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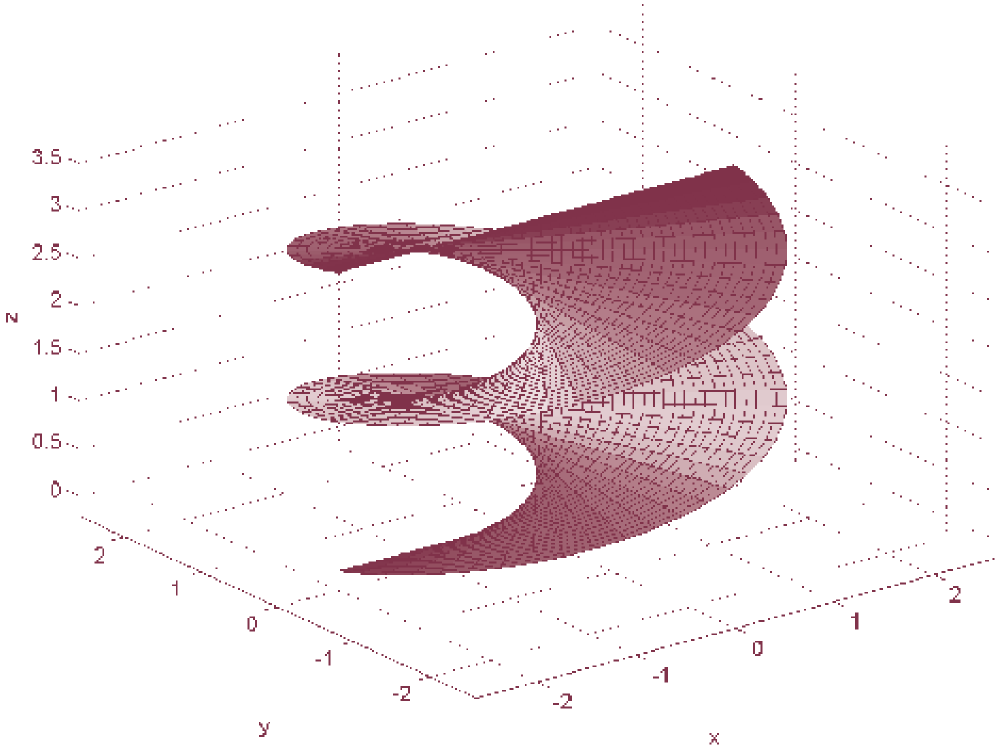

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On Hadamard-type inequalities for m-convex functions via Riemann-Liouville fractional integrals

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Abstract. In this paper we prove the Hadamard-type inequalities for m-convex functions via Riemann-Liouville fractional integrals and the Hadamard-type inequalities for convex functions via Riemann-Liouville fractional integral are deduced. Also we find connections with some well known results related to the Hadamard inequality.

Mathematics Subject Classification (2010): 26A51, 26A33, 26D10.

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1. Introduction

Following L'Hospital's and Leibniz's first inquiries, fractional calculus was primarily a study reserved for the best minds in mathematics. Fourier, Euler, Laplace were among those who were interested in fractional calculus and its mathematical consequences [15]. Euler and Liouville developed their thoughts about the computation of non-integer order integrals and derivatives. Many initiate, using their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative. The most well-known of these definitions that have been popularized in the subject of fractional calculus are the Riemann-Liouville and the Grunwald-Letnikov definition [4, 12]. In [18] Riemann-Liouville fractional integrals are defined as follows:

Definition 1.1. Let $f \in L_1[a,b]$. Then Riemann-Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined as:

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a \tag{1.1}$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b. \tag{1.2}$$

For further details one may see [15, 16, 17, 9, 8, 13, 19].
Convex functions play a vital role in the mathematical analysis. They have been considered for defining and finding new dimensions of analysis. In [20] Toader define the concept of m-convexity, an intermediate between usual convexity and star shape function.

Definition 1.2. A function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

If we take $m = 1$, then we recapture the concept of convex functions defined on $[0, b]$ and if we take $m = 0$, then we get the concept of starshaped functions on $[0, b]$. We recall that $f : [0, b] \to \mathbb{R}$ is called *starshaped* if

$$f(tx) \leq tf(x) \text{ for all } t \in [0, 1] \text{ and } x \in [0, b].$$

Denote by $K_m(b)$ the set of the m-convex functions on $[0, b]$ for which $f(0) < 0$, then one has

$$K_1(b) \subset K_m(b) \subset K_0(b),$$

whenever $m \in (0, 1)$. Note that in the class $K_1(b)$ are only convex functions $f : [0, b] \to \mathbb{R}$ for which $f(0) \leq 0$ (see [5]).

Example 1.3. [14] The function $f : [0, \infty) \to \mathbb{R}$, given by

$$f(x) = \frac{1}{12}\left(4x^3 - 15x^2 + 18x - 5\right)$$

is $\frac{16}{17}-$convex function but it is not convex function.

For more results and inequalities related to m-convex functions one can consult for example [7, 5, 11, 2, 16] along with references.
Let $f : I \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

is well known in literature as the Hadamard inequality.

For more refinements, generalizations and inequalities related to (1.3), see [1, 2, 3, 16, 6].

In [19], Sarikaya et al. proved the following Hadamard-type inequalities for Riemann-Liouville fractional integrals.

Theorem 1.4. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on [a,b], then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{(\frac{a+b}{2})+}f(b) + J^\alpha_{(\frac{a+b}{2})-}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.4}$$

with $\alpha > 0$.

Theorem 1.5. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}[J^\alpha_{(\frac{a+b}{2})+}f(b) + J^\alpha_{(\frac{a+b}{2})-}f(a)] - f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}}\left[((\alpha+1)|f'(a)|^q + (\alpha+3)|f'(b)|^q)^{\frac{1}{q}}\right. \\
&\left. + ((\alpha+3)|f'(a)|^q + (\alpha+1)|f'(b)|^q)^{\frac{1}{q}}\right].
\end{aligned} \tag{1.5}$$

Theorem 1.6. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q > 1$, then the following inequality for fractional integral holds:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}[J^\alpha_{(\frac{a+b}{2})+}f(b) + J^\alpha_{(\frac{a+b}{2})-}f(a)] - f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q + 3|f'(b)|q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}[|f'(a)| + |f'(b)|],
\end{aligned} \tag{1.6}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In this paper we generalize the fractional Hadamard-type inequalities (1.4), (1.5) and (1.6) for m-convex function via Riemann-Liouville fractional integrals and show that these inequalities are the special cases of our results. Also we find some well known results.

2. Hadamard-type inequalities for m-convex functions via fractional integrals

Start with the following result.

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a m-convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned}
&f\left(\frac{a+mb}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^\alpha}\left[J^\alpha_{(\frac{a+mb}{2})+}f(mb) + m^{\alpha+1}J^\alpha_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right] \\
&\leq \frac{\alpha}{4(\alpha+1)}\left[f(a) - m^2 f\left(\frac{a}{m^2}\right)\right] + \frac{m}{2}\left[f(b) + mf\left(\frac{a}{m^2}\right)\right]
\end{aligned} \tag{2.1}$$

with $\alpha > 0$.

Proof. From m-convexity of f we have,

$$f\left(\frac{x+my}{2}\right) \leq \frac{f(x) + mf(y)}{2}. \tag{2.2}$$

Put $x = \frac{t}{2}a + m\frac{(2-t)}{2}b, y = \frac{(2-t)}{2m}a + \frac{t}{2}b$ for $t \in [0,1]$. Then $x, y \in [a,b]$ and above inequality gives,

$$2f\left(\frac{a+mb}{2}\right) \leq f\left(\frac{t}{2}a + m\frac{2-t}{2}b\right) + mf\left(\frac{2-t}{2m}a + \frac{t}{2}b\right), \tag{2.3}$$

multiplying both sides of above inequality with $t^{\alpha-1}$, and integrating over $[0,1]$ we have,

$$\begin{aligned}
&\frac{2}{\alpha}f\left(\frac{a+mb}{2}\right)\\
&\leq \int_0^1 t^{\alpha-1}f\left(\frac{t}{2}a + m\frac{2-t}{2}b\right)dt + m\int_0^1 t^{\alpha-1}f\left(\frac{2-t}{2m}a + \frac{t}{2}b\right)dt\\
&= \int_{mb}^{\frac{a+mb}{2}}\left(\frac{2}{mb-a}(mb-u)\right)^{\alpha-1} f(u)\frac{2du}{a-mb}\\
&+ m^2\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\left(\frac{2}{b-\frac{a}{m}}(v-\frac{a}{m})\right)^{\alpha-1} f(v)\frac{2dv}{mb-a}\\
&= \frac{2^\alpha\Gamma(\alpha)}{(mb-a)^\alpha}\left[J^\alpha_{(\frac{a+mb}{2})+}f(mb) + m^{\alpha+1}J^\alpha_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right],
\end{aligned}$$

from which one has

$$f\left(\frac{a+mb}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^\alpha}\left[J^\alpha_{(\frac{a+mb}{2})+}f(mb) + m^{\alpha+1}J^\alpha_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]. \tag{2.4}$$

On the other hand m-convexity of f gives

$$\begin{aligned}
&f\left(\frac{t}{2}a + m\frac{2-t}{2}b\right) + mf\left(\frac{2-t}{2m}a + \frac{t}{2}b\right)\\
&\leq \frac{t}{2}\left[f(a) - m^2f\left(\frac{a}{m^2}\right)\right] + m\left[f(b) + mf\left(\frac{a}{m^2}\right)\right],
\end{aligned}$$

multiplying both sides of above inequality with $t^{\alpha-1}$, and integrating over $[0,1]$ we have,

$$\begin{aligned}
&\int_0^1 t^{\alpha-1}f\left(\frac{t}{2}a + m\frac{2-t}{2}b\right)dt + m\int_0^1 t^{\alpha-1}f\left(\frac{2-t}{2m}a + \frac{t}{2}b\right)dt\\
&\leq \frac{1}{2}\left[f(a) - m^2f\left(\frac{a}{m^2}\right)\right]\int_0^1 t^\alpha dt + m\left[f(b) + mf\left(\frac{a}{m^2}\right)\right]\int_0^1 t^{\alpha-1}dt\\
&\int_{mb}^{\frac{a+mb}{2}}\left(\frac{2}{mb-a}(mb-u)\right)^{\alpha-1} f(u)\frac{2du}{a-mb}\\
&+ m^2\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\left(\frac{2}{b-\frac{a}{m}}(v-\frac{a}{m})\right)^{\alpha-1} f(v)\frac{2dv}{mb-a}\\
&\leq \frac{1}{2(\alpha+1)}\left[f(a) - m^2f\left(\frac{a}{m^2}\right)\right] + \frac{m}{\alpha}\left[f(b) + mf\left(\frac{a}{m^2}\right)\right]
\end{aligned}$$

from which one has

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+mb}{2}\right)+}f(mb)+m^{\alpha+1}J^{\alpha}_{\left(\frac{a+mb}{2m}\right)-}f\left(\frac{a}{m}\right)\right] \\
&\leq \frac{\alpha}{4(\alpha+1)}\left[f(a)-m^{2}f\left(\frac{a}{m^{2}}\right)\right]+\frac{m}{2}\left[f(b)+mf\left(\frac{a}{m^{2}}\right)\right].
\end{aligned} \tag{2.5}$$

Combining inequality (2.4) and inequality (2.5) we get inequality (2.1) . □

Remark 2.2. If we take $m = 1$, Theorem 2.1 gives inequality (1.4) of Theorem 1.4 and putting $\alpha = 1$ along with $m = 1$ in Theorem 2.1 we get the classical Hadamard inequality.

For next results we need the following lemma.

Lemma 2.3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f^{'} \in L[a, b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+mb}{2}\right)+}f(mb)+m^{\alpha+1}J^{\alpha}_{\left(\frac{a+mb}{2m}\right)-}f\left(\frac{a}{m}\right)\right] \\
&-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right] \\
&=\frac{mb-a}{4}\left[\int_{0}^{1}t^{\alpha}f'\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)dt-\int_{0}^{1}t^{\alpha}f'\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)dt\right].
\end{aligned} \tag{2.6}$$

Proof. One can note that

$$\begin{aligned}
&\frac{mb-a}{4}\left[\int_{0}^{1}t^{\alpha}f'\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)dt\right] \\
&=\frac{mb-a}{4}\left[-\frac{2}{mb-a}f\left(\frac{a+mb}{2}\right)\right. \\
&\left.-\frac{2\alpha}{(a-mb)}\int_{mb}^{\frac{a+mb}{2}}\left(\frac{2}{mb-a}(mb-x)\right)^{\alpha-1}\frac{2}{a-mb}f(x)dx\right]
\end{aligned} \tag{2.7}$$

$$=\frac{mb-a}{4}\left[-\frac{2}{mb-a}f\left(\frac{a+mb}{2}\right)+\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(mb-a)^{\alpha+1}}J^{\alpha}_{\left(\frac{a+mb}{2}\right)-}f(mb)\right]. \tag{2.8}$$

Similarly

$$\begin{aligned}
&-\frac{mb-a}{4}\left[\int_{0}^{1}t^{\alpha}f'\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)dt\right] \\
&=-\frac{mb-a}{4}\left[\frac{2m}{mb-a}f\left(\frac{a+mb}{2m}\right)-\frac{2^{\alpha+1}m^{\alpha+1}\Gamma(\alpha+1)}{(mb-a)^{\alpha+1}}J^{\alpha}_{\left(\frac{a+mb}{2m}\right)+}f\left(\frac{a}{m}\right)\right].
\end{aligned} \tag{2.9}$$

Adding (2.7) and (2.9) one has (2.6). □

Using the above lemma we give the following Hadamard-type inequality.

Theorem 2.4. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f'|^q$ is m-convex on $[a, b]$ for $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\right.\\
&\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\\
&\leq \frac{mb-a}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}}\left[\left((\alpha+1)\left|f'(a)\right|^q+m\left(\alpha+3\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(m\left(\alpha+3\right)\left|f'\left(\frac{a}{m^2}\right)\right|^q+(\alpha+1)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{2.10}$$

with $\alpha > 0$.

Proof. From Lemma 2.3 and m-convexity of $|f'|^q$ and for $q = 1$ we have

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\right.\\
&\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\\
&\leq \frac{mb-a}{4}\int_0^1 t^{\alpha}\left(\left|f'\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|dt+\left|f'\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|\right)dt.\\
&=\frac{mb-a}{4}\left(\frac{m}{\alpha+1}\left[|f'(b)|+\left|f'\left(\frac{a}{m^2}\right)\right|\right]\right.\\
&\left.+\left[|f'(a)|-m\left|f'\left(\frac{a}{m^2}\right)\right|+|f'(b)|-m|f'(b)|\right]\right).
\end{aligned}$$

For $q > 1$ we proceed as follows. Using Lemma 2.3 we have

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\right.\\
&\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\\
&\leq \frac{mb-a}{4}\int_0^1 t^{\alpha}\left|f'\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|dt+\int_0^1 t^{\alpha}\left|f'\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|dt.
\end{aligned}$$

Using power mean inequality we get

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\right.\\
&\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|\\
&\leq \frac{mb-a}{4}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1 t^{\alpha}\left|f'\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left[\int_0^1 t^\alpha \left|f'\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\Bigg].$$

m-convexity of $|f'|^q$ gives

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^\alpha}\left[J^\alpha_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^\alpha_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq \frac{mb-a}{4}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1 t^\alpha\left(\frac{t}{2}|f'(a)|^q+m\frac{2-t}{2}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1 t^\alpha\left(m\frac{2-t}{2}|f'\left(\frac{a}{m^2}\right)|^q+\frac{t}{2}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}\right]$$
$$=\frac{mb-a}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}}\left[((\alpha+1)|f'(a)|^q+m(\alpha+3)|f'(b)|^q)^{\frac{1}{q}}\right.$$
$$\left.+\left(m(\alpha+3)|f'\left(\frac{a}{m^2}\right)|^q+(\alpha+1)|f'(b)|^q\right)^{\frac{1}{q}}\right].$$

Hence the proof is complete. □

Remark 2.5. If we take $m=1$ in Theorem 2.4, we get inequality (1.5) of Theorem 1.5 and if we take $\alpha=q=1$ along with $m=1$ in Theorem 2.4, then inequality (2.10) gives the following result.

Corollary 2.6. *With the assumptions of Theorem* 2.4 *we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right). \tag{2.11}$$

Theorem 2.7. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $a<b$. If $|f'|^q$ is m-convex on $[a,b]$ for $q>1$, then the following inequality for fractional integral holds:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^\alpha}\left[J^\alpha_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^\alpha_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q+3m|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3m|f'\left(\frac{a}{m^2}\right)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{mb-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|+3m\left(|f'\left(\frac{a}{m^2}\right)|+|f'(b)|\right)\right], \tag{2.12}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.3 we have

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{4}\left[\int_0^1 t^{\alpha}\left|f'\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|dt+\int_0^1 t^{\alpha}\left|f'\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|dt\right].$$

From the *Hölder's* inequality we get

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{4}\left[\left[\int_0^1 t^{\alpha p}dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1 t^{\alpha p}dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right].$$

m-convexity of $|f'|^q$ gives

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1\left(\frac{t}{2}|f'(a)|^q+m\frac{2-t}{2}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1\left(m\frac{2-t}{2}|f'(\frac{a}{m^2})|^q+\frac{t}{2}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}\right]$$
$$=\frac{mb-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left[\frac{|f'(a)|^q+3m|f'(b)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3m|f'\left(\frac{a}{m^2}\right)|^q+|f'(b)|^q}{4}\right]^{\frac{1}{q}}\right].$$

For the second inequality of (2.12) we use Minkowski's inequality as

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[J^{\alpha}_{(\frac{a+mb}{2})+}f(mb)+m^{\alpha+1}J^{\alpha}_{(\frac{a+mb}{2m})-}f\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f\left(\frac{a+mb}{2}\right)+mf\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{16}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[[|f'(a)|^q+3m|f'(b)|^q]^{\frac{1}{q}}+\left[3m|f'\left(\frac{a}{m^2}\right)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\right]$$

$$\leq \frac{mb-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|+3m\left(\left|f'\left(\frac{a}{m^2}\right)\right|+|f'(b)|\right)\right]. \qquad \square$$

Remark 2.8. If we take $m=1$ in Theorem 2.7, we get inequality (1.6) of Theorem 1.6 and if we take $\alpha=1$ along with $m=1$ in Theorem 2.7, then inequality (2.12) gives the following result.

Corollary 2.9. *With the assumptions of Theorem* 2.7 *we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[(|f'(a)|^q+3|f'(b)|^q)^{\frac{1}{q}}+(3|f'(a)|^q+|f'(b)|^q)^{\frac{1}{q}}\right].\end{aligned} \tag{2.13}$$

References

[1] Azpeitia, A.G., *Convex functions and the Hadamard inequality*, Revista Colombiana Mat., **28**(1994), 7-12.

[2] Bakula, M.K., Ozdemir, M.E., Pečarić, J., *Hadamard type inequalities for m-convex and (α,m)-convex functions*, J. Ineq. Pure Appl. Math., **9**(2008), no. 4, art. 96.

[3] Bakula, M.K., Pečarić, J., *Note on some Hadamard-type inequalities*, J. Ineq. Pure Appl. Math., **5**(2004), no. 3, art. 74.

[4] David, S.A., Linares, J.L., Pallone, E.M.J.A., *Fractional order calculus: historical apologia, basic concepts and some applications*, Rev. Bras. Ensino Fs., **33**(2011), no. 4, 4302-4302.

[5] Dragomir, S.S., *On some new inequalities of Hermite-Hadamard type for m-convex functions,* Tamkang J. Math., **33**(2002), no. 1, 45-56.

[6] Farid, G., Rehman, A.U., Zahra, M., *On Hadamard inequalities for relative convex functions via fractional integrals*, Nonlinear Anal. Forum, **21**(2016), no. 1, 77-86.

[7] Farid, G., Marwan, M., Rehman, A.U., *New mean value theorems and generalization of Hadamard inequality via coordinated m-convex functions*, J. Inequal. Appl. 2015, Article ID 283, 2015, 11p.

[8] Gorenflo, R., Mainardi, F., *Fractional calculus: integral and differential equations of fractional order*, Springer Verlag, Wien, 1997, 223-276.

[9] Hilfer, R., (ed.), *Applications of fractional calculus in physics*, World Scientific Publishing Co. 2000.

[10] Kirmaci, U.S., Bakula, M.K., Ozdemir, M.E., Pečarić, J., *Hadamard-tpye inequalities for s-convex functions*, Appl. Math. Comput., **193**(2007), 26-35.

[11] Bakula, M.K., Pečarić, J., Ribicic, M., *Companion inequalities to Jensen's inequality for m-convex and (α,m)-convex functions*, J. Inequal. Pure Appl. Math., **7**(2006), art. 194, online:htpp://jipam.vu.edu.au.

[12] Loverro, A., *Fractional calculus: history, definitions and applications for the engineer*, Department of Aerospace and Mechanical Engineering, University of Notre Dame, 2004.

[13] Miller, S., Ross, B., *An introduction to fractional calculus and fractional differential equations*, John Wiley And Sons, USA, 1993.

[14] Mocanu, P.T., Serb, I., Toader, G., *Real star-convex functions*, Studia Univ. Babeş-Bolyai Math., **42**(1997), no. 3, 65-80.

[15] Nishimoto, K., *An essence of Nishimoto's fractional calculus*, Descartes Press Co., 1991.

[16] Özdemir, M.E., Avci, M., Set, E., *On some inequalities of Hermite-Hadamard type via m-convexity*, Appl. Math. Lett., **23**(2010), no. 9, 1065-1070.

[17] Podlubny, I., *Fractional differential equations*, Mathematics in Science and Engineering V198, Academic Press 1999.

[18] Sarikaya, M.Z., Set, E., Yaldiz, H., Basak, N., *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, J. Math. Comput. Model., **57**(2013), 2403-2407.

[19] Sarikaya, M.Z., Yildirim, H., *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals*, RGMIA Research Report Collection, **17**(98)(2014), 10 pp.

[20] Toader, G.H., *Some generalizations of convexity*, Proc. Colloq. Approx. Optim., 1984, 329-338.

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The first Zolotarev case in the Erdös-Szegö solution to a Markov-type extremal problem of Schur

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Abstract. Schur's [14] Markov-type extremal problem asks to find the maximum (1) $\sup\limits_{-1\leq\xi\leq 1}\ \sup\limits_{P_n\in\mathbf{B}_{n,\xi,2}} |P_n^{(1)}(\xi)|$, where $\mathbf{B}_{n,\xi,2} = \{P_n \in \mathbf{B}_n : P_n^{(2)}(\xi) = 0\} \subset \mathbf{B}_n = \{P_n : |P_n(x)| \leq 1 \text{ for } |x| \leq 1\}$ and P_n is an algebraic polynomial of degree $\leq n$. Erdös and Szegö [3] found that for $n \geq 4$ this maximum is attained if $\xi = \pm 1$ and $P_n \in \mathbf{B}_{n,\xi,2}$ is a (unspecified) member of the 1-parameter family of hard-core Zolotarev polynomials $Z_{n,t}$. Our first result centers around the proof in [3] for the initial case $n = 4$ and is three-fold: (i) the numerical value for (1) in ([3], (7.9)) is not correct, but sufficiently precise; (ii) from preliminary work in [3] can in fact be deduced a closed analytic expression for (1) if $n = 4$, allowing numerical evaluation to any precision; (iii) even the explicit power form representation of an extremal $Z_{4,t} = Z_{4,t^*}$ can be deduced from [3], thus providing an exemplification of Schur's problem that seems to be novel. Additionally, we cross-check its validity twice: firstly by deriving Z_{4,t^*} conversely from a general formula for $Z_{4,t}$ that we have given in [12], and secondly by applying Theorem 5.10 in [11]. We then turn to a generalized solution of Schur's problem, to k -th derivatives, by Shadrin [16]. Again we provide in explicit form the corresponding maximum as well as an extremizer polynomial for the first non-trivial degree $n = 4$. Finally, we contribute to the fuller description of $Z_{4,t}$ by providing its critical points in explicit form.

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1. Introduction

The famous A. A. Markov inequality of 1889 [8] asserts an estimate on the size of the first derivative of an algebraic polynomial P_n of degree $\leq n$ and can be restated as follows:

$$\sup_{\xi \in \mathbf{I}} \sup_{P_n \in \mathbf{B}_n} |P_n^{(1)}(\xi)| = n^2 = T_n^{(1)}(1), \tag{1.1}$$

where $\mathbf{I} = [-1, 1]$ and $\mathbf{B}_n = \{P_n : |P_n(x)| \leq 1 \text{ for } x \in \mathbf{I}\}$. As indicated, this maximum will be attained if, up to the sign, $P_n = T_n \in \mathbf{B}_n$ is the n-th Chebyshev polynomial of the first kind on $\mathbf{I}$ (defined by $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ with $T_1(x) = x$, $T_0(x) = 1$) and if $\xi = \pm 1$, see e.g. ([10], p. 529), ([13], p. 123).

In 1919 I. Schur ([14], §2), inspired by (1.1), was led to the problem of finding the maximum of $|P_n^{(1)}(\xi)|$ under the additional restriction $P_n^{(2)}(\xi) = 0$: Determine $P_n = P_n^*$ which attains, for $n \geq 3$,

$$\sup_{\xi \in \mathbf{I}} \sup_{P_n \in \mathbf{B}_{n,\xi,2}} |P_n^{(1)}(\xi)| = n^2 M_n, \tag{1.2}$$

where $\mathbf{B}_{n,\xi,2} = \{P_n \in \mathbf{B}_n : P_n^{(2)}(\xi) = 0\}$ and M_n is a constant (depending on n). Schur ([14], (9)) proved that there holds

$$\sup_{\xi \in \mathbf{I}} \sup_{P_n \in \mathbf{B}_{n,\xi,2}} |P_n^{(1)}(\xi)| < \frac{1}{2} n^2, \text{so that } M_n < \frac{1}{2}. \tag{1.3}$$

In 1942 P. Erdös and G. Szegö addressed this problem of Schur and they provided the following solution ([3], Theorem 2):

*The maximum (*1.2*) will be attained, for $n \geq 4$, only if $\xi = 1$ and $P_n = P_n^*$ is a (unspecified) member of the* 1*-parameter family (with parameter t) of hard-core Zolotarev polynomials $\pm Z_{n,t}$; or if $\xi = -1$ and $P_n = P_n^*$ is a (unspecified) member of the family $\pm Z_{n,t}^-$, where $Z_{n,t}^-(x) = Z_{n,t}(-x)$.*

We leave aside the simple case $n = 3$ (with solution $\xi = 0$ and $P_3 = P_3^* = \pm T_3$ ([3], p. 466)). Henceforth we will confine ourselves to specify only one extremal polynomial P_n^* for a given problem on $\mathbf{I}$, but will keep in mind that $-P_n^*$ as well as $\pm Q_n^*$, where $\pm Q_n^*(x) = \pm P_n^*(-x)$, may likewise be extremal. The solutions to (1.1) and (1.2) have in common that the maximum is attained at the endpoints $\xi = \pm 1$ of the unit interval $\mathbf{I}$. But, on the other hand, the solutions differ greatly when it comes to exhibit an explicit extremal polynomial from $\mathbf{B}_n$ resp. $\mathbf{B}_{n,\xi,2}$: Whereas in (1.1) an extremizer is, for all $n \geq 1$, the well-known n -th Chebyshev polynomial T_n [13], the explicit power form of the intricate extremizers $Z_{n,t}$ in (1.2) remained arcane for all $n \geq 4$. This is due to the fact that for a general degree n the explicit power form of a hard-core Zolotarev polynomial $Z_{n,t}$ is not known ([16], p. 1185). Rather, $Z_{n,t}$ can be expressed with the aid of elliptic functions (see ([1], pp. 280), ([10], p. 407), [18]) which amounts to *an extremely complicated concoction of elliptic quantities* ([17], p. 52).

It is a purpose of this note to provide, nearly one hundred years after the origin of

Schur's problem, the explicit power form of a particular hard-core Zolotarev polynomial $Z_{n,t} = Z_{n,t^*}$ which is extremal for (1.2), at least for the first nontrivial case $n = 4$. Such a solution was coined *Schur polynomial* in ([11], Section 5d), where a numerical method (solution of a system of nonlinear equations) is advised in order to determine it.

We will first tackle the explicit analytic expression for (1.2) if $n = 4$. Once it has been established, to calculate its numerical value to arbitrary precision becomes immediate. Incidentally, we notice that the numerical value for $16M_4$ as given in ([3], (7.9)) is not correct from the third decimal place on. We then deduce, in three alternative fashions, an extremal hard-core Zolotarev polynomial $P_4^* = Z_{4,t^*}$ with optimal value t^* of the parameter t. This Schur polynomial P_4^* may well serve as illustrative example of the result in ([3], Theorem 2). Finally, we will consider a recent generalization of Schur's problem (1.2), due to A. Shadrin [16], to higher derivatives of P_n, and again we will exemplify the quartic case $n = 4$. In a closing remark we reveal the critical points of $Z_{4,t}$ to get a fuller picture of the quartic hard-core Zolotarev polynomial.

2. Analytical and numerical value of the maximum in the quartic case

To determine the value in (1.2) for $n = 4$ we rely on preliminary work in ([3], Section 7) and will therefore retain, for the reader's convenience, the notation used there. A sought-for extremal hard-core Zolotarev polynomial P_4^* which solves (1.2) can be assumed to be from class $\mathbf{B}_{4,1,2}$ and be represented as, see ([3], (7.3)),

$$P_4^*(x) = 1 - \lambda(1-x)(B_4 - x)(y_1 - x)^2, \tag{2.1}$$

where λ, B_4, y_1 are parameters which reflect properties of P_4^*, such as:

$$P_4^*(-1) = -1,\ P_4^*(y_1) = 1,\ P_4^{*(1)}(y_1) = 0,\ P_4^*(1) = P_4^*(B_4) = 1.$$

The first and second derivative of P_4^* at $x = 1$ read:

$$P_4^{*(1)}(1) = \lambda(B_4 - 1)(1 - y_1)^2 \text{ and } P_4^{*(2)}(1) = 2\lambda(y_1 - 1)(2(1 - B_4) - (y_1 - 1)), \tag{2.2}$$

so that the condition $P_4^{*(2)}(1) = 0$ yields $y_1 = 3 - 2B_4$ which, when inserted into $P_4^{*(1)}(1)$, eliminates there the parameter y_1. From $P_4^*(-1) = -1$ one deduces, upon inserting the said value of y_1, that

$$\lambda = \frac{1}{(B_4 + 1)(4 - 2B_4)^2},$$

see (2.1). This implies

$$P_4^{*(1)}(1) = \frac{(B_4 - 1)^3}{(B_4 - 2)^2(B_4 + 1)}.$$

The identity

$$\frac{2}{B_4 - 1} = \frac{11 - \sqrt{33} + 2\sqrt{5(5 + \sqrt{33})}}{8},$$

which is given in an equivalent form in ([3], (7.8)), allows to evaluate B_4 (see (3.2) below). Inserting this value of B_4 into the preceding expression for $P_4^{*(1)}(1)$ eventually yields the analytical expression for the maximum, which can be evaluated numerically to any desired precision:

$$
\begin{aligned}
P_4^{*(1)}(1) &= \sup_{\xi\in\mathbf{I}}\sup_{P_4\in\mathbf{B}_{4,\xi,2}}\left|P_4^{(1)}(\xi)\right| = 16M_4 \\
&= \frac{-561+161\sqrt{33}+\sqrt{30(15215+3329\sqrt{33})}}{288} \\
&= 4.7876468942...,
\end{aligned} \tag{2.3}
$$

being a root of $P_4(x) = -65536 - 39424x - 1915x^2 + 1683x^3 + 216x^4$.

By contrast, Formula (7.9) in [3] states that

$$P_4^{*(1)}(1) = 4.7881... \tag{2.4}$$

holds, a value which is now seen to be biased in the third and fourth decimal place. But that bias does not harm the argument in [3] for $n = 4$ since the first two valid decimal places are sufficiently conclusive for P_4^* to be the extremal element (as a comparison is drawn with competitor polynomial T_4 and value $\left|T_4^{(1)}\left(\frac{1}{\sqrt{6}}\right)\right| = 4.3546...$, see ([3], (7.2))).

The constant M_4 itself can thus be represented as

$$
\begin{aligned}
M_4 &= \frac{P_4^{*(1)}(1)}{16} = \sup_{\xi\in\mathbf{I}}\sup_{P_4\in\mathbf{B}_{4,\xi,2}}\frac{\left|P_4^{(1)}(\xi)\right|}{4^2} \\
&= \frac{-561+161\sqrt{33}+\sqrt{30(15215+3329\sqrt{33})}}{4608} \\
&= 0.2992279308... .
\end{aligned} \tag{2.5}
$$

We note that according to ([3], (1.3), (1.4)) there holds $\lim_{n\to\infty} M_n = 0.3124...$. Schur ([14], p. 277) had obtained the weaker result $0.217... \leq \limsup_{n\to\infty} M_n \leq 0.465...$.

3. Explicit power form representation of an extremal hard-core Zolotarev polynomial in the quartic case

Having expressed the parameters $\lambda = \lambda(B_4)$ and $y_1 = y_1(B_4)$ as functions of B_4 alone and knowing the value of the constant B_4, it is possible to even retrieve the explicit power form of an extremal P_4^*. In fact, according to the preceding Section we have

$$
\begin{aligned}
P_4^*(x) &= 1 - \lambda(1-x)(B_4-x)(y_1-x)^2 \\
&= 1 - \frac{(1-x)(B_4-x)(3-2B_4-x)^2}{(B_4+1)(4-2B_4)^2}
\end{aligned} \tag{3.1}
$$

Inserting now

$$\begin{aligned} B_4 &= \frac{177 - 17\sqrt{33} + \sqrt{30(527 + 97\sqrt{33})}}{144} \\ &= 1.8034303689... \end{aligned} \tag{3.2}$$

and expanding (3.1) leads us, after some algebraic manipulations, to the explicit power form representation of an extremal quartic hard-core Zolotarev polynomial P_4^* with

$$P_4^*(x) = \sum_{i=0}^{4} a_i^* x^i$$

and with coefficients

$$\begin{aligned} a_0^* &= \frac{21297 - 2081\sqrt{33} - \sqrt{30(3160847 + 628577\sqrt{33})}}{9216} = -0.5328330303... \\ a_1^* &= \frac{291 - 1139\sqrt{33} - \sqrt{30(-1236313 + 427337\sqrt{33})}}{4608} = -2.6688925571... \\ a_2^* &= \frac{-849 + 161\sqrt{33} + \sqrt{30(15215 + 3329\sqrt{33})}}{384} = 2.8407351706... \\ a_3^* &= \frac{4317 + 1139\sqrt{33} + \sqrt{30(-1236313 + 427337\sqrt{33})}}{4608} = 3.6688925571... \\ a_4^* &= \frac{-921 - 1783\sqrt{33} - \sqrt{330(-59555 + 64243\sqrt{33})}}{9216} = -2.3079021403... \, . \end{aligned} \tag{3.3}$$

These optimal coefficients a_i^* are roots of the following respective quartic polynomials $P_{4,i}$ with integer coefficients:

$$\begin{aligned} P_{4,0}(x) &= -7951932 - 7463259x + 11697424x^2 - 4089024x^3 + 442368x^4 \\ P_{4,1}(x) &= 12221 + 273251x - 7120x^2 - 3492x^3 + 13824x^4 \\ P_{4,2}(x) &= -236196 - 112023x + 17720x^2 + 13584x^3 + 1536x^4 \\ P_{4,3}(x) &= 288684 - 303831x + 65348x^2 - 51804x^3 + 13824x^4 \\ P_{4,4}(x) &= 314928 + 2644083x - 861584x^2 + 176832x^3 + 442368x^4. \end{aligned} \tag{3.4}$$

This result constitutes, to the best of our knowledge, the first explicit example of an extremal P_n^* which solves Schur's problem according to Erdös-Szegö ([3], Theorem 2) (here for the first Zolotarev case $n = 4$). It is therefore worth summarizing the properties of that Schur polynomial $P_4^* \in \mathbf{B_4}$:

(i) The equiripple property on $\mathbf{I}$, i.e., 4 alternation points, including the endpoints ± 1:

$$\begin{aligned}
P_4^*(-1) &= -1,\\
P_4^*(y_1) &= 1 \text{ and } P_4^{*(1)}(y_1) = 0, \text{where}\\
y_1 &= \tfrac{1}{72}(39 + 17\sqrt{33} - \sqrt{30(527 + 97\sqrt{33})}) = -0.6068607378...,\\
P_4^*(y_2) &= -1 \text{ and } P_4^{*(1)}(y_2) = 0, \text{where}\\
y_2 &= \tfrac{1}{72}(105 - \sqrt{33} - \sqrt{30(95 + 17\sqrt{33})}) = 0.3226516930...,\\
P_4^*(1) &= 1.
\end{aligned} \tag{3.5}$$

(ii) The Zolotarev property at three points $A_4 < B_4 < C_4$ to the right of $\mathbf{I}$ (of which B_4 and C_4 are two additional alternation points)

$$\begin{aligned}
P_4^{*(1)}(A_4) &= 0, \text{ where}\\
A_4 &= \frac{279 + 25\sqrt{33} + \sqrt{30(2879 + 561\sqrt{33})}}{576} = 1.4764907146...,\\
P_4^*(B_4) &= 1, \text{ where } B_4 \text{ is given in (3.2)},\\
P_4^*(C_4) &= -1, \text{ where}\\
C_4 &= \frac{201 + 55\sqrt{33} - \sqrt{330(61 + 19\sqrt{33})}}{144} = 1.9444055070... .
\end{aligned} \tag{3.6}$$

Additionally, by construction, P_4^* satisfies

$$\begin{aligned}
P_4^{*(2)}(1) &= 2(a_2^* + 3a_3^* + 6a_4^*) = 0, \text{i.e.}, P_4^* \in \mathbf{B}_{4,1,2}\\
P_4^{*(1)}(1) &= a_1^* + 2a_2^* + 3a_3^* + 4a_4^* = 16M_4, \text{ see (2.3)},
\end{aligned} \tag{3.7}$$

and we add, by inspection, that

$$a_3^* = 1 - a_1^* \text{ and } a_4^* = -a_0^* - a_2^*. \tag{3.8}$$

That particular hard-core Zolotarev polynomial P_4^* may well serve as elucidating example to provide for explanation purposes in lectures or expository writings on Schur's problem, respectively on its solution by Erdös-Szegö, see e.g. [4].

4. Alternative deductions of an explicit extremal hard-core Zolotarev polynomial in the quartic case

In ([12], p. 357) we have provided explicit expressions for the parameterized coefficients of an arbitrary fourth-degree hard-core Zolotarev polynomial on $\mathbf{I}$. But since the assumption was made there that it attains the value 1 at $x = -1$, we prefer to consider here the negative form of that polynomial in order to be compliant with [3]. We hence set

$$Z_{4,t}(x) = \sum_{i=0}^{4} -a_i(t)x^i, \text{ with } 1 < t < 1 + \sqrt{2} \tag{4.1}$$

where the coefficients $a_i(t)$ read as follows:

$$\begin{array}{rcl} a_0(t) & = & \big(-a^5 - b^3 + a^4(-2+3b) + a^3(-1+6b-3b^2)+ \\ & & +a(3b^2-2b^3) + a^2(3b+2b^2+b^3)\big)\,\kappa, \\ a_1(t) & = & (a^2(-16b+8b^2) + a(-12b+8b^2-4b^3))\kappa, \\ a_2(t) & = & (a^2(8-16b) + 6b - 4b^2 + 2b^3 + a(6-4b+2b^2))\kappa, \\ a_3(t) & = & (-4+8a^2+8b+8ab-4b^2)\kappa, \\ a_4(t) & = & (-4-6a+2b)\kappa \end{array} \tag{4.2}$$

with

$$\begin{array}{rcl} \kappa & = & \dfrac{1}{(1+a)^2(-a+b)^3} \\ a & = & \dfrac{1-3t-t^2-t^3}{(1+t)^3} \\ b & = & \dfrac{1+t+3t^2-t^3}{(1+t)^3}. \end{array} \tag{4.3}$$

Here a and b with $a < b$ are the alternation points of $Z_{4,t}$ in the interior of $\mathbf{I}$. We now proceed to determine the optimal parameter $t = t^*$ and the corresponding explicit coefficients $-a_i(t^*)$ of an extremal polynomial Z_{4,t^*} with $Z_{4,t^*}(x) = \sum_{i=0}^{4} -a_i(t^*)x^i$ which, according to the general result in ([3], Theorem 2), solves Schur's problem (1.2) for $n = 4$.

The assumption $Z_{4,t} \in \mathbf{B}_{4,1,2}$, i.e., $Z_{4,t}^{(2)}(1) = 0$, implies

$$a_2(t) + 3a_3(t) + 6a_4(t) = 0. \tag{4.4}$$

Employing the definition of $a_i(t)$ in (4.2),(4.3) this amounts to the following equation, after some algebraic manipulations:

$$\frac{(1+t)^3(3+t(2+t))(-2+t(-7+t(1+3(-1+t)t)))}{4(t+t^3)^2} = 0. \tag{4.5}$$

The numerator vanishes, for $1 < t < 1+\sqrt{2}$, only if we choose

$$t = t^* = \frac{3+\sqrt{33}+\sqrt{30(-1+\sqrt{33})}}{12} = 1.7229220588..., \tag{4.6}$$

which is a root of the polynomial $P_4(x) = -2 - 7x + x^2 - 3x^3 + 3x^4$. Inserting the optimal parameter (4.6) into the coefficients $-a_i(t)$ of $Z_{4,t}$, see (4.2), (4.3), shows that $-a_i(t^*)$ indeed coincides for $i = 0,1,2,3,4$ with a_i^* as given in (3.3). We check only the coefficient $-a_4(t)$ and leave it to the reader to check the remaining coefficients:

$$-a_4(t) = \frac{4+6a-2b}{(1+a)^2(-a+b)^3} = \frac{(1-t)(1+t)^9}{32t^3(1+t^2)^2}, \tag{4.7}$$

and inserting now $t = t^*$ according to (4.6) indeed yields $-a_4(t^*) = a_4^*$ as given in (3.3). After all, we so obtain an alternative and independent deduction of the extremal hard-core Zolotarev polynomial $P_4^* = Z_{4,t^*}$ which we had already found in Section 3, based on preliminary work in [3].

A third argument can be brought forward to prove that $P_4^* = Z_{4,t^*}$ is a sought-for extremizer in (1.2) for $n = 4$, see ([11], Theorem 5.10): It suffices to verify that the following five equations hold true

$$-1 + 2(-y_1 + y_2) - (1 + B_4 - C_4) = 0 \tag{4.8}$$

$$1 + 2(-y_1^2 + y_2^2) - (1 + B_4^2 - C_4^2) = 0 \tag{4.9}$$

$$-1 + 2(-y_1^3 + y_2^3) - (1 + B_4^3 - C_4^3) = 0 \tag{4.10}$$

$$\frac{16(A_4 - 1)^2}{(B_4 - 1)(C_4 - 1)} = 1 + 2\left(\frac{2}{A_4 - 1} - \frac{1}{B_4 - 1} - \frac{1}{C_4 - 1}\right) \tag{4.11}$$

$$A_4 = \frac{3}{8}(B_4 + C_4) - \frac{1}{4}(y_1 + y_2), \tag{4.12}$$

where y_1 and y_2 are defined in (3.5), A_4 and C_4 are defined in (3.6), and B_4 is defined in (3.2). We leave it to the reader to check the validity of equations (4.8) - (4.12). Summarizing, we have thus established

Proposition 4.1. *Polynomial P_4^* with $P_4^*(x) = \sum_{i=0}^{4} a_i^* x^i$ and explicit coefficients a_i^* $(i = 0, 1, 2, 3, 4)$ according to (3.3) is a sought-for extremal hard-core Zolotarev polynomial of degree four which solves, according to Erdös-Szegö ([3], Theorem 2), Schur's problem (1.2) for $n = 4$. The corresponding maximum*

$$\sup_{\xi \in \mathbf{I}} \sup_{P_4 \in \mathbf{B}_{4,\xi,2}} \left| P_4^{(1)}(\xi) \right| = 16 M_4$$

is explicitly given in (2.3), so that M_4 equals the constant given in (2.5).

5. A generalized Schur problem and its solution for the quartic case

A. A. Markov's inequality (1.1) for the first derivative of P_n was generalized in 1892 by his half-brother V. A. Markov ([9], p. 93) to the k -th derivative and can be restated as follows, see also ([10], p. 545), ([13], Theorem 2.24):

$$\sup_{\xi \in \mathbf{I}} \sup_{P_n \in \mathbf{B}_n} \left| P_n^{(k)}(\xi) \right| = \prod_{j=0}^{k-1} \frac{n^2 - j^2}{2j + 1} = T_n^{(k)}(1), (1 \le k \le n), \tag{5.1}$$

indicating that the maximum is attained if $P_n = T_n$ and $\xi = 1$. Shadrin [16] has analogously generalized Schur's problem (1.2) to the k -th derivative. This generalized problem can be stated as follows:
Determine, for $1 \le k \le n - 2$ and $n \ge 4$, an algebraic polynomial $P_n = P_n^*$ of degree $\le n$ which attains the maximum

$$\sup_{\xi \in \mathbf{I}} \sup_{P_n \in \mathbf{B}_{n,\xi,k+1}} \left| P_n^{(k)}(\xi) \right| = \prod_{j=0}^{k-1} \frac{n^2 - j^2}{2j + 1} M_{n,k} = T_n^{(k)}(1) M_{n,k}, \tag{5.2}$$

where $\mathbf{B}_{n,\xi,k+1} = \{P_n \in \mathbf{B}_n : P_n^{(k+1)}(\xi) = 0\}$ and $M_{n,k}$ is a constant (depending on n and k). Shadrin ([16], Proposition 4.4) found that, for $k \ge 2$, this maximum is attained if $\xi = 1$ and $P_n = P_n^* \in \mathbf{B}_{n,1,k+1}$ is a Zolotarev polynomial Z_n (not necessarily a hard-core one), or if $\xi = \omega_{k,n}$, the rightmost zero of $T_n^{(k+1)}$, and $P_n = P_n^* = T_n$, so that altogether there holds:

$$\sup_{\xi \in \mathbf{I}} \sup_{P_n \in \mathbf{B}_{n,\xi,k+1}} \left| P_n^{(k)}(\xi) \right| = \max\{|Z_n^{(k)}(1)|, |T_n^{(k)}(\omega_{k,n})|\}. \tag{5.3}$$

We are now going to determine that maximum as well as an extremizer polynomial for the quartic case $n = 4$ and for the second derivative, i.e., $k = 2 = n - 2$ (the case $k = 1$ is settled in Proposition 4.1). It is well known that Zolotarev polynomials Z_n of degree $n \geq 4$ on $\mathbf{I}$ satisfy $||Z_n||_\infty = 1$ (maximum-norm) and exhibit at least n equiripple points on $\mathbf{I}$ where the values ± 1 are attained alternately, see ([16], p. 1190). Apart from sign and reflection, the Zolotarev polynomial Z_4 takes on the role (see also ([1], pp. 280), ([10], p. 406)):

(i) $Z_4 = T_3$, with $T_3(x) = -3x + 4x^3$,

(ii) $Z_4 = T_4$, with $T_4(x) = 1 - 8x^2 + 8x^4$,

(iii) $Z_4 = T_{4,\beta}$, with $T_{4,\beta}(x) = T_4\left(\dfrac{2x - \beta + 1}{1 + \beta}\right)$
where $1 < \beta \leq 1 + 2\tan^2\left(\frac{\pi}{8}\right) = 7 - 4\sqrt{2} = 1.3431457505...$,

(iv) $Z_4 = Z_{4,t}$, the hard-core Zolotarev polynomial, as given in (4.1).

We first calculate $|Z_4^{(2)}(1)|$, subject to the constraint $Z_4^{(3)}(1) = 0$, and observe that polynomials (i), (ii), (iii) cannot be extremal due to $T_3^{(3)}(1) = 24 \neq 0$, resp. $T_4^{(3)}(1) = 192 \neq 0$, resp. $T_{4,\beta}^{(3)}(1) = \dfrac{1536(3 - \beta)}{(1 + \beta)^4} \neq 0$ if $1 < \beta \leq 7 - 4\sqrt{2}$. For polynomial (iv) we get, after some algebraic manipulations,

$$|Z_{4,t}^{(3)}(1)| = \left| \frac{3(1 + t)^6(-1 + t(-8 + 2t + 3t^3))}{8t^3(1 + t^2)^2} \right|. \tag{5.4}$$

The numerator vanishes for $1 < t < 1 + \sqrt{2}$ only if

$$t = t^{**} = \frac{1 + \sqrt{2(-1 + \sqrt{3})}}{\sqrt{3}} = 1.2759444802... . \tag{5.5}$$

Inserting this parameter t^{**} into $|Z_{4,t}^{(2)}(1)|$ yields, again after some manipulations,

$$|Z_{4,t^{**}}^{(2)}(1)| = \left| -12 - \frac{22}{\sqrt{3}} + 4\sqrt{\frac{10}{3} + 2\sqrt{3}} \right| = 14.2729495641... . \tag{5.6}$$

In view of (5.3), we have next to compare (5.6) to $|T_4^{(2)}(\omega_{2,4})|$. Since the only, and hence the rightmost, zero of $T_4^{(3)}$ is $\omega_{2,4} = 0$, we get

$$|T_4^{(2)}(0)| = |-16| = 16 > |Z_{4,t^{**}}^{(2)}(1)|.$$

So eventually we arrive at the identity

$$\begin{aligned}\sup_{\xi\in\mathbf{I}}\sup_{P_4\in\mathbf{B}_{4,\xi,3}}|P_4^{(2)}(\xi)| &= \max\{|Z_4^{(2)}(1)|,|T_4^{(2)}(0)|\}=16\\ &= \prod_{j=0}^{1}\frac{4^2-j^2}{2j+1}M_{4,2}=80M_{4,2},\end{aligned}\tag{5.7}$$

yielding $M_{4,2}=\dfrac{1}{5}=0.2$. Summarizing, we have thus established

Proposition 5.1. *Polynomial $P_4^*=T_4$ with $T_4(x)=1-8x^2+8x^4$ is a sought-for extremal polynomial of degree four which solves, according to Shadrin (*[16]*, Proposition 4.4), the generalized Schur problem (*5.2*) for $n=4$ and $k=2$. The corresponding maximum $\sup_{\xi\in\mathbf{I}}\sup_{P_4\in\mathbf{B}_{4,\xi,3}}|P_4^{(2)}(\xi)|=80M_{4,2}$ is* 16*, so that $M_{4,2}$ equals the constant $\frac{1}{5}$.*

Shadrin ([16], Theorem 7.1) has added to (5.3) the following estimate which can be viewed as an extension, to the k-th derivative, of Schur's estimate (1.3):

$$\sup_{\xi\in\mathbf{I}}\sup_{P_n\in\mathbf{B}_{n,\xi,k+1}}|P_n^{(k)}(\xi)|\le\prod_{j=0}^{k-1}\frac{n^2-j^2}{2j+1}\lambda_{n,k}=T_n^{(k)}(1)\lambda_{n,k}\quad(1\le k\le n-2),\tag{5.8}$$

where $\lambda_{n,k}=\dfrac{1}{k+1}\cdot\dfrac{n-1}{n-1+k}$.

Thus for $k=2$ and $n=4$ we get $\lambda_{4,2}=\dfrac{1}{3}\cdot\dfrac{3}{5}=\dfrac{1}{5}=0.2=M_{4,2}$, see (5.7). However, for $k=1$ and $n=4$ we get $\lambda_{4,1}=\dfrac{1}{2}\cdot\dfrac{3}{4}=\dfrac{3}{8}=0.375>M_4=0.299...$, see (2.5) and ([16], Remark 5.5).

6. Concluding remarks

1. In deducing (2.3) we have been guided by a computer algebra system which the authors of [3], who have paved the way, certainly did not have at their disposal.

2. Our explicit power form representation ([12], p. 357) for the fourth hard-core Zolotarev polynomial $Z_{4,t}$ remained unnoticed, and several related formulas have been published afterwards, e.g. ([2], p. 184), ([15], p. 242), ([18], p. 721). Shadrin [15] attributes his formula (with a different range of the parameter t) to V. A. Markov [9] and remarks: *But already for $n=4$ it seems that nobody really believed that an explicit form can be found. As a matter of fact it was, by V. Markov in 1892.* In a private communication Professor Shadrin kindly called our attention to p. 73 in [9] from which his formula can be recovered. However, one has first to exploit the relation $4z=t^3+t$ (see p. 71 in [9]), then fix the parameter α and finally rearrange the Taylor form of the given fourth-degree polynomial, centered at $x_0=2z$, to the usual power form centered at $x_0=0$. It is under these side conditions that priority for the power form representation of $Z_{4,t}$ belongs indeed to V. A. Markov [9].

3. In Section 4 we have alternatively deduced the Schur polynomial P_4^* from the explicit power form $Z_{4,t}(x)=...$ as given, up to the sign, in ([12], p. 357). P_4^* can

likewise be deduced from the explicit power form $Z_4(x,t) = ...$ as given in ([15], p. 242), however instead of $Z_{4,t}^{(2)}(1) = 0$ (see (4.4)) one has then to set $Z_4^{(2)}(-1,t) = 0$.

4. For the quartic Schur polynomial $P_4^* = Z_{4,t^*}$ we have determined its five critical points $y_1, y_2 \in \mathbf{I}$ and A_4, B_4, C_4 with $1 < A_4 < B_4 < C_4$. As Z_{4,t^*} is a special case of the general quartic hard-core Zolotarev polynomial $Z_{4,t}$ it is desirable to know the corresponding five (general) critical points of $Z_{4,t}$ as well. These are, as can be verified by insertion: $y_1(t) = a(t) = a$ and $y_2(t) = b(t) = b$ as given in (4.3), and furthermore

$$A_4(t) = \frac{1+4t+2t^2+4t^3+t^4}{2(-1+t)(1+t)^3} \tag{6.1}$$

$$B_4(t) = \frac{1+2t+6t^3-t^4}{(-1+t)(1+t)^3} \tag{6.2}$$

$$C_4(t) = \frac{-1+6t+2t^3+t^4}{(-1+t)(1+t)^3}. \tag{6.3}$$

Choosing $t = t^*$ according to (4.6) takes us back to the five critical points of $P_4^* = Z_{4,t^*}$.

5. The optimal parameter $t = t^*$ according to (4.6) which selects the quartic Schur polynomial Z_{4,t^*} among all $Z_{4,t}$ with $1 < t < 1+\sqrt{2}$ can alternatively be determined as follows: In (4.11) replace A_4 by $A_4(t)$, B_4 by $B_4(t)$ and C_4 by $C_4(t)$ according to (6.1) - (6.3). Then solve this generalized equation (4.11) for the unknown number t. The solution will turn out as $t = t^*$.

6. As some progress has been achieved in the computation of $Z_{n,t}$ for the next higher polynomial degrees $n \geq 5$ (see [5], [6], [7], [11]), we hope that we will be able to extend our results to some $n \geq 5$. Meanwhile, we have succeeded so for the case $n = 5$. (The second Zolotarev case in the Erdös-Szegö solution to a Markov-type extremal problem of Schur, J. Numer. Anal. Approx. Theory 46(2017), to appear).

References

[1] Achieser, N.I., *Theory of approximation*, Dover Publications, Mineola N.Y., 2003 (Russian 1947).

[2] Collins, G.E., *Application of quantifier elimination to Solotareff's approximation problem*, in Stability Theory: Hurwitz Centenary Conference, (R. Jeltsch et al., eds.), Ascona 1995, Birkhäuser Verlag, Basel, 1996 (ISNM 121), 181-190.

[3] Erdös, P., Szegö, G., *On a problem of I. Schur*, Ann. Math., **43**(1942), 451-470.

[4] Finch, S.R., *Zolotarev-Schur Constant*, in Finch, S.R., Mathematical Constants, Cambridge University Press, Cambridge (UK), 2003, 229-231.

[5] Grasegger, G., Vo, N. Th., *An algebraic-geometric method for computing Zolotarev polynomials*, Technical Report no. 16-02, RISC Report Series, Johannes Kepler University, Linz, Austria, 2016, 1-17.

[6] Lazard, D., *Solving Kaltofen's challenge on Zolotarev's approximation problem*, in Proceedings of International Symposium on Symbolic and Algebraic Computation (ISSAC), (J.-G. Dumas, ed.), Genova 2006, ACM, New York, 2006, 196-203.

[7] Malyshev, V.A., *The Abel equation*, St. Petersburg Math. J., **13**(2002), 893-938 (Russian 2001).

[8] Markov, A.A., *On a question of D.I. Mendeleev*, Zapiski. Imper. Akad. Nauk., St. Petersburg, **62**(1889), 1-24 (Russian).

[9] Markov, V.A., *On functions deviating least from zero*, Izdat. Akad. Nauk., St. Petersburg, 1892, iv + 111 (Russian).

[10] Milovanović, G.V., Mitrinović, D.S., Rassias, Th.M., *Topics in polynomials: Extremal problems, inequalities, zeros*, World Scientific, Singapore, 1994.

[11] Peherstorfer, F., Schiefermayr, K., *Description of extremal polynomials on several intervals and their computation. II*, Acta Math. Hungar., **83**(1999), 59-83.

[12] Rack, H.-J., *On polynomials with largest coefficient sums*, J. Approx. Theory, **56**(1989), 348-359.

[13] Rivlin, Th.J., *Chebyshev polynomials*, 2nd ed., Wiley, New York, 1990.

[14] Schur, I., *Über das Maximum des absoluten Betrages eines Polynoms in einem gegebenen Intervall*, Math. Z., **4**(1919), 271-287.

[15] Shadrin, A., *Twelve proofs of the Markov inequality*, in Approximation Theory: A volume dedicated to Borislav Bojanov, (D.K. Dimitrov et al., eds.), M. Drinov Acad. Publ. House, Sofia, 2004, 233-298.

[16] Shadrin, A., *The Landau-Kolmogorov inequality revisited*, Discrete Contin. Dyn. Syst., **34**(2014), 1183-1210.

[17] Todd, J., *A legacy from E.I. Zolotarev (1847-1878)*, Math. Intell., **10**(1988), 50-53.

[18] Vlček, M., Unbehauen, R., *Zolotarev polynomials and optimal FIR filters*, IEEE Trans. Signal Process., **47**(1999), 717-730. Corrections: IEEE Trans. Signal Process., **48**(2000), 2171.

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An inequality of Ostrowski-Grüss type for double integrals

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Abstract. In this study, we establish Ostrowski-Grüss type involving functions of two independent variables for double integrals. Cubature formula is also provided.

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Keywords: Ostrowski-Grüss type inequality, double integrals, two independent variables.

1. Introduction

In 1935, G. Grüss [7] proved the following inequality:

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \frac{1}{b-a} \int_a^b g(x)dx \right| \\ \leq\ & \frac{1}{4}(\Phi_1 - \varphi_1)(\Phi_2 - \varphi_2), \end{aligned} \tag{1.1}$$

provided that f and g are two integrable function on $[a, b]$ satisfying the condition

$$\varphi_1 \leq f(x) \leq \Phi_1 \text{ and } \varphi_2 \leq g(x) \leq \Phi_2 \text{ for all } x \in [a, b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible.

In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [9]:

Theorem 1.1 (Ostrowski inequality). *Let $f : [a, b] \to R$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \to R$ is bounded on (a, b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, we have the inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty, \tag{1.3}$$

for all $x \in [a,b]$. *The constant* $\frac{1}{4}$ *is the best possible.*

In 1882, P. L. Čebyšev [2] gave the following inequality:

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty , \tag{1.4}$$

where $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous function, whose first derivatives f' and g' are bounded,

$$\begin{aligned} & T(f,g) \\ = \ & \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \end{aligned} \tag{1.5}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty = \underset{t\in[a,b]}{ess\sup} |p(t)|$.

The following result of Grüss type was proved by Dragomir and Fedotov [4]:

Theorem 1.2. *Let* $f, u : [a,b] \to \mathbb{R}$ *be such that* u *is* L*-Lipshitzian on* $[a,b]$, *i.e,*

$$|u(x) - u(y)| \leq L|x-y| \quad \text{for all } x \in [a,b], \tag{1.6}$$

f *is Riemann integrable on* $[a,b]$ *and there exist the real numbers* m, M *so that*

$$m \leq f(x) \leq M \quad \text{for all } x \in [a,b]. \tag{1.7}$$

Then we have the inequality,

$$\left| \int_a^b f(x)du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b f(x)dx \right| \leq \frac{1}{2}L(M-m)(b-a).$$

From [8], if $f : [a,b] \to \mathbb{R}$ is differentiable on (a,b) with the first derivative f' integrable on $[a,b]$, then Montgomery identity holds:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \int_a^b P(x,t)f'(t)dt, \tag{1.8}$$

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \leq t \leq x \\ \frac{t-b}{b-a}, & x < t \leq b. \end{cases}$$

In [5], Dragomir and Wang proved following Ostrowski-Grüss type inequality using the inequality (1.1) and Montgomery identity (1.8):

Theorem 1.3. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differantiable mapping in* I° *and let* $a, b \in I^\circ$ *with* $a < b$. *If* $f \in L_1[a,b]$ *and*

$$\varphi_3 \leq f'(x) \leq \Phi_3, \ \forall x \in [a,b],$$

then we have the following inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4}(b-a)(\Phi_3 - \varphi_3), \tag{1.9}$$

for all $x \in [a,b]$.

Barnett and Dragomir established following Ostrowski inequality for double integrals in [1]:

Theorem 1.4. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be a continuous on* $[a,b] \times [c,d]$, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$ *exists on* $(a,b) \times (c,d)$, *and is bounded, i.e.,*

$$\|f_{xy}\|_\infty = \sup_{(x,y)\in(a,b)\times(c,d)} \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| < \infty$$

then we have the inequality

$$\begin{aligned} &\left| \int_a^b \int_c^d f(t,s)dsdt - \left[(b-a) \int_c^d f(x,s)ds \right.\right. \\ &\left.\left. + (d-c) \int_a^b f(t,y)dt - (b-a)(d-c) f(x,y) \right] \right| \\ \leq\; & \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2} \right)^2 \right] \|f_{xy}\|_\infty \end{aligned} \tag{1.10}$$

for all $(x,y) \in [a,b] \times [c,d]$.

In [1], the inequality (1.10) is established by the use of integral identity involving Peano kernels. In [10], Pachpatte obtained an inequality in the view (1.10) by using elementary analysis. The interested reader is also refered to ([1], [6], [10],[11],[13]-[15]) for Ostrowski type inequalities in several independent variables.

Recently, Sarikaya and Kiris have proved the following Grüss type inequality for double integrals in [12]:

Theorem 1.5. *Let* $f, g : [a,b] \times [c,d] \to \mathbb{R}$ *be two functions defined and integrable on* $[a,b] \times [c,d]$. *Then for*

$$\varphi \leq f(x,y) \leq \Phi \text{ and } \gamma \leq g(x,y) \leq \Gamma. \text{ for all } (x,y) \in [a,b] \times [c,d]$$

we have

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx\right. \qquad (1.11)\\
&\left.-\left(\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right)\left(\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)dydx\right)\right|\\
\leq\ & \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma).
\end{aligned}$$

Moreover, Cerone and Dragomir [3] extended Gruss type inequalities for Lebesgue integrals on measurable spaces. This includes domaind from the plane provided in [12].

In this work, using the inequality (1.11), we will obtain an Ostrowski-Grüss type inequality for functions of two independent variables.

2. Main results

First, we give the following notations to simplify the presentation of some intervals.

$$\begin{aligned}
\Delta_1 &= [a,x]\times[c,y],\ \Delta_2=[a,x]\times[y,d],\\
\Delta_3 &= [x,b]\times[c,y],\ \Delta_4=[x,b]\times[y,d].
\end{aligned}$$

Theorem 2.1. *Let $f:\Delta:[a,b]\times[c,d]\to\mathbb{R}$ be a continuous on Δ, $f_{xy}=\frac{\partial^2 f}{\partial x\partial y}$ exists on Δ°. If f integrable and*

$$\varphi\leq f_{xy}(x,y)\leq\Phi,\ \forall(x,y)\in\Delta$$

then we have the following inequality

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt-\left[\frac{1}{(d-c)}\int_c^d f(x,s)ds\right.\right. \qquad (2.1)\\
&\left.+\frac{1}{(b-a)}\int_a^b f(t,y)dt-f(x,y)\right]\\
&\left.-\frac{f(b,d)-f(b,c)-f(a,d)+f(a,c)}{(b-a)(d-c)}\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\right|\\
\leq\ & \frac{1}{4}(P-p)(\Phi-\varphi)
\end{aligned}$$

where

$$P=\max\{(x-a)(y-c),(b-x)(d-y)\}$$

and

$$p = \min\left\{(x-a)(y-d), (x-b)(y-c)\right\}$$

for all $(x, y) \in \Delta$.

Proof. Define the kernel $p(x, t; y, s)$ by

$$p(x,t;y,s) := \begin{cases} (t-a)(s-c), & \text{if } (t,s) \in [a,x]\times[c,y] \\ (t-a)(s-d), & \text{if } (t,s) \in [a,x]\times(y,d] \\ (t-b)(s-c), & \text{if } (t,s) \in (x,b]\times[c,y] \\ (t-b)(s-d), & \text{if } (t,s) \in (x,b]\times(y,d]\,. \end{cases}$$

Then, we have

$$\int_a^b \int_c^d p(x,t;y,s) f_{ts}(t,s)\,dsdt \tag{2.2}$$

$$\begin{aligned} &= \int_a^x \int_c^y (t-a)(s-c) f_{ts}(t,s)\,dsdt + \int_a^x \int_y^d (t-a)(s-d) f_{ts}(t,s)\,dsdt \\ &\quad + \int_x^b \int_c^y (t-b)(s-c) f_{ts}(t,s)\,dsdt + \int_x^b \int_y^d (t-b)(s-d) f_{ts}(t,s)\,dsdt \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Let us calculate the integrals I_1, I_2, I_3 and I_4. Firstly, we have the equality

$$I_1 = \int_a^x \int_c^y (t-a)(s-c) f_{ts}(t,s)\,dsdt \tag{2.3}$$

$$= \int_a^x (t-a)\left[(y-c) f_t(t,y) - \int_c^y f_t(t,s)\,ds\right] dt$$

$$= (y-c)\int_a^x (t-a) f_t(t,y)\,dt - \int_c^y \left(\int_a^x (t-a) f_t(t,s)\,dt\right) ds$$

$$= (y-c)\left[(x-a) f(x,y) - \int_a^x f(t,y)\,dt\right] - \int_c^y \left[(x-a) f(x,s) - \int_a^x f(t,s)\,dt\right] ds$$

$$= (x-a)(y-c) f(x,y) - (y-c)\int_a^x f(t,y)\,dt - (x-a)\int_c^y f(x,s)\,ds + \int_a^x \int_c^y f(t,s)\,dsdt.$$

Also, similar computations we have the equalities

$$I_2 = \int_a^x \int_y^d (t-a)(s-d) f_{ts}(t,s) ds dt \tag{2.4}$$

$$= (x-a)(d-y)f(x,y) - (d-y)\int_a^x f(t,y)dt - (x-a)\int_y^d f(x,s)ds + \int_a^x \int_y^d f(t,s)dsdt,$$

$$I_3 = \int_x^b \int_c^y (t-b)(s-c) f_{ts}(t,s) ds dt \tag{2.5}$$

$$= (b-x)(y-c)f(x,y) - (y-c)\int_x^b f(t,y)dt - (b-x)\int_c^y f(x,s)ds + \int_x^b \int_c^y f(t,s)dsdt,$$

and

$$I_4 = \int_x^b \int_y^d (t-b)(s-d) f_{ts}(t,s) ds dt \tag{2.6}$$

$$= (b-x)(d-y)f(x,y) - (d-y)\int_x^b f(t,y)dt - (b-x)\int_y^d f(x,s)ds + \int_x^b \int_y^d f(t,s)dsdt.$$

If we substitute the equalities (2.3)-(2.6) in (2.2), then we have

$$\int_a^b \int_c^d p(x,t;y,s) f_{ts}(t,s) ds dt \tag{2.7}$$

$$= (b-a)\,(d-c)\,f(x,y) - (b-a)\int_c^d f(x,s)ds - (d-c)\int_a^b f(t,y)dt + \int_a^b \int_c^d f(t,s)dsdt.$$

Applying Theorem 1.5 to mappings $p(x,.;y,.)$ and $f_{ts}(.,.)$, we establish

$$\begin{aligned} &\Bigg| \frac{1}{(b-a)\,(d-c)} \int_a^b \int_c^d p(x,t;y,s) f_{ts}(t,s) ds dt \\ &- \left(\frac{1}{(b-a)\,(d-c)} \int_a^b \int_c^d p(x,t;y,s) ds dt \right) \\ &\times \left(\frac{1}{(b-a)\,(d-c)} \int_a^b \int_c^d f_{ts}(t,s) ds dt \right) \Bigg| \\ \leq \quad & \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma). \end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
\Gamma &= \sup_{(t,s)\in\Delta} p(x,t;y,s) \qquad (2.9)\\
&= \max\left\{ \sup_{(t,s)\in\Delta_1} (t-a)(s-c), \sup_{(t,s)\in\Delta_2} (t-a)(s-d), \right.\\
&\left. \sup_{(t,s)\in\Delta_3} (t-b)(s-c), \sup_{(t,s)\in\Delta_4} (t-b)(s-d) \right\}\\
&= \max\{(x-a)(y-c),(b-x)(d-y)\} = P,
\end{aligned}$$

and

$$\begin{aligned}
\gamma &= \inf_{(t,s)\in\Delta} p(x,t;y,s) \qquad (2.10)\\
&= \min\left\{ \inf_{(t,s)\in\Delta_1} (t-a)(s-c), \inf_{(t,s)\in\Delta_2} (t-a)(s-d), \right.\\
&\left. \inf_{(t,s)\in\Delta_3} (t-b)(s-c), \inf_{(t,s)\in\Delta_4} (t-b)(s-d) \right\}\\
&= \min\{(x-a)(y-d),(x-b)(y-c)\} = p.
\end{aligned}$$

Also, we have the equalities

$$\begin{aligned}
&\int_a^b\int_c^d p(x,t;y,s)dsdt \qquad (2.11)\\
&= \int_a^x\int_c^y (t-a)(s-c)dsdt + \int_a^x\int_y^d (t-a)(s-d)dsdt\\
&+ \int_x^b\int_c^y (t-b)(s-c)dsdt + \int_x^b\int_y^d (t-b)(s-d)dsdt\\
&= \frac{(x-a)^2(y-c)^2}{4} - \frac{(x-a)^2(d-y)^2}{4}\\
&- \frac{(b-x)^2(y-c)^2}{4} + \frac{(b-x)^2(d-y)^2}{4}\\
&= \frac{\left[(x-a)^2-(b-x)^2\right]\left[(y-c)^2-(d-y)^2\right]}{4}\\
&= (b-a)(d-c)\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)
\end{aligned}$$

and

$$\int_a^b \int_c^d f_{ts}(t,s)dsdt = f(b,d) - f(b,c) - f(a,d) + f(a,c). \tag{2.12}$$

If we put the equalities (2.7) and (2.9)-(2.12) in (2.8), then we obtain the desired inequality (2.1). □

Corollary 2.2. *With the assumptions in Theorem* 2.1*, if* $|f_{xy}(x,y)| \leq M$ *for all* $(x,y) \in [a,b] \times [c,d]$ *and some positive constant* M*, then we have*

$$\begin{aligned}
&\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right. \\
&- \left[\frac{1}{(d-c)} \int_c^d f(x,s)ds + \frac{1}{(b-a)} \int_a^b f(t,y)dt - f(x,y) \right] \\
&\left. - \frac{f(b,d) - f(b,c) - f(a,d) + f(a,c)}{(b-a)(d-c)} \left(x - \frac{a+b}{2}\right)\left(y - \frac{c+d}{2}\right) \right| \\
\leq\ & \frac{1}{2}(P-p)M
\end{aligned}$$

where

$$P = \max\{(x-a)(y-c), (b-x)(d-y)\}$$

and

$$p = \min\{(x-a)(y-d), (x-b)(y-c)\}$$

for all $(x,y) \in [a,b] \times [c,d]$*.*

Corollary 2.3. *Under assumptions of Theorem* 2.1 *with* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$*, we have the following inequality*

$$\begin{aligned}
&\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt - \left[\frac{1}{(d-c)} \int_c^d f\left(\frac{a+b}{2}, s\right) ds \right. \right. \\
&\left. \left. + \frac{1}{(b-a)} \int_a^b f\left(t, \frac{c+d}{2}\right) dt - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right] \right| \\
\leq\ & \frac{1}{8}(b-a)(d-c)(\Phi - \varphi).
\end{aligned}$$

Corollary 2.4. *Under assumption of Theorem* 2.1 *with* $x = b$ *and* $y = d$*, we get the inequality*

$$\begin{aligned}
&\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right. \\
&- \left[\frac{1}{(d-c)} \int_c^d f(b,s)ds + \frac{1}{(b-a)} \int_a^b f(t,d)dt - f(b,d) \right] \\
&\left. - \frac{f(b,d) - f(b,c) - f(a,d) + f(a,c)}{4} \right| \\
\leq\ & \frac{1}{4}(b-a)(d-c)(\Phi - \varphi).
\end{aligned}$$

3. Applications for cubature formulae

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$, and $J_m : c = y_0 < y_1 < ... < y_m = d$, $h_i := x_{i+1} - x_i$ $(i = 0, ..., n-1)$, and $l_j := y_{j+1} - y_j$ $(j = 0, ..., m-1)$,

$$\begin{aligned}
\upsilon(h) &:= \max\{h_i |\ i = 0, ..., n-1\}, \\
\mu(l) &:= \max\{l_j |\ j = 0, ..., m-1\}.
\end{aligned}$$

Then, the following theorem holds.

Theorem 3.1. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be as in Theorem* 2.1 *and* $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$ *be intermediate points. Then we have the cubature formula:*

$$\begin{aligned}
&\int_a^b \int_c^d f(t,s)dsdt \qquad (3.1)\\
=\ & \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f(\xi_i, s)ds + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f(t, \eta_j)dt \\
&- \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j f(\xi_i, \eta_j) \\
&+ \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j) \right] \\
&\times \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) \left(\eta_j - \frac{y_j + y_{j+1}}{2} \right) \\
&+ R(\xi, \eta, I_n, J_m, f).
\end{aligned}$$

where the remainer term $R(\xi,\eta,I_n,J_m,f)$ satisfies the estimation

$$|R(\xi,\eta,I_n,J_m,f)| \leq \frac{1}{4}\upsilon(h)\mu(l)\max_{i,j}\left(P_{ij}-p_{ij}\right)\left(\Phi-\varphi\right) \tag{3.2}$$

where

$$P_{ij}=\max\left\{\left(\xi_i-x_i\right)\left(\eta_j-y_j\right),\left(x_{i+1}-\xi_i\right)\left(y_{j+1}-\eta_j\right)\right\},$$

and

$$p_{ij}=\min\left\{\left(\xi_i-x_i\right)\left(\eta_j-y_{j+1}\right),\left(\xi_i-x_{i+1}\right)\left(\eta_j-y_j\right)\right\}.$$

Proof. Aplying Theorem 2.1 on the bidimentional interval $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, we get

$$\begin{aligned}
&\left|\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f(t,s)dsdt \right. \qquad (3.3)\\
&-\left[h_i\int_{y_j}^{y_{j+1}} f(\xi_i,s)ds+l_j\int_{x_i}^{x_{i+1}} f(t,\eta_j)dt-h_il_jf(\xi_i,\eta_j)\right]\\
&-\left[f\left(x_{i+1},y_{j+1}\right)-f\left(x_{i+1},y_j\right)-f\left(x_i,y_{j+1}\right)+f\left(x_i,y_j\right)\right]\\
&\left.\times\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)\right|\\
\leq\ &\frac{1}{4}h_il_j\left(P_{ij}-p_{ij}\right)\left(\Phi_{ij}-\varphi_{ij}\right)
\end{aligned}$$

where

$$\Phi_{ij}:=\sup_{(t,s)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]}\left|f_{ts}(t,s)\right|,\quad \varphi_{ij}:=\inf_{(t,s)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]}\left|f_{ts}(t,s)\right|$$

for all $i=0,1,...,n-1$; $j=0,1,...,m-1$.

Summing the inequality (3.3) over i from 0 to $n-1$ and j from 0 to $m-1$ and using the generalized triangle inequality, we get

$$\begin{aligned}
|R(\xi,\eta,I_n,J_m,f)| &\leq \frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}h_il_j\left(P_{ij}-p_{ij}\right)\left(\Phi_{ij}-\varphi_{ij}\right)\\
&\leq \frac{1}{4}\upsilon(h)\mu(l)\max_{i,j}\left(P_{ij}-p_{ij}\right)\max_{ij}\left(\Phi_{ij}-\varphi_{ij}\right)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}1\\
&= \frac{nm}{4}\upsilon(h)\mu(l)\max_{i,j}\left(P_{ij}-p_{ij}\right)\left(\Phi-\varphi\right).
\end{aligned}$$

This completes the proof. $\square$

References

[1] Barnett, N.S., Dragomir, S.S., *An Ostrowski type inequality for double integrals and applications for cubature formulae*, Soochow J. Math., **27**(2001), no. 1, 109-114.

[2] Čebyšev, P.L., *Sur less expressions approximatives des integrales definies par les autres prises entre les memes limites,* Proc. Math. Soc. Charkov, **2**(1882), 93-98.

[3] Cerone, P., Dragomir, S.S., *A refinement of the Grüss inequality and applications*, Tamkang J. Math., **38**(2007), no. 1, 37-49.

[4] Dragomir, S.S., Fedotov, I., *An inequality of Grüss type for Riemann-Stieltjes integral and applications for special means,* Tamkang J. of Math., **29**(1998), no. 4, 287-292.

[5] Dragomir, S.S., Wang, S., *An inequality of Ostrowski-Grüss' type and its applications to the estimation of error bounds for some special means and for some sumerical quadrature rules*, Computers Math. Applic., **33**(1997), no. 11, 15-20.

[6] Dragomir, S.S., Barnett, N.S., Cerone, P., *An n-dimensional version of Ostrowski's inequality for mappings of Hölder type*, RGMIA Res. Pep. Coll., **2**(1999), no. 2, 169-180.

[7] Grüss, G., *Über das maximum des absoluten Betrages von* $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$, Math. Z., **39**(1935), 215-226.

[8] Mitrinovic, D.S., Pecaric, J.E., Fink, A.M., *Inequalities involving functions and their integrals and derivatives*, Kluwer Academic Publishers, Dordrecht, 1991.

[9] Ostrowski, A.M., *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv., **10**(1938), 226-227.

[10] Pachpatte, B.G., *On a new Ostrowski type inequality in two independent variables*, Tamkang J. Math., **32**(2001), no. 1, 45-49.

[11] Pachpatte, B.G., *A new Ostrowski type inequality for double integrals*, Soochow J. Math., **32**(2006), no. 2, 317-322.

[12] Sarikaya, M.Z., Kiris, M.E., *On Čebysev-Grüss type inequalities for double integrals*, T.J.M.M., **7**(2015), no. 1, 75-83.

[13] Sarikaya, M.Z., *On the Ostrowski type integral inequality*, Acta Math. Univ. Comenianae, **79**(2010), no. 1, 129-134.

[14] Sarikaya, M.Z., *On the Ostrowski type integral inequalityfor double integrals*, Demostratio Mathematica, **45**(2012), no. 3, 533-540.

[15] Ujević, N., *Some double integral inequalities and applications*, Appl. Math. E-Notes, **7**(2007), 93-101.

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Majorization for certain classes of analytic functions defined by convolution structure

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Abstract. In this paper, we investigate majorization properties for certain classes of analytic functions defined by convolution structure. Also we point out some new and known consequences of our main result.

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Keywords: Analytic functions, starlike function, Hadamard product, majorization.

1. Introduction

Let $f(z)$ and $g(z)$ be analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

For analytic function $f(z)$ and $g(z)$ in U , we say that $f(z)$ is majorized by $g(z)$ in U (see [10]) and write

$$f(z) << g(z) \quad (z \in U), \tag{1.1}$$

if there exists a function $\varphi(z)$, analytic in U such that

$$|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in U). \tag{1.2}$$

It may be noted that (1.1) is closely related to the concept of quasi-subordination between analytic functions.

If $f(z)$ and $g(z)$ are analytic functions in U, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as $f(z) \prec g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)),\ z \in U$. Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence, (see [11, p.4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $A(p)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \qquad (\ p \in \mathbb{N} = \{1, 2, ...\}) \tag{1.3}$$

which are analytic and $p-$valent in the open unit disc. We note that $A(1) = A$. Let $g(z) \in A(p)$, be given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.4}$$

For $\lambda, \ell \geqslant 0, p \in \mathbb{N}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f(z)$, $g(z) \in A(p)$, A. O. Mostafa, [12] defined the linear operator $D^m_{\lambda,\ell,p}(f * g)$ as follows:

$$D^m_{p,\ell,\lambda}(f * g) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p + \ell + \lambda(k-p)}{p+\ell}\right]^m a_k b_k z^k. \tag{1.5}$$

From (1.5), it is easy to verify that (see [12]),

$$\lambda z \left(D^m_{\lambda,\ell,p}(f * g)(z)\right)' = (\ell + p) D^{m+1}_{\lambda,\ell,p}(f*g)(z) - [p(1-\lambda) + \ell] D^m_{\lambda,\ell,p}(f*g)(z). \tag{1.6}$$

We note that:

(i) For $b_k = 1$ or $g(z) = \frac{z^p}{1-z}$ we have $D^m_{\lambda,\ell,p} f(z) = I^m_p(\lambda, \ell) f(z)$, where the operator $I^m_p(\lambda, \ell)$ was introduced and studied by Cătaş [4], which contains intern the operators D^m_p, (see [2] and [8]) and D^m_λ (see [1]).

(ii) For $b_k = \frac{(\alpha_1)_{k-p} \ldots (\alpha_q)_{k-p}}{(\beta_1)_{k=p} \ldots (\beta_s)_{k-p} (1)_{k-p}}$, the operator

$$D^m_{\lambda,\ell,p}(f * g)(z) = I^{m,\ell}_{p,q,r,\lambda}(\alpha_1, \beta_1) f(z),$$

where the operator $I^{m,\ell}_{p,q,r,\lambda}(\alpha_1, \beta_1) f(z)$ was introduced and studied by El-Ashwah and Aouf [6], $\alpha_1, \alpha_2, ..., \alpha_q$ and $\beta_1, \beta_2, ..., \beta_s$ are real or complex number $(\beta_j \in \mathbb{C} \backslash \mathbb{Z}^-_0 = \{0, -1, -2, ...\}; j = 1, ..., s;)(q \leq s+1; q, s \in \mathbb{N}_0, p \in \mathbb{N}; z \in U)$ and

$$(\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}), \\ \theta(\theta - 1)...(\theta + \nu - 1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases}$$

Also, for many special operators of the operator $I^{m,\ell}_{p,q,r,\lambda}(\alpha_1, \beta_1) f(z)$ (see [6]).

(iii) For $m = 0$, $b_k = \frac{(\alpha_1)_{k-p} \ldots (\alpha_q)_{k-p}}{(\beta_1)_{k=p} \ldots (\beta_s)_{k-p} (1)_{k-p}}$, the operator

$$D^m_{\lambda,\ell,p}(f * g)(z) = S^j_{p,q,s}(\gamma; \alpha_1) f(z),$$

where the operator $S^j_{p,q,s}(\gamma; \alpha_1) f(z)$, was introduced and studied by El-Ashwah [5].

(iv) For $m = 0$ and $b_k = \frac{\Gamma(p+\alpha+\beta)\Gamma(k+\beta)}{\Gamma(p+\beta)\Gamma(k+\alpha+\beta)}$, the operator $D^m_{p,\ell,\lambda}(f*g)(z) = Q^\alpha_{p,\beta}(f)$ $(\alpha \geq 0, \beta > -1, p \in \mathbb{N})$, where the operator $Q^\alpha_{p,\beta}$ was introduced by Liu and Owa [9].

For $h(z)$ given by

$$h(z) = z^p + \sum_{k=p+1}^{\infty} c_k z^k$$

A function $f(z) \in A(p)$ is said to be in the class $S^{m,j}_{\lambda,\ell,p}(\gamma)$ of $p-$valent functions of complex order $\gamma \neq 0$ in U, if and only if

$$\operatorname{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z(D^m_{\lambda,\ell,p}(f*h)(z))^{(j+1)}}{(D^m_{\lambda,\ell,p}(f*h)(z))^{(j)}}-p+j\right)\right\}>0$$

$$(p \in \mathbb{N};\ j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \ell, \lambda \geq 0; \gamma \in \mathbb{C}^*;\ z \in U). \tag{1.7}$$

Clearly, we have the following relationships:

(i) $S^{0,0}_{\lambda,\ell,1}(\gamma) = S(\gamma)(\gamma \in \mathbb{C}^*)$,

(ii) $S^{0,1}_{\lambda,\ell,1}(\gamma) = \kappa(\gamma)\ (\gamma \in \mathbb{C}^*)$,

(iii) $S^{0,0}_{\lambda,\ell,1}(1-\alpha) = S^*(\alpha)\ (0 \leq \alpha < 1)$.

The classes $S(\gamma)$ and $\kappa(\gamma)$ are classes of starlike and convex functions of complex order $\gamma \neq 0$ in U which were studied by Nasr and Aouf [13] and $S^*(\alpha)$ is the class of starlike functions of order α in U.

Also, for $m = 0$ the operator $S^j_p(h;\gamma)$ was introduced and studied by El-Ashwah and Aouf [7].

Definition 1.1. Let $-1 \leq B < A \leq 1, p \in \mathbb{N}; j \in \mathbb{N}_0, \gamma \in \mathbb{C}^*$,

$$|\gamma(A-B)+(p-j)B| < (p-j),\ \ f \in A(p)\,.$$

Then $f \in S^{m,j}_{\lambda,\ell,p}(\gamma; A, B)$, the class of $p-$valent functions of complex order γ in U if and only if

$$\left\{1+\frac{1}{\gamma}\left(\frac{z(D^m_{\lambda,\ell,p}(f*h)(z))^{(j+1)}}{(D^m_{\lambda,\ell,p}(f*h)(z))^{(j)}}-p+j\right)\right\} \prec \frac{1+Az}{1+Bz}. \tag{1.8}$$

A majorization problem for the subclasses of analytic function has recently been investigated by Altintas et al. [3] and MacGregor [11]. In this paper we investigate majorization problem for the class $S^{m,j}_{\lambda,\ell,p}(\gamma; A, B)$ and some related subclasses.

2. Main results

Unless otherwise mentioned we shall assume throughout the paper that, $-1 \leq B < A \leq 1$, $\gamma \in \mathbb{C}^*$, $\ell, \lambda \geq 0$, $p \in \mathbb{N}$ and $m, j \in \mathbb{N}_0$.

Theorem 2.1. *Let the function $f \in A(p)$ and suppose that $g \in S^{m,j}_{\lambda,\ell,p}(\gamma; A, B)$. If $(D^m_{\lambda,\ell,p}(f*h)(z))^{(j)}$ is majorized by $(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}$ in U, then*

$$\left|(D^{m+1}_{\lambda,\ell,p}(f*h)(z))^{(j)}\right| \leq \left|(D^{m+1}_{\lambda,\ell,p}(g*h)(z))^{(j)}\right| \qquad (|z| < r_1)\,, \tag{2.1}$$

where $r_1 = r_1(p, \gamma, \lambda, \ell, A, B)$ is the smallest positive root of the equation

$$|\gamma\lambda(A-B)+(p+\ell)B|\,r^3 - [2\lambda\,|B| + (p+\ell)]\,r^2 -$$
$$[|\gamma\lambda(A-B)+(p+\ell)B| + 2\lambda]\,r + (p+\ell) = 0. \tag{2.2}$$

Proof. Since $(g*h)(z) \in S^{m,j}_{\lambda,\ell,p}(\gamma; A, B)$, we find from (1.8) that

$$1+\frac{1}{\gamma}\left(\frac{z(D^m_{\lambda,\ell,p}(g*h)(z))^{(j+1)}}{(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}}-p+j\right) = \frac{1+Aw(z)}{1+Bw(z)}, \tag{2.3}$$

where w is analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z\in U)$. From (2.3), we have

$$\frac{z(D^m_{\lambda,\ell,p}(g*h)(z))^{(j+1)}}{(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}}=\frac{(p-j)+[\gamma(A-B)+(p-j)B]w(z)}{1+Bw(z)}. \tag{2.4}$$

In view of

$$\begin{aligned}\lambda z\left(D^m_{\lambda,\ell,p}(f*g)(z)\right)^{(j+1)}&=(p+\ell)\left(D^{m+1}_{\lambda,\ell,p}(f*g)(z)\right)^{(j)}\\&-\left[p\left(1-\lambda\right)+\lambda j+\ell\right]\left(D^m_{\lambda,\ell,p}(f*g)(z)\right)^{(j)}\\&0\le j\le p;\ p\in\mathbb{N},\lambda>0;\ z\in U,\end{aligned} \tag{2.5}$$

(2.4) immediately yields the following inequality:

$$\left|(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}\right|\le\frac{(p+\ell)(1+|B|\,|z|)}{(p+\ell)-|\gamma\lambda(A-B)+(p+\ell)B|\,|z|}\left|(D^{m+1}_{\lambda,\ell,p}(g*h)(z))^{(j)}\right|. \tag{2.6}$$

Next, since $(D^m_{\lambda,\ell,p}(f*h)(z))^{(j)}$ is majorized by $(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}$ in U, from (1.2), we have

$$(D^m_{\lambda,\ell,p}(f*h)(z))^{(j)}=\varphi(z)(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}. \tag{2.7}$$

Differentiating (2.7) with respect to z, we have

$$z(D^m_{\lambda,\ell,p}(f*h)(z))^{(j+1)}=z\varphi^{'}(z)(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}+z\varphi(z)(D^m_{\lambda,\ell,p}(g*h)(z))^{(j+1)}. \tag{2.8}$$

From (2.5) and (2.8), we have

$$(D^{m+1}_{\lambda,\ell,p}(f*h)(z))^{(j)}=\frac{\lambda z}{p+\ell}\varphi^{'}(z)(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}+\varphi(z)(D^{m+1}_{\lambda,\ell,p}(g*h)(z))^{(j)}. \tag{2.9}$$

Thus, by noting that $\varphi(z)$ satisfies the inequality (see [14]),

$$\left|\varphi^{'}(z)\right|\le\frac{1-|\varphi(z)|^2}{1-|z|^2}\quad(z\in U),$$

we see that

$$\begin{aligned}&\left|(D^{m+1}_{\lambda,\ell,p}(f*h)(z))^{(j)}\right|\\&\le\left(|\varphi(z)|+\frac{1-|\varphi(z)|^2}{1-|z|^2}.\frac{\lambda\,|z|\,(1+|B|\,|z|)}{(p+\ell)-|\gamma\lambda(A-B)+(p+\ell)B|\,|z|}\right)\left|(D^{m+1}_{\lambda,\ell,p}(g*h)(z))^{(j)}\right|,\end{aligned} \tag{2.10}$$

which upon setting

$$|z|=r\quad\text{and}\quad|\varphi(z)|=\rho\quad(0\le\rho\le1),$$

leads us to the inequality

$$\begin{aligned}&\left|(D^{m+1}_{\lambda,\ell,p}(f*h)(z))^{(j)}\right|\\&\le\frac{\Theta(\rho)}{(1-r^2)((p+\ell)-|\gamma\lambda(A-B)+(p+\ell)B|\,r)}\left|(D^{m+1}_{\lambda,\ell,p}(g*h)(z))^{(j)}\right|,\end{aligned}$$

where

$$\begin{aligned}\Theta(\rho)=-r\lambda\left(1+|B|\,r\right)\rho^2+\left(1-r^2\right)\left[(p+\ell)-|\gamma\lambda(A-B)+(p+\ell)B|\,r\right]\rho\\+r\lambda\left(1+|B|\,r\right),\end{aligned} \tag{2.11}$$

takes its maximum value at $\rho = 1$, with $r_1 = r_1(p,\gamma,\lambda,\ell,A,B)$, where $r_1(p,\gamma,\lambda,\ell,A,B)$ is the smallest positive root of (2.2). Therefore the function $\Phi(\rho)$ defined by

$$\Phi(\rho) = -\sigma\lambda\,(1+|B|\,\sigma)\,\rho^2 + (1-\sigma^2)\,[(p+\ell) - |\gamma\lambda(A-B)+(p+\ell)\,B|\,\sigma]\,\rho \\ +\sigma\lambda\,(1+|B|\,\sigma) \tag{2.12}$$

is an increasing function on the interval $0 \le \rho \le 1$, so that

$$\Phi(\rho) \le \Phi(1) = (1-\sigma^2)\,[(p+\ell) - |\gamma(A-B)+(p+\ell)B|\,\sigma] \tag{2.13}$$

$$(0 \le \rho \le 1;\ 0 \le \sigma \le r_0(p,\gamma,j,A,B))\,.$$

Hence upon setting $\rho = 1$ in (2.12), we conclude that (2.1) holds true for $|z| \le r_1 = r_1(p,\gamma,\lambda,\ell,A,B)$, where $r_1(p,\gamma,\lambda,\ell,A,B)$, is the smallest positive root of (2.2). This completes the proof of Theorem 1.

Putting $A = 1$ and $B = -1$ in Theorem 1, we obtain the following result.

Corollary 2.2. *Let the function $f \in A(p)$ and suppose that $g \in S^{m,j}_{\lambda,\ell,p}(\gamma)$.*

If $(D^m_{\lambda,\ell,p}(f*h)(z))^{(j)}$ is majorized by $(D^m_{\lambda,\ell,p}(g*h)(z))^{(j)}$ in U, then

$$\left|(D^{m+1}_{\lambda,\ell,p}(f*h)(z))^{(j)}\right| \le \left|(D^{m+1}_{\lambda,\ell,p}(g*h)(z))^{(j)}\right| \qquad (|z| < r_1)\,,$$

where $r_1 = r_1(p,\gamma,\lambda,\ell)$ is given by

$$r_1 = r_1(p,\gamma,\lambda,\ell) = \frac{k - \sqrt{k^2 - 4(p+\ell)\,|2\gamma\lambda-(p+\ell)|}}{2\,|2\gamma\lambda-(p+\ell)|}, \tag{2.14}$$

where $k = 2\lambda + (p+\ell)) + |2\gamma\lambda - (p+\ell))|\,.$

Putting $A = 1$, $B = -1$ and $p = j = 1$ in Theorem 1, we obtain the following result.

Corollary 2.3. *Let the function $f \in A$ and suppose that $g \in S^{m,0}_{\lambda,\ell}(\gamma)$.*

If $(D^m_{\lambda,\ell}(f*h)(z))$ is majorized by $(D^m_{\lambda,\ell}(g*h)(z))$ in U, then

$$\left|(D^{m+1}_{\lambda,\ell}(f*h)(z))\right| \le \left|(D^{m+1}_{\lambda,\ell}(g*h)(z))\right| \qquad (|z| < r_2)\,,$$

where $r_2 = r_2(\gamma,\lambda,\ell)$ is given by

$$r_2 = r_2(\gamma,\lambda,\ell) = \frac{k - \sqrt{k^2 - 4(1+\ell)\,|2\gamma\lambda-(1+\ell)|}}{2\,|2\gamma\lambda-(1+\ell)|}, \tag{2.15}$$

where $k = 2\lambda + (1+\ell)) + |2\gamma\lambda - (1+\ell))|\,.$

Putting $A = \lambda = 1$, $B = -1, m = \ell = 0$, and $h(z) = \frac{z^p}{1-z}$ (or $c_{k+p} = 1$) in Theorem 1, we obtain the following result.

Corollary 2.4. *Let the function $f \in A(p)$* and suppose that $g \in S_p$. If $f(z)$ is majorized by $g(z)$ in U, then

$$|f'(z)| \le |g'(z)| \quad (|z| < r_3)\,,$$

where $r_3 = r_3(p,\gamma)$ is given by

$$r_3 = r_3(p;\gamma) = \frac{k - \sqrt{k^2 - 4p\,|2\gamma-p|}}{2\,|2\gamma-p|},$$

where $k = 2 + p + |2\gamma - p|\,.$

Putting $\gamma = 1$ in Corollary 3, we obtain the following result.

Corollary 2.5. *Let the function* $f \in A(p)$ and suppose that $g \in S_p(\gamma)$. If $f(z)$ is majorized by $g(z)$ in U, then

$$|f'(z)| \leq |g'(z)| \qquad (|z| < r_4),$$

where r_4 is given by

$$r_4 = r_4(p) = \frac{k - \sqrt{k^2 - 4p|2-p|}}{2|2-p|},$$

where $k = 2 + p + |2-p|$

Remarks 2.6. (i) Putting $p = 1$ in Corollary 3 we obtain the results obtained by Altintas et al. [3],

(ii) Putting $p = 1$ in Corollary 4 we obtain the results obtained by MacGregor [10].

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References

[1] Al-Oboudi, F.M., *On univalent function defined by a generalized Salagean operator*, Internat. J. Math. Math. Sci., **27**(2004), 1429-1436.

[2] Aouf, M.K., Mostaafa, A.O., *On a subclass of* $n-p-$*valent prestarlike functions*, Comput. Math. Appl., (2008), no. 55, 851-861.

[3] Altintas, O., Ozkan, O., Srivastava, H.M., *Majorization by starlike functions of complex order*, Complex Var., **46**(2001), 207-218.

[4] Catas, A., *On certain classes of p-valent functions defined by multiplier transformations*, in Proceedings of the International Symposium on Geometric Function Theory and Applications: GFTA 2007 Proceedings (Istanbul, Turkey; 20-24 August 2007) (S. Owa and Y. Polatoglu, eds.), TC Istanbul Kűltűr University Publications, Vol. 91, TC Istanbul Kűltűr University, Istanbul, Turkey, 2008, 241-250,

[5] El-Ashwah, R.M., *Majorization properties for subclass of analytic p-valent functions defined by the generalized hypergeometric function*, Tamsui. Oxf. J. Infor. Math. Scie., **28**(2012), no. 4, 395-405.

[6] El-Ashwah, R.M., Aouf, M.K., *Differential subordination and superordination for certain subclasses of p-valent functions*, Math. Comput Modelling, **51**(2010), 349-360.

[7] El-Ashwah, R.M., Aouf, M.K., *Majorization properties for subclasses of analytic p-valent functions defined by convolution*, Kyungpook Math. J., **53**(2013), 615-624.

[8] Kamali, M., Orhan, H., *On a subclass of certian starlike functions with negative coefficients*, Bull. Korean Math. Soc., **41**(2004), no. 1, 53-71.

[9] Liu, J.L., Owa, S., *Properties of certain integral operators*, Internat. J. Math. Math. Sci., **3**(2004), no. 1, 69-75.

[10] MacGregor, T.H., *Majorization by univalent functions*, Duke Math. J., **34**(1967), 95-102.

[11] Miller, S.S., Mocanu, P.T., *Differential Subordinations: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York. and Basel, 2000.

[12] Mostafa, A.O., *Differential sandwich theorem for* $p-$*valent functions related to certain operator*, Acta Univ. Apulensis, **30**(2012), 221-235.

[13] Nasr, M.A., Aouf, M.K., *Starlike function of complex order*, J. Natur. Sci. Math., **25**(1985), no. 1, 1-12.

[14] Nehari, Z., *Conformal Mapping*, MacGraw-Hill Book Company, New York, Toronto and London, 1952.

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Bounds on third Hankel determinant for certain classes of analytic functions

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Abstract. In this paper, the estimate for the third Hankel determinant $H_{3,1}(f)$ of Taylor coefficients of function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, belonging to certain classes of analytic functions in the open unit disk $\mathbb{D}$, are investigated.

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1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the class of analytic functions in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and $\mathcal{A}$ be the class of functions $f \in \mathcal{H}(\mathbb{D})$, having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}, \tag{1.1}$$

with the standard normalization $f(0) = 0$, $f'(0) = 1$. We denote by $\mathcal{S}$, the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{D}$, and $\mathcal{P}$ denotes the class of functions $p \in \mathcal{H}(\mathbb{D})$ with $\Re(p(z)) > 0$, $z \in \mathbb{D}$.

A function $f \in \mathcal{A}$ is called starlike (with respect to origin 0), if f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is a starlike domain. We denote this class of starlike functions by $\mathcal{S}^*$. A function $f \in \mathcal{S}$ maps the unit disk $\mathbb{D}$ onto a convex domain is called convex function, and this class of functions is denoted by $\mathcal{K}$. Let $\mathcal{M}(\lambda)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < \lambda, \quad z \in \mathbb{D}, \tag{1.2}$$

for some $\lambda\,(\lambda > 1)$. And let $\mathcal{N}(\lambda)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ if and only if $zf'(z) \in \mathcal{M}(\lambda)$, i.e. $f(z)$ satisfy the inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \lambda, \quad z \in \mathbb{D}, \tag{1.3}$$

for some $\lambda\,(\lambda > 1)$. These classes $\mathcal{M}(\lambda)$ and $\mathcal{N}(\lambda)$ were investigated recently by Nishiwaki and Owa [19] (see also [23]). For $1 < \lambda \leq 4/3$, the classes $\mathcal{M}(\lambda)$ and $\mathcal{N}(\lambda)$ were investigated by Uralegaddi *et al.* [32].

Throughout the present paper, by $\mathcal{M}$ we always mean the class of functions $\mathcal{M}(3/2)$, and by $\mathcal{N}$ we always mean the class of functions $\mathcal{N}(3/2)$. Ozaki [24] proved that functions in $\mathcal{N}$ are univalent in $\mathbb{D}$. Moreover, if $f \in \mathcal{N}$, then (see e.g. [11, Theorem 1] and [21, p. 196]) one have

$$\frac{zf'(z)}{f(z)} \prec g(z) = \frac{2(1-z)}{2-z}, \quad z \in \mathbb{D},$$

where $\prec$ denotes the subordination [18]. We see that g above is univalent in $\mathbb{D}$ and maps $\mathbb{D}$ onto the disk $|w - (2/3)| < 2/3$. Thus, functions in $\mathcal{M}$ are starlike in $\mathbb{D}$.

For $f \in \mathcal{A}$ of the form (1.1), a classical problem settled by Fekete and Szegö [9] is to find the maximum value of the coefficient functional $\Phi_\lambda(f) := a_3 - \lambda a_2^2$ for each $\lambda \in [0,1]$, over the function $f \in \mathcal{S}$. By applying the Löewner method they proved that

$$\max_{f\in\mathcal{S}} |\Phi_\lambda(f)| = \begin{cases} 1 + 2\exp\left(\frac{-2\lambda}{1-\lambda}\right), & \lambda \in [0,1), \\ 1, & \lambda = 1. \end{cases}$$

The problem of calculating the maximum of the coefficient functional $\Phi_\lambda(f)$ for various compact subfamilies of $\mathcal{A}$, as well as λ being an arbitrary real or complex number, has been studied by many authors (see e.g. [1, 12, 13, 17, 30, 31]).

We denote by $H_{q,n}(f)$ where n, $q \in \mathbb{N} = \{1, 2, \cdots\}$, the Hankel determinant of functions $f \in \mathcal{A}$ of the form (1.1), which is defined by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix} \quad (a_1 = 1). \tag{1.4}$$

The Hankel determinant $H_{q,n}(f)$ has been studied by several authors including Cantor [6], Noonan and Thomas [20], Pommerenke [26, 25], Hayman [10], Ehrenborg [8], which are useful, in showing that a function of bounded characteristic in $\mathbb{D}$.

Indeed, $H_{2,1}(f) = \Phi_1(f)$ is the Fekete-Szegö coefficient functional. Many authors have studied the problem of calculating $\max\limits_{f\in\mathcal{F}} |H_{2,2}(f)|$ for various subfamily $\mathcal{F}$ of the class $f \in \mathcal{A}$ (see e.g. [2, 4, 14]). Recently, several authors including Babalola [3], Bansal *et al.* [5], Prajapat *et al.* [28], Raza and Malik [29] have obtained the bounds on the third Hankel determinant $H_{3,1}(f)$ for certain families of analytic functions, which is defined by

$$\begin{aligned} H_{3,1}(f) &= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \\ &= a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2). \end{aligned} \tag{1.5}$$

In the present paper, we investigate the bounds on $H_{3,1}(f)$ for the functions belonging to the classes $\mathcal{M}$ and $\mathcal{N}$ defined above. In order to get the main results, we need the following known results.

Lemma 1.1. ([16]) *If $p \in \mathcal{P}$ be of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, then*

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

and

$$4c_3 = c_1^3 + 2c_1x(4 - c_1^2) - c_1x^2(4 - c_1^2) + 2(4 - c_1^2)(1 - |x|^2)z,$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

Lemma 1.2. ([22, Theorem 1]) *If $f \in \mathcal{N}$ be given by* (1.1), *then*

$$|a_n| \leq \frac{1}{n(n-1)}, \quad n \geq 2.$$

The result is sharp for the function f_n such that $f_n'(z) = (1 - z^{n-1})^{1/(n-1)}$, $n \geq 2$.

As it is known that, if $f(z) \in \mathcal{N}$ then $zf'(z) \in \mathcal{M}$, therefore from Lemma 1.2, we conclude that

Lemma 1.3. *If $f(z) \in \mathcal{M}$ be given by* (1.1), *then*

$$|a_n| \leq \frac{1}{n-1}, \quad n \geq 2.$$

The result is sharp for the function $g_n(z) = z(1 - z^{n-1})^{1/(n-1)}$, $n \geq 2$.

Lemma 1.4. ([22, Corollary 2]) *If $f \in \mathcal{N}$ be given by* (1.1), *then*

$$|a_3 - a_2^2| \leq 1/4.$$

Equality is attained for the function f such that $f'(z) = (1 - z^2e^{i\theta})^{1/2}$, $\theta \in [0, 2\pi]$.

2. Main results

Our first main result is contained in the following theorem:

Theorem 2.1. *Let the function $f \in \mathcal{M}$ be given by* (1.1), *then*

$$|a_3 - a_2^2| \leq 1. \tag{2.1}$$

The result (2.1) *is sharp and equality in* (2.1) *is attained for the function*

$$e_1(z) = z - z^2.$$

Proof. If the function $f \in \mathcal{M}$ be given by (1.1), then we may write

$$\frac{zf'(z)}{f(z)} = \frac{3}{2} - \frac{1}{2}p(z), \tag{2.2}$$

where $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ is analytic in $\mathbb{D}$ and $\Re(p(z)) > 0$ in $\mathbb{D}$. Also, we have $|c_n| \leq 2$ for all $n \geq 1$ (see [7]). In terms of power series expansion, the last identity is equivalent to

$$\sum_{n=1}^{\infty} na_n z^n = \left(1 - \frac{1}{2}\sum_{n=1}^{\infty} c_n z^n\right)\left(\sum_{n=1}^{\infty} a_n z^n\right),$$

where $a_1 = 1$. Equating the coefficients of z^n on both sides, we deduce that

$$a_2 = -\frac{1}{2}c_1, \quad a_3 = \frac{1}{8}(c_1^2 - 2c_2), \quad a_4 = \frac{1}{48}(6c_1c_2 - 8c_3 - c_1^3). \tag{2.3}$$

Now using Lemma 1.1 for some x such that $|x| \leq 1$, we have

$$|a_3 - a_2^2| = \left|\frac{1}{8}(c_1^2 - 2c_2) - \frac{1}{4}c_1^2\right| = \frac{1}{8}|2c_1^2 + x(4 - c_1^2)|.$$

As $|c_1| \leq 2$, taking $c_1 = c$, assume without restriction that $c \in [0, 2]$. Hence applying the triangle inequality with $\mu = |x|$, we obtain

$$\begin{aligned} |a_3 - a_2^2| &\leq \frac{1}{8}[2c^2 + \mu(4 - c^2)] \\ &= F_1(c, \mu). \end{aligned}$$

Let $\Omega = \{(c, \mu) : 0 \leq c \leq 2 \text{ and } 0 \leq \mu \leq 1\}$. Differentiating F_1 with respect to μ, we get

$$\frac{\partial F_1}{\partial \mu} = \frac{1}{8}(4 - c^2) \geq 0 \quad \text{for} \quad 0 \leq \mu \leq 1.$$

Therefore $F_1(c, \mu)$ is a non-decreasing function of μ on the closed interval $[0, 1]$. Thus, it attains maximum value at $\mu = 1$. Let

$$\max_{0 \leq \mu \leq 1} F_1(c, \mu) = F_1(c, 1) = \frac{c^2 + 4}{8} = G_1(c).$$

We observe that $G_1(c)$ is an increasing function in $[0, 2]$, so it will attains maximum value at $c = 2$. Next, to find the critical point on the boundary of Ω, we examine all the four line segments of Ω. Along the line segment $c = 2$ with $0 \leq \mu \leq 1$, we have $F_1(c, \mu) = F_1(2, \mu) = 1$, which is a constant, thus every point on the line segment is the critical point. For the line segment $c = 0$ with $0 \leq \mu \leq 1$, we have $F_1(c, \mu) = F_1(0, \mu) = \mu/2$. For the line segment $\mu = 0$ with $0 \leq c \leq 2$, we have $F_1(c, \mu) = F_1(c, 0) = c^2/4$, which gives the critical point $(0, 0)$ and $F_1(0, 0) = 0$. Also, for the line segment $\mu = 1$ with $0 \leq c \leq 2$, we have $F_1(c, \mu) = F_1(c, 1) = (c^2 + 4)/8$, which gives another critical point $(0, 1)$ and $F_1(0, 1) = 1/2$.

Putting this all together we can conclude that the maximum of $F_1(c, \mu)$ lie at each point along the line segment $c = 2$ with $0 \leq \mu \leq 1$, which can also be verified through the mathematica plot of $F_1(c,\mu)$ over the region Ω given below in the Figure 1. Hence

$$\max_{\Omega} F_1(c,\mu) = F_1(2,\mu) = 1.$$

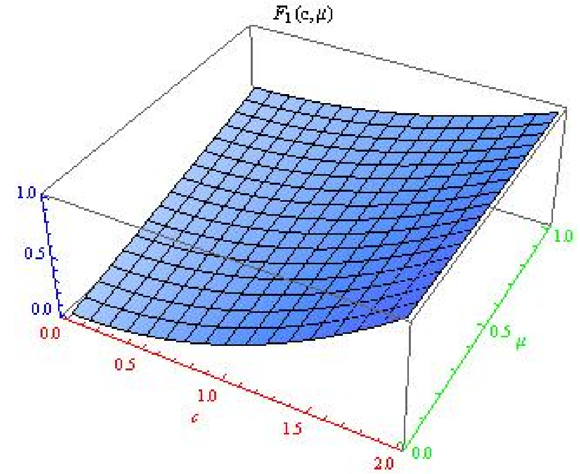

FIGURE 1. Mapping of $F_1(c,\mu)$ over Ω

To find the extremal function, setting $c_1 = 2$ and $x = 1$ in Lemma 1.1, we find that $c_2 = c_3 = 2$, using these values in (2.3), we get that $a_2 = -1$ and $a_3 = a_4 = 0$, therefore the extremal function would be $e_1(z) = z - z^2$. A simple calculation shows that $e_1(z) \in \mathcal{M}$. This complete the proof of Theorem 2.1. □

Theorem 2.2. *Let the function $f \in \mathcal{M}$ be given by* (1.1)*, then*

$$|a_2a_4 - a_3^2| \le \frac{1}{4}. \tag{2.4}$$

The result (2.4) *is sharp and equality is attained for the function*

$$e_2(z) = z - \frac{1}{2}z^3 \text{ and } e_3(z) = z(1-z^2)^{1/2}.$$

Proof. Using (2.3) and applying Lemma 1.1 for some x and z such that $|x| \le 1$ and $|z| \le 1$, we have

$$\begin{aligned} |a_2a_4 - a_3^2| &= \left| -\frac{1}{96}c_1(6c_1c_2 - 8c_3 - c_1^3) - \frac{1}{64}\left(c_1^2 - 2c_2\right)^2 \right| \\ &= \frac{1}{192}\left| -3x^2(4-c_1^2)^2 + 2c_1^2x(4-c_1^2) - 4c_1^2x^2(4-c_1^2) \right. \\ &\quad \left. +8c_1(4-c_1^2)(1-|x|^2)z \right|. \end{aligned}$$

As $|c_1| \leq 2$, taking $c_1 = c$, assume without restriction that $c \in [0, 2]$. Thus applying the triangle inequality with $\mu = |x|$, we obtain

$$\begin{aligned} |a_2a_4 - a_3^2| &\leq \frac{1}{192}\left[(4-c^2)\{3\mu^2(4-c^2) + 2c^2\mu + 4\mu^2c^2 + 8c(1-\mu^2)\}\right] \\ &= \frac{1}{192}\left[(4-c^2)\{(12-8c+c^2)\mu^2 + 2c^2\mu + 8c\}\right] \\ &= F_2(c,\mu). \end{aligned}$$

Differentiating $F_2(c, \mu)$ in the above equation with respect to μ, we get

$$\frac{\partial F_2}{\partial \mu} = \frac{(4-c^2)}{96}\left\{(12-8c+c^2)\mu + c^2\right\} \geq 0 \quad \text{for} \quad 0 \leq \mu \leq 1.$$

Therefore $F_2(c, \mu)$ is a non-decreasing function of μ on closed interval $[0, 1]$. Thus, it attains maximum value at $\mu = 1$. Let

$$\max_{0\leq\mu\leq 1} F_2(c,\mu) = F_2(c,1) = \frac{16-c^4}{64} = G_2(c).$$

We observe that $G_2(c)$ is a decreasing function in $[0, 2]$, so it will attains maximum value at $c = 0$. Next, to find the critical point on the boundary of Ω, we examine all the four line segments of Ω by the earlier method used in Theorem 2.1, and we are getting $(0, 0)$, $(2/\sqrt{3}, 0)$ and $(0, 1)$ are the critical points and $F_2(0, 0) = 0$, $F_2(2/\sqrt{3}, 0) = 2/9\sqrt{3}$ *and* $F_2(0, 1) = 1/4$. Therefore maximum value of $F_2(c, \mu)$ is obtained by putting $c = 0$ and $\mu = 1$, which can also verified through the mathematica plot of $F_2(c, \mu)$ over Ω given below in Figure 2. Hence

$$\max_{\Omega} F_2(c,\mu) = F_2(0,1) = \frac{1}{4}.$$

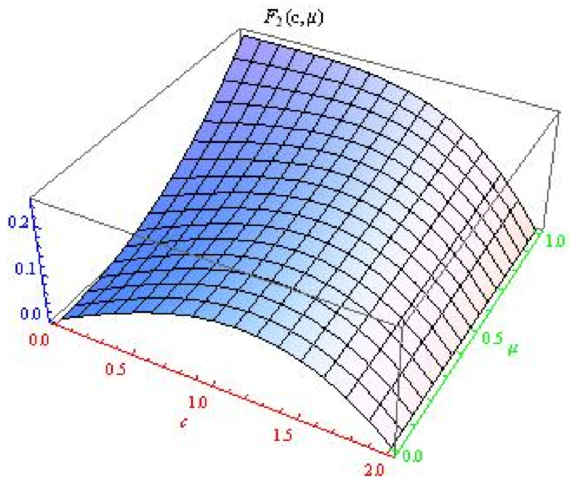

FIGURE 2. Mapping of $F_2(c, \mu)$ over Ω

Now, to find extremal function, set $c_1 = 0$ and selecting $x = 1$ in Lemma 1.1, we find that $c_2 = 2$ and $c_3 = 0$. Using these values in (2.3), we get $a_2 = a_4 = 0$ and $a_3 = 1/2$, therefore one of the extremal function of (2.4) would be $e_2(z) = z - \frac{1}{2}z^3$. We can also see that equality in (2.4) is attended for the function $e_3(z) = z(1-z^2)^{1/2} \in \mathcal{M}$. A simple calculation shows that $e_2 \in \mathcal{M}$ and $e_3 \in \mathcal{M}$. This complete the proof of Theorem 2.2. □

Theorem 2.3. *Let the function $f \in \mathcal{M}$ be given by* (1.1), *then*

$$|a_2a_3 - a_4| \leq \frac{2\sqrt{3}}{9}. \tag{2.5}$$

Proof. Using (2.3) and applying Lemma 1.1 for some x and z such that $|x| \leq 1$ and $|z| \leq 1$, we have

$$\begin{aligned} |a_2a_3 - a_4| &= \left|\frac{1}{16}c_1(c_1^2 - 2c_2) + \frac{1}{48}(6c_1c_2 - 8c_3 - c_1^3)\right| \\ &= \frac{1}{24}\left|2c_1x(4-c_1^2) - c_1x^2(4-c_1^2) + 2(4-c_1^2)(1-|x|^2)z\right|. \end{aligned}$$

As $|c_1| \leq 2$, letting $c_1 = c$, assume without restriction that $c \in [0,2]$. Hence applying the triangle inequality with $\mu = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{(4-c^2)}{24}[2 + 2c\mu + (c-2)\mu^2] \\ &= F_3(c,\mu). \end{aligned}$$

To find the maximum of F_3 over the region Ω, differentiating F_3 with respect to μ and c, we get

$$\frac{\partial F_3}{\partial \mu} = \frac{(4-c^2)}{12}\left[c + (c-2)\mu\right] \tag{2.6}$$

$$\frac{\partial F_3}{\partial c} = \frac{1}{24}\left[-4c + (8-6c^2)\mu + \left(4 + 4c - 3c^2\right)\mu^2\right]. \tag{2.7}$$

A critical point of $F_3(c,\mu)$ must satisfy $\frac{\partial F_3}{\partial \mu} = 0$ and $\frac{\partial F_3}{\partial c} = 0$. The condition $\frac{\partial F_3}{\partial \mu} = 0$ gives $c = \pm 2$ or $\mu = -c/(c-2)$. The interior point (c,μ) of Ω satisfying such condition in only $(0,0)$, and at that point $(0,0)$, we have

$$\left(\frac{\partial^2 F_3}{\partial \mu^2}\right)\left(\frac{\partial^2 F_3}{\partial c^2}\right) - \left(\frac{\partial^2 F_3}{\partial c\,\partial \mu}\right)^2 = 0.$$

Hence, it is not certain that at $(0,0)$ function have maximum value in Ω. Since Ω is closed and bounded and F_3 is continuous, the maximum of F_3 shall be attained on the boundary of Ω. Along the line segment $c = 2$ with $0 \leq \mu \leq 1$, we have $F_3(c,\mu) = F_3(2,\mu) = 0$, which is a constant. For the line segment $c = 0$ with $0 \leq \mu \leq 1$, we have $F_3(c,\mu) = F_3(0,\mu) = (1-\mu^2)/3$, which gives the same critical point $(0,0)$ and $F_3(0,0) = 1/3$. For the line segment $\mu = 0$ with $0 \leq c \leq 2$, we have $F_3(c,\mu) = F_3(c,0) = (4-c^2)/12$, which gives the same critical point $(0,0)$. Also, for the line segment $\mu = 1$ with $0 \leq c \leq 2$, we have $F_3(c,\mu) = F_3(c,1) = (4c - c^3)/8$, which gives another critical point $(2/\sqrt{3}, 1)$ on this line and $F_3(2/\sqrt{3}, 1) = 2\sqrt{3}/9$. Therefore, the point $(0, 0)$ and $(2/\sqrt{3}, 1)$ are the only critical points of F_3 over Ω. Hence, the largest value of $F_3(c, \mu)$ over the region Ω lies at $(2/\sqrt{3}, 1)$ and

$$\max_{\Omega} F_3(c, \mu) = F_3(2/\sqrt{3}, 1) = \frac{2\sqrt{3}}{9}. \qquad \square$$

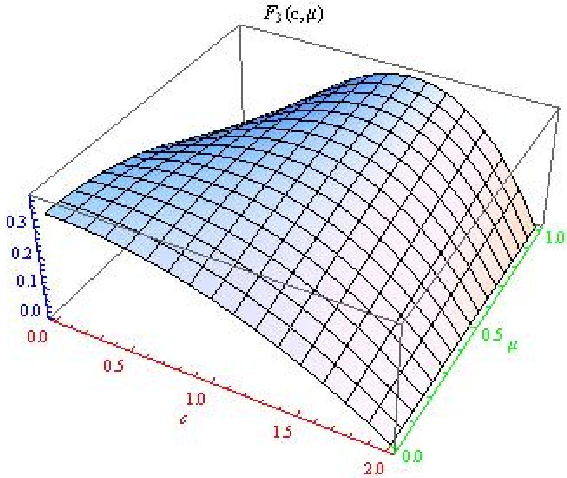

FIGURE 3. Mapping of $F_3(c, \mu)$ over Ω

Theorem 2.4. *Let the function $f \in \mathcal{M}$ be given by* (1.1), *then*

$$|H_{3,1}(f)| \leq \frac{81 + 16\sqrt{3}}{216}.$$

Proof. Using Lemma 1.3, Theorem 2.1, Theorem 2.2, Theorem 2.3 and the triangle inequality on $H_{3,1}(f)$, we get

$$\begin{aligned} |H_{3,1}(f)| &\leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2| \\ &\leq \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{2\sqrt{3}}{9} + \frac{1}{4} \cdot 1 = \frac{81 + 16\sqrt{3}}{216}. \end{aligned}$$

This completes the proof of Theorem 2.4. $\square$

Theorem 2.5. *Let the function $f \in \mathcal{N}$ be given by* (1.1), *then*

$$|a_2a_3 - a_4| \leq \frac{1}{12}. \tag{2.8}$$

The result (2.8) *is sharp and equality in* (2.8) *is attained for the function e_4 where $e_4'(z) = (1 - z^3)^{1/3}$.*

Proof. Let the function $f \in \mathcal{N}$ be given by (1.1), then by definitions it is clear that $f(z) \in \mathcal{N}$ *if and only if* $zf'(z) \in \mathcal{M}$, thus replacing a_n by na_n in (2.3), we get

$$a_2 = -\frac{1}{4}c_1, \quad a_3 = \frac{1}{24}(c_1^2 - 2c_2), \quad a_4 = \frac{1}{192}(6c_1c_2 - 8c_3 - c_1^3). \tag{2.9}$$

Now using (2.9) and applying Lemma 1.1 for some x and z such that $|x| \leq 1$ and $|z| \leq 1$, we have

$$\begin{aligned} |a_2a_3 - a_4| &= \left| -\frac{1}{96}c_1(c_1^2 - 2c_2) - \frac{1}{192}(6c_1c_2 - 8c_3 - c_1^3) \right| \\ &= \frac{1}{192}\left|3c_1x(4 - c_1^2) - 2c_1x^2(4 - c_1^2) + 4(4 - c_1^2)(1 - |x|^2)z\right|. \end{aligned}$$

As $|c_1| \leq 2$, taking $c_1 = c$, assume without restriction that $c \in [0, 2]$. Hence applying the triangle inequality with $\mu = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{(4 - c^2)}{192}[4 + 3c\mu + 2(c - 2)\mu^2] \\ &= F_4(c, \mu). \end{aligned}$$

Following the earlier method used in Theorem 2.3, we can show that the global maximum of $F_4(c, \mu)$ over the region Ω is achieved at $(0, 0)$ and $F_4(0, 0) = 1/12$. This can also be verified through the mathematica plot of $F_4(c, \mu)$ over Ω given below in Figure 4.

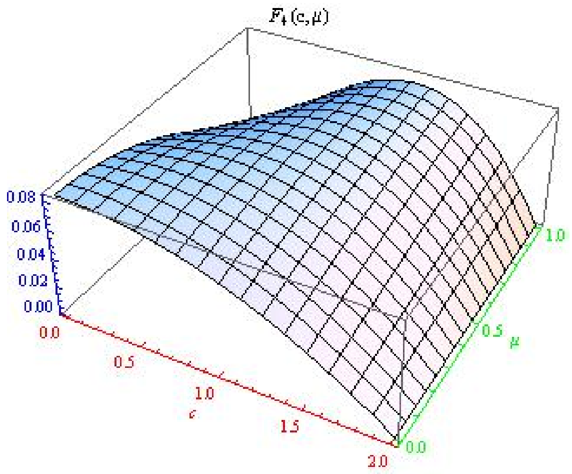

FIGURE 4. Mapping of $F_4(c, \mu)$ over Ω

Also observe that equality in (2.8) is attained for the function e_4 where

$$e_4'(z) = (1 - z^3)^{1/3}.$$

A computation shows that $e_4 \in \mathcal{N}$. Hence the result is obtained. $\square$

Theorem 2.6. *Let the function $f \in \mathcal{N}$ be given by* (1.1), *then*

$$|a_2a_4 - a_3^2| \leq \frac{9}{320}. \tag{2.10}$$

Proof. Using (2.9) and applying Lemma 1.1 for some x and z such that $|x| \leq 1$ and $|z| \leq 1$, we have

$$\begin{aligned}
|a_2a_4 - a_3^2| &= \frac{1}{192}\left|-\frac{1}{4}c_1(6c_1c_2 - 8c_3 - c_1^3) - \frac{1}{3}\left(c_1^2 - 2c_2\right)^2\right| \\
&= \frac{1}{192}\left|\frac{1}{12}c_1^4 + \frac{1}{6}c_1^2c_2 + \frac{4}{3}c_2^2 - 2c_1c_3\right| \\
&= \frac{1}{2304}\left|3xc_1^2(4-c_1^2) - 6x^2c_1^2(4-c_1^2) + 12zc_1(4-c_1^2)(1-|x|^2)\right. \\
&\quad \left. -4(4-c_1^2)^2x^2\right|.
\end{aligned}$$

As $|c_1| \leq 2$, taking $c_1 = c$, assume without restriction that $c \in [0, 2]$. Thus applying the triangle inequality with $\mu = |x|$, we obtain

$$\begin{aligned}
|a_2a_4 - a_3^2| &\leq \frac{(4-c^2)}{2304}\left\{12c + 3c^2\mu + 2(8 - 6c + c^2)\mu^2\right\} \\
&= F_5(c, \mu).
\end{aligned}$$

Differentiating $F_5(c, \mu)$ with respect to μ, we get

$$\frac{\partial F_5}{\partial \mu} = \frac{(4-c^2)}{2304}\left\{4\mu(c^2 - 6c + 8) + 3c^2\right\} \geq 0 \quad \text{for} \quad 0 \leq \mu \leq 1.$$

Therefore $F_5(c, \mu)$ is a non-decreasing function of μ on closed interval $[0, 1]$. Thus, it attains maximum value at $\mu = 1$. Let

$$\max_{0 \leq \mu \leq 1} F_5(c, \mu) = F_5(c, 1) = \frac{1}{2304}(64 + 4c^2 - 5c^4) = G_5(c).$$

We can see that $G_5(c)$ is an increasing function in $[0, \sqrt{2/5}]$, so $G_5(c)$ attains maximum value at $c = \sqrt{2/5}$. Next, to find the critical points on the boundary of Ω, we examine all the four line segments of Ω by the earlier method used in Theorem 2.1 and 2.3, and we get $(0, 0)$, $(2/\sqrt{3}, 0)$ and $(0, 1)$ are the critical points and $F_5(0, 0) = 0$, $F_5(2/\sqrt{3}, 0) = 1/36\sqrt{3}$ and $F_5(0, 1) = 1/36$. Therefore $F_5(c, \mu)$ have maximum value at $\mu = 1$ and $c = \sqrt{2/5}$ in the region Ω. Thus

$$\max_{\Omega} F_5(c, \mu) = F_5(\sqrt{2/5}, 1) = \frac{9}{320}.$$

This completes the proof of Theorem 2.6. □

Remark 2.7. For $f \in \mathcal{S}$, Thomas [27, p. 166] conjectured that

$$|H_{2,n}(f)| = |a_na_{n+2} - a_{n+1}^2| \leq 1, \quad n = 2, 3, 4 \cdots .$$

Subsequently, Li and Srivastava [15, p. 1040] shown that this conjecture is not valid for $n \geq 4$, *i.e.* conjecture is valid only for $n = 2, 3$. From Theorem 2.6, we found that, if function f is member of class $\mathcal{N}$ and having form (1.1), then $|H_{2,2}(f)| \leq 9/320$.

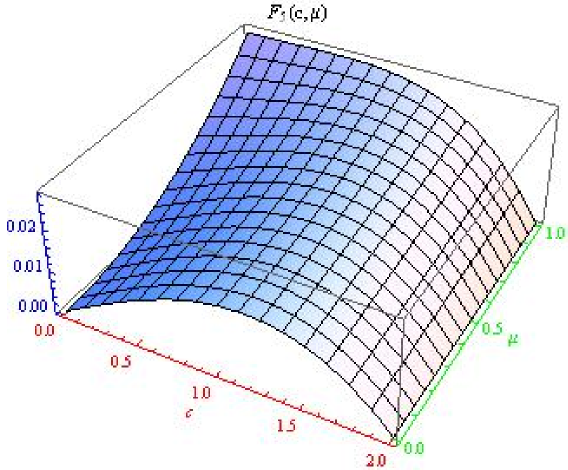

FIGURE 5. Mapping of $F_5(c,\mu)$ over Ω

Since all functions in $\mathcal{N}$ are univalent in $\mathbb{D}$. Therefore, Theorem 2.6 validates the Thomas conjecture when $n = 2$ for the function belonging to the classes $\mathcal{N}$.

Theorem 2.8. *Let the function $f \in \mathcal{N}$ be given by* (1.1)*, then*

$$|H_{3,1}(f)| \leq \frac{139}{5760}.$$

Proof. Using Lemma 1.2, Lemma 1.4, Theorem 2.5, Theorem 2.6 and the triangle inequality on $H_{3,1}(f)$, we get

$$\begin{aligned} |H_{3,1}(f)| &\leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2| \\ &\leq \frac{1}{6}\frac{9}{320} + \frac{1}{12}\frac{1}{12} + \frac{1}{20}\frac{1}{4} = \frac{139}{5760}. \end{aligned}$$

This completes the proof of Theorem 2.8. □

References

[1] Abdet Gawad, H.R., Thomas, D.K., *The Fekete-Szegö problem for strongly close-to-convex functions*, Proc. Amer. Math. Soc., **114**(1992), 345-349.

[2] Al-Abbadi, M.H., Darus, M., *Hankel determinant for certain class of analytic function defined by generalized derivative operators*, Tamkang J. Math., **43**(2012), 445-453.

[3] Babalola, K.O., *On third order Hankel determinant for some classes of univalent functions*, Inequal. Theory Appl., **6**(2010), 1-7.

[4] Bansal, D., *Upper bound of second Hankel determinant for a new class of analytic functions*, Appl. Math. Lett., **26**(2013), 103-107.

[5] Bansal, D., Maharana, S., Prajapat, J.K., *Third order Hankel determinant for certain univalent functions*, J. Korean Math. Soc., **52**(2015), no. 6, 1139-1148.

[6] Cantor, D.G., *Power series with the integral coefficients*, Bull. Amer. Math. Soc., **69**(1963), 362-366.

[7] Duren, P.L., *Univalent Functions*, Springer Verlag, New Yark Inc. 1983.

[8] Ehrenborg, R., *The Hankel determinant of exponential polynomials*, Amer. Math. Monthly, **107**(2000), 557-560.

[9] Fekete, M., Szegö, G., *Eine Benberkung uber ungerada Schlichte funktionen*, J. Lond. Math. Soc., **8**(1933), 85-89.

[10] Hayman, W.K., *On second Hankel determinant of mean univalent functions*, Proc. Lond. Math. Soc., **18**(1968), 77-94.

[11] Jovanovic, I., Obradoviȉ, M., *A note on certain classes of univalent functions*, Filomat, **9**(1995), 69-72.

[12] Keogh, F.R., Merkes, E.P., *A Coefficient Inequality for Certain Classes of Analytic Functions*, Proc. Amer. Math. Soc., **20**(1969), 8-12.

[13] Koepf, W., *On the Fekete-Szegö problem for close-to-convex functions*, Proc. Amer. Math. Soc., **101**(1987), 85-95.

[14] Lee, S.K., Ravichandran, V., Supramaniam, S., *Bounds for the second Hankel determinant of certain univalent functions*, J. Inequal. Appl., **2013**(2013), art. 281.

[15] Li, J.L., Srivastava, H.M., *Some questions and conjectures in the theory of univalent functions*, Rocky Mountain J. Math., **28**(1998), 1035-1041.

[16] Libera, R.J., Zlotkiewicz, E.J., *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc., **85**(1982), 225-230.

[17] London, R.R., *Fekete-Szegö inequalities for close-to-convex functions*, Proc. Amer. Math. Soc., **117**(1993), 947-950.

[18] Miller, S.S., Mocanu, P.T., *Differential Subordinations. Theory and Applications*, New York and Basel, Marcel Dekker, 2000.

[19] Nishiwaki, J., Owa, S., *Coefficient inequalities for certain analytic functions*, Int. J. Math. Math. Sci., **29**(2002), 285-290.

[20] Noonan, J.W., Thomas, D.K., *On the second Hankel determinant of areally mean p-valent functions*, Trans. Amer. Math. Soc., **223**(1976), 337-346.

[21] Obradović, M., Ponnusamy, S., *Injectivity and starlikeness of section of a class of univalent functions*, Contemporary Math., **591**(2013), 195-203.

[22] Obradović, M., Ponnusamy, S., Wirths, K.-J., *Coefficient characterizations and sections for some univalent functions*, Sib. Math. J., **54**(2013), 679-696.

[23] Owa, S., Srivastava, H.M., *Some generalized convolution properties associated with certain subclasses of analytic functions*, J. Inequal. Pure Appl. Math., **3**(2002), no. 3, art. 42, 13 pages.

[24] Ozaki, S., *On the theory of multivalent functions II*, Sci. Rep. Tokyo Bunrika Daigaku. Sect A, **4**(1941), 45-87.

[25] Pommerenke, C., *On the coefficients and Hankel determinant of univalent functions*, J. Lond. Math. Soc., **41**(1966), 111-122.

[26] Pommerenke, C., *On the Hankel determinant of univalent functions*, Mathematika, **14**(1967), 108-112.

[27] Parvatham, R., Ponnusamy, S., (eds.) *New trends in geometric function theory and application*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1981.

[28] Prajapat, J.K., Bansal, D., Singh, A., Mishra, A.K., *Bounds on third Hankel determinant for close-to-convex functions*, Acta Univ. Sapientiae Math., **7**(2015), no. 2, 210-219.

[29] Raza, M., Malik, S.N., *Upper bound of the third Hankel determinant for a class of analytic functions related with Lemniscate of Bernoulli*, J. Inequal. Appl., **2013**(2013), art. 412.

[30] Shanmugam, T.N., Ramachandran, C., Ravichandran, V., *Fekete-Szegö problem for subclass of starlike functions with respect to symmetric points*, Bull. Korean Math. Soc., **43**(2006), no. 3, 589-598.

[31] Srivastava, H.M., Mishra, A.K., Das, M.K., *Fekete-Szegö problem for a subclass of close-to-convex functions*, Complex Var. Theory Appl., **44**(2001), 145-163.

[32] Uralegaddi, B.A., Ganigi, M.D., Sarangi, S.M., *Univalent functions with positive coefficients*, Tamkang J. Math., **25**(1994), 225-230.

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Schwarzian derivative and Janowski convexity

Nisha Bohra and V. Ravichandran

Abstract. Sufficient conditions relating the Schwarzian derivative to the Janowski convexity of a normalized analytic function f are obtained. As a consequence, sufficient conditions are determined for the function f to be Janowski convex and convex of order α. Also, some equivalent sharp inequalities are proved for f to be Janowski convex.

Mathematics Subject Classification (2010): 30C45, 30C80.

Keywords: Schwarzian derivative, Janowski convexity, subordination.

1. Introduction and main results

Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}$ be the class of univalent functions in $\mathcal{A}$. An analytic function f is subordinate to an analytic function g, written as $f(z) \prec g(z)$, provided there is an analytic function w defined on $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. For $-1 \leq B < A \leq 1$, let $\mathcal{P}[A, B]$ be the class consisting of normalized analytic functions $p(z) = 1 + c_1 z + \cdots$ in $\mathbb{D}$ satisfying

$$p(z) \prec \frac{1 + Az}{1 + Bz}.$$

The class $K[A, B]$ of Janowski convex functions [2] consists of functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}[A, B].$$

For $0 \leq \alpha < 1$, $K[1 - 2\alpha, -1] \equiv K(\alpha)$ is the usual class of convex functions of order α. For $f \in \mathcal{S}$, the Schwarzian derivative of f is defined as

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

The Schwarzian derivative has the property that it is invariant with respect to Möbius transformations, that is, $S(Mof,z) \equiv S(f,z)$ for any Mobius transformation $M(z)$, and $S(M,z) \equiv 0$ if and only if $M(z)$ is a Möbius transformation. There are several sufficient conditions relating the Schwarzian derivative to the univalency of f (see [5] and [6]). Miller and Mocanu in [3] determined sufficient conditions relating the Schwarzian derivative to the convexity of f. In this paper, we find the sufficient conditions for Janowski convexity of f. Also, Harmelin in [1] derived sharp bounds for $|(1-|z|)^2 f''(z)/f'(z) - 2\overline{z}|$ and for $(1-|z|^2)^2|S_f(z)|$, obtaining the refinement of Nehari's result [7] for convex functions of order α. Here, we further extend this result for the class $K[A,B]$ of Janowski convex functions. Our first result gives a general condition for a function to be Janowski convex.

Theorem 1.1. *Let* $\Phi : \mathbb{C}^2 \to \mathbb{C}$ *satisfy* $\operatorname{Re}\, \Phi\left(\dfrac{(1+A)\rho i + (1-A)}{(1+B)\rho i + (1-B)}, \tau + i\eta\right) \leq 0$ *when* ρ, τ, $\eta \in \mathbb{R}$ *and*

$$2\tau(\rho^2(1+B)^2 + (1-B)^2)^2 + (A^2-B^2)(\rho^2(1+B) - (1-B))^2 - 4\rho^2(A^2-B^2) \leq 0. \tag{1.1}$$

Let $f \in \mathcal{A}$ *with* $f'(z) \neq 0$ *and* $(A-B)f'(z) - (1+B)zf''(z) \neq 0$. *If*

$$\operatorname{Re}\Phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2 S_f(z)\right) > 0, \tag{1.2}$$

where $z \in \mathbb{D}$, *then* $f \in K[A,B]$.

Remark 1.2. For $A = 1$ and $B = -1$, Theorem 1.1 reduces to [4, Theorem 4.6 b.].

Remark 1.3. The following functions satisfies the condition (1.1) of Theorem 1.1.

(1) $\Phi_1(u,v) = (A+B)(u-1)^2 + 2(A-B)v$,
(2) $\Phi_2(u,v) = 2(A-B)v - (A+B)(\operatorname{Im} u)^2$.

Thus, we have the following.

Corollary 1.4. *Let* $f \in \mathcal{A}$ *with* $f'(z) \neq 0$ *and* $(A-B)f'(z) - (1+B)zf''(z) \neq 0$. *Then each of the following is a sufficient condition for* f *to be in* $K[A,B]$.

(1) $\operatorname{Re}\left((A+B)\left(\dfrac{zf''(z)}{f'(z)}\right)^2 + 2(A-B)z^2 S_f(z)\right) > 0,$

(2) $\operatorname{Re}\left(2(A-B)z^2 S_f(z) - (A+B)\left(\operatorname{Im}\left(1 + \dfrac{zf''(z)}{f'(z)}\right)\right)^2\right) > 0.$

For $A = 1 - 2\alpha$, $B = -1$, Theorem 1.1 gives the following sufficient condition for a function $f \in \mathcal{A}$ to be convex of order α.

Corollary 1.5. *Let* $\Phi : \mathbb{C}^2 \to \mathbb{C}$ *satisfy* $\operatorname{Re}\, \Phi((1-\alpha)\rho i + \alpha, \tau + i\eta) \leq 0$ *when* ρ, τ, $\eta \in \mathbb{R}$, $0 \leq \alpha < 1$ *and*

$$2\tau - \alpha(1-\alpha)(1-\rho^2) \leq 0. \tag{1.3}$$

Let $f \in \mathcal{A}$ *with* $f'(z) \neq 0$. *If*

$$\operatorname{Re}\Phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2 S_f(z)\right) > 0, \quad \textit{where} \quad z \in \mathbb{D},$$

then $f \in K(\alpha)$.

Remark 1.6. The following functions satisfies the condition (1.3) of Corollary 1.5.

(1) $\Phi_1(u,v) = 2v - \alpha$,
(2) $\Phi_2(u,v) = 2v + u^2 - \alpha$,
(3) $\Phi_3(u,v) = 2v(1-\alpha) - \alpha(u-1)^2$.

Corollary 1.7. *Let* $f \in \mathcal{A}$ *with* $f'(z) \neq 0$. *Then each of the following is a sufficient condition for* f *to be in* $K(\alpha)$.

(1) $\operatorname{Re}\left(2z^2 S_f(z) - \alpha\right) > 0$,
(2) $\operatorname{Re}\left(2z^2 S_f(z) + \left(1 + \dfrac{zf''(z)}{f'(z)}\right)^2 - \alpha\right) > 0$,
(3) $\operatorname{Re}\left(2(1-\alpha)z^2 S_f(z) - \alpha\left(\dfrac{zf''(z)}{f'(z)}\right)^2\right) > 0$.

The next theorem gives necessary and sufficient conditions for a function $f \in \mathcal{A}$ to be Janowski convex.

Theorem 1.8. *Let* $f \in \mathcal{A}$. *The following statements are equivalent:*

(1) $f \in K[A,B]$.
(2) $\left|2B\overline{z} + \dfrac{2(1-B^2r^2) - (1-r^2)|A+B|}{A-B}\dfrac{f''(z)}{f'(z)}\right|^2 \leq 2(2 - (1-r^2)|A+B|)$.
(3) $\left|1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{1-ABr^2}{1-B^2r^2}\right| \leq \dfrac{(A-B)r}{1-B^2r^2}$.
(4) $$\frac{2(1-B^2r^2)(1-r^2) - (1-r^2)^2|A+B|}{A-B}|S_f(z)| + \frac{1}{2}\left|2B\overline{z} + \frac{2(1-B^2r^2) - (1-r^2)|A+B|}{A-B}\frac{f''(z)}{f'(z)}\right|^2 \leq 2 - (1-r^2)|A+B|,$$
where $|z| = r < 1$.

Moreover, the inequalities (3) *and* (4) *are sharp.*

Inequalities (3) and (4) gives the following coefficient bounds.

Corollary 1.9. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K[A,B]$. *Then*

$$|a_2| \leq \frac{A-B}{2}, \qquad |a_3| \leq \frac{1}{6}(A-B)(A-B+1).$$

Moreover, the bounds are sharp.

2. Proofs of main theorems

We will use the following lemma.

Lemma 2.1. [4] *Let* $\Omega \subset \mathbb{C}$ *and* $\Psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ *satisfy* $\Psi(i\rho, \sigma; z) \notin \Omega$ *whenever* $z \in \mathbb{D}$, ρ *real and* $\sigma \leq -(1+\rho^2)/2$. *If* p *is analytic in* $\mathbb{D}$ *with* $p(0) = 1$, *and* $\Psi(p(z), zp'(z); z) \in \Omega$ *for* $z \in \mathbb{D}$, *then* $\operatorname{Re} p(z) > 0$ *in* $\mathbb{D}$.

Proof of Theorem 1.1. Let $p : \mathbb{D} \to \mathbb{C}$ be defined as

$$p(z) = \frac{(A-B)f'(z) + (1-B)zf''(z)}{(A-B)f'(z) - (1+B)zf''(z)}. \tag{2.1}$$

Then p is analytic and $p(0) = 1$. Also, a calculation using equation (2.1) shows that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{(1+A)p(z) + (1-A)}{(1+B)p(z) + (1-B)}$$

and

$$z^2 S_f(z) = \frac{(A-B)(4zp'(z) - (A+B+2)p^2(z) + 2(A+B)p(z) + 2 - B - A)}{2((1+B)p(z) + (1-B))^2}.$$

We define a transformation from $\mathbb{C}^2 \to \mathbb{C}^2$ as

$$u = \frac{(1+A)r + (1-A)}{(1+B)r + (1-B)}$$

$$v = \frac{(A-B)(4s - (A+B+2)r^2 + 2(A+B)r + 2 - B - A)}{2((1+B)r + (1-B))^2}.$$

Let $\Psi(r, s) = \Phi(u, v)$
$$= \Phi\left(\frac{(1+A)r + (1-A)}{(1+B)r + (1-B)}, \frac{(A-B)(4s - (A+B+2)r^2 + 2(A+B)r + 2 - B - A)}{2((1+B)r + (1-B))^2}\right).$$
Then

$$\Psi(p(z), zp'(z)) = \Phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2 S_f(z)\right).$$

Hence, according to (1.2), we have $\operatorname{Re} \Psi(p(z), zp'(z)) > 0$. We will use Lemma 2.1 to prove that $\operatorname{Re} p(z) > 0$.
Taking $r = i\rho$ and $s = \sigma$, we obtain

$$u = \frac{(1+A)\rho i + (1-A)}{(1+B)\rho i + (1-B)}$$

$$v = \frac{(A-B)(4\sigma + (A+B+2)\rho^2 + 2(A+B)\rho i + 2 - B - A)}{2((1+B)\rho i + (1-B))^2}.$$

The condition $\sigma \leq -(1+\rho^2)/2$ is equivalent to

$$\frac{2\tau((1-B)^2 + \rho^2(1+B)^2)^2 - 4\rho^2(1-B^2)(A^2-B^2)}{(A-B)((1-B)^2 - \rho^2(1+B)^2)} + (A+B)(1-\rho^2) \leq 0,$$

where τ is real part of v. On simplification, we have

$$\rho^4(2\tau(1+B)^4 + (A^2-B^2)(1+B)^2) + \rho^2(4\tau(1-B^2)^2 - 2(A^2-B^2)(3-B^2)) \\ +2\tau(1-B)^4 + (A^2-B^2)(1-B)^2 \leq 0,$$

which is equivalent to

$$2\tau(\rho^2(1+B)^2 + (1-B)^2)^2 + (A^2-B^2)(\rho^2(1+B) - (1-B))^2 - 4\rho^2(A^2-B^2) \leq 0.$$

Hence Re $\Phi\left(\dfrac{(1+A)\rho i+(1-A)}{(1+B)\rho i+(1-B)}, \tau+i\eta\right)$ = Re $\Phi(u,v) \le 0$ using (1.1). This gives Re $\Psi(\rho i, \sigma) \le 0$ whenever $\sigma \le -(1+\rho^2)/2$.
From Lemma 2.1, we get Re $p(z) > 0$ or equivalently

$$\frac{(A-B)f' + (1-B)zf''}{(A-B)f' - (1+B)zf''} \prec \frac{1+z}{1-z}.$$

By definition of subordination, there exists an analytic map $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ and

$$\frac{(A-B)f' + (1-B)zf''}{(A-B)f' - (1+B)zf''} = \frac{1+w(z)}{1-w(z)}.$$

A simple computation gives

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1+Aw(z)}{1+Bw(z)},$$

and hence

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz}, \quad \text{or} \quad f \in K[A,B]. \qquad \square$$

Proof of Theorem 1.8. Clearly (1) $\Leftrightarrow$ (3). We show that (1) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Rightarrow$ (1).
Let $f \in K[A,B]$. Then there exists an analytic function $w : \mathbb{D} \to \mathbb{D}$ with $|w(z)| \le |z|$ such that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1+Aw(z)}{1+Bw(z)}.$$

This gives

$$\frac{f''(z)}{f'(z)} = \frac{(A-B)\phi(z)}{1+Bz\phi(z)} \quad \text{or} \quad \phi(z) = \frac{f''(z)/f'(z)}{(A-B) - Bzf''(z)/f'(z)},$$

where $\phi(z) = w(z)/z$ is analytic and satisfies $|\phi(z)| \le 1$ in $\mathbb{D}$. A simple computation gives

$$\phi'(z) = \frac{2(A-B)S_f(z) + \left(\dfrac{f''(z)}{f'(z)}\right)^2 (A+B)}{2((A-B) - Bzf''(z)/f'(z))^2}.$$

But $|\phi'(z)| \le (1-|\phi(z)|^2)/(1-|z|^2)$ by the invariant form of Schwarz lemma, so we get

$$\frac{(1-|z|^2)}{2} \frac{\left|2(A-B)S_f(z) + \left(\dfrac{f''(z)}{f'(z)}\right)^2 (A+B)\right|}{|(A-B) - Bzf''(z)/f'(z)|^2} \le 1 - \left|\frac{f''(z)/f'(z)}{(A-B) - Bzf''(z)/f'(z)}\right|^2.$$

This gives

$$\begin{aligned}(1-|z|^2)(A-B)|S_f(z)| &- \frac{(1-|z|^2)}{2}\left|\left(\frac{f''(z)}{f'(z)}\right)^2 (A+B)\right| \\ &\le (A-B)^2 - (1-B^2|z|^2)\left|\frac{f''(z)}{f'(z)}\right|^2 - 2B(A-B)\,\mathrm{Re}\,\frac{zf''(z)}{f'(z)}.\end{aligned}$$

After simplification, we have the desired inequality given by (4).
Clearly (4) $\Rightarrow$ (2). We show that (2) $\Rightarrow$ (1). Opening the square in (2) yields

$$\left((1-B^2r^2)-\frac{(1-r^2)}{2}|A+B|\right)\left|\frac{f''(z)}{f'(z)}\right|^2+2B(A-B)\operatorname{Re}\frac{zf''(z)}{f'(z)}\leq(A-B)^2. \quad (2.2)$$

Adding and subtracting $(1-B^2)r^2\left|\frac{f''(z)}{f'(z)}\right|^2$ in the left hand side of (2.2), we get

$$(1-B^2)r^2\left|\frac{f''(z)}{f'(z)}\right|^2+\frac{(1-r^2)}{2}(2-|A+B|)\left|\frac{f''(z)}{f'(z)}\right|^2+2B(A-B)\operatorname{Re}\frac{zf''(z)}{f'(z)}\leq(A-B)^2.$$

Since $\frac{(1-r^2)}{2}(2-|A+B|)\left|\frac{f''(z)}{f'(z)}\right|^2\geq 0$ for all $z\in\mathbb{D}$, we get

$$(1-B^2)r^2\left|\frac{f''(z)}{f'(z)}\right|^2+2B(A-B)\operatorname{Re}\frac{zf''(z)}{f'(z)}\leq(A-B)^2. \quad (2.3)$$

Now, if $B\neq-1$, the above equation gives

$$r^2\left|\frac{f''(z)}{f'(z)}\right|^2+\frac{2B(A-B)}{1-B^2}\operatorname{Re}\frac{zf''(z)}{f'(z)}\leq\frac{(A-B)^2}{1-B^2}.$$

Upon simplification, we have

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{1-AB}{1-B^2}\right|\leq\frac{A-B}{1-B^2},$$

which means $f\in K[A,B]$. For $B=-1$, inequality (2.3) reduces to

$$-2(A+1)\operatorname{Re}\frac{zf''(z)}{f'(z)}\leq(A+1)^2.$$

This gives

$$1+\operatorname{Re}\frac{zf''(z)}{f'(z)}\geq\frac{1-A}{2},$$

which means $f\in K[A,-1]$.

To verify the sharpness for inequality (3), let $1+\frac{zf''(z)}{f'(z)}=\frac{1+Az}{1+Bz}$. We show that $\left|\frac{1+Az}{1+Bz}-\frac{1-ABr^2}{1-B^2r^2}\right|=\frac{(A-B)r}{1-B^2r^2}$. Let $w=\frac{1+Az}{1+Bz}$. Then

$$|w|^2-\frac{2\operatorname{Re}w(1-ABr^2)}{1-B^2r^2}=\frac{A^2r^2-1}{1-B^2r^2}.$$

Adding $\left(\frac{1-ABr^2}{1-B^2r^2}\right)^2$ both sides, we have the desired equality.
To verify the sharpness of inequality (4), we substitute

$$\frac{f''(z)}{f'(z)}=\frac{A-B}{1+Bz}\quad\text{and}\quad S_f(z)=\frac{-(A^2-B^2)}{2(1+Bz)^2}$$

in the left hand side of inequality (4). We show that

$$\frac{2(1-B^2r^2)(1-r^2)-(1-r^2)^2|A+B|}{|1+Bz|^2}|A+B|$$
$$+\left|2B\overline{z}+\frac{2(1-B^2r^2)-(1-r^2)|A+B|}{1+Bz}\right|^2-4+2(1-r^2)|A+B|=0. \qquad (2.4)$$

Simplifying the left hand side of equation (2.4), we get

$$\frac{4(1-B^2r^2)^2-2(1-B^2r^2)(1-r^2)|A+B|}{|1+Bz|^2}-4(1-B^2r^2)+2(1-r^2)|A+B|$$
$$+4B(2(1-B^2r^2)-(1-r^2)|A+B|)\operatorname{Re}\left(\frac{z}{1+Bz}\right),$$
$$=(2(1-B^2r^2)-(1-r^2)|A+B|)\left(\frac{2(1-B^2r^2)}{|1+Bz|^2}-2\right)+4B(2(1-B^2r^2)$$
$$-(1-r^2)|A+B|)\operatorname{Re}\left(\frac{z}{1+Bz}\right),$$
$$=(2(1-B^2r^2)-(1-r^2)|A+B|)\left(\frac{-4B(\operatorname{Re}z+Br^2)}{|1+Bz|^2}\right)+4B(2(1-B^2r^2)$$
$$-(1-r^2)|A+B|)\operatorname{Re}\left(\frac{z}{1+Bz}\right),$$
$$=-4B(2(1-B^2r^2)-(1-r^2)|A+B|)\operatorname{Re}\left(\frac{z}{1+Bz}\right)+4B(2(1-B^2r^2)$$
$$-(1-r^2)|A+B|)\operatorname{Re}\left(\frac{z}{1+Bz}\right),$$
$$=0.$$

This completes the proof. □

References

[1] Harmelin, R., *Locally convex functions and the Schwarzian derivative*, Israel J. Math., **67**(1989), no. 3, 367-379.

[2] Janowski, W., *Some extremal problems for certain families of analytic functions*, Ann. Polon. Math., **28**(1973), 297-326.

[3] Miller, S.S., Mocanu, P.T., *Second-order differential inequalities in the complex plane*, J. Math. Anal. Appl., **65**(1978), no. 2, 289-305.

[4] Miller, S.S., Mocanu, P.T., *Differential subordinations*, Monographs and Textbooks in Pure and Applied Mathematics, 225, Dekker, New York, 2000.

[5] Nehari, Z., *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc., **55**(1949), 545-551.

[6] Nehari, Z., *Some criteria of univalence*, Proc. Amer. Math. Soc., **5**(1954), 700-704.

[7] Nehari, Z., *A property of convex conformal maps*, J. Analyse Math., **30**(1976), 390-393.

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Admissible classes of analytic functions associated with generalized Struve functions

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Abstract. In the present paper, by considering suitable classes of admissible functions we investigate some strong differential subordination as well as superordination results for analytic functions associated with normalized form of the generalized Struve functions. As a consequence of these results, new strong differential sandwich-type results are obtained.

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1. Introduction and motivations

Denote by $\mathcal{H}(\mathbb{U})$, the class of functions which are analytic in the open unit disk

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}.$$

For $a \in \mathbb{C},\ n \in \mathbb{N} := \{1, 2, 3, \cdots\}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\},$$

with $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H} \equiv \mathcal{H}[1, 1]$.
Let $\mathcal{A}$ denote the class of all normalized analytic functions in $\mathbb{U}$ of the form:

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1} \quad (z \in \mathbb{U}). \tag{1.1}$$

For $f,\ g \in \mathcal{A}$, where f given by (1.1) and g is defined by

$$g(z) = z + \sum_{n=1}^{\infty} b_{n+1} z^{n+1} \quad (z \in \mathbb{U}),$$

the Hadamard product (or convolution) of f and g, denoted by $f * g$ is defined as

$$(f * g)(z) = z + \sum_{n=1}^{\infty} a_{n+1} b_{n+1} z^{n+1} = (g * f)(z).$$

Let $f, F \in \mathcal{H}(\mathbb{U})$. The function f is said to be subordinate to F, or equivalently F is said to be superordinate to f, if there exists a function $\omega(z)$ analytic in $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = F(w(z))$ $(z \in \mathbb{U})$. In such a case, we write $f(z) \prec F(z)$ $(z \in \mathbb{U})$. Furthermore, if the function F is univalent in $\mathbb{U}$, then $f(z) \prec F(z) \iff f(0) = F(0)$ and $f(\mathbb{U}) \subset F(U)$ (see [4]).
Now we consider the following second-order linear non-homogeneous differential equation [14, page 341]):

$$z^2 w''(z) + z w'(z) + (z^2 - p^2) w(z) = \frac{4\left(\frac{z}{2}\right)^{p+1}}{\sqrt{\pi}\,\Gamma(p + \frac{1}{2})}, \tag{1.2}$$

where $z \in \mathbb{C}$ and Γ stands for the Euler's gamma function. The solution of the homogeneous part is the Bessel's function of order p, where p is a real or complex number. The particular integral of (1.2) is called the Struve function of order p, given by

$$H_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n + \frac{3}{2})\Gamma(p + n + \frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p+1} \qquad (z \in \mathbb{C}). \tag{1.3}$$

The differential equation

$$z^2 w''(z) + z w'(z) - (z^2 + p^2) w(z) = \frac{4\left(\frac{z}{2}\right)^{p+1}}{\sqrt{\pi}\,\Gamma(p + \frac{1}{2})}, \tag{1.4}$$

differs from (1.2) only in the coefficient of w. The particular integral of (1.4) is called the modified Struve function of order p and is given by [14, page 353]:

$$\begin{aligned} L_p(z) &= -\, i e^{-\frac{ip\pi}{2}} H_p(iz) \\ &= \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n+p+1}}{\Gamma(n + \frac{3}{2})\Gamma(p + n + \frac{3}{2})} \qquad (z \in \mathbb{C}). \end{aligned} \tag{1.5}$$

Further, let us consider the second-order linear non-homogeneous differential equation of the form (see [6]):

$$z^2 w''(z) + b z w'(z) + [c z^2 - p^2 + (1-b)p] w(z) = \frac{4\left(\frac{z}{2}\right)^{p+1}}{\sqrt{(\pi)}\,\Gamma(p + \frac{b}{2})} \quad (b, c, p \in \mathbb{C}). \tag{1.6}$$

Taking $b = c = 1$ and $b = 1, c = -1$ in equation (1.6), we get (1.2) and (1.4) respectively. Thus, (1.6) is the generalizes (1.2) and (1.4). This permits us to study the Struve and modified Struve function together. The function $w_{p,b,c}(z)$, called the generalized Sturve function of order p is defined to be the particular integral of (1.6). Moreover, the function $w_{b,p,c}(z)$ has the following familiar representation:

$$w_{p,b,c}(z) := \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(n + \frac{3}{2})\Gamma(p + n + \frac{(b+2)}{2})} \left(\frac{z}{2}\right)^{2n+p+1} \qquad (z \in \mathbb{C}). \tag{1.7}$$

Note that, the series (1.7) is convergent everywhere but generally not univalent in the open unit disk $\mathbb{U}$.
Recently, Raza and Yağmur [12] (see also, [13]) considered the function $\varphi_{p,b,c}(z)$ defined in terms of the generalized Struve function $w_{p,b,c}(z)$ by the transformation

$$\begin{aligned}\varphi_{p,b,c}(z) &= 2^p\sqrt(\pi)\Gamma(p+\frac{b+2}{2})z^{\frac{-p+1}{2}}w_{p,b,c}(\sqrt{z})\\ &= z+\sum_{n=1}^{\infty}\frac{\left(\frac{-c}{4}\right)^n}{\left(\frac{3}{2}\right)_n(\nu)_n}z^{n+1}\end{aligned}$$

$$\left(\nu = p+\frac{b+2}{2}\notin\mathbb{Z}_0^- := \{0,-1,-2,\cdots\}, b,p,c\in\mathbb{C}\right),$$

where $(\lambda)_n$ denotes the Pochhammer (or Appell) symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases}1 & (n=0,\ \lambda\in\mathbb{C}\setminus\{0\})\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N},\ \lambda\in\mathbb{C}).\end{cases}$$

For convenience of notation, we write $\varphi_{\nu,c}(z) = \varphi_{p,b,c}(z)$. Now, we introduce a new operator $\mathcal{J}_\nu^c : \mathcal{A}\longrightarrow\mathcal{A}$ which is defined by means of Hadamard product as

$$\mathcal{J}_\nu^c f(z) = \varphi_{\nu,c}(z)*f(z) = z+\sum_{n=1}^{\infty}\frac{(-c)^n}{4^n\left(\frac{3}{2}\right)_n(\nu)_n}a_{n+1}z^{n+1}\quad (z\in\mathbb{U}). \tag{1.8}$$

It is easy to verify from (1.8) that

$$z(\mathcal{J}_{\nu+1}^c f(z))' = \nu\mathcal{J}_\nu^c f(z)-(\nu-1)\mathcal{J}_{\nu+1}^c f(z). \tag{1.9}$$

We need the following definitions and lemmas in order to investigate our main results.

Definition 1.1. (see [7, 8]) Let $H(z,\xi)$ be analytic in $\mathbb{U}\times\bar{\mathbb{U}}$ and let $f(z)$ be analytic and univalent in $\mathbb{U}$. Then the function $H(z,\xi)$ is said to be strongly subordinate to $f(z)$, or $f(z)$ is said to be strongly superordinate to $H(z,\xi)$, written as $H(z,\xi)\prec\prec f(z)$, if for $\xi\in\bar{\mathbb{U}}$, $H(z,\xi)$ as the function of z is subordinate to $f(z)$. We note that (see [1, 2, 11])

$$H(z,\xi)\prec\prec f(z)\quad (z\in\mathbb{U},\ \xi\in\bar{\mathbb{U}}) \Longleftrightarrow H(0,\xi)=f(0)\text{ and }H(\mathbb{U}\times\bar{\mathbb{U}})\subset f(\mathbb{U}).$$

Definition 1.2. (see [4, 11]) Let $\phi : \mathbb{C}^3\times\mathbb{U}\times\bar{\mathbb{U}}\longrightarrow\mathbb{C}$ and let $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the following (second-order) differential subordination:

$$\phi(p(z), zp'(z), z^2p''(z), z;\xi)\prec\prec h(z)\quad (z\in\mathbb{U};\xi\in\bar{\mathbb{U}}), \tag{1.10}$$

then $p(z)$ is called a solution of the strong differential subordination. The univalent function $q(z)$ is called a dominant of the solutions of the strong differential subordination or more simply a dominant, if $p(z)\prec q(z)$ $(z\in\mathbb{U})$ for all $p(z)$ satisfying (1.10). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z)\prec q(z)$ $(z\in\mathbb{U})$ for all dominants $q(z)$ of (1.10) is said to be the best dominant.

Recently, Oros [9] introduced the following notion of strong differential superordination as the dual concept of strong differential subordination.

Definition 1.3. (see [5, 9]) Let $\varphi : \mathbb{C}^3 \times \mathbb{U} \times \bar{\mathbb{U}} \longrightarrow \mathbb{C}$ and let $h(z)$ be analytic in $\mathbb{U}$. If $p(z)$ and $\varphi(p(z)zp'(z), z^2p''(z); z, \xi)$ are univalent in $\mathbb{U}$ for $\xi \in \bar{\mathbb{U}}$ and satisfy the following (second-order) strong differential superordination:

$$h(z) \prec\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi) \quad (z \in \mathbb{U},\ \xi \in \bar{\mathbb{U}}), \tag{1.11}$$

then $p(z)$ is called a solution of the strong differential superordination. An analytic function $q(z)$ is called a subordinant of the solution of the strong differential superordination or more simply a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.11). A univalent subordinant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (1.11) is said to be the best subordinant.

Denote by Q, the class of functions q that are analytic and injective on $\bar{\mathbb{U}} \setminus E(q)$, where

$$E(q) = \{\zeta \in \partial\mathbb{U} : lim_{z\longrightarrow\zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \setminus E(q)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$.

Definition 1.4. (see [11]) Let Ω be a set in $\mathbb{C}$, $q(z) \in Q$ and $n \in \mathbb{N}$. The class of admissible functions $\psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \mathbb{U} \times \bar{\mathbb{U}} \longrightarrow \mathbb{C}$ that satisfy the admissiblility condition $\psi(r, s, t; z, \xi) \notin \Omega$ whenever

$$r = q(\zeta),\ s = \alpha\zeta q'(\zeta) \text{ and } \Re\left(\frac{t}{s} + 1\right) \geq \alpha\Re\left\{1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\},$$

for $z \in \mathbb{U},\ \zeta \in \partial\mathbb{U} \setminus E(q),\ \xi \in \bar{\mathbb{U}};\ \alpha \geq n$. In particular, for $n = 1$, we write $\psi_1[\Omega, q]$ as $\psi[\Omega, q]$.

Definition 1.5. (see [9]) Let Ω be a set in $\mathbb{C}$ and $q \in H[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\psi'_n[\Omega, q]$ consists of those functions

$$\psi : \mathbb{C}^3 \times \mathbb{U} \times \bar{\mathbb{U}} \longrightarrow \mathbb{C}$$

that satisfy the admissibility condition:

$$\psi(r, s, t; \zeta, \xi) \in \Omega$$

whenever

$$r = q(z), s = \frac{zq'(z)}{m} \text{ and } \Re\left(\frac{t}{s} + 1\right) \leq \frac{1}{m}\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\},$$

for $z \in \mathbb{U},\ \xi \in \bar{\mathbb{U}},\ \zeta \in \partial\mathbb{U}$ and $m \geq n \geq 1$. In particular, for $n = 1$, we denote $\psi'_1[\Omega, q]$ as $\psi'[\Omega, q]$.

For the above two classes of admissible functions, the following results have been proved by earlier authors (see, for details [9, 11]).

Lemma 1.6. (see [11]) *Let $\psi \in \psi_n[\Omega, q]$ with $q(0) = a$. If $p \in H[a, n]$ satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z, \xi) \in \Omega,$$

then

$$p(z) \prec q(z)\ (z \in \mathbb{U}).$$

Lemma 1.7. (see [9]) *Let* $\psi \in \psi'_n[\Omega, q]$ *with* $q(0) = a$. *If the analytic function* $p(z) \in Q(a)$ *and*

$$\psi(p(z), zp'(z), z^2p''(z); z, \xi)$$

is univalent in $\mathbb{U}$ *for* $\xi \in \bar{\mathbb{U}}$, *then*

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z, \xi) : z \in \mathbb{U}, \xi \in \bar{\mathbb{U}}\}$$

implies the following subordination relationship:

$$q(z) \prec p(z) \ (z \in \mathbb{U}).$$

Results dealing with the first-order and the second-order strong differential subordination and strong differential superordination for analytic functions in the open unit disk are available in literature. In recent years, several authors obtained many interesting results involving various linear and non-linear operators associated with strong differential subordination and superordination (see [1, 3, 8, 9, 10, 11, 14]). By making use of the strong differential subordination and superordination results of Oros and Oros [9, 11], under certain classes of admissible functions we investigate some strong differential subordination and strong differential superordination results of analytic functions associated with the operator $\mathcal{J}_\nu^c$ defined by (1.8). Further, we find sufficient conditions for suitable classes of admissible functions so that

$$q_1(z) \prec \mathcal{J}_{\nu+1}^c f(z) \prec q_2(z)$$

holds true for suitable univalent functions q_1 and q_2 with $q_1(0) = q_2(z) = 0$.

2. Subordination results

We need the following class of admissible functions in order to prove the subordination results associated with the operator $\mathcal{J}_\nu^c$ defined by (1.8).

Definition 2.1. Let Ω be a set in $\mathbb{C}$, $q \in Q_0 \cap H_0$, $\Re(\nu) > 0$. The class of admissible functions $\phi_{\mathcal{J}}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \times \bar{\mathbb{U}} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z, \xi) \notin \Omega,$$

whenever

$$u = q(\zeta), \ v = \frac{\alpha\zeta q'(\zeta) + (\nu - 1)q(\zeta)}{\nu},$$

and

$$\Re\left\{\frac{\nu(\nu-1)w - (\nu-1)(\nu-2)u}{\nu v - (\nu-1)u} + (3 - 2\nu)\right\} \geq \alpha\Re\left(1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right),$$
$$(z \in \mathbb{U}; \zeta \in \partial\mathbb{U} \setminus E(q), \ \xi \in \bar{\mathbb{U}}, \alpha \geq 1).$$

Theorem 2.2. *Let* $\phi \in \phi_{\mathcal{J}}[\Omega, q]$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left\{\phi\left(\mathcal{J}_{\nu+1}^c f(z), \mathcal{J}_\nu^c f(z), \mathcal{J}_{\nu-1}^c f(z)\right) : z \in \mathbb{U}, \xi \in \bar{\mathbb{U}}\right\} \subset \Omega, \tag{2.1}$$

then

$$\mathcal{J}_{\nu+1}^c f(z) \prec q(z) \quad (z \in \mathbb{U}).$$

Proof. Define the function $p(z)$ by

$$p(z) = \mathcal{J}_{\nu+1}^{c} f(z) \quad (z \in \mathbb{U}). \tag{2.2}$$

Clearly $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 0$. Differentiating (2.2) with respect to z and making use of identity (1.9) in the resulting equation, we get

$$\mathcal{J}_{\nu}^{c} f(z) = \frac{zp'(z) + (\nu - 1)p(z)}{\nu}. \tag{2.3}$$

Further, a simple calculation shows that

$$\mathcal{J}_{\nu-1}^{c} f(z) = \frac{z^2 p''(z) + 2(\nu - 1)zp'(z) + (\nu - 1)(\nu - 2)p(z)}{\nu(\nu - 1)} \tag{2.4}$$

Now, define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \ v = \frac{(\nu - 1)r + s}{\nu}, \ w = \frac{(\nu - 1)(\nu - 2)r + 2(\nu - 1)s + t}{\nu(\nu - 1)}. \tag{2.5}$$

Let

$$\begin{aligned} \psi(r, s, t; z, \xi) &= \phi(u, v, w; z, \xi) \\ &= \phi\left(r, \frac{(\nu - 1)r + s}{\nu}, \frac{(\nu - 1)(\nu - 2)r + 2(\nu - 1)s + t}{\nu(\nu - 1)}; z, \xi\right). \end{aligned} \tag{2.6}$$

If we use equations (2.2) -(2.4), we find from (2.6) that

$$\psi(p(z), zp'(z), z^2 p''(z); z, \xi) = \phi(\mathcal{J}_{\nu+1} f(z), \mathcal{J}_{\nu}^{c} f(z), \mathcal{J}_{\nu-1}^{c}; z, \xi).$$

Hence (2.1) becomes

$$\psi(p(z), zp'(z), z^2 p''(z); z, \xi) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \phi_{\mathcal{J}}[\Omega, q]$ in Definition 2.1 is equivalent to the the admissibility condition for ψ as given in Definition 1.4.
From (2.5), it follows that

$$\frac{t}{s} + 1 = \frac{\nu(\nu - 1)w - (\nu - 1)(\nu - 2)u}{\nu v - (\nu - 1)u} + (3 - 2\nu),$$

and hence $\psi \in \psi[\Omega, q]$. By Lemma 1.6 we have

$$p(z) \prec q(z) \quad (z \in \mathbb{U}),$$

or, equivalently,

$$\mathcal{J}_{\nu+1}^{c} f(z) \prec q(z) \quad (z \in \mathbb{U}).$$

Thus, the proof of Theorem 2.2 is completed. □

Corollary 2.3. *The conclusion of Theorem* 2.2 *can be written in the generalized form as:*

$$\{\phi(\mathcal{J}_{\nu+1}^{c} f(z), \mathcal{J}_{\nu}^{c} f(z), \mathcal{J}_{\nu-1}^{c} f(z); \omega(z), \xi)\} \subset \Omega,$$

then

$$\mathcal{J}_{\nu+1}^{c} f(z) \prec q(z) \quad (z \in \mathbb{U}),$$

where $\omega(z)$ is any mapping from $\mathbb{U}$ onto $\mathbb{U}$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain and $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω, then the class $\phi_{\mathcal{J}}[h(\mathbb{U}, q)]$ is written as $\phi_{\mathcal{J}}[h, q]$.
The following result is an immediate consequence of Theorem 2.2.

Theorem 2.4. *Let $\phi \in \phi_{\mathcal{J}}[h, q]$. If $f \in \mathcal{A}$ satisfies*

$$\phi(\mathcal{J}^c_{\nu+1} f(z), \mathcal{J}^c_{\nu} f(z), \mathcal{J}^c_{\nu-1} f(z); z, \xi) \prec\prec h(z) \quad (z \in \mathbb{U}, \xi \in \bar{\mathbb{U}}), \tag{2.7}$$

then

$$\mathcal{J}_{\nu+1} f(z) \prec q(z) \quad (z \in \mathbb{U}).$$

The following result is an extension of Theorem 2.2 where the behaviour of q on $\partial\mathbb{U}$ is not known.

Corollary 2.5. *Let $\Omega \subset \mathbb{C}$ and q be univalent in $\mathbb{U}$ with $q(0) = 0$. Let $\phi \in \phi_{\mathcal{J}}[\Omega, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}$ satisfies*

$$\phi(\mathcal{J}^c_{\nu+1} f(z), \mathcal{J}^c_{\nu} f(z), \mathcal{J}^c_{\nu-1} f(z), z, \xi) \in \Omega,$$

the

$$\mathcal{J}_{\nu+1} f(z) \prec q(z) \quad (z \in \mathbb{U}).$$

Proof. From Theorem 2.2, it follows that

$$\mathcal{J}^c_{\nu+1} f(z) \prec q_\rho f(z).$$

Since $q_\rho(z) \prec q(z)$, hence the result follows. □

Theorem 2.6. *Let h and q be univalent in $\mathbb{U}$ with $q(0) = 0$. Set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times \mathbb{U} \times \bar{\mathbb{U}} \longrightarrow \mathbb{C}$ satisfies one of the following conditions:*
(i) $\phi \in \phi_{\mathcal{J}}[h, q_\rho]$ for some $\rho \in (0, 1)$ or
(ii) there exists $\rho_0 \in (0, 1)$ such that $\phi \in \phi_{\mathcal{J}}[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.
If $f \in \mathcal{A}$ satisfies (2.7), then

$$\mathcal{J}^c_{\nu+1} f(z) \prec q(z) \quad (z \in \mathbb{U}).$$

Proof. Case(i). By applying Theorem 2.2, we obtain $p(z) \prec q_\rho(z)$. Since $q_\rho \prec q$, we have $p(z) \prec q(z)$.i.e

$$\mathcal{J}^c_{\nu+1} f(z) \prec q(z) \quad (z \in \mathbb{U}).$$

Case(ii). If we let $p_\rho(z) = p(\rho z)$, then

$$\begin{aligned}
&\phi(\mathcal{J}^c_{\nu+1} f(z), \mathcal{J}^c_{\nu} f(z), \mathcal{J}^c_{\nu-1} f(z); z, \xi)\\
&= \phi\left(p_\rho(z), \tfrac{(\nu-1)p_\rho(z)+zp'_\rho(z)}{\nu}, \tfrac{z^2 p''_\rho(z)+2(\nu-1)zp'_\rho(z)+(\nu-1)(\nu-2)p_\rho(z)}{\nu(\nu-1)}; z, \xi\right)\\
&= \phi\left(p(\rho z), \tfrac{(\nu-1)p(\rho z)+zp'(\rho z)}{\nu}, \tfrac{z^2 p''(\rho z)+2(\nu-1)zp'(\rho z)+(\nu-1)(\nu-2)p(\rho z)}{\nu(\nu-1)}; \rho z, \xi\right) \in h_\rho(\mathbb{U}).
\end{aligned}$$

By making use of Corollary 2.3 with $\omega(z) = \rho z$, we obtain $p_\rho(z) \prec q_\rho(z)$ for $\rho \in (\rho_0, 1)$. By letting $\rho \longrightarrow 1$ we obtain

$$p(z) \prec q(z)$$

i.e.

$$\mathcal{J}^c_{\nu+1} f(z) \prec q(z) \quad (z \in \mathbb{U}).$$ □

Next theorem gives the best dominant of the strong differential subordination (2.7).

Theorem 2.7. *Let $h(z)$ be univalent in $\mathbb{U}$ and $\phi : \mathbb{C}^3 \times \mathbb{U} \times \bar{\mathbb{U}} \longrightarrow \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \tfrac{zq'(z)+(\nu-1)q(z)}{\nu}, \tfrac{z^2q''(z)+2(\nu-1)zq'(z)+(\nu-1)(\nu-2)q(z)}{\nu(\nu-1)}; z, \xi\right) = h(z) \tag{2.8}$$

has a solution $q(z)$ with $q(0) = 0$ and satisfies any one of the following conditions:

(i) $q \in Q_0$ and $\phi \in \phi_{\mathcal{J}}[h, q]$.

(ii) q is univalent in $\mathbb{U}$ and $\phi \in \phi_{\mathcal{J}}[h, q_\rho]$ for some $\rho \in (0, 1)$, or

(iii) q is univalent in $\mathbb{U}$ and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \phi_{\mathcal{J}}[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}$ satisfies (2.7) and $\phi(\mathcal{J}^c_{\nu+1} f(z), \mathcal{J}^c_\nu f(z), \mathcal{J}^c_{\nu-1} f(z); z, \xi)$ is analytic in $\mathbb{U}$, then

$$\mathcal{J}^c_{\nu+1} f(z) \prec q(z) \quad (z \in \mathbb{U}),$$

and q is the best dominant.

Proof. By applying Theorem 2.4 and Theorem 2.6 we deduce that q is dominant of (2.7). Since q satisfies (2.8), it is also a solution of (2.7) and therefore q will be the dominant of all dominants of (2.7). Hence q will be the best dominant of (2.7). □

In the particular case when $q(z) = Mz (M > 0)$ and in view of Definition 2.1, the class of admissible functions $\phi_{\mathcal{J}}[\Omega, q]$ denoted by $\phi_{\mathcal{J}}[\Omega, M]$ is described below.

Definition 2.8. Let Ω be a set in $\mathbb{C}$, $\nu \in \mathbb{C}$ with $\nu \neq 0, 1$ and $M > 0$. The class of admissible functions $\phi_{\mathcal{J}}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \times \bar{\mathbb{U}} \longrightarrow \mathbb{C}$ such that

$$\phi\left(Me^{i\theta}, \frac{\alpha+\nu-1}{\nu} Me^{i\theta}, \frac{L + [2(\nu-1)\alpha + (\nu-1)(\nu-2)]Me^{i\theta}}{\nu(\nu-1)}; z, \xi\right) \notin \Omega, \tag{2.9}$$

$(z \in \mathbb{U},\ \xi \in \bar{\mathbb{U}},\ \Re(Le^{-i\theta}) \geq \alpha(\alpha-1)M,\ \theta \in \mathbb{R},\ \alpha \geq 1)$.

Corollary 2.9. *Let $\phi \in \phi_{\mathcal{J}}[\Omega, M]$. If $f \in \mathbb{A}$ satisfies*

$$\phi(\mathcal{J}^c_{\nu+1} f(z), \mathcal{J}^c_\nu f(z), \mathcal{J}_{\nu-1} f(z); z, \xi) \in \Omega \quad (z \in \mathbb{U}, \xi \in \bar{\mathbb{U}}),$$

then

$$\mathcal{J}^c_{\nu+1} f(z) \prec Mz \quad (z \in \mathbb{U}).$$

In the special case $\Omega = q(\mathbb{U}) = \{w : |w| < M\}$, the class $\phi_{\mathcal{J}}[\Omega, M]$ is simply denoted by $\phi_{\mathcal{J}}[M]$.

Corollary 2.10. *Let $\phi \in \phi_{\mathcal{J}}[M]$. If $f \in \mathcal{A}$ satisfies*

$$|\phi(\mathcal{J}^c_{\nu+1} f(z), \mathcal{J}^c_\nu f(z), \mathcal{J}^c_{\nu-1} f(z); z, \xi)| < M \quad (z \in \mathbb{U}, \xi \in \bar{\mathbb{U}}),$$

then

$$|\mathcal{J}^c_{\nu+1} f(z)| < M \quad (z \in \mathbb{U}).$$

3. Superordination and sandwich-type results

In this section, strong differential superordination, the dual problem of strong differential subordination for generalized Struve function defined as (1.8) is investigated. For this purpose, we define the class of admissible functions as follows:

Definition 3.1. Let Ω be a set in $\mathbb{C}$, $q \in H[0,1]$ with $q'(z) \neq 0$, $\Re(\nu) > 0$. The class of admissible functions $\phi'_{\mathcal{J}}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \times \bar{\mathbb{U}} \longrightarrow \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta, \xi) \in \Omega,$$

whenever

$$u = q(z), v = \frac{zq'(z) + m(\nu - 1)}{m\nu} q(z),$$

and

$$\Re\left[\frac{\nu(\nu-1)w - (\nu-1)(\nu-2)u}{\nu v - (\nu-1)u} + (3 - 2\nu)\right] \leq \frac{1}{m}\Re\left(1 + \frac{zq''(z)}{q'(z)}\right)$$

$$(z \in \mathbb{U}, \xi \in \bar{\mathbb{U}}, \zeta \in \partial\mathbb{U},\ m \geq 1).$$

Theorem 3.2. *Let $\phi \in \phi'_{\mathcal{J}}[\Omega, q]$. If $f \in \mathcal{A}$, $\mathcal{J}^c_{\nu+1} f(z) \in Q_0$ and*

$$\phi(\mathcal{J}^c_{\nu+1} f(z), \mathcal{J}^c_{\nu} f(z), \mathcal{J}^c_{\nu-1} f(z); z, \xi)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\phi(\mathcal{J}^c_{\nu+1} f(z), \mathcal{J}^c_{\nu} f(z), \mathcal{J}^c_{\nu-1} f(z); z, \xi) : z \in \mathbb{U}, \xi \in \bar{\mathbb{U}}\right\} \tag{3.1}$$

implies the following subordination result holds

$$q(z) \prec \mathcal{J}^c_{\nu+1} f(z) \quad (z \in \mathbb{U}).$$

Proof. Let

$$p(z) = \mathcal{J}^c_{\nu+1} f(z).$$

Then, from (2.6) and (3.1) we obtain

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2 p''(z); z, \xi) : z \in \mathbb{U}, \xi \in \bar{\mathbb{U}}\right\}.$$

Since

$$\frac{t}{s} + 1 = \frac{\nu(\nu-1)w - (\nu-1)(\nu-2)u}{\nu v - (\nu-1)u} + (3 - 2\nu),$$

the admissibility condition for $\phi \in \phi'_{\mathcal{J}}[\Omega, q]$ in Definition 3.1 is equivalent to the admissibility condition for ψ as given in Definition 1.5. Hence $\psi \in \psi'[\Omega, q]$ and by Lemma 1.7,

$$q(z) \prec p(z) \quad (z \in \mathbb{U}),$$

or equivalently,

$$q(z) \prec \mathcal{J}^c_{\nu+1} f(z) \quad (z \in \mathbb{U}).$$ □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\phi'_{\mathcal{J}}[h(\mathbb{U}), q]$ is written as $\phi'_{\mathcal{J}}[h, q]$. The following result is the immediate consequence of Theorem 3.2.

Theorem 3.3. *Let $q \in H[0,1]$, h be analytic in $\mathbb{U}$ and $\phi \in \phi'_{\mathcal{J}}[h,q]$. If $f \in \mathcal{A}$, $\mathcal{J}^c_{\nu+1}f(z) \in Q_0$ and*

$$\phi(\mathcal{J}^c_{\nu+1}f(z), \mathcal{J}^c_{\nu}f(z), \mathcal{J}^c_{\nu-1}f(z); z, \xi)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec\prec \phi(\mathcal{J}^c_{\nu+1}f(z), \mathcal{J}^c_{\nu}f(z), \mathcal{J}^c_{\nu-1}f(z); z, \xi) \quad (z \in \mathbb{U},\ \xi \in \bar{\mathbb{U}}), \tag{3.2}$$

implies

$$q(z) \prec \mathcal{J}^c_{\nu+1}f(z) \quad (z \in \mathbb{U}).$$

Theorem 3.2 and Theorem 3.3 can only be used to obtain subordinants of differential superordination of the form (3.1) or (3.2). The following theorem proves the existence of the best dominant of (3.2) for an appropriate ϕ.

Theorem 3.4. *Let h be analytic in $\mathbb{U}$ and $\phi : \mathbb{C}^3 \times \mathbb{U} \times \bar{\mathbb{U}} \longrightarrow \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq^(z)+(\nu-1)q(z)}{\nu}, \frac{z^2q''(z)+2(\nu-1)zq'(z)+(\nu-1)(\nu-2)q(z)}{\nu(\nu-1)}; z, \xi\right) = h(z) \tag{3.3}$$

has a solution $q \in Q_0$. If $\phi \in \phi'_{\mathcal{J}}[h,q]$, $f \in \mathcal{A}$, $\mathcal{J}^c_{\nu+1}f(z) \in Q_0$ and

$$\phi(\mathcal{J}^c_{\nu+1}f(z), \mathcal{J}^c_{\nu}f(z), \mathcal{J}^c_{\nu-1}f(z); z, \xi) \tag{3.4}$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec\prec \phi(\mathcal{J}^c_{\nu+1}f(z), \mathcal{J}^c_{\nu}f(z), \mathcal{J}^c_{\nu-1}f(z); z, \xi) \quad (z \in \mathbb{U},\ \xi \in \bar{\mathbb{U}}), \tag{3.5}$$

implies

$$q(z) \prec \mathcal{J}^c_{\nu+1}f(z) \quad (z \in \mathbb{U}).$$

and $q(z)$ is the best subordinant.

Proof. By applying Theorem 3.3, we deduce that q is a dominate of (3.2). Since q satisfies (3.3), it is also a solution of (3.2) and therefore q will be dominated by all dominates of (3.2). Hence q is the best dominates of (3.2). □

Combining Theorem 2.4 and Theorem 3.3, we obtain the following sandwich-type theorem.

Theorem 3.5. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $\mathbb{U}$, h_2 be univalent function in $\mathbb{U}$, $q_2 \in Q_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \phi_{\mathcal{J}}[h_2, q_2] \cap \phi'_{\mathcal{J}}[h_1, q_1]$. If $f \in \mathcal{A}$, $\mathcal{J}^c_{\nu+1}f(z) \in H[0,1] \cap Q_0$ and*

$$\phi(\mathcal{J}^c_{\nu+1}f(z), \mathcal{J}^c_{\nu}f(z), \mathcal{J}^c_{\nu-1}f(z); z, \xi)$$

is univalent function in $\mathbb{U}$, then

$$h_1(z) \prec\prec \phi(\mathcal{J}^c_{\nu+1}f(z), \mathcal{J}^c_{\nu}f(z), \mathcal{J}^c_{\nu-1}f(z); z, \xi) \prec\prec h_2(z),$$

implies $q_1(z) \prec \mathcal{J}^c_{\nu+1}f(z) \prec q_2(z)$, $(z \in \mathbb{U})$.

References

[1] Antonino, J.A., Romaguera, S., *Strong differential subordination to Briot-Bouquet differential equations*, J. Differential Equations, **114**(1994), 101-105.

[2] Antonino, J.A., *Strong differential subordination and applications to univalency conditions*, J. Korean Math. Soc., **43**(2006), 311-322.

[3] Cho, N.E., Kwon, O.S., Srivastava, H.M., *Strong differential subordination and superordination for multivalently meromorphic function involving the Liu-Srivastava operatior*, Integral Transforms and Spec. Funct., **21**(8)(2010), 589-601.

[4] Miller, S.S., Mocanu, P.T., *Differential Subordinations: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, No. 225, Marcel Dekker Incorporated, New York and Bessel, 2000.

[5] Miller, S.S., Mocanu, P.T., *Subordinants of differential superordinations*, Complex Var. Theory Appl., **48**(10)(2003), 815-826.

[6] Orhan, H., Yağmur, N., *Geometric properties of generalized Struve functions*, in International Congress in Honour of Professor Hari M. Srivastava, Bursa, Turkey, August, 2012.

[7] Oros, G.I., *First order differential superordinations using the Dziok-Srivastava linear operator*, Math. Reports, **12**(62)(2001), 37-44.

[8] Oros, G.I., *First order strong differential superordinaton*, Gen. Math., **15**(2007), no. 2-3, 77-87.

[9] Oros, G.I., *Strong differential superordination*, Acta Univ. Apulensis Math. Inform., **19**(2009), 101-106.

[10] Oros, G.I., Oros, G., *First order linear strong differential subordination*, Gen. Math., **15**(2007), no. 2-3, 98-107.

[11] Oros, G.I., Oros, G., *Strong differential subordination*, Turk. J. Math., **33**(2009), 249-257.

[12] Raza, M., Yağmur, N., *Some properties of a class of analytic functions defined by generalized Struve functions*, Turk. J. Math., **39**(2015), 931-944.

[13] Yağmur, N., Orhan, H., *Starlikeness and convexity of generalized Struve functions*, Abst. Appl. Anal., (2013), doi:10.1155/2013/954513.

[14] Zhang, S., Jin, J., *Computation of Special Functions*, A Wiley Interscience Publication, John Wiley and Sons, New York, NY, USA, 1996.

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Mellin transform in bicomplex space and its application

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Abstract. Motivated by the recent applications of bicomplex theory to the study of functions of large class, in this paper, we define bicomplex Mellin transform of bicomplex-valued functions. Also, we derive some of it's basic properties and inversion theorem in bicomplex space. Application of bicomplex Mellin transform in networks with time-varying parameters problem has been illustrated.

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1. Introduction

In this paper, we extend the Mellin transform of complex-valued function in complex variable to Mellin transform of bicomplex-valued function in bicomplex variable. In 1892, Segre Corrado [18] defined bicomplex numbers as

$$C_2 = \{\xi : \xi = x_0 + i_1x_1 + i_2x_2 + jx_3 |\ x_0, x_1, x_2, x_3 \in C_0\},$$

or

$$C_2 = \{\xi : \xi = z_1 + i_2z_2 |\ z_1, z_2 \in C_1\}.$$

where i_1 and i_2 are imaginary units such that $i_1^2 = i_2^2 = -1$, $i_1i_2 = i_2i_1 = j$, $j^2 = 1$ and C_0, C_1 and C_2 are sets of real numbers, complex numbers and bicomplex numbers, respectively. The set of bicomplex numbers is a commutative ring with unit and zero divisors. Hence, contrary to quaternions, bicomplex numbers are commutative with some non-invertible elements situated on the null cone.

In 1928 and 1932, Futagawa Michiji originated the concept of holomorphic functions of a bicomplex variable in a series of papers [10], [11]. In 1934, Dragoni [8] gave some basic results in the theory of bicomplex holomorphic functions while Price G.B. [16] and Rönn S. [17] have developed the bicomplex algebra and function theory.

In recent developments, authors have done efforts to extend Polygamma function [13], inverse Laplace transform, it's convolution theorem [2], Stieltjes transform [1], Tauberian Theorem of Laplace-Stieltjes transform [3] and Bochner Theorem of Fourier-Stieltjes transform [4] in the bicomplex variable from their complex counterpart. In their procedure, the idempotent representation of bicomplex numbers plays a vital role.

Hjalmar Mellin (1854-1933, see, e.g. [15]) gave his name to the Mellin transform that associates to a complex-valued function $f(t)$ defined over the interval $(0, \infty)$, the function of complex variable s, as

$$\bar{f}(s) = \int_0^\infty t^{s-1} f(t) dt.$$

The change of variables $t = e^{-x}$ shows that the Mellin transform is closely related to the Laplace transform. General properties of the Mellin transform are usually treated in detail in books on integral transforms, like those of Poularikas A.D. [15] and Davies B. [6]. In 1959, Francis R.G. [12] discussed the application of complex Mellin transform to networks with time-varying parameters. In 1995, Flajolet P. et al. [9] used Mellin transform for the asymptotic analysis of harmonic sums.

For solving the large class of bicomplex partial differential equations, we need integral transforms defined for large class. In this process we derive bicomplex Mellin transform with convergence conditions that can be capable of transferring the signals from real-valued t domain to bicomplexified frequency ξ domain.

Idempotent Representation: Every bicomplex number can be uniquely expressed as a complex combination of e_1 and e_2, viz.

$$\xi = (z_1 + i_2 z_2) = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2,$$

(where $e_1 = \frac{1+j}{2}, e_2 = \frac{1-j}{2}; e_1 + e_2 = 1$ and $e_1 e_2 = e_2 e_1 = 0$).

This representation of a bicomplex number is known as Idempotent Representation of ξ. The coefficients $(z_1 - i_1 z_2)$ and $(z_1 + i_1 z_2)$ are called the Idempotent Components of the bicomplex number $\xi = z_1 + i_2 z_2$ and $\{e_1, e_2\}$ is called Idempotent Basis.

Cartesian Set: The Auxiliary complex spaces A_1 and A_2 are defined as follows:

$$A_1 = \{w_1 = z_1 - i_1 z_2, \ \forall\, z_1, z_2 \in C_1\}, \ A_2 = \{w_2 = z_1 + i_1 z_2, \ \forall\, z_1, z_2 \in C_1\}.$$

A cartesian set $X_1 \times_e X_2$ determined by $X_1 \subseteq A_1$ and $X_2 \subseteq A_2$ and is defined as:

$$X_1 \times_e X_2 = \{z_1 + i_2 z_2 \in C_2 : z_1 + i_2 z_2 = w_1 e_1 + w_2 e_2, \ w_1 \in X_1, w_2 \in X_2\}.$$

With the help of idempotent representation, we define projection mappings $P_1 : C_2 \to A_1 \subseteq C_1$, $P_2 : C_2 \to A_2 \subseteq C_1$ as follows:

$$P_1(z_1 + i_2 z_2) = P_1[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 - i_1 z_2) \in A_1, \ \forall\, z_1 + i_2 z_2 \in C_2,$$

$$P_2(z_1 + i_2 z_2) = P_2[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 + i_1 z_2) \in A_2, \ \forall\, z_1 + i_2 z_2 \in C_2.$$

In the following theorem, Price G.B. discussed the convergence of bicomplex function with respect to it's idempotent complex component functions. This theorem is useful in proving our results.

Theorem 1.1. (*Price G.B.* [16]). *$F(\xi) = F_{e_1}(\xi_1)e_1 + F_{e_2}(\xi_2)e_2$ is convergent in domain $D \subseteq C_2$ iff $F_{e_1}(\xi_1)$ and $F_{e_2}(\xi_2)$ under functions $P_1 : D \to D_1 \subseteq C_1$ and $P_2 : D \to D_2 \subseteq C_1$ are convergent in domains D_1 and D_2, respectively.*

The organization of this paper is as follows:
In Section 2, we establish bicomplex Mellin transform with convergence conditions. In Section 3, we present some useful properties of bicomlex Mellin transform. In Section 4, we establish the inversion theorem for bicomplex Mellin transform. In section 5, we discuss application of bicomplex Mellin transform in finding the solution of bicomplex partial differential equation generated by network model and last Section 6 contains the conclusion.

2. Bicomplex Mellin transform

Let $f_1(t)$ be a complex-valued continuous function on the interval $(0,\infty)$ with $f_1(t) = O\left(t^{-\alpha_1}\right)$ as $t \to 0^+$ and $f_1(t) = O\left(t^{-\beta_1}\right)$ as $t \to \infty$, where $\alpha_1 < \beta_1$. Then Mellin transform of $f_1(t)$ is

$$\mathfrak{M}\left[f_1(t); s_1\right] = \int_0^\infty t^{s_1-1} f_1(t)dt = \bar{f}_1(s_1), \quad s_1 \in C_1 \tag{2.1}$$

where $\bar{f}_1(s_1)$ is analytic and convergent in the vertical strip

$$\Omega_1 = \left\{s_1 \in C_1 : \alpha_1 < \mathrm{Re}(s_1) < \beta_1\right\}. \tag{2.2}$$

Similarly,$f_2(t)$ be a complex-valued continuous function on the interval $(0,\infty)$ with $f_2(t) = O\left(t^{-\alpha_2}\right)$ as $t \to 0^+$ and $f_2(t) = O\left(t^{-\beta_2}\right)$ as $t \to \infty$, where $\alpha_2 < \beta_2$. Then Mellin transform of $f_2(t)$ is

$$\mathfrak{M}\left[f_2(t); s_2\right] = \int_0^\infty t^{s_2-1} f_2(t)dt = \bar{f}_2(s_2), \quad s_2 \in C_1 \tag{2.3}$$

where $\bar{f}_2(s_2)$ is analytic and convergent in the vertical strip

$$\Omega_2 = \left\{s_2 \in C_1 : \alpha_2 < \mathrm{Re}(s_2) < \beta_2\right\}. \tag{2.4}$$

Since $\bar{f}_1(s_1)$ and $\bar{f}_2(s_2)$ are complex functions which are analytic and convergent in the strips Ω_1 and Ω_2 respectively. Now, we have linear combination of $\bar{f}_1(s_1)$ and $\bar{f}_2(s_2)$ w.r.t. e_1 and e_2 respectively

$$\begin{aligned}
\bar{f}_1(s_1)e_1 + \bar{f}_2(s_2)e_2 &= \left(\int_0^\infty t^{s_1-1} f_1(t)dt\right) e_1 + \left(\int_0^\infty t^{s_2-1} f_2(t)dt\right) e_2 \\
\bar{f}(\xi) &= \int_0^\infty t^{(s_1e_1+s_2e_2)-1} \left(f_1(t)e_1 + f_2(t)e_2\right) dt \\
\bar{f}(\xi) &= \int_0^\infty t^{\xi-1} f(t)dt
\end{aligned} \tag{2.5}$$

where $\xi = s_1e_1 + s_2e_2$ and $\bar{f}(\xi)$ is analytic and convergent in the strip

$$\begin{aligned}
\Omega = \{\xi : \xi = s_1e_1 + s_2e_2 \in C_2; \alpha < \mathrm{Re}(P_1 : \xi) < \beta; \alpha < \mathrm{Re}(P_2 : \xi) < \beta; \\
\alpha = \max(\alpha_1, \alpha_2) \text{ and } \beta = \min(\beta_1, \beta_2)\}.
\end{aligned} \tag{2.6}$$

$\because\ \alpha < \mathrm{Re}(s_1) = x_1 < \beta$ and $\alpha < \mathrm{Re}(s_2) = x_2 < \beta$, we have

$$\begin{aligned}\xi &= (x_1 + i_1 y_1)e_1 + (x_2 + i_1 y_2)e_2 = (x_1 + i_1 y_1)\left(\frac{1 + i_1 i_2}{2}\right) \\ &\quad + (x_2 + i_1 y_2)\left(\frac{1 - i_1 i_2}{2}\right) \\ &= \frac{x_1 + x_2}{2} + \left(\frac{y_1 + y_2}{2}\right) i_1 + \left(\frac{y_2 - y_1}{2}\right) i_2 + \left(\frac{x_1 - x_2}{2}\right) i_1 i_2.\end{aligned}$$

Now, there are three possible cases:

1. If $x_1 = x_2 = a_0$ (say) then $\frac{x_1 - x_2}{2} = 0$ and $\frac{x_1+x_2}{2} = a_0$.
   Hence, if $\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2$, then $\alpha < a_0 < \beta$ and $a_3 = 0$.
2. If $x_1 > x_2$, then $\frac{x_1 - x_2}{2} > 0$,
   $\frac{x_1+x_2}{2} < \frac{\beta+x_2}{2} < \frac{\beta+x_2}{2} + \frac{\beta-x_1}{2} = \beta - \frac{x_1-x_2}{2}$
   and $\frac{x_1+x_2}{2} > \frac{\alpha+x_1}{2} > \frac{\alpha+x_1}{2} + \frac{\alpha-x_2}{2} = \alpha + \frac{x_1-x_2}{2}$.
   Thus, $\alpha + a_3 < a_0 < \beta - a_3$ and $a_3 > 0$.
3. If $x_1 < x_2$, then $\frac{x_1 - x_2}{2} < 0$,
   $\frac{x_1+x_2}{2} < \frac{\beta+x_1}{2} < \frac{\beta+x_1}{2} + \frac{\beta-x_2}{2} = \beta + \frac{x_1-x_2}{2}$
   and $\frac{x_1+x_2}{2} > \frac{\alpha+x_2}{2} > \frac{\alpha+x_2}{2} + \frac{\alpha-x_1}{2} = \alpha - \frac{x_1-x_2}{2}$.
   Thus, $\alpha - a_3 < a_0 < \beta + a_3$ and $a_3 < 0$.

These three conditions can be written in the following set builder form

$\Omega_1 = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \ :\ \alpha < a_0 < \beta \text{ and } a_3 = 0\}$,
$\Omega_2 = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \ :\ \alpha + a_3 < a_0 < \beta - a_3 \text{ and } a_3 > 0\}$,
$\Omega_3 = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \ :\ \alpha - a_3 < a_0 < \beta + a_3 \text{ and } a_3 < 0\}$.

Thus, $\alpha < \mathrm{Re}(P_1 : \xi) < \beta$ and $\alpha < \mathrm{Re}(P_2 : \xi) < \beta$ implies $\xi \in \Omega_1 \cup \Omega_2 \cup \Omega_3 = \Omega$ which can be defined as:

$$\Omega = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \in C_2 : \alpha + |a_3| < a_0 < \beta - |a_3|\} \tag{2.7}$$

or equivalently,

$$\Omega = \{\xi \in C_2 : \alpha + |\mathrm{Im}_j(\xi)| < \mathrm{Re}(\xi) < \beta - |\mathrm{Im}_j(\xi)|\}$$

where $\mathrm{Im}_j(\xi)$ denotes the imaginary part w.r.t. j unit of a bicomplex number.

Conversely, the existence condition of bicomplex Mellin transform $\bar{f}(\xi)$ can be obtained in the following way:

If $\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \in \Omega$,

$$\alpha + |a_3| < a_0 < \beta - |a_3|. \tag{2.8}$$

Now, in idempotent components, ξ can be expressed as

$$\begin{aligned}\xi &= a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \\ &= [(a_0 + a_3) + i_1(a_1 - a_2)]\, e_1 + [(a_0 - a_3) + i_1(a_1 + a_2)]\, e_2 \\ &= s_1 e_1 + s_2 e_2.\end{aligned}$$

Depending on the value of a_3, there arises three cases:

1. $a_3 = 0$ and $\alpha < a_0 < \beta$ which trivially leads $\alpha < a_0 + a_3 < \beta$ and $\alpha < a_0 - a_3 < \beta$.

2. When $a_3 > 0$, from the inequality (2.8) $\alpha + a_3 < a_0 < \beta - a_3$, we get $\alpha < a_0 - a_3$ and $a_0 + a_3 < \beta$. This result can be interpreted as $\alpha < a_0 - a_3 < a_0 + a_3 < \beta$.
3. When $a_3 < 0$, from the inequality (2.8) $\alpha - a_3 < a_0 < \beta + a_3$, we get $\alpha < a_0 + a_3$ and $a_0 - a_3 < \beta$. This result can be interpreted as $\alpha < a_0 + a_3 < a_0 - a_3 < \beta$.

Hence the result.

Now, we define the Mellin transform in the bicomplex space as follows:

Definition 2.1. Let $f(t)$ be a bicomplex-valued continuous function on the interval $(0, \infty)$ with $f(t) = O\left(t^{-\alpha}\right)$ as $t \to 0^+$ and $f(t) = O\left(t^{-\beta}\right)$ as $t \to \infty$, where $\alpha < \beta$. Then bicomplex Mellin transform of $f(t)$ defined as

$$\mathfrak{M}[f(t); \xi] = \int_0^\infty t^{\xi - 1} f(t) dt = \bar{f}(\xi), \quad \xi \in \Omega$$

where $\bar{f}(\xi)$ is analytic and convergent in Ω defined in

$$\Omega = \{\xi \in C_2 : \alpha + |\mathrm{Im}_j(\xi)| < \mathrm{Re}(\xi) < \beta - |\mathrm{Im}_j(\xi)|\} \tag{2.9}$$

where $\mathrm{Im}_j(\xi)$ denotes the imaginary part w.r.t. j unit of a bicomplex number.

Following is the illustration to explain the process of finding the bicomplex Mellin transform of a bicomplex valued function.

Example 2.2. Let $f(t) = t^a U(t - t_0)$, where $U(t - t_0)$ is unit-step function, then

$$\mathfrak{M}[f(t); \xi] = -\frac{t_0^{\xi + a}}{\xi + a}, \quad \mathrm{Re}(\xi + a) < -\left|\mathrm{Im}_j(\xi + a)\right|.$$

Solution. By applying the definition of bicomplex Mellin transform

$$\begin{aligned} \mathfrak{M}[f(t); \xi] &= \int_0^\infty t^{\xi - 1} t^a U(t - t_0) dt \\ &= \int_{t_0}^\infty t^{\xi + a - 1} dt \\ &= -\frac{t_0^{\xi + a}}{\xi + a}. \end{aligned}$$

Table 1. Bicomplex Mellin transform of some basic functions

S.No.	$f(t)$	Bicomplex Hankel Transform $F(\xi)$	Region of Convergence
1.	$(1+t)^{-a}$	$\frac{\Gamma(\xi)\Gamma(a-\xi)}{\Gamma(a)}$	$\lvert\mathrm{Im}_j(a-\xi)\rvert < \mathrm{Re}(a-\xi)$
2.	$(1+t)^{-1}$	$\frac{\pi}{\sin(\pi\xi)}$	$\lvert\mathrm{Im}_j(\xi)\rvert < \mathrm{Re}(\xi) < 1 - \lvert\mathrm{Im}_j(\xi)\rvert$
3.	$e^{nt},\ n > 0$	$\frac{\Gamma(\xi)}{n^\xi}$	$\mathrm{Re}(\xi) > \lvert\mathrm{Im}_j(\xi)\rvert$
4.	$\sin(at),\ a > 0$	$\frac{\Gamma(\xi)\sin\left(\frac{\pi\xi}{2}\right)}{a^\xi}$	$-1+\lvert\mathrm{Im}_j(\xi)\rvert < \mathrm{Re}(\xi) < 1-\lvert\mathrm{Im}_j(\xi)\rvert$
5.	$\cos(at),\ a > 0$	$\frac{\Gamma(\xi)\cos\left(\frac{\pi\xi}{2}\right)}{a^\xi}$	$\lvert\mathrm{Im}_j(\xi)\rvert < \mathrm{Re}(\xi) < 1 - \lvert\mathrm{Im}_j(\xi)\rvert$
6.	$\log(1+t)$	$\frac{\pi}{\xi\sin(\pi\xi)}$	$-1 + \lvert\mathrm{Im}_j(\xi)\rvert < \mathrm{Re}(\xi) < -\lvert\mathrm{Im}_j(\xi)\rvert$
7.	t^{-a}	$-\frac{1}{\xi - a}$	$\mathrm{Re}(\xi - a) < -\lvert\mathrm{Im}_j(\xi - a)\rvert$

3. Properties of bicomplex Mellin transform

In this section, we discuss the basic properties of bicomplex Mellin transform viz. linearity property, change of scale property, shifting property, Mellin transform of derivatives and operators, relation with bicomplex Laplace transform and some other properties.

Theorem 3.1. *(Linearity Property). Let $f(t)$ and $g(t)$ are bicomplex-valued functions with $f(t) = O\left(t^{-\alpha_1}\right)$, $g(t) = O\left(t^{-\alpha_2}\right)$ as $t \to 0^+$ and $f(t) = O\left(t^{-\beta_1}\right)$, $g(t) = O\left(t^{-\beta_2}\right)$ as $t \to \infty$, with $\max(\alpha_1, \alpha_2) + |Im_j(\xi)| < Re(\xi) < \min(\beta_1, \beta_2) - |Im_j(\xi)|$, then*

$$\mathfrak{M}[c_1 f(t) + c_2 g(t); \xi] = c_1 \mathfrak{M}[f(t); \xi] + c_2 \mathfrak{M}[g(t); \xi] \tag{3.1}$$

where c_1 and c_2 are arbitrary constants.

Proof. By applying the definition of bicomplex Mellin transform

$$\begin{aligned}\mathfrak{M}[c_1 f(t) + c_2 g(t); \xi] &= \int_0^\infty t^{\xi-1}[c_1 f(t) + c_2 g(t)]dt \\ &= c_1 \int_0^\infty t^{\xi-1} f(t)dt + c_2 \int_0^\infty t^{\xi-1} g(t)dt \\ &= c_1 \mathfrak{M}[f(t); \xi] + c_2 \mathfrak{M}[g(t); \xi].\end{aligned}$$

□

Theorem 3.2. *(Change of scale property). Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$, then*

$$\mathfrak{M}[f(at); \xi] = a^{-\xi} \bar{f}(\xi), \quad \xi \in \Omega,\ a > 0 \tag{3.2}$$

where Ω is defined in (2.9).

Proof. By applying the definition of bicomplex Mellin transform

$$\begin{aligned}\mathfrak{M}[f(at); \xi] &= \int_0^\infty t^{\xi-1} f(at)dt, \qquad [\text{where } \xi = s_1 e_1 + s_2 e_2] \\ &= \left(\int_0^\infty t^{s_1-1} f_1(at)dt\right) e_1 + \left(\int_0^\infty t^{s_2-1} f_2(at)dt\right) e_2\end{aligned}$$

Put $at = u$, to obtain

$$\begin{aligned}&= \frac{1}{a^{s_1}} \left(\int_0^\infty t^{s_1-1} f_1(u)dt\right) e_1 + \frac{1}{a^{s_2}} \left(\int_0^\infty t^{s_2-1} f_2(u)dt\right) e_2 \\ &= \frac{1}{a^{s_1e_1+s_2e_2}} \int_0^\infty t^{s_1e_1+s_2e_2-1} \left(f_1(u)e_1 + f_2(u)e_2\right) dt \\ &= \frac{1}{a^\xi} \int_0^\infty t^{\xi-1} f(u)dt \\ &= \frac{\bar{f}(\xi)}{a^\xi}.\end{aligned}$$

□

Theorem 3.3. *(Bicomplex Mellin Transform of Derivatives). Let $\bar{f}(\xi)$ be bicomplex Mellin transform of bicomplex-valued function $f(t)$, then*

$$\mathfrak{M}\left[f^{(n)}(t);\xi\right] = (-1)^n \frac{\Gamma(\xi)}{\Gamma(\xi-n)}\bar{f}(\xi-n), \quad (\xi-n)\in\Omega \tag{3.3}$$

where Ω is defined in (2.9) and provided $t^{\xi-r-1}f^{(r)}(t)$ vanishes as $t\to 0$ and as $t\to\infty$ for $r=0,1,2,\cdots,(n-1)$.

Proof. For $n=1$, according to the definition of bicomplex Mellin transform,

$$\mathfrak{M}\left[f^{'}(t);\xi\right] = \int_0^\infty t^{\xi-1}f^{'}(t)dt$$

which on integration by parts, gives

$$\begin{aligned}\mathfrak{M}\left[f^{'}(t);\xi\right] &= t^{\xi-1}f(t)|_0^\infty - (\xi-1)\int_0^\infty t^{\xi-2}f(t)dt\\ &= -(\xi-1)\bar{f}(\xi-1).\end{aligned}$$

Therefore, the result is true for $n=1$. Let the the above result is true for $n=m$

$$\mathfrak{M}\left[f^{(m)}(t);\xi\right] = (-1)^m \frac{\Gamma(\xi)}{\Gamma(\xi-m)}\bar{f}(\xi-m). \tag{3.4}$$

Now, for $n=m+1$

$$\mathfrak{M}\left[f^{(m+1)}(t);\xi\right] = \int_0^\infty t^{\xi-1}f^{(m+1)}(t)dt$$

Integrating by parts, we get

$$\begin{aligned}\mathfrak{M}\left[f^{(m+1)}(t);\xi\right] &= t^{\xi-1}f^{(m)}(t)|_0^\infty - (\xi-1)\int_0^\infty t^{\xi-2}f^{(m)}(t)dt\\ &= -(\xi-1)(-1)^m\frac{\Gamma(\xi-1)}{\Gamma(\xi-m-1)}\bar{f}(\xi-m-1), \text{ [using (3.4)]}\\ &= (-1)^{m+1}\frac{\Gamma(\xi)}{\Gamma(\xi-m-1)}\bar{f}(\xi-m-1).\end{aligned}$$

Therefore, the result is true for $n=m+1$. Hence, by the principal of mathematical induction the result is true for all $n=1,2,\cdots$. Therefore,

$$\mathfrak{M}\left[f^{(n)}(t);\xi\right] = (-1)^n \frac{\Gamma(\xi)}{\Gamma(\xi-n)}\bar{f}(\xi-n). \qquad \square$$

Theorem 3.4. *(Shifting Property). Let $\bar{f}(\xi)$ be bicomplex Mellin transform of bicomplex-valued function $f(t)$. Then*

$$\mathfrak{M}\left[t^a f(t);\xi\right] = \bar{f}(\xi+a), \quad (\xi+a)\in\Omega,\ a\in C_2 \tag{3.5}$$

where Ω is defined in (2.9).

Proof. By applying the definition of bicomplex Mellin transform,

$$\mathfrak{M}\left[t^a f(t);\xi\right] = \int_0^\infty t^{\xi-1}t^a f(t)dt = \int_0^\infty t^{\xi+a-1}f(t)dt = \bar{f}(\xi+a). \qquad \square$$

Theorem 3.5. *Let $\bar{f}(\xi)$ be bicomplex Mellin transform of bicomplex-valued function $f(t)$. Then*

$$\mathfrak{M}\left[f(t^a);\xi\right] = \frac{1}{a}\bar{f}\left(\frac{\xi}{a}\right), \quad \frac{\xi}{a} \in \Omega,\ 0 \neq a \in C_0 \tag{3.6}$$

where Ω is defined in (2.9).

Proof. By applying the definition of bicomplex Mellin transform,

$$\begin{aligned}\mathfrak{M}\left[f(t^a);\xi\right] &= \int_0^\infty t^{\xi-1} f(t^a)dt \\ &= \frac{1}{a}\int_0^\infty u^{\frac{\xi}{a}-1} f(u)du \qquad [\text{substituting } t^a = u] \\ &= \frac{1}{a}\bar{f}\left(\frac{\xi}{a}\right).\end{aligned}$$

□

Theorem 3.6. *Let $\bar{f}(\xi)$ be bicomplex Mellin transform of bicomplex-valued function $f(t)$. Then*

$$\mathfrak{M}\left[t^n f^{(n)}(t);\xi\right] = (-1)^n \frac{\Gamma(\xi+n)}{\Gamma(\xi)}\bar{f}(\xi), \quad \xi \in \Omega \tag{3.7}$$

where Ω is defined in (2.9) and provided $t^{\xi-r}f^{(r)}(\xi)$ vanishes as $t \to 0$ and as $t \to \infty$ for $r = 0, 1, 2, \cdots, (n-1)$.

Proof. By applying the definition of bicomplex Mellin transform,

$$\begin{aligned}\mathfrak{M}\left[t^n f^{(n)}(t);\xi\right] &= \int_0^\infty t^{\xi-1} t^n f^{(n)}(t)dt, \qquad [\text{where } \xi = s_1e_1 + s_2e_2] \\ &= \left(\int_0^\infty t^{s_1-1}t^n f_1^{(n)}(t)dt\right)e_1 + \left(\int_0^\infty t^{s_2-1}t^n f_2^{(n)}(t)dt\right)e_2 \\ &= (-1)^n\frac{\Gamma(s_1+n)}{\Gamma(s_1)}\bar{f}_1(s_1)e_1 + (-1)^n\frac{\Gamma(s_2+n)}{\Gamma(s_2)}\bar{f}_2(s_2)e_2, \\ &\qquad\qquad [\text{using [7, Equation (8.3.12)]}] \\ &= (-1)^n\frac{\Gamma(s_1e_1+s_2e_2+n)}{\Gamma(s_1e_1+s_2e_2)}\left(\bar{f}_1(s_1)e_1 + \bar{f}_2(s_2)e_2\right) \\ &= (-1)^n\frac{\Gamma(\xi+n)}{\Gamma(\xi)}\bar{f}(\xi).\end{aligned}$$

□

Theorem 3.7. *(Bicomplex Mellin Transform of Differential Operators). Let $\bar{f}(\xi)$ be bicomplex Mellin transform of bicomplex-valued function $f(t)$. Then*

$$\mathfrak{M}\left[\left(t\frac{d}{dt}\right)^2 f(t);\xi\right] = \mathfrak{M}\left[t^2f^{''}(t) + tf^{'}(t);\xi\right] = (-1)^2\xi^2\bar{f}(\xi),\ \xi \in \Omega \tag{3.8}$$

where Ω is defined in (2.9).

Proof. By applying the definition of bicomplex Mellin transform,

$$\begin{aligned}\mathfrak{M}\left[\left(t\frac{d}{dt}\right)^2 f(t);\xi\right] &= \mathfrak{M}\left[t^2 f^{''}(t) + tf^{'}(t);\xi\right]\\ &= \mathfrak{M}\left[t^2 f^{''}(t);\xi\right] + \mathfrak{M}\left[tf^{'}(t);\xi\right]\\ &= \xi(\xi+1)\bar{f}(\xi) - \xi\bar{f}(\xi)\\ &= (-1)^2\xi^2\bar{f}(\xi).\end{aligned}$$

In general,

$$\mathfrak{M}\left[\left(t\frac{d}{dt}\right)^n f(t);\xi\right] = (-1)^n\xi^n\bar{f}(\xi).$$

□

Theorem 3.8. *(Bicomplex Mellin Transform of Integrals). Let $\bar{f}(\xi)$ be bicomplex Mellin transform of bicomplex-valued function $f(t)$. Then*

$$\mathfrak{M}\left[\int_0^t f(x)dx;\xi\right] = -\frac{1}{\xi}\bar{f}(\xi+1), \quad (\xi+1)\in\Omega \tag{3.9}$$

where Ω is defined in (2.9).

Proof. We write

$$g(t) = \int_0^t f(x)dx$$

so that $g^{'}(t) = f(t)$ with $g(0) = 0$. Taking the bicomplex Mellin transform of $g^{'}(t)$ and using Theorem 3.3 therein, we get

$$\begin{aligned}\mathfrak{M}\left[g^{'}(t);\xi\right] &= -(\xi-1)\mathfrak{M}[g(t);\xi-1]\\ &= -(\xi-1)\mathfrak{M}\left[\int_0^t f(x)dx;\xi-1\right]\end{aligned}$$

Replacing ξ by $\xi+1$, we get the desired result (3.9). □

3.1. Relation with Bicomplex Laplace Transform

The bicomplex Laplace transform and its properties are discussed by Kumar A. and Kumar P. [14]. It is defined as

Definition 3.9. Let $f(t)$ be a bicomplex-valued function of exponential order $\alpha \in C_0$. Then Laplace Transform of $f(t)$ for $t \geq 0$ can be defined as:

$$L\{f(t)\} = \int_0^\infty f(t)e^{-\xi t}dt = F(\xi)$$

Here $F(\xi)$ exist and is convergent for all $\xi \in D = D_1 \cup D_2 \cup D_3$
or

$$D = \{\xi \in C_2 : H_\rho(\xi) \text{ represent a Right half-plane } a_0 > \alpha + |a_3|\},$$

where

$$D_1 = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 : a_0 > \alpha,\ a_3 = 0\},$$
$$D_2 = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 : a_0 > \alpha + a_3,\ a_3 > 0\}$$

and

$$D_3 = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 : a_0 > \alpha - a_3,\ a_3 < 0\}.$$

In D, there are infinite ξ which have same H_ρ hyperbolic projection because a_1 and a_2 are free from restriction.

Therefore, the usual right-sided bicomplex Laplace transform is analytic in half-plane $\mathrm{Re}(\xi) > \alpha + |\mathrm{Im}_j(\xi)|$. In the same way, left-sided bicomplex Laplace transform is analytic in the region $\mathrm{Re}(\xi) < \beta - |\mathrm{Im}_j(\xi)|$. If the two half-planes overlap, the region of analyticity of the two-sided bicomplex Laplace transform is thus the strip

$$D = \{\xi \in C_2 : \alpha + |\mathrm{Im}_j(\xi)| < \mathrm{Re}(\xi) < \beta - |\mathrm{Im}_j(\xi)|\}.$$

Hence, D is equivalent to Ω defined in (2.9).

Theorem 3.10. *Let $\bar{f}(\xi)$ be bicomplex Mellin transform of bicomplex-valued function $f(t)$. Then*

$$\mathfrak{M}[f(t);\xi] = \int_{-\infty}^{\infty} e^{\xi x} f(e^{-x})dx = L\left[f(e^{-x});\xi\right], \quad \xi \in \Omega \tag{3.10}$$

where Ω is defined in (2.9).

Proof. Taking $t = e^{-x}$ in the definition of bicomplex Mellin transform

$$\mathfrak{M}[f(t);\xi] = \int_0^{\infty} t^{\xi-1} f(t)dt,$$

we get

$$\mathfrak{M}[f(t);\xi] = \int_{-\infty}^{\infty} e^{\xi x} f(e^{-x})dx = L\left[f(e^{-x});\xi\right]. \qquad \square$$

4. Inversion of bicomplex Mellin transform

In this section, we discuss the inversion of bicomplex Mellin transform. Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued continuous function $f(t)$. Then $\bar{f}(\xi) = \bar{f}_1(s_1)e_1 + \bar{f}_2(s_2)e_2$ is analytic in the strip Ω, which is defined in (2.6). The inverse formula for complex mellin transform (see, e.g. Poularikas A.D. [15, chapter 11] and Davies B. [6, p. 195-210]) is

$$\begin{aligned} f_1(t) &= \frac{1}{2\pi i_1}\int_{c_1 - i_1\infty}^{c_1 + i_1\infty} t^{-s_1}\bar{f}_1(s_1)ds_1, \quad \alpha_1 < c_1 < \beta_1 \\ &= \frac{1}{2\pi i_1}\int_{\Omega_1} t^{-s_1}\bar{f}_1(s_1)ds_1 \end{aligned} \tag{4.1}$$

where, Ω_1 is defined in (2.2). Similarly, another inverse formula for complex Mellin transform is

$$\begin{aligned} f_2(t) &= \frac{1}{2\pi i_1}\int_{c_2-i_1\infty}^{c_2+i_1\infty} t^{-s_2}\bar{f}_2(s_2)ds_2, \quad \alpha_2 < c_2 < \beta_2 \\ &= \frac{1}{2\pi i_1}\int_{\Omega_2} t^{-s_2}\bar{f}_2(s_2)ds_2 \end{aligned} \tag{4.2}$$

where, Ω_2 is defined in (2.3).

Now, using complex inversions (4.1) and (4.2), we obtain the bicomplex-valued function as

$$\begin{aligned} f(t) &= f_1(t)e_1 + f_2(t)e_2 \\ &= \left(\frac{1}{2\pi i_1}\int_{\Omega_1} t^{-s_1}\bar{f}_1(s_1)ds_1\right)e_1 + \left(\frac{1}{2\pi i_1}\int_{\Omega_2} t^{-s_2}\bar{f}_2(s_2)ds_2\right)e_2 \\ &= \frac{1}{2\pi i_1}\left(\int_{(\Omega_1,\Omega_2)} t^{-(s_1e_1+s_2e_2)}\left(\bar{f}_1(s_1)e_1 + \bar{f}_2(s_2)e_2\right)d(s_1e_1+s_2e_2)\right) \\ &= \frac{1}{2\pi i_1}\int_{\Omega} t^{-s_1}\bar{f}(\xi)d\xi \end{aligned} \tag{4.3}$$

where, Ω is defined in (2.9).

Consider the problem of asymptotically expanding $f(t)$ as $t \to 0^+$, when $\bar{f}(\xi)$ is known to be continuable in $-M + |\mathrm{Im}_j(\xi)| \le \mathrm{Re}(\xi) \le \alpha - |\mathrm{Im}_j(\xi)|$ for some $M > 0$. We also postulate that $\bar{f}(\xi)$ has finitely many poles λ_k such that $\mathrm{Re}(\lambda_k) > -M + |\mathrm{Im}_j(\lambda_k)|$. Then

$$f(t) = \sum_{\lambda_k\in\mathcal{K}} \mathrm{Res}\left[t^{-\xi}\bar{f}(\xi),\ \xi = \lambda_k\right] + O\left(t^M\right), \quad \text{as } t \to 0^+$$

where $\mathcal{K}$ is the set of singularities and M is as large as we want. Similarly, for problem of asymptotically expanding $f(t)$ as $t \to \infty$. Then contour taken in right and side of the fundamental strip, we have

$$f(t) = -\sum_{\lambda_k\in\mathcal{K}} \mathrm{Res}\left[t^{-\xi}\bar{f}(\xi),\ \xi = \lambda_k\right] + O\left(t^{-M}\right), \quad \text{as } t \to \infty.$$

Following is the illustration to explain the process of finding the inverse bicomplex Mellin transform.

Example 4.1. Let $\bar{f}(\xi) = \frac{1}{(\xi-a)(\xi-b)}$, for $\mathrm{Re}(\xi - a) < -|\mathrm{Im}_j(\xi - a)|$ and $\mathrm{Re}(a - b) < -|\mathrm{Im}_j(a - b)|$. Then find the inverse bicomplex Mellin transform $f(t)$ of $\bar{f}(\xi)$.

Solution. By applying the inverse bicomplex Mellin transform on $\bar{f}(\xi)$

$$\begin{aligned} f(t) &= \frac{1}{2\pi i_1}\int_{\Omega} t^{-\xi}\bar{f}(\xi)d\xi \\ &= -\left[\operatorname{Res}\left(t^{-\xi}\frac{1}{(\xi-a)(\xi-b)},\ \xi=a\right)+\operatorname{Res}\left(t^{-\xi}\frac{1}{(\xi-a)(\xi-b)},\ \xi=b\right)\right] \\ &= \frac{1}{b-a}\left(t^{-a}-t^{-b}\right). \end{aligned}$$

5. Application of bicomplex Mellin transform

In this paper, we are interested in determining the extent to which the output voltage V and current I using by bicomplex concept differs from their input values as the length of the transmission line tends to a very small value.

Now, let us define bicomplex scalar field as

$$F \equiv V + i_2 I \tag{5.1}$$

where voltage V and current I are complex scalar fields. Now, we consider an equivalent circuit of a transmission line of small length Δx containing resistance $R\Delta x$, capacitance $C\Delta x$, and inductance $L\Delta x$ as shown in Figure 1.

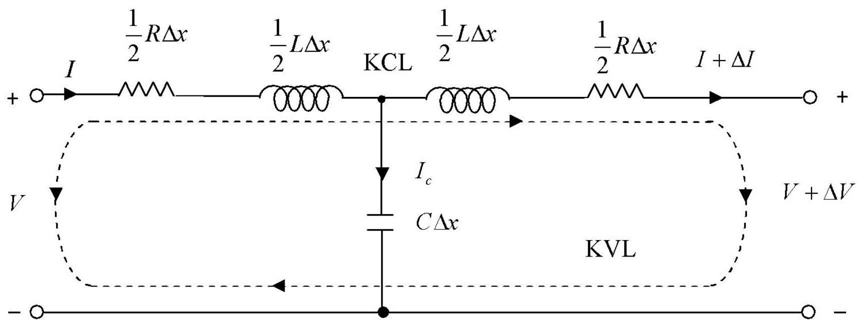

Figure 1. Equivalent circuit of a transmission line

The above figure is a symmetrical network. By using the Kirchhoff's voltage law (KVL), we have

$$V = \frac{1}{2}RI\Delta x + \frac{1}{2}L\frac{\partial I}{\partial t}\Delta x + \frac{1}{2}L\frac{\partial}{\partial t}(I+\Delta I)\Delta x + \frac{1}{2}R(I+\Delta I)\Delta x + V + \Delta V. \tag{5.2}$$

Dividing (5.2) by Δx and simplifying, we get

$$\frac{\Delta V}{\Delta x} = -\left[RI + L\frac{\partial I}{\partial t} + \left(\frac{L}{2}\frac{\partial}{\partial t}\frac{\Delta I}{\Delta x} + \frac{R}{2}\frac{\Delta I}{\Delta x}\right)\Delta x\right]. \tag{5.3}$$

Taking limit as $\Delta x \to 0$, we get

$$\frac{\partial V}{\partial x} = -\left[RI + L\frac{\partial I}{\partial t}\right]. \tag{5.4}$$

By applying Kirchhoff's current law (KCL) on the equivalent circuit of the transmission line, we get

$$
\begin{aligned}
I &= I_c + I + \Delta I \\
&= C\frac{\partial}{\partial t}\left(V + \frac{\Delta V}{2}\right)\Delta x + I + \Delta I. \qquad (5.5)
\end{aligned}
$$

Dividing (5.5) by Δx and simplifying, we get

$$
\frac{\Delta I}{\Delta x} = -\left[C\frac{\partial V}{\partial t} + \frac{C}{2}\frac{\partial}{\partial t}\left(\frac{\Delta V}{\Delta x}\right)\Delta x\right]. \qquad (5.6)
$$

Taking limit as $\Delta x \to 0$, we get

$$
\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}. \qquad (5.7)
$$

The differential equations in (5.4) and (5.7) describes the evaluation of current and voltage in a lossy transmission line. Differentiating (5.4) w.r.t. x and simplifying using (5.7), we get

$$
\frac{\partial^2 V}{\partial x^2} = CL\frac{\partial^2 V}{\partial t^2} + CR\frac{\partial V}{\partial t}. \qquad (5.8)
$$

Similarly, differentiating (5.7) w.r.t. x and simplifying using (5.4), we get

$$
\frac{\partial^2 I}{\partial x^2} = CL\frac{\partial^2 I}{\partial t^2} + CR\frac{\partial I}{\partial t}. \qquad (5.9)
$$

Equations (5.8) and (5.9) are hyperbolic partial differential equations which describes the voltage and current along power transmission lines.

Combining equation (5.8) and (5.9) with the help of bicomplex unit i_2 as

$$
\begin{aligned}
& \frac{\partial^2 V}{\partial x^2} + i_2\frac{\partial^2 I}{\partial x^2} = CL\left(\frac{\partial^2 V}{\partial t^2} + i_2\frac{\partial^2 I}{\partial t^2}\right) + CR\left(\frac{\partial V}{\partial t} + i_2\frac{\partial I}{\partial t}\right) \\
\Rightarrow\quad & \frac{\partial^2}{\partial x^2}(V + i_2 I) = CL\frac{\partial^2}{\partial t^2}(V + i_2 I) + CR\frac{\partial}{\partial t}(V + i_2 I) \\
\Rightarrow\quad & \frac{\partial^2}{\partial x^2}F(x,t) = CL\frac{\partial^2}{\partial t^2}F(x,t) + CR\frac{\partial}{\partial t}F(x,t) \qquad (5.10)
\end{aligned}
$$

where $F(x,t)$ is bicomplex-valued function defined by (5.1).

In particular, a circuit which has resistance $R = \frac{1}{t}$, capacitance $C = t^2$ and inductance $L = 1$. The differential equation (5.10) of bicomplex-valued function becomes

$$
\frac{\partial^2}{\partial x^2}F(x,t) = t^2\frac{\partial^2}{\partial t^2}F(x,t) + t\frac{\partial}{\partial t}F(x,t). \qquad (5.11)
$$

For finding the solution of partial differential equation (5.11), we assume boundary conditions as

$$
F(0,t) = 0 \quad \text{and} \quad F(1,t) = A\left(\frac{1}{t^a} + \frac{1}{t^b}\right) \qquad (5.12)
$$

where $A \in C_2$, $\mathrm{Re}(b-a) > |\mathrm{Im}_j(b-a)|$. By taking the bicomplex Mellin transform of (5.11) w.r.t. t and making use of Theorem 3.7, we get

$$\frac{d^2}{dx^2}\bar{F}(x,\xi) = \xi^2 \bar{F}(x,\xi). \tag{5.13}$$

Therefore, by taking the bicomplex Mellin transform of (5.12) and using in solution of (5.13), we get

$$\bar{F}(x,\xi) = A\left[\frac{(-2\xi+a+b)\left(e^{\xi x}-e^{-\xi x}\right)}{(\xi-a)(\xi-b)\left(e^{\xi}-e^{-\xi}\right)}\right]. \tag{5.14}$$

By taking the inverse bicomplex Mellin transform (5.14), we get

$$F(x,t) = \frac{1}{2\pi i_1}\int_{\Omega} t^{-\xi}\bar{F}(x,\xi)d\xi \tag{5.15}$$

where $\bar{F}(x,\xi)$ is analytic in $\mathrm{Re}(\xi-a) > |\mathrm{Im}_j(\xi-a)|$. Then taking a semi-circle on the right-hand side of a large radius and using by residue theorem, we have

$$\begin{aligned} F(x,t) &= A\left[\frac{\sinh(ax)}{\sinh(a)}t^{-a} + \frac{\sinh(bx)}{\sinh(b)}t^{-b}\right] \\ &= A_1\left[\frac{\sinh(a_1x)}{\sinh(a_1)}t^{-a_1} + \frac{\sinh(b_1x)}{\sinh(b_1)}t^{-b_1}\right]e_1 \\ &\quad + A_2\left[\frac{\sinh(a_2x)}{\sinh(a_2)}t^{-a_2} + \frac{\sinh(b_2x)}{\sinh(b_2)}t^{-b_2}\right]e_2 \end{aligned}$$

where $A = A_1e_1 + A_2e_2$, $a = a_1e_1 + a_2e_2$ and $b = b_1e_1 + b_2e_2$. Therefore,

$$\begin{aligned} F(x,t) &\equiv V + i_2 I \\ &= \frac{1}{2}\left\{A_1\left[\frac{\sinh(a_1x)}{\sinh(a_1)}t^{-a_1} + \frac{\sinh(b_1x)}{\sinh(b_1)}t^{-b_1}\right]\right. \\ &\quad \left. +A_2\left[\frac{\sinh(a_2x)}{\sinh(a_2)}t^{-a_2} + \frac{\sinh(b_2x)}{\sinh(b_2)}t^{-b_2}\right]\right\} \\ &\quad + i_2\frac{i_1}{2}\left\{A_1\left[\frac{\sinh(a_1x)}{\sinh(a_1)}t^{-a_1} + \frac{\sinh(b_1x)}{\sinh(b_1)}t^{-b_1}\right]\right. \\ &\quad \left. -A_2\left[\frac{\sinh(a_2x)}{\sinh(a_2)}t^{-a_2} + \frac{\sinh(b_2x)}{\sinh(b_2)}t^{-b_2}\right]\right\}. \end{aligned} \tag{5.16}$$

Separating the bi-real and bi-imaginary parts of (5.16), we obtain the voltage and current of above model as

$$\begin{aligned} V(x,t) &= \frac{1}{2}\left\{A_1\left[\frac{\sinh(a_1x)}{\sinh(a_1)}t^{-a_1} + \frac{\sinh(b_1x)}{\sinh(b_1)}t^{-b_1}\right]\right. \\ &\quad \left. +A_2\left[\frac{\sinh(a_2x)}{\sinh(a_2)}t^{-a_2} + \frac{\sinh(b_2x)}{\sinh(b_2)}t^{-b_2}\right]\right\} \end{aligned}$$

and

$$I(x,t) = \frac{i_1}{2}\left\{A_1\left[\frac{\sinh(a_1x)}{\sinh(a_1)}t^{-a_1} + \frac{\sinh(b_1x)}{\sinh(b_1)}t^{-b_1}\right]\right.$$
$$\left. -A_2\left[\frac{\sinh(a_2x)}{\sinh(a_2)}t^{-a_2} + \frac{\sinh(b_2x)}{\sinh(b_2)}t^{-b_2}\right]\right\}.$$

6. Conclusion

The concept of bicomplex numbers has been applied for finding the solution of differential equations of bicomplex-valued function generated by network diagram. In this paper, we derive Mellin transform and its inverse in bicomplex space which is the generalization of complex Mellin transform. The application has been illustrated to find the solution of partial differential equation of bicomplex-valued function generated by a network. The bicomplex analysis has great advantage that it separates the voltage and current as complex components.

References

[1] Agarwal, R., Goswami, M.P., Agarwal, R.P., *Bicomplex version of Stieltjes transform and applications,* Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms, **21**(2014), no. 4-5, 229-246.

[2] Agarwal, R., Goswami, M.P., Agarwal, R.P., *Convolution theorem and applications of bicomplex Laplace transform,* Advances in Mathematical Sciences and Applications, **24**(2014), no. 1, 113-127.

[3] Agarwal, R., Goswami, M.P., Agarwal, R.P., *Tauberian theorem and applications of bicomplex Laplace-Stieltjes transform,* Dynamics of Continuous, Discrete and Impulsive Systems, Series B: Applications & Algorithms, **22**(2015), 141-153.

[4] Agarwal, R., Goswami, M.P., Agarwal, R.P., *Bochner theorem and applications of bicomplex Fourier-Stieltjes transform,* Advanced Studies in Contemporary Mathematics, **26**(2016), no. 2, 355-369.

[5] Banerjee, A., Datta, S.K., Hoque, A., *Fourier transform and its inverse for functions of bicomplex variables,* arXiv.org math arXiv:1404.4236, 2014, 1-20.

[6] Davies, B., *Integral transforms and their applications,* Springer, 2001.

[7] Debnath Loknath, Bhatta Dambaru, *Integral transforms and their applications*, Chaman & Hall/CRC Taylor & Francis Group, 2006.

[8] Dragoni, G.S., *Sulle funzioni olomorfe di una variable bicomplessa,* Reale Acad. d'Italia Mem. Class Sci. Fic. Mat. Nat., **5**(1934), 597-665.

[9] Flajolet, P., Gourdon, X., Dumas, P., *Mellin transforms and asymptotics: Harmonic sums,* Theoretical Computer Science, **144**(1995), 3-58.

[10] Futagawa, M., *On the theory of functions of quaternary variable - I,* Tohoku Math. J., **29**(1928), 175-222.

[11] Futagawa, M., *On the theory of functions of quaternary variable - II,* Tohoku Math. J., **35**(1932), 69-120.

[12] Gerardi, F.R., *Application of Mellin and Hankel transforms to networks with time-varying parameters,* IRE Transaction on Circuit Theory, **6**(1959), no. 2, 197-208.

[13] Goyal, R., *Bicomplex polygamma function,* Tokyo Journal of Mathematics, **30**(2007), no. 2, 523-530.

[14] Kumar, A., Kumar, P., *Bicomplex version of Laplace transform,* International Journal of Engg. and Tech., **3**(2011), no. 3, 225-232.

[15] Poularikas, A.D., *The transforms and applications handbook,* CRC Press, 1999.

[16] Price, G.B., *An introduction to multicomplex spaces and functions*, Marcel Dekker Inc., New York, 1991.

[17] Rönn, S., *Bicomplex algebra and function theory,* arXiv:math/0101200v1 [math.CV], (2001), 1-71.

[18] Segre, C., *Le rappresentazioni reale delle forme complesse e gli enti iperalgebrici,* Math. Ann., **40**(1892), 413-467.

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Properties of m-complex symmetric operators

Muneo Chō, Eungil Ko and Ji Eun Lee

Dedicated to the memory of Professor Takayuki Furuta in deep sorrow

Abstract. In this paper, we study several properties of m-complex symmetric operators. In particular, we prove that if $T \in \mathcal{L}(\mathcal{H})$ is an m-complex symmetric operator and N is a nilpotent operator of order $n > 2$ with $TN = NT$, then $T+N$ is a $(2n+m-2)$-complex symmetric operator. Moreover, we investigate the decomposability of $T + A$ and TA where T is an m-complex symmetric operator and A is an algebraic operator. Finally, we provide various spectral relations of such operators. As some applications of these results, we discuss Weyl type theorems for such operators.

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Keywords: Conjugation, m-complex symmetric operator, nilpotent perturbations, decomposable, Weyl type theorems.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. A *conjugation* on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. For any conjugation C, there is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Ce_n = e_n$ for all n (see [14] for more details). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$.

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In 1970, J. W. Helton [18] initiated the study of operators $T \in \mathcal{L}(\mathcal{H})$ which satisfy an identity of the following form;

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}T^{m-j}=0. \tag{1.1}$$

In the light of complex symmetric operators, using the identity (1.1), we define m-complex symmetric operators as follows; an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an *m-complex symmetric operator* if there exists some conjugation C such that

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}CT^{m-j}C=0$$

for some positive integer m. In this case, we say that T is an m-complex symmetric operator with conjugation C. In particular, if $m=1$, T is called a 1-complex symmetric operator (simply a complex symmetric operator). The authors have studied spectral properties and local spectral properties of m-complex symmetric operators. In particular, they have shown that if T is an m-complex symmetric operator with the conjugation C, then T is decomposable if and only if T^* has the property (β) (see [9]). Set $\Delta_m(T) := \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}CT^{m-j}C$. Then T is an m-complex symmetric operator with conjugation C if and only if $\Delta_m(T)=0$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a *strict m-complex symmetric operator* if T is an m-complex symmetric operator but it is not an $(m-1)$-complex symmetric operator. Note that

$$T^*\Delta_m(T)-\Delta_m(T)(CTC)=\Delta_{m+1}(T). \tag{1.2}$$

Hence, if T is an m-complex symmetric operator with conjugation C, then T is an n-complex symmetric operator with conjugation C for all $n \geq m$. In sequel, it was shown from [10] that if m is even, then $\Delta_m(T)$ is complex symmetric with the conjugation C, and if m is odd, then $\Delta_m(T)$ is skew complex symmetric with the conjugation C. Moreover, we investigate conditions for $(m+1)$-complex symmetric operators to be m-complex symmetric operators and characterize the spectrum of $\Delta_m(T)$. All normal operators, algebraic operators of order 2, Hankel matrices, finite Toeplitz matrices, all truncated Toeplitz operators, some Volterra integration operators, nilpotent operators of order k, and nilpotent perturbations of Hermitian operators are included in the class of m-complex symmetric operators (see [14], [15], [16], [19], and [9] for more details). The class of m-complex symmetric operators is surprisingly large class.

Many authors have studied Hermitian, isometric, unitary, and normal operators perturbed by nilpotent operators (see [2], [6], [8], and [21], etc). In 2014, T. Bermudez, A. Martinon, V. Muller, and J. Noda ([6]) have been studied the perturbation of m-isometries by nilpotent operators. In light of m-complex symmetric operators, we consider the nilpotent perturbations of m-complex symmetric operators. In particular, we prove that if $T \in \mathcal{L}(\mathcal{H})$ is an m-complex symmetric operator and N is a nilpotent operator of order $n > 2$ with $TN = NT$, then $T+N$ is a $(2n+m-2)$-complex symmetric operator. Moreover, we investigate the decomposability of $T+A$

or TA where T is m-complex symmetric operators. Finally, we provide various spectral relations of such operators. As some applications of these results, we focus on Weyl type theorems for such operators.

2. Preliminaries

If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, $\sigma_{su}(T)$, $\Gamma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, $\sigma_e(T)$, $\sigma_{le}(T)$, $\sigma_{re}(T)$, $\sigma_b(T)$, $\sigma_w(T)$, $\sigma_{se}(T)$, and $\sigma_{es}(T)$ for the spectrum, the surjective spectrum, the compression spectrum, the point spectrum, the approximate point spectrum, the essential spectrum, the left essential spectrum, the right essential spectrum, Browder spectrum, Weyl spectrum, the semi-regular spectrum, and the essentially semi-regular spectrum of T, respectively.

An operator $T \in \mathcal{L}(H)$ is said to have *the single-valued extension property* (or SVEP) if for every open subset G of $\mathbb{C}$ and any $\mathcal{H}$-valued analytic function f on G such that $(T-\lambda)f(\lambda) \equiv 0$ on G, we have $f(\lambda) \equiv 0$ on G. For an operator $T \in \mathcal{L}(\mathcal{H})$ and for a vector $x \in \mathcal{H}$, the *local resolvent set* $\rho_T(x)$ of T at x is defined as the union of every open subset G of $\mathbb{C}$ on which there is an analytic function $f : G \to \mathcal{H}$ such that $(T-\lambda)f(\lambda) \equiv x$ on G. The *local spectrum* of T at x is given by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. We define the *local spectral subspace* of $T \in \mathcal{L}(\mathcal{H})$ by $H_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ for a subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Dunford's property* (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $\{f_n\}$ of $\mathcal{H}$-valued analytic functions on G such that $(T-\lambda)f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G, we get that $f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$ there are T-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that

$$\mathcal{H} = \mathcal{X} + \mathcal{Y}, \ \sigma(T|_{\mathcal{X}}) \subset \overline{U}, \text{ and } \sigma(T|_{\mathcal{Y}}) \subset \overline{V}.$$

It is well-known that

$$\begin{aligned} \text{Decomposable} &\Rightarrow \text{Bishop's property } (\beta) \\ &\Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}. \end{aligned}$$

The converse implications do not hold, in general (see [20] for more details).

We say that *Weyl's theorem holds* for $T \in \mathcal{L}(\mathcal{H})$ if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

where $\pi_{00}(T) = \{\lambda \in iso\sigma(T) : 0 < dim\ ker(T-\lambda) < \infty\}$ and $iso\Delta$ denotes the set of all isolated points of Δ. We say that *Browder's theorem holds* for $T \in \mathcal{L}(\mathcal{H})$ if $\sigma_b(T) = \sigma_w(T)$. We recall the definitions of some spectra;

$$\sigma_{ea}(T) := \cap\{\sigma_a(T+K) : K \in \mathcal{K}(\mathcal{H})\}$$

is the essential approximate point spectrum, and

$$\sigma_{ab}(T) := \cap\{\sigma_a(T+K) : TK = KT \text{ and } K \in \mathcal{K}(\mathcal{H})\}$$

is the Browder essential approximate point spectrum. We put

$$\pi_{00}(T) := \{\lambda \in \text{iso }\sigma(T) : 0 < dim\ker(T-\lambda) < \infty\}$$

and

$$\pi_{00}^a(T) := \{\lambda \in \text{iso }\sigma_{ap}(T) : 0 < dim\ker(T-\lambda) < \infty\}.$$

For an operator $T \in \mathcal{L}(\mathcal{H})$, we say that

(i) *a-Browder's theorem holds for* T if $\sigma_{ea}(T) = \sigma_{ab}(T)$;

(ii) *a-Weyl's theorem holds for* T if $\sigma_{ap}(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T)$;

(iii) T has *the property* (w) if $\sigma_{ap}(T) \setminus \sigma_{ea}(T) = \pi_{00}(T)$.

It is known that

$$\begin{array}{ccc} \text{Property } (w) & \Longrightarrow & a\text{-Browder's theorem} \\ \Downarrow & & \Uparrow \\ \text{Weyl's theorem} & \Longleftarrow & a\text{-Weyl's theorem.} \end{array}$$

We refer the reader to [1] for more details.

Let $T_n = T|_{\text{ran}(T^n)}$ for each nonnegative integer n; in particular, $T_0 = T$. If T_n is upper semi-Fredholm for some nonnegative integer n, then T is called a *upper semi-B-Fredholm* operator. In this case, by [7], T_m is a upper semi-Fredholm operator and $ind(T_m) = ind(T_n)$ for each $m \geq n$. Thus, one can consider the *index* of T, denoted by $ind_B(T)$, as the index of the semi-Fredholm operator T_n. Similarly, we define lower semi-B-Fredholm operators. We say that $T \in \mathcal{L}(\mathcal{H})$ is *B-Fredholm* if it is both upper and lower semi-B-Fredholm. In [7], Berkani proved that $T \in \mathcal{L}(\mathcal{H})$ is B-Fredholm if and only if $T = T_1 \oplus T_2$ where T_1 is Fredholm and T_2 is nilpotent. Let $SBF_+^-(\mathcal{H})$ be the class of all upper semi-B-Fredholm operators such that $ind_B(T) \leq 0$, and let

$$\sigma_{SBF_+^-}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(\mathcal{H})\}.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *B-Weyl* if it is B-Fredholm of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a B-Weyl operator }\}.$$

We say that $\lambda \in \sigma_{ap}(T)$ is a *left pole* of T if it has finite ascent, i.e., $a(T) < \infty$ and $\text{ran}(T^{a(T)+1})$ is closed where $a(T) = dim\ker(T)$. The notation $p_0(T)$ (respectively, $p_0^a(T)$) denotes the set of all poles (respectively, left poles) of T, while $\pi_0(T)$ (respectively, $\pi_0^a(T)$) is the set of all eigenvalues of T which is an isolated point in $\sigma(T)$ (respectively, $\sigma_{ap}(T)$).

3. Main Results

In this section, we study several properties of m-complex symmetric operators. Recall that an operator $N \in \mathcal{L}(\mathcal{H})$ is said to be *nilpotent* of order n if $N^n = 0$ and $N^{n-1} \neq 0$ for some positive integer n. It is well-known from [13, Theorem 5] that every nilpotent of order 2 is a complex symmetric (or 1-complex symmetric in our definition) operator. However if T is nilpotent of order n with $n > 2$, then T may not be a complex symmetric operator. We first give the following example of (strict) m-complex symmetric operators.

Example 3.1. *Let C be a conjugation given by $C(z_1, z_2, z_3) = (\overline{z_3}, \overline{z_2}, \overline{z_1})$ on $\mathbb{C}^3$. If $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix}$ on $\mathbb{C}^3$, then N is nilpotent of order 3 and $N^* \neq CNC$. Since $N^{*2} = CN^2C$, it follows that*

$$\begin{aligned} \Delta_3(N) = \sum_{j=0}^{3} (-1)^{3-j} \binom{3}{j} N^{*j} C N^{3-j} C &= -3N^{*2}CNC + 3N^*CN^2C \\ &= -3CN^3C + 3N^{*3} = 0. \end{aligned}$$

Hence N is a strict 3-complex symmetric operator with conjugation C.

On the other hand, let J be a conjugation given by $J(z_1, z_2, z_3) = (\overline{z_1}, \overline{z_2}, \overline{z_3})$ on $\mathbb{C}^3$. Then N is a 5-complex symmetric operator with conjugation J from [9]. Since $N^3 = 0$, we have

$$\sum_{j=0}^{4} (-1)^{4-j} \binom{4}{j} N^{*j} J N^{4-j} J = 6N^{*2}JN^2J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 294 \end{pmatrix} \neq 0.$$

Thus N is not a 4-complex symmetric operator. Hence N is a strict 5-complex symmetric operator with conjugation J.

In the following theorem, we examine conditions for the operator $T + N$ to be a $(2n + m - 2)$-complex symmetric operator.

Theorem 3.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be strict m-complex symmetric with a conjugation C and let N be nilpotent of order $n > 2$ with $TN = NT$. Then $T + N$ is a $(2n+m-2)$-complex symmetric operator with conjugation C.*

Proof. Let $R = T + N$ and $k = 2n + m - 2$. Since

$$\begin{aligned} [(a+b) - (c+d)]^k &= [\{(a-c) + b\} - d)]^k \\ &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} [(a-c) + b]^{k-i} d^i \\ &= \sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^i \binom{k}{i} \binom{k-i}{j} b^j (a-c)^{k-i-j} d^i \\ &= \sum_{k_1+k_2+k_3=m} \binom{k}{k_1, k_2, k_3} b^{k_3} (a-c)^{k_1} d^{k_2}, \end{aligned}$$

it follows that

$$\begin{aligned}\Delta_k(R) &= \sum_{k_1+k_2+k_3=k} \binom{k}{k_1, k_2, k_3} N^{*k_3}\Delta_{k_1}(T)CN^{k_2}C \\ &= \sum_{i=0}^{k}\sum_{j=0}^{k-i}(-1)^i\binom{k}{i}\binom{k-i}{j} N^{*j}\Delta_{k-i-j}(T)CN^iC. \end{aligned} \tag{3.1}$$

(i) If $j \geq n$ or $i \geq n$, then $N^{*j} = 0$ and $N^i = 0$. Hence (3.1) implies that $\Delta_k(R) = 0$ due to the fact that $N^n = 0$.
(ii) If $j < n$ and $i < n$, then

$$\begin{aligned} k - i - j &= (2n + m - 2) - i - j \\ &\geq 2n + m - 2 - (n - 1) - (n - 1) = m. \end{aligned}$$

Thus $\Delta_{k-i-j}(T) = 0$ and so $\Delta_k(R) = 0$ from (3.1). Hence $T + N$ is a $(2n + m - 2)$-complex symmetric operator with conjugation C. $\square$

From Theorem 3.2, we also know that $T+N$ is not necessarily a *strict* $(2n+m-2)$-complex symmetric operator. For example, if T is a complex symmetric operator and N is nilpotent of order $n > 2$ with $TN = NT$, then $T = T + N + (-N)$ is not a strict $(4n - 3)$-complex symmetric operator.

Example 3.3. *Let N be a nilpotent operator of order $n > 2$ with $N^* \neq CNC$. Then $I + N$ is an $(2n - 1)$-complex symmetric operator from Theorem 3.2. In particular, assume that C is a conjugation given by $C(z_1, z_2, z_3) = (\overline{z_1}, \overline{z_2}, \overline{z_3})$ on $\mathbb{C}^3$. If $R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = I + N$ where $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ on $\mathbb{C}^3$, then $N^3 = 0$ and $N^2 \neq 0$. Then we have $\Delta_4(R) = \Delta_4(N) = 6N^{*2}CN^2C \neq 0$. Hence R is a strict 5-complex symmetric operator from the previous note.*

Remark 3.4. *If we omit "strict" in Theorem 3.2, it is not necessarily that $T + N$ is a $(2n + m - 2)$-complex symmetric operator. For example, if $T = A \oplus 0$ and $N = 0 \oplus Q$ where A is an m-complex symmetric operator and Q is a nilpotent operator of order n, then it is clear that T is an m-complex symmetric operator, N is a nilpotent operator of order n, and T commutes with N. Hence $T+N = A\oplus Q$ is an k-complex symmetric operator for $k = max\{m, 2n - 1\}$.*

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$. We next study some properties of $\Delta_m(T)$.

Proposition 3.5. *Let T be $(m+1)$-complex symmetric with a conjugation C. If $\Delta_m(T)$ is hyponormal, then $ker(\Delta_m(T) - \lambda) \cap ker(\Delta_1(T) - \lambda) = \{0\}$ for any nonzero $\lambda \in \mathbb{C}$.*

Proof. If $x \in ker(\Delta_m(T) - \lambda) \cap ker(\Delta_1(T) - \lambda)$, then $\Delta_m(T)x = \Delta_1(T)x = \lambda x$. Since $ker(\Delta_m(T) - \lambda) \subset ker(\Delta_m(T) - \lambda)^*$, it follows from (1.2) that

$$\begin{aligned} 0 &= \langle \Delta_{m+1}(T)x, x\rangle = \langle [T^*\Delta_m(T) - \Delta_m(T)CTC]x, x\rangle \\ &= \langle \Delta_m(T)x, Tx\rangle - \langle CTCx, \Delta_m(T)^*x\rangle \end{aligned}$$

$$= \lambda\left(\langle T^*x,x\rangle - \langle CTCx,x\rangle\right)$$
$$= \lambda\langle \Delta_1(T)x,x\rangle = \lambda^2\|x\|^2.$$

Hence we have $x = 0$. $\square$

Corollary 3.6. *Let C be a conjugation operator on $\mathcal{H}$. Suppose that H and K are Hermitian operators which satisfy $HCK = KCH$ and $CSC \geq S$, where $S = i(HK - KH)$. For an operator $T = H + iK$, if T is 2-complex symmetric with the conjugation C, then $ker(\Delta_1(T) - \lambda) = \{0\}$ for any nonzero $\lambda \in \mathbb{C}$.*

Proof. If $T = H + iK$, then

$$\Delta_1(T) = T^* - CTC = (H - iK) - C(H + iK)C = \Delta_1(H) - i\Delta_1(K). \tag{3.2}$$

Since $\Delta_1(H)$ and $\Delta_1(K)$ are Hermitian, $HCK = KCH$, and $CSC \geq S$, it follows from (3.2) that

$$\begin{aligned}
\Delta_1(T)^*\Delta_1(T) - \Delta_1(T)\Delta_1(T)^* &= 2i[\Delta_1(K)\Delta_1(H) - \Delta_1(H)\Delta_1(K)]\\
&= 2i[-(HK - KH) + (HCK - KCH)C\\
&\quad +C(HCK - KCH) - C(HK - KH)C]\\
&= -2i(HK - KH) + C[2i(HK - KH)C]\\
&= 2(CSC - S) \geq 0.
\end{aligned}$$

Hence, $\Delta_1(T)$ is hyponormal and the proof follows by Proposition 3.5. $\square$

Lemma 3.7. *Let T be in $\mathcal{L}(\mathcal{H})$ and let C be a conjugation on $\mathcal{H}$. If T commutes with N and CN^*C, then*

$$\Delta_m(T + N) = \sum_{j=0}^{m} \binom{m}{j} \Delta_j(T)\cdot\Delta_{m-j}(N) \tag{3.3}$$

where $\Delta_0(T) = \Delta_0(N) = I$. In particular, if T is complex symmetric with the conjugation C, then

$$\Delta_m(T + N) = \Delta_m(N) \tag{3.4}$$

for any $m \in \mathbb{N}$.

Proof. Let $R = T + N$. If T commutes with N and CN^*C, then it holds

$$T\cdot CN^{*j}C = CN^{*j}C\cdot T \text{ and } N\cdot CT^{*j}C = CT^{*j}C\cdot N$$

for every positive integers j. Then (3.3) obviously holds for $m = 1$. Suppose that (3.3) holds for m. Then (1.2) and (3.3) imply

$$\begin{aligned}
\Delta_{m+1}(R) &= (T^* + N^*)\cdot\Delta_m(R) - \Delta_m(R)\cdot(CTC + CNC)\\
&= \sum_{j=0}^{m}\binom{m}{j}(T^* + N^*)\cdot\Delta_j(T)\cdot\Delta_{m-j}(N)\\
&\quad -\sum_{j=0}^{m}\binom{m}{j}\Delta_j(T)\cdot\Delta_{m-j}(N)\cdot(CTC + CNC)\\
&= \sum_{j=0}^{m}\binom{m}{j}\Big(T^*\cdot\Delta_j(T) - \Delta_j(T)\cdot CTC\Big)\Delta_{m-j}(N)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{m}\binom{m}{j}\Delta_j(T)\Big(N^*\cdot\Delta_{m-j}(N)-\Delta_{m-j}(N)\cdot CNC\Big)\\
=&\sum_{j=0}^{m}\binom{m}{j}\Delta_{j+1}(T)\cdot\Delta_{m-j}(N)+\sum_{j=0}^{m}\binom{m}{j}\Delta_j(T)\cdot\Delta_{m+1-j}(N)\\
=&\sum_{j=0}^{m+1}\binom{m+1}{j}\Delta_j(T)\cdot\Delta_{m+1-j}(N).
\end{aligned}
$$

Hence (3.3) holds for any positive integer m.

We will show the second statement. Suppose that T is complex symmetric with the conjugation C. By induction, we prove that $\Delta_m(R)=\Delta_m(N)$ for any $m\in\mathbb{N}$. If $m=1$, it is obvious. Assume that $\Delta_{m-1}(R)=\Delta_{m-1}(N)$. Since N and CN^*C commute with T, it follows that

$$
\begin{aligned}
T^*\Delta_{m-1}(N) &= T^*[\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}N^{*j}CN^{m-1-j}C]\\
&= [\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}N^{*j}CN^{m-1-j}C]T^*=\Delta_{m-1}(N)T^*.
\end{aligned}
$$

Moreover, since $CTC=T^*$ and T^* commutes with $\Delta_{m-1}(R)$, we obtain from (1.2) that

$$
\begin{aligned}
\Delta_m(R) &= R^*\Delta_{m-1}(R)-\Delta_{m-1}(R)CRC\\
&= (T^*+N^*)\Delta_{m-1}(N)-\Delta_{m-1}(N)(CTC+CNC)\\
&= (T^*+N^*)\Delta_{m-1}(N)-\Delta_{m-1}(N)(T^*+CNC)\\
&= N^*\Delta_{m-1}(N)-\Delta_{m-1}(N)CNC=\Delta_m(N).
\end{aligned}
$$

So this completes the proof. □

Proposition 3.8. *Let $T\in\mathcal{L}(\mathcal{H})$ commute with N and CN^*C where C is a conjugation on $\mathcal{H}$. If T is k-complex symmetric for all k with $0\le k\le(2l+k-2)$ and N is a nilpotent of order l, then $T+N$ is $(2l+k-2)$-complex symmetric. In particular, if T is complex symmetric with the conjugation C, then $T+N$ is $(2n-1)$-complex symmetric if and only if N is a nilpotent of order n.*

Proof. If T is m-complex symmetric and N is a nilpotent of order n, then $\Delta_m(T)=0$ and $\Delta_{2n-1}(N)=0$ from [9]. Thus (3.3) and (1.2) implies $\Delta_{2n+m-2}(T+N)=0$. Hence $T+N$ is $(2n+m-2)$-complex symmetric. The remaining cases also hold by a similar method.

For the second statement, if T is complex symmetric, then by (3.4), $T+N$ is $(2n-1)$-complex symmetric if and only if N is a nilpotent of order n. □

We next consider the decomposability of $T+A$ and TA where T is m-complex symmetric operator and A is an algebraic operator. For any set $G\subset\mathbb{C}$, we denote $G^*=\{\overline{z}:z\in G\}$.

Theorem 3.9. *Let $T \in \mathcal{L}(\mathcal{H})$ be an m-complex symmetric operator and A be an algebraic operator of order k. If $R = T + A$ or $R = TA$ where T commutes with A, then the following statements are equivalent:*
(i) *T is decomposable.*
(ii) *T^* has the property (β).*
(iii) *R is decomposable.*
(iv) *R^* has the property (β).*

Proof. Since the proof of (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) follow from [9, Theorem 4.7], we only consider the following implication (ii) $\Leftrightarrow$ (iv).

(1) In the case $R = T + A$. Assume that T^* has the property (β). Since A is an algebraic operator of order k, there exists a nonconstant polynomial $p(\lambda) = (\lambda - \gamma_1)(\lambda - \gamma_2)(\lambda - \gamma_3)\cdots(\lambda - \gamma_k)$ such that $p(A) = 0$. Set $p_0(\lambda) = 1$ and $p_j(\lambda) = (\lambda - \gamma_1)(\lambda - \gamma_2)\cdots(\lambda - \gamma_j)$ for $j = 1, 2, \cdots, k$. Let G be an open set in $\mathbb{C}$ and $f_n : G \to \mathcal{H}$ be a sequence of analytic functions such that

$$\lim_{n\to\infty} \|(T^* + A^* - z)f_n(z)\|_K = 0 \tag{3.5}$$

for every compact set K in D. Fix any compact subset K of D. Since

$$(A^* - \overline{\gamma_1})(A^* - \overline{\gamma_2})(A^* - \overline{\gamma_3})\cdots(A^* - \overline{\gamma_k}) = 0,$$

$p_{k-1}(A)^*A^* = \overline{\gamma_k}p_{k-1}(A)^*$. This gives that

$$\begin{aligned} &\lim_{n\to\infty} \|(T^* + \overline{\gamma_k} - z)p_{k-1}(A)^* f_n(z)\|_K \\ = &\lim_{n\to\infty} \|p_{k-1}(A)^*(T^* + A^* - z)f_n(z)\|_K = 0. \end{aligned} \tag{3.6}$$

Moreover, since $T^* + \overline{\gamma_k}$ has the property (β), we have

$$\lim_{n\to\infty} \|p_{k-1}(A)^* f_n(z)\|_K = 0. \tag{3.7}$$

Equations (3.5) and (3.7) imply that

$$\begin{aligned} &\lim_{n\to\infty} \|(T^* + \overline{\gamma_{k-1}} - z)p_{k-2}(A)^* f_n(z)\|_K \\ = &\lim_{n\to\infty} \|p_{k-2}(A)^*(T^* + A^* - z)f_n(z)\|_K = 0. \end{aligned}$$

Since $T^* + \overline{\gamma_{k-1}}$ has the property (β), we get that $\lim_{n\to\infty} \|p_{k-2}(A)^* f_n(z)\|_K = 0$. Hence, by induction we get that $\lim_{n\to\infty} \|f_n(z)\|_K = 0$. Therefore, R^* has the property (β).

(2) In the case $R = TA$. Assume that T^* has the property (β). Let G be an open set in $\mathbb{C}$ and $f_n : G \to \mathcal{H}$ be a sequence of analytic functions such that

$$\lim_{n\to\infty} \|(R^* - z)f_n(z)\|_K = \lim_{n\to\infty} \|(T^*A^* - z)f_n(z)\|_K = 0 \tag{3.8}$$

for every compact set K in D. Thus, it holds that

$$\lim_{n\to\infty} \|(A^* - \overline{\gamma_k})T^* f_n(z) + \overline{\gamma_k}T^* f_n(z) - z f_n(z)\|_K = 0. \tag{3.9}$$

Since $T^*A^* = A^*T^*$ and $p(A)^* = 0$, we obtain from (3.9) that

$$\lim_{n\to\infty} \|(\overline{\gamma_k}T^* - z)p_{k-1}(A)^* f_n(z)\|_K = 0. \tag{3.10}$$

In addition, since $\overline{\gamma_k}T^*$ has the property (β), (3.10) implies that

$$\lim_{n\to\infty} \|p_{k-1}(A)^* f_n(z)\|_K = 0. \tag{3.11}$$

Then we get from (3.8) that

$$\lim_{n\to\infty} \|(A^* - \overline{\gamma_{k-1}})T^* f_n(z) + \overline{\gamma_{k-1}}T^* f_n(z) - z f_n(z)\|_K = 0. \tag{3.12}$$

Since $T^*A^* = A^*T^*$ and $p(A)^* = 0$, we obtain from (3.12) that

$$\lim_{n\to\infty} \|(\overline{\gamma_{k-1}}T^* - z)p_{k-2}(A)^* f_n(z)\|_K = 0. \tag{3.13}$$

Moreover, since $\overline{\gamma_{k-1}}T^*$ has the property (β), (3.13) implies that

$$\lim_{n\to\infty} \|p_{k-2}(A)^* f_n(z)\|_K = 0. \tag{3.14}$$

Hence, by induction we get $\lim_{n\to\infty} \|f_n(z)\|_K = 0$, and so R^* has the property (β). The converse implication holds by similar arguments above. So this completes the proof. □

We observe that the order k of A played a role to eliminate A in the proof of Theorem 3.9. Moreover, we need an m-complex symmetric operator to prove (i) $\Leftrightarrow$ (ii)(see [9, Theorem 4.7]).

Corollary 3.10. *Let $T \in \mathcal{L}(\mathcal{H})$ be a complex symmetric operator and A be an algebraic operator of order k. If $R = T + A$ or $R = TA$ where T commutes with A, then the following statements are equivalent:*
(i) *T is decomposable.*
(ii) *T^* has the property (β).*
(iii) *T has the property (β).*
(iv) *R is decomposable.*
(v) *R^* has the property (β).*
(vi) *R has the property (β).*

Proof. Suppose that T is a complex symmetric operator. Since the implications (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii) hold by [20, Theorems 1.2.29 and 2.2.5], we consider the reverse implications. If T^* has the property (β), then T is decomposable from [9]. If T has the property (β), then T is decomposable from [19]. Therefore, we have (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii). Moreover, we get that (iii)$\Leftrightarrow$(vi) by a similar method. Hence we get this result from Theorem 3.9. □

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is called a 2-*normal* operator if T is unitarily equivalent to an operator matrix of the form $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ where T_i are mutually commuting normal operators.

Example 3.11. *Let $R \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ be a 2-normal operator. Then R is complex symmetric from* [16] *and R is unitarily equivalent to $\begin{pmatrix} N_1 & N_2 \\ 0 & N_3 \end{pmatrix}$. If $N_1N_2 = N_2N_3$, then $\begin{pmatrix} N_1 & 0 \\ 0 & N_3 \end{pmatrix}$ and $\begin{pmatrix} 0 & N_2 \\ 0 & 0 \end{pmatrix}$ commute and $\begin{pmatrix} 0 & N_2 \\ 0 & 0 \end{pmatrix}$ is nilpotent of order 2. Moreover,* *since ${N_1}^* \oplus {N_3}^*$ has the property (β), it follows that* $\begin{pmatrix} N_1 & N_2 \\ 0 & N_3 \end{pmatrix}$ *is decomposable from Theorem* 3.9*. Hence R is decomposable.*

Let us recall that for an operator $T \in \mathcal{L}(\mathcal{H})$, a closed subspace $\mathcal{M} \subset \mathcal{H}$ is *invariant* for T if $T\mathcal{M} \subset \mathcal{M}$, and it is *hyperinvariant* for T if it is invariant for every operator in the commutant $\{T\}^{'} = \{S \in \mathcal{L}(\mathcal{H}) : TS = ST\}$ of T. A subspace $\mathcal{M}$ of $\mathcal{H}$ is *nontrivial* if it is different from $\{0\}$ and $\mathcal{H}$. As some applications of Theorem 3.9, we get the following corollary.

Corollary 3.12. *Let $R = T + A$ or $R = TA$ be in $\mathcal{L}(\mathcal{H})$ where T is an m-complex symmetric operator and A is an algebriac operator of order k with $TA = AT$. If T^* has the property (β), then the following statements hold:*
(i) *R and R^* have the property (β) and the single-valued extension property.*
(ii) *If $\sigma(R)$ has nonempty interior, then R has a nontrivial invariant subspace.*
(iii) *$H_R(F)$ is a hyperinvariant subspace for R.*
(iv) *If f is any function analytic on a neighborhood of $\sigma(R)$, then both Weyl's and Browder's theorems hold for $f(R)$ and*

$$\sigma_w(f(R)) = \sigma_b(f(R)) = f(\sigma_w(R)) = f(\sigma_b(R)).$$

Proof. (i) From [20], we know that R is decomposable if and only if R and R^* have the property (β). Hence this completes the proof.

(ii) Since T^* has the property (β), it follows from Theorem 3.9 that R is decomposable. Moreover, since R has the property (β) by [20] and $\sigma(R)$ has nonempty interior, the proof follows from [12, Theorem 2.1].

(iii) If T^* has the property (β), then R is decomposable from Theorem 3.9. Therefore $H_R(F)$ is a spectral maximal space of R by [11, Proposition 3.8] and [20, Theorem 1.2.29]. Hence $H_R(F)$ is a hyperinvariant subspace for R.

(iv) Since $f(R)$ is decomposable from [20, p 145], it follows that $f(R)$ is clearly subscalar. Hence $f(R)$ satisfies Weyl's theorem from [1, p 175]. Moreover, since $f(R)$ has the single-valued extension property, Browder's theorem holds for $f(R)$ and the last relations are satisfied from [1, Theorem 3.71]. □

Proposition 3.13. *Let $R = T+N$ where $T \in \mathcal{L}(\mathcal{H})$ is an m-complex symmetric operator with a conjugation C and N is a nilpotent operator of order n with $TN = NT$. Then the following arguments hold;*
(i) *If T^* has the single-valued extension property, then R and R^* has the single-valued extension property.*
(ii) *If T has Dunford's property (C) and $\sigma_T(x) \subset \sigma_R(N^{n-1}x) \cap \sigma_R(x)$ for all $x \in \mathcal{H}$, then R has Dunford's property (C).*

Proof. (i) Let $R = T + N$. If T is m-complex symmetric and T^* has the single-valued extension property, then T has the single-valued extension property from [9, Theorem 4.10]. Let G be an open set in $\mathbb{C}$ and let $f : G \to \mathcal{H}$ be an analytic function such that $(R - z)f(z) \equiv 0$ on G, which implies

$$(T - z)f(z) + Nf(z) = 0. \tag{3.15}$$

Since $N^n = 0$ and $TN = NT$, it follows that $(T - z)N^{n-1}f(z) = 0$. Since T has the single-valued extension property, we have $N^{n-1}f(z) = 0$. Moreover, (3.15) implies $(T - z)N^{n-2}f(z) = 0$. Since T has the single-valued extension property, we get that $N^{n-2}f(z) = 0$. By similar process, we obtain that $f(z) = 0$. Hence R has the single-valued extension property. Similarly, we get that R^* have the single-valued extension property. Hence R and R^* have the single-valued extension property.

(ii) Let T have Dunford's property (C) and $\sigma_T(x) \subset \sigma_R(N^{n-1}x)$ for all $x \in \mathcal{H}$. Then it suffices to show that $\sigma_R(N^{n-1}x) \subset \sigma_T(x)$. Indeed, we assume $z_0 \in \rho_T(x)$. Then there is an $\mathcal{H}$-valued analytic function $f(z)$ in a neighborhood D of z_0 such that $(T - z)f(z) = x$ for every $z \in D$. Since $TN = NT$ and $N^n = 0$, it follows that

$$(R - z)N^{n-1}f(z) = (T - z)N^{n-1}f(z) \equiv N^{n-1}x \text{ on } D.$$

Since $N^{n-1}f(z)$ is analytic on D, we get $z_0 \in \rho_R(N^{n-1}x)$. Hence $\sigma_R(N^{n-1}x) \subset \sigma_T(x)$. Thus $\sigma_T(x) = \sigma_R(N^{n-1}x)$. Therefore, we have $N^{n-1}H_R(F) = H_T(F)$. Since $N^{n-1}H_R(F) \subset H_R(F)$, it follows that $H_T(F) \subset H_R(F)$ where F is a closed subset of $\mathbb{C}$. Moreover, since $\sigma_T(x) \subset \sigma_R(x)$ for all $x \in \mathcal{H}$, it follows that $H_R(F) \subset H_T(F)$ and so $H_R(F) = H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. Hence R has Dunford's property (C). This completes the proof. □

For an operator $T \in \mathcal{L}(\mathcal{H})$, the *quasinilpotent part* of T is defined by

$$H_0(T) := \{x \in \mathcal{H} : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Then $H_0(T)$ is a linear (not necessarily closed) subspace of $\mathcal{H}$. We remark from [3] that if T has the single-valued extension property, then

$$H_0(T - \lambda) = \{x \in \mathcal{H} : \lim_{n\to\infty} \|(T - \lambda)^n x\|^{\frac{1}{n}} = 0\} = H_T(\{\lambda\})$$

for all $\lambda \in \mathbb{C}$. It is well known from [1] and [3] that if $H_0(T - \lambda) = \{0\}$ for all $\lambda \in \mathbb{C}$, then T has the single-valued extension property.

Corollary 3.14. Let $R = T + N$ be in $\mathcal{L}(\mathcal{H})$ with the same hypotheses as in Proposition 3.13. If T^* has the single-valued extension property, then the following properties hold:
(i) $\sigma(R) = \sigma_{su}(R) = \sigma_{ap}(R) = \sigma_{se}(R)$.
(ii) $\sigma_{es}(R) = \sigma_b(R) = \sigma_w(R) = \sigma_e(R)$.
(iii) $H_0(R - \lambda) = H_R(\{\lambda\})$ and $H_{R^*}(\{\lambda\}) = H_0(R^* - \lambda)$ for all $\lambda \in \mathbb{C}$.

Proof. Since T^* has the single-valued extension property, it follows that R and R^* have the single-valued extension property from Proposition 3.13. Hence the proof follows from [1, Corollaries 2.45 and 3.53], and [3, Theorem 1.5]. □

We next state various spectral relations of m-complex symmetric operators.

Lemma 3.15. *If T is an m-complex symmetric operator, then the following relations hold;*
(i) $\sigma_p(T) \subseteq \sigma_p(T^*)^*$, $\sigma_{ap}(T) \subset \sigma_{ap}(T^*)^*$, $\Gamma(T^*)^* \subseteq \Gamma(T)$, $\sigma_{su}(T^*)^* \subseteq \sigma_{su}(T)$, *and*

$$\sigma(T) = \sigma_{ap}(T^*)^* = \sigma_{su}(T).$$

(ii) $\sigma_{le}(T) \subseteq \sigma_{le}(T^*)^*$, $\sigma_{re}(T^*)^* \subseteq \sigma_{re}(T)$, *and* $\sigma_e(T) = \sigma_{re}(T)$.
(iii) *If* T^* *has the single-valued extension property, then*

$$\sigma(T) = \sigma_{ap}(T) = \sigma_{ap}(T^*)^* = \sigma(T^*)^*.$$

Proof. (i) From [9, Theorem 4.1], $\sigma_p(T) \subseteq \sigma_p(T^*)^*$ and $\sigma_{ap}(T) \subset \sigma_{ap}(T^*)^*$. Since $\Gamma(S)^* = \sigma_p(S^*)$ and $\sigma_{su}(S)^* = \sigma_{ap}(S^*)$ for any $S \in \mathcal{L}(\mathcal{H})$, $\Gamma(T^*)^* \subseteq \Gamma(T)$ and $\sigma_{su}(T^*)^* \subseteq \sigma_{su}(T)$. On the other hand, since T is an m-complex symmetric operator, it follows from [17, Corollary, page 222] that $\sigma(T) = \sigma_{ap}(T) \cup \sigma_{ap}(T^*)^* \subseteq \sigma_{ap}(T^*)^*$. Since $\sigma_{su}(S)^* = \sigma_{ap}(S^*)$ for any $S \in \mathcal{L}(\mathcal{H})$, we get that $\sigma(T) \subseteq \sigma_{ap}(T^*)^* = \sigma_{su}(T) \subset \sigma(T)$. Hence we obtain

$$\sigma(T) = \sigma_{ap}(T^*)^* = \sigma_{su}(T).$$

(ii) If $\lambda \in \sigma_{le}(T)$, then there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $\{x_n\}$ weakly converges to 0 and $\lim_{n\to\infty} \|(T-\lambda)x_n\| = 0$ for any $T \in \mathcal{L}(\mathcal{H})$. Then we have $\lim_{n\to\infty}(CTC - \overline{\lambda})Cx_n = 0$. Since T is an m-complex symmetric operator with conjugation C, it follows that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left\| \left(\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} C T^{m-j} C \right) Cx_n \right\| \\ &= \lim_{n\to\infty} \left\| \left(\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} \overline{\lambda}^{m-j} \right) Cx_n \right\| \\ &= \lim_{n\to\infty} \|(T^* - \overline{\lambda})^m Cx_n\|. \end{aligned}$$

Moreover, since $\{x_n\}$ weakly converges to 0, $\{Cx_n\}$ weakly converges to 0. Hence we get that $\sigma_{le}(T) \subseteq \sigma_{le}(T^*)^*$. Since $\sigma_{re}(S)^* = \sigma_{le}(S^*)$ for any $S \in \mathcal{L}(\mathcal{H})$, it follows that $\sigma_{re}(T^*)^* \subseteq \sigma_{re}(T)$. Moreover, since $\sigma_e(S) = \sigma_{le}(S) \cup \sigma_{re}(S)$ for any $S \in \mathcal{L}(\mathcal{H})$, we obtain that

$$\sigma_e(T) = \sigma_{le}(T) \cup \sigma_{re}(T) \subseteq \sigma_{le}(T^*)^* \cup \sigma_{re}(T) = \sigma_{re}(T).$$

Since $\sigma_{re}(S) \subseteq \sigma_e(S)$ for any $S \in \mathcal{L}(\mathcal{H})$, we obtain that $\sigma_{re}(T) = \sigma_e(T)$.

(iii) If T^* has the single-valued extension property, then T has the single-valued extension property from [9]. Note that $\sigma(S)^* = \sigma(S^*)$ and $\sigma_{su}(S)^* = \sigma_{ap}(S^*)$ for any $S \in \mathcal{L}(\mathcal{H})$. Since T and T^* have the single-valued extension property, it follows from [20] that $\sigma(T)^* = \sigma(T^*) = \sigma_{su}(T^*) = \sigma_{ap}(T)^*$. Moreover, since $\sigma_{ap}(T) \subset \sigma_{ap}(T^*)^*$ by (i), it follows that $\sigma(T) = \sigma_{ap}(T) \subseteq \sigma_{ap}(T^*)^* \subseteq \sigma(T^*)^* = \sigma(T)$. Hence we get

$$\sigma(T) = \sigma_{ap}(T) = \sigma_{ap}(T^*)^* = \sigma(T^*)^*.$$

This completes the proof. □

Proposition 3.16. *Let* $T \in \mathcal{L}(\mathcal{H})$ *be an* m*-complex symmetric operator and* N *be a nilpotent operator of order* n *with* $TN = NT$. *If* $R = T + N$, *then the following properties hold:*
(i) $\sigma_p(R) \subset \sigma_p(T^*)^* \cup \{0\}$, $\Gamma(R^*)^* \subset \Gamma(T) \cup \{0\}$, $\sigma_{ap}(R) \subseteq \sigma_{ap}(T^*)^* \cup \{0\}$, *and* $\sigma_{ap}(R) \subseteq \sigma(T) \cup \{0\}$.
(ii) $\sigma_{le}(R) \subset \sigma_{le}(T)$ *and* $\sigma_{re}(R^*)^* \subset \sigma_{re}(T^*)^*$. *In addition, if* T^* *is an* m*-complex symmetric operator, then* $\sigma_e(R) \subseteq \sigma_e(T)$.

Proof. (i) Assume that $R = T + N$ where T is an m-complex symmetric operator, $N^n = 0$, and $TN = NT$. Since T commutes with N, it follows from Lemma 3.15 and [20, Page 256] that

$$\sigma_{ap}(R) \subseteq \sigma_{ap}(T) + \sigma_{ap}(N) \subseteq \sigma_{ap}(T^*)^* \cup \{0\}.$$

Hence $\sigma_{ap}(R) \subseteq \sigma(T) \cup \{0\}$ from Lemma 3.15. By the similar method, we get that $\sigma_p(R) \subset \sigma_p(T^*)^* \cup \{0\}$. On the other hand, since $\Gamma(S)^* = \sigma_p(S^*)$ for any $S \in \mathcal{L}(\mathcal{H})$ and the previous result, we conclude that $\Gamma(R^*)^* \subset \Gamma(T) \cup \{0\}$.

(ii) If $\lambda \in \sigma_{le}(R)$, then there exists a sequence $\{x_i\}$ of unit vectors in $\mathcal{H}$ such that $\{x_i\}$ weakly converges to 0 and $\lim_{i\to\infty} \|(R-\lambda)x_i\| = 0$. Put $y_i = \frac{N^{n-1}x_i}{\|N^{n-1}x_i\|}$ for some $n \geq 1$. Since T commutes with N and $N^n = 0$, it follows that

$$\begin{aligned}\lim_{i\to\infty} \|(T-\lambda)y_i\| &= \lim_{i\to\infty} \|(T-\lambda)\frac{N^{n-1}x_i}{\|N^{n-1}x_i\|}\| \\ &= \lim_{i\to\infty} \|N^{n-1}(T+N-\lambda)\frac{x_i}{\|N^{n-1}x_i\|}\| \\ &= \lim_{i\to\infty} \|N^{n-1}(R-\lambda)\frac{x_i}{\|N^{n-1}x_i\|}\| = 0.\end{aligned}$$

In addition, if $\{x_i\}$ weakly converges to 0, then $\{y_i\}$ weakly converges to 0. Therefore $\lambda \in \sigma_{le}(T)$. So, $\sigma_{le}(R) \subseteq \sigma_{le}(T)$. Since $\sigma_{re}(S)^* = \sigma_{le}(S^*)$ for any $S \in \mathcal{L}(\mathcal{H})$, we obtain $\sigma_{re}(R^*)^* \subset \sigma_{re}(T^*)^*$. If T^* is an m-complex symmetric operator, then we get $\sigma_{le}(R^*) \subset \sigma_{le}(T^*)$ in a similar way. Thus $\sigma_e(R) = \sigma_{le}(R) \cup \sigma_{re}(R) \subseteq \sigma_{le}(T) \cup \sigma_{re}(T) = \sigma_e(T)$. Hence $\sigma_e(R) \subseteq \sigma_e(T)$. This completes the proof. □

Finally, we deal with Weyl type theorems for m-complex symmetric operators.

Theorem 3.17. *Let $T \in \mathcal{L}(\mathcal{H})$ be m-complex symmetric. Suppose that T^* has the single-valued extension property. Then the following statements are equivalent:*
(i) *T^* satisfies a-Weyl's theorem.*
(ii) *T^* satisfies Weyl's theorem.*
(iii) *T^* has the property (w).*
In addition, the following statements are equivalent.
(iv) *T^* satisfies generalized a-Weyl's theorem.*
(v) *T^* satisfies generalized Weyl's theorem.*

Proof. (a) Suppose that T^* satisfies Weyl's theorem. Since T is m-complex symmetric, it follows from Lemma 3.15 that $\sigma_{ap}(T^*)^* = \sigma(T) = \sigma(T^*)^*$ and so $\sigma_{ap}(T^*) = \sigma(T^*)$. On the other hand, since $\sigma_{ea}(T^*) \subset \sigma_w(T^*)$ is obvious, it suffices to show $\sigma_w(T^*) \subset \sigma_{ea}(T^*)$. Indeed, if $\lambda \notin \sigma_{ea}(T^*)$, then $T^* - \lambda$ is semi-Fredholm and $ind(T^*-\lambda) \leq 0$. Since T^* has the single-valued extension property, it follows from [9] and [1] that $T = (T^*)^*$ has the single-valued extension property and $ind(T^*-\lambda) \geq 0$ for every $\lambda \notin \sigma_{ea}(T^*)$, respectively. Therefore $ind(T^*-\lambda) = 0$ for every $\lambda \notin \sigma_{ea}(T^*)$. Thus $\lambda \notin \sigma_w(T^*)$. Hence $\sigma_{ea}(T^*) = \sigma_w(T^*)$. This gives that

$$\pi_{00}^a(T^*) = \pi_{00}(T^*) = \sigma(T^*) \setminus \sigma_w(T^*) = \sigma_{ap}(T^*) \setminus \sigma_{ea}(T^*).$$

Hence a-Weyl's theorem holds for T^*. Similarly, since $\pi_{00}^a(T^*) = \pi_{00}(T^*)$, we can show that (i) $\Leftrightarrow$ (iii). It is clear that (i) $\Rightarrow$ (ii). So we have this result.

(b) By [5, Theorem 3.7], it suffices to prove that (ii) $\Rightarrow$ (i). Suppose that T^* satisfies generalized Weyl's theorem. Then $\sigma_{BW}(T^*) = \sigma(T^*) \setminus \pi_0(T^*)$. Since T is m-complex symmetric, it follows from Lemma 3.15 that $\sigma_{ap}(T^*) = \sigma(T^*)$ and so

$$\sigma_{BW}(T^*) = \sigma(T^*) \setminus \pi_0(T^*) = \sigma_{ap}(T^*) \setminus \pi_0^a(T^*).$$

Hence it suffices to show that $\sigma_{SBF_+^-}(T^*) = \sigma_{BW}(T^*)$. If $\lambda \notin \sigma_{SBF_+^-}(T^*)$, then $T^* - \lambda$ is semi-B-Fredholm and $ind_B(T^* - \lambda) \leq 0$. Since T is m-complex symmetric operator and T^* has the single-valued extension property, it follows from [1] that $ind_B(T^* - \lambda) \geq 0$ for every $\lambda \notin \sigma_{SBF_+^-}(T^*)$. Thus $ind_B(T^* - \lambda) = 0$ for every $\lambda \notin \sigma_{SBF_+^-}(T^*)$. Therefore $\sigma_{SBF_+^-}(T^*) \supset \sigma_{BW}(T^*)$. Since $\sigma_{SBF_+^-}(T^*) \subset \sigma_{BW}(T^*)$ is clear, we obtain that

$$\sigma_{SBF_+^-}(T^*) = \sigma_{BW}(T^*) = \sigma_{ap}(T^*) \setminus \pi_{00}^a(T^*).$$

Hence the generalized a-Weyl's theorem holds for T^*. □

Corollary 3.18. *Let $T \in \mathcal{L}(\mathcal{H})$ be an m-complex symmetric operator. Then the following arguments are equivalent:*
(i) *T^* satisfies Browder's theorem.*
(ii) *T^* satisfies a-Browder's theorem.*
(iii) *T^* satisfies the generalized Browder's theorem.*
(iv) *T^* satisfies the generalized a-Browder's theorem.*

Proof. Since it is well known that (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv) from [4, Theorem 2.1 and Theorem 2.2], we only consider (iii) $\Leftrightarrow$ (iv). Since $\sigma(T^*) = \sigma_{ap}(T^*)$ from Lemma 3.15, we have $p_0(T^*) = p_0^a(T^*)$. Moreover, $\sigma_{SBF_+^-}(T^*) = \sigma_{BW}(T^*)$ as in the proof of Theorem 3.17. Using these results, we get that (iii) $\Leftrightarrow$ (iv). This completes the proof. □

References

[1] Aiena, P., *Fredholm and local spectral theory with applications to multipliers*, Kluwer Academic Publ., 2004.

[2] Agler, J., *Sub-Jordan operators: Bishop's theorem, spectral inclusion, and spectral sets*, J. Oper. Theory, **7**(1982), no. 2, 373-395.

[3] Aiena, P., Colasante, M.L., González, M., *Operators which have a closed quasi-nilpotent part*, Proc. Amer. Math. Soc., **130**(2002), 2701-2710.

[4] Amouch, M., Zguitti, H., *On the equivalence of Browder's and generalized Browder's theorem*, Glasgow Math. J., **48**(2006), 179-185.

[5] Berkani, M., Koliha, J.J., *Weyl type theorems for bounded linear operators*, Acta Sci. Math., **69**(2003), 359-376.

[6] Bermúdez, T., Martinón, A., Müller, V., Noda, J., *Perturbation of m-isometries by nilpotent operators*, Abstr. Appl. Anal., **2**(2014), 313-328.

[7] Berkani, M., *On a class of quasi-Fredholm operators*, Int. Eq. Op. Th., **34**(1999), 244-249.

[8] Chō, M., Ota, S., Tanahashi, K., Uchiyama, M., *Spectral properties of m-isometric operators*, Functional Analysis, Application and Computation, **4**(2012), no. 2, 33-39.

[9] Chō, M., Ko, E., Lee, J., *On m-complex symmetric operators*, Mediterranean Journal of Mathematics, **13**(2016), no. 4, 2025-2038.

[10] Chō, M., Ko, E., Lee, J., *On m-complex symmetric operators, II*, Mediterranean Journal of Mathematics, **13**(2016), no. 5, 3255-3264.

[11] Colojoara, I., Foias, C., *Theory of generalized spectral operators*, Gordon and Breach, New York, 1968.

[12] Eschmeier, J., *Invariant subspaces for operators with Bishop's property (β) and thick spectrum*, J. Funct. Anal., **94**(1990), 196-222.

[13] Garcia, S.R., *Aluthge transforms of complex symmetric operators and applications*, Int. Eq. Op. Th., **60**(2008), 357-367.

[14] Garcia, S.R., Putinar, M., *Complex symmetric operators and applications*, Trans. Amer. Math. Soc., **358**(2006), 1285-1315.

[15] Garcia, S.R., Putinar, M., *Complex symmetric operators and applications, II*, Trans. Amer. Math. Soc., **359**(2007), 3913-3931.

[16] Garcia, S.R., Wogen, W.R., *Some new classes of complex symmetric operators*, Trans. Amer. Math. Soc., **362**(2010), 6065-6077.

[17] Halmos, P.R., *A Hilbert space problem book*, Springer-Verlag, Berlin - Heidelberg - New York, 1980.

[18] Helton, J.W., *Operators with a representation as multiplication by x on a Sobolev space*, Colloquia Math. Soc. Janos Bolyai, **5**, Hilbert Space Operators, Tihany, Hungary, 1970, 279-287.

[19] Jung, S., Ko, E., Lee, M., Lee, J., *On local spectral properties of complex symmetric operators*, J. Math. Anal. Appl., **379**(2011), 325-333.

[20] Laursen, K., Neumann, M., *An introduction to local spectral theory*, Clarendon Press, Oxford, 2000.

[21] McCullough, S., Rodman, L., *Hereditary classes of operators and matrices*, Amer. Math. Monthly, **104**(1997), no. 5, 415-430.

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A study of tube-like surfaces according to type 2 Bishop frame in Euclidean space

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Abstract. The main goal of this paper is the study of the classical differential geometry of a special kind of tube surfaces, so-called tube-like surface in 3-dimensional Euclidean space $\mathbf{E}^3$. It is generated by sweeping a space curve along another central space curve. In particular, the type 2 Bishop frame is considered and some important theorems are obtained for that one. Finally, an application is presented and plotted using computer aided geometric design.

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1. Introduction

In the study of the differential geometry of submanifolds, it is prevalent to examine different types of curvature conditions. Accurately, one is excited to determine all submanifolds satisfying such a condition. A delectable curvature property to study for a surface $\Sigma : \phi = \phi(u, v)$ in an Euclidean space $\mathbf{E}^3$ which requires the existence a functional relationship $\Gamma(k_1, k_2) = 0$ between the principal curvatures is called Weingarten surfaces or $W-surfaces$. With the use of the Gaussian and mean curvatures K and H respectively, we can redefine $W-surfaces$, as surfaces satisfying $\Gamma(K, H) = 0$, or, equivalently, the corresponding Jacobian determinant is identically zero, i.e.,

$$\Gamma(K, H) = \left| \frac{\partial(K, H)}{\partial(u, v)} \right| = 0.$$

Besides, if ϕ satisfies a linear equation $aK + bH = c$, $a, b, c \in \mathbb{R}$, $(a, b, c) \neq (0, 0, 0)$, then it said to be a linear Weingarten surface or $LW-surface$.

Here, when $a = 0$, a $LW-surface$ ϕ becomes a surface with constant mean curvature. Also, when $b = 0$, a $LW-surface$ ϕ will be a surface with constant Gaussian curvature. From this point, the linear Weingarten surfaces represent a natural generalization of surfaces which have constant mean curvature or constant Gaussian curvature.

As well known, in the differential geometry of curves, the curves are investigated by the well know Frenet-Serret equations because they are considered as the path of a moving particle in the Euclidean space. On the other hand, some researchers aimed to determine another moving frame for a regular curve. In 1975, Bishop pioneered "Bishop frame" by means of parallel vector fields. This frame is also called a "parallel" or "alternative" frame of the curves [4]. The important application of Bishop frame is that it is used in the area of biology and computer graphics. For example, it may be possible to compute information about the shape of sequences of DNA using a curve defined by Bishop frame. Also, it may provide a new way to control virtual cameras in computer animations [20]. In the present time a good deal of research has been done using Bishop frames [5, 6, 7, 10, 25]. Because of the importance of this frame, the authors in [29] introduced a new version of the Bishop frame and called it a type 2 Bishop frame which was studied in [11, 19].

Beside the above some geometries were interested with the study of Weingarten surfaces. For example in [27, 28], the Weingarten surfaces in Euclidean space were introduced by J. Weingarten in the context of the problem of finding all surfaces isometric to a given surface of revolution. Further, applications of these surfaces on computer aided design and shape investigation can be presented in [26]. Also, in the three dimensional Euclidean space, Munteanu and Nistor [17] and Lopez [14, 15] studied polynomial translation and cyclic linear Weingarten surfaces, respectively. In addition, Ro and Yoon [21] studied a tube of Weingarten types satisfying some equation in terms of the Gaussian curvature, mean curvature and second Gaussian curvature. Kim and Yoon [13] classified quadric surfaces in Euclidean 3-space in terms of the Gaussian curvature and the mean curvature while Yoon and Jun [31] classified non-degenerate quadric surfaces in Euclidean 3-space in terms of the isometric immersion and the Gauss map. Recently, in [23], the author was studied Weingarten tube-like surfaces in Euclidean 3-space. In a Minkowski 3-space $\mathbf{E}_1^3$, a classification of these surfaces is given in [1, 2, 8, 12, 16, 24].

This paper is devoted to use the new version of type 2 Bishop frame which was given in [29] to introduce a study for parametrization of a tube-like surface satisfying the Jacobi condition in Euclidean 3-space $\mathbf{E}^3$. Moreover, for $A, Q \in \{K, H, K_{II}\}$, we discuss the (A, Q)-Weingarten and linear Weingarten for that one. Thus, the geometry of such surface in terms of its intrinsic geometric formulas is established. An application of this surface is considered and plotted.

2. Geometric preliminaries

Let $\mathbf{E}^3$ be a Euclidean 3-space with the scalar product given by

$$g = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a standard rectangular coordinate system of $\mathbf{E}^3$. In particular, the norm of a vector $U \in \mathbf{E}^3$ is given by $\|U\| = \sqrt{\langle u, u\rangle}$. If $u = (u_1, u_2, u_3)$ and

$v = (v_1, v_2, v_3)$ are arbitrary vectors in $\mathbf{E}^3$, we define the vector product of u and v as the following

$$u \wedge v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)\,. \tag{2.1}$$

Let $\alpha = \alpha(t) : I \to \mathbf{E}^3$ be a space curve in $\mathbf{E}^3$. Denote by $\{e(t), p(t), q(t)\}$ the moving Frenet frame along the curve α, then the Frenet formulas are given by [22]

$$\frac{\partial}{\partial t}\begin{bmatrix} e(t) \\ p(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix}\begin{bmatrix} e(t) \\ p(t) \\ q(t) \end{bmatrix}, \tag{2.2}$$

where $e = \frac{\alpha'(t)}{\|\alpha'(t)\|}$, $p = \frac{e'(t)}{\|e'(t)\|}$ and $q = e(t) \wedge p(t)$ are the tangent, the principal normal and the binormal vector fields of the curve α, respectively. The functions κ and τ are called curvature and torsion of α, respectively. The prime ′ denotes the differentiation with respect to the $t-$parameter.

The type 2 Bishop formulas of α are defined by

$$\frac{\partial}{\partial t}\begin{bmatrix} N_1(t) \\ N_2(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\kappa_1(t) \\ 0 & 0 & -\kappa_2(t) \\ \kappa_1(t) & \kappa_2(t) & 0 \end{bmatrix}\begin{bmatrix} N_1(t) \\ N_2(t) \\ q(t) \end{bmatrix}. \tag{2.3}$$

For this frame, the vectors N_1, N_2 and q are the tangent, the principal normal, and the binormal vector fields of the curve α.

Here, the type 2 Bishop curvatures are defined by

$$\kappa_1(t) = -\tau \cos\theta(t), \tag{2.4}$$

$$\kappa_2(t) = -\tau \sin\theta(t). \tag{2.5}$$

It can be also shown that

$$\theta' = \kappa = \frac{f'}{1 + (f)^2}, \quad f = \frac{\kappa_2}{\kappa_1}.$$

We shall call the set $\{N_1, N_2, q, \kappa_1, \kappa_2\}$ as type 2 Bishop invariants of the curve $\alpha = \alpha(t)$.

The Bishop frame or parallel transport frame is an alternative to the Frenet frame. Thus, the matrix relation between type 2 Bishop and Frenet-Serret frames can be expressed as

$$\begin{bmatrix} e(t) \\ p(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} \sin\theta(t) & -\cos\theta(t) & 0 \\ \cos\theta(t) & \sin\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} N_1(t) \\ N_2(t) \\ q(t) \end{bmatrix}.$$

We denote a surface M in $\mathbf{E}^3$ by

$$\phi = \phi(s, t).$$

Let ζ be the standard unit normal vector field on the surface M defined by

$$\zeta = \frac{\phi_s \wedge \phi_t}{\|\phi_s \wedge \phi_t\|}, \quad \phi_s = \frac{\partial \phi}{\partial s}, \quad \phi_t = \frac{\partial \phi}{\partial t}. \tag{2.6}$$

Thus, we have the metric g_{ij} and the coefficients of the second fundamental form $h_{ij}, i,j=1,2$

$$g_{11}=\langle \phi_s,\phi_s\rangle,\ g_{12}=\langle \phi_s,\phi_t\rangle,\ g_{22}=\langle \phi_t,\phi_t\rangle. \tag{2.7}$$

$$h_{11}=\langle \phi_{ss},\zeta\rangle,\ h_{12}=\langle \phi_{st},\zeta\rangle,\ h_{22}=\langle \phi_{tt},\zeta\rangle, \tag{2.8}$$

where $\langle,\rangle$ is the Euclidean inner product.

Under this parametrization of the surface M, the Gaussian curvature K and the mean curvature H have the following forms [18]

$$K=\frac{Det\ (h_{ij})}{Det\ (g_{ij})}, \tag{2.9}$$

$$H=\frac{1}{2}\mathrm{tr}(g^{ij}h_{jk}), \tag{2.10}$$

where (g^{kl}) is the associated contravariant metric tensor field of the covariant metric tensor field (g_{kl}); that is, $g^{ik}g_{jk}=\delta^i_j$.

A surface M in a three-dimensional Euclidean space $\mathbf{E}^3$ with positive Gaussian curvature K possesses a positive definite second fundamental form II if appropriately orientated. Therefore, the second fundamental form defines a new Riemannian metric on M. In turn, we can consider the Gaussian curvature K_{II} of the second fundamental form which is regarded as a Riemannian metric. If a surface has non-zero Gaussian curvature everywhere, K_{II} can be defined formally and it is the curvature of the Riemannian manifold (M,II).

Definition 2.1. Given a surface M in the three- dimensional Euclidean space $\mathbf{E}^3$, the second Gaussian curvature is defined by [3]

$$K_{II}=\frac{1}{\left(h_{11}h_{22}-h_{12}^2\right)^2} \tag{2.11}$$

$$\times\left\{\begin{vmatrix} -\frac{1}{2}(h_{11})_{vv}+(h_{12})_{uv}-\frac{1}{2}(h_{22})_{uu} & \frac{1}{2}(h_{11})_u & (h_{12})_u-\frac{1}{2}(h_{11})_v \\ (h_{12})_v-\frac{1}{2}(h_{22})_u & h_{11} & h_{12} \\ \frac{1}{2}(h_{22})_v & h_{12} & h_{22} \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}(h_{11})_v & \frac{1}{2}(h_{22})_u \\ \frac{1}{2}(h_{11})_v & h_{11} & h_{12} \\ \frac{1}{2}(h_{22})_u & h_{12} & h_{22} \end{vmatrix}\right\}$$

Now, to serve our study it is important to consider the following definition:

Definition 2.2. [30] (1) A regular surface is flat (developable) if and only if its Gaussian curvature is identically zero.

(2) A regular surface for which the mean curvature vanishes identically is minimal surface.

(3) A non -developable surface is said to be II-flat if the second Gaussian curvature is equal to zero.

(4) A non- developable surface is called II -minimal if the second mean curvature is vanished.

3. Tube-like surface with type 2 Bishop frame in $\mathbf{E}^3$

In this section, we study a special case of surfaces in 3 dimensions, i.e., a tube-like surface that is generated by sweeping a space curve along another central space curve.

The tube-like surface can be obtained from the tube surface which is a special kind of the canal surface.

A canal surface is the envelope of a moving sphere with varying radius defined by the trajectory $\alpha(t)$ (center curve) of its center and a radius function $r(t)$. If the radius function $r(t)$ is a constant, then the canal surface is called a tube [9].

For a sufficiently small parameter $r > 0$ and by $\alpha(t)$ as a center curve with nonzero curvature, the tube-like surface of radius r with type 2 Bishop formulas (2.3) can be written as

$$\phi(s,t) = \alpha(t) + r[\cos s\ N_2(t) - \sin s\ q(t)], \tag{3.1}$$

where in general r can be a function of t. For fixed t, when s runs from 0 to 2π, we have a circle around the point $\alpha(t)$ in the N_2B plane. As we change t, this circle moves along the space curve α, and we will generate a tube-like surface along α.

Then, the two tangent vectors and the unit normal vector to the surface are given by

$$\begin{cases} \phi_s = -r[\sin s\ N_2 + \cos s\ q], \\ \phi_t = \Omega N_1 - r\kappa_2[\sin s\ N_2 + \cos s\ q], \\ \zeta = -\cos s\ N_2 + \sin s\ q, \quad \Omega = 1 - r\kappa_1 \sin s, \end{cases} \tag{3.2}$$

respectively. From (2.2) and (2.7) it is easily checked that the coefficients of the first fundamental form g_{11}, g_{12} and g_{22} of ϕ are given by

$$g_{11} = r^2, \quad g_{12} = r^2\kappa_2, \quad g_{22} = \Omega^2 + r^2\kappa_2^2,$$

From this, we have

$$g = r^2(\Omega^2 + r^2\kappa_2^2) - (r^2\kappa_2)^2. \tag{3.3}$$

This leads to the coefficients of the second fundamental form h_{11}, h_{12} and h_{22} of ϕ given by

$$h_{11} = r,\ h_{12} = r\kappa_2,\ h_{22} = r\kappa_2^2 - \kappa_1 \sin s + r\kappa_1^2 \sin^2 s.$$

It follows that

$$h = r(r\kappa_2^2 - \kappa_1 \sin s + r\kappa_1^2 \sin^2 s) - (r\kappa_2)^2. \tag{3.4}$$

Besides, the Gaussian curvature K and the mean curvature H of (3.1) are respectively, given by

$$K = -\frac{\kappa_1 \cos s}{\Omega r}, \tag{3.5}$$

$$H = \frac{1 - 2r\kappa_1 \cos s}{2\Omega r}. \tag{3.6}$$

If the second fundamental form of ϕ is non-degenerate, i.e., $h_{11}h_{22} - (h_{12})^2 \neq 0$, then the second Gaussian curvature K_{II} on $\phi(s,t)$ can be obtained

$$K_{II} = \frac{1}{4\Omega^4 r \sin^2 s}[1 + \sin^2 s - 6r\kappa_1 \sin^3 s + 4r^2\kappa_1^2 \sin^4 s]. \tag{3.7}$$

3.1. Tube-like surface of W- type

In the following, we study the tube-like surface ϕ in $\mathbf{E}^3$ satisfying the Jacobi equation $\Gamma(X,Y)=0$, $X\neq Y$, of the curvatures K, H and K_{II} of ϕ and we formulate the main results in the next theorems.

Theorem 3.1. *Let M be a tube-like surface in $\mathbf{E}^3$ defined by Eq. (3.1), then M is a (K,H) $W-$surface.*

Proof. Let M be a tube-like surface in $\mathbf{E}^3$. Differentiating K and H with respect to s and t respectively, then we obtain

$$K_s=-\frac{\kappa_1\cos s}{r\Omega^2},\quad K_t=-\frac{\kappa_1'\sin s}{r\Omega^2}, \tag{3.8}$$

$$H_s=-\frac{\kappa_1\cos s}{2\Omega^2},\quad H_t=-\frac{\kappa_1'\sin s}{2\Omega^2}. \tag{3.9}$$

By using (3.8) and (3.9), M satisfies identically the Jacobi equation

$$\phi(K,H)=K_sH_t-K_tH_s=0.$$

Therefore M is a $W-$*surface.*

Theorem 3.2. *Let M be a tube-like surface parameterized by (3.1) with non-degenerate second fundamental form in the Euclidean 3-space E^3. If M is a (K,K_{II}) $W-$surface, then $\kappa_1'=0$, i.e., the curvature of $\alpha(t)$ is a non-zero constant.*

Proof. Let M be a tube-like surface in $\mathbf{E}^3$ parameterized by (3.1). If we take derivative of K_{II} given by (3.7) with respect to s and t respectively, and using Eq. (3.8) then we have

$$(K_{II})_s=\frac{-1}{2r\Omega^3\sin^3 s}[1-r\kappa_1(2\cos^2 s+r\kappa_1\sin^3 s)\sin s]\cos s, \tag{3.10}$$

$$(K_{II})_t=\frac{\kappa_1'}{2\Omega^3\sin s}[\cos^2 s-\sin^2 s+r\kappa_1\sin^3 s]. \tag{3.11}$$

We consider the tube-like surface (3.1) in $\mathbf{E}^3$ satisfying the Jacobi equation

$$\phi(K,K_{II})=K_s(K_{II})_t-K_t(K_{II})_s=0, \tag{3.12}$$

with respect to the Gaussian curvature K and the second Gaussian curvature K_{II}. Then, substituting from (3.10) and (3.11) into (3.12), we get

$$\kappa_1'\cos s=0.$$

Since this polynomial is equal to zero for every s, its coefficient must be zero. Therefore, we conclude that $\kappa_1'=0$.

Theorem 3.3. *Let M be a tube-like surface parameterized by (3.1) with non-degenerate second fundamental form in the Euclidean 3-space $\mathbf{E}^3$. If M is a (H,K_{II}) $W-$surface, then $\kappa_1'=0$. This means that the curvature of $\alpha(t)$ is a non-zero constant.*

Proof. We assume that the tube-like surface given by (3.1) with non-degenerate second fundamental form in $\mathbf{E}^3$ is a (H, K_{II}) $W-$surface. Then, it satisfies the Jacobi equation

$$\phi(H, K_{II}) = H_s(K_{II})_t - H_t(K_{II})_s = 0. \tag{3.13}$$

Equations (3.9), (3.10), (3.11) and (3.13) lead to

$$\kappa_1' \cos s = 0. \tag{3.14}$$

From (3.14)*, one can get* $\kappa_1' = 0$. Thus, the curvature of $\alpha(t)$ is a non-zero constant.

4. Tube-like surface of LW- type

Now, to examine the linear Weingarten property of the tube-like surface ϕ defined along the space curve $\alpha(t)$. Let us analyze the following theorems.

Theorem 4.1. *Suppose that the tube-like surface defined by (*3.1*) in* $\mathbf{E}^3$ *is a* $LW-$*surface satisfying* $aK + bH = c$*, then* $\kappa_1 = 0$ *and* M *is an open part of a circular-like cylinder.*

Proof. Consider the parametrization (3.1) with K and H given by (3.5) and (3.6) respectively, then the relation

$$aK + bH = c,$$

implies

$$2\kappa_1(a + br - cr^2)\sin s - b + 2cr = 0. \tag{4.1}$$

Since $\sin s$ and 1 are linearly independent, we have

$$2\kappa_1(a + br - cr^2) = 0, \quad b = 2cr.$$

This leads to

$$\kappa_1(a + cr^2) = 0.$$

If $a + cr^2 \neq 0$, then $\kappa_1 = 0$. Thus, M is an open part of a circular-like cylinder.

Theorem 4.2. *Let* $(A, Q) \in \{(K, K_{II}), (H, K_{II})\}$*, then there are no* (A, Q) $LW-$*tube-like surfaces in Euclidean 3-space* $\mathbf{E}^3$.

Proof. Firstly, we suppose that the tube-like surface (3.1) with non-degenerate second fundamental form in $\mathbf{E}^3$. satisfies the equation

$$aK + bK_{II} = c. \tag{4.2}$$

By the aid of (3.5) and (3.7), the equation (4.2) takes the form

$$\frac{-1}{4r\Omega^2\sin^2 s}[-4r\kappa_1^2(a + br - cr^2)\sin^4 s$$
$$+2\kappa_1(2a + 3br - 4cr^2)\sin^3 s - (b - 4cr)\sin^2 s - b] = 0.$$

Since the identity holds for every s, all the coefficients must be zero. Therefore, we obtain

$$4r\kappa_1^2(a + br - cr^2) = 0,$$
$$2\kappa_1(2a + 3br - 4cr^2) = 0,$$
$$(b - 4cr) = 0, \quad b = 0.$$

Thus, we get $b = 0,\ c = 0$ and $\kappa_1 = 0$. In this case, the second fundamental form of M is degenerate.

Secondly, let the tube-like surface (3.1) with non-degenerate second fundamental form in $\mathbf{E}^3$ satisfy the relation

$$aH + bK_{II} = c. \tag{4.3}$$

From Equations (3.6), (3.7) and (4.3), we get

$$\begin{aligned}\frac{-1}{4r\Omega^2 \sin^2 s}[&-4r^2\kappa_1^2(a+b-cr)\sin^4 s\\ &+2r\kappa_1(3a+3b-4cr)\sin^3 s\\ &-(2a+b-4cr)\sin^2 s - b] = 0.\end{aligned}$$

Based on the above, one can obtain $b = 0,\ c = 0$ and $\kappa_1 = 0$. It indicates that the second fundamental form of the tube-like surface is degenerate. Then, there are no (H, K_{II})-linear Weingarten tube-like surfaces in $\mathbf{E}^3$.

5. Application

Now, as an application of our main results, we give the following example

Example 5.1. Consider the surface given by

$$\phi(s,t) = \alpha(t) + r(\cos s\ N_2(t) - \sin s\ q(t)), \tag{5.1}$$

where $\alpha(t)$ is given by

$$\alpha(t) = (\cos t, \sin t, t). \tag{5.2}$$

The Bishop frame $\{N_1(t), N_2(t), q(t)\}$ of the curve α is expressed as

$$\left\{\begin{aligned} N_1(t) &= \tfrac{1}{\sqrt{2}}(-\sin t, \cos t, 1),\\ N_2(t) &= -(\cos t, \sin t, 0),\\ q(t) &= \tfrac{1}{\sqrt{2}}(\sin t, -\cos t, 1)\end{aligned}\right. \tag{5.3}$$

Thus, the parametric form of the tube-like surface $\phi(s,t)$ can be written as

$$\phi(s,t) = (\lambda_1(s,t), \lambda_2(s,t), \lambda_3(s,t)), \tag{5.4}$$

where

$$\left\{\begin{aligned} &\lambda_1 = ((1 - r\cos s)\cos t - \tfrac{1}{\sqrt{2}} r \sin s \sin t),\\ &\lambda_2 = ((1 - r\cos s)\sin t + \tfrac{1}{\sqrt{2}} r \sin s \cos t),\\ &\lambda_3 = (t - \tfrac{1}{\sqrt{2}} r \sin s).\end{aligned}\right. \tag{5.5}$$

For this parametrization surface, the components of the first fundamental form are given by

$$\left\{\begin{aligned} &g_{11} = r^2, \quad g_{12} = \tfrac{-1}{2} r^2 \sin\theta(t),\\ &g_{22} = \tfrac{1}{4}[(2 + r\sin s\cos\theta(t))^2 + r^2\sin^2\theta(t)].\end{aligned}\right. \tag{5.6}$$

The unit normal vector of ϕ is obtained from (2.6) as

$$\zeta = \cos s\ N_2(t) - \sin s\ q(t). \tag{5.7}$$

Then the second fundamental form components of ϕ are as follows:

$$\begin{cases} h_{11} = -r, \quad h_{12} = \frac{1}{2} r \sin\theta(t), \\ h_{22} = \frac{1}{4}(-2\sin s\cos\theta(t) - r(\sin^2 s + \cos^2 \sin^2\theta(t)). \end{cases} \tag{5.8}$$

In addition, the Gaussian curvature K and the mean curvature H of ϕ are respectively, given by

$$K = \frac{\sin \mathrm{s} \cos\theta(t)}{2r + r^2 \sin \mathrm{s} \cos\theta(t)}, \tag{5.9}$$

$$H = \frac{-1}{r} + \frac{1}{r(2 + r\sin s\cos\theta(t))}. \tag{5.10}$$

Since $h_{11}h_{22} - h_{12}^2 \neq 0$, then we can get the second Gaussian curvature K_{II} on $\phi(s,t)$ as follows:

$$K_{II} = -\frac{\cos^2 \mathrm{s} + 2\sin^2 \mathrm{s} + 3r\sin^3 \mathrm{s}\cos\theta(t) + r^2\sin^4 \mathrm{s}\cos\theta(t)}{r(2 + r\sin \mathrm{s}\cos\theta(t))^2 \sin^2 \mathrm{s}}. \tag{5.11}$$

From aforementiomed data, one can deduce that when $\sin s = 0$, then from (5.9) and (5.10), we get $K = 0$ and $H = \frac{-1}{2r} =$ const., respectively.

Therefore, in the three dimensional Euclidean space $\mathbf{E}^3$, equations (5.9)-(5.11) show that:

The surface (5.4) is a (K, H) $W-$*surface* (Theorem 3.1.).

Besides, it is (K, K_{II}) and (H, K_{II}) $W-$*surface* (Theorems 3.2 and 3.3).

Moreover, it is an open part of a circular-like cylinder (Theorem 4.1).

In addition, there are no tube-like surfaces of types (K, K_{II}) and (H, K_{II}) $LW-$*surface* (Theorem 4.2).

We can easily see the graph of some tube-like surfaces generated by circular helix in Figures 1, 2, 3.

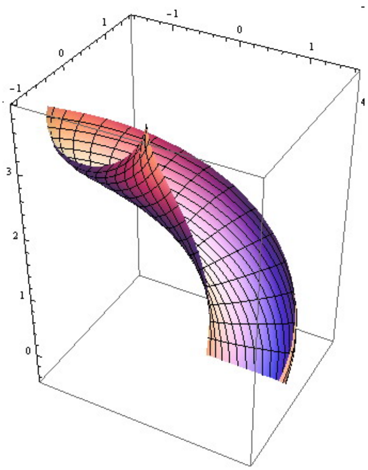

Figure 1. Tube-like surface generated by circular helix with $s \in [0, 1.1\pi]$, $t \in [0, 1.2\pi]$

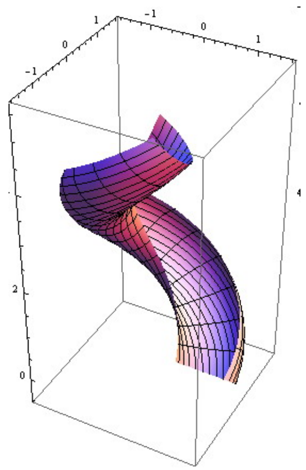

Figure 2. Tube-like surface generated by circular helix with $s \in [0, 1.2\pi]$, $t \in [0, 1.7\pi]$

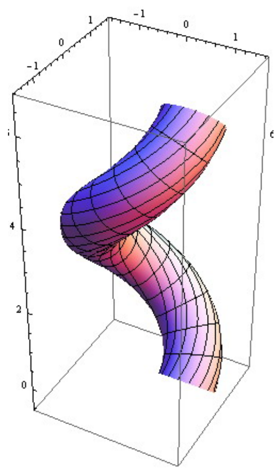

Figure 3. Tube-like surface generated by circular helix with $s \in [0, 2\pi]$, $t \in [0, 2\pi]$

6. Conclusion

In this paper, we proposed a definition of a tube-like surface in the three-dimensional Euclidean space E^3. It is generated by sweeping a space curve along another central space curve. We investigated the meant surface on satisfying some equations in terms of the Gaussian curvature K, the mean curvature H and the second Gaussian curvatures K_{II} using a new version of Bishop frame. As an application to demonstrate our theoretical results, we have given an example.

References

[1] Abdel-Aziz, H.S., Khalifa Saad, M., *Weingarten timelike tube surfaces around a spacelike curve*, Int. J. of Math. Analysis, **25**(2011), no. 5, 1225-1236.

[2] Abdel-Aziz, H.S., Khalifa Saad, M., Sezai Kiziltug, *Parallel Surfaces of Weingarten Type in Minkowski 3-Space*, Int. Math. Forum, **4**(2012), no. 7, 2293-2302.

[3] Baikoussis, C., Koufogiorgos, T., *On the inner curvature of the second fundamental form of helicoidal surfaces*, Arch. Math., **68**(1997), no. 2, 169-176.

[4] Bishop, L.R., *There is more than one way to frame a curve*, The American Mathematical Monthly, **82**(1975), no. 3, 246-251.

[5] Bükcü, B., Karacan, M.K., *Special Bishop motion and Bishop Darboux rotation axis of the space curve*, The Journal of Dynamical Systems and Geometric Theories, **6**(2008), 27-34.

[6] Bükcü, B., Karacan, M.K., *On the slant helices according to Bishop frame of the timelike curve in Lorentzian space*, Tamkang Journal of Mathematics, **39**(2008), no. 3, 255-262.

[7] Bükcü, B., Karacan, M.K., *The slant helices according to Bishop frame*, International Journal of Computational and Mathematical Sciences, **3**(2009), no. 2, 67-70.

[8] Dillen, F., Kühnel, W., *Ruled Weingarten surfaces in a Minkowski 3-space*, Manuscripta Math, **98**(1999), 307-320.

[9] Doğan, F., *A note on tubes*, Int. J. of Phys. and Math. Sciences, **3**(2013), no. 1, 98-105.

[10] Karacan, M.K., Bükcü, B., Yuksel, N., *On the dual Bishop Darboux rotation axis of the dual space curve*, Applied Sciences, **10**(2008), 115-120.

[11] Kiziltuğ, S., *On characterization of inextensible flows of curves according to type-2 Bishop frame E^3*, Mathematical and Computational Applications, **19**(2014), no. 1, 69-77.

[12] Kim, H.Y., Yoon, D.W., *Classification of ruled surfaces in a Minkowski 3-space*, J. Geom. Phys., **49**(2004), 89-100.

[13] Kim, H.Y., Yoon, D.W., *Weingarten quadric surfaces in a Euclidean 3-space*, Turk. J. Math, **35** (2011), 479-485.

[14] Lopez, R., *On linear Weingarten surfaces*, International J. Math., **19**(2008), 439-448.

[15] Lopez, R., *Special Weingarten surfaces foliated by circles*, Monatsh. Math., **154**(2008), 289-302.

[16] Lopez, R., *Parabolic Weingarten surfaces in hyperbolic space*, Publ. Math. Debrecen, **74**(2009), 59-80.

[17] Munteanu, M.I., Nistor, A.I., *Polynomial translation Weingarten surfaces in 3-dimensional Euclidean space*, Differential geometry, Worled Sci. Publ., Hackensack, NJ, (2009), 316-320.

[18] O'Neill, B., *Elementary Differential Geometry*, Academic Press, New York, 1966.

[19] Ozyilmaz, E., *Classical differential geometry of curves according to type-2 Bishop trihedra*, Mathematical & Computational Applications, **16**(2011), no. 4, 858-867.

[20] Petrović, M., Verstraelen, J., Verstraelen, L., *Principal normal spectral variations of space curves*, Proyecciones, **19**(2000), no. 2, 141-155.

[21] Ro, J., Yoon, D.W., *Tubes of Weingarten types in a Euclidean 3-space*, J. Chungcheong Math. Soc., **22**(2009), no. 3, 359-366.

[22] Shifrin, T., *Differential geometry: A first course in curves and surfaces*, Preliminary Version, Springer, 2015.

[23] Souror, A.H., *Weingarten tube-like surfaces in Euclidean 3-space*, Stud. Univ. Babeş-Bolyai Math., **61**(2016), no. 2, 239-250.

[24] Sodsiri, W., *Ruled surfaces of Weingarten type in Minkowski 3-space*, PhD. thesis, Katholieke Universiteit Leuven, Belgium, 2005.

[25] Süha, Y., *Bishop spherical images of a spacelike curve in Minkowski 3-space*, International Journal of Physical Sciences, **5**(2010), no. 6, 898-905.

[26] Van-Brunt, Grant, K., *Potential applications of Weingarten surfaces in CAGD, Part I: Weingarten surfaces and surface shape investigation*, Comput. Aided Geom. Des, **13**(1996), 569-582.

[27] Weingarten, J., *Ueber eine Klasse auf einander abwickelbarer Flaachen*, J. Reine Angew. Math., **59**(1861), 382-393.

[28] Weingarten, J., *Ueber die Flachen derer Normalen eine gegebene Flache beruhren*, J. Reine Angew. Math., **62**(1863), 61-63.

[29] Yılmaz, S., Turgut, M., *A new version of Bishop frame and an application to spherical images*, Journal of Mathematical Analysis and Applications, **371**(2010), no. 2, 764-776.

[30] Yilmaz, T., Murat, K.K., *Differential Geometry of three dimensions*, Syndic of Cambridge University Press, 1981.

[31] Yoon, D.W., Jun, J.S., *Non-degenerate quadric surfaces in Euclidean 3-space*, Int. J. Math. Anal., **6**(2012), no. 52, 2555-2562.

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Linear Weingarten factorable surfaces in isotropic spaces

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Abstract. In this paper, we deal with a certain factorable surface in the isotropic 3-space satisfying $aK+bH=c$, where K is the relative curvature, H the isotropic mean curvature and $a,b,c \in \mathbb{R}$. We obtain a complete classification for such surfaces. As a further study, we prove that a certain graph surface with $K=H^2$ is either a non-isotropic plane or a parabolic sphere.

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1. Introduction

Let M^2 be a regular surface of a Euclidean 3-space $\mathbb{R}^3$ and κ_1, κ_2 its principal curvatures. Then M^2 is called a *Weingarten surface* if the following non-trivial functional relation occurs:

$$\phi(\kappa_1, \kappa_2) = 0 \tag{1.1}$$

for a smooth function ϕ of two variables. (1.1) immediately yields

$$\delta(K, H) = 0, \tag{1.2}$$

where K and H are respectively the Gaussian and mean curvatures of M^2. (1.2) is equivalent to the vanishing of the corresponding Jacobian determinant, i.e. $|\partial(K,H)/\partial(u,v)| = 0$ for a coordinate pair (u,v) on M^2. If M^2 is a surface that satisfies

$$aH + bK = c,\ \ a,b,c \in \mathbb{R},\ \ (a,b,c) \neq (0,0,0), \tag{1.3}$$

then it is called a *linear Weingarten surface* (*LW-surface*). If $a=0$ or $b=0$ in (1.3), then the LW-surfaces reduce to the surfaces with constant curvature. Such surfaces have been extensively studied, see [7, 8], [13]-[17], [30].

On the other hand, a surface in $\mathbb{R}^3$ that is the graph of the function $z(x,y) = f(x)g(y)$ is said to be *factorable* or *homothetical*. In various ambient spaces, these surfaces have been desribed in terms of their curvatures and Laplace operator in [4, 9, 10, 12, 18, 19, 28, 29]. As distinct from the Euclidean case, a graph surface in the isotropic space $\mathbb{I}^3$ is said to be *factorable* if it is graph of either $z(x,y) = f(x)g(y)$ or $x(y,z) = f(y)g(z)$. We call them the *factorable surface of type 1* and *type 2*, respectively. Note that the factorable surface of one type cannot be carried into that of another type by the isometries of $\mathbb{I}^3$. These surfaces of both type in $\mathbb{I}^3$ with $K, H = const.$ were obtained in [1]-[3].

The main purpose of this paper is to obtain LW-factorable surfaces of type 1 in $\mathbb{I}^3$. As a further study, we classify the graph surfaces of the function $z = z(x,y)$ in $\mathbb{I}^3$ with $K = H^2$.

2. Preliminaries

For general references of the isotropic geometry, see [5], [23]-[27]. The isotropic 3-space $\mathbb{I}^3$ is a Cayley-Klein space defined from a 3-dimensional projective space $P(\mathbb{R}^3)$ with the absolute figure (ω, f_1, f_2), where ω is a plane in $P(\mathbb{R}^3)$ and f_1, f_2 are two complex-conjugate straight lines in ω. The homogeneous coordinates in $P(\mathbb{R}^3)$ are introduced in such a way that the *absolute plane* ω is given by $X_0 = 0$ and the *absolute lines* f_1, f_2 by $X_0 = X_1 + iX_2 = 0$, $X_0 = X_1 - iX_2 = 0$. The intersection point $F(0:0:0:1)$ of these two lines is called the *absolute point*. The affine coordinates in $P(\mathbb{R}^3)$ are given by $x_1 = \frac{X_1}{X_0}$, $x_2 = \frac{X_2}{X_0}$, $x_3 = \frac{X_3}{X_0}$. The group of motions of $\mathbb{I}^3$ is defined by

$$(x_1, x_2, x_3) \longmapsto (x_1', x_2', x_3') : \begin{cases} x_1' = a_1 + x_1 \cos\phi - x_2 \sin\phi, \\ x_2' = a_2 + x_1 \sin\phi + x_2 \cos\phi, \\ x_3' = a_3 + a_4 x_1 + a_5 x_2 + x_3, \end{cases}$$

where $a_1, ..., a_5, \phi \in \mathbb{R}$.

Consider the points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *isotropic distance* $d_{\mathbb{I}}(x,y)$ of two points x and y is defined as

$$d_{\mathbb{I}}(x,y) = (y_1 - x_1)^2 + (y_2 - x_2)^2.$$

The lines in x_3-direction are called *isotropic* lines. The plane containing an isotropic line is called an *isotropic plane.* Other planes are *non-isotropic.*

Let M^2 be a graph surface immersed in $\mathbb{I}^3$ corresponding to a real-valued smooth function $z = z(x,y)$ on an open domain $D \subseteq \mathbb{R}^2$. Then it is parameterized as follows:

$$r : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{I}^3, \ \ (x,y) \longmapsto (x, y, z(x,y)). \tag{2.1}$$

It follows from (2.1) that M^2 is an admissible (i.e. without isotropic tangent planes). The metric on M^2 induced from $\mathbb{I}^3$ is given by $g_* = dx^2 + dy^2$. This implies that M^2 is always flat with respect to the induced metric g_* and thus its Laplacian is of the form $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The *relative* (or *isotropic Gaussian*) *curvature* K and the *isotropic mean curvature* H of M^2 are defined by

$$K = z_{xx} z_{yy} - (z_{xy})^2 \tag{2.2}$$

and

$$H = \frac{\triangle z}{2} = \frac{z_{xx} + z_{yy}}{2}. \tag{2.3}$$

M^2 is called *isotropic minimal* (resp. *isotropic flat*) if H (resp. K) vanishes.

3. LW-factorable surfaces of type 1

Let M^2 be a factorable surface of type 1 in $\mathbb{I}^3$, i.e., the graph of $z(x,y) = f(x)g(y)$. By (2.2) and (2.3), we get

$$K = (f''f)(g''g) - (f')^2(g')^2 \tag{3.1}$$

and

$$2H = f''g + fg'', \tag{3.2}$$

where $f' = \frac{df}{dx}$ and $g' = \frac{dg}{dy}$, etc. We mainly aim to classify the LW-factorable surfaces of type 1 in $\mathbb{I}^3$. For this, let M^2 satisfy the relation (1.3). Since at least one of a, b and c is nonzero in (1.3), without loss of generality, we may assume $b \neq 0$. By dividing both sides of (1.3) with b and putting $\frac{a}{b} = 2m_0$ and $\frac{c}{b} = n_0$, we write

$$2m_0H + K = n_0, \;\; m_0, n_0 \in \mathbb{R}. \tag{3.3}$$

If $m_0 = 0$, M^2 turns to be a factorable surface of type 1 in $\mathbb{I}^3$ with $K = const.$ however such surfaces were already provided in [1]. In our framework, it is meaningful to take $m_0 \neq 0$. By $(3.1) - (3.3)$, we get

$$(f''f)(g''g) - (f')^2(g')^2 + m_0(f''g + fg'') = n_0. \tag{3.4}$$

We have to distinguish several cases in order to solve (3.4). Remark that the roles of f and g are symmetric, so discussing on the cases based on f shall be sufficient. From now on, we use the notation c_i to denote nonzero constants and d_i to denote some constants, $i = 1, 2, 3, ...$

Case 1. $f(x) = f_0 \in \mathbb{R} - \{0\}$. By (3.4), we find

$$g(y) = \frac{n_0}{2f_0m_0}y^2 + d_1y + d_2. \tag{3.5}$$

Thereby, M^2 is isotropic flat factorable surface of type 1 with $H = \frac{n_0}{2m_0}$.

Case 2. f is a linear function, i.e. $f(x) = c_1x + d_3$. It follows from (3.4) that

$$m_0d_3g'' - c_1^2(g')^2 + (m_0c_1g'')x = n_0. \tag{3.6}$$

(3.6) implies that $g'' = 0$, namely $g(y) = c_2y + d_4$. In this case, M^2 is isotropic minimal factorable surface of type 1 with $K = -(c_1c_2)^2$.

Case 3. f is a non-linear function. From the symmetry, g is also a non-linear function. By dividing (3.4) with the product ff'', we get

$$g''g - \frac{(f')^2}{ff''}(g')^2 + m_0\frac{g}{f} + m_0\frac{g''}{f''} = \frac{n_0}{ff''}. \tag{3.7}$$

By taking partial derivative (3.7) with respect to y and then dividing with $g'g''$, we deduce

$$1+\frac{gg'''}{g'g''}-2\frac{(f')^2}{ff''}+\left(\frac{m_0}{f}\right)\frac{1}{g''}+\left(\frac{m_0}{f''}\right)\frac{g'''}{g'g''}=0. \tag{3.8}$$

We have two cases:

Case 3.1. $g'''=0$, i.e.

$$g(y)=c_3y^2+d_5y+d_6. \tag{3.9}$$

Up to suitable translations of y, we may assume $d_5=d_6=0$. Then (3.8) reduces to

$$1-2\frac{(f')^2}{ff''}+\left(\frac{m_0}{2c_3}\right)\frac{1}{f}=0. \tag{3.10}$$

(3.10) can be rewritten as

$$\left(\frac{m_0}{2c_3}+f\right)f''-2(f')^2=0. \tag{3.11}$$

After solving (3.11), we find

$$f(x)=-\left(\frac{1}{c_4x+d_7}+\frac{m_0}{2c_3}\right). \tag{3.12}$$

Considering (3.9) and (3.12) into (3.4) gives the contradiction

$$x=-\frac{1}{c_4}\left(\frac{2m_0c_3}{n_0+m_0^2}+d_7\right)$$

due to the fact that x is an independent variable.

Case 3.2. $g'''\neq 0$. By taking partial derivatives of (3.8) with respect to x and y, we conclude

$$\left(\frac{f'}{f^2}\right)\frac{g'''}{(g'')^2}-\frac{f'''}{(f'')^2}\left(\frac{g'''}{g'g''}\right)'=0. \tag{3.13}$$

Due to $f'g'''\neq 0$, neither f''' nor $\left(\frac{g'''}{g'g''}\right)'$ can vanish in (3.13). Then (3.13) can be rewritten as

$$\frac{f'(f'')^2}{f^2f'''}=\frac{(g'')^2}{g'''}\left(\frac{g'''}{g'g''}\right)'. \tag{3.14}$$

Since the left side of (3.14) is a function of x, however the right side is a function of y. Then both sides have to be equal a nonzero constant, namely

$$\frac{f'(f'')^2}{f^2f'''}=c_5=\frac{(g'')^2}{g'''}\left(\frac{g'''}{g'g''}\right)'. \tag{3.15}$$

From the left side of (3.15), we write

$$\frac{f'''}{(f'')^2}=\frac{1}{c_5}\frac{f'}{f^2} \tag{3.16}$$

or, by taking once integral with respect to x,

$$f''=\frac{c_5f}{c_5d_8f+1}. \tag{3.17}$$

Likewise, by the right side of (3.15), we deduce

$$\frac{g'''}{g'g''} = \frac{-c_5}{g''} + d_9. \tag{3.18}$$

Substituting (3.17) and (3.18) into (3.8) yields

$$\begin{aligned} &1 + (m_0 d_8 + g)\, d_9 - \frac{c_5 (m_0 d_8 + g)}{g''} + \frac{m_0 d_9}{c_5 f} - \\ &- \frac{2 (c_5 d_8 f + 1) (f')^2}{c_5 f^2} = 0. \end{aligned} \tag{3.19}$$

Taking partial derivative of (3.19) with respect to y and considering (3.18) leads to

$$g'' = -c_5 (g + m_0 d_8). \tag{3.20}$$

After substituting (3.20) into (3.19), we conclude

$$2 + m_0 d_8 d_9 + d_9 g + \frac{m_0 d_9}{c_5 f} - \frac{2 (f')^2}{f f''} = 0,$$

which yields $d_9 = 0$ and $f f'' = (f')^2$. Solving this one gives $f(x) = c_6 \exp(c_7 x)$. By putting this in (3.4) we derive the polynomial equation on (f):

$$c_7^2 \left[g g'' - (g')^2 \right] f^2 + m_0 \left(c_7^2 g + g'' \right) f - n_0 = 0,$$

which implies that the coefficients must be zero; namely $n_0 = 0$,

$$g g'' - (g')^2 = 0 \text{ and } c_7^2 g + g'' = 0. \tag{3.21}$$

(3.21) leads to the contradiction $c_7^2 g^2 + (g')^2 = 0$ and therefore we have proved the following:

Theorem 3.1. *Let M^2 be a LW-factorable surface of type 1 which is the graph of $z(x,y) = f(x) g(y)$ in $\mathbb{I}^3$. Then we have either*
(A) $f(x) = f_0 \in \mathbb{R} - \{0\}$, $g(y) = c_6 y^2 + d_{10} y + d_{11}$;
(B) *or* $z(x,y) = (c_7 x + d_{12})(c_8 y + d_{13})$.

4. Graph surfaces with $K = H^2$

Let M^2 be a surface of the Euclidean 3-space $\mathbb{R}^3$. The Euler inequality for M^2 including the Gaussian and mean curvature follows

$$K \leq H^2. \tag{4.1}$$

The equality sign of (4.1) holds on M^2 if and only if it is totally umbilical, i.e. a part of a plane or a two sphere in $\mathbb{E}^3$. For more generalizations, see [6, 11], [20]-[22]. Now we are interested in the factorable surfaces of type 1 in $\mathbb{I}^3$ satisfying $K = H^2$. For this, let us reconsider (3.1) and (3.2). If $K = H^2$, then

$$(f'' g - f g'')^2 + 4 (f' g')^2 = 0. \tag{4.2}$$

(4.2) immediately implies that

$$f'' g - f g'' = 0 \text{ and } f' g' = 0. \tag{4.3}$$

By (4.3) we conclude that either $f = const.$ and $g(y) = c_1 y + d_1$ or $g = const.$ and $f(x) = c_2 x + d_2$, which yields the following result:

Proposition 4.1. *The factorable surfaces of type 1 in $\mathbb{I}^3$ satisfying $K = H^2$ are only non-isotropic planes.*

As a generalization, we are able to investigate the graph surfaces of type 1 in $\mathbb{I}^3$ satisfying $K = H^2$. More precisely, let M^2 be a graph surface of $z = z(x, y)$ in $\mathbb{I}^3$. If $K = H^2$ on M^2, then we get

$$(z_{xx} - z_{yy})^2 + 4(z_{xy})^2 = 0, \tag{4.4}$$

which yields that

$$z_{xy} = 0 \tag{4.5}$$

and

$$z_{xx} = z_{yy}. \tag{4.6}$$

By (4.5), we derive

$$z(x, y) = \alpha(x) + \beta(y) \tag{4.7}$$

and considering (4.7) into (4.6) gives

$$\frac{d^2\alpha}{dx^2} = \frac{d^2\beta}{dy^2} = d_3, \ d_3 \in \mathbb{R}. \tag{4.8}$$

By solving (4.8), we find

$$\alpha(x) = \frac{d_3}{2}x^2 + d_4 x + d_5, \ \beta(y) = \frac{d_3}{2}y^2 + d_6 y + d_7. \tag{4.9}$$

(4.9) implies that M^2 is either a non-isotropic plane $(d_3 = 0)$ or a parabolic sphere $(d_3 \neq 0)$. Consequently, we have

Theorem 4.2. *A graph surface of a function $z = z(x, y)$ in $\mathbb{I}^3$ with $K = H^2$ is either (a piece of) a non-isotropic plane or (a piece of) a parabolic sphere given by*

$$z(x, y) = c_3\left(x^2 + y^2\right) + d_8 x + d_9 y + d_{10}.$$

References

[1] Aydin, M.E., Ergut, M., *Isotropic geometry of graph surfaces associated with product production functions in economics*, Tamkang J. Math., **47**(4)(2016), 433-443.

[2] Aydin, M.E., *Constant curvature factorable surfaces in 3-dimensional isotropic space*, arXiv:1612.02458 [math.DG].

[3] Aydin, M.E., Ogrenmis, A.O., *Homothetical and translation hypersurfaces with constant curvature in the isotropic space*, BSG Proceedings, **23**(2016), 1-10.

[4] Bekkar, M., Senoussi, B., *Factorable surfaces in the three-dimensional Euclidean and Lorentzian spaces satisfying $\triangle r_i = \lambda_i r_i$*, J. Geom., **103**(2012), 17-29.

[5] Chen, B.Y., Decu, S., Verstraelen, L., *Notes on isotropic geometry of production models*, Kragujevac J. Math., **38**(2014), no. 1, 23-33.

[6] Chen, B.Y., *Mean curvature and shape operator of isometric immersions in real-space-forms*, Glasg. Math. J., **38**(1996), 87-97.

[7] Dillen, F., Kuhnel, W., *Ruled Weingarten surfaces in Minkowski 3-space*, Manuscripta Math., **98**(1999), 307-320.

[8] Galvez, J.A., Martinez, A., Milan, F., *Linear Weingarten surfaces in R^3*, Monatsh. Math., **138**(2003), 133-144.

[9] Goemans, W., Van de Woestyne, I., *Translation and homothetical lightlike hypersurfaces of semi-Euclidean space*, Kuwait J. Sci. Eng., **38**(2011), no. 2A, 35-42.

[10] Gray, A., Abbena, E., Salamon, S., *Modern differential geometry of curves and surfaces with mathematica*, Third Edition, Chapman and Hall/CRC Press, New York, 2006.

[11] Hou, Z.H., Ji, F., *Helicoidal surfaces with $H^2 = K$ in Minkowski 3-space*, J. Math. Anal. Appl., **325**(2007), 101-113.

[12] Jiu, L., Sun, H., *On minimal homothetical hypersurfaces*, Colloq. Math., **109**(2007), 239-249.

[13] Kim, M.H., Yoon, D.W., *Weingarten quadric surfaces in a Euclidean 3-space*, Turk. J. Math., **35**(2011), 479-485.

[14] Kuhnel, W., *Ruled W-surfaces*, Arch. Math., **62**(1994), 475-480.

[15] Lee, C.W., *Linear Weingarten rotational surfaces in pseudo-Galilean 3-space*, Int. J. Math. Anal., **9**(50)(2015), 2469-2483.

[16] Liu, H., Liu, G., *Weingarten rotation surfaces in 3-dimensional de Sitter space*, J. Geom., **79**(2004), 156-168.

[17] Lopez, R., *Rotational linear Weingarten surfaces of hyperbolic type*, Israel J. Math., **167**(2008), 283-301.

[18] Lopez, R., Moruz, M., *Translation and homothetical surfaces in Euclidean space with constant curvature*, J. Korean Math. Soc., **52**(2015), no. 3, 523-535.

[19] Meng, H., Liu, H., *Factorable surfaces in Minkowski space*, Bull. Korean Math. Soc., **46**(2009), no. 1, 155-169.

[20] Mihai, I., *On the generalized Wintgen inequality for Lagrangian submanifolds in complex space forms*, Nonlinear Analysis, **95**(2014), 714-720.

[21] Mihai, I., *On the generalized Wintgen inequality for Legendrian submanifolds in Sasakian space forms*, Tohoku J. Math, to appear.

[22] Mihai, A., *Geometric inequalities for purely real submanifolds in complex space forms*, Results Math., **55**(2009), 457-468.

[23] Milin-Sipus, Z., *Translation surfaces of constant curvatures in a simply isotropic space*, Period. Math. Hung., **68**(2014), 160-175.

[24] Pottmann, H., Opitz, K., *Curvature analysis and visualization for functions defined on Euclidean spaces or surfaces*, Comput. Aided Geom. Design, **11**(1994), 655-674.

[25] Pottmann, H., Grohs, P., Mitra, N.J., *Laguerre minimal surfaces, isotropic geometry and linear elasticity*, Adv. Comput. Math., **31**(2009), 391-419.

[26] Sachs, H., *Isotrope Geometrie des Raumes*, Vieweg-Verlag, Braunschweig, Wiesbaden, 1990.

[27] Strubecker, K., *Differentialgeometrie des isotropen Raumes III*, Flachentheorie, Math. Zeitsch., **48**(1942), 369-427.

[28] Van de Woestyne, I., *Minimal homothetical hypersurfaces of a semi-Euclidean space*, Results Math., **27**(1995), 333-342.

[29] Yu, Y., Liu, H., *The factorable minimal surfaces*, Proceedings of The Eleventh International Workshop on Diff. Geom., **11**(2007), 33-39.

[30] Yoon, D.W., Tuncer, Y., Karacan, M.K., *Non-degenerate quadric surfaces of Weingarten type*, Annales Polonici Math., **107**(2013), 59-69.

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Book reviews

Keith Devlin; Finding Fibonacci. The quest to rediscover the forgotten mathematical genius who changed the world.

Princeton University Press, Princeton NJ, 2017, vi+241 p., ISBN: 978-0-691-17486-0/hbk; 978-1-4008-8553-4/ebook

Keith Devlin is a professional mathematician and a successful popular science writer with 35 published books, both academic and for the general public as well, commentator to a weekly emission of the National Public Radio (USA), known as "the Math Guy". In 2002, intrigued by the scarcity of known facts about the distinguished Italian mathematician of the Middle Ages, Leonardo da Pisa (or Pisano), known also as Fibonacci (cca. 1175-1250), the author embarked on a quest to fill in this gap. After several visits in Italy, consultations with some Italian mathematicians and historians and manuscript hunting over several Italian archives, the conclusions were published in two books: *The Man of Numbers: Fibonacci's Arithmetic Revolution*, Walker Books (2011), 192 pp, and *Leonardo and Steve: the young genius who beat Apple to market by 800 Years*, Ted Weinstein (2011), e-book. In these books he analyzes the great influence the books written by Fibonacci (particularly *Liber Abbaci*) had on the development of knowledge and economy (mainly the trade) in that period. The revolutionary idea of Fibonacci was the introduction of the Hindu-Arabic decimal system of numeration, the rules for the arithmetic operations done using this system, and practical applications to everyday life. The author arrives at the conclusion that some shorter, practical versions (devoted to general public, written in local dialects) of the book that circulated in that period and later, have all at their origin a short version written by Fibonnaci himself.

In the present book the author tells the story of this quest - the people he met and who helped him, descriptions of places he visited and some happy events that made his plan realizable. The author mentions three major lucky events of this kind:

- the meeting in 2001 with the Italian historian of medieval mathematics at the University of Siena - Rafaella Franci;
- the translation in English (the first and the only) of *Liber Abbaci* by Laurence Sigler, completed after his death by his wife Judith Sigler, published with Springer in 2002;
- a paper, *Fibonacci and the financial revolution* (20 pp), published by William Goetzmann in 2004 (discussed in Ch. 15, *Leonardo and the modern finance*).

All these are described in the book, along with some information on some geographic and touristic aspects from Italy (with photographes) and details of the discussions and on the people he met. A chapter, (Ch. 14, *This will change the world*) is dedicated to the parallelism between the revolution done by Fibonacci and that done by Steve Jobs with the introduction of personal computers, in particular of Apple Macintosh in 1984 - both were done by a single person and involved computation with target to the marketplace. This is treated at large in the above mentioned book on Leonardo and Steve.

Written in the alert and attractive style characteristic to all popular writings of the author, with a lot of information of various various kind - personal, about people and places, historical, mathematical - this book, based on a diary kept by the author, will attract a large audience interested to know the story of this genius of the Middle Ages whose books influenced so much de development of the modern Western civilization up to our days, unfairly forgotten and neglected until the sixtieth of the last century.

S. Cobzaş

Petro-Luciano Buono; Advanced Calculus. Differential Calculus and Stokes' Theorem, De Gruyter Textbook, De Gruyter, Berlin/Boston 2016, x+303 p., (ISBN 978-3-11-043821-5/pbk; 978-3-11-043822-2/ebook.

The book is based on the notes of a one-semester Calculus III course at the University of Ontario Institute of Technology starting with 2012. Its aim is to give a unified treatment of Green's, Stokes and Gauss' theorems (in $\mathbb{R}^2$ and $\mathbb{R}^3$), paving the way to more advanced topics from differential geometry. The approach proposed by the author has a geometric flavor, the tangent space being introduced early in the study of differentiability of functions of one variable, differential forms and pullbacks. The main advantage of this approach, based on tools from linear algebra, consists in the possibility to define the differential of a function properly, as acting on tangent vectors, and from there the study of differential forms and pullbacks in the context of line integrals. As the author mentions in the Preface, one starts with the introduction of terminology in the context of curves (one-dimensional geometric objects, easier to understand) and then, after the introduction of the differentials of vector functions of several variables and of the Jacobian, one extends the differential form concepts to higher dimensions.

The one dimensional case is treated in Chapters 2, *Calculus of vector functions*, 3, *Tangent spaces and 1-forms*, and 4, *Line integrals*. The first chapter of the book contains some preliminary results form set theory, linear algebra, curves and surfaces (with illustrations in $\mathbb{R}^2$ and $\mathbb{R}^3$).

The general case of differential calculus for mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ is considered in Chapters 5, *Differential calculus of mappings*, and 6, *Applications of differential calculus* (including the study of extrema, parametrizations of curves and surfaces). Chapter 7, *Double and triple integrals*, contains a presentation of Riemann integral for domains in $\mathbb{R}^2$ and $\mathbb{R}^3$ and a proof of Green's theorem in $\mathbb{R}^2$.

General k-forms a treated in Chapter 8, *Wedge products and exterior derivatives*, and their integration in Chapter 9, *Integration of forms* (pullbacks, change of variables, orientation of surfaces). Stokes' theorem (in $\mathbb{R}^3$) is proved in Chapter 10, *Stokes' theorem and applications* (including a version for vector fields).

The characteristic features of the book are the abundance of worked examples, illustrated by nicely drawn suggestive figures and the excellent layout (the author promises to make available to the mathematical community the codes of the figures). There are also exercises at the end of each section.

The book is clearly written, in a pleasant style, and can be recommended as a textbook for advanced calculus courses.

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