in V'$ there exist $v \in V$ such that $T(v) = b$.

4. Proof of theorem

Study of finite dimension problem

Since H is separable, (because H is a closed vector subspace of $\left(W^{1,p(x)}(\Omega)\right)^3$, and $\left(W^{1,p(x)}(\Omega)\right)^3$ is a Banach space separable) then there exist a countable family $(f_n)_{n\in\mathbb{N}^*}$ dense in H. Let $V_n = Vect\{f_i, \ i = 1, ..., n\}$ be the vector space generated by the first n functions of this family. So we have $\dim V_n \leq n$, $V_n \subset V_{n+1}$ for all $n \in \mathbb{N}^*$ and we have $\overline{\bigcup_{n\in\mathbb{N}} V_n} = H$. We deduce that for all $v \in H$ there exist a sequence $v_n \in V_n$, such that $v_n \to v$ in H when $n \to +\infty$.
In the first step, we fix $n \in \mathbb{N}^*$ and look for u_n solution of the following problem, posed in finite dimension:

$$\begin{cases} u_n \in V_n, \\ \displaystyle\int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u_n(x)) \frac{\partial v_i}{\partial x_j} dx = \langle f, v\rangle_{H',H}, \ \forall v \in V_n. \end{cases} \tag{3.4}$$

The application $v \to \langle f, v\rangle_{H',H}$ is a linear mapping of V_n to $\mathbb{R}$ (it is also continuous because $\dim V_n < +\infty$). We denote by b_n this application. So $b_n \in V_n'$ and

$$\langle b_n, v\rangle_{V_n',V_n} = \langle f, v\rangle_{H',H}.$$

Let $u \in V_n$. We denote by $T_n(u)$ the mapping of V_n into V_n' which has $v \in V_n$ associated

$$\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u(x))\frac{\partial v_i}{\partial x_j}dx.$$

This application is linear, so it is also an element of $V_n^{'}$ and we have

$$\langle T_n(u), v\rangle_{V_n',V_n} = \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u(x))\frac{\partial v_i}{\partial x_j}dx.$$

We have thus defined an application T of V_n to V_n'. We shall show that T is continuous and coercive. We can thus deduce by the lemma (3.5), that T is surjective, and therefore that there exist $u_n \in V_n$ satisfying $T(u_n) = b_n$, that is to say u_n is the solution of the problem (3.4).

Continuity of T_n. To ease the writing, we note $V = V_n$ equipped with $\|u\|_V = \|u\|_H$ and note $T = T_n$. Let u, $\overline{u} \in V$, we have:

$$\begin{aligned}
\|T(u) - T(\overline{u})\|_{V'} &= \max_{v\in V,\ \|v\|_V=1} \langle T(u) - T(\overline{u}), v\rangle_{V',V} \\
&= \max_{v\in V,\ \|v\|_H=1} \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)(E_{kh}(\nabla u) - E_{kh}(\nabla \overline{u}))\frac{\partial v_i}{\partial x_j}dx, \\
&\leq \max_{v\in H,\ \|v\|_H=1} \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)(E_{kh}(\nabla u) - E_{kh}(\nabla \overline{u}))\frac{\partial v_i}{\partial x_j}dx.
\end{aligned}$$

Putting

$$a = \|a_{ijkh}\|_{L^\infty(\Omega)},$$

we obtain by Hölder inequality

$$\begin{aligned}
&\|T(u) - T(\overline{u})\|_{V'} \\
&\leq \max_{v\in H, \|v\|_H=1} 2a \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} \|E_{kh}(\nabla u) - E_{kh}(\nabla \overline{u})\|_{L^{p'(x)}} \left\|\frac{\partial v_i}{\partial x_j}\right\|_{L^{p(x)}(\Omega)} \\
&\leq 2a \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} \|E_{kh}(\nabla u) - E_{kh}(\nabla \overline{u})\|_{L^{p'(x)}(\Omega)}.
\end{aligned}$$

Thus if $(u_n)_{n\in\mathbb{N}}$ is a sequence of V such that $u_n \to \overline{u}$ in V, we have

$$\|T(u_n) - T(\overline{u})\|_{V'} \leq 2a \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} \|E_{kh}(\nabla u_n) - E_{kh}(\nabla \overline{u})\|_{L^{p'(x)}(\Omega)}.$$

As the norm in H equivalent to the norm in $\left(W^{1,p(x)}(\Omega)\right)^3$, then $u_n \to \overline{u}$ in V involves $u_n \to \overline{u}$ in $\left(W^{1,p(x)}(\Omega)\right)^3$.

In view of lemma (3.3), we obtain $E_{kh}(\nabla u_n) \to E_{kh}(\nabla \overline{u})$ in $L^{p'(x)}(\Omega)$, $\forall k, h = 1$

to 3. We have thus shown that $T(u_n) \to T(\overline{u})$ in V', so T is continuous.
Coercivity of T_n. Taking into account, definition and assumptions (3.2), we obtain:

$$\begin{aligned}
\langle T(u).u\rangle_{V',V} &= \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u(x))\frac{\partial u_i}{\partial x_j}dx,\\
&\geq \alpha_0 \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} E_{kh}(\nabla u(x))\frac{\partial u_i}{\partial x_j}dx,\\
&\geq \alpha_0\alpha C_1 \int_\Omega |\nabla u|^{p(x)}\,dx,\\
&\geq \alpha_0\alpha C_1 \min\left\{\|\nabla u\|^{p_-}_{L^{p(x)}(\Omega)}, \|\nabla u\|^{p_+}_{L^{p(x)}(\Omega)}\right\}\\
&\geq \alpha_0\alpha C_1 \min\left\{\|u\|^{p_-}_{V}, \|u\|^{p_+}_{V}\right\}.
\end{aligned}$$

Consequently, the operator T is coercive. This yields the existence of solution for problem (3.4).

Study of infinite dimension problem

The solution of the problem (3.4) is obtained.
So to show the existence of u a solution of (3.3), we will estimate u_n the solution of (3.4) and then by crossing to the limit when $n \to +\infty$ we will have the solution u of our problem (3.3).
Therefore that technique used to show that the limit of the nonlinear term is the desired term.
a. Estimation on u_n
In view of coercivity, if we substitute v by u_n in (3.4), we obtain:

$$\alpha_0\alpha C_1 \int_\Omega |\nabla u_n|^{p(x)}\,dx \leq \|f\|_{H'}\|u_n\|_H,$$

on the other hand

$$\alpha_0\alpha C_1 \min\left\{\|u_n\|^{p_-}_{H}, \|u_n\|^{p_+}_{H}\right\} \leq \|f\|_{H'}\|u_n\|_H.$$

b. Passage to the limit
Since $(u_n)_{n\in\mathbb{N}}$ is bounded in H, which is reflexive (because H is a closed vector subspace of $\left(W^{1,p(x)}(\Omega)\right)^3$, and $\left(W^{1,p(x)}(\Omega)\right)^3$ is a reflexive Banach space), we deduce that there exist a subsequence denoted again $(u_n)_{n\in\mathbb{N}}$ such that $u_n \to u$ weakly in H.
By hypothesis (3), the sequence $(E_{kh}(\nabla u_n))_{n\in\mathbb{N}}$ is bounded in $L^{p'(x)}(\Omega)$, hence there exist $\rho \in L^{p'(x)}(\Omega)$ such that, with a close subsequence,

$$E_{kh}(\nabla u_n) \to \rho \text{ weakly in } L^{p'(x)}(\Omega).$$

Let $v \in H$, then there exist $v_n \in V_n$, $n \in \mathbb{N}^*$ such that

$$\begin{aligned}
&v_n \to v \text{ in } H,\\
&\nabla v_n \to \nabla v \text{ in } \left(L^{p(x)}(\Omega)\right)^9.
\end{aligned}$$

We substitute v by v_n in (3.4), we obtain:

$$\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u_n(x))\frac{\partial v_{ni}}{\partial x_j}dx = \langle f, v_n\rangle_{H',H}\,,\ \forall v \in V_n.$$

Since $\langle f, v_n\rangle \to \langle f, v\rangle$, $E_{kh}(\nabla u_n) \to \rho$ weakly in $L^{p'(x)}(\Omega)$ and $\frac{\partial v_{ni}}{\partial x_j} \to \frac{\partial v_i}{\partial x_j}$ for $i = 1$ to 3 strongly in $L^{p(x)}(\Omega)$ (because $\nabla v_n \to \nabla v$ in $\left(L^{p(x)}(\Omega)\right)^9$strongly), using the lemma (3.2), we obtain

$$\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)\rho\frac{\partial v_i}{\partial x_j}dx = \langle f, v\rangle_{H',H}\,,\ \forall v \in H. \tag{3.5}$$

We tend to conclude that ρ is equal to $E_{kh}(\nabla u)$. Unfortunately, this is not obvious because the E_{kh} are nonlinear.

c. Limit of nonlinear term

Finally, it remains to prove that

$$\begin{cases} \displaystyle\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)\rho\frac{\partial v_i}{\partial x_j}dx = \\ \displaystyle\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u(x))\frac{\partial v_i}{\partial x_j}dx,\ \forall v \in H. \end{cases} \tag{3.6}$$

(I) First, we have

$$\lim_{n\to\infty}\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u_n(x))\frac{\partial u_{ni}}{\partial x_j}dx = \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)\rho\frac{\partial u_i}{\partial x_j}dx.$$

Indeed

$$\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u_n(x))\frac{\partial u_{ni}}{\partial x_j}dx = \langle f, u_n\rangle \to \langle f, u\rangle$$

(II) Proof of (3.6)

Let $v \in H$, there exist $(v_n)_{n\in\mathbb{N}}$ such that $v_n \in V_n$ for all $n \in \mathbb{N}$ and $v_n \to v$ in H when $n \to +\infty$. We will pass to the limit in the term

$$\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u_n(x))\frac{\partial v_{ni}}{\partial x_j}dx,$$

thanks to the hypothesis (4) of (3.2).
Indeed,

$$\begin{aligned}
0 &\leq \int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x)(E_{kh}(\nabla u_n) - E_{kh}(\nabla v_n)) \left(\frac{\partial u_{ni}}{\partial x_j} - \frac{\partial v_{ni}}{\partial x_j} \right) dx \\
&= \int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u_n) \frac{\partial u_{ni}}{\partial x_j} dx - \int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u_n) \frac{\partial v_{ni}}{\partial x_j} dx \\
&- \int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla v_n) \frac{\partial u_{ni}}{\partial x_j} dx + \int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla v_n) \frac{\partial v_{ni}}{\partial x_j} dx \\
&= T_{1,n} - T_{2,n} - T_{3,n} + T_{4,n}.
\end{aligned}$$

It has been seen that in **(I)**:

$$\lim_{n \to +\infty} T_{1,n} = \int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) \rho \frac{\partial u_i}{\partial x_j} dx,$$

we have

$$\lim_{n \to +\infty} T_{2,n} = \int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) \rho \frac{\partial v_i}{\partial x_j} dx,$$

by a product of a strong convergence in $L^{p(x)}(\Omega)$ and a weak convergence in $L^{p'(x)}(\Omega)$ (lemma (3.2)).
The same

$$\lim_{n \to +\infty} T_{3,n} = \int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla v) \frac{\partial u_i}{\partial x_j} dx,$$

by a product of a strong convergence in $L^{p'(x)}(\Omega)$ and a weak convergence in $L^{p(x)}(\Omega)$.
Finally, we have

$$\lim_{n \to +\infty} T_{4,n} = \int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla v) \frac{\partial v_i}{\partial x_j} dx,$$

by the product of a strong convergence in $L^{p'(x)}(\Omega)$ and a strong convergence in $L^{p(x)}(\Omega)$.
The passage to the limit in inequality thus gives:

$$\int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) (\rho - E_{kh}(\nabla v)) \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j} \right) dx \geq 0 \text{ for all } v \in H.$$

The function test v is now astutely chosen. We take $v = u + \frac{1}{n} w$ with $w \in H$ and $n \in \mathbb{N}^*$. We obtain

$$-\frac{1}{n} \int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) \left(\rho - E_{kh} \left(\nabla u + \frac{1}{n} \nabla w \right) \right) \frac{\partial w_i}{\partial x_j} dx \geq 0,$$

so

$$\int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) \left(\rho - E_{kh} \left(\nabla u + \frac{1}{n} \nabla w \right) \right) \frac{\partial w_i}{\partial x_j} dx \leq 0,$$

but $u + \frac{1}{n} w \to u$ in H, thus by the lemma (3.3),

$$E_{kh} \left(\nabla u + \frac{1}{n} \nabla w \right) \to E_{kh} \left(\nabla u \right) \text{ in } L^{p'(x)} \left(\Omega \right).$$

By passing to the limit when $n \to +\infty$, we obtain then

$$\int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) \left(\rho - E_{kh} \left(\nabla u \right) \right) \frac{\partial w_i}{\partial x_j} dx \leq 0, \ \forall w \in H.$$

By the linearity (we can change w in $-w$), we get:

$$\int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) \left(\rho - E_{kh} \left(\nabla u \right) \right) \frac{\partial w_i}{\partial x_j} dx = 0, \ \forall w \in H,$$

we deduce that

$$\int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) \rho \frac{\partial w_i}{\partial x_j} dx = \int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh} \left(\nabla u \right) \frac{\partial w_i}{\partial x_j} dx, \ \forall w \in H.$$

We have thus proved that u is a solution of (3.3).

Uniqueness

We suppose that $\left(E_{kh} \left(\xi \right) - E_{kh} \left(\eta \right) \right) \left(\xi_{ij} - \eta_{ij} \right) > 0$, if $\xi_{ij} \neq \eta_{ij}$, and f does not depend to u. Let u_1 and u_2 be two solutions:

$$\int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u_l(x)) \frac{\partial v_i}{\partial x_j} dx = \langle f, v \rangle_{H',H}, \ l = 1, 2; \ \forall v \in H.$$

Subtracting term to term and substituting v by $u_1 - u_2$, we obtain:

$$\int_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) (E_{kh} \left(\nabla u_1 \right) - E_{kh}(\nabla u_2)) (\frac{\partial u_{1i}}{\partial x_j} - \frac{\partial u_{2i}}{\partial x_j}) dx = 0.$$

Since

$$M = \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) (E_{kh} \left(\nabla u_1 \right) - E_{kh}(\nabla u_2)) (\frac{\partial u_{1i}}{\partial x_j} - \frac{\partial u_{2i}}{\partial x_j}) \geq 0,$$

and $M > 0$ if $\frac{\partial u_{1i}}{\partial x_j} \neq \frac{\partial u_{2i}}{\partial x_j}$; we get $\frac{\partial u_{1i}}{\partial x_j} = \frac{\partial u_{2i}}{\partial x_j}$ a.e. $\forall i, j = 1$ to 3, and thus $u_1 = u_2$ a.e.

5. Conclusion

In this work, we consider the nonlinear elasticity system as Leray–Lions's operators with variable exponents, to study the existence and uniqueness of Neumann's problem solution by Galerkin techniques and monotone operator theory. It has been found that these techniques adapt well to this type of problems with different boundary conditions.
From a perspective of this work, first, we will consider the same problem with the boundary conditions Robin, Tresca, and secondly, the boundary conditions no homogeneous of Dirichlet, Neumann, mixed and Robin.

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Bounds for blow-up time in a semilinear parabolic problem with variable exponents

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Abstract. This report deals with a blow-up of the solutions to a class of semilinear parabolic equations with variable exponents nonlinearities. Under some appropriate assumptions on the given data, a more general lower bound for a blow-up time is obtained if the solutions blow up. This result extends the recent results given by Baghaei Khadijeh et al. [8], which ensures the lower bounds for the blow-up time of solutions with initial data $\varphi(0) = \int_\Omega {u_0}^k dx$, $k =$ constant.

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Keywords: Parabolic equation, variable nonlinearity, bounds of the blow-up time.

1. Introduction

In this paper, we are concerned with the following semilinear parabolic equation

$$\begin{cases} u_t - \Delta u = u^{p(x)}, \quad x \in \Omega, \ t > 0 \\ u = 0 \text{ on } \Gamma, \ \ t \geq 0 \\ u(x,0) = u_0(x) \geq 0, \ x \in \Omega, \end{cases} \tag{1.1}$$

where Ω be a bounded domain in $\mathbb{R}^n$, with a smooth boundary $\Gamma = \partial\Omega$, $T \in (0, +\infty]$, and the initial value $u_0 \in H_0^1(\Omega)$, the exponent $p(.)$ is given measurable function on $\overline{\Omega}$ such that:

$$1 < p_1 = \underset{x\in\Omega}{ess\inf} p(x) \leq p(x) \leq p_2 = \underset{x\in\Omega}{ess\sup} p(x) < \infty, \tag{1.2}$$

and satisfy the following Zhikov-Fan uniform local continuity condition:

$$|p(x) - p(y)| \leq \frac{M}{|\log|x-y||}, \text{ for all } x, \ y \text{ in } \Omega \text{ with } |x-y| < \frac{1}{2}, \ M > 0. \tag{1.3}$$

The problem (1.1) arises from many important mathematical models in engineering and physical sciences. For example, nuclear science, chemical reactions, heat transfer,

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population dynamics, biological sciences, etc., and have interested a great deal of attention in the research, see [4, 7, 9] and the references therein. For problem (1.1), Hua Wang et al. [10] established a blow-up result with positive initial energy under some suitable assumptions on the parameters $p(.)$ and u_0. In [9], the authors proved that there are non-negative solutions with a blow-up in finite time if and only if $p_2 > 1$. The authors in [11] obtained the solution of problem (1.1) blows up in finite time when the initial energy is positive. The following problem was considered by R. Abita in [3]

$$u_t - \Delta u_t - \Delta u = u^{p(x)}, \quad x \in \Omega, \ t > 0.$$

The author proved that the nonnegative classical solutions blow-up in finite time with arbitrary positive initial energy and suitable large initial values. Also, he employed a differential inequality technique to obtain an upper bound for blow-up time if $p(.)$ and the initial value satisfies some conditions. In [8], the authors based exactly on the idea on the one in [6], derived the lower bounds for the time of blow-up, if the solutions blow-up. In order to declare the main results of this paper, we need to add the following energy functional corresponding to the problem (1.1) (see [2])

$$E(t) = \frac{1}{2}\int_\Omega |\nabla u(t)|^2 \, dx - \int_\Omega \frac{1}{p(x)+1} u(x,t)^{p(x)+1} \, dx. \tag{1.4}$$

2. Lower bounds of the blow-up time

In this section, we investigate the lower bound for the blow-up time T in some suitable measure. The idea of the proof of the following theorem is inspired by on the one in [6]. For this goal, we start by the following lemma concerning the energy of the solution.

Lemma 2.1. *Let $u(x,t)$ be a weak solution of* (1.1)*, then $E(t)$ is a nonincreasing function on $[0,T]$, that is*

$$\frac{dE(t)}{dt} = -\int_\Omega u_t^2(x,t)\, dx \le 0 \tag{2.1}$$

and the inequality $E(t) \le E(0)$ is satisfied.

We consider the following partition of Ω,

$$\Omega^- = \{x \in \Omega \mid 1 > (k(x)-1)\ln|u|\}, \quad \Omega^+ = \{x \in \Omega \mid 1 \le (k(x)-1)\ln|u|\}, \ \forall t > 0$$

where each $\Omega^\pm$ depends on t, and setting

$$\widetilde{E}(0) = \frac{1}{2}\int_{\Omega^-} |\nabla u_0|^2 \, dx - \int_{\Omega^-} \frac{1}{p(x)+1} u_0^{p(x)+1} dx.$$

Now, we are in a position to affirm our principal theorem results.

Theorem 2.2. *Assume $u_0 \in L^{k(.)}(\Omega)$, and the nonnegative weak solution $u(x,t)$ of problem* (1.1) *blows up in finite time T, then T has a lower bound by:*

$$\int_{\varphi(0)}^{+\infty} \frac{d\gamma}{C_1 + C_2\gamma^{\frac{3n-6}{3n-8}}}, \tag{2.2}$$

where

$$\varphi(0) = \int_{\Omega} \frac{1}{k(x)(k(x)-1)} {u_0}^{k(x)} dx, \tag{2.3}$$

where $k(.)$ *is a measurable function on* $\overline{\Omega}$ *such that*

$$\begin{aligned} \max(1, 2(n-2)(p_2-1)) < k_1 = ess \inf_{x\in\Omega} k(x) \leq k(x) \leq k_2 \\ = ess \sup_{x\in\Omega} k(x) < \infty, \end{aligned} \tag{2.4}$$

and

$$\sqrt{C_k} = \sup_{x\in\overline{\Omega}} |\nabla k(x)| \in L^2(\Omega), \ C_k > 0 \tag{2.5}$$

and C_i $(i = 1,\ 2)$ *are positive constants will be described later.*

Notation 2.3. We note that the presence of the variable-exponent nonlinearities in (2.6) below, makes analysis in the paper somewhat harder than that in the related ones, we will establish and give a precise estimate for the lifespan T of the solution in this case. The method used here is the differential inequality technique. However, our argument is considerably different and it is more abbreviated.

Proof of Theorem (2.2). Set

$$\varphi(t) = \int_{\Omega} \frac{1}{k(x)(k(x)-1)} u(x,t)^{k(x)} dx. \tag{2.6}$$

Multiplying the equation Eq. (1.1) by u and integrating by parts, we see

$$\begin{aligned} \varphi'(t) &= \int_{\Omega} \frac{1}{k(x)-1} u^{k(x)-1} u_t dx = \int_{\Omega} \frac{1}{k(x)-1} u^{k(x)-1} \left(\Delta u + u^{p(x)}\right) dx \\ &= \int_{\Omega} \frac{1}{k(x)-1} u^{k(x)-1} \Delta u dx + \int_{\Omega} \frac{1}{k(x)-1} u^{k(x)+p(x)-1} dx \\ &= -\int_{\Omega} \nabla \left(\frac{1}{k(x)-1} u^{k(x)-1}\right) \nabla u dx + \int_{\Omega} \frac{1}{k(x)-1} |u|^{k(x)+p(x)-1} dx \end{aligned}$$

where we have used the divergence theorem, the boundary condition on u.
It is straightforward to check that

$$\nabla \left(\frac{1}{k(x)-1} u^{k(x)-1}\right) = u^{k(x)} |u|^{-2} \nabla u + \frac{\nabla k(x)}{k(x)-1} u^{k(x)-1} \left(\ln|u| - \frac{1}{k(x)-1}\right)$$

then, we get

$$\varphi'(t) = -\int_{\Omega} u^{k(x)} |u|^{-2} |\nabla u|^2 dx + \int_{\Omega} \frac{1}{k(x)-1} u^{k(x)+p(x)-1} dx + \mathcal{Q} \tag{2.7}$$

where

$$\mathcal{Q} = \int_{\Omega} u^{k(x)-1} \left(\frac{1}{(k(x)-1)^2} - \frac{1}{(k(x)-1)} \ln|u|\right) \nabla k(x) . \nabla u dx$$

Considering the following properties of the function $\mathcal{G}$,

$$\mathcal{G}(\lambda) = \frac{\lambda^{\gamma}}{\gamma^2}(1-\gamma\ln\lambda), \quad 0 \le \lambda \le e^{\frac{1}{\gamma}};$$

$$\mathcal{G}(0) = \mathcal{G}\left(e^{\frac{1}{\gamma}}\right) = 0,\ \mathcal{G}'(\lambda) = -\lambda^{\gamma-1}\ln\lambda, \quad \max_{0\le\lambda\le e^{\frac{1}{\gamma}}} \mathcal{G}(\lambda) = \mathcal{G}(1) = \frac{1}{\gamma^2},$$

and using the fact that

$$\int_{\Omega^-} |\nabla u|^2\, dx \le 2\widetilde{E}(0) + 2\int_{\Omega^-} \frac{1}{p(x)+1} u(x,t)^{p(x)+1}\, dx, \text{ (by (1.4) and (2.1))}$$

applying the Hölder, Young inequalities and (2.5), $\mathcal{Q}$ is evaluated as follows:

$$\mathcal{Q} = \int_{\Omega} u^{k(x)-1}\left(\frac{1}{(k(x)-1)^2} - \frac{1}{k(x)-1}\ln|u|\right)\nabla k(x).\nabla u dx$$

$$= \int_{\Omega\cap(1>(k(x)-1)\ln|u(x,t)|)} u^{k(x)-1}\left(\frac{1}{(k(x)-1)^2} - \frac{1}{k(x)-1}\ln|u|\right)\nabla k(x).\nabla u dx$$

$$\int_{\Omega\cap(1\le(k(x)-1)\ln|u(x,t)|)} u^{k(x)-1}\left(\frac{1}{(k(x)-1)^2} - \frac{1}{k(x)-1}\ln|u|\right)\nabla k(x).\nabla u dx$$

$$\le \int_{\Omega^-} \frac{1}{(k(x)-1)^2}|u|^{k(x)-1}(1-(k(x)-1)\ln|u|)|\nabla u||\nabla k(x)|\, dx$$

$$\le \int_{\Omega^-} \frac{1}{(k(x)-1)^2}|\nabla k(x)||\nabla u|\, dx \le \frac{1}{2(k_1-1)^2}\left(C_k + \int_{\Omega^-}|\nabla u|^2\, dx\right)$$

$$\le \frac{1}{2(k_1-1)^2}\left(C_k + 2E(0) + 2\int_{\Omega^-}\frac{1}{p(x)+1}u(x,t)^{p(x)+1}\, dx\right)$$

$$\le \frac{1}{2(k_1-1)^2}\left(C_k + 2E(0) + \frac{2}{p_1+1}\max\left(\int_{\Omega^-}|u|^{p_2+1}\, dx, \int_{\Omega^-}|u|^{p_1+1}\, dx\right)\right)$$

$$\le \frac{1}{(k_1-1)^2}\left(\frac{1}{2}C_k + E(0) + \frac{1}{p_1+1}e^{\frac{p_2+1}{k_1-1}}|\Omega|\right). \tag{2.8}$$

Because in Ω^+, we have

$$\int_{\Omega^+} |u|^{k(x)-1}\left(\frac{1}{(k(x)-1)^2} - \frac{1}{k(x)-1}\ln|u|\right)|\nabla k(x)|\, dx \le 0,$$

while that of the first term in the right-hand side of (2.7) was estimated as follows

$$-\int_{\Omega}|u|^{k(x)-2}|\nabla u|^2\, dx \le -\min\left(\int_{\Omega}|u|^{k_2-2}|\nabla u|^2\, dx, \int_{\Omega}|u|^{k_1-2}|\nabla u|^2\, dx\right).$$

Using the fact

$$|\nabla u^{\gamma}| = \gamma u^{\gamma-1}|\nabla u|$$

to get

$$-\int_{\Omega}|u|^{k(x)-2}|\nabla u|^{2}dx\leq-\min\left(\frac{4}{(k_{2})^{2}}\int_{\Omega}\left|\nabla u^{\frac{k_{2}}{2}}\right|^{2}dx,\frac{4}{(k_{1})^{2}}\int_{\Omega}\left|\nabla u^{\frac{k_{1}}{2}}\right|^{2}dx\right)\quad(2.9)$$

Plugging this estimate (2.8) and (2.9) into (2.7), we obtain

$$\begin{aligned}\varphi'(t)\leq\min&\left(\frac{-4}{(k_{2})^{2}}\int_{\Omega}\left|\nabla u^{\frac{k_{2}}{2}}\right|^{2}dx,\frac{-4}{(k_{1})^{2}}\int_{\Omega}\left|\nabla u^{\frac{k_{1}}{2}}\right|^{2}dx\right)\\&+\frac{1}{k_{1}-1}\int_{\Omega}u^{k(x)+p_{2}-1}dx+\frac{1}{k_{1}-1}\int_{\Omega}u^{k(x)+p_{1}-1}dx\\&+\frac{1}{(k_{1}-1)^{2}}\left(\frac{1}{2}C_{k}+E(0)+\frac{1}{p_{1}+1}e^{\frac{p_{2}+1}{k_{1}-1}}|\Omega|\right)\end{aligned}\quad(2.10)$$

By using (2.4), we can apply the Hölder and Young inequalities to get

$$\begin{aligned}\int_{\Omega}u^{k(x)+p_{2}-1}dx&\leq\int_{\Omega}1.\alpha_{1}dx+\int_{\Omega}\alpha_{2}.u^{\frac{k(x)(2n-3)}{2(n-2)}}dx\\&\leq(\sup\alpha_{1})|\Omega|+(\sup\alpha_{2})\left(\int_{\Omega}u^{\frac{k(x)(2n-3)}{2(n-2)}}dx\right),\end{aligned}\quad(2.11)$$

and

$$\begin{aligned}\int_{\Omega}u^{k(x)+p_{1}-1}dx&\leq\int_{\Omega}1.\alpha_{3}dx+\int_{\Omega}\alpha_{4}.u^{\frac{k(x)(2n-3)}{2(n-2)}}dx\\&\leq\left(\sup_{\Omega}\alpha_{3}\right)|\Omega|+\left(\sup_{\Omega}\alpha_{4}\right)\left(\int_{\Omega}u^{\frac{k(x)(2n-3)}{2(n-2)}}dx\right),\end{aligned}\quad(2.12)$$

where

$$\alpha_{1}=1-\frac{2(n-2)(k(x)+p_{2}-1)}{(2n-3)k(x)},\quad\alpha_{2}=\frac{2(n-2)(k(x)+p_{2}-1)}{(2n-3)k(x)},$$
$$\alpha_{3}=1-\frac{2(n-2)(k(x)+p_{1}-1)}{(2n-3)k(x)},\quad\alpha_{4}=\frac{2(n-2)(k(x)+p_{1}-1)}{(2n-3)k(x)};$$

observe that $\alpha_{2}\geq\alpha_{4}$ and $\alpha_{1}\leq\alpha_{3}$.

Combining (2.11) and (2.12) with (2.10) give

$$\begin{aligned}\varphi'(t)\leq\frac{-1}{2}\frac{4}{(k_{2})^{2}}&\left(\int_{\Omega}\left|\nabla u^{\frac{k_{2}}{2}}\right|^{2}dx+\int_{\Omega}\left|\nabla u^{\frac{k_{1}}{2}}\right|^{2}dx\right)\\&+\frac{2}{k_{1}-1}\left(\sup_{\Omega}\alpha_{2}\right)\int_{\Omega}u^{\frac{k(x)(2n-3)}{2(n-2)}}dx\\+\frac{1}{(k_{1}-1)^{2}}&\left(\frac{1}{2}C_{k}+E(0)+\frac{1}{p_{1}+1}e^{\frac{p_{2}+1}{k_{1}-1}}|\Omega|\right)+\frac{|\Omega|}{k_{1}-1}\sup_{\Omega}(\alpha_{3}+\alpha_{1})\end{aligned}\quad(2.13)$$

We now make use of Schwarz's inequality to the second term on the right-hand side of (2.13) as follows

$$\int_\Omega u^{\frac{k(x)(2n-3)}{2(n-2)}}dx \leq \left(\int_\Omega u^{k(x)}dx\right)^{\frac{1}{2}}\left(\int_\Omega u^{\frac{k(x)(n-1)}{n-2}}dx\right)^{\frac{1}{2}} \tag{2.14}$$
$$\leq \left(\int_\Omega u^{k(x)}dx\right)^{\frac{3}{4}}\left(\int_\Omega \left(u^{\frac{k(x)}{2}}\right)^{\frac{2n}{n-2}}dx\right)^{\frac{1}{4}},$$

Next, by using the Sobolev inequality (see [5]), for $n \geq 3$, we get

$$\begin{aligned}\left\|u^{\frac{k(x)}{2}}\right\|_{\frac{2n}{n-2}}^{\frac{n}{2(n-2)}} &\leq B^{\frac{n}{2(n-2)}}\max\left(\left\|\nabla u^{\frac{k_2}{2}}\right\|_2^{\frac{n}{2(n-2)}}, \left\|\nabla u^{\frac{k_1}{2}}\right\|_2^{\frac{n}{2(n-2)}}\right) \\ &\leq B^{\frac{n}{2(n-2)}}\left(\left\|\nabla u^{\frac{k_2}{2}}\right\|_2^{\frac{n}{2(n-2)}} + \left\|\nabla u^{\frac{k_1}{2}}\right\|_2^{\frac{n}{2(n-2)}}\right),\end{aligned} \tag{2.15}$$

where B is the best constant in the Sobolev inequality.
By inserting the last inequality in (2.14) and (2.15), we have

$$\int_\Omega u^{\frac{k(x)(2n-3)}{2(n-2)}}dx \leq$$
$$\leq B^{\frac{n}{2(n-2)}}\left(\int_\Omega u^{k(x)}dx\right)^{\frac{3}{4}}\left(\left(\int_\Omega \left|\nabla u^{\frac{k_2}{2}}\right|^2 dx\right)^{\frac{n}{4(n-2)}} + \left(\int_\Omega \left|\nabla u^{\frac{k_1}{2}}\right|^2 dx\right)^{\frac{n}{4(n-2)}}\right),$$

Now, we can use the Young inequality to get

$$\int_\Omega u^{\frac{k(x)(2n-3)}{2(n-2)}}dx \leq 2B^{\frac{2n}{3n-8}}\frac{3n-8}{4(n-2)\varepsilon^{\frac{n}{3n-8}}}\left(\int_\Omega u^{k(x)}dx\right)^{\frac{3(n-2)}{3n-8}} \tag{2.16}$$
$$+\frac{\varepsilon n}{4(n-2)}\left(\int_\Omega \left|\nabla u^{\frac{k_1}{2}}\right|^2 dx + \int_\Omega \left|\nabla u^{\frac{k_2}{2}}\right|^2 dx\right)$$

where ε is a positive constant to be determined later. Combining (2.16) with (2.13), we obtain

$$\varphi'(t) \leq C_1 + C_2\varphi(t)^{\frac{3(n-2)}{3n-8}} + C_3\left(\int_\Omega \left|\nabla u^{\frac{k_2}{2}}\right|^2 dx + \int_\Omega \left|\nabla u^{\frac{k_1}{2}}\right|^2 dx\right),$$

where

$$C_1 = \frac{1}{(k_1-1)^2}\left(\frac{1}{2}C_k + E(0) + \frac{1}{p_1+1}e^{\frac{p_2+1}{k_1-1}}|\Omega|\right) + \frac{|\Omega|}{k_1-1}\sup_\Omega(\alpha_3+\alpha_1)$$
$$C_2 = \frac{4}{k_1-1}\left(\sup_\Omega \alpha_2\right)B^{\frac{2n}{3n-8}}\frac{3n-8}{4(n-2)\varepsilon^{\frac{n}{3n-8}}},$$
$$C_3 = \frac{2}{k_1-1}\frac{\varepsilon n}{4(n-2)}\left(\sup_\Omega \alpha_2\right) - \frac{2}{(k_2)^2}$$

If we choose $\varepsilon > 0$ such that

$$0 < \varepsilon \leq \frac{4\left(n-2\right)\left(k_1-1\right)}{\left(\sup\limits_{\Omega}\alpha_2\right) n\left(k_2\right)^2}$$

then, we obtain the differential inequality

$$\varphi'\left(t\right) \leq C_1 + C_2\varphi\left(t\right)^{\frac{3(n-2)}{3n-8}} \tag{2.17}$$

Integration of the differential inequality (2.17) from 0 to t leads to

$$\int_{\varphi(0)}^{\varphi(t)} \frac{d\gamma}{C_1 + C_2\gamma^{\frac{3(n-2)}{3n-8}}} \leq t \tag{2.18}$$

In fact, let $t \to T^-$, (2.18) leads to

$$\int_{\varphi(0)}^{+\infty} \frac{d\gamma}{C_1 + C_2\gamma^{\frac{3(n-2)}{3n-8}}} \leq T.$$

where

$$\varphi\left(0\right) = \int_{\Omega} \frac{1}{k\left(x\right)\left(k\left(x\right)-1\right)} {u_0}^{k(x)} dx.$$

Thus, the proof is achieved. □

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Some applications of Maia's fixed point theorem for Fredholm integral equation systems

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Abstract. The aim of this paper is to study the existence and uniqueness of solutions for some Fredholm integral equation systems by applying the vectorial form of Maia's fixed point theorem. Some abstract Gronwall lemmas and an abstract comparison lemma are also obtained.

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Keywords: Space of continuous functions, vector-valued metric, matrix convergent to zero, A-contraction, fixed point, Picard operator, weakly Picard operator, integral equation, Fredholm integral equation system, vectorial Maia's fixed point theorem, abstract Gronwall lemma, abstract comparison lemma.

1. Introduction

Let $a, b \in \mathbb{R}_+$, with $a < b$. Let $C[a, b]$ be the set of all real valued functions which are continuous on the interval $[a, b]$. Using a vectorial form of Maia's fixed point theorem, we study the existence and uniqueness of solutions $(x_1, x_2) \in (C[a, b])^2$ for the following Fredholm integral equation systems:

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t, s, x_1(s))ds + \int_a^b H_1(t, s, x_1(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t, s, x_2(s))ds + \int_a^b H_2(t, s, x_2(s))ds \end{cases} \tag{1.1}$$

and

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t, s, x_1(s))ds + \int_a^b H_1(t, s, x_2(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t, s, x_2(s))ds + \int_a^b H_2(t, s, x_1(s))ds \end{cases} \tag{1.2}$$

where $g_1, g_2 \in C[a, b]$, $K_1, K_2, H_1, H_2 \in C([a, b] \times [a, b] \times \mathbb{R}, \mathbb{R})$ are given functions.

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2. Preliminaries

We recall here some notions, notations and results which will be used in the sequel of this paper.

2.1. L-space

The notion of L-space was introduced in 1906 by M. Fréchet ([4]). It is an abstract space in which works one of the basic tools in the theory of operatorial equations, especially in the fixed point theory: the sequence of successive approximations method.

Let X be a nonempty set. Let $s(X) := \big\{\{x_n\}_{n\in\mathbb{N}} \mid x_n \in X,\ n \in \mathbb{N}\big\}$. Let $c(X)$ be a subset of $s(X)$ and $Lim : c(X) \to X$ be an operator. By definition, the triple $(X, c(X), Lim)$ is called L-space (denoted by $(X, \to)$) if the following conditions are satisfied:

(i) if $x_n = x$, for all $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}} \in c(X)$ and $Lim\{x_n\}_{n\in\mathbb{N}} = x$.

(ii) if $\{x_n\}_{n\in\mathbb{N}} \in c(X)$ and $Lim\{x_n\}_{n\in\mathbb{N}} = x$, then for all subsequences $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$, we have that $\{x_{n_i}\}_{i\in\mathbb{N}} \in c(X)$ and $Lim\{x_{n_i}\}_{i\in\mathbb{N}} = x$.

A simple example of an L-space is the pair $(X, \xrightarrow{d})$, where X is a nonempty set and $\xrightarrow{d}$ is the convergence structure induced by a metric d on X.

In general, an L-space is any nonempty set endowed with a structure implying a notion of convergence for sequences. Other examples of L-spaces are: Hausdorff topological spaces, generalized metric spaces in Perov' sense (i.e. $d(x, y) \in \mathbb{R}^m_+$), generalized metric spaces in Luxemburg' sense (i.e. $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$), K-metric spaces (i.e. $d(x, y) \in K$, where K is a cone in an ordered Banach space), gauge spaces, 2-metric spaces, D-R-spaces, probabilistic metric spaces, syntopogenous spaces.

2.2. Picard operators and weakly Picard operators on L-spaces

Let $(X, \to)$ be an L-space. An operator $f : X \to X$ is called weakly Picard operator (WPO) if the sequence of successive approximations, $\{f^n(x)\}_{n\in\mathbb{N}}$, converges for all $x \in X$ and its limit (which generally depend on x) is a fixed point of f.

If an operator f is WPO and the fixed point set of f is a singleton, $F_f = \{x^*\}$, then by definition, f is called Picard operator (PO).

For a WPO, $f : X \to X$, we define the operator $f^\infty : X \to X$, by

$$f^\infty(x) := \lim_{n\to\infty} f^n(x).$$

Notice that, $f^\infty(X) = F_f$, i.e., f^∞ is a set retraction of X on F_f.

If X is a nonempty set, then the triple $(X, \to, \leq)$ is an ordered L-space if $(X, \to)$ is an L-space and $\leq$ is a partial order relation on X which is closed with respect to the convergence structure of the L-space.

In the setting of ordered L-spaces, we have some properties concerning WPOs and POs.

Theorem 2.2.1 (Abstract Gronwall Lemma). *Let $(X, \to, \leq)$ be an ordered L-space and $f : X \to X$ be an increasing WPO. Then:*

(i) $x \in X,\ x \leq f(x) \Rightarrow x \leq f^\infty(x)$;

(ii) $x \in X,\ x \geq f(x) \Rightarrow x \geq f^\infty(x)$.

In particular, if f is a PO and we denote $F_f = \{x^*\}$, then:

(i') $\forall\ x \in X,\ x \leq f(x) \Rightarrow x \leq x^*$;
(ii') $\forall\ x \in X,\ x \geq f(x) \Rightarrow x \geq x^*$.

Theorem 2.2.2 (Abstract Comparison Lemma). *Let $(X, \to, \leq)$ be an ordered L-space and the operators $f, g, h : X \to X$ be such that:*

(1) *$f \leq g \leq h$;*
(2) *f, g, h are $WPOs$;*
(3) *g is increasing.*

Then:

$$x, y, z \in X,\ x \leq y \leq z \Rightarrow f^\infty(x) \leq g^\infty(y) \leq h^\infty(z).$$

In particular, if f, g, h are POs and we denote $F_f = \{x^*\}$, $F_g = \{y^*\}$, $F_h = \{z^*\}$, then

$$\forall\ x, y, z \in X,\ x \leq y \leq z \Rightarrow x^* \leq y^* \leq z^*.$$

Regarding the theory of $WPOs$ and POs see [12], [13], [15], [16], [18], [11], [17], [3].

2.3. Maia's fixed point theorem

The following result was proved by M.G. Maia in [5].

Theorem 2.3.1. *Let X be a nonempty set, d and ρ be two metrics on X and $V : X \to X$ be an operator. We suppose that:*

(1) *there exists $c > 0$ such that, $d(x, y) \leq c\rho(x, y)$, $\forall\ x, y \in X$;*
(2) *(X, d) is a complete metric space;*
(3) *$V : (X, d) \to (X, d)$ is continuous;*
(4) *$V : (X, \rho) \to (X, \rho)$ is an l-contraction, i.e.,*

$$\exists\ l \in [0, 1) \text{ such that } \rho(V(x), V(y)) \leq l\rho(x, y),\ \forall\ x, y \in X.$$

Then:

(i) $F_V = \{x^*\}$;
(ii) $V : (X, d) \to (X, d)$ *is PO.*

Maia's Theorem 2.3.1 remains true if we replace the condition (1) with the following one:

$(1')$ there exists $c > 0$ such that, $d(V(x), V(y)) \leq c\rho(x, y)$, $\forall\ x, y \in X$.

Hence, we obtain the so called Rus' variant of Maia's fixed point theorem. More considerations can be found in [11], [9], [10], [14].

2.4. Matrices which converge to zero

We denote by $M_m(\mathbb{R}_+)$ the set of all $m \times m$ square matrices with positive real elements, by I_m the identity $m \times m$ matrix and by O_m the zero $m \times m$ matrix.

$A \in M_m(\mathbb{R}_+)$ is said to be convergent to zero if $A^n \to O_m$ as $n \to \infty$.

Some examples of matrices that converge to zero are the following:

a) $A = \begin{pmatrix} a & a \\ b & b \end{pmatrix} \in M_2(\mathbb{R}_+)$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

b) $A = \begin{pmatrix} a & b \\ a & b \end{pmatrix} \in M_2(\mathbb{R}_+)$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

c) $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in M_2(\mathbb{R}_+)$, where $a, b, c \in \mathbb{R}_+$ and $\max\{a, c\} < 1$.

A classical result in matrix analysis is the following theorem (see [19], [1]), which characterizes the matrices that converge to zero.

Theorem 2.4.1. *Let $A \in M_m(\mathbb{R}_+)$. The following assertions are equivalent:*

(1) *A is convergent to zero;*
(2) *its spectral radius $\rho(A)$ is strictly less than 1; that is, $|\lambda| < 1$, for any $\lambda \in \mathbb{C}$ with $\det(A - \lambda I_m) = 0$;*
(3) *the matrix $(I_m - A)$ is nonsingular and*
$$(I_m - A)^{-1} = I_m + A + A^2 + \ldots + A^n + \ldots;$$
(4) *the matrix $(I_m - A)$ is nonsingular and $(I_m - A)^{-1}$ has nonnegative elements.*

Throughout this paper, we will make an identification between row and column vectors in $\mathbb{R}^m$.

2.5. Vector-valued metric spaces

Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^m_+$ is called a vector-valued metric on X if the following conditions are satisfied:

(1) $d(x, y) = 0 \in \mathbb{R}^m \Leftrightarrow x = y$, for all $x, y \in X$;
(2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

On $\mathbb{R}^m_+$, the relation $\leq$ is defined in the component-wise sense.

Some examples of vector-valued metrics are the following:

Example 2.5.1. Let $X := (C[a, b])^2$ and $d : (C[a, b])^2 \times (C[a, b])^2 \to \mathbb{R}^2_+$, defined by
$$d(x, y) := \left(\max_{t \in [a,b]} |x_1(t) - y_1(t)|, \max_{t \in [a,b]} |x_2(t) - y_2(t)| \right),$$
for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in (C[a, b])^2$.

Example 2.5.2. Let $X := (C[a, b])^2$ and $\rho : (C[a, b])^2 \times (C[a, b])^2 \to \mathbb{R}^2_+$, defined by
$$\rho(x, y) := \left(\left(\int_a^b |x_1(t) - y_1(t)|^2 dt \right)^{\frac{1}{2}}, \left(\int_a^b |x_2(t) - y_2(t)|^2 dt \right)^{\frac{1}{2}} \right),$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in (C[a, b])^2$.

A nonempty set X endowed with a vector-valued metric d is called a generalized metric space in Perov' sense (or a $\mathbb{R}_+^m$-metric space) and it is denoted by the pair (X, d). The notions of convergent sequence, Cauchy sequence, completeness, open and closed subset and so forth are similar to those defined for usual metric spaces. The basic fixed point result which holds in generalized metric spaces in Perov' sense is the following (see [6], [7]).

Theorem 2.5.3 (Perov's fixed point theorem). *Let (X, d) be a complete generalized metric space, where $d : X \times X \to \mathbb{R}_+^m$. Let $f : X \to X$ be an A-contraction, i.e. there exists a matrix $A \in M_m(\mathbb{R}_+)$ convergent to zero, such that*

$$d(f(x), f(y)) \leq Ad(x, y), \forall\ x, y \in X.$$

Then f is PO in the L-space $(X, \xrightarrow{d})$.

Remark 2.5.4. It would be of interest to extend the study from [8] and [2] to the case of vector-valued metric spaces.

3. Vectorial Maia's fixed point theorems

In this section we present the Rus' variant of Maia's fixed point theorem in the setting of generalized metric spaces in Perov's sense.

Theorem 3.1. *Let X be a nonempty set, endowed with two vector-valued metrics, $d, \rho : X \times X \to \mathbb{R}_+^m$. Let $T : X \to X$ be an operator. We assume that:*

(1) *there exists a matrix $C \in M_m(\mathbb{R}_+)$ such that*

$$d(T(x), T(y)) \leq C\rho(x, y),\ \forall\ x, y \in X;$$

(2) *(X, d) is a complete generalized metric space;*
(3) *$T : (X, d) \to (X, d)$ is continuous;*
(4) *$T : (X, \rho) \to (X, \rho)$ is an A-contraction, i.e. there exists a matrix $A \in M_m(\mathbb{R}_+)$ convergent to zero, such that*

$$\rho(T(x), T(y)) \leq A\rho(x, y),\ \forall\ x, y \in X.$$

Then T is PO in the L-spaces $(X, \xrightarrow{d})$ and $(X, \xrightarrow{\rho})$.

Proof. Let $x_0 \in X$. By (4), the sequence of successive approximations $\{T^n(x_0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, ρ). Indeed, for $n, p \in \mathbb{N}$ we have

$$\begin{aligned}\rho(T^n(x_0), T^{n+p}(x_0)) &\leq \sum_{k=n}^{n+p-1} \rho(T^k(x_0), T^{k+1}(x_0)) \leq \sum_{k=n}^{n+p-1} A^k \rho(x_0, T(x_0)) \\ &\leq A^n (I_m - A)^{-1} \rho(x_0, T(x_0)) \to 0 \text{ as } n, p \to \infty.\end{aligned}$$

By (1), we get that $\{T^n(x_0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d). By (2), there exists $x^*\in X$, such that $T^n(x_0)\xrightarrow{d}x^*$ as $n\to\infty$. By (3), it follows that $x^*\in F_T$, since

$$\begin{aligned} d(x^*,T(x^*)) &\le d(x^*,T^n(x_0))+d(T^n(x_0),T(x^*)) \\ &= d(x^*,T^n(x_0))+d(T(T^{n-1}(x_0)),T(x^*)) \\ &\to d(x^*,x^*)+d(T(x^*),T(x^*))=0, \text{ as } n\to\infty. \end{aligned}$$

By (4), we obtain the uniqueness of the fixed point x^*. Hence T is PO in $(X,\xrightarrow{d})$.
We show next that T is PO in $(X,\xrightarrow{\rho})$.
For any $x_0\in X$, since $x^*\in F_T$, by (4) we have

$$\rho(x^*,T^n(x_0))=\rho(T^n(x^*),T^n(x_0))\le A^n\rho(x^*,x_0)\to 0 \text{ as } n\to\infty$$

which implies that $T^n(x_0)\xrightarrow{\rho}x^*$ as $n\to\infty$. Since x^* is the unique fixed point, we get that T is PO in $(X,\xrightarrow{\rho})$. □

Remark 3.2. Notice that, in the proof of the above result, Perov's Theorem cannot be applied for $T:(X,\rho)\to(X,\rho)$, because the lack of completeness of the generalized metric space (X,ρ).

Remark 3.3. From the proof of the above result, we can deduce the following weak Perov's contraction principle:

Theorem 3.4. *Let (X,ρ) be a generalized metric space, where $\rho:X\times X\to\mathbb{R}^m_+$. Let $T:X\to X$ be an operator. We assume that:*

(i) $F_T\neq\emptyset$;
(ii) there exists a matrix $A\in M_m(\mathbb{R}_+)$ which converges to zero, such that $\rho(T(x),T(y))\le A\rho(x,y)$, for all $x,y\in X$.

Then T is PO in the L-space $(X,\xrightarrow{\rho})$.

Another fixed point result of Maia type in vectorial form is the following.

Theorem 3.5. *Let X be a nonempty set, endowed with two vector-valued metrics, $d,\rho:X\times X\to\mathbb{R}^m_+$. Let $T:X\to X$ be an operator. We assume that:*

(1) *$F_T\neq\emptyset$;*
(2) *there exists a matrix $C\in M_m(\mathbb{R}_+)$ such that*

$$d(T(x),T(y))\le C\rho(x,y),\ \forall\ x,y\in X;$$

(3) *$T:(X,\rho)\to(X,\rho)$ is an A-contraction, i.e. there exists a matrix $A\in M_m(\mathbb{R}_+)$ convergent to zero, such that*

$$\rho(T(x),T(y))\le A\rho(x,y),\ \forall\ x,y\in X.$$

Then T is PO in the L-spaces $(X,\xrightarrow{d})$ and $(X,\xrightarrow{\rho})$.

Proof. By applying Theorem 3.4, T is PO in $(X,\xrightarrow{\rho})$. So $F_T=\{x^*\}$. For any $x_0\in X$,

$$\begin{aligned} d(x^*,T^{n+1}(x_0)) &= d(T^{n+1}(x^*),T^{n+1}(x_0)) \\ &\le C\rho(T^n(x^*),T^n(x_0)) \\ &\le CA^n\rho(x^*,x_0)\to 0 \end{aligned}$$

as $n \to \infty$. So T is PO in $(X, \xrightarrow{d})$. $\square$

4. Applications of vectorial Maia's fixed point theorem

In this section we study the existence and uniqueness of solutions for Fredholm integral equations systems (1.1) and (1.2), by applying the vectorial Maia's fixed point theorem.

First, let us consider the system (1.1)

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_1(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_2(s))ds \end{cases}$$

where $g_1, g_2 \in C[a,b]$, $K_1, K_2, H_1, H_2 \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$, are given functions. We are searching the conditions in which the system (1.1) has a unique solution $(x_1, x_2) \in (C[a,b])^2$.

We assume that there exist $L_{K_j}, L_{H_j} > 0$, $j \in \{1,2\}$ such that:

$$|K_j(t,s,u) - K_j(t,s,v)| \leq L_{K_j}|u-v|,$$
$$|H_j(t,s,u) - H_j(t,s,v)| \leq L_{H_j}|u-v|,$$

for all $t, s \in [a,b]$, $u, v \in \mathbb{R}$, $j \in \{1,2\}$.

On $X := (C[a,b])^2$ we consider the metrics $d, \rho : X \times X \to \mathbb{R}_+^2$, where

$$d(x,y) := \begin{pmatrix} \max\limits_{t\in[a,b]} |x_1(t) - y_1(t)| \\ \max\limits_{t\in[a,b]} |x_2(t) - y_2(t)| \end{pmatrix} \tag{4.1}$$

and

$$\rho(x,y) := \begin{pmatrix} \left(\int_a^b |x_1(t) - y_1(t)|^2 dt\right)^{\frac{1}{2}} \\ \left(\int_a^b |x_2(t) - y_2(t)|^2 dt\right)^{\frac{1}{2}} \end{pmatrix}, \tag{4.2}$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in (C[a,b])^2$.

We consider the operator $T : (C[a,b])^2 \to (C[a,b])^2$, defined by

$$\begin{aligned} T(x)(t) &= \begin{pmatrix} T_1(x)(t) \\ T_2(x)(t) \end{pmatrix} \\ &:= \begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_1(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_2(s))ds \end{pmatrix} \end{aligned} \tag{4.3}$$

for all $x = (x_1, x_2) \in (C[a,b])^2$.

We have,

$$\rho(T(x), T(y)) = \begin{pmatrix} \left(\int_a^b |T_1(x)(t) - T_1(y)(t)|^2 dt\right)^{\frac{1}{2}} \\ \left(\int_a^b |T_2(x)(t) - T_2(y)(t)|^2 dt\right)^{\frac{1}{2}} \end{pmatrix}$$

and

$$\begin{pmatrix} |T_1(x)(t) - T_1(y)(t)| \\ |T_2(x)(t) - T_2(y)(t)| \end{pmatrix} \leq \begin{pmatrix} \int_a^b \big|K_1(t,s,x_1(s)) - K_1(t,s,y_1(s))\big| ds \\ \int_a^b \big|K_2(t,s,x_2(s)) - K_2(t,s,y_2(s))\big| ds \end{pmatrix}$$
$$+ \begin{pmatrix} \int_a^b \big|H_1(t,s,x_1(s)) - H_1(t,s,y_1(s))\big| ds \\ \int_a^b \big|H_2(t,s,x_2(s)) - H_2(t,s,y_2(s))\big| ds \end{pmatrix}$$
$$\leq \begin{pmatrix} \int_a^b L_{K_1} |x_1(s) - y_1(s)| ds \\ \int_a^b L_{K_2} |x_2(s) - y_2(s)| ds \end{pmatrix} + \begin{pmatrix} \int_a^b L_{H_1} |x_1(s) - y_1(s)| ds \\ \int_a^b L_{H_2} |x_2(s) - y_2(s)| ds \end{pmatrix}$$
$$\overset{\substack{\text{Hölder's} \\ \text{inequality}}}{\leq} \begin{pmatrix} \big[\big(\int_a^b |L_{K_1}|^2 ds\big)^{\frac{1}{2}} + \big(\int_a^b |L_{H_1}|^2 ds\big)^{\frac{1}{2}}\big]\big(\int_a^b |x_1(s) - y_1(s)|^2 ds\big)^{\frac{1}{2}} \\ \big[\big(\int_a^b |L_{K_2}|^2 ds\big)^{\frac{1}{2}} + \big(\int_a^b |L_{H_2}|^2 ds\big)^{\frac{1}{2}}\big]\big(\int_a^b |x_2(s) - y_2(s)|^2 ds\big)^{\frac{1}{2}} \end{pmatrix}$$
$$= \begin{pmatrix} \big(L_{K_1} + L_{H_1}\big)\sqrt{b-a}\,\tilde{\rho}(x_1,y_1) \\ \big(L_{K_2} + L_{H_2}\big)\sqrt{b-a}\,\tilde{\rho}(x_2,y_2) \end{pmatrix},$$

where

$$\tilde{\rho}(x_1,y_1) := \left(\int_a^b |x_1(s) - y_1(s)|^2 ds\right)^{\frac{1}{2}}, \ \tilde{\rho}(x_2,y_2) := \left(\int_a^b |x_2(s) - y_2(s)|^2 ds\right)^{\frac{1}{2}}.$$

Hence,

$$\rho(T(x),T(y)) \leq \begin{pmatrix} \big(\int_a^b [(L_{K_1} + L_{H_1})\sqrt{b-a}\,\tilde{\rho}(x_1,y_1)]^2 dt\big)^{\frac{1}{2}} \\ \big(\int_a^b [(L_{K_2} + L_{H_2})\sqrt{b-a}\,\tilde{\rho}(x_2,y_2)]^2 dt\big)^{\frac{1}{2}} \end{pmatrix}$$
$$= \begin{pmatrix} (L_{K_1} + L_{H_1})(b-a)\tilde{\rho}(x_1,y_1) \\ (L_{K_2} + L_{H_2})(b-a)\tilde{\rho}(x_2,y_2) \end{pmatrix} = A\rho(x,y),$$

where

$$A := \begin{pmatrix} (L_{K_1} + L_{H_1})(b-a) & 0 \\ 0 & (L_{K_2} + L_{H_2})(b-a) \end{pmatrix} \in M_2(\mathbb{R}_+)$$

is a matrix that converges to zero if $(L_{K_1}+L_{H_1})(b-a) < 1$ and $(L_{K_2}+L_{H_2})(b-a) < 1$. So, if we add these two conditions, T becomes an A-contraction with respect to ρ. In addition, for all $x, y \in C[a,b]$, we have $d(T(x),T(y)) \leq C\rho(x,y)$, where

$$C := \begin{pmatrix} (L_{K_1} + L_{H_1})\sqrt{b-a} & 0 \\ 0 & (L_{K_2} + L_{H_2})\sqrt{b-a} \end{pmatrix} \in M_2(\mathbb{R}_+).$$

By applying Theorem 3.1, the system (1.1) has a unique solution in $(C[a,b])^2$. Hence, we have obtained the following result:

Theorem 4.1. *Let $a, b \in \mathbb{R}_+$ with $a < b$. We consider the system of Fredholm integral equations*

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_1(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_2(s))ds \end{cases}$$

where $g_1, g_2 \in C[a,b]$, $K_1, K_2, H_1, H_2 \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$ *are given functions.*

We assume that:

(*i*) *there exist* $L_{K_j}, L_{H_j} > 0$, $j \in \{1,2\}$ *such that:*

$$|K_j(t,s,u) - K_j(t,s,v)| \leq L_{K_j}|u-v|,$$
$$|H_j(t,s,u) - H_j(t,s,v)| \leq L_{H_j}|u-v|,$$

for all $t, s \in [a,b]$, $u, v \in \mathbb{R}$, $j \in \{1,2\}$;

(*ii*) $(L_{K_1} + L_{H_1})(b-a) < 1$ *and* $(L_{K_2} + L_{H_2})(b-a) < 1$.

Then the system has a unique solution $x^* = (x_1^*, x_2^*) \in (C[a,b])^2$.

Remark 4.2. By Theorem 3.1, the operator T defined in (4.3) is PO. Hence, for all $t \in [a,b]$ we have $x^*(t) = \lim\limits_{n\to\infty} x_n(t)$, for each $x_0 = (x_0^1, x_0^2) \in (C[a,b])^2$, where $\{x_n\}_{n\in\mathbb{N}} \subset (C[a,b])^2$ is defined by

$$x_{n+1}(t) = \begin{pmatrix} x_{n+1}^1(t) \\ x_{n+1}^2(t) \end{pmatrix} = \begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_n^1(s))ds + \int_a^b H_1(t,s,x_n^1(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_n^2(s))ds + \int_a^b H_2(t,s,x_n^2(s))ds \end{pmatrix}.$$

Corollary 4.3. *Let* $a, b \in \mathbb{R}_+$ *with* $a < b$. *We consider the system of Fredholm integral equations*

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds \end{cases}$$

where $g_1, g_2 \in C[a,b]$, $K_1, K_2 \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$ *are given functions.*

We assume that:

(*i*) *there exist* $L_{K_j} > 0$, $j \in \{1,2\}$ *such that:*

$$|K_j(t,s,u) - K_j(t,s,v)| \leq L_{K_j}|u-v|,$$

for all $t, s \in [a,b]$, $u, v \in \mathbb{R}$, $j \in \{1,2\}$;

(*ii*) $L_{K_1}(b-a) < 1$ *and* $L_{K_2}(b-a) < 1$.

Then the system has a unique solution $x^* = (x_1^*, x_2^*) \in (C[a,b])^2$.

Proof. We apply Theorem 4.1, by considering H_1 and H_2 as zero functions and by taking $L_{H_1} = 0$ and $L_{H_2} = 0$. □

Now, let us consider the system (1.2)

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_2(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_1(s))ds \end{cases}$$

where $g_1, g_2 \in C[a,b]$, $K_1, K_2, H_1, H_2 \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$ are given functions.

We assume that there exist $L_{K_j}, L_{H_j} > 0$, $j \in \{1,2\}$ such that:

$$|K_j(t,s,u) - K_j(t,s,v)| \leq L_{K_j}|u-v|,$$
$$|H_j(t,s,u) - H_j(t,s,v)| \leq L_{H_j}|u-v|,$$

for all $t, s \in [a,b]$, $u, v \in \mathbb{R}$, $j \in \{1,2\}$.

On $X := (C[a,b])^2$ we consider the metrics $d, \rho : X \times X \to \mathbb{R}_+^2$ defined as in (4.1) and (4.2). Also, we consider the operator $T : (C[a,b])^2 \to (C[a,b])^2$, defined by

$$\begin{aligned} T(x)(t) &= \begin{pmatrix} T_1(x)(t) \\ T_2(x)(t) \end{pmatrix} \\ &:= \begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_2(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_1(s))ds \end{pmatrix} \end{aligned} \tag{4.4}$$

for all $x = (x_1, x_2) \in (C[a,b])^2$.
In a similar manner as shown for the system (1.1), we get $\rho(T(x), T(y)) \leq A\rho(x,y)$, for all $x, y \in (C[a,b])^2$, where

$$A := \begin{pmatrix} L_{K_1}(b-a) & L_{H_1}(b-a) \\ L_{H_2}(b-a) & L_{K_2}(b-a) \end{pmatrix} \in M_2(\mathbb{R}_+).$$

The matrix A converges to zero if

$$\frac{\left|(L_{K_1} + L_{K_2}) \pm \sqrt{(L_{K_1} + L_{K_2})^2 - 4(L_{K_1}L_{K_2} - L_{H_1}L_{H_2})}\right|}{2}(b-a) < 1.$$

So, if we add this condition, T becomes an A-contraction with respect to ρ.
In addition, for all $x, y \in (C[a,b])^2$, we obtain $d(T(x), T(y)) \leq C\rho(x,y)$, where

$$C := \begin{pmatrix} L_{K_1}\sqrt{b-a} & L_{H_1}\sqrt{b-a} \\ L_{H_2}\sqrt{b-a} & L_{K_2}\sqrt{b-a} \end{pmatrix} \in M_2(\mathbb{R}_+).$$

By applying Theorem 3.1, the system (1.2) has a unique solution in $(C[a,b])^2$. Hence, we have obtained the following result:

Theorem 4.4. *Let $a, b \in \mathbb{R}_+$ with $a < b$. We consider the system of Fredholm integral equations*

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_2(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_1(s))ds \end{cases}$$

where $g_1, g_2 \in C[a,b]$, $K_1, K_2, H_1, H_2 \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$ are given functions.

We assume that:

(i) there exist $L_{K_j}, L_{H_j} > 0$, $j \in \{1,2\}$ such that:

$$\begin{aligned} |K_j(t,s,u) - K_j(t,s,v)| &\leq L_{K_j}|u-v|, \\ |H_j(t,s,u) - H_j(t,s,v)| &\leq L_{H_j}|u-v|, \end{aligned}$$

for all $t, s \in [a,b]$, $u, v \in \mathbb{R}$, $j \in \{1,2\}$;

(ii) $\frac{b-a}{2}\left|(L_{K_1} + L_{K_2}) \pm \sqrt{(L_{K_1} + L_{K_2})^2 - 4(L_{K_1}L_{K_2} - L_{H_1}L_{H_2})}\right| < 1.$

Then the system has a unique solution $x^* = (x_1^*, x_2^*) \in (C[a,b])^2$.

Remark 4.5. By Theorem 3.1, the operator T defined in (4.4) is PO. Hence, for all $t \in [a,b]$ we have $x^*(t) = \lim_{n\to\infty} x_n(t)$, for each $x_0 = (x_0^1, x_0^2) \in (C[a,b])^2$, where $\{x_n\}_{n\in\mathbb{N}} \subset (C[a,b])^2$ is defined by

$$x_{n+1}(t) = \begin{pmatrix} x_{n+1}^1(t) \\ x_{n+1}^2(t) \end{pmatrix} = \begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_n^1(s))ds + \int_a^b H_1(t,s,x_n^2(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_n^2(s))ds + \int_a^b H_2(t,s,x_n^1(s))ds \end{pmatrix}.$$

Corollary 4.6. *Let $a, b \in \mathbb{R}_+$ with $a < b$. We consider the system of Fredholm integral equations*

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b H_1(t,s,x_2(s))ds \\ x_2(t) = g_2(t) + \int_a^b H_2(t,s,x_1(s))ds \end{cases}$$

where $g_1, g_2 \in C[a,b]$, $H_1, H_2 \in C([a,b]\times[a,b]\times\mathbb{R},\mathbb{R})$ are given functions.

We assume that:

(*i*) *there exist $L_{H_j} > 0$, $j \in \{1,2\}$ such that:*

$$|H_j(t,s,u) - H_j(t,s,v)| \le L_{H_j}|u-v|,$$

for all $t, s \in [a,b]$, $u, v \in \mathbb{R}$, $j \in \{1,2\}$;

(*ii*) $(b-a)\sqrt{L_{H_1}L_{H_2}} < 1$.

Then the system has a unique solution $x^* = (x_1^*, x_2^*) \in (C[a,b])^2$.

Proof. We apply Theorem 4.4, by considering K_1 and K_2 as zero functions and by taking $L_{K_1} = 0$ and $L_{K_2} = 0$. □

5. Abstract Gronwall lemmas

Since the operators T, defined in (4.3) and (4.4), are POs, by using Theorem 2.2.1 we can establish the following abstract Gronwall lemmas for our systems (1.1) and (1.2).

Theorem 5.1. *Let $a, b \in \mathbb{R}_+$ with $a < b$. We consider the system of Fredholm integral equations*

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_1(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_2(s))ds \end{cases}$$

where $g_1, g_2 \in C[a,b]$, $K_1, K_2, H_1, H_2 \in C([a,b]\times[a,b]\times\mathbb{R},\mathbb{R})$ are given functions.

We assume that:

(*i*) *there exist $L_{K_j}, L_{H_j} > 0$, $j \in \{1,2\}$ such that:*

$$|K_j(t,s,u) - K_j(t,s,v)| \le L_{K_j}|u-v|,$$
$$|H_j(t,s,u) - H_j(t,s,v)| \le L_{H_j}|u-v|,$$

for all $t, s \in [a,b]$, $u, v \in \mathbb{R}$, $j \in \{1,2\}$;

(*ii*) $(L_{K_1} + L_{H_1})(b-a) < 1$ *and* $(L_{K_2} + L_{H_2})(b-a) < 1$;

(iii) $K_j(t,s,\cdot), H_j(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ *are increasing functions, for all* $t,s \in [a,b]$ *and* $j \in \{1,2\}$.

Let $x^* = (x_1^*, x_2^*) \in (C[a,b])^2$ *be the unique solution of the system.*
Then the following implications hold:

(1) *for all* $x = (x_1, x_2) \in (C[a,b])^2$ *with*

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \leq \begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_1(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_2(s))ds \end{pmatrix},$$

for all $t \in [a,b]$, *we have* $x \leq x^*$;

(2) *for all* $x = (x_1, x_2) \in (C[a,b])^2$ *with*

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \geq \begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_1(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_2(s))ds \end{pmatrix},$$

for all $t \in [a,b]$, *we have* $x \geq x^*$.

Theorem 5.2. *Let* $a, b \in \mathbb{R}_+$ *with* $a < b$. *We consider the system of Fredholm integral equations*

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_2(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_1(s))ds \end{cases}$$

where $g_1, g_2 \in C[a,b]$, $K_1, K_2, H_1, H_2 \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$ *are given functions.*

We assume that:

(i) *there exist* $L_{K_j}, L_{H_j} > 0$, $j \in \{1,2\}$ *such that:*

$$|K_j(t,s,u) - K_j(t,s,v)| \leq L_{K_j}|u-v|,$$
$$|H_j(t,s,u) - H_j(t,s,v)| \leq L_{H_j}|u-v|,$$

for all $t,s \in [a,b]$, $u,v \in \mathbb{R}$, $j \in \{1,2\}$;

(ii) $\frac{b-a}{2}\left|(L_{K_1} + L_{K_2}) \pm \sqrt{(L_{K_1} + L_{K_2})^2 - 4(L_{K_1}L_{K_2} - L_{H_1}L_{H_2})}\right| < 1$;

(iii) $K_j(t,s,\cdot), H_j(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ *are increasing functions, for all* $t,s \in [a,b]$ *and* $j \in \{1,2\}$.

Let $x^* = (x_1^*, x_2^*) \in (C[a,b])^2$ *be the unique solution of the system.*
Then the following implications hold:

(1) *for all* $x = (x_1, x_2) \in (C[a,b])^2$ *with*

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \leq \begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_2(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_1(s))ds \end{pmatrix},$$

for all $t \in [a,b]$, *we have* $x \leq x^*$;

(2) *for all* $x = (x_1, x_2) \in (C[a,b])^2$ *with*

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \geq \begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_2(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_1(s))ds \end{pmatrix},$$

for all $t \in [a,b]$, *we have* $x \geq x^*$.

6. Abstract comparison lemmas

We can establish also some abstract comparison results, taking into account Theorem 2.2.2. One of them is the following.

Theorem 6.1. *Let $a, b \in \mathbb{R}_+$ with $a < b$. We consider the systems of Fredholm integral equations*

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_1(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_2(s))ds \end{cases} \tag{6.1}$$

$$\begin{cases} y_1(t) = g_3(t) + \int_a^b K_3(t,s,y_1(s))ds + \int_a^b H_3(t,s,y_1(s))ds \\ y_2(t) = g_4(t) + \int_a^b K_4(t,s,y_2(s))ds + \int_a^b H_4(t,s,y_2(s))ds \end{cases} \tag{6.2}$$

$$\begin{cases} z_1(t) = g_5(t) + \int_a^b K_5(t,s,z_1(s))ds + \int_a^b H_5(t,s,z_1(s))ds \\ z_2(t) = g_6(t) + \int_a^b K_6(t,s,z_2(s))ds + \int_a^b H_6(t,s,z_2(s))ds \end{cases} \tag{6.3}$$

where $g_i \in C[a,b]$, for all $i = \overline{1,6}$ and $K_j, H_j \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$, for all $j = \overline{1,6}$, are given functions.

We assume that:

(i) there exist $L_{K_j}, L_{H_j} > 0$, $j = \overline{1,6}$ such that:

$$|K_j(t,s,u) - K_j(t,s,v)| \le L_{K_j}|u-v|,$$
$$|H_j(t,s,u) - H_j(t,s,v)| \le L_{H_j}|u-v|,$$

for all $t, s \in [a,b]$, $u, v \in \mathbb{R}$, $j = \overline{1,6}$;

(ii) $(L_{K_j} + L_{H_j})(b-a) < 1$, for all $j = \overline{1,6}$;

(iii) $K_j(t,s,\cdot), H_j(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ are increasing functions, for all $t, s \in [a,b]$ and $j = \overline{3,4}$;

(iv) for all $t \in [a,b]$,

$$\begin{aligned} &\begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_1(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_2(s))ds \end{pmatrix} \\ \le &\begin{pmatrix} g_3(t) + \int_a^b K_3(t,s,x_1(s))ds + \int_a^b H_3(t,s,x_1(s))ds \\ g_4(t) + \int_a^b K_4(t,s,x_2(s))ds + \int_a^b H_4(t,s,x_2(s))ds \end{pmatrix} \\ \le &\begin{pmatrix} g_5(t) + \int_a^b K_5(t,s,x_1(s))ds + \int_a^b H_5(t,s,x_1(s))ds \\ g_6(t) + \int_a^b K_6(t,s,x_2(s))ds + \int_a^b H_6(t,s,x_2(s))ds \end{pmatrix}. \end{aligned}$$

Let $x^* = (x_1^*, x_2^*), y^* = (y_1^*, y_2^*), z^* = (z_1^*, z_2^*) \in (C[a,b])^2$ be the unique solutions of the systems (6.1), (6.2) and respectively (6.3) .

Then for any $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in (C[a,b])^2$ we have

$$x \le y \le z \Rightarrow x^* \le y^* \le z^*.$$

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