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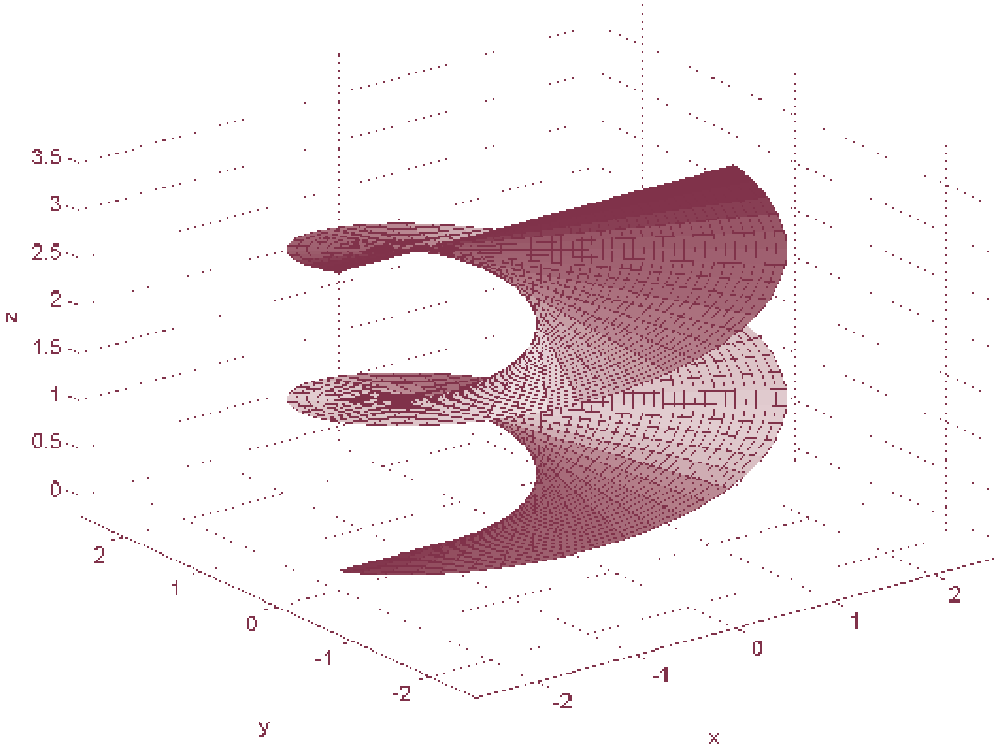

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Coplexes in abelian categories

Flaviu Pop

Abstract. Starting with a pair $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$ of additive contravariant functors which are adjoint on the right, between abelian categories, and with a class $\mathcal{U}$, we define the notion of $(\mathrm{F},\mathcal{U})$-coplex. Considering a reflexive object U of $\mathcal{A}$ with $\mathrm{F}(U) = V$ projective object in $\mathcal{B}$, we construct a natural duality between the category of all $(\mathrm{F}, \mathrm{add}(U))$-coplexes in $\mathcal{A}$ and the subcategory of $\mathcal{B}$ consisting in all objects in $\mathcal{B}$ which admit a projective resolution with all terms in the class $\mathrm{add}(V)$.

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1. Introduction

The study of dualities between subcategories of the module categories, induced by Hom contravariant functors associated to a given bimodule, is very important in the Module Theory in order to compare some special classes of modules. Also, is very useful to generalize such dualities, between module categories, to dualities induced by a pair of adjoint functors between abelian (or, Grothendieck) categories, because they could be applied to different pairs of adjoint functors. In [7], Castaño-Iglesias generalized the notion of costar module, introduced by Colby and Fuller in [8], to the notion of costar object in Grothedieck categories. In [5], the authors extends the notion of f-cotilting module (see, for example, [16]) to the notion of f-cotilting pair of contravariant functors. In [14], it is constructed a natural duality, induced by a pair of adjoint contravariant functors between abelian categories and, applying this result to some special classes of objects, the author generalizes some of the results related to the notion of finitistic n-self cotilting module, introduced by Breaz in [4]. A particular case of finitistic n-self cotilting module is also generalized in [6]. Starting with a pair of adjoint covariant functors $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$, between abelian categories, in [15] it is studied, inspired by some of the results obtained by Fuller in [12] on module categories, some closure properties of some full subcategories $\mathcal{C}$ and $\mathcal{D}$ such that the restrictions $\mathrm{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathrm{G}$ induce an equivalence. In [1] and [2], it is generalized the concepts of r-costar module and Co-$\star^n$-module to the concepts of r-costar pair and Co-$\star^n$-tuple of contravariant functors between abelian categories. Moreover, in [3], the author generalizes $\star^s$-modules and $\star^n$-modules to $\star^s$-tuples and $\star^n$-tuples of covariant functors between abelian categories.

In this paper, we extend the notion of G-coplex, introduced by Faticoni in [10] (see also [11, Chapter 9]) in module categories, to the notion of $(\mathrm{F},\mathcal{U})$-coplex in arbitrary abelian categories. More exactly, starting with a pair $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$ of additive contravariant functors, between two arbitrary abelian categories, which are adjoint on the right and with a class $\mathcal{U}$ of objects in $\mathcal{A}$, we define the notion of $(\mathrm{F},\mathcal{U})$-coplex, associated to this pair of functors and to the considered class. Then, setting the class $\mathcal{U}$ to be the class $\mathrm{add}(U)$, i.e. the class of all direct summands of finite direct sums of copies of U, for some reflexive object U of $\mathcal{A}$ with $\mathrm{F}(U) = V$ being projective object in $\mathcal{B}$, we construct a natural duality between the category of all $(\mathrm{F}, \mathrm{add}(U))$-coplexes in $\mathcal{A}$ and the subcategory of $\mathcal{B}$ consisting in all objects in $\mathcal{B}$ which admit a projective resolution with all terms in the class $\mathrm{add}(V)$.

2. Preliminaries

Throughout this paper, we consider a pair $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$ of additive contravariant functors, between two abelian categories, which are adjoint on the right with the natural transformations of right adjunction $\delta : 1_{\mathcal{A}} \to \mathrm{GF}$ and $\zeta : 1_{\mathcal{B}} \to \mathrm{FG}$. We note that the natural transformations of right adjunction, δ and ζ, satisfy the identities $\mathrm{F}(\delta_X) \circ \zeta_{\mathrm{F}(X)} = 1_{\mathrm{F}(X)}$ and $\mathrm{G}(\zeta_Y) \circ \delta_{\mathrm{G}(Y)} = 1_{\mathrm{G}(Y)}$ for all $X \in \mathcal{A}$ and for all $Y \in \mathcal{B}$. Moreover, we mention that the functors F and G are left exact.

The classical example of such a pair of functors is the following (see, for example, [9, Chapter 4]).

Example 2.1. Let R and S be two unital associative rings and let U be an (S, R)-bimodule. If we denote by Mod-R (respectively, by S-Mod) the category of all right R-(respectively, left S-) modules, then the pair of Hom contravariant functors induced by U,

$$\Delta = \mathrm{Hom}_R(-, U) : \text{Mod-}R \rightleftarrows S\text{-Mod} : \mathrm{Hom}_S(-, U) = \Delta',$$

is a pair of right adjoint contravariant functors via the adjunction

$$\mu_{XY} : \mathrm{Hom}_R(X, \mathrm{Hom}_S(Y, U)) \to \mathrm{Hom}_S(Y, \mathrm{Hom}_R(X, U))$$

with

$$\mu_{XY}(f)(y) : x \mapsto f(x)(y)$$

where $X \in \text{Mod-}R, Y \in S\text{-Mod}, x \in X, y \in Y, f \in \mathrm{Hom}_R(X, \mathrm{Hom}_S(Y, U))$. Associated to this adjunction, the natural transformations δ and ζ are in fact the evaluation maps

$$\delta_X : X \to \mathrm{Hom}_S(\mathrm{Hom}_R(X, U), U); \delta_X(x) : f \mapsto f(x)$$

and

$$\zeta_Y : Y \to \mathrm{Hom}_R(\mathrm{Hom}_S(Y, U), U); \zeta_Y(y) : g \mapsto g(y),$$

where $X \in \text{Mod-}R$, $Y \in S\text{-Mod}$, $x \in X, y \in Y, f \in \mathrm{Hom}_R(X, U), g \in \mathrm{Hom}_S(Y, U)$. □

Castaño-Iglesias, in [7], gives an example of a pair of right adjoint contravariant functors between the categories of all G-graded unital right R-modules and of all G-graded unital left S-modules, where G is a group and R and S are two G-graded unital rings (see also [13]). Other examples of such pairs of functors could be found in [14].

An object X in $\mathcal{A}$ (respectively, in $\mathcal{B}$) is called *δ-faithful* (respectively, *ζ-faithful*) if δ_X (respectively, ζ_X) is a monomorphism and we will denote by Faith_δ (respectively, by Faith_ζ) the class of all δ-faithful (respectively, ζ-faithful) objects. An object X in $\mathcal{A}$ (respectively, in $\mathcal{B}$) is called *δ-reflexive* (respectively, *ζ-reflexive*) if δ_X (respectively, ζ_X) is an isomorphism and we will denote by Refl_δ (respectively, by Refl_ζ) the class of all δ-reflexive (respectively, ζ-reflexive) objects.

We have the following basic results related to the closure properties of the classes of all faithful objects (see [5] for the proof).

Lemma 2.2. *The following statements hold:*

(a) $\mathrm{F}(\mathcal{A}) \subseteq \mathrm{Faith}_\zeta$ *and* $\mathrm{G}(\mathcal{B}) \subseteq \mathrm{Faith}_\delta$*;*
(b) *The classes* Faith_δ *and* Faith_ζ *are closed with respect to subobjects.*

Recall that, for a given object X, $\mathrm{add}(X)$ denotes the class of all direct summands of finite direct sums of copies of X. The following basic results are often used in this paper.

Lemma 2.3. *Let U be a δ-reflexive object with* $\mathrm{F}(U) = V$*. Then:*

(a) *V is ζ-reflexive;*
(b) $\mathrm{add}(U) \subseteq \mathrm{Refl}_\delta$ *and* $\mathrm{add}(V) \subseteq \mathrm{Refl}_\zeta$*;*
(c) $\mathrm{F}(\mathrm{add}(U)) = \mathrm{add}(V)$ *and* $\mathrm{G}(\mathrm{add}(V)) = \mathrm{add}(U)$*.*

We recall that, a *complex* $(\mathcal{C}, d)$ in $\mathcal{A}$ is a sequence of objects and morphisms in $\mathcal{A}$

$$\mathcal{C} : \dots \xrightarrow{d_{n-1}} C_{n-1} \xrightarrow{d_n} C_n \xrightarrow{d_{n+1}} C_{n+1} \xrightarrow{d_{n+2}} \dots$$

such that $d_{n+1}d_n = 0$, for all $n \in \mathbb{Z}$. The morphisms d_n are called *differentiations*. We will shorten the notation $(\mathcal{C}, d)$ to $\mathcal{C}$. We mention that the equation $d_{n+1}d_n = 0$ is equivalent to $\mathrm{Im}(d_n) \subseteq \mathrm{Ker}(d_{n+1})$. Moreover, the complex $\mathcal{C}$ is said to be *bounded below* (respectively, *bounded above*), if $C_n = 0$, for all $n < 0$ (respectively, for all $n > 0$). If $\mathcal{C}$ and $\mathcal{C}'$ are two complexes in $\mathcal{A}$, a sequence of morphisms $f = (\dots, f_{n-1}, f_n, f_{n+1}, \dots)$, where $f_n \in \mathrm{Hom}_\mathcal{A}(C_n, C'_n)$, is called *chain map between complexes $\mathcal{C}$ and $\mathcal{C}'$* if the following diagram is commutative

$$\begin{array}{ccccccccc}
\mathcal{C} : \dots & \xrightarrow{d_{n-1}} & C_{n-1} & \xrightarrow{d_n} & C_n & \xrightarrow{d_{n+1}} & C_{n+1} & \xrightarrow{d_{n+2}} & \dots \\
 & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \\
\mathcal{C}' : \dots & \xrightarrow{d'_{n-1}} & C'_{n-1} & \xrightarrow{d'_n} & C'_n & \xrightarrow{d'_{n+1}} & C'_{n+1} & \xrightarrow{d'_{n+2}} & \dots
\end{array}$$

i.e. $f_n d_n = d'_n f_{n-1}$, for all integers $n \in \mathbb{Z}$. By $\mathrm{Comp}_\mathcal{A}$ will be denoted the category of all complexes in $\mathcal{A}$, defined as follows: the class of objects consist in the class of all complexes in $\mathcal{A}$ and the set of morphisms between two complexes $\mathcal{C}$ and $\mathcal{C}'$ consist in the set of all chain maps between $\mathcal{C}$ and $\mathcal{C}'$.

If $f = (\dots, f_{n-1}, f_n, f_{n+1}, \dots) : \mathcal{C} \to \mathcal{C}'$ is a chain map between complexes $\mathcal{C}$ and $\mathcal{C}'$, then we say that f *is null homotopic* (or, f *is homotopic to zero*) if there are, for all integers $n \in \mathbb{Z}$, the morphisms $s_n : C_n \to C'_{n-1}$ in $\mathcal{A}$ such that $f_n = s_{n+1}d_{n+1} + d'_n s_n$, for all integers $n \in \mathbb{Z}$. The sequence $s = (\dots, s_{n-1}, s_n, s_{n+1}, \dots)$ is called *a homotopy of* f (or, *a homotopy between* f *and* 0). The morphisms are illustrated in the following diagram

$$\begin{array}{ccccccccc}
\mathcal{C} : \dots & \xrightarrow{d_{n-1}} & C_{n-1} & \xrightarrow{d_n} & C_n & \xrightarrow{d_{n+1}} & C_{n+1} & \xrightarrow{d_{n+2}} & \dots \\
 & \swarrow{\scriptstyle s_{n-1}} & \downarrow{\scriptstyle f_{n-1}} & \swarrow{\scriptstyle s_n} & \downarrow{\scriptstyle f_n} & \swarrow{\scriptstyle s_{n+1}} & \downarrow{\scriptstyle f_{n+1}} & \swarrow{\scriptstyle s_{n+2}} & \\
\mathcal{C}' : \dots & \xrightarrow{d'_{n-1}} & C'_{n-1} & \xrightarrow{d'_n} & C'_n & \xrightarrow{d'_{n+1}} & C'_{n+1} & \xrightarrow{d'_{n+2}} & \dots
\end{array}$$

The condition for s to be a homotopy of f says that each vertical morphism is the sum of the sides of the parallelogram containing it. If $f = (\dots, f_{n-1}, f_n, f_{n+1}, \dots) : \mathcal{C} \to \mathcal{C}'$ and $g = (\dots, g_{n-1}, g_n, g_{n+1}, \dots) : \mathcal{C} \to \mathcal{C}'$ are two chain maps, then we say that f *and* g *are homotopic* (or, f *is homotopic to* g), written $f \simeq g$, if

$$f - g = (\dots, f_{n-1} - g_{n-1}, f_n - g_n, f_{n+1} - g_{n+1}, \dots) : \mathcal{C} \to \mathcal{C}'$$

is a null homotopic chain map. A homotopy between $f - g$ and 0 is also called *a homotopy between* f *and* g. The homotopic relation " $\simeq$ " is an equivalence relation on the set of chain maps $f : \mathcal{C} \to \mathcal{C}'$. We denote by $[f]$ the homotopy (equivalence) class of f.

For a complex $\mathcal{C} \in \mathrm{Comp}_{\mathcal{A}}$ and for some integer $n \in \mathbb{Z}$, we denote by $H_n(\mathcal{C})$ the *n-th homology of* $\mathcal{C}$, i.e. $H_n(\mathcal{C}) = \mathrm{Ker}(d_{n+1})/\mathrm{Im}(d_n)$.

Definition 2.4. Let $\mathcal{U}$ be a class of objects in $\mathcal{A}$. A bounded below complex $\mathcal{C}$ in $\mathrm{Comp}_{\mathcal{A}}$

$$\mathcal{C} : C_0 \xrightarrow{\sigma_1} C_1 \xrightarrow{\sigma_2} C_2 \xrightarrow{\sigma_3} \dots$$

is called $(\mathrm{F},\mathcal{U})$*-coplex* if the following conditions are satisfied:

(1) $C_k \in \mathcal{U}$, for all $k \geq 0$;
(2) The induced complex

$$\mathrm{F}(\mathcal{C}) : \dots \xrightarrow{\mathrm{F}(\sigma_3)} \mathrm{F}(C_2) \xrightarrow{\mathrm{F}(\sigma_2)} \mathrm{F}(C_1) \xrightarrow{\mathrm{F}(\sigma_1)} \mathrm{F}(C_0)$$

is an exact sequence in $\mathcal{B}$.

Now, for a class $\mathcal{U}$ of objects in $\mathcal{A}$, we define the category of all $(\mathrm{F},\mathcal{U})$-coplexes, denoted by $(\mathrm{F},\mathcal{U})$-coplex, as follows:

(A) the class of objects consists in the class of all $(\mathrm{F},\mathcal{U})$-coplexes $\mathcal{C}$;
(B) the set of morphisms between two $(\mathrm{F},\mathcal{U})$-coplexes $\mathcal{C}$ and $\mathcal{C}'$, consists in the set of all homotopy classes of chain maps $f : \mathcal{C} \to \mathcal{C}'$.

For the rest of the paper, we set a δ-reflexive object U in $\mathcal{A}$ such that $V = \mathrm{F}(U)$ is a projective object in $\mathcal{B}$. Moreover, we suppose that all considered subcategories of $\mathcal{A}$ and $\mathcal{B}$ are isomorphically closed.

Let Y and B be two objects in $\mathcal{B}$ and let n be a positive integer. A projective resolution $\cdots \to P_1 \to P_0 \to Y \to 0$ of Y is called *finitely-B-generated* if $P_i \in \mathrm{add}(B)$ for all $i \geq 0$. We will denote by $\mathrm{gen}^\bullet(B)$ the class of all objects $X \in \mathcal{B}$ such that there exists a finitely-B-generated projective resolution of X. A projective resolution $\cdots \to P_{n+1} \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to Y \to 0$ of Y is called *n-finitely-B-generated* if $P_i \in \mathrm{add}(B)$ for all $i = \overline{0, n}$. We will denote by n-$\mathrm{gen}^\bullet(B)$ the class of all objects $X \in \mathcal{B}$ for which there exists an n-finitely-B-generated projective resolution of X.

Lemma 2.5. *Let $\mathcal{C} : C_0 \xrightarrow{\sigma_1} C_1 \xrightarrow{\sigma_2} C_2 \xrightarrow{\sigma_3} \ldots$ be a complex in $\mathrm{Comp}_{\mathcal{A}}$, with $C_k \in \mathrm{add}(U)$, for all $k \geq 0$. Then $\mathcal{C}$ is an $(\mathrm{F}, \mathrm{add}(U))$-coplex if and only if $\mathrm{F}(\mathcal{C})$ is a finitely-V-generated projective resolution of $H_0(\mathrm{F}(\mathcal{C}))$.*

Proof. Suppose that $\mathcal{C}$ is an $(\mathrm{F}, \mathrm{add}(U))$-coplex. Then, by definition, the induced sequence

$$\mathrm{F}(\mathcal{C}) : \ldots \xrightarrow{\mathrm{F}(\sigma_3)} \mathrm{F}(C_2) \xrightarrow{\mathrm{F}(\sigma_2)} \mathrm{F}(C_1) \xrightarrow{\mathrm{F}(\sigma_1)} \mathrm{F}(C_0) \xrightarrow{\varepsilon_0} \mathrm{Coker}(\mathrm{F}(\sigma_1)) \to 0$$

is an exact sequence in $\mathcal{B}$. Since all $C_k \in \mathrm{add}(U)$, we have, by Lemma 2.3, that all $\mathrm{F}(C_k) \in \mathrm{add}(V)$. We also have that all $\mathrm{F}(C_k)$ are projective in $\mathcal{B}$, because V is projective in $\mathcal{B}$. Therefore $\mathrm{F}(\mathcal{C})$ is a finitely-V-generated projective resolution of $\mathrm{Coker}(\mathrm{F}(\sigma_1))$.

Conversely, if the induced sequence $\mathrm{F}(\mathcal{C})$ is a finitely-V-generated projective resolution of $\mathrm{Coker}(\mathrm{F}(\sigma_1))$, then $\mathrm{F}(\mathcal{C})$ is an exact sequence in $\mathcal{B}$. From hypothesis, $C_k \in \mathrm{add}(U)$, for all $k \geq 0$. It follows that $\mathcal{C}$ is an $(\mathrm{F}, \mathrm{add}(U))$-coplex. □

It is well known that, if $f, g : \mathcal{C} \to \mathcal{C}'$ are two homotopic chain maps between complexes $\mathcal{C}$ and $\mathcal{C}'$, then $H_0(\mathrm{F}(f)) = H_0(\mathrm{F}(g))$. Therefore, the functor F^U from the following definition is well-defined.

Definition 2.6. The contravariant functor $\mathrm{F}^U : (\mathrm{F}, \mathrm{add}(U))\text{-coplex} \to \mathrm{gen}^\bullet(V)$ is defined as follows:

(A) On objects, we set $\mathrm{F}^U(\mathcal{C}) = H_0(\mathrm{F}(\mathcal{C}))$, for each $\mathcal{C} \in (\mathrm{F}, \mathrm{add}(U))$-coplex.
(B) On morphisms, we take $\mathrm{F}^U([f]) = H_0(\mathrm{F}(f))$, for each morphism $[f] : \mathcal{C} \to \mathcal{C}'$ of $(\mathrm{F}, \mathrm{add}(U))$-coplexes.

Definition 2.7. The contravariant functor $\mathrm{G}^U : \mathrm{gen}^\bullet(V) \to (\mathrm{F}, \mathrm{add}(U))$-coplex is defined as follows:

(A) On objects. Let $Y \in \mathrm{gen}^\bullet(V)$. Then Y has a finitely-V-generated projective resolution

$$\mathcal{P}(Y) : \ldots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} Y \to 0.$$

We mention that the chosen projective resolution $\mathcal{P}(Y)$ is unique up to a homotopy. Applying the functor G to the projective resolution $\mathcal{P}(Y)$, we obtain the following complex in $\mathcal{A}$

$$\mathrm{G}(\mathcal{P}(Y)) : \mathrm{G}(P_0) \xrightarrow{\mathrm{G}(\partial_1)} \mathrm{G}(P_1) \xrightarrow{\mathrm{G}(\partial_2)} \mathrm{G}(P_2) \xrightarrow{\mathrm{G}(\partial_3)} \ldots$$

Since $\mathcal{P}(Y)$ is finitely-V-generated, we have $P_k \in \operatorname{add}(V)$, for all $k \geq 0$, and, since $\zeta : 1_{\operatorname{add}(V)} \to \mathrm{FG}$ is a natural isomorphism, the following diagram is commutative with the vertical maps isomorphisms

$$\begin{array}{ccccccc}
\dots & \xrightarrow{\partial_3} & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 \\
 & & \downarrow{\scriptstyle \zeta_{P_2}} & & \downarrow{\scriptstyle \zeta_{P_1}} & & \downarrow{\scriptstyle \zeta_{P_0}} \\
\dots & \xrightarrow{\mathrm{FG}(\partial_3)} & \mathrm{FG}(P_2) & \xrightarrow{\mathrm{FG}(\partial_2)} & \mathrm{FG}(P_1) & \xrightarrow{\mathrm{FG}(\partial_1)} & \mathrm{FG}(P_0)
\end{array}$$

Since the top row is an exact sequence, it follows that the bottom row is an exact sequence. By Lemma 2.3, $\mathrm{G}(P_k) \in \operatorname{add}(U)$, for all $k \geq 0$. Thus $\mathrm{G}(\mathcal{P}(Y))$ is a complex in $\mathcal{A}$ with all $\mathrm{G}(P_k) \in \operatorname{add}(U)$ and the induced sequence $\mathrm{FG}(\mathcal{P}(Y))$ is an exact sequence. Therefore $\mathrm{G}(\mathcal{P}(Y))$ is an $(\mathrm{F}, \operatorname{add}(U))$-coplex. We set

$$\mathrm{G}^U(Y) = \mathrm{G}(\mathcal{P}(Y)).$$

(B) On morphisms. Let $\phi \in \operatorname{Hom}_{\mathrm{gen}^\bullet(V)}(Y, Y')$. Then ϕ lifts to a chain map

$$f = (\dots, f_2, f_1, f_0) : \mathcal{P}(Y) \to \mathcal{P}(Y')$$

where $\mathcal{P}(Y)$ and $\mathcal{P}(Y')$ are finitely-V-generated projective resolutions associated to Y and Y', respectively.

$$\begin{array}{ccccccccccc}
\dots & \xrightarrow{\partial_3} & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\partial_0} & Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \phi} & & \\
\dots & \xrightarrow{\partial'_3} & P'_2 & \xrightarrow{\partial'_2} & P'_1 & \xrightarrow{\partial'_1} & P'_0 & \xrightarrow{\partial'_0} & Y' & \longrightarrow & 0
\end{array}$$

Applying the functor G, we get a chain map in $\mathcal{A}$,

$$\mathrm{G}(f) = (\mathrm{G}(f_0), \mathrm{G}(f_1), \mathrm{G}(f_2), \dots) : \mathrm{G}(\mathcal{P}(Y')) \to \mathrm{G}(\mathcal{P}(Y))$$

illustrated in the following diagram

$$\begin{array}{ccccccc}
\mathrm{G}(P'_0) & \xrightarrow{\mathrm{G}(\partial'_1)} & \mathrm{G}(P'_1) & \xrightarrow{\mathrm{G}(\partial'_2)} & \mathrm{G}(P'_2) & \xrightarrow{\mathrm{G}(\partial'_3)} & \dots \\
{\scriptstyle \mathrm{G}(f_0)}\downarrow & & {\scriptstyle \mathrm{G}(f_1)}\downarrow & & {\scriptstyle \mathrm{G}(f_2)}\downarrow & & \\
\mathrm{G}(P_0) & \xrightarrow{\mathrm{G}(\partial_1)} & \mathrm{G}(P_1) & \xrightarrow{\mathrm{G}(\partial_2)} & \mathrm{G}(P_2) & \xrightarrow{\mathrm{G}(\partial_3)} & \dots
\end{array}$$

Since $\mathrm{G}(\mathcal{P}(Y))$ and $\mathrm{G}(\mathcal{P}(Y'))$ are $(\mathrm{F}, \operatorname{add}(U))$-coplexes, it follows that the homotopy class $[\mathrm{G}(f)]$ is a morphism in the category $(\mathrm{F}, \operatorname{add}(U))$-coplex. We set

$$\mathrm{G}^U(\phi) = [\mathrm{G}(f)].$$

3. Main result

The main result of the paper is the following theorem.

Theorem 3.1. *The functors* F^U *and* G^U *induce the following duality*

$$\mathrm{F}^U : (\mathrm{F}, \mathrm{add}(U))\text{-coplex} \rightleftarrows \mathrm{gen}^{\bullet}(V) : \mathrm{G}^U$$

Proof. First, we show that the composition $\mathrm{F}^U \circ \mathrm{G}^U$ is natural isomorphic to the identity functor $1_{\mathrm{gen}^{\bullet}(V)}$.

Let $Y \in \mathrm{gen}^{\bullet}(V)$. Then Y has a finitely-V-generated projective resolution

$$\mathcal{P}(Y) : \ldots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} Y \to 0.$$

Applying the functor G, we obtain the following $(\mathrm{F}, \mathrm{add}(U))$-coplex

$$\mathrm{G}(\mathcal{P}(Y)) : \mathrm{G}(P_0) \xrightarrow{\mathrm{G}(\partial_1)} \mathrm{G}(P_1) \xrightarrow{\mathrm{G}(\partial_2)} \mathrm{G}(P_2) \xrightarrow{\mathrm{G}(\partial_3)} \ldots$$

and then $\mathrm{G}^U(Y) = \mathrm{G}(\mathcal{P}(Y))$. Applying the functor F, we have the exact sequence

$$\mathrm{FG}(\mathcal{P}(Y)) : \ldots \xrightarrow{\mathrm{FG}(\partial_3)} \mathrm{FG}(P_2) \xrightarrow{\mathrm{FG}(\partial_2)} \mathrm{FG}(P_1) \xrightarrow{\mathrm{FG}(\partial_1)} \mathrm{FG}(P_0) \xrightarrow{\varepsilon_0} \mathrm{Coker}(\mathrm{FG}(\partial_1)) \to 0$$

and then $\mathrm{F}^U(\mathrm{G}(\mathcal{P}(Y))) = \mathrm{Coker}(\mathrm{FG}(\partial_1))$. Thus $(\mathrm{F}^U \circ \mathrm{G}^U)(Y) = \mathrm{Coker}(\mathrm{FG}(\partial_1))$.

Since all $P_k \in \mathrm{add}(V)$ and since $\zeta : 1_{\mathrm{add}(V)} \to \mathrm{FG}$ is a natural isomorphism, the following diagram is commutative with the vertical maps isomorphisms.

$$\begin{array}{ccccccccc}
\ldots \xrightarrow{\partial_3} & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\partial_0} & Y & \longrightarrow 0 \\
& \downarrow{\scriptstyle \zeta_{P_2}} & & \downarrow{\scriptstyle \zeta_{P_1}} & & \downarrow{\scriptstyle \zeta_{P_0}} & & {\scriptstyle \beta_Y}\downarrow\uparrow{\scriptstyle \gamma_Y} & \\
\ldots \xrightarrow{\mathrm{FG}(\partial_3)} & \mathrm{FG}(P_2) & \xrightarrow{\mathrm{FG}(\partial_2)} & \mathrm{FG}(P_1) & \xrightarrow{\mathrm{FG}(\partial_1)} & \mathrm{FG}(P_0) & \xrightarrow{\varepsilon_0} & \mathrm{Coker}(\mathrm{FG}(\partial_1)) & \longrightarrow 0
\end{array}$$

Since $(\varepsilon_0 \circ \zeta_{P_0}) \circ \partial_1 = 0$ and Y is the cokernel of ∂_1, there is a unique morphism $\beta_Y : Y \to \mathrm{Coker}(\mathrm{FG}(\partial_1))$ such that $\varepsilon_0 \circ \zeta_{P_0} = \beta_Y \circ \partial_0$. Also, since $(\partial_0 \circ \zeta_{P_0}^{-1}) \circ \mathrm{FG}(\partial_1) = 0$, there is a unique morphism $\gamma_Y : \mathrm{Coker}(\mathrm{FG}(\partial_1)) \to Y$ such that $\partial_0 \circ \zeta_{P_0}^{-1} = \gamma_Y \circ \varepsilon_0$. It it easy to see that $\beta_Y \circ \gamma_Y = 1_{\mathrm{Coker}(\mathrm{FG}(\partial_1))}$ and $\gamma_Y \circ \beta_Y = 1_Y$. Thus $\beta_Y : Y \to (\mathrm{F}^U \circ \mathrm{G}^U)(Y)$ is an isomorphism.

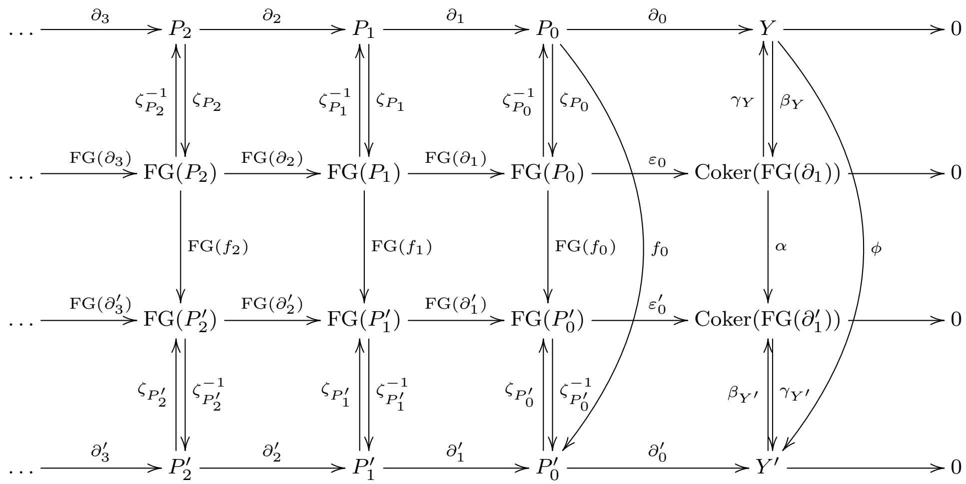
Let $\phi \in \mathrm{Hom}_{\mathrm{gen}^{\bullet}(V)}(Y, Y')$. Then ϕ lifts to a chain map $f : \mathcal{P}(Y) \to \mathcal{P}(Y')$, where $\mathcal{P}(Y)$ and $\mathcal{P}(Y')$ are the finitely-V-generated projective resolutions of Y and Y', respectively, as we see in the following diagram:

$$\begin{array}{ccccccccc}
\mathcal{P}(Y) : \ldots \xrightarrow{\partial_3} & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\partial_0} & Y & \longrightarrow 0 \\
& \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \phi} & \\
\mathcal{P}(Y') : \ldots \xrightarrow{\partial'_3} & P'_2 & \xrightarrow{\partial'_2} & P'_1 & \xrightarrow{\partial'_1} & P'_0 & \xrightarrow{\partial'_0} & Y' & \longrightarrow 0
\end{array}$$

By definition, we have $\mathrm{G}^U(\phi) = [\mathrm{G}(f)] : \mathrm{G}^U(Y') \to \mathrm{G}^U(Y)$.

$$\begin{array}{ccccccc}
G(P'_0) & \xrightarrow{G(\partial'_1)} & G(P'_1) & \xrightarrow{G(\partial'_2)} & G(P'_2) & \xrightarrow{G(\partial'_3)} & \ldots \\
\downarrow{\scriptstyle G(f_0)} & & \downarrow{\scriptstyle G(f_1)} & & \downarrow{\scriptstyle G(f_2)} & & \\
G(P_0) & \xrightarrow{G(\partial_1)} & G(P_1) & \xrightarrow{G(\partial_2)} & G(P_2) & \xrightarrow{G(\partial_3)} & \ldots
\end{array}$$

Since $\varepsilon'_0 \circ \mathrm{FG}(f_0) \circ \mathrm{FG}(\partial_1) = 0$, there is a unique morphism $\alpha : \mathrm{Coker}(\mathrm{FG}(\partial_1)) \to \mathrm{Coker}(\mathrm{FG}(\partial'_1))$ such that $\varepsilon'_0 \circ \mathrm{FG}(f_0) = \alpha \circ \varepsilon_0$. Then $\mathrm{F}^U([\mathrm{G}(f)]) = \alpha$, and thus $(\mathrm{F}^U \circ \mathrm{G}^U)(\phi) = \alpha$.



From the fact that $\zeta : 1_{\mathcal{B}} \to \mathrm{FG}$ is a natural transformation, we have $\mathrm{FG}(f_0) \circ \zeta_{P_0} = \zeta_{P'_0} \circ f_0$. It follows that we have the following equalities

$$\alpha \circ \beta_Y \circ \partial_0 = \alpha \circ \varepsilon_0 \circ \zeta_{P_0} =$$
$$\varepsilon'_0 \circ \mathrm{FG}(f_0) \circ \zeta_{P_0} = \varepsilon'_0 \circ \zeta_{P'_0} \circ f_0 =$$
$$\beta_{Y'} \circ \partial'_0 \circ f_0 = \beta_{Y'} \circ \phi \circ \partial_0.$$

Hence $\alpha \circ \beta_Y = \beta_{Y'} \circ \phi$, because ∂_0 is an epimorphism. Therefore we have the equality $(\mathrm{F}^U \circ \mathrm{G}^U)(\phi) \circ \beta_Y = \beta_{Y'} \circ \phi$, i.e. the following diagram is commutative

$$\begin{array}{ccc}
Y & \xrightarrow{\phi} & Y' \\
\downarrow{\scriptstyle \beta_Y} & & \downarrow{\scriptstyle \beta_{Y'}} \\
(\mathrm{F}^U \circ \mathrm{G}^U)(Y) & \xrightarrow{(\mathrm{F}^U \circ \mathrm{G}^U)(\phi)} & (\mathrm{F}^U \circ \mathrm{G}^U)(Y')
\end{array}$$

Second, we show that the composition $\mathrm{G}^U \circ \mathrm{F}^U$ is natural isomorphic with the identity functor $1_{(\mathrm{F},\mathrm{add}(U))\text{-coplex}}$.

Let $\mathcal{C} \in (\mathrm{F}, \mathrm{add}(U))$-coplex. Then

$$\mathcal{C} : C_0 \xrightarrow{\sigma_1} C_1 \xrightarrow{\sigma_2} C_2 \xrightarrow{\sigma_3} \dots$$

is a complex in $\mathcal{A}$, with $C_k \in \mathrm{add}(U)$, for all $k \geq 0$, and the induced sequence

$$\mathrm{F}(\mathcal{C}) : \dots \xrightarrow{\mathrm{F}(\sigma_3)} \mathrm{F}(C_2) \xrightarrow{\mathrm{F}(\sigma_2)} \mathrm{F}(C_1) \xrightarrow{\mathrm{F}(\sigma_1)} \mathrm{F}(C_0) \xrightarrow{\varepsilon_0} \mathrm{Coker}(\mathrm{F}(\sigma_1)) \to 0$$

is a finitely-V-generated projective resolution of $\mathrm{Coker}(\mathrm{F}(\sigma_1))$. By definition $\mathrm{F}^U(\mathcal{C}) = \mathrm{Coker}(\mathrm{F}(\sigma_1))$. Moreover, $\mathrm{G}^U(\mathrm{Coker}(\mathrm{F}(\sigma_1))) = \mathrm{GF}(\mathcal{C})$, hence $(\mathrm{G}^U \circ \mathrm{F}^U)(\mathcal{C}) = \mathrm{GF}(\mathcal{C})$.

Since $\delta : 1_{\mathcal{A}} \to \mathrm{GF}$ is a natural transformation, we have that

$$\delta_{\mathcal{C}} = (\delta_{C_0}, \delta_{C_1}, \delta_{C_2}, \dots)$$

is a chain map between $(\mathrm{F}, \mathrm{add}(U))$-coplexes $\mathcal{C}$ and $\mathrm{GF}(\mathcal{C})$, hence we have $[\delta_{\mathcal{C}}] \in \mathrm{Hom}_{(\mathrm{F},\mathrm{add}(U))\text{-coplex}}(\mathcal{C}, \mathrm{GF}(\mathcal{C}))$. On the other hand, since $C_k \in \mathrm{add}(U)$, the morphisms $\delta_{C_k} : C_k \to \mathrm{GF}(C_k)$ are isomorphisms, hence

$$\delta_{\mathcal{C}}^{-1} = (\delta_{C_0}^{-1}, \delta_{C_1}^{-1}, \delta_{C_2}^{-1}, \dots)$$

is a chain map between $(\mathrm{F}, \mathrm{add}(U))$-coplexes $\mathrm{GF}(\mathcal{C})$ and $\mathcal{C}$ and thus we have $[\delta_{\mathcal{C}}^{-1}] \in \mathrm{Hom}_{(\mathrm{F},\mathrm{add}(U))\text{-coplex}}(\mathrm{GF}(\mathcal{C}), \mathcal{C})$.

$$\begin{array}{ccccccc}
C_0 & \xrightarrow{\sigma_1} & C_1 & \xrightarrow{\sigma_2} & C_2 & \xrightarrow{\sigma_3} & \dots \\
\downarrow{\scriptstyle \delta_{C_0}} & & \downarrow{\scriptstyle \delta_{C_1}} & & \downarrow{\scriptstyle \delta_{C_2}} & & \\
\mathrm{GF}(C_0) & \xrightarrow{\mathrm{GF}(\sigma_1)} & \mathrm{GF}(C_1) & \xrightarrow{\mathrm{GF}(\sigma_2)} & \mathrm{GF}(C_2) & \xrightarrow{\mathrm{GF}(\sigma_3)} & \dots \\
\downarrow{\scriptstyle \delta_{C_0}^{-1}} & & \downarrow{\scriptstyle \delta_{C_1}^{-1}} & & \downarrow{\scriptstyle \delta_{C_2}^{-1}} & & \\
C_0 & \xrightarrow{\sigma_1} & C_1 & \xrightarrow{\sigma_2} & C_2 & \xrightarrow{\sigma_3} & \dots
\end{array}$$

Since $\delta_{C_k}^{-1} \circ \delta_{C_k} = 1_{C_k}$ and $\delta_{C_k} \circ \delta_{C_k}^{-1} = 1_{\mathrm{GF}(C_k)}$ in $\mathcal{A}$, for all $k \geq 0$, we have $[\delta_{\mathcal{C}}^{-1}] \circ [\delta_{\mathcal{C}}] = [1_{\mathcal{C}}]$ and $[\delta_{\mathcal{C}}] \circ [\delta_{\mathcal{C}}^{-1}] = [1_{\mathrm{GF}(\mathcal{C})}]$ in $(\mathrm{F}, \mathrm{add}(U))$-coplex, hence $[\delta_{\mathcal{C}}] : \mathcal{C} \to (\mathrm{G}^U \circ \mathrm{F}^U)(\mathcal{C})$ is an isomorphism in $(\mathrm{F}, \mathrm{add}(U))$-coplex.

Let $[f] \in \mathrm{Hom}_{(\mathrm{F},\mathrm{add}(U))\text{-coplex}}(\mathcal{C}, \mathcal{C}')$. Then

$$f = (f_0, f_1, f_2, \dots) : \mathcal{C} \to \mathcal{C}'$$

is a chain map between $(\mathrm{F}, \mathrm{add}(U))$-coplexes $\mathcal{C}$ and $\mathcal{C}'$, as illustrated below:

$$\begin{array}{ccccccc}
C_0 & \xrightarrow{\sigma_1} & C_1 & \xrightarrow{\sigma_2} & C_2 & \xrightarrow{\sigma_3} & ... \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\
C_0' & \xrightarrow{\sigma_1'} & C_1' & \xrightarrow{\sigma_2'} & C_2' & \xrightarrow{\sigma_3'} & ...
\end{array}$$

It follows that

$$\mathrm{F}(f) = (\dots, \mathrm{F}(f_2), \mathrm{F}(f_1), \mathrm{F}(f_0)) : \mathrm{F}(\mathcal{C}') \to \mathrm{F}(\mathcal{C})$$

is a chain map between exact sequences $\mathrm{F}(\mathcal{C}')$ and $\mathrm{F}(\mathcal{C})$

$$\begin{array}{ccccccccccc}
\ldots & \xrightarrow{\mathrm{F}(\sigma_3')} & \mathrm{F}(C_2') & \xrightarrow{\mathrm{F}(\sigma_2')} & \mathrm{F}(C_1') & \xrightarrow{\mathrm{F}(\sigma_1')} & \mathrm{F}(C_0') & \xrightarrow{\varepsilon_0'} & \mathrm{Coker}(\mathrm{F}(\sigma_1')) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mathrm{F}(f_2)} & & \Big\downarrow{\scriptstyle \mathrm{F}(f_1)} & & \Big\downarrow{\scriptstyle \mathrm{F}(f_0)} & & \Big\downarrow{\scriptstyle \phi} & & \\
\ldots & \xrightarrow{\mathrm{F}(\sigma_3)} & \mathrm{F}(C_2) & \xrightarrow{\mathrm{F}(\sigma_2)} & \mathrm{F}(C_1) & \xrightarrow{\mathrm{F}(\sigma_1)} & \mathrm{F}(C_0) & \xrightarrow{\varepsilon_0} & \mathrm{Coker}(\mathrm{F}(\sigma_1)) & \longrightarrow & 0
\end{array}$$

Since $(\varepsilon_0 \circ \mathrm{F}(f_0)) \circ \mathrm{F}(\sigma_1') = 0$, there is a unique morphism $\phi : \mathrm{Coker}(\mathrm{F}(\sigma_1')) \to \mathrm{Coker}(\mathrm{F}(\sigma_1))$ in $\mathcal{B}$ such that $\varepsilon_0 \circ \mathrm{F}(f_0) = \phi \circ \varepsilon_0'$ and then, by definition, $\mathrm{F}^U([f]) = \phi$. Moreover, by definition of G^U, we have $\mathrm{G}^U(\phi) = [\mathrm{GF}(f)]$. Thus $(\mathrm{G}^U \circ \mathrm{F}^U)([f]) = [\mathrm{GF}(f)]$.

Since $\delta : 1_{\mathcal{A}} \to \mathrm{GF}$ is a natural transformation, we have $\mathrm{GF}(f_k) \circ \delta_{C_k} = \delta_{C_k'} \circ f_k$, for all $k \geq 0$, hence $[\mathrm{GF}(f) \circ \delta_{\mathcal{C}}] = [\delta_{\mathcal{C}'} \circ f]$. Thus $[\mathrm{GF}(f)] \circ [\delta_{\mathcal{C}}] = [\delta_{\mathcal{C}'}] \circ [f]$ and therefore $(\mathrm{G}^U \circ \mathrm{F}^U)([f]) \circ [\delta_{\mathcal{C}}] = [\delta_{\mathcal{C}'}] \circ [f]$. So, the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{[f]} & \mathcal{C}' \\
\Big\downarrow{\scriptstyle [\delta_{\mathcal{C}}]} & & \Big\downarrow{\scriptstyle [\delta_{\mathcal{C}'}]} \\
(\mathrm{G}^U \circ \mathrm{F}^U)(\mathcal{C}) & \xrightarrow{(\mathrm{G}^U \circ \mathrm{F}^U)([f])} & (\mathrm{G}^U \circ \mathrm{F}^U)(\mathcal{C}')
\end{array}$$

□

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References

[1] Al-Nofayee, S., *r-Costar pair of contravariant functors*, International Journal of Mathematics and Mathematical Sciences, 2012, Article ID 481909, 8 pages.

[2] Al-Nofayee, S., *Co-$\star^n$-tuple of contravariant functors*, British Journal of Mathematics and Computer Science, **4**(2014), no. 12, 1701-1709.

[3] Al-Nofayee, S., *$\star^s$-tuple and $\star^n$-tuple of covariant functors*, British Journal of Mathematics and Computer Science, **7**(2015), no. 6, 439-449.

[4] Breaz, S., *Finitistic n-self-cotilting modules*, Communications in Algebra, **37**(2009), no. 9, 3152-3170.

[5] Breaz, S., Modoi, C., Pop, F., *Natural Equivalences and Dualities*, Proceedings of the International Conference on Modules and Representation Theory (7-12 July 2008, Cluj-Napoca, Romania), Cluj University Press, 2009, 25-40.

[6] Breaz, S., Pop, F., *Dualities Induced by Right Adjoint Contravariant Functors*, Stud. Univ. Babeş-Bolyai Math., **55**(2010), no. 1, 75-83.

[7] Castaño-Iglesias, F., *On a natural duality between Grothendieck categories*, Communications in Algebra, **36**(2008), no. 6, 2079-2091.

[8] Colby, R.R., Fuller, K.R., *Costar modules*, Journal of Algebra, **242**(2001), no. 1, 146-159.

[9] Colby, R.R., Fuller, K.R., *Equivalence and Duality for Module Categories. With Tilting and Cotilting for Rings*, Cambridge University Press, Cambridge, 2004.

[10] Faticoni, T.G., *A duality for self-slender modules*, Communications in Algebra, **35**(2007), no. 12, 4175-4182.

[11] Faticoni, T.G., *Modules Over Endomorphism Rings*, Cambridge University Press, Cambridge, 2010.

[12] Fuller, K.R., *Natural and doubly natural dualities*, Communications in Algebra, **34**(2006), no. 2, 749-762.

[13] Năstăsescu, C., Van Oystaeyen, F., *Methods of Graded Rings*, Lectures Notes in Mathematics 1836, Springer-Verlag, 2004.

[14] Pop, F., *Natural dualities between abelian categories*, Central European Journal of Mathematics, **9**(2011), no. 5, 1088-1099.

[15] Pop, F., *Closure properties associated to natural equivalences*, Indagationes Mathematicae, **24**(2013), no. 2, 403-414.

[16] Wisbauer, R., *Cotilting objects and dualities*, Representations of algebras, Sao Paulo, 1999, 215-233, Lecture Notes in Pure and Appl. Math., 224, Dekker, New York, 2002.

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A note on the Laplace operator for holomorphic functions on complex Lie groups

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Abstract. In this note we obtain the local expression of the Laplace operator acting on holomorphic functions defined on a complex Lie group. Also, some applications to the theory of holomorphic last multipliers are given.

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1. Preliminaries on complex Lie groups

Let G be a complex Lie group of dimension n. Its Lie algebra, $\mathfrak{g}$, has as underlying vector space the holomorphic tangent space $T_e^{1,0}G$ at the identity $e \in G$. As known, an element $A \in T_e^{1,0}G$ determines a unique left invariant vector field which takes the value A at e; moreover, these vector fields are the elements of $\mathfrak{g}$.

Following the ideas from [1], let $\{E_\alpha\},\ \alpha = 1, \ldots, n$, be a base of the Lie algebra $\mathfrak{g}$ and $\chi^\alpha,\ \alpha = 1, \ldots, n$ the dual base for the 1-*forms of Maurer-Cartan*, that is, $\chi^\alpha(E_\beta) = \delta^\alpha_\beta,\ (\alpha, \beta = 1, \ldots, n)$. It is known ([11], Lemma 1.6) that E_α are holomorphic vector fields (as they are left-invariant) and also χ^α are holomorphic left-invariant 1-forms.

A differential form η is said to be *left-invariant* if it is invariant by every left translation L_a, $(a \in G)$, that is, if $L_a^*\eta = \eta$ for every $a \in G$, where L_a^* is the holomorphic cotangent map of L_a. It follows that any left invariant form must be holomorphic. For an element $U \in \mathfrak{g}$ and an element η in the dual space $\mathfrak{g}^*$, $\eta(U)$ is constant on G. Since

$$\partial\eta(U, V) = U\eta(V) - V\eta(U) - \eta([U, V]),$$

where $d = \partial + \bar{\partial}$ is the usual decomposition of the exterior derivative, one obtains

$$\partial\eta(U, V) = -\eta([U, V]), \tag{1.1}$$

where U, V are elements of $\mathfrak{g}$ and η is any element of the dual space. By setting

$$[E_\beta, E_\gamma] = C_{\beta\ \gamma}^{\ \alpha} E_\alpha, \tag{1.2}$$

the relation (1.1) yields

$$\partial \chi^\alpha = -\frac{1}{2} C_{\beta\ \gamma}^{\ \alpha} \chi^\beta \wedge \chi^\gamma. \tag{1.3}$$

The complex constants $C_{\beta\ \gamma}^{\ \alpha}$ are called the *constants of structure* of $\mathfrak{g}$ with respect to the holomorphic base $\{E_1, \ldots, E_n\}$. These constants are not arbitrary since they must satisfy the relations

$$[E_\alpha, E_\beta] + [E_\beta, E_\alpha] = 0 \tag{1.4}$$

and

$$[E_\alpha, [E_\beta, E_\gamma]] + [E_\beta, [E_\gamma, E_\alpha]] + [E_\gamma, [E_\alpha, E_\beta]] = 0 \tag{1.5}$$

for all $\alpha, \beta, \gamma = 1, \ldots, n$, that is

$$C_{\beta\ \gamma}^{\ \alpha} + C_{\gamma\ \beta}^{\ \alpha} = 0 \tag{1.6}$$

and

$$C_{\alpha\ \beta}^{\ \rho} C_{\gamma\ \rho}^{\ \delta} + C_{\beta\ \gamma}^{\ \rho} C_{\alpha\ \rho}^{\ \delta} + C_{\gamma\ \alpha}^{\ \rho} C_{\beta\ \rho}^{\ \delta} = 0. \tag{1.7}$$

Equations (1.3) are called the *holomorphic Maurer-Cartan equations.*

Equation (1.2) indicates that the structure constants are the components of a holomorphic tensor on $T_e^{1,0}G$ of type $(1, 2)$. A new holomorphic tensor on $T_e^{1,0}G$ can be defined by setting

$$C_{\alpha\beta} = C_{\alpha\ \sigma}^{\ \rho} C_{\rho\ \beta}^{\ \sigma} \tag{1.8}$$

with respect to the holomorphic left invariant base $\{E_\alpha\}$ $(\alpha = 1, \ldots, n)$ of $\mathfrak{g}$. It is easily verified that this holomorphic tensor is symmetric. Also, it can be shown that a necessary and sufficient condition for the complex Lie group G to be semi-simple is that the complex matrix $(C_{\alpha\beta})_{n\times n}$ is invertible.

The holomorphic tensor defined by the equations (1.8) can now be used to raise and lower indices and, for this purpose, the inverse matrix $(C^{\alpha\beta})_{n\times n}$ will be considered.

In terms of a system of local complex coordinates $(u^1, \ldots, u^n)$ on G, the holomorphic vector fields E_α, $\alpha = 1, \ldots, n$, can be expressed as $E_\alpha = \chi_\alpha^i \dfrac{\partial}{\partial u^i}$. Since G is complex parallelizable (see [14]), the $n \times n$ matrix (χ_α^i) has rank n and so, by setting

$$g^{ij} = \chi_\alpha^i \chi_\beta^j C^{\alpha\beta}, \tag{1.9}$$

a positive definite and symmetric matrix $(g^{ij})_{n\times n}$ is obtained. Hence, a *holomorphic Riemannian metric* g on G can be defined by means of the complex quadratic form

$$ds^2 = g_{jk} du^j \otimes du^k, \tag{1.10}$$

where $(g_{jk})_{n\times n}$ denotes the matrix inverse to $(g^{jk})_{n\times n}$, that is, $g_{jk} = C_{\beta\gamma}\chi_j^\beta \chi_k^\gamma$.

Moreover, the holomorphic metric tensor g can be also used to raise and lower indices in the usual manner, and this holomorphic metric is completely determined by the complex Lie group G.

In the following, we define n holomorphic covariant vector fields χ^α ($\alpha = 1, \ldots, n$) on G, with local components $\chi^\alpha_i (i = 1, \ldots, n)$ given by

$$\chi^\alpha_i = C^{\alpha\beta} \chi^j_\beta g_{ij}. \tag{1.11}$$

It easily follows that

$$\chi^i_\alpha \chi^\alpha_j = \delta^i_j \quad \text{and} \quad \chi^i_\alpha \chi^\beta_i = \delta^\alpha_\beta. \tag{1.12}$$

Also, we consider the set of n^2 linear holomorphic 1-forms $\omega^i_j = \Gamma^{\,i}_{jk} du^k$ defined locally by setting

$$\Gamma^{\,i}_{jk} = \chi^i_\alpha \frac{\partial \chi^\alpha_j}{\partial u^k}. \tag{1.13}$$

By virtue of the equations (1.12), the holomorphic coefficients $\Gamma^{\,i}_{jk}$ can also be expressed as

$$\Gamma^{\,i}_{jk} = -\chi^\alpha_j \frac{\partial \chi^i_\alpha}{\partial u^k} \tag{1.14}$$

and they represent the local coefficients of a left invariant holomorphic connection ∇ on G, that is, ∇ is absolutely parallel with respect to every left-invariant holomorphic vector field $U = U^\alpha E_\alpha \in \mathfrak{g}$.

It is easily verified that in the overlap $U \cap U'$ of two local charts, the holomorphic 1-forms ω^i_j change by the rule

$$\frac{\partial u'^k}{\partial u^j} \omega'^i_k = \frac{\partial u'^i}{\partial u^k} \omega^k_j - \frac{\partial^2 u'^i}{\partial u^l \partial u^j} du^l.$$

The next natural step is to consider the torsion of this connection. As in the case of real Lie groups (see [1, 13]), the holomorphic torsion tensor will be written as

$$T^{\,i}_{jk} = \frac{1}{2} \chi^i_\alpha \left(\frac{\partial \chi^\alpha_j}{\partial u^k} - \frac{\partial \chi^\alpha_k}{\partial u^j} \right). \tag{1.15}$$

Since the equations (1.2) can be expressed in terms of the local coordinates (u^i) in the form

$$\chi^r_\beta \frac{\partial \chi^i_\gamma}{\partial u^r} - \chi^r_\gamma \frac{\partial \chi^i_\beta}{\partial u^r} = C^{\;\alpha}_{\beta\;\gamma} \chi^i_\alpha, \tag{1.16}$$

by using the holomorphic Maurer-Cartan equations (1.3) it easily follows that

$$T^{\,i}_{jk} = \frac{1}{2} C^{\;\alpha}_{\beta\;\gamma} \chi^i_\alpha \chi^\beta_j \chi^\gamma_k. \tag{1.17}$$

Also, if we consider the local coefficients of the holomorphic Levi-Civita connection $\overset{\circ}{\nabla}$ with respect to the holomorphic metric $g = ds^2$ from (1.10) on G, they can be expressed as

$$\overset{\circ}{\Gamma}{}^{\,i}_{jk} = \frac{1}{2} \chi^i_\alpha \left(\frac{\partial \chi^\alpha_j}{\partial u^k} + \frac{\partial \chi^\alpha_k}{\partial u^j} \right), \tag{1.18}$$

from which follows that

$$\Gamma^{\,i}_{jk} = \overset{\circ}{\Gamma}{}^{\,i}_{jk} + T^{\,i}_{jk}. \tag{1.19}$$

We have

Lemma 1.1. *The elements of the Lie algebra $\mathfrak{g}$ of G define holomorphic translations in G.*

Proof. It follows in a similar manner to the case of real Lie groups, see [1]. □

2. Laplace operators for holomorphic functions on G

In this section, we introduce the Laplace operator acting on holomorphic functions on the complex Lie group G, depending on the given holomorphic metric tensor on G.

Denote by $\omega = \chi^1 \wedge \cdots \wedge \chi^n$, where $\chi^i, i = 1, \ldots, n$ are the elements of the base of holomorphic 1-forms defined in Section 1. Then ω is a nowhere vanishing holomorphic left-invariant n-form, called the *holomorphic volume element*, and it can be used to define the *divergence* of a holomorphic vector field $U = U^\alpha E_\alpha$ by setting

$$\operatorname{div}(U)\omega = \partial(i_U \omega). \tag{2.1}$$

Note that the divergence can also be defined by means of the Lie derivative L_U with respect with a left invariant holomorphic vector field U:

$$\operatorname{div}(U)\omega = L_U \omega, \tag{2.2}$$

where

$$L_U \eta = \frac{d}{dt}\bigg|_{t=0} (\varphi_U^t)^* \eta$$

for an arbitrary holomorphic tensor η. The equivalence between definitions (2.1) and (2.2) is due to Cartan's formula $L_U \eta = \partial(i_U \eta) + i_U \partial \eta$ for $\eta = \omega$. The first definition is more convenient for computations, though. Another property of the divergence is

$$\operatorname{div}(fU) = Uf + f \operatorname{div} U$$

for a holomorphic vector field U and a holomorphic function f defined on G.

Also, for a given holomorphic vector field $U = U^\alpha E_\alpha$ on G, we have

$$\operatorname{div} U = E_\alpha(U^\alpha). \tag{2.3}$$

Let G be a semi-simple complex Lie group with the holomorphic Riemannian metric

$$g = g_{ij} du^i \otimes du^j, \quad g_{ij} = C_{\alpha\beta} \chi_i^\alpha \chi_j^\beta. \tag{2.4}$$

A simple computation gives $g(E_\alpha, E_\beta) = C_{\alpha\beta}$ and the holomorphic metric tensor g will now be used to define the *gradient* of a holomorphic function f on G. If $\operatorname{grad} f = V^\beta E_\beta$ is a holomorphic vector field defined in a local chart, then the classical definition

$$g(U, \operatorname{grad} f) = Uf$$

for $U = U^\alpha E_\alpha$ yields $V^\beta = C^{\beta\alpha}(E_\alpha f)$, hence

$$\operatorname{grad} f = C^{\beta\alpha}(E_\alpha f) E_\beta. \tag{2.5}$$

A *Laplace operator for holomorphic functions* on G can now be introduced by

$$\Delta f = (\operatorname{div} \circ \operatorname{grad}) f = C^{\beta\alpha} E_\beta(E_\alpha f). \tag{2.6}$$

In local coordinates, this reads

$$\begin{aligned}\Delta f &= C^{\beta\alpha}\chi^i_\beta \frac{\partial}{\partial u^i}\left(\chi^j_\alpha \frac{\partial f}{\partial u^j}\right)\\ &= C^{\beta\alpha}\chi^i_\beta \frac{\partial \chi^j_\alpha}{\partial u^i}\frac{\partial f}{\partial u^j} + C^{\beta\alpha}\chi^i_\beta\chi^j_\alpha \frac{\partial^2 f}{\partial u^i \partial u^j}.\end{aligned}$$

But

$$\chi^i_\beta \frac{\partial \chi^j_\alpha}{\partial u^i} = -\chi^i_\beta \chi^k_\alpha \Gamma^{\,j}_{ki},$$

where $\Gamma^{\,i}_{jk} = -\chi^\beta_j \dfrac{\partial \chi^i_\beta}{\partial u^k}$ are the coefficients of the holomorphic connection written in the form (1.14), such that

$$\Delta f = g^{ij}\left(\frac{\partial^2 f}{\partial u^i \partial u^j} - \Gamma^{\,k}_{ji}\frac{\partial f}{\partial u^k}\right). \tag{2.7}$$

Since

$$\frac{\partial^2 f}{\partial u^i \partial u^j} - \Gamma^{\,k}_{ji}\frac{\partial f}{\partial u^k} = \nabla_i \nabla_j f,$$

where ∇_k is the covariant derivative with respect to the left invariant holomorphic connection ∇ defined in the previous section, this leads to the following formula for the Laplace operator of holomorphic functions on G:

$$\Delta f = g^{ij}\nabla_i\nabla_j f. \tag{2.8}$$

Remark 2.1. If G is not semi-simple then a holomorphic Riemannian metric on G can be defined by setting

$$h = h_{ij}du^i \otimes du^j, \quad h_{ij} = \delta_{\alpha\beta}\chi^\alpha_i \chi^\beta_j, \tag{2.9}$$

and similar computations as above lead to the following local expression of the Laplacian:

$$\Delta f = E^2_\alpha f = h^{ij}\nabla_i\nabla_j f. \tag{2.10}$$

Let us compute the local expression of the Laplacian in two particular cases.

Example 2.2. Consider the standard 4-dimensional complex manifold $\mathbb{C}^4$ with the holomorphic coordinates (z^1, z^2, z^3, z^4) and the following multiplication rule:

$$\begin{aligned}&(z^1, z^2, z^3, z^4)\cdot(w^1, w^2, w^3, w^4) = \\ &\quad =(z^1 e^{\lambda w^3} + w^1, z^2 e^{-\lambda w^3} + w^2, z^3 + w^3, z^4 + w^4 - \lambda z^1 w^2 e^{\lambda w^3}),\end{aligned} \tag{2.11}$$

where λ is a nonzero complex parameter. The above multiplication rule endows $\mathbb{C}^4$ with a non-abelian complex Lie structure. For $\lambda = 0$, we obtain the usual abelian Lie group $\mathbb{C}^4$, therefore we will consider here $\lambda \neq 0$. We denote by G the non-abelian complex Lie group $\mathbb{C}^4$ endowed with the multiplication rule (2.11).

It is easy to see that the following left-invariant holomorphic vector fields given by

$$Z_1 = \frac{\partial}{\partial z^1}, \quad Z_2 = \frac{\partial}{\partial z^2} - \lambda z^1 \frac{\partial}{\partial z^4}, \quad Z_3 = \lambda z^1 \frac{\partial}{\partial z^1} - \lambda z^2 \frac{\partial}{\partial z^2} + \frac{\partial}{\partial z^3}, \quad Z_4 = \frac{\partial}{\partial z^4} \tag{2.12}$$

form a basis of the holomorphic Lie algebra $\mathfrak{g}$ of G. If we compute the Lie brackets of these holomorphic vector fields, we obtain

$$[Z_1, Z_2] = -\lambda Z_4, \quad [Z_1, Z_3] = \lambda Z_1, \quad [Z_2, Z_3] = -\lambda Z_2,$$
$$[Z_1, Z_4] = [Z_2, Z_4] = [Z_3, Z_4] = 0,$$

therefore, the components of the Lie brackets are constant. Hence, they are the structure constants of $\mathfrak{g}$ with respect to the basis $\{Z_1, Z_2, Z_3, Z_4\}$. We have $C_{\alpha\beta}^{\gamma} = 0$, $\alpha, \beta, \gamma = \overline{1,4}$ with the following exceptions:

$$C_{12}^{4} = -\lambda, \qquad C_{13}^{1} = \lambda, \qquad C_{23}^{2} = -\lambda.$$

The tensor field introduced by (1.8) will consequently vanish, i.e., $C_{\alpha\beta} = 0$ for all $\alpha, \beta = \overline{1,4}$, which means that G is not semi-simple. Then, according to (2.10), the Laplace operator Δ acting on holomorphic functions $f \in Hol(\mathbb{C}^4)$ is

$$\Delta f = \sum_{\alpha} Z_{\alpha}^2 f = Z_1^2 f + Z_2^2 f + Z_3^2 f + Z_4^2 f.$$

Now, a basic computation using (2.12) gives

$$\begin{aligned}\Delta f = &\big(1 + \lambda^2 (z^1)^2\big) \frac{\partial^2 f}{\partial (z^1)^2} + \big(1 + \lambda^2 (z^2)^2\big) \frac{\partial^2 f}{\partial (z^2)^2} + \frac{\partial^2 f}{\partial (z^3)^2} \\ &+ \big(1 + \lambda^2 (z^1)^2\big) \frac{\partial^2 f}{\partial (z^4)^2} - 2\lambda^2 z^1 z^2 \frac{\partial^2 f}{\partial z^1 \partial z^2} + 2\lambda z^1 \frac{\partial^2 f}{\partial z^1 \partial z^3} \\ &- 2\lambda z^2 \frac{\partial^2 f}{\partial z^2 \partial z^3} - 2\lambda z^1 \frac{\partial^2 f}{\partial z^2 \partial z^4} + \lambda^2 z^1 \frac{\partial f}{\partial z^1} + \lambda^2 z^2 \frac{\partial f}{\partial z^2}.\end{aligned}$$

Example 2.3. Let $G = \mathbb{C}^* \times \mathbb{C}$ with the multiplication

$$(z^1, z^2) \circ (w^1, w^2) = \big(z^1 w^1, \frac{1}{2} w z^1 w^2 + z^2 (w^1)^2\big)$$

and consider the vector fields $Z_1 = z^1 \frac{\partial}{\partial z^1} + 2z^2 \frac{\partial}{\partial z^2}$, $Z_2 = z^1 \frac{\partial}{\partial z^2}$. Then, $(G, \circ)$ is a complex Lie group with the holomorphic Lie algebra $\mathfrak{g} = \operatorname{span}\{Z_1, Z_2\}$. Moreover, G is not semi-simple, as it can be easily shown by computating the tensor $C_{\alpha\beta}$, as in the previous example. We therefore have $\Delta f = Z_1^2 f + Z_2^2 f$, which yields the Laplacian in the form

$$\begin{aligned}\Delta f = &(z^1)^2 \frac{\partial^2 f}{\partial (z^1)^2} + \big((z^1)^2 + 4(z^2)^2\big) \frac{\partial^2 f}{\partial (z^2)^2} \\ &+ 4z^1 z^2 \frac{\partial^2 f}{\partial z^1 \partial z^2} + z^1 \frac{\partial f}{\partial z^1} + 4z^2 \frac{\partial f}{\partial z^2}.\end{aligned}$$

We will use this example later for illustrating another property of the Laplace operator.

A straightforward computation gives an interesting property of the Laplacian introduced above in the general case.

Proposition 2.4. *The following identity holds:*

$$[\Delta, E_\alpha] = 2(h^{ij}\chi^k_\alpha - h^{ik}\chi^j_\alpha)\Gamma^{\,l}_{jk}\frac{\partial^2}{\partial u^i \partial u^l}, \tag{2.13}$$

where $h^{ij} = \delta^{\alpha\beta}\chi^i_\alpha\chi^j_\beta$ *and* $\Gamma^{\,l}_{jk}$ *are the local coefficients of the holomorphic connection* ∇.

Let us check the result in the case of the Lie group $G = \mathbb{C}^* \times \mathbb{C}$ from Example 2.3.

Example 2.5. First, we compute

$$[\Delta, Z_1]f = 2(z^1)^2\frac{\partial^2 f}{\partial (z^2)^2}; \tag{2.14}$$
$$[\Delta, Z_2]f = -2(z^1)^2\frac{\partial^2 f}{\partial z^1 \partial z^2} - 4z^1z^2\frac{\partial^2 f}{\partial (z^2)^2}.$$

Then, from $Z_\alpha = \chi^i_\alpha\dfrac{\partial}{\partial z^i}$ we get

$$\chi^1_1 = z^1, \quad \chi^2_1 = 2z^2, \quad \chi^1_2 = 0, \quad \chi^2_2 = z^1,$$

such that, using (1.13), we can compute the coefficients $\Gamma^{\,l}_{jk}$:

$$\Gamma^{\,1}_{11} = -\frac{1}{z^1}, \quad \Gamma^{\,1}_{12} = \Gamma^{\,1}_{21} = \Gamma^{\,1}_{22} = 0,$$
$$\Gamma^{\,2}_{11} = \frac{2z^2}{(z^1)^2}, \quad \Gamma^{\,2}_{12} = -\frac{2}{z^1}, \quad \Gamma^{\,2}_{21} = -\frac{1}{z^1}, \quad \Gamma^{\,2}_{22} = 0.$$

We also need $h^{ij} = \delta^{\alpha\beta}\chi^i_\alpha\chi^j_\beta$, that is,

$$h^{11} = (z^1)^2, \quad h^{12} = h^{21} = 2z^1z^2, \quad h^{22} = (z^1)^2 + 4(z^2)^2.$$

Hence, replacing the nonzero terms in the left-hand side of the first identity (2.14) and doing a straightforward computation yields

$$\begin{aligned}
[\Delta, Z_1]f &= 2(h^{ij}\chi^k_1 - h^{ik}\chi^j_1)\Gamma^{\,l}_{jk}\frac{\partial^2}{\partial z^i \partial z^l} \\
&= 2\big[(h^{11}\chi^2_1 - h^{12}\chi^1_1)\Gamma^{\,2}_{12} + (h^{12}\chi^1_1 - h^{11}\chi^2_1)\Gamma^{\,2}_{21}\big]\frac{\partial^2 f}{\partial z^1 \partial z^2} \\
&\quad + 2\big[(h^{21}\chi^2_1 - h^{22}\chi^1_1)\Gamma^{\,2}_{12} + (h^{22}\chi^1_1 - h^{21}\chi^2_1)\Gamma^{\,2}_{21}\big]\frac{\partial^2 f}{\partial (z^2)^2} \\
&= 2(z^1)^2\frac{\partial^2 f}{\partial (z^2)^2},
\end{aligned}$$

since the first term vanishes. The second identity from (2.14) follows analogously.

We shall also illustrate the property from Proposition 2.4 in the case of the complex Lie group $GL(n, \mathbb{C})$.

Example 2.6. As $\dim(GL(n, \mathbb{C})) = n^2$, all the indices from the general case will be replaced by pairs of indices, for instance α becomes $\binom{\alpha}{\beta}$, i becomes $\binom{i}{m}$, etc. As a convention, these pairs will be rewritten in a manner that should be clear from the text below.

First, let $u \in GL(n,\mathbb{C})$ be a complex matrix with elements $\{A_i^\alpha\}$, such that a left-invariant holomorphic vector field will be denoted by

$$E_\alpha^\beta := E_{\binom{\alpha}{\beta}} = \chi_{\binom{\alpha}{\beta}}^{\binom{i}{m}} \frac{\partial}{\partial u^{\binom{i}{m}}} =: \chi_{\alpha m}^{i\beta} \frac{\partial}{\partial u_m^i},$$

where $\chi_{\alpha m}^{i\beta} \dfrac{\partial}{\partial u_m^i} = \delta_\alpha^i A_m^\beta$ (see [7] for more details). The holomorphic Riemannian metric is

$$h^{\binom{i}{m}\binom{j}{n}} =: h_{mn}^{ij} = \delta^{\alpha\beta} \delta_{\nu\mu} \chi_{\alpha m}^{i\nu} \chi_{\beta n}^{j\mu}$$

(the group $GL(n,\mathbb{C})$ is *not* semi-simple). The local coefficients of the holomorphic connection defined in Section 1 are

$$\Gamma_{\binom{j}{n}\binom{k}{p}}^{\binom{l}{q}} =: \Gamma_{jkq}^{lnp} = \chi_{j\tau}^{\varepsilon n} \frac{\partial \chi_{\varepsilon q}^{l\tau}}{\partial u_p^k}.$$

These yield

$$\begin{aligned}
[\Delta, E_\gamma] f &= 2\big(h_{mn}^{ij} \chi_{\gamma p}^{k\sigma} - h_{mp}^{ik} \chi_{\gamma n}^{j\sigma}\big) \Gamma_{jkq}^{lnp} \frac{\partial^2 f}{\partial u_m^i \partial u_q^l} \\
&= -2\big(\delta^{\alpha\beta} \delta_{\nu\mu} \chi_{\alpha m}^{i\nu} \chi_{\beta n}^{j\mu} \chi_{\gamma p}^{k\sigma} - \delta^{\alpha\beta} \delta_{\nu\mu} \chi_{\alpha m}^{i\nu} \chi_{\beta p}^{k\mu} \chi_{\gamma n}^{j\sigma}\big) \chi_{j\tau}^{\varepsilon n} \frac{\partial \chi_{\varepsilon q}^{l\tau}}{\partial u_p^k} \frac{\partial^2 f}{\partial u_m^i \partial u_q^l} \\
&= -2\Bigg(\delta^{\alpha\beta} \delta_{\nu\mu} \delta_\alpha^i A_m^\nu \delta_\beta^j A_n^\mu \delta_\gamma^k A_p^\sigma \delta_j^\varepsilon A_\tau^n \delta_\varepsilon^l \frac{\partial A_q^\tau}{\partial u_p^k} \\
&\quad - \delta^{\alpha\beta} \delta_{\nu\mu} \delta_\alpha^i A_m^\nu \delta_\beta^k A_p^\mu \delta_\gamma^j A_n^\sigma \delta_j^\varepsilon A_\tau^n \delta_\varepsilon^l \frac{\partial A_q^\tau}{\partial u_p^k} \Bigg) \frac{\partial^2 f}{\partial u_m^i \partial u_q^l} \\
&= 2\Bigg(\delta_{\nu\mu} A_m^\nu A_p^\sigma \frac{\partial A_q^\mu}{\partial u_p^k} - \delta_{\nu\mu} A_m^\nu A_p^\mu \frac{\partial A_q^\sigma}{\partial u_p^k} \Bigg) \frac{\partial^2 f}{\partial u_m^i \partial u_q^l}.
\end{aligned}$$

Remark 2.7. Denoting by $\overset{\circ}{\nabla}$ the covariant derivative with respect to the Levi-Civita connection, the substitution of (1.19) in (2.7) yields

$$\begin{aligned}
\Delta f &= g^{ij} \left(\frac{\partial^2 f}{\partial u^i \partial u^j} - \overset{\circ}{\Gamma}{}_{ji}^{k} \frac{\partial f}{\partial u^k} \right) - T_{ji}^k \frac{\partial f}{\partial u^k} \\
&= g^{ij} \, \overset{\circ}{\nabla}_i \overset{\circ}{\nabla}_j \, f - T_{ji}^k \frac{\partial f}{\partial u^k},
\end{aligned}$$

such that a harmonic holomorphic function f on G must satisfy the identity

$$T_{ji}^k \frac{\partial f}{\partial u^k} = g^{ij} \, \overset{\circ}{\nabla}_i \overset{\circ}{\nabla}_j \, f. \tag{2.15}$$

Note that T_{jk}^i is the holomorphic torsion tensor of the holomorphic connection from (1.13).

3. Holomorphic last multipliers for holomorphic vector fields on G

The holomorphic volume element ω on G defined in Section 2 will now be used to introduce the notion of holomorphic last multipliers. The computations are similar to the case of smooth manifolds, [2, 3, 4, 5], or complex manifolds [6]. More precisely, consider a holomorphic vector field of the form $U = U^i \frac{\partial}{\partial u^i}$, $\theta = i_U \omega$ and let

$$\frac{du^i}{dt} = U^i(u^1(t), \dots, u^n(t)),\ 1 \leq i \leq n,\ t \in \mathbb{R}$$

be a complex ODE system on G defined by the holomorphic vector field U. The classical definition of a last multiplier function for a vector field on smooth manifolds, [2, 3], can now be applied to the case of the complex Lie group G.

Definition 3.1. A holomorphic function μ on G is called a holomorphic last multiplier of the complex ODE system generated by U (or holomorphic last multiplier for U) if

$$\partial(\mu\theta) := \partial\mu \wedge \theta + \mu \cdot \partial\theta = 0. \tag{3.1}$$

Note that for every holomorphic function μ on G, $\partial\mu\wedge\omega = 0$, such that for every holomorphic vector field U on G we have

$$0 = i_U(\partial\mu \wedge \omega) = (i_U \partial\mu) \cdot \omega - \partial\mu \wedge (i_U\omega)$$

or, equivalently,

$$U(\mu) \cdot \omega = \partial\mu \wedge (i_U\omega) = \partial\mu \wedge \theta.$$

Now, definitions (2.1) and (3.1) yield the following result.

Proposition 3.2. *A holomorphic function μ on G is a holomorphic last multiplier for the holomorphic vector field U if and only if*

$$U(\mu) + \mu \cdot \operatorname{div} U = 0. \tag{3.2}$$

Remark 3.3. Relation (3.2) indicates that if ν is a holomorphic non-zero function on G which satisfies the equation

$$L_U(\nu) := U(\nu) = (\operatorname{div} U) \cdot \nu, \tag{3.3}$$

then $1/\nu$ is a holomorphic last multiplier for U and the holomorphic function ν which satisfies (3.3) will be called an inverse holomorphic multiplier for U.

Proposition 3.4. *Let μ be a holomorphic function on G. The set of holomorphic vector fields for which μ is a holomorphic last multiplier is a Lie subalgebra in the algebra of holomorphic vector fields on G.*

Proof. The proof follows as in [6]. □

It is now interesting to search for a holomorphic last multiplier for a holomorphic vector field U of divergence type, that is, $\mu = \operatorname{div} V$ for some holomorphic vector field V on G. From (3.2),

$$U(\operatorname{div} V) + \operatorname{div} V \cdot \operatorname{div} U = 0. \tag{3.4}$$

Multiplying (3.4) by ω gives

$$L_U(\operatorname{div} V) \cdot \omega + \operatorname{div} V \cdot L_U\omega = 0$$

or, equivalently,

$$L_U(\operatorname{div} V \cdot \omega) = L_U L_V \omega = 0.$$

Hence, we have

Proposition 3.5. *If V is a holomorphic vector field which satisfies $L_U L_V \omega = 0$, then $\mu = \operatorname{div} V$ is a holomorphic last multiplier for the holomorphic vector field U.*

The next step is to study holomorphic last multipliers for holomorphic gradient vector fields on the complex Lie group G endowed with a holomorphic Riemannian metric (for instance g or h from Section 2). Such a metric g defines a *holomorphic metric volume form* ω_g (see [10]), as a holomorphic n-form on G such that

$$\omega_g(U_1, \ldots, U_n) = \pm 1,$$

where $\{U_i\}$, $i = 1, \ldots, n$, is an orthonormal holomorphic frame on (G, g), that is, $g(U_j, U_k) = \delta_{jk}$, $j, k = 1, \ldots, n$. As a complex manifold, if (G, g) admits such a volume element, it admits precisely two of them.

If f is a holomorphic function on G, $U = \operatorname{grad} f$ is the gradient vector field of f defined in Section 2 and α is a holomorphic last multiplier for U, then relation (3.2) becomes

$$g(\operatorname{grad} f, \operatorname{grad} \mu) + \mu \Delta f = 0. \tag{3.5}$$

A straightforward computation in local complex coordinates on G yields a similar identity to the case of holomorphic Riemannian manifolds, [6]:

$$g(\operatorname{grad} f, \operatorname{grad} \mu) = \frac{1}{2}\big(\Delta(f\mu) - f\Delta\mu - \mu\Delta f\big). \tag{3.6}$$

Hence,

$$\Delta(f\alpha) + \mu\Delta f = f\Delta\alpha, \tag{3.7}$$

which leads to the following result.

Proposition 3.6. *Let G be a complex Lie group endowed with a holomorphic metric g. If f, μ are holomorphic functions on G such that f is a holomorphic last multiplier for* $\operatorname{grad} \mu$ *and μ is a holomorphic last multiplier for* $\operatorname{grad} f$*, then $f\alpha$ is a holomorphic harmonic function on G.*

Corollary 3.7. *If G is a complex Lie group endowed with a holomorphic metric g and f is a holomorphic function on G, then μ is a holomorphic last multiplier for $U = \operatorname{grad} \mu$ if and only if μ^2 is a holomorphic harmonic function on G.*

Corollary 3.8. *If G is a complex Lie group endowed with a holomorphic metric g and f is a holomorphic function on G, then μ^2 is a holomorphic harmonic function on G if and only if*

$$\mu\Delta\mu + g(\operatorname{grad} \mu, \operatorname{grad} \mu) = 0.$$

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References

[1] Goldberg, S.I., *Curvature and Homology*, Revised Edition, Dover Publication, Inc. Mineola, New-York, 1998.

[2] Crâşmăreanu, M., *Last multipliers theory on manifolds*, Tensor, **66**(2005), no. 1, 18–25.

[3] Crâşmăreanu, M., *Last multipliers as autonomous solutions of the Liouville equation of transport*, Houston J. Math., **34**(2008), no. 2, 455–466.

[4] Crâşmăreanu, M., *Last multipliers for multivectors with applications to Poisson geometry*, Taiwanese J. Math., **13**(2009), no. 5, 1623–1636.

[5] Crâşmăreanu, M., *Last multipliers on weighted manifolds and the weighted Liouville equation*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **77** (2015), no. 3 (in press).

[6] Crâşmăreanu, M., Ida, C., Popescu, P., *Holomorphic last multipliers on complex manifolds*, Preprint available to arXiv:submit/1295734 [math.DG] 4 July 2015.

[7] Ida, C., Ionescu, A., *On a metric holomorphic connection in complex Lie groups*, BSG Proceedings, **21**(2014).

[8] Ionescu, A., *On lifts of left-invariant holomorphic vector fields on complex Lie groups*, Bulletin of Transilvania University, Series III, **7**(2014), no. 2.

[9] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry II*, Wiley Interscience, New-York, 1969.

[10] LeBrun, C. R., *$\mathcal{H}$-Space with a cosmological constant*, Proc. R. Soc. Lond. A, **380**(1982), 171-185.

[11] Lee, D. H., *The structure of complex Lie groups*, Research Notes in Mathematics Series, Chapman and Hall/CRC, 232 pag.

[12] Toth, A., Varolin, D., *Holomorphic diffeomorphisms of complex semisimple Lie groups*, Inventiones Mathematicae, **139**(2000), 351–369.

[13] Rund, H., *Local differential-geometric structures on Lie groups*, Tensor, N.S., **48**(1998), 64–87.

[14] Wang, H.-C., *Complex Parallelisable Manifolds*, Proceedings of the American Math. Soc., **5**(1954), no. 5, 71–776.

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Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative

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Abstract. The purpose of this paper is to establish some types of Ulam stability: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for a class of implicit Hadamard fractional-order differential equation.

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1. Introduction

The concept of fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order. See, for example, the books [1, 2, 3, 5, 17, 34] and references therein. Fractional differential equations arise naturally in various fields such as viscoelastic materials, polymer science, fractals, chaotic dynamics, nonlinear control, signal processing, bioengineering and chemical engineering, etc. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. We refer the reader, for example, to the books such as [6, 18, 20, 22, 30], and references therein.

On the other hand, the stability problem of functional equations (of group homomorphisms) was formulated in 1940 by Ulam, in a talk given at Wisconsin University [31, 32]. In 1941, Hyers [13] gave the partial answer to the question of Ulam (for the approximately additive mappings) in the case Banach spaces. Hyers's theorem was generalized by Aoki [4] (for additive mappings). Between 1978 and 1998, Th. M. Rassias established the Hyers-Ulam stability of linear and nonlinear mappings [21, 23, 24]. In 1997, Obloza is the first author who has investigated the Hyers-Ulam stability of linear differential equations [19]. During the last two decades, many papers

[7, 12, 14, 15, 16, 33] and books [11, 25, 26, 27] on this subject have been published in order to generalize the results of Hyers in many directions. Recently in [8, 9, 10] Benchohra and Lazreg considered some existence and Ulam stability results for various classes of implicit differential equations involving the Caputo fractional derivative.

The purpose of this paper is to establish existence, uniqueness and stability results of solutions for the following initial value problem for implicit fractional-order differential equation

$$^{H}D^{\alpha}y(t) = f(t, y(t), ^{H}D^{\alpha}y(t)), \text{ for each } t \in J, \ 0 < \alpha \leq 1, \tag{1.1}$$

$$y(1) = y_1, \tag{1.2}$$

where $^{H}D^{\alpha}$ is the Hadamard fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function space, $y_1 \in \mathbb{R}$ and $J = [1, T]$, $T > 1$.

The paper is organized as follows. In Section 2 we introduce some definitions, notations, and lemmas which are used throughout the paper. In Section 3, we will prove an existence and uniqueness results concerning the problem (1.1)-(1.2). Section 4 is devoted to Ulam-Hyers stabilities for the problem (1.1)-(1.2). Finally, in the last section, we give two examples to illustrate our main results.

This paper initiates the existence and Ulam stability of implicit differential equations involving the Hadamard fractional derivative.

2. Preliminaries

Definition 2.1. ([17]) The Hadamard fractional integral of order α for a continuous function $g : [1, \infty) \to \mathbb{R}$ is defined as

$$_{H}I^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds, \ \alpha > 0,$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt, \ \alpha > 0.$

Definition 2.2. ([17]) The Hadamard derivative of fractional order α for a continuous function $g : [1, \infty) \to \mathbb{R}$ is defined as

$$^{H}D^{\alpha}g(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \frac{g(s)}{s} ds, \quad n-1 < \alpha < n, \ n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number α and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.3. ([17]) The Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \quad \Re(\alpha) > 0.$$

The general Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \quad \Re(\alpha) > 0, \ \Re(\beta) > 0.$$

Thus,

$$E_\alpha(z) = E_{\alpha,1}(z),$$
$$E_1(z) = E_{1,1}(z) = e^z,$$
$$E_2(z) = \cosh\sqrt{z},$$
$$E_{1,2}(z) = \frac{e^z - 1}{z}$$

and

$$E_{2,2}(z) = \frac{\sinh\sqrt{z}}{\sqrt{z}}.$$

We state the following generalization of Gronwall's inequality.

Lemma 2.4. *([29]) For any $t \in [1, T]$,*

$$u(t) \leq a(t) + b(t)\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{u(s)}{s}ds,$$

where all the functions are not negative and continuous. The constant $\alpha > 0$, b is a bounded and monotonic increasing function on $[1, T]$, then,

$$u(t) \leq a(t) + \int_1^t \left[\sum_{n=1}^{\infty}\frac{(b(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\left(\log\frac{t}{s}\right)^{n\alpha-1}a(s)\right]\frac{ds}{s}, \quad t \in [1, T].$$

We adopt the definitions in Rus [28]: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for the equation, for the implicit fractional-order differential equation (1.1).

Definition 2.5. The equation (1.1) is **Ulam-Hyers stable** if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^1(J, \mathbb{R})$ of the inequality

$$|^H D^\alpha z(t) - f(t, z(t), ^H D^\alpha z(t))| \leq \epsilon, \; t \in J, \tag{2.1}$$

there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (1.1) with

$$|z(t) - y(t)| \leq c_f \epsilon, \; t \in J.$$

Definition 2.6. The equation (1.1) is **generalized Ulam-Hyers stable** if there exists $\psi_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\psi_f(0) = 0$, such that for each solution $z \in C^1(J, \mathbb{R})$ of the inequality (2.1) there exists a solution $y \in C^1(J, \mathbb{R})$ of the equation (1.1) with

$$|z(t) - y(t)| \leq \psi_f(\epsilon), \; t \in J.$$

Definition 2.7. The equation (1.1) is **Ulam-Hyers-Rassias stable** with respect to $\varphi \in C(J, \mathbb{R}_+)$ if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^1(J, \mathbb{R})$ of the inequality

$$|^H D^\alpha z(t) - f(t, z(t), ^H D^\alpha z(t))| \leq \epsilon\varphi(t), \; t \in J, \tag{2.2}$$

there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (1.1) with

$$|z(t) - y(t)| \leq c_f \epsilon\varphi(t), \; t \in J.$$

Definition 2.8. The equation (1.1) is **generalized Ulam-Hyers-Rassias stable** with respect to $\varphi \in C(J, \mathbb{R}_+)$ if there exists a real number $c_{f,\varphi} > 0$ such that for each solution $z \in C^1(J, \mathbb{R})$ of the inequality

$$|^H D^\alpha z(t) - f(t, z(t), ^H D^\alpha z(t))| \leq \varphi(t),\ t \in J, \tag{2.3}$$

there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (1.1) with

$$|z(t) - y(t)| \leq c_{f,\varphi}\varphi(t),\ t \in J.$$

Remark 2.9. A function $z \in C^1(J, \mathbb{R})$ is a solution of of the inequality (2.1) if and only if there exists a function $g \in C(J, \mathbb{R})$ (which depend on y) such that

(i) $|g(t)| \leq \epsilon$, $\forall t \in J$.
(ii) $^H D^\alpha z(t) = f(t, z(t), ^H D^\alpha z(t)) + g(t)$, $t \in J$.

Remark 2.10. Clearly,

(i) Definition 2.5 $\Longrightarrow$ Definition 2.6.
(ii) Definition 2.7 $\Longrightarrow$ Definition 2.8.

Remark 2.11. A solution of the implicit differential inequality (2.1) is called an fractional ϵ-solution of the implicit fractional differential equation (1.1).

So, the Ulam stabilities of the implicit differential equations with fractional order are some special types of data dependence of the solutions of fractional implicit differential equations.

3. Existence and uniqueness of solutions

By a solution of the problem (1.1) − (1.2) we mean a function $u \in C^1(J, \mathbb{R})$ satisfying equation (1.1) on J and condition (1.2).

Lemma 3.1. *Let a function $f(t, u, v) : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous. Then the problem* (1.1) − (1.2) *is equivalent to the problem*

$$y(t) = y_1 +_H I^\alpha g(t), \tag{3.1}$$

where $g \in C(J, \mathbb{R})$ satisfies the functional equation

$$g(t) = f(t, y_1 +_H I^\alpha g(t), g(t)).$$

Proof. If $^H D^\alpha y(t) = g(t)$ then $_H I^\alpha\ ^H D^\alpha y(t) =_H I^\alpha g(t)$. So we obtain

$$y(t) = y_1 +_H I^\alpha g(t).$$

□

Theorem 3.2. *Assume*

(H1) *The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.*
(H2) *There exist constants $k > 0$ and $l > 0$ such that*

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq k|u - \bar{u}| + l|v - \bar{v}| \text{ for any } u, v, \bar{u}, \bar{v} \in \mathbb{R} \text{ and } t \in J.$$

If

$$\frac{k(\log T)^{\alpha}}{\Gamma(\alpha+1)}+l<1, \tag{3.2}$$

then there exists a unique solution for the IVP (1.1) − (1.2) *on* J.

Proof. Define the operator $N : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by:

$$N(z)(t)=f\left(t, y_1+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}z(s)\frac{ds}{s}, z(t)\right),\ \text{for each } t\in J \tag{3.3}$$

Let $u, w \in C(J,\mathbb{R})$. Then for $t \in J$, we have

$$\begin{aligned}
|(Nu)(t)-(Nw)(t)| &\leq \frac{k}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|u(s)-w(s)|\frac{ds}{s}\\
&\quad +l|u(t)-w(t)|\\
&\leq \left(\frac{k}{\Gamma(\alpha)}\int_1^t(\log t)^{\alpha-1}\frac{ds}{s}+l\right)||u-w||_{\infty}\\
&\leq \left(\frac{k(\log T)^{\alpha}}{\Gamma(\alpha+1)}+l\right)||u-w||_{\infty}.
\end{aligned}$$

Then

$$||Nu-Nw||_{\infty}\leq\left(\frac{k(\log T)^{\alpha}}{\Gamma(\alpha+1)}+l\right)||u-w||_{\infty}. \tag{3.4}$$

By (3.2), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point $z \in C(J,\mathbb{R})$, i.e $z = N(z)$.
Therefore

$$z(t)=f\left(t, y_1+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}z(s)\frac{ds}{s}, z(t)\right),\ \text{for each } t\in J$$

Set

$$y(t)=y_1+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}z(s)\frac{ds}{s}.$$

This implies that ${}^H D^{\alpha}y(t)=z(t)$ and hence

$${}^H D^{\alpha}y(t)=f(t,y(t),{}^H D^{\alpha}y(t)),\ \text{for each } t\in J.$$

□

4. Ulam-Hyers stability

Theorem 4.1. *Assume that the assumptions (H1), (H2) and* (3.2) *hold. Then the equation* (1.1) *is Ulam-Hyers stable.*

Proof. Let $z \in C(J,\mathbb{R})$ be a solution of the inequality (2.1), i.e.

$$|{}^H D^{\alpha}z(t)-f(t,z(t),{}^H D^{\alpha}z(t))|\leq\epsilon,\ t\in J. \tag{4.1}$$

Let us denote by $y \in C(J,\mathbb{R})$ the unique solution of the Cauchy problem

$${}^H D^{\alpha}y(t)=f(t,y(t),{}^H D^{\alpha}y(t)),\ \text{for each } t\in J,\ 0<\alpha\leq 1,$$

$$y(1) = z(1).$$

By using Lemma 3.1, we have

$$y(t) = z(1) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_y(s) \frac{ds}{s},$$

where $g_y \in C(J, \mathbb{R})$ satisfies the functional equation

$$g_y(t) = f(t, y(1) +_H I^\alpha g_y(t), g_y(t)).$$

By integration of (4.1) we obtain

$$\begin{aligned} \left| z(t) - z(1) - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_z(s) \frac{ds}{s} \right| &\leq \frac{\epsilon (\log t)^\alpha}{\Gamma(\alpha+1)} \\ &\leq \frac{\epsilon (\log T)^\alpha}{\Gamma(\alpha+1)}, \end{aligned} \tag{4.2}$$

where $g_z \in C(J, \mathbb{R})$ satisfies the functional equation

$$g_z(t) = f(t, z(1) +_H I^\alpha g_z(t), g_z(t)).$$

On the other hand, we have, for each $t \in J$

$$\begin{aligned} |z(t) - y(t)| &= \left| z(t) - z(1) - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_y(s) \frac{ds}{s} \right| \\ &= \left| z(t) - z(1) - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_z(s) \frac{ds}{s} \right. \\ &+ \left. \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (g_z(s) - g_y(s)) \frac{ds}{s} \right| \\ &\leq \left| z(t) - z(1) - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_z(s) \frac{ds}{s} \right| \\ &+ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |g_z(s) - g_y(s)| \frac{ds}{s}, \end{aligned} \tag{4.3}$$

where

$$g_y(t) = f(t, y(t), g_y(t)),$$

and

$$g_z(t) = f(t, z(t), g_z(t)).$$

By $(H2)$, we have, for each $t \in J$

$$\begin{aligned} |g_z(t) - g_y(t)| &= |f(t, z(t), g_z(t)) - f(t, y(t), g_y(t))| \\ &\leq k|z(t) - y(t)| + l|g_z(t) - g_y(t)|. \end{aligned}$$

Then

$$|g_z(t) - g_y(t)| \leq \frac{k}{1-l} |z(t) - y(t)|. \tag{4.4}$$

Thus, by (4.2), (4.3), (4.4), and Lemma 2.4 we get

$$\begin{aligned}
|z(t)-y(t)| &\leq \frac{\epsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{k}{(1-l)\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|z(s)-y(s)|\frac{ds}{s} \\
&\leq \frac{\epsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad +\int_1^t\left[\sum_{n=1}^{\infty}\left(\frac{k}{(1-l)}\right)^n\frac{1}{\Gamma(n\alpha)}\left(\log\frac{t}{s}\right)^{n\alpha-1}\frac{\epsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right]\frac{ds}{s} \\
&\leq \frac{\epsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)}\left[1+\sum_{n=1}^{\infty}\left(\frac{k}{1-l}\right)^n\frac{1}{\Gamma(n\alpha)}\frac{(\log T)^{n\alpha}}{n\alpha}\right] \\
&= \frac{\epsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)}\left[1+\sum_{n=1}^{\infty}\left(\frac{k}{1-l}\right)^n\frac{(\log T)^{n\alpha}}{\Gamma(n\alpha+1)}\right] \\
&= \frac{\epsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)}\left[1+\sum_{n=1}^{\infty}\frac{\left(\frac{k}{1-l}(\log T)^{\alpha}\right)^n}{\Gamma(n\alpha+1)}\right] \\
&= \frac{\epsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha}\left(\frac{k}{1-l}(\log T)^{\alpha}\right).
\end{aligned}$$

Then, for each $t\in J$

$$|z(t)-y(t)| \leq \frac{\epsilon(\log T)^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha}\left(\frac{k}{1-l}(\log T)^{\alpha}\right):=c_f\epsilon. \tag{4.5}$$

So, the equation (1.1) is Ulam-Hyers stable. This completes the proof. By putting $\psi(\epsilon)=c\epsilon$, $\psi(0)=0$ yields that the equation (1.1) is generalized Ulam-Hyers stable. □

5. Ulam-Hyers-Rassias stability

Theorem 5.1. *Assume (H1), (H2), (3.2) and*

(H3) *The function $\varphi\in C(J,\mathbb{R}_+)$ is increasing and there exists $\lambda_\varphi>0$ such that, for each $t\in J$, we have*

$$_HI^{\alpha}\varphi(t)\leq\lambda_\varphi\varphi(t).$$

Then the equation (1.1) is Ulam-Hyers-Rassias stable with respect to φ.

Proof. Let $z\in C(J,\mathbb{R})$ be a solution of the inequality (2.2), i.e.

$$|^HD^{\alpha}z(t)-f(t,z(t),^HD^{\alpha}z(t))|\leq\epsilon\varphi(t),\ t\in J,\ \epsilon>0. \tag{5.1}$$

Let us denote by $y\in C(J,\mathbb{R})$ the unique solution of the Cauchy problem

$$^HD^{\alpha}y(t)=f(t,y(t),^HD^{\alpha}y(t)),\ \text{for each},\ t\in J,\ 0<\alpha\leq 1,$$

$$y(1)=z(1).$$

By using Lemma 3.1, we have

$$y(t) = z(1) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_y(s) \frac{ds}{s},$$

where $g_y \in C(J, \mathbb{R})$ satisfies the functional equation

$$g_y(t) = f(t, y(1) +_H I^\alpha g_y(t), g_y(t)).$$

By integration of (5.1) and from (H3), we obtain

$$\begin{aligned} \left| z(t) - z(1) - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_z(s) \frac{ds}{s} \right| &\leq \frac{\epsilon}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \varphi(s) \frac{ds}{s} \\ &\leq \epsilon \lambda_\varphi \varphi(t), \qquad (5.2) \end{aligned}$$

where $g_z \in C(J, \mathbb{R})$ satisfies the functional equation

$$g_z(t) = f(t, z(1) + I^\alpha g_z(t), g_z(t)).$$

On the other hand, we have, for each $t \in J$

$$\begin{aligned} |z(t) - y(t)| &= \left| z(t) - z(1) - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_y(s) \frac{ds}{s} \right| \\ &= \left| z(t) - z(1) - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_z(s) \frac{ds}{s} \right. \\ &+ \left. \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (g_z(s) - g_y(s)) \frac{ds}{s} \right| \\ &\leq \left| z(t) - z(1) - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_z(s) \frac{ds}{s} \right| \\ &+ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |g_z(s) - g_y(s)| \frac{ds}{s}, \qquad (5.3) \end{aligned}$$

where

$$g_y(t) = f(t, y(t), g_y(t)),$$

and

$$g_z(t) = f(t, z(t), g_z(t)).$$

Then, by (4.4), (5.2), and (5.3)

$$\begin{aligned} |z(t) - y(t)| &\leq \epsilon \lambda_\varphi \varphi(t) + \frac{k}{(1-l)\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |z(s) - y(s)| \frac{ds}{s} \\ &\leq \epsilon \lambda_\varphi \varphi(t) + \frac{k||z-y||_\infty}{(1-l)\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\leq \epsilon \lambda_\varphi \varphi(t) + \frac{k||z-y||_\infty}{(1-l)\Gamma(\alpha)} \frac{(\log T)^\alpha}{\alpha}. \end{aligned}$$

Thus,

$$||z-y||_\infty \left[1-\frac{k(\log T)^\alpha}{(1-l)\Gamma(\alpha+1)}\right] \le \epsilon\lambda_\varphi\varphi(t).$$

We obtain, by (3.2)

$$||z-y||_\infty \le \frac{\epsilon\lambda_\varphi\varphi(t)}{\left[1-\frac{k(\log T)^\alpha}{(1-l)\Gamma(\alpha+1)}\right]}.$$

Then, for each $t\in J$

$$|z(t)-y(t)| \le \left[1-\frac{k(\log T)^\alpha}{(1-l)\Gamma(\alpha+1)}\right]^{-1}\lambda_\varphi\epsilon\varphi(t) := c_f\epsilon\varphi(t). \quad (5.4)$$

So, the equation (1.1) is Ulam-Hyers-Rassias stable. □

6. Examples

Example 6.1. Consider the following Cauchy problem

$${}^H D^{\frac{1}{2}} y(t) = \frac{1}{200}(t\sin y(t) - y(t)\cos(t)) + \frac{1}{100}\sin {}^H D^{\frac{1}{2}} y(t), \text{ for each } t\in[1,e], \quad (6.1)$$

$$y(1) = 1. \quad (6.2)$$

Set

$$f(t,u,v) = \frac{1}{200}(t\sin u - u\cos(t)) + \frac{1}{100}\sin v, \quad t\in[1,e], \ u,v\in\mathbb{R}.$$

Clearly, the function f is jointly continuous.
For any $u,v,\bar{u},\bar{v}\in\mathbb{R}$ and $t\in[1,e]:$

$$\begin{aligned}|f(t,u,v)-f(t,\bar{u},\bar{v})| &\le \frac{1}{200}|t||\sin u-\sin\bar{u}| + \frac{1}{200}|\cos t||u-\bar{u}| \\ &\quad + \frac{1}{100}|\sin v-\sin\bar{v}| \\ &\le \frac{e}{200}|u-\bar{u}| + \frac{1}{200}|u-\bar{u}| + \frac{1}{100}|v-\bar{v}| \\ &= \frac{e+1}{200}|u-\bar{u}| + \frac{1}{100}|v-\bar{v}|.\end{aligned}$$

Hence condition (H2) is satisfied with $k=\frac{e+1}{200}$ and $l=\frac{1}{100}$.
Thus condition

$$\frac{k(\log T)^\alpha}{(1-l)\Gamma(\alpha+1)} = \frac{\frac{e+1}{200}}{(1-\frac{1}{100})\Gamma(\frac{3}{2})} = \frac{e+1}{99\sqrt{\pi}} < 1,$$

is satisfied. It follows from Theorem 3.2 that the problem (6.1)-(6.2) as a unique solution, and from Theorem 4.1 the equation (6.1) is Ulam-Hyers stable.

Example 6.2. Consider the following Cauchy problem

$$^{H}D^{\frac{1}{2}}y(t) = \frac{2+|y(t)|+|^{H}D^{\frac{1}{2}}y(t)|}{120e^{t+10}(1+|y(t)|+|^{H}D^{\frac{1}{2}}y(t)|)}, \text{ for each } t\in[1,e], \tag{6.3}$$

$$y(1) = 1. \tag{6.4}$$

Set

$$f(t,u,v) = \frac{2+|u|+|v|}{120e^{t+10}(1+|u|+|v|)}, \quad t\in[1,e],\ u,v\in\mathbb{R}.$$

Clearly, the function f is jointly continuous.
For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t\in[1,e]$

$$|f(t,u,v)-f(t,\bar{u},\bar{v})| \le \frac{1}{120e^{10}}(|u-\bar{u}|+|v-\bar{v}|).$$

Hence condition (H2) is satisfied with $K = L = \frac{1}{120e^{10}}$.
Let $\varphi(t) = (\log t)^{\frac{1}{2}}$. We have

$$\begin{aligned} {}_{H}I^{\alpha}\varphi(t) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_1^t \left(\log\frac{t}{s}\right)^{\frac{1}{2}-1}(\log t)^{\frac{1}{2}}\frac{ds}{s} \\ &\le \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_1^t \left(\log\frac{t}{s}\right)^{\frac{1}{2}-1}\frac{ds}{s} \\ &= \frac{2\omega(t)}{\sqrt{\pi}}. \end{aligned}$$

Thus

$${}_{H}I^{\alpha}\varphi(t) \le \frac{2}{\sqrt{\pi}}(\log t)^{\frac{1}{2}} := \lambda_{\varphi}\varphi(t).$$

Thus condition (H3) is satisfied with $\varphi(t) = (\log t)^{\frac{1}{2}}$ and $\lambda_{\varphi} = \frac{2}{\sqrt{\pi}}$ It follows from Theorem 3.2 that the problem (6.3)-(6.4) as a unique solution on J, and from Theorem 5.1 the equation (6.3) is Ulam-Hyers-Rassias stable.

References

[1] Abbas, S., Benchohra, M., N'Guérékata, G.M., *Topics in Fractional Differential Equations*, Springer-Verlag, New York, 2012.

[2] Abbas, S., Benchohra, M., N'Guérékata, G.M., *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[3] Anastassiou, G.A., *Advances on Fractional Inequalities*, Springer, New York, 2011.

[4] Aoki, T., *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2**(1950), 64-66.

[5] Baleanu, D., Diethelm, K., Scalas, E., Trujillo, J.J., *Fractional Calculus Models and Numerical Methods*, World Scientific Publishing, New York, 2012.

[6] Baleanu, D., Güvenç, Z.B., Machado, J.A.T., *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer, New York, 2010.

[7] Benchohra, M., Bouriah, S., *Existence and stability results for nonlinear boundary value problem for implicit differential equations of fractional order*, Moroccan J. Pure Appl. Anal., **1**(1)(2015), 22-36.

[8] Benchohra, M., Lazreg, J.E., *Nonlinear fractional implicit differential equations*, Commun. Appl. Anal., **17**(2013), 471-482.

[9] Benchohra, M., Lazreg, J.E., *Existence and uniqueness results for nonlinear implicit fractional differential equations with boundary conditions*, Rom. J. Math. Comput. Sc., **4**(1)(2014), 60-72.

[10] Benchohra, M., Lazreg, J.E., *On stability for nonlinear implicit fractional differential equations*, Matematiche (Catania), **70**(2015), 49-61.

[11] Cho, Y.J., Rassias, Th.M., Saadati, R., *Stability of Functional Equations in Random Normed Spaces*, Springer, New York, 2013.

[12] Gavruta, P., *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184**(1994), 431-436.

[13] Hyers, D.H., *On the stability of the linear functional equation*, Natl. Acad. Sci. U.S.A., **27**(1941), 222-224.

[14] Jung, S.M., *On the Hyers-Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal. Appl., **222**(1998), 126-137.

[15] Jung, S.M., *Hyers-Ulam stability of linear differential equations of first order, II*, Appl. Math. Lett., **19**(2006), 854-858.

[16] Jung, S.M., Lee, K.S., *Hyers-Ulam stability of first order linear partial differential equations with constant coefficients*, Math. Inequal. Appl., **10**(2007), 261-266.

[17] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[18] Lakshmikantham, V., Leela, S., Vasundhara, J., *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[19] Obloza, M., *Hyers stability of the linear differential equation*, Rocznik Nauk-Dydakt. Prace Mat., **13**(1993), 259-270.

[20] Ortigueira, M.D., *Fractional Calculus for Scientists and Engineers*, Lecture Notes in Electrical Engineering, 84, Springer, Dordrecht, 2011.

[21] Park, C., Rassias, Th.M., *Homomorphisms and derivations in proper JCQ^*-triples*, J. Math. Anal. Appl., **337**(2008), 1404-1414.

[22] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[23] Rassias, Th.M., *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297-300.

[24] Rassias, Th.M., *On a modified Hyers-Ulam sequence*, J. Math. Anal. Appl., **158**(1991), 106-113.

[25] Rassias, Th.M., *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht 2003.

[26] Rassias, Th.M., *Functional Equations, Difference Inequalities and Ulam Stability Notions (F.U.N.)*, Nova Science Publishers, New York, 2010.

[27] Rassias, Th.M., Brzdek, J., *Functional Equations in Mathematical Analysis*, Springer, New York, 2012.

[28] Rus, I.A., *Ulam stabilities of ordinary differential equations in a Banach space*, Carpathian J. Math., **26**(2010), 103-107.

[29] Lin, S.-Y., *Generalized Gronwall inequalities and their applications to fractional differential equations*, J. Inequal. Appl., **2013**, 2013:549, 9 pp.

[30] Tarasov, V.E., *Fractional Dynamics: Application of Fractional Calculus to Dynamics of particles, Fields and Media*, Springer, Heidelberg; Higher Education Press, Beijing, 2010.

[31] Ulam, S.M., *Problems in Modern Mathematics*, Chapter 6, John Wiley and Sons, New York, USA, 1940.

[32] Ulam, S.M., *A Collection of Mathematical Problems*, Interscience, New York, 1960.

[33] Wang, L., Lv, L., Zhou, Y., *Ulam stability and data dependence for fractional differential equations with Caputo derivative*, Electron. J. Qual. Theory Differ. Equat., **63**(2011), 1-10.

[34] Zhou, Y., *Basic Theory of Fractional Differential Equations*, World Scientific Publishing, 2014.

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Self adjoint operator harmonic polynomials induced Chebyshev-Grüss inequalities

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Abstract. We present here very general self adjoint operator harmonic Chebyshev-Grüss inequalities with applications.

Mathematics Subject Classification (2010): 26D10, 26D20, 47A60, 47A67.

Keywords: Self adjoint operator, Hilbert space, Chebyshev-Grüss inequalities, harmonic polynomials.

1. Motivation

Here we mention the following inspiring and motivating result.

Theorem 1.1. (Čebyšev, 1882, [3]) *Let $f, g : [a, b] \to \mathbb{R}$ absolutely continuous functions. If $f', g' \in L_\infty([a, b])$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right| \tag{1.1}$$
$$\leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty .$$

Also we mention

Theorem 1.2. (Grüss, 1935, [9]) *Let f, g integrable functions from $[a, b]$ into $\mathbb{R}$, such that $m \leq f(x) \leq M$, $\rho \leq g(x) \leq \sigma$, for all $x \in [a, b]$, where $m, M, \rho, \sigma \in \mathbb{R}$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right| \tag{1.2}$$
$$\leq \frac{1}{4} (M-m)(\sigma - \rho) .$$

Next we follow [1], pp. 132-152.

We make

Brief Assumption 1.3. Let $f : \prod_{i=1}^{m} [a_i, b_i] \to \mathbb{R}$ with $\frac{\partial^l f}{\partial x_i^l}$ for $l = 0, 1, ..., n$; $i = 1, ..., m$, are continuous on $\prod_{i=1}^{m} [a_i, b_i]$.

Definition 1.4. We put

$$q(x_i, s_i) = \begin{cases} s_i - a_i, \text{ if } s_i \in [a_i, x_i], \\ s_i - b_i, \text{ if } s_i \in (x_i, b_i], \end{cases} \tag{1.3}$$

$x_i \in [a_i, b_i]$, $i = 1, ..., m$.

Let $(P_n)_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials, that is $P_n' = P_{n-1}$, $n \in \mathbb{N}$, $P_0 = 1$.

Let functions f_λ, $\lambda = 1, ..., r \in \mathbb{N} - \{1\}$, as in Brief Assumption 1.3, and $n_\lambda \in \mathbb{N}$ associated with f_λ.

We set

$$A_{i\lambda}(x_i, ..., x_m) := \frac{n_\lambda^{i-1}}{\prod_{j=1}^{i-1} (b_j - a_j)}$$

$$\times \left[\sum_{k=1}^{n_\lambda - 1} (-1)^{k+1} P_k(x_i) \int_{a_1}^{b_1} ... \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^k f_\lambda(s_1, ..., s_{i-1}, x_i, ..., x_m)}{\partial x_i^k} ds_1 ... ds_{i-1} \right. \tag{1.4}$$

$$+ \sum_{k=1}^{n_\lambda - 1} \frac{(-1)^k (n_\lambda - k)}{b_i - a_i}$$

$$\times \left[P_k(b_i) \int_{a_1}^{b_1} ... \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^{k-1} f_\lambda(s_1, ..., s_{i-1}, b_i, x_{i+1}, ..., x_m)}{\partial x_i^{k-1}} ds_1 ... ds_{i-1} \right.$$

$$\left.\left. - P_k(a_i) \int_{a_1}^{b_1} ... \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^{k-1} f_\lambda(s_1, ..., s_{i-1}, a_i, x_{i+1}, ..., x_m)}{\partial x_i^{k-1}} ds_1 ... ds_{i-1} \right] \right],$$

and

$$B_{i\lambda}(x_i, ..., x_m) := \frac{n_\lambda^{i-1} (-1)^{n_\lambda + 1}}{\prod_{j=1}^{i} (b_j - a_j)} \tag{1.5}$$

$$\times \left[\int_{a_1}^{b_1} ... \int_{a_i}^{b_i} P_{n_\lambda - 1}(s_i) q(x_i, s_i) \frac{\partial^{n_\lambda} f_\lambda(s_1, ..., s_i, x_{i+1}, ..., x_m)}{\partial x_i^{n_\lambda}} ds_1 ... ds_i \right],$$

for all $i = 1, ..., m$; $\lambda = 1, ..., r$.

We also set

$$A_1 := \left(\frac{\left(\prod_{j=1}^{m} (b_j - a_j) \right)}{3} \right) \cdot \left[\sum_{\lambda=1}^{r} \left\{ \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^{r} \|f_\rho\|_{\infty, \prod_{j=1}^{m} [a_j, b_j]} \right) \right. \right. \tag{1.6}$$

$$\times\left(\sum_{i=1}^{m}\left[\left(b_i-a_i\right)n_\lambda^{i-1}\left\|P_{n_\lambda-1}\right\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_\lambda}f_\lambda}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right]\right)\Bigg\}\Bigg],$$

(let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$)

$$A_2:=\sum_{\lambda=1}^{r}\sum_{i=1}^{m}\left\|\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_\rho\right\|_{L_p\left(\prod_{j=1}^{m}[a_j,b_j]\right)}\left\|B_{i\lambda}\right\|_{L_q\left(\prod_{j=i}^{m}[a_j,b_j]\right)}\left(\prod_{j=1}^{i-1}\left(b_j-a_j\right)\right)^{\frac{1}{q}}, \tag{1.7}$$

and

$$A_3:=\frac{1}{2}\left\{\sum_{\lambda=1}^{r}\left\{\left\|\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_\rho\right\|_{L_1\left(\prod_{j=1}^{m}[a_j,b_j]\right)}\left[\sum_{i=1}^{m}\left[\left(b_i-a_i\right)n_\lambda^{i-1}\right.\right.\right.\right. \tag{1.8}$$

$$\left.\left.\left.\left.\times\ \left\|P_{n_\lambda-1}\right\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_\lambda}f_\lambda}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right]\right]\right\}\right\}.$$

We finally set

$$W:=r\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\prod_{\rho=1}^{r}f_\rho\left(x\right)\right)dx \tag{1.9}$$

$$-\frac{1}{\prod_{j=1}^{n}\left(b_j-a_j\right)}\sum_{\lambda=1}^{r}n_\lambda^m\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_\rho\left(x\right)\right)dx\right)\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}f_\lambda\left(s\right)ds\right)$$

$$-\sum_{\lambda=1}^{r}\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_\rho\left(x\right)\right)\left(\sum_{i=1}^{m}A_{i\lambda}\left(x_i,...,x_m\right)\right)\right)dx.$$

We mention

Theorem 1.5. ([1], p. 151-152) *It holds*

$$\left|W\right|\leq\min\left\{A_1,A_2,A_3\right\}. \tag{1.10}$$

2. Background

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. The Gelfand map establishes a $*-$isometrically isomorphism Φ between the set $C\left(Sp\left(A\right)\right)$

of all continuous functions defined on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see e.g. [8, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ (the operation composition is on the right) and $\Phi(\overline{f}) = (\Phi(f))^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f), \text{ for all } f \in C(Sp(A)),$$

and we call it the continuous functional calculus for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$ then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

(P) $f(t) \geq g(t)$ for any $t \in Sp(A)$, implies that $f(A) \geq g(A)$ in the operator order of $B(H)$.

Equivalently, we use (see [6], pp. 7-8):

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family.

Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieljes integral:

$$\langle f(U)x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y \rangle), \tag{2.1}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of bounded variation on the interval $[m, M]$, and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle,$$

for any $x, y \in H$. Furthermore, it is known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is increasing and right continuous on $[m, M]$.

An important formula used a lot here is

$$\langle f(U)x, x \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, x \rangle), \quad \forall\, x \in H. \tag{2.2}$$

As a symbol we can write

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda. \tag{2.3}$$

Above,

$$m = \min\{\lambda | \lambda \in Sp(U)\} := \min Sp(U),\ M = \max\{\lambda | \lambda \in Sp(U)\} := \max Sp(U).$$

The projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, are called the spectral family of A, with the properties:

(a) $E_\lambda \leq E_{\lambda'}$ for $\lambda \leq \lambda'$;

(b) $E_{m-0} = 0_H$ (zero operator), $E_M = 1_H$ (identity operator) and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$.

Furthermore

$$E_\lambda := \varphi_\lambda (U), \ \forall \lambda \in \mathbb{R}, \tag{2.4}$$

is a projection which reduces U, with

$$\varphi_\lambda (s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

The spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ determines uniquely the self-adjoint operator U and vice versa.

For more on the topic see [10], pp. 256-266, and for more detalis see there pp. 157-266. See also [5].

Some more basics are given (we follow [6], pp. 1-5):

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{C}$. A bounded linear operator A defined on H is selfjoint, i.e., $A = A^*$, iff $\langle Ax, x\rangle \in \mathbb{R}$, $\forall x \in H$, and if A is selfadjoint, then

$$\|A\| = \sup_{x\in H:\|x\|=1} |\langle Ax, x\rangle| . \tag{2.5}$$

Let A, B be selfadjoint operators on H. Then $A \leq B$ iff $\langle Ax, x\rangle \leq \langle Bx, x\rangle$, $\forall x \in H$.

In particular, A is called positive if $A \geq 0$.

Denote by

$$\mathcal{P} := \left\{ \varphi (s) := \sum_{k=0}^{n} \alpha_k s^k | n \geq 0, \, \alpha_k \in \mathbb{C}, \, 0 \leq k \leq n \right\}. \tag{2.6}$$

If $A \in \mathcal{B}(H)$ (the Banach algebra of all bounded linear operators defined on H, i.e. from H into itself) is selfadjoint, and $\varphi (s) \in \mathcal{P}$ has real coefficients, then $\varphi (A)$ is selfadjoint, and

$$\|\varphi (A)\| = \max \{|\varphi (\lambda)| , \lambda \in Sp (A)\} . \tag{2.7}$$

If φ is any function defined on $\mathbb{R}$ we define

$$\|\varphi\|_A := \sup \{|\varphi (\lambda)| , \lambda \in Sp (A)\} . \tag{2.8}$$

If A is selfadjoint operator on Hilbert space H and φ is continuous and given that $\varphi (A)$ is selfadjoint, then $\|\varphi (A)\| = \|\varphi\|_A$. And if φ is a continuous real valued function so it is $|\varphi|$, then $\varphi (A)$ and $|\varphi| (A) = |\varphi (A)|$ are selfadjoint operators (by [6], p. 4, Theorem 7).

Hence it holds

$$\begin{aligned} \||\varphi (A)|\| &= \||\varphi|\|_A = \sup \{||\varphi (\lambda)|| , \lambda \in Sp (A)\} \\ &= \sup \{|\varphi (\lambda)| , \lambda \in Sp (A)\} = \|\varphi\|_A = \|\varphi (A)\| , \end{aligned}$$

that is

$$\||\varphi (A)|\| = \|\varphi (A)\| . \tag{2.9}$$

For a selfadjoint operator $A \in \mathcal{B}(H)$ which is positive, there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$, that is $\left(\sqrt{A}\right)^2 = A$. We call B the square root of A.

Let $A \in \mathcal{B}(H)$, then A^*A is selfadjoint and positive. Define the "operator absolute value" $|A| := \sqrt{A^*A}$. If $A = A^*$, then $|A| = \sqrt{A^2}$.

For a continuous real valued function φ we observe the following:

$$|\varphi(A)| \text{ (the functional absolute value) } = \int_{m-0}^{M} |\varphi(\lambda)| \, dE_\lambda$$

$$= \int_{m-0}^{M} \sqrt{(\varphi(\lambda))^2} dE_\lambda = \sqrt{(\varphi(A))^2} = |\varphi(A)| \text{ (operator absolute value)},$$

where A is a selfadjoint operator.

That is we have

$$|\varphi(A)| \text{ (functional absolute value) } = |\varphi(A)| \text{ (operator absolute value)}. \tag{2.10}$$

Let $A, B \in \mathcal{B}(H)$, then

$$\|AB\| \leq \|A\| \, \|B\|, \tag{2.11}$$

by Banach algebra property.

3. Main results

Let $(P_n)_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials, that is $P_n' = P_{n-1}$, $n \in \mathbb{N}$, $P_0 = 1$. Furthermore, let $[a, b] \subset \mathbb{R}$, $a \neq b$, and $h : [a, b] \to \mathbb{R}$ be such that $h^{(n-1)}$ is absolutely continuous function for some $n \in \mathbb{N}$.

We set

$$q(x, t) = \begin{cases} t - a, & \text{if } t \in [a, x], \\ t - b, & \text{if } t \in (x, b], \end{cases} \quad x \in [a, b]. \tag{3.1}$$

By [4], and [1], p. 133, we get the generalized Fink type representation formula

$$h(x) = \sum_{k=1}^{n-1} (-1)^{k+1} P_k(x) h^{(k)}(x)$$

$$+ \sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{b-a} \left[P_k(b) h^{(k-1)}(b) - P_k(a) h^{(k-1)}(a)\right] \tag{3.2}$$

$$+ \frac{n}{b-a} \int_a^b h(t) \, dt + \frac{(-1)^{n+1}}{b-a} \int_a^b P_{n-1}(t) q(x, t) h^{(n)}(t) \, dt,$$

$\forall\, x \in [a, b]$, $n \in \mathbb{N}$, when $n = 1$ the above sums are zero.

For the harmonic sequence of polynomials $P_k(t) = \frac{(t-x)^k}{k!}$, $k \in \mathbb{Z}_+$, (3.2) reduces to Fink formula, see [7].

Next we present very general harmonic Chebyshev-Grüss operator inequalities based on (3.2). Then we specialize them for $n = 1$.

We give

Theorem 3.1. *Let $n \in \mathbb{N}$ and $f, g \in C^n([a,b])$ with $[m, M] \subset (a,b)$, $m < M$. Here A is a selfadjoint linear bounded operator on the Hilbert space H with spectrum $Sp(A) \subseteq [m, M]$. We consider any $x \in H : \|x\| = 1$.*

Then

$$\langle (\Delta(f,g))(A)x, x\rangle := |\langle f(A)\,g(A)x, x\rangle - \langle f(A)x, x\rangle \langle g(A)x, x\rangle$$
$$-\frac{1}{2}\left[\sum_{k=1}^{n-1}(-1)^{k+1}\left\{\left[\left\langle P_k(A)\left(g(A)\,f^{(k)}(A) + f(A)\,g^{(k)}(A)\right)x, x\right\rangle\right]\right.\right. \tag{3.3}$$
$$-\left.\left.\left[\left\langle P_k(A)\,f^{(k)}(A)x, x\right\rangle \langle g(A)x, x\rangle + \left\langle P_k(A)\,g^{(k)}(A)x, x\right\rangle \langle f(A)x, x\rangle\right]\right\}\right]\Bigg|$$
$$\leq \frac{\left[\|g(A)\|\,\|f^{(n)}\|_{\infty,[m,M]} + \|f(A)\|\,\|g^{(n)}\|_{\infty,[m,M]}\right]}{2(M-m)}$$
$$\|P_{n-1}\|_{\infty,[m,M]}\left[\left\|(M1_H - A)^2\right\| + \left\|(A - m1_H)^2\right\|\right].$$

Proof. Here $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of A. Set

$$k(\lambda, t) := \begin{cases} t - m, & m \leq t \leq \lambda, \\ t - M, & \lambda < t \leq M. \end{cases} \tag{3.4}$$

where $\lambda \in [m, M]$.

Hence by (3.2) we obtain

$$f(\lambda) = \sum_{k=1}^{n-1}(-1)^{k+1}P_k(\lambda)\,f^{(k)}(\lambda) \tag{3.5}$$
$$+\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{M-m}\left[P_k(M)\,f^{(k-1)}(M) - P_k(m)\,f^{(k-1)}(m)\right]$$
$$+\frac{n}{M-m}\int_m^M f(t)\,dt + \frac{(-1)^{n+1}}{M-m}\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt,$$

and

$$g(\lambda) = \sum_{k=1}^{n-1}(-1)^{k+1}P_k(\lambda)\,g^{(k)}(\lambda) \tag{3.6}$$
$$+\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{M-m}\left[P_k(M)\,g^{(k-1)}(M) - P_k(m)\,g^{(k-1)}(m)\right]$$
$$+\frac{n}{M-m}\int_m^M g(t)\,dt + \frac{(-1)^{n+1}}{M-m}\int_m^M P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt,$$

$\forall\ \lambda \in [m, M]$.

By applying the spectral representation theorem on (3.5), (3.6), i.e. integrating against E_λ over $[m, M]$, see (2.3), (ii), we obtain:

$$f(A) = \sum_{k=1}^{n-1}(-1)^{k+1}P_k(A)\,f^{(k)}(A) \tag{3.7}$$

$$+\left(\sum_{k=1}^{n-1}\frac{(-1)^{k}(n-k)}{M-m}\left[P_{k}(M)f^{(k-1)}(M)-P_{k}(m)f^{(k-1)}(m)\right]\right)1_{H}$$

$$+\left(\frac{n}{M-m}\int_{m}^{M}f(t)\,dt\right)1_{H}+\frac{(-1)^{n+1}}{M-m}\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_{\lambda},$$

and

$$g(A)=\sum_{k=1}^{n-1}(-1)^{k+1}P_{k}(A)\,g^{(k)}(A) \tag{3.8}$$

$$+\left(\sum_{k=1}^{n-1}\frac{(-1)^{k}(n-k)}{M-m}\left[P_{k}(M)g^{(k-1)}(M)-P_{k}(m)g^{(k-1)}(m)\right]\right)1_{H}$$

$$+\left(\frac{n}{M-m}\int_{m}^{M}g(t)\,dt\right)1_{H}+\frac{(-1)^{n+1}}{M-m}\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_{\lambda}.$$

We notice that

$$g(A)\,f(A)=f(A)\,g(A) \tag{3.9}$$

to be used next.

Then it holds

$$g(A)\,f(A)=\sum_{k=1}^{n-1}(-1)^{k+1}g(A)\,P_{k}(A)\,f^{(k)}(A) \tag{3.10}$$

$$+\left(\sum_{k=1}^{n-1}\frac{(-1)^{k}(n-k)}{M-m}\left[P_{k}(M)f^{(k-1)}(M)-P_{k}(m)f^{(k-1)}(m)\right]\right)g(A)$$

$$+\left(\frac{n}{M-m}\int_{m}^{M}f(t)\,dt\right)g(A)$$

$$+\frac{(-1)^{n+1}}{M-m}g(A)\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_{\lambda},$$

and

$$f(A)\,g(A)=\sum_{k=1}^{n-1}(-1)^{k+1}f(A)\,P_{k}(A)\,g^{(k)}(A) \tag{3.11}$$

$$+\left(\sum_{k=1}^{n-1}\frac{(-1)^{k}(n-k)}{M-m}\left[P_{k}(M)g^{(k-1)}(M)-P_{k}(m)g^{(k-1)}(m)\right]\right)f(A)$$

$$+\left(\frac{n}{M-m}\int_{m}^{M}g(t)\,dt\right)f(A)$$

$$+\frac{(-1)^{n+1}}{M-m}f(A)\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_{\lambda}.$$

Here from now on we consider $x \in H : \|x\| = 1$; immediately we get

$$\int_{m-0}^{M} d\langle E_\lambda x, x\rangle = 1.$$

Then it holds (see (2.2))

$$\langle f(A)x, x\rangle = \sum_{k=1}^{n-1} (-1)^{k+1} \left\langle P_k(A) f^{(k)}(A)x, x\right\rangle \tag{3.12}$$

$$+\sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{M-m} \left[P_k(M) f^{(k-1)}(M) - P_k(m) f^{(k-1)}(m)\right]$$

$$+\frac{n}{M-m}\int_m^M f(t)\,dt + \frac{(-1)^{n+1}}{M-m}\int_{m-0}^{M}\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right) d\langle E_\lambda x, x\rangle,$$

and

$$\langle g(A)x, x\rangle = \sum_{k=1}^{n-1} (-1)^{k+1} \left\langle P_k(A) g^{(k)}(A)x, x\right\rangle \tag{3.13}$$

$$+\sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{M-m} \left[P_k(M) g^{(k-1)}(M) - P_k(m) g^{(k-1)}(m)\right]$$

$$+\frac{n}{M-m}\int_m^M g(t)\,dt + \frac{(-1)^{n+1}}{M-m}\int_{m-0}^{M}\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right) d\langle E_\lambda x, x\rangle.$$

Then we get

$$\langle f(A)x, x\rangle \langle g(A)x, x\rangle = \sum_{k=1}^{n-1} (-1)^{k+1} \left\langle P_k(A) f^{(k)}(A)x, x\right\rangle \langle g(A)x, x\rangle \tag{3.14}$$

$$+\left(\sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{M-m} \left[P_k(M) f^{(k-1)}(M) - P_k(m) f^{(k-1)}(m)\right]\right) \langle g(A)x, x\rangle$$

$$+\left(\frac{n}{M-m}\int_m^M f(t)\,dt\right)\langle g(A)x, x\rangle$$

$$+\frac{(-1)^{n+1}\langle g(A)x, x\rangle}{M-m}\int_{m-0}^{M}\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right) d\langle E_\lambda x, x\rangle,$$

and

$$\langle g(A)x, x\rangle \langle f(A)x, x\rangle = \sum_{k=1}^{n-1} (-1)^{k+1} \left\langle P_k(A) g^{(k)}(A)x, x\right\rangle \langle f(A)x, x\rangle \tag{3.15}$$

$$+\left(\sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{M-m} \left[P_k(M) g^{(k-1)}(M) - P_k(m) g^{(k-1)}(m)\right]\right) \langle f(A)x, x\rangle$$

$$+\left(\frac{n}{M-m}\int_m^M g(t)\,dt\right)\langle f(A)x, x\rangle$$

$$+\frac{(-1)^{n+1}\langle f(A)x,x\rangle}{M-m}\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)d\langle E_{\lambda}x,x\rangle.$$

Furthermore we obtain

$$\langle f(A)\,g(A)\,x,x\rangle \overset{(3.10)}{=} \sum_{k=1}^{n-1}(-1)^{k+1}\left\langle g(A)\,P_k(A)\,f^{(k)}(A)\,x,x\right\rangle \tag{3.16}$$

$$+\left(\sum_{k=1}^{n-1}\frac{(-1)^{k}(n-k)}{M-m}\left[P_k(M)\,f^{(k-1)}(M)-P_k(m)\,f^{(k-1)}(m)\right]\right)\langle g(A)x,x\rangle$$

$$+\left(\frac{n}{M-m}\int_{m}^{M}f(t)\,dt\right)\langle g(A)x,x\rangle$$

$$+\frac{(-1)^{n+1}}{M-m}\left\langle\left(g(A)\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_{\lambda}\right)x,x\right\rangle,$$

and

$$\langle f(A)\,g(A)\,x,x\rangle \overset{(3.11)}{=} \sum_{k=1}^{n-1}(-1)^{k+1}\left\langle f(A)\,P_k(A)\,g^{(k)}(A)\,x,x\right\rangle \tag{3.17}$$

$$+\left(\sum_{k=1}^{n-1}\frac{(-1)^{k}(n-k)}{M-m}\left[P_k(M)\,g^{(k-1)}(M)-P_k(m)\,g^{(k-1)}(m)\right]\right)\langle f(A)x,x\rangle$$

$$+\left(\frac{n}{M-m}\int_{m}^{M}g(t)\,dt\right)\langle f(A)x,x\rangle$$

$$+\frac{(-1)^{n+1}}{M-m}\left\langle\left(f(A)\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_{\lambda}\right)x,x\right\rangle.$$

By (3.14) and (3.16) we obtain

$$E := \langle f(A)\,g(A)\,x,x\rangle-\langle f(A)x,x\rangle\langle g(A)x,x\rangle \tag{3.18}$$

$$=\sum_{k=1}^{n-1}(-1)^{k+1}\left[\left\langle g(A)\,P_k(A)\,f^{(k)}(A)\,x,x\right\rangle-\left\langle P_k(A)\,f^{(k)}(A)\,x,x\right\rangle\langle g(A)x,x\rangle\right]$$

$$+\frac{(-1)^{n+1}}{M-m}\left[\left\langle\left(g(A)\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_{\lambda}\right)x,x\right\rangle\right.$$

$$\left.-\langle g(A)x,x\rangle\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)d\langle E_{\lambda}x,x\rangle\right],$$

and by (3.15) and (3.17) we derive

$$E := \langle f(A)\,g(A)\,x,x\rangle-\langle f(A)x,x\rangle\langle g(A)x,x\rangle \tag{3.19}$$

$$=\sum_{k=1}^{n-1}(-1)^{k+1}\left[\left\langle f(A)\,P_k(A)\,g^{(k)}(A)\,x,x\right\rangle-\left\langle P_k(A)\,g^{(k)}(A)\,x,x\right\rangle\langle f(A)x,x\rangle\right]$$

$$+\frac{(-1)^{n+1}}{M-m}\left[\left\langle\left(f(A)\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_{\lambda}\right)x,x\right\rangle\right.$$

$$\left.=\langle f(A)\,x,x\rangle\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)d\langle E_{\lambda}x,x\rangle\right].$$

Consequently, we get that

$$\begin{aligned}2E=\sum_{k=1}^{n-1}(-1)^{k+1}&\left\{\left[\left\langle g(A)\,P_k(A)\,f^{(k)}(A)\,x,x\right\rangle+\left\langle f(A)\,P_k(A)\,g^{(k)}(A)\,x,x\right\rangle\right]\right.\\ &-\left.\left[\left\langle P_k(A)\,f^{(k)}(A)\,x,x\right\rangle\langle g(A)\,x,x\rangle+\left\langle P_k(A)\,g^{(k)}(A)\,x,x\right\rangle\langle f(A)\,x,x\rangle\right]\right\}\\ &+\frac{(-1)^{n+1}}{M-m}\left\{\left[\left\langle\left(g(A)\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_{\lambda}\right)x,x\right\rangle\right.\right.\\ &+\left.\left\langle\left(f(A)\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_{\lambda}\right)x,x\right\rangle\right]\\ &-\left[\langle g(A)\,x,x\rangle\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)d\langle E_{\lambda}x,x\rangle\right.\\ &+\left.\left.\langle f(A)\,x,x\rangle\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)d\langle E_{\lambda}x,x\rangle\right]\right\}.\end{aligned}\tag{3.20}$$

We find that

$$\begin{aligned}&\langle f(A)\,g(A)\,x,x\rangle-\langle f(A)\,x,x\rangle\langle g(A)\,x,x\rangle\\ &-\frac{1}{2}\left[\sum_{k=1}^{n-1}(-1)^{k+1}\left\{\left[\left\langle P_k(A)\left(g(A)\,f^{(k)}(A)+f(A)\,g^{(k)}(A)\right)x,x\right\rangle\right]\right.\right.\\ &-\left.\left.\left[\left\langle P_k(A)\,f^{(k)}(A)\,x,x\right\rangle\langle g(A)\,x,x\rangle+\left\langle P_k(A)\,g^{(k)}(A)\,x,x\right\rangle\langle f(A)\,x,x\rangle\right]\right\}\right]\\ &=\frac{(-1)^{n+1}}{2(M-m)}\left\{\left[\left\langle\left(g(A)\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_{\lambda}\right)x,x\right\rangle\right.\right.\\ &+\left.\left\langle\left(f(A)\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_{\lambda}\right)x,x\right\rangle\right]\\ &-\left[\langle g(A)\,x,x\rangle\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)d\langle E_{\lambda}x,x\rangle\right.\\ &+\left.\left.\langle f(A)\,x,x\rangle\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)d\langle E_{\lambda}x,x\rangle\right]\right\}=:R.\end{aligned}\tag{3.21}$$

Therefore it holds

$$|R|\leq\frac{1}{2(M-m)}\left\{\left[\|g(A)\|\left\|\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_{\lambda}\right\|\right.\right.$$

$$+\|f(A)\|\left\|\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_{\lambda}\right\|\Bigg] \tag{3.22}$$

$$+\left[\|g(A)\|\left\|\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_{\lambda}\right\|\right.$$

$$\left.+\|f(A)\|\left\|\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_{\lambda}\right\|\right]\Bigg\}$$

$$=\frac{1}{(M-m)}\Bigg\{\|g(A)\|\left\|\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_{\lambda}\right\|$$

$$+\|f(A)\|\left\|\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_{\lambda}\right\|\Bigg\}=:(\xi_1). \tag{3.23}$$

We notice the following:

$$\left\|\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_{\lambda}\right\|$$

$$=\sup_{x\in H:\|x\|=1}\left|\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)d\langle E_{\lambda}x,x\rangle\right|$$

$$\leq\sup_{x\in H:\|x\|=1}\left(\int_{m-0}^{M}\left(\int_{m}^{M}|P_{n-1}(t)|\,|k(\lambda,t)|\left|f^{(n)}(t)\right|dt\right)d\langle E_{\lambda}x,x\rangle\right) \tag{3.24}$$

$$\leq\left(\|P_{n-1}\|_{\infty,[m,M]}\left\|f^{(n)}\right\|_{\infty,[m,M]}\right)$$

$$\sup_{x\in H:\|x\|=1}\left(\int_{m-0}^{M}\left(\int_{m}^{M}|k(\lambda,t)|\,dt\right)d\langle E_{\lambda}x,x\rangle\right)=:(\xi_2).$$

(Notice that

$$\int_{m}^{M}|k(\lambda,t)|\,dt=\int_{m}^{\lambda}(t-m)\,dt+\int_{\lambda}^{M}(M-t)\,dt=\frac{(\lambda-m)^2+(M-\lambda)^2}{2}.) \tag{3.25}$$

Hence it holds

$$(\xi_2)\overset{(3.25)}{=}\left(\frac{\|P_{n-1}\|_{\infty,[m,M]}\left\|f^{(n)}\right\|_{\infty,[m,M]}}{2}\right)$$

$$\times\sup_{x\in H:\|x\|=1}\left[\left\langle(M1_H-A)^2x,x\right\rangle+\left\langle(A-m1_H)^2x,x\right\rangle\right]$$

$$\leq\left(\frac{\|P_{n-1}\|_{\infty,[m,M]}\left\|f^{(n)}\right\|_{\infty,[m,M]}}{2}\right)\left[\left\|(M1_H-A)^2\right\|+\left\|(A-m1_H)^2\right\|\right]. \tag{3.26}$$

We have proved that

$$\left\|\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_{\lambda}\right\| \tag{3.27}$$

$$\leq \left(\frac{\|P_{n-1}\|_{\infty,[m,M]} \left\|f^{(n)}\right\|_{\infty,[m,M]}}{2}\right) \left[\left\|(M1_H - A)^2\right\| + \left\|(A - m1_H)^2\right\|\right].$$

Similarly, it holds

$$\left\|\int_{m-0}^{M} \left(\int_{m}^{M} P_{n-1}(t)\, k(\lambda,t)\, g^{(n)}(t)\, dt\right) dE_\lambda\right\|$$

$$\leq \left(\frac{\|P_{n-1}\|_{\infty,[m,M]} \left\|g^{(n)}\right\|_{\infty,[m,M]}}{2}\right) \left[\left\|(M1_H - A)^2\right\| + \left\|(A - m1_H)^2\right\|\right]. \quad (3.28)$$

Next we apply (3.27), (3.28) into (3.23), we get

$$(\xi_1) \leq \frac{1}{(M-m)} \left\{ \|g(A)\| \left(\frac{\|P_{n-1}\|_{\infty,[m,M]} \left\|f^{(n)}\right\|_{\infty,[m,M]}}{2}\right.\right. \quad (3.29)$$

$$\left.\times \left[\left\|(M1_H - A)^2\right\| + \left\|(A - m1_H)^2\right\|\right]\right) + \|f(A)\|$$

$$\left.\times \left(\frac{\|P_{n-1}\|_{\infty,[m,M]} \left\|g^{(n)}\right\|_{\infty,[m,M]}}{2}\right) \left[\left\|(M1_H - A)^2\right\| + \left\|(A - m1_H)^2\right\|\right]\right\}$$

$$= \frac{1}{2(M-m)} \left\{ \left[\|g(A)\| \left\|f^{(n)}\right\|_{\infty,[m,M]} + \|f(A)\| \left\|g^{(n)}\right\|_{\infty,[m,M]}\right] \right.$$

$$\left. \|P_{n-1}\|_{\infty,[m,M]} \left[\left\|(M1_H - A)^2\right\| + \left\|(A - m1_H)^2\right\|\right]\right\}. \quad (3.30)$$

We have proved that

$$|R| \leq \frac{\left(\|g(A)\| \left\|f^{(n)}\right\|_{\infty,[m,M]} + \|f(A)\| \left\|g^{(n)}\right\|_{\infty,[m,M]}\right)}{2(M-m)}$$

$$\|P_{n-1}\|_{\infty,[m,M]} \left[\left\|(M1_H - A)^2\right\| + \left\|(A - m1_H)^2\right\|\right]. \quad (3.31)$$

The theorem is proved. □

It follows the case $n = 1$.

Corollary 3.2. (to Theorem 3.1) *Let $f, g \in C^1([a,b])$ with $[m,M] \subset (a,b)$, $m < M$. Here A is a selfadjoint bounded linear operator on the Hilbert space H with spectrum $Sp(A) \subseteq [m,M]$. We consider any $x \in H : \|x\| = 1$.*

Then

$$|\langle f(A)\, g(A)\, x, x\rangle - \langle f(A)\, x, x\rangle \langle g(A)\, x, x\rangle| \quad (3.32)$$

$$\leq \frac{\left[\|g(A)\| \|f'\|_{\infty,[m,M]} + \|f(A)\| \|g'\|_{\infty,[m,M]}\right]}{2(M-m)}$$

$$\left[\left\|(M1_H - A)^2\right\| + \left\|(A - m1_H)^2\right\|\right].$$

We continue with

Theorem 3.3. *All as in Theorem* 3.1. *Let* $\alpha, \beta, \gamma > 1 : \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1$. *Then*

$$\langle (\Delta (f,g)) (A) x, x \rangle \leq \frac{\|P_{n-1}\|_{\alpha,[m,M]}}{(M-m)(\beta+1)^{\frac{1}{\beta}}}$$
$$\left[\|g(A)\| \left\| f^{(n)} \right\|_{\gamma,[m,M]} + \|f(A)\| \left\| g^{(n)} \right\|_{\gamma,[m,M]} \right] \tag{3.33}$$
$$\left[\left\| (A - m1_H)^{1+\frac{1}{\beta}} \right\| + \left\| (M1_H - A)^{1+\frac{1}{\beta}} \right\| \right].$$

Proof. As in (3.24) we have

$$\left\| \int_{m-0}^{M} \left(\int_m^M P_{n-1}(t) k(\lambda,t) f^{(n)}(t) dt \right) dE_\lambda \right\|$$
$$= \sup_{x \in H: \|x\|=1} \left| \int_{m-0}^{M} \left(\int_m^M P_{n-1}(t) k(\lambda,t) f^{(n)}(t) dt \right) d(\langle E_\lambda x, x \rangle) \right| =: \psi_1. \tag{3.34}$$

Here $\alpha, \beta, \gamma > 1 : \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1$. By Hölder's inequality for three functions we get

$$\left| \int_m^M P_{n-1}(t) k(\lambda,t) f^{(n)}(t) dt \right| \leq \int_m^M |P_{n-1}(t)| |k(\lambda,t)| \left| f^{(n)}(t) \right| dt$$
$$\leq \|P_{n-1}\|_\alpha \left\| f^{(n)} \right\|_\gamma \left(\int_m^M |k(\lambda,t)|^\beta dt \right)^{\frac{1}{\beta}}$$
$$= \|P_{n-1}\|_\alpha \left\| f^{(n)} \right\|_\gamma \left(\int_m^\lambda (t-m)^\beta dt + \int_\lambda^M (M-t)^\beta dt \right)^{\frac{1}{\beta}} \tag{3.35}$$
$$= \|P_{n-1}\|_\alpha \left\| f^{(n)} \right\|_\gamma \left[\frac{(\lambda-m)^{\beta+1} + (M-\lambda)^{\beta+1}}{\beta+1} \right]^{\frac{1}{\beta}}$$
$$\leq \frac{\|P_{n-1}\|_\alpha \left\| f^{(n)} \right\|_\gamma}{(\beta+1)^{\frac{1}{\beta}}} \left[(\lambda-m)^{\frac{\beta+1}{\beta}} + (M-\lambda)^{\frac{\beta+1}{\beta}} \right].$$

I.e. it holds

$$\left| \int_m^M P_{n-1}(t) k(\lambda,t) f^{(n)}(t) dt \right|$$
$$\leq \frac{\|P_{n-1}\|_\alpha \left\| f^{(n)} \right\|_\gamma}{(\beta+1)^{\frac{1}{\beta}}} \left[(\lambda-m)^{1+\frac{1}{\beta}} + (M-\lambda)^{1+\frac{1}{\beta}} \right], \quad \forall \lambda \in [m,M]. \tag{3.36}$$

Therefore we get

$$\psi_1 \leq \sup_{x \in H: \|x\|=1} \int_{m-0}^{M} \left| \int_m^M P_{n-1}(t) k(\lambda,t) f^{(n)}(t) dt \right| d\langle E_\lambda x, x \rangle$$
$$\leq \left(\sup_{x \in H: \|x\|=1} \int_{m-0}^{M} \left[(\lambda-m)^{1+\frac{1}{\beta}} + (M-\lambda)^{1+\frac{1}{\beta}} \right] d\langle E_\lambda x, x \rangle \right)$$

$$\frac{\|P_{n-1}\|_{\alpha,[m,M]}\left\|f^{(n)}\right\|_{\gamma,[m,M]}}{(\beta+1)^{\frac{1}{\beta}}}$$

$$\leq\left(\frac{\|P_{n-1}\|_{\alpha,[m,M]}\left\|f^{(n)}\right\|_{\gamma,[m,M]}}{(\beta+1)^{\frac{1}{\beta}}}\right) \tag{3.37}$$

$$\left[\left\|(A-m1_H)^{1+\frac{1}{\beta}}\right\|+\left\|(M1_H-A)^{1+\frac{1}{\beta}}\right\|\right].$$

We have proved that

$$\left\|\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_\lambda\right\| \tag{3.38}$$

$$\leq\frac{\|P_{n-1}\|_{\alpha,[m,M]}\left\|f^{(n)}\right\|_{\gamma,[m,M]}}{(\beta+1)^{\frac{1}{\beta}}}\left[\left\|(A-m1_H)^{1+\frac{1}{\beta}}\right\|+\left\|(M1_H-A)^{1+\frac{1}{\beta}}\right\|\right].$$

Similarly, it holds

$$\left\|\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_\lambda\right\|$$

$$\leq\frac{\|P_{n-1}\|_{\alpha,[m,M]}\left\|g^{(n)}\right\|_{\gamma,[m,M]}}{(\beta+1)^{\frac{1}{\beta}}}\left[\left\|(A-m1_H)^{1+\frac{1}{\beta}}\right\|+\left\|(M1_H-A)^{1+\frac{1}{\beta}}\right\|\right]. \tag{3.39}$$

Using (3.23) we derive

$$|R|\leq\frac{1}{(M-m)}\left\{\|g(A)\|\frac{\|P_{n-1}\|_{\alpha,[m,M]}\left\|f^{(n)}\right\|_{\gamma,[m,M]}}{(\beta+1)^{\frac{1}{\beta}}}\right. \tag{3.40}$$

$$\left[\left\|(A-m1_H)^{1+\frac{1}{\beta}}\right\|+\left\|(M1_H-A)^{1+\frac{1}{\beta}}\right\|\right]$$

$$+\|f(A)\|\frac{\|P_{n-1}\|_{\alpha,[m,M]}\left\|g^{(n)}\right\|_{\gamma,[m,M]}}{(\beta+1)^{\frac{1}{\beta}}}$$

$$\left.\left[\left\|(A-m1_H)^{1+\frac{1}{\beta}}\right\|+\left\|(M1_H-A)^{1+\frac{1}{\beta}}\right\|\right]\right\}$$

$$=\frac{1}{(M-m)}\left[\|g(A)\|\left\|f^{(n)}\right\|_{\gamma,[m,M]}+\|f(A)\|\left\|g^{(n)}\right\|_{\gamma,[m,M]}\right]\frac{\|P_{n-1}\|_{\alpha,[m,M]}}{(\beta+1)^{\frac{1}{\beta}}} \tag{3.41}$$

$$\left[\left\|(A-m1_H)^{1+\frac{1}{\beta}}\right\|+\left\|(M1_H-A)^{1+\frac{1}{\beta}}\right\|\right],$$

proving the claim. □

The case $n=1$ follows.

Corollary 3.4. (to Theorem 3.3) *All as in Theorem* 3.3*. It holds*

$$\begin{aligned}&|\langle f(A)g(A)x,x\rangle-\langle f(A)x,x\rangle\langle g(A)x,x\rangle|\\ \leq&\frac{1}{(M-m)(\beta+1)^{\frac{1}{\beta}}}\left[\|g(A)\|\|f'\|_{\gamma,[m,M]}+\|f(A)\|\|g'\|_{\gamma,[m,M]}\right]\\ &\left[\left\|(A-m1_H)^{1+\frac{1}{\beta}}\right\|+\left\|(M1_H-A)^{1+\frac{1}{\beta}}\right\|\right].\end{aligned}\tag{3.42}$$

We also give

Theorem 3.5. *All as in Theorem* 3.1*. It holds*

$$\begin{aligned}&\langle(\Delta(f,g))(A)x,x\rangle\leq\|P_{n-1}\|_{\infty,[m,M]}\\ &\left[\|g(A)\|\left\|f^{(n)}\right\|_{1,[m,M]}+\|f(A)\|\left\|g^{(n)}\right\|_{1,[m,M]}\right].\end{aligned}\tag{3.43}$$

Proof. We have that

$$\begin{aligned}\left|\int_m^M P_{n-1}(t)k(\lambda,t)f^{(n)}(t)\,dt\right|&\leq\int_m^M|P_{n-1}(t)||k(\lambda,t)|\left|f^{(n)}(t)\right|dt\\ &\leq\|P_{n-1}\|_{\infty,[m,M]}(M-m)\int_m^M\left|f^{(n)}(t)\right|dt\\ &=\|P_{n-1}\|_{\infty,[m,M]}(M-m)\left\|f^{(n)}\right\|_{1,[m,M]}.\end{aligned}\tag{3.44}$$

So that

$$\begin{aligned}&\left|\int_m^M P_{n-1}(t)k(\lambda,t)f^{(n)}(t)\,dt\right|\\ &\leq(M-m)\|P_{n-1}\|_{\infty,[m,M]}\left\|f^{(n)}\right\|_{1,[m,M]}.\end{aligned}$$

Hence

$$\begin{aligned}&\left\|\int_{m-0}^M\left(\int_m^M P_{n-1}(t)k(\lambda,t)f^{(n)}(t)\,dt\right)dE_\lambda\right\|\\ =&\sup_{x\in H:\|x\|=1}\left|\int_{m-0}^M\left(\int_m^M P_{n-1}(t)k(\lambda,t)f^{(n)}(t)\,dt\right)d\langle E_\lambda x,x\rangle\right|\\ &\leq(M-m)\|P_{n-1}\|_{\infty,[m,M]}\left\|f^{(n)}\right\|_{1,[m,M]},\end{aligned}\tag{3.45}$$

and similarly,

$$\begin{aligned}&\left\|\int_{m-0}^M\left(\int_m^M P_{n-1}(t)k(\lambda,t)g^{(n)}(t)\,dt\right)dE_\lambda\right\|\\ &\leq(M-m)\|P_{n-1}\|_{\infty,[m,M]}\left\|g^{(n)}\right\|_{1,[m,M]}.\end{aligned}\tag{3.46}$$

Using (3.23) we obtain

$$|R|\leq\frac{1}{(M-m)}\left\{\|g(A)\|(M-m)\|P_{n-1}\|_{\infty,[m,M]}\left\|f^{(n)}\right\|_{1,[m,M]}\right.$$

$$+ \|f(A)\| (M-m) \|P_{n-1}\|_{\infty,[m,M]} \left\| g^{(n)} \right\|_{1,[m,M]} \Bigg\}$$

$$= \|P_{n-1}\|_{\infty,[m,M]} \left[\|g(A)\| \left\| f^{(n)} \right\|_{1,[m,M]} + \|f(A)\| \left\| g^{(n)} \right\|_{1,[m,M]} \right], \tag{3.47}$$

proving the claim. □

The case $n = 1$ follows.

Corollary 3.6. (to Theorem 3.5) *It holds*

$$|\langle f(A) g(A) x, x\rangle - \langle f(A) x, x\rangle \langle g(A) x, x\rangle|$$
$$\leq \left[\|g(A)\| \|f'\|_{1,[m,M]} + \|f(A)\| \|g'\|_{1,[m,M]} \right]. \tag{3.48}$$

Comment 3.7. The case of harmonic sequence of polynomials $P_k(t) = \frac{(t-x)^k}{k!}$, $k \in \mathbb{Z}_+$, was completely studied in [2], and this work generalizes it.

Another harmonic sequence of polynomials related to this work is

$$P_k(t) = \frac{1}{k!}\left(t - \frac{m+M}{2}\right)^k, \quad k \in \mathbb{Z}_+, \tag{3.49}$$

see also [4].

The Bernoulli polynomials $B_n(t)$ can be defined by the formula (see [4])

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!} x^n, \quad |x| < 2\pi, \ t \in \mathbb{R}. \tag{3.50}$$

They satisfy the relation

$$B'_n(t) = nB_{n-1}(t), \quad n \in \mathbb{N}.$$

The sequence

$$P_n(t) = \frac{1}{n!} B_n(t), \quad n \in \mathbb{Z}_+, \tag{3.51}$$

is a harmonic sequence of polynomials, $t \in \mathbb{R}$.

The Euler polynomials are defined by the formula (see [4])

$$\frac{2e^{tx}}{e^x + 1} = \sum_{n=0}^{\infty} \frac{E_n(t)}{n!} x^n, \quad |x| < \pi, \ t \in \mathbb{R}. \tag{3.52}$$

They satisfy

$$E'_n(t) = nE_{n-1}(t), \quad n \in \mathbb{N}.$$

The sequence

$$P_n(t) = \frac{1}{n!} E_n(t), \quad n \in \mathbb{Z}_+, \ t \in \mathbb{R}, \tag{3.53}$$

is a harmonic sequence of polynomials.

Finally:

Comment 3.8. One can apply (3.3), (3.33) and (3.43), for the harmonic sequences of polynomials defined by (3.49), (3.51) and (3.53).

In particular, when (see (3.49))

$$P_n(t) = \frac{1}{n!}\left(t - \frac{m+M}{2}\right)^n, \quad n \in \mathbb{Z}_+, \tag{3.54}$$

we get

$$\|P_{n-1}\|_{\infty,[m,M]} = \frac{1}{(n-1)!}\left(\frac{M-m}{2}\right)^{n-1}, \tag{3.55}$$

and

$$\|P_{n-1}\|_{\alpha,[m,M]} = \frac{1}{(n-1)!\,(\alpha(n-1)+1)^{\frac{1}{\alpha}}}\left(\frac{(M-m)^{\alpha(n-1)+1}}{2^{\alpha(n-1)}}\right), \tag{3.56}$$

where $\alpha, \beta, \gamma > 1 : \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1$.

References

[1] Anastassiou, G.A., *Intelligent Comparisons: Analytic Inequalities*, Springer, New York, Heidelberg, 2016.

[2] Anastassiou, G.A., *Self Adjoint Operator Chebyshev-Grüss type inequalities*, submitted, 2016.

[3] Čebyšev, P.L., *Sur les expressions approximatives des intégrales définies par les autres proses entre les mêmes limites*, Proc. Math. Soc. Charkov, **2**(1882), 93-98.

[4] Dedić, Lj., Pečarić, J.E., Ujević, N., *On generalizations of Ostrowski inequality and some related results*, Czechoslovak Mathematical Journal, **53**(2003), no. 1, 173-189.

[5] Dragomir, S.S., *Inequalities for functions of selfadjoint operators on Hilbert Spaces*, ajmaa.org/RGMIA/monographs/InFuncOp.pdf, 2011.

[6] Dragomir, S.S., *Operator inequalities of Ostrowski and Trapezoidal type*, Springer, New York, 2012.

[7] Fink, A.M., *Bounds on the deviation of a function from its averages*, Czechoslovak Mathematical Journal, **42**(117)(1992), 289-310.

[8] Furuta, T., Mićić Hot, J., Pečaric, J., Seo, Y., *Mond-Pečaric Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[9] Grüss, G., *Über das Maximum des absoluten Betrages von* $\left[\left(\frac{1}{b-a}\right)\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx\right)\right]$, Math. Z., **39**(1935), 215-226.

[10] Helmberg, G., *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc., New York, 1969.

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Some new estimates for Fejér type inequalities in quantum analysis

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Abstract. In this paper we derive some new quantum estimates of Fejér type inequalities which involve Riemann type of quantum integrals via some classes of convex functions. We also discuss some special cases which can be deduced from the main results of this paper.

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1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, I is an interval, is said to be a convex function on I if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. If the reversed inequality in (1.1) holds, then f is said to be concave. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I and $a, b \in I$ with $a < b$. Then the following double inequality holds:

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx \leq \int_a^b f(x)p(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b p(x)dx, \tag{1.2}$$

where $p : [a,b] \to \mathbb{R}$ is non-negative, integrable, and symmetric about $x = \frac{a+b}{2}$. This inequality is known as the Fejér inequality for convex functions (see [2, 3, 16, 17]).

Theory of convexity plays an important role in different fields of pure and applied sciences. Due to its importance in recent years several new generalizations of classical convexity have been proposed in the literature. Breckner [1] introduced the notion of s-convex function, as

Definition 1.1 ([1]). Let $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex in the second sense if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{1.3}$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$. This class of s-convex functions is usually denoted by K_s^2.

For more information on s-convex functions, see [4].

In 2007, Varošanec [15] introduced the notion of h-convex functions, which not only generalizes the class of convex functions but also some other classes of convex functions, see [15]. Thus it was noticed that the class of h-convex functions is quite unifying one. This class is defined as:

Definition 1.2 ([15]). Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function and $[0, 1] \subset J$. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex function, or that f belong to the class $SX(h, I)$, if f is nonnegative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y). \tag{1.4}$$

If inequality (1.4) is reversed, then f is said to be h-concave, i.e. $f \in SV(h, I)$. Obviously, if $h(t) = t$, then all nonnegative convex functions belong to $SX(h, I)$ and all nonnegative concave functions belong to $SV(h, I)$;and if $h(t) = t^s$, where $s \in (0, 1)$, then $SX(h, I) \supseteq K_s^2$.

In [13], G. H. Toader defined the concept of m-convexity as the following:

Definition 1.3. The function $f : [0, b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have:

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y). \tag{1.5}$$

Denote by $K_m(b)$ the set of the m-convex functions on $[0, b]$.
In [6], V. G. Miheşan introduced the class of (s, m)-convex functions as the following:

Definition 1.4. The function $f : [0, b] \to \mathbb{R}$ is said to be (s, m)-convex, where $(s, m) \in (0, 1]^2$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t^s)f(y). \tag{1.6}$$

Denote by $K_m^s(b)$ the set of the (s, m)-convex functions on $[0, b]$.
In [17], Yang and Tseng established the following theorem

Theorem 1.5 (see [17], Remark 6). *Let $f : [a, b] \to \mathbb{R}$ be a convex function and $p : [a, b] \to \mathbb{R}$ be a nonnegative, integrable and symmetric about $x = \frac{a+b}{2}$. If H and F are defined on $[0, 1]$ by*

$$H(t) = \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) p(x)dx,$$

and

$$F(t) = \int_a^b \frac{1}{2}\left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) p\left(\frac{x+a}{2}\right) + f\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right) p\left(\frac{x+b}{2}\right)\right] dx,$$

then H, F are convex and increasing on $[0, 1]$ and for all $t \in [0, 1]$

$$f\left(\frac{a+b}{2}\right) \int_a^b p(x)dx = H(0) \leq H(t) \leq H(1) = \int_a^b f(x)p(x)dx$$

and

$$\int_a^b f(x)p(x)dx = F(0) \leq F(t) \leq F(1) = \frac{f(a)+f(b)}{2}\int_a^b p(x)dx.$$

In [12] , M. Z. Sarikaya, E. Set and M. E. Özdemir established the following inequality:

Theorem 1.6. *Let $f \in SX(h, I)$, $a, b \in I$ with $a < b$, $f \in L_1([a, b])$ and $p : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then, for $h(\frac{1}{2}) \neq 0$, we have*

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx &\leq \int_a^b f(x)p(x)dx \\ &\leq \frac{f(a)+f(b)}{2}\left(h(t)+h(1-t)\right)\int_a^b p(x)dx. \qquad (1.7)\end{aligned}$$

In [14], the following inequalities of Fejér type via s-convex function was derived:

Theorem 1.7. *Let $f \in K_s^2$, $a, b \in [0, \infty[$ with $a < b$ and $p : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then*

$$\begin{aligned}2^{s-1} f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx &\leq \int_a^b f(x)p(x)dx \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b \left(\left(\frac{b-x}{b-a}\right)^s + \left(\frac{x-a}{b-a}\right)^s\right)p(x)dx. \qquad (1.8)\end{aligned}$$

Again, in [14], the authors proved the following theorems:

Theorem 1.8 ([14])**.** *Let $f : [0, \infty[\to \mathbb{R}$ be an m-convex function with $m \in]0, 1]$. If $0 \leq a < b < \infty$ and $f \in L_1[0, b]$, then*

$$\int_a^b f(x)p(x)dx \leq \min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2}, \frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}\int_a^b p(x)dx.$$

Theorem 1.9 ([14])**.** *Let $f : [0, \infty[\to \mathbb{R}$ be an m-convex function with $m \in]0, 1]$. If $0 \leq a < b < \infty$ and $f \in L_1[0, b]$, then*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx \leq \int_a^b \frac{f(x)+mf(\frac{x}{m})}{2}p(x)dx \\ &\leq \frac{1}{8}\left(f(a)+f(b)+2m\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)+m^2\left(f\left(\frac{a}{m^2}\right)+f\left(\frac{b}{m^2}\right)\right)\right)\int_a^b p(x)dx.\end{aligned}$$

The aim of this work is to establish the q-analogue of Fejér inequalities for some convex type functions. For this we recall some basic concepts of quantum calculus. Let $0 < q < 1$, the q-Jackson integral from 0 to b is defined by [5] as:

$$\int_0^b f(x)d_qx = (1-q)b\sum_{n=0}^{\infty} f(bq^n)q^n \qquad (1.9)$$

provided the sum converge absolutely.

The q-Jackson integral in a generic interval $[a,b]$ is given by [5]

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{1.10}$$

In [10], the authors presented a Riemann-type q-integral by:

$$R_q(f;a,b) = (b-a)(1-q)\sum_{k=0}^{\infty} f(a+(b-a)q^k)q^k. \tag{1.11}$$

We can get another definition from the Riemman-type q-integral:

$$\begin{aligned}&\frac{2}{b-a}\int_a^b f(x)d_q^Rx \\ &= (1-q)\sum_{k=0}^{\infty}\left(f\left(\frac{a+b}{2}+q^k\left(\frac{b-a}{2}\right)\right)+f\left(\frac{a+b}{2}-q^k\left(\frac{b-a}{2}\right)\right)\right)q^k\end{aligned}$$

From the q-Jackson integral we can write:

$$\begin{aligned}\frac{2}{b-a}\int_a^b f(x)d_q^Rx &= \int_{-1}^1 f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)d_qt \\ &= \int_{-1}^1 f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)d_qt.\end{aligned} \tag{1.12}$$

Contrary to the q-Jackson integral, if

$$f(x) \le g(x), \quad x \in [a,b]$$

then

$$\int_a^b f(x)d_q^Rx \le \int_a^b g(x)d_q^Rx. \tag{1.13}$$

In [11], the authors established the q-analogue of Hermite-Hadamard inequalities for convex function

Theorem 1.10. *Let $f:[a,b]\to\mathbb{R}$ be a convex function. Then one has the inequalities:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)d_q^Rt \le \frac{f(a)+f(b)}{2}.$$

For some recent studies on quantum integral inequalities, see [7, 8, 9].

2. Main results

In this section, we discuss main results of the paper. For this we need the following Lemma:

Lemma 2.1. *If $f:[a,b]\to\mathbb{R}$ is a convex function. Then the following inequality holds for all $s,t,u,v\in[0,1]$ with $s\le t\le u\le v$ and $t+u=s+v$*

$$f(t)+f(u) \le f(s)+f(v) \tag{2.1}$$

Proof. Since f is convex function, for all $s, t, u, v \in [0,1]$ with $s \leq t \leq u \leq v$ and $t + u = s + v$, we obtain

$$\frac{f(v) - f(u)}{v - u} \geq \frac{f(u) - f(t)}{u - t} \geq \frac{f(t) - f(s)}{t - s} = \frac{f(t) - f(s)}{v - u}$$

then, we have

$$f(v) - f(u) \geq f(t) - f(s).$$

The proof is completed. □

Theorem 2.2. *Let $f : [a, b] \to \mathbb{R}$ be a convex function and $p : [a, b] \to \mathbb{R}$ be a non-negative, integrable and symmetric about $x = \frac{a+b}{2}$. If H and F are defined on $[0, 1]$ by*

$$H(t) = \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) p(x) d_q^R x,$$

and

$$F(t) = \int_a^b \frac{1}{2}\left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) p\left(\frac{x+a}{2}\right) + f\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right) p\left(\frac{x+b}{2}\right)\right] d_q^R x,$$

then H, F are convex and increasing on $[0, 1]$ and for all $t \in [0, 1]$

$$f\left(\frac{a+b}{2}\right) \int_a^b p(x) d_q^R x = H(0) \leq H(t) \leq H(1) = \int_a^b f(x) p(x) d_q^R x \tag{2.2}$$

and

$$\int_a^b f(x) p(x) d_q^R x = F(0) \leq F(t) \leq F(1) = \frac{f(a) + f(b)}{2} \int_a^b p(x) d_q^R x. \tag{2.3}$$

Proof. For all $t_1,\ t_2,\ \lambda \in [0, 1]$ and $x \in [a, b]$, by convexity of f , we have

$$\begin{aligned} & f\left((\lambda t_1 + (1-\lambda)t_2)x + (1 - (\lambda t_1 + (1-\lambda)t_2))\frac{a+b}{2}\right) \\ &= f\left(\lambda\left(t_1 x + (1-t_1)\frac{a+b}{2}\right) + (1-\lambda)\left(t_2 x + (1-t_2)\frac{a+b}{2}\right)\right) \\ &\leq \lambda f\left(t_1 x + (1-t_1)\frac{a+b}{2}\right) + (1-\lambda) f\left(t_2 x + (1-t_2)\frac{a+b}{2}\right). \end{aligned} \tag{2.4}$$

Utilizing the inequality (1.13) and by $p(\cdot)$ a nonnegative function, we get

$$\begin{aligned} & H(\lambda t_1 + (1-\lambda)t_2) \\ &= \int_a^b f\left((\lambda t_1 + (1-\lambda)t_2)x + (1 - (\lambda t_1 + (1-\lambda)t_2))\frac{a+b}{2}\right) p(x) d_q^R x \\ &\leq \lambda \int_a^b f\left(t_1 x + (1-t_1)\frac{a+b}{2}\right) p(x) d_q^R x \\ &+ (1-\lambda) \int_a^b f\left(t_2 x + (1-t_2)\frac{a+b}{2}\right) p(x) d_q^R x \\ &= \lambda H(t_1) + (1-\lambda) H(t_2), \end{aligned}$$

which shows that H is convex on $[0,1]$.
Now, let $0 \le s \le t \le 1$ and $w \in [0,1]$, we have

$$\begin{aligned}
& t\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)+(1-t)\frac{a+b}{2} \\
\le\, & s\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)+(1-s)\frac{a+b}{2} \\
\le\, & s\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)+(1-s)\frac{a+b}{2} \\
\le\, & t\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)+(1-t)\frac{a+b}{2},
\end{aligned}$$

and for all $w \in [-1,0]$, we have

$$\begin{aligned}
& t\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)+(1-t)\frac{a+b}{2} \\
\le\, & s\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)+(1-s)\frac{a+b}{2} \\
\le\, & s\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)+(1-s)\frac{a+b}{2} \\
\le\, & t\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)+(1-t)\frac{a+b}{2},
\end{aligned}$$

for all $w \in [-1,1]$, we get

$$\begin{aligned}
& \left(s\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)+(1-s)\frac{a+b}{2}\right) \\
+ & \left(s\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)+(1-s)\frac{a+b}{2}\right) \\
= & \left(t\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)+(1-t)\frac{a+b}{2}\right) \\
+ & \left(t\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)+(1-t)\frac{a+b}{2}\right).
\end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned}
& f\left(s\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)+(1-s)\frac{a+b}{2}\right) \\
+ & f\left(s\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)+(1-s)\frac{a+b}{2}\right) \\
\le & f\left(t\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)+(1-t)\frac{a+b}{2}\right) \\
+ & f\left(t\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)+(1-t)\frac{a+b}{2}\right).
\end{aligned}$$

Then,

$$
\begin{aligned}
H(s) &= \int_a^b f\left(sx+(1-s)\frac{a+b}{2}\right)p(x)d_q^R x \\
&= \frac{b-a}{2}\int_{-1}^{1} f\left(s\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)+(1-s)\frac{a+b}{2}\right) \\
&\times p\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)d_q w \\
&= \frac{b-a}{4}\int_{-1}^{1} f\left(s\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)+(1-s)\frac{a+b}{2}\right) \\
&\times p\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)d_q w \\
&+ \frac{b-a}{4}\int_{-1}^{1} f\left(s\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)+(1-s)\frac{a+b}{2}\right) \\
&\times p\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)d_q w.
\end{aligned}
$$

Since $p(\cdot)$ is nonnegative, integrable and symmetric about $x=\frac{a+b}{2}$, we get

$$
\begin{aligned}
&\frac{b-a}{4}\int_{-1}^{1} f\left(s\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)+(1-s)\frac{a+b}{2}\right) \\
&\times p\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)d_q w \\
&= \frac{b-a}{4}\int_{-1}^{1} f\left(s\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)+(1-s)\frac{a+b}{2}\right) \\
&\times p\left(a+b-(\frac{1-w}{2}a+\frac{1+w}{2}b)\right)d_q w \\
&= \frac{b-a}{4}\int_{-1}^{1} f\left(s\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)+(1-s)\frac{a+b}{2}\right) \\
&\times p\left(\frac{1+w}{2}a+\frac{1-w}{2}b\right)d_q w \\
&= \frac{b-a}{4}\int_{-1}^{1} f\left(s\left(\frac{1+w}{2}a+\frac{1-w}{2}b\right)+(1-s)\frac{a+b}{2}\right) \\
&\times p\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)d_q w \\
&= \frac{b-a}{4}\int_{-1}^{1} f\left(s\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)+(1-s)\frac{a+b}{2}\right) \\
&\times p\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)d_q w,
\end{aligned}
$$

then, we obtain

$$
\begin{aligned}
H(s) = &\frac{b-a}{4}\int_{-1}^{1} f\left(s\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)+(1-s)\frac{a+b}{2}\right) \\
&\times p\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right) d_q w \\
&+\frac{b-a}{4}\int_{-1}^{1} f\left(s\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)+(1-s)\frac{a+b}{2}\right) \\
&\times p\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right) d_q w \\
&\leq \frac{b-a}{4}\int_{-1}^{1} f\left(t\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)+(1-t)\frac{a+b}{2}\right) \\
&\times p\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right) d_q w \\
&+\frac{b-a}{4}\int_{-1}^{1} f\left(t\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)+(1-t)\frac{a+b}{2}\right) \\
&\times p\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right) d_q w \\
&= H(t).
\end{aligned}
$$

Thus, H is increasing on $[0,1]$ and the inequality (2.2) holds for all $t\in[0,1]$. For all $t_1,\ t_2,\ \lambda\in[0,1]$ and $x\in[a,b]$, by convexity of f , we get

$$
\begin{aligned}
&f\left(\frac{1+\lambda t_1+(1-\lambda)t_2}{2}a+\frac{1-\lambda t_1-(1-\lambda)t_2}{2}x\right) \\
&= f\left(\frac{\lambda(1+t_1)+(1-\lambda)(1+t_2)}{2}a+\frac{\lambda(1-t_1)+(1-\lambda)(1-t_2)}{2}x\right) \\
&\leq \lambda f\left(\frac{(1+t_1)}{2}a+\frac{(1-t_1)}{2}x\right)+(1-\lambda)f\left(\frac{(1+t_2)}{2}a+\frac{(1-t_2)}{2}x\right). \qquad (2.5)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&f\left(\frac{1+\lambda t_1+(1-\lambda)t_2}{2}b+\frac{1-\lambda t_1-(1-\lambda)t_2}{2}x\right) \\
&\leq \lambda f\left(\frac{(1+t_1)}{2}b+\frac{(1-t_1)}{2}x\right)+(1-\lambda)f\left(\frac{(1+t_2)}{2}b+\frac{(1-t_2)}{2}x\right), \qquad (2.6)
\end{aligned}
$$

then, using (2.5), (2.6), (1.13) and the fact that $p(\cdot)$ is nonnegative function, we obtain

$$\begin{aligned}
&F(\lambda t_1+(1-\lambda)t_2)\\
&\le \lambda\int_a^b f\left(\frac{1+t_1}{2}a+\frac{1-t_1}{2}x\right)p\left(\frac{x+a}{2}\right)d_q^R x\\
&+(1-\lambda)\int_a^b f\left(\frac{1+t_2}{2}a+\frac{1-t_2}{2}x\right)p\left(\frac{x+a}{2}\right)d_q^R x\\
&+\lambda\int_a^b f\left(\frac{1+t_1}{2}b+\frac{1-t_1}{2}x\right)p\left(\frac{x+b}{2}\right)d_q^R x\\
&+(1-\lambda)\int_a^b f\left(\frac{1+t_2}{2}b+\frac{1-t_2}{2}x\right)p\left(\frac{x+b}{2}\right)d_q^R x\\
&=\lambda F(t_1)+(1-\lambda)F(t_2).
\end{aligned}$$

Thus, F is convex on $[0,1]$.

For all $w\in[-1,1]$, and $0\le s\le t\le 0$, we have

$$\begin{aligned}
&\frac{1+t}{2}a+\frac{1-t}{2}\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\\
&\le \frac{1+s}{2}a+\frac{1-s}{2}\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\\
&\le \frac{1+s}{2}b+\frac{1-s}{2}\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)\\
&\le \frac{1+t}{2}b+\frac{1-t}{2}\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
&\frac{1+s}{2}a+\frac{1-s}{2}\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\\
&+\frac{1+s}{2}b+\frac{1-s}{2}\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)\\
&=\frac{1+t}{2}a+\frac{1-t}{2}\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\\
&+\frac{1+t}{2}b+\frac{1-t}{2}\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right).
\end{aligned}$$

Using Lemma 2.1, and the fact that $p(\cdot)$ is nonnegative function, we have

$$\begin{aligned}
& f\left(\frac{1+s}{2}a+\frac{1-s}{2}\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right) \\
& \times p\left(\frac{1}{2}\left(a+\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right) \\
& + f\left(\frac{1+s}{2}b+\frac{1-s}{2}\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)\right) \\
& \times p\left(\frac{1}{2}\left(a+\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right) \\
& \leq f\left(\frac{1+t}{2}a+\frac{1-t}{2}\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right) \\
& \times p\left(\frac{1}{2}\left(a+\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right) \\
& + f\left(\frac{1+t}{2}b+\frac{1-t}{2}\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)\right) \\
& \times p\left(\frac{1}{2}\left(a+\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right).
\end{aligned}$$

Integrating with respect to w on $[-1,1]$, we have

$$\begin{aligned}
& \int_{-1}^{1} f\left(\frac{1+s}{2}a+\frac{1-s}{2}\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right) \\
& \times p\left(\frac{1}{2}\left(a+\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right) d_q w \\
& + \int_{-1}^{1} f\left(\frac{1+s}{2}b+\frac{1-s}{2}\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)\right) \\
& \times p\left(\frac{1}{2}\left(a+\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right) d_q w \\
& \leq \int_{-1}^{1} f\left(\frac{1+t}{2}a+\frac{1-t}{2}\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right) \\
& \times p\left(\frac{1}{2}\left(a+\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right) d_q w \\
& + \int_{-1}^{1} f\left(\frac{1+t}{2}b+\frac{1-t}{2}\left(a+b-\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right)\right) \\
& \times p\left(\frac{1}{2}\left(a+\frac{1-w}{2}a+\frac{1+w}{2}b\right)\right) d_q w,
\end{aligned}$$

where, using the fact that $p(\cdot)$ is nonnegative, integrable and symmetric about $x = \frac{a+b}{2}$, we get

$$\begin{aligned}
&\int_{-1}^{1} f\left(\frac{1+s}{2}b + \frac{1-s}{2}\left(a+b-\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right)\right) \\
&\times p\left(\frac{1}{2}\left(a + \frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) d_q w \\
&= \int_{-1}^{1} f\left(\frac{1+s}{2}b + \frac{1-s}{2}\left(a+b-\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right)\right) \\
&\times p\left(a+b-\frac{1}{2}\left(a + \frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) d_q w \\
&= \int_{-1}^{1} f\left(\frac{1+s}{2}b + \frac{1-s}{2}\left(\frac{1+w}{2}a + \frac{1-w}{2}b\right)\right) \\
&\times p\left(\frac{1}{2}\left(b + \frac{1+w}{2}a + \frac{1-w}{2}b\right)\right) d_q w \\
&= \int_{a}^{b} f\left(\frac{1+s}{2}b + \frac{1-s}{2}x\right) p\left(\frac{b+x}{2}\right) d_q^R x.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_{-1}^{1} f\left(\frac{1+t}{2}b + \frac{1-t}{2}\left(a+b-\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)\right)\right) \\
&\times p\left(\frac{1}{2}\left(a + \frac{1-w}{2}a + \frac{1+w}{2}b\right)\right) d_q w \\
&= \int_{a}^{b} f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) p\left(\frac{b+x}{2}\right) d_q^R x,
\end{aligned}$$

then,

$$\begin{aligned}
F(s) &= \frac{1}{2}\int_{a}^{b} \frac{1}{2} f\left(\frac{1+s}{2}a + \frac{1-s}{2}x\right) \\
&\times p\left(\frac{a+x}{2}\right) d_q^R x + \int_{a}^{b} \frac{1}{2} f\left(\frac{1+s}{2}b + \frac{1-s}{2}x\right) p\left(\frac{b+x}{2}\right) d_q^R x \\
&\leq \int_{a}^{b} \frac{1}{2} f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) \\
&\times p\left(\frac{a+x}{2}\right) d_q^R x + \int_{a}^{b} \frac{1}{2} f\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right) p\left(\frac{b+x}{2}\right) d_q^R x \\
&= F(t).
\end{aligned}$$

Thus, F is increasing on $[0, 1]$ and the inequality (2.3) holds for all $t \in [0, 1]$. □

Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}$ be a h-convex function and $p : [a, b] \to \mathbb{R}$ be positive, integrable, and symmetric about $x = \frac{a+b}{2}$. Then the following inequalities hold:*

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \int_a^b p(x) d_q^R x \\
&\leq \int_a^b f(x)p(x) d_q^R x \\
&\leq f(a) \int_a^b h\left(\frac{b-x}{b-a}\right) p(x) d_q^R x + f(b) \int_a^b h\left(\frac{x-a}{b-a}\right) p(x) d_q^R x
\end{aligned}$$

Proof. Since f is h-convex function, we have

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2}\right)(f(x) + f(y)), \tag{2.7}$$

for all $x, y \in [a, b]$.
In (2.7), if we choose $x = \frac{1-w}{2}a + \frac{1+w}{2}b$ and $y = \frac{1+w}{2}a + \frac{1-w}{2}b$, $w \in [-1, 1]$ and by p is positive, we get

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \\
&= f\left(\frac{1}{2}\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) + \frac{1}{2}\left(\frac{1+w}{2}a + \frac{1-w}{2}b\right)\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \\
&\leq h\left(\frac{1}{2}\right) f\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \\
&+ h\left(\frac{1}{2}\right) f\left(\frac{1+w}{2}a + \frac{1-w}{2}b\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right)
\end{aligned}$$

Integrating with respect w over $[-1, 1]$, we obtain

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) \int_{-1}^1 p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) d_q w \\
&\leq h\left(\frac{1}{2}\right) \int_{-1}^1 f\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) d_q w \\
&+ h\left(\frac{1}{2}\right) \int_{-1}^1 f\left(\frac{1+w}{2}a + \frac{1-w}{2}b\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) d_q w,
\end{aligned}$$

Then, by p is symmetric function about $\frac{a+b}{2}$, we have

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) \frac{2}{b-a} \int_a^b p(x) d_q^R x \\
&\leq h\left(\frac{1}{2}\right) \frac{2}{b-a} \int_a^b f(x)p(x) d_q^R x + h\left(\frac{1}{2}\right) \frac{2}{b-a} \int_a^b f(x) p\left(\frac{a+b}{2} + \frac{a+b}{2} - x\right) d_q^R x \\
&= h\left(\frac{1}{2}\right) \frac{2}{b-a} \int_a^b f(x)p(x) d_q^R x + h\left(\frac{1}{2}\right) \frac{2}{b-a} \int_a^b f(x)p(x) d_q^R x,
\end{aligned}$$

the first inequality is proved.

The proof of second inequality is given as

$$\begin{aligned}
&f\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)p\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\\
&\leq f(a)h\left(\frac{1-w}{2}\right)p\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)\\
&+f(b)h\left(\frac{1+w}{2}\right)p\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)
\end{aligned}$$

we integrate w on $[-1,1]$, we obtain

$$\begin{aligned}
&\int_{-1}^{1}f\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)p\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)d_qw\\
&\leq\int_{-1}^{1}f(a)h\left(\frac{1-w}{2}\right)p\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)d_qw\\
&+\int_{-1}^{1}f(b)h\left(\frac{1+w}{2}\right)p\left(\frac{1-w}{2}a+\frac{1+w}{2}b\right)d_qw.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{2}{b-a}\int_a^b f(x)p(x)d_q^Rx &\leq f(a)\frac{2}{b-a}\int_a^b h\left(\frac{b-x}{b-a}\right)p(x)d_q^Rx\\
&+\frac{2}{b-a}f(b)\int_a^b h\left(\frac{x-a}{b-a}\right)p(x)d_q^Rx.
\end{aligned}$$

The proof is complete. □

Remark 2.4. If we choose $p(x)=1$ and $h(t)=t$, then Theorem(2.3) reduces to Theorem(1.10).

Corollary 2.5. *Let $f\in K_s^2$, $a,b\in[0,\infty[$ with $a<b$ and $p:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then*

$$\begin{aligned}
&2^{s-1}f\left(\frac{a+b}{2}\right)\int_a^b p(x)d_q^Rx\leq\int_a^b f(x)p(x)d_q^Rx\\
&\leq f(a)\int_a^b\left(\frac{b-x}{b-a}\right)^s p(x)d_q^Rx+f(b)\int_a^b\left(\frac{x-a}{b-a}\right)^s p(x)d_q^Rx
\end{aligned}$$

Remark 2.6. If we choose $s=1$, $h(t)=t$ and $p(x)=1$, then Corollary (2.5) reduces to Theorem(1.10).

Theorem 2.7. *Let $f:[0,\infty[\to\mathbb{R}$ be an m-convex function with $m\in]0,1]$. If $0\leq a<b<\infty$ and $f\in L_1[0,b]$, then*

$$\int_a^b f(x)p(x)d_q^Rx\leq\min\left\{\int_a^b L_x(a,b)p(x)d_q^Rx,\int_a^b L_x(b,a)p(x)d_q^Rx\right\}.$$

with $L_x(a,b)=\left(f(a)\left(\frac{b-x}{b-a}\right)+mf\left(\frac{b}{m}\right)\left(\frac{x-a}{b-a}\right)\right)$.

Proof. Since f is m-convex and for all $w \in [-1,1]$, we have

$$\begin{aligned} & f\left(\frac{1-w}{2}a + m\frac{1+w}{2m}b\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \\ \le & \left(\frac{1-w}{2}\right) f(a) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right) \\ & + m\left(\frac{1+w}{2}\right) f\left(\frac{b}{m}\right) p\left(\frac{1-w}{2}a + \frac{1+w}{2}b\right), \end{aligned}$$

and

$$\begin{aligned} & f\left(\frac{1-w}{2}b + m\frac{1+w}{2m}a\right) p\left(\frac{1-w}{2}b + \frac{1+w}{2}a\right) \\ \le & \left(\frac{1-w}{2}\right) f(b) p\left(\frac{1-w}{2}b + \frac{1+w}{2}a\right) \\ & + m\left(\frac{1+w}{2}\right) f\left(\frac{a}{m}\right) p\left(\frac{1-w}{2}b + \frac{1+w}{2}a\right). \end{aligned}$$

Then, by integrating both sides with respect w on $[-1,1]$, we have

$$\begin{aligned} \frac{2}{b-a}\int_a^b f(x)p(x)d_q^R x \le & f(a)\frac{2}{b-a}\int_a^b \left(\frac{b-x}{b-a}\right) p(x)d_q^R x \\ & + mf\left(\frac{b}{m}\right)\frac{2}{b-a}\int_a^b \left(\frac{x-a}{b-a}\right) p(x)d_q^R x \end{aligned}$$

and

$$\begin{aligned} \frac{2}{b-a}\int_a^b f(x)p(x)d_q^R x \le & f(b)\frac{2}{b-a}\int_a^b \left(\frac{a-x}{a-b}\right) p(x)d_q^R x \\ & + mf\left(\frac{a}{m}\right)\frac{2}{b-a}\int_a^b \left(\frac{x-b}{a-b}\right) p(x)d_q^R x. \end{aligned}$$

This completes the proof. □

Theorem 2.8. *Let $f : [0,\infty[\to \mathbb{R}$ be a m-convex function with $m \in]0,1]$ and $p : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then*

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right)\int_a^b p(x)d_q^R x \le & \int_a^b \left(f(x) + mf\left(\frac{x}{m}\right)\right) p(x)d_q^R x \\ \le & \left(f(a) + mf\left(\frac{a}{m}\right)\right)\int_a^b \left(\frac{b-x}{b-a}\right) p(x)d_q^R x \\ & + \left(mf\left(\frac{b}{m}\right) + m^2 f\left(\frac{b}{m^2}\right)\right)\int_a^b \left(\frac{x-a}{b-a}\right) p(x)d_q^R x. \end{aligned}$$

Proof. According to the definition of m-convex function, for all $t \in [-1,1]$, we have

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right) \\
&= f\left(\frac{1}{2}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+\frac{m}{2m}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right) p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right) \\
&\leq \frac{1}{2} f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right) p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right) \\
&+ \frac{m}{2} f\left(\frac{\frac{1+t}{2}a+\frac{1-t}{2}b}{m}\right) p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right).
\end{aligned}$$

Integrating with respect to t on $[-1,1]$, we have

$$\begin{aligned}
& \int_{-1}^{1} f\left(\frac{a+b}{2}\right) p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right) d_q t \\
&\leq \frac{1}{2}\int_{-1}^{1} f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right) p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right) d_q t \\
&+ \int_{-1}^{1} \frac{m}{2} f\left(\frac{\frac{1+t}{2}a+\frac{1-t}{2}b}{m}\right) p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right) d_q t.
\end{aligned}$$

Since $p(\cdot)$ is symmetric function, so, we have

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) \frac{2}{b-a}\int_a^b p(x) d_q^R x \\
&\leq \frac{2}{2(b-a)}\int_a^b f(x)p(x) d_q^R x + \frac{2m}{2(b-a)}\int_a^b f\left(\frac{x}{m}\right) p(a+b-x) d_q^R x \\
&= \frac{1}{(b-a)}\int_a^b f(x)p(x) d_q^R x + \frac{m}{(b-a)}\int_a^b f\left(\frac{x}{m}\right) p(x) d_q^R x.
\end{aligned}$$

Then

$$f\left(\frac{a+b}{2}\right) 2\int_a^b p(x) d_q^R x \leq \int_a^b f(x)p(x) d_q^R x + m\int_a^b f\left(\frac{x}{m}\right) p(x) d_q^R x.$$

the first inequality is proved.
Now the proof of second inequality is given by

$$\begin{aligned}
& f\left(\frac{1-t}{2}a+m\frac{1+t}{2m}b\right) p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right) \\
&\leq f(a)\frac{1-t}{2} p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right) \\
&+ m\frac{1+t}{2} f\left(\frac{b}{m}\right) p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right),
\end{aligned}$$

and

$$\begin{aligned}&mf\left(\frac{1-t}{2m}a+\frac{1+t}{2m^2}mb\right)p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\\&\leq mf\left(\frac{a}{m}\right)\frac{1-t}{2}p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\\&+m^2\frac{1+t}{2}f\left(\frac{b}{m^2}\right)p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right).\end{aligned}$$

Integrating both sides with respect to t on $[-1,1]$, we obtain

$$\begin{aligned}\frac{2}{b-a}\int_a^b f(x)p(x)d_q^Rx &\leq f(a)\frac{2}{b-a}\int_a^b\left(\frac{b-x}{b-a}\right)p(x)d_q^Rx\\&+f\left(\frac{b}{m}\right)\frac{2m}{b-a}\int_a^b\left(\frac{x-a}{b-a}\right)p(x)d_q^Rx,\end{aligned}$$

and

$$\begin{aligned}m\frac{2}{b-a}\int_a^b f\left(\frac{x}{m}\right)p(x)d_q^Rx &\leq mf\left(\frac{a}{m}\right)\frac{2}{b-a}\int_a^b\frac{b-x}{b-a}p(x)d_q^Rx\\&+m^2f\left(\frac{b}{m^2}\right)\frac{2}{b-a}\int_a^b\frac{x-a}{b-a}p(x)d_q^Rx.\end{aligned}$$

Then, the proof of Theorem (2.8) is completed. □

Theorem 2.9. *Let $f:[0,\infty[\to\mathbb{R}$ be an (s,m)-convex function with $(s,m)\in]0,1]^2$ and $p:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then*

$$\begin{aligned}2^sf\left(\frac{a+b}{2}\right)\int_a^b p(x)d_q^Rx &\leq \int_a^b f(x)p(x)d_q^Rx+m(2^s-1)\int_a^b f\left(\frac{x}{m}\right)p(x)d_q^Rx\\&\leq\left(f(a)+m(2^s-1)f\left(\frac{a}{m}\right)\right)\int_a^b\left(\frac{b-x}{b-a}\right)^s p(x)d_q^Rx\\&+m\left(f\left(\frac{b}{m}\right)+m(2^s-1)f\left(\frac{b}{m^2}\right)\right)\int_a^b\left(\frac{x-a}{b-a}\right)^s p(x)d_q^Rx\end{aligned}$$

Proof. Since f is (s,m)-convex function on $[a,b]$, we can write

$$\begin{aligned}&f\left(\frac{1}{2}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+\frac{m}{2m}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right)p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\\&\leq\left(\frac{1}{2}\right)^s f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\\&+m\left(1-\left(\frac{1}{2}\right)^s\right)f\left(\frac{1}{m}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right)p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right).\end{aligned}$$

Integrating with respect to t on $[-1,1]$, we get

$$f\left(\frac{a+b}{2}\right)\frac{2}{b-a}\int_a^b p(x)d_q^R x \le \left(\frac{1}{2}\right)^s \frac{2}{b-a}\int_a^b f(x)p(x)d_q^R x$$
$$\le m\left(1-\left(\frac{1}{2}\right)^s\right)\frac{2}{b-a}\int_a^b f\left(\frac{x}{m}\right)p(a+b-x)d_q^R x.$$

Since $p(\cdot)$ is symmetric function, so, we have

$$2^s f\left(\frac{a+b}{2}\right)\int_a^b p(x)d_q^R x \le \int_a^b f(x)p(x)d_q^R x + m(2^s-1)\int_a^b f\left(\frac{x}{m}\right)p(x)d_q^R x.$$

Also, we have

$$\begin{aligned}
&f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\\
&= f\left(\frac{1-t}{2}a+\frac{m}{m}\left(1-\left(\frac{1-t}{2}\right)\right)b\right)p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\\
&\le \left(\frac{1-t}{2}\right)^s f(a)p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\\
&+ m\left(1-\left(\frac{1-t}{2}\right)^s\right)f\left(\frac{b}{m}\right)p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right),
\end{aligned}$$

and

$$\begin{aligned}
&f\left(\frac{\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)}{m}\right)p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\\
&= f\left(\frac{1-t}{2}\frac{a}{m}+\frac{m}{m^2}\left(1-\left(\frac{1-t}{2}\right)\right)b\right)p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\\
&\le \left(\frac{1-t}{2}\right)^s f\left(\frac{a}{m}\right)p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\\
&+ m\left(1-\left(\frac{1-t}{2}\right)^s\right)f\left(\frac{b}{m^2}\right)p\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right).
\end{aligned}$$

Integrating both sides with respect to t on $[-1,1]$, we get

$$\begin{aligned}
&\frac{2}{b-a}\int_a^b f(x)p(x)d_q^R x\\
&\le \frac{2}{b-a}f(a)\int_a^b\left(\frac{b-x}{b-a}\right)^s p(x) + mf\left(\frac{b}{m}\right)\frac{2}{b-a}\int_a^b\left(1-\left(\frac{b-x}{b-a}\right)^s\right)p(x)d_q^R x,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{2}{b-a}\int_a^b f\left(\frac{x}{m}\right)p(x)d_q^R x\\
&\le \frac{2}{b-a}f\left(\frac{a}{m}\right)\int_a^b\left(\frac{b-x}{b-a}\right)^s p(x)d_q^R x + mf\left(\frac{b}{m^2}\right)\frac{2}{b-a}\int_a^b\left(1-\left(\frac{b-x}{b-a}\right)^s\right)p(x)d_q^R x.
\end{aligned}$$

The proof of Theorem (2.9) is completed. □

References

[1] Breckner, W.W., *Stetigkeitsaussagen fiir eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen,* Publ. Inst. Math., **23**(1978), 13-20.

[2] Féjer, L., *Über die Fourierreihen II,* (in Hungarian), Math. Naturwiss. Anz Ungar. Akad. Wiss., **24**(1906), 369-390.

[3] Dragomir, S.S., *Two mappings in connection to Hadamard's inequalities,* J. Math. Anal. Appl., **167**(1992), 49-56.

[4] Hudzik, H., Maligranda, L., *Some remarks on s-convex functions,* Aequationes Math., **48**(1994), 100-111.

[5] Jackson, F.H., *On a q-definite integrals,* Quarterly J. Pure Appl. Math., **41**(1910) 193-203.

[6] Miheşan, V.G., *A generalizations of the convexity,* Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, Romania, 1993.

[7] Noor, M.A., Noor, K.I., Awan, M.U., *Some Quantum estimates for Hermite-Hadamard inequalities,* Appl. Math. Comput., **251**(2015), 675-679.

[8] Noor, M.A., Noor, K.I., Awan, M.U., *Some quantum integral inequalities via preinvex functions,* Appl. Math. Comput., **269**(2015), 242-251.

[9] Noor, M.A., Noor, K.I., Awan, M.U., *Quantum analogues of Hermite-Hadamard type inequalities for generalized convexity,* in: N. Daras and M.T. Rassias (Ed.), Computation, Cryptography and Network Security, 2015.

[10] Rajković, P.M., Stanković, M.S., Marinković, S.D., *The zeros of polynomials orthogonal with respect to q-integral on several intervals in the complex plane,* Proceedings of The Fifth International Conference on Geometry, Integrability and Quantization, 2003, Varna, Bulgaria (ed. I.M. Mladenov, A.C. Hirsshfeld), 178-188.

[11] Taf, S., Brahim, K., Riahi, L., *Some results for Hadamard-type inequalities in quantum calculs*, Le Matematiche, **69**(2014), no. 2, 243-258.

[12] Sarikaya, M.Z., *On some new inequalities of Hadamard type involving h-convex functions,* Acta Math. Univ. Comenianae, **79**(2010), no. 2, 265-272.

[13] Toader, G.H., *Some generalisations of the convexity,* Proc. Colloq. Approx. Optim, (1984), 329-338.

[14] Tseng, K.L., Hwang, S.R., Dragomir, S.S., *On some new inequalities of Hermite-Hadamard-Fejér type involving convex functions,* Demonstratio Math., **40**(2007), no. 1, 51-64.

[15] Varošanec, S., *On h-convexity,* J. Math. Anal. Appl., **326**(2007), no. 1, 303-311.

[16] Yang, G.S., Tseng, K.L., *A note On Hadamard's inequality,* Tamkang. J. Math., **28**(1997) 33-37.

[17] Yang, G.S., Tseng, K.L., *On certain integral inequalities related to Hermite-Hadamard inequalites,* J. Math. Anal. Appll, **239**(1999), 180-187.

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Starlikeness and related properties of certain integral operator for multivalent functions

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Abstract. In this paper, the authors introduce a new general integral operator for multivalent functions. The new sufficient conditions for the operator $\mathcal{J}_{p,\gamma,g}^{\alpha_i,\,\beta_i}(f_1, f_2, ..., f_n)$ when $\gamma = 1$ is determined for the class of p-valently starlike, p-valently close-to-convex, uniformly p-valent close-to-convex and strongly starlike of order δ $(0 < \delta \leq 1)$ in $\mathbb{U}$. Our results generalize the results of Frasin [7].

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1. Introduction and definitions

Let $\mathcal{A}(p, n)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \qquad (p,\ n \in \mathbb{N} := \{1,\ 2,\ 3 \cdots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. We write $\mathcal{A}(p,\ 1) = \mathcal{A}_p$ and $\mathcal{A}_1 = \mathcal{A}$.

A function $f \in \mathcal{A}_p$ is said to be p-valently starlike of order δ $(0 \leq \delta < p)$, denoted by the class $\mathcal{S}_p^*(\delta)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta \quad (z \in \mathbb{U}). \tag{1.2}$$

Also, we say that a function $f \in \mathcal{A}_p$ is said to be p-valently convex of order δ $(0 \leq \delta < p)$ and belong to the class $\mathcal{K}_p(\delta)$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta \quad (z \in \mathbb{U}). \tag{1.3}$$

Further, a function $f \in \mathcal{A}_p$ is said to be p-valently close-to-convex of order δ $(0 \leq \delta < p)$, denoted by $\mathcal{C}_p(\delta)$ if and only if

$$\Re\left(\frac{f'(z)}{z^{p-1}}\right) > \delta \quad (z \in \mathbb{U}). \tag{1.4}$$

We note that

$$f(z) \in \mathcal{K}_p(\delta) \Longleftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}_p^*(\delta).$$

Furthermore, $\mathcal{S}_p^*(0) = \mathcal{S}_p^*,\ \mathcal{K}_p(0) = \mathcal{K}_p$ and $\mathcal{C}_p(0) = \mathcal{C}_p$ are respectively denote the class of p-valently starlike, p-valently convex and p-valently close-to-convex functions in $\mathbb{U}$. Also, $\mathcal{S}_1^* = \mathcal{S}^*,\ \mathcal{K}_1 = \mathcal{K}$ and $\mathcal{C}_1 = \mathcal{C}$ are respectively the usual classes of starlike, convex and close-to-convex functions in $\mathbb{U}$.

A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{UC}_p(\beta)$ of uniformly p-valent close-to-convex functions of order β $(0 \leq \beta < p)$ in $\mathbb{U}$ if and only if

$$\Re\left(\frac{zf'(z)}{g(z)} - \beta\right) \geq \left|\frac{zf'(z)}{g(z)} - p\right| \quad (z \in \mathbb{U}) \tag{1.5}$$

for some $g(z) \in \mathcal{US}_p(\beta)$ where $\mathcal{US}_p(\beta)$ is the class of uniformly p-valent starlike functions of order β $(-1 \leq \beta < p)$ in $\mathbb{U}$ that satisfies

$$\Re\left(\frac{zf'(z)}{f(z)} - \beta\right) \geq \left|\frac{zf'(z)}{f(z)} - p\right| \quad (z \in \mathbb{U}).$$

The uniformly p-valent starlike functions were first introduced in [9].

For functions f given by (1.1) and g belong to the class $\mathcal{A}_p$ given by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_{k+p} z^{k+p} \quad (z \in \mathbb{U}),$$

the Hadamard product (or convolution) of f and g denoted by $f * g$ is given by

$$(f * g)(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} b_{k+p} z^{k+p} \quad (z \in \mathbb{U}). \tag{1.6}$$

Note that for $g(z) = \frac{z^p}{1-z}$, $f * g = f$.

Analogous to the integral operator defined by Goswami and Bulut [10] on p-valent meromorphic functions, we now define the following general integral operator on the space of p-valent analytic functions in the class $\mathcal{A}_p$.

Definition 1.1. Let $n \in \mathbb{N}$, $\alpha_i,\ \beta_i \in \mathbb{R}_+ \cup \{0\}$ for all $i = 1, 2, 3, ..., n,\ \gamma \in \mathbb{C}$ with $\Re(\gamma) > 0$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n),\ \beta = (\beta_1, \beta_2, ..., \beta_n)$. For $f_i,\ g \in \mathcal{A}_p$ $(1 \leq i \leq n)$, we introduce a new general integral operator $\mathcal{J}_{p,\ \gamma,\ g}^{\alpha_i,\ \beta_i} : \mathcal{A}_p^n \longrightarrow \mathcal{A}_p$ by

$$\mathcal{J}_{p,\ \gamma,\ g}^{\alpha_i,\ \beta_i}(f_1, f_2, .., f_n)(z) = \left[\int_0^z \gamma p t^{\gamma p - 1} \prod_{i=1}^{n} \left(\frac{(f_i * g)(t)}{t^p}\right)^{\beta_i} \left(\frac{(f_i * g)'(t)}{p t^{p-1}}\right)^{\alpha_i} dt\right]^{\frac{1}{\gamma}}. \tag{1.7}$$

Here and throughout in the sequel every many-valued function is taken with the principal branch.

Note that, the integral operator $\mathcal{J}_{p,\ \gamma,\ g}^{\alpha_i,\ \beta_i}(f_1, f_2, , ..., f_n)(z)$ generalizes several previously studied operators as follows:

- For $p = 1,\ g(z) = \frac{z}{1-z}$, we obtain the integral operator

$$\mathcal{I}_{\gamma}^{\alpha_i,\beta_i}(f_1, f_2, ..., f_n)(z) = \left\{ \int_0^z \gamma t^{\gamma-1} \prod_{i=1}^{n} (f_i'(t))^{\alpha_i} \left(\frac{f_i(t)}{t} \right)^{\beta_i} dt \right\}^{\frac{1}{\gamma}} \tag{1.8}$$

introduced and studied by Frasin [8].

- For $p = 1,\ g(z) = \frac{z}{1-z}$ and $\alpha = (0, 0, ..., 0)$, we obtain the integral operator

$$\mathcal{I}_{\gamma}(f_1, f_2, ..., f_n)(z) = \left\{ \int_0^z \gamma t^{\gamma-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\beta_i} dt \right\}^{\frac{1}{\gamma}} \tag{1.9}$$

introduced and studied by Breaz and Breaz [3].

- For $p = 1,\ g(z) = \frac{z}{1-z},\ \beta = (0, 0, ..., 0)$ and $\gamma = 1$, we obtain the integral operator $\mathcal{F}_{\alpha_1,\alpha_2,...,\alpha_n}(z)$ where

$$\mathcal{F}_{\alpha_1,\alpha_2,...,\alpha_n}(z) = \int_0^z \prod_{i=1}^{n} (f_i'(t))^{\alpha_i}\, dt \tag{1.10}$$

introduced and studied by Breaz et al. [4].

- For $p = 1,\ n = 1,\ g(z) = \frac{z}{1-z},\ \alpha_1 = \alpha,\ \beta_1 = \beta,\ f_1 = f$ and $\gamma = 1$, we obtain the integral operator

$$\mathcal{F}_{\alpha,\ \beta}(z) = \int_0^z (f'(t))^{\alpha} \left(\frac{f(t)}{t} \right)^{\beta} dt \tag{1.11}$$

studied in [5].

- For $p = 1,\ n = 1,\ g(z) = \frac{z}{1-z},\ \alpha_1 = 0,\ \beta_1 = \beta,\ f_1 = f$ and $\gamma = 1$, we obtain the integral operator

$$\mathcal{F}_{\beta}(z) = \int_0^z \left(\frac{f(t)}{t} \right)^{\beta} dt \tag{1.12}$$

studied in [11]. In particular, for $\beta = 1$, the above operator reduces to

$$\mathcal{I}(z) = \int_0^z \frac{f(t)}{t} dt \tag{1.13}$$

known as Alexander integral operator (see [1]).

- For $p = 1,\ n = 1,\ g(z) = \frac{z}{1-z},\ \beta_1 = 0,\ \alpha_1 = \alpha, f_1 = f$ and $\gamma = 1$, we obtain the integral operator

$$\mathcal{G}_{\alpha}(z) = \int_0^z (f'(t))^{\alpha}\, dt \tag{1.14}$$

studied in [15] (also see [16]).

- For $g(z) = \frac{z^p}{1-z},\ \alpha = (0, 0, ..., 0)$ and $\gamma = 1$, we obtain the integral operator

$$\mathcal{F}_p(z) = \int_0^z p t^{p-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t^p} \right)^{\beta_i} dt \tag{1.15}$$

introduced and studied by Frasin [7].

- For $g(z) = \frac{z^p}{1-z}$, $\beta = (0,0,...,0)$ for $i = 1,2,3,..,n$ and $\gamma = 1$, we obtain the integral operator

$$\mathcal{G}_p(z) = \int_0^z pt^{p-1} \prod_{i=1}^{n} \left(\frac{f_i'(t)}{pt^{p-1}} \right)^{\alpha_i} dt \tag{1.16}$$

introduced and studied by Frasin [7].

Various sufficient conditions for convexity and starlikeness of multivalent functions corresponding to different integral operators have been obtained by various authors. Motivated by the aforementioned work, in this paper the authors derive various sufficient conditions for the operator defined in (1.7) when $\gamma = 1$ to be p-valently starlike, p-valently close-to-convex, uniformly close-to-convex and strongly starlike of order δ $(0 < \delta \leq 1)$ in $\mathbb{U}$.

2. Preliminaries

In order to derive our main results, we need the following lemmas.

Lemma 2.1. *(see [12]) If $f \in \mathcal{A}_p$ satisfies*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < p + \frac{1}{4} \quad (z \in \mathbb{U}), \tag{2.1}$$

then f is p-valently starlike in $\mathbb{U}$.

Lemma 2.2. *(see [2]) If $f \in \mathcal{A}_p$ satisfies*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < p + \frac{1}{3} \quad (z \in \mathbb{U}), \tag{2.2}$$

then f is uniformly p-valent close-to-convex in $\mathbb{U}$.

Lemma 2.3. *(see [17]) If $f \in \mathcal{A}_p$ satisfies*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < p + \frac{a+b}{(1+a)(1-b)} \quad (z \in \mathbb{U}), \tag{2.3}$$

where $a > 0$, $b \geq 0$ and $a + 2b \leq 1$, then f is p-valently close-to-convex in $\mathbb{U}$.

Lemma 2.4. *(see [14]) If $f \in \mathcal{A}_p$ satisfies*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{p}{4} - 1 \quad (z \in \mathbb{U}), \tag{2.4}$$

then

$$\Re\sqrt{\frac{zf'(z)}{f(z)}} > \frac{\sqrt{p}}{2} \quad (z \in \mathbb{U}). \tag{2.5}$$

Lemma 2.5. *(see [13]) If $f \in \mathcal{A}_p$ satisfies*

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > p-\frac{\delta}{2} \quad (z \in \mathbb{U}), \tag{2.6}$$

then

$$\left|arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\delta \quad (0<\delta\le 1; z\in\mathbb{U}), \tag{2.7}$$

or f is strongly starlike of order δ in $\mathbb{U}$.

Lemma 2.6. *(see [6]) If $f \in \mathcal{A}_p$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}+1-p\right| < p+1 \quad (z \in \mathbb{U}), \tag{2.8}$$

then f is p-valently starlike in $\mathbb{U}$.

3. Main results

In this section, we investigate sufficient conditions for the integral operator $\mathcal{J}_{p,1,g}^{\alpha_i,\ \beta_i}(f_1, f_2, ..., f_n)(z)$ to be in the class $\mathcal{S}_p^*$. For the sake of simplicity, we shall write $\mathcal{J}_{p,g}(z)$ instead of $\mathcal{J}_{p,1,g}^{\alpha_i,\ \beta_i}(f_1, f_2, ..., f_n)(z)$.

Theorem 3.1. *Let $\alpha_i,\ \beta_i \in \mathbb{R}_+ \cup \{0\}$ for all $i = 1, 2, 3, ..., n$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)$. If $f_i,\ g \in \mathcal{A}_p$ $(1 \le i \le n)$ satisfies*

$$\Re\left\{\alpha_i\frac{z\,(f_i*g)''(z)}{(f_i*g)'(z)}+\beta_i\frac{z\,(f_i*g)'(z)}{(f_i*g)(z)}\right\} < (p-1)\alpha_i+p\beta_i+\frac{1}{4n} \quad (z\in\mathbb{U}), \tag{3.1}$$

then the general integral operator $\mathcal{J}_{p,g}(z) \in \mathcal{S}_p^*$.

Proof. From (1.7), it is easy to see that

$$\mathcal{J}_{p,g}'(z) = pz^{p-1}\left(\frac{(f_1*g)'(z)}{pz^{p-1}}\right)^{\alpha_1}\left(\frac{(f_1*g)(z)}{z^p}\right)^{\beta_1}.... \\ \left(\frac{(f_n*g)'(z)}{pz^{p-1}}\right)^{\alpha_n}\left(\frac{(f_n*g)(z)}{z^p}\right)^{\beta_n}. \tag{3.2}$$

Differentiating (3.2) logarithmically with respect to z and multiply by z, we obtain

$$\frac{z\mathcal{J}_{p,g}''(z)}{\mathcal{J}_{p,g}'(z)} = (p-1)+\sum_{i=1}^{n}\alpha_i\left(\frac{z\,(f_i*g)''(z)}{(f_i*g)'(z)}-p+1\right)+\sum_{i=1}^{n}\beta_i\left(\frac{z\,(f_i*g)'(z)}{(f_i*g)(z)}-p\right),$$

which implies

$$1+\frac{z\mathcal{J}_{p,g}''(z)}{\mathcal{J}_{p,g}'(z)} = \sum_{i=1}^{n}\left(\alpha_i\frac{z\,(f_i*g)''(z)}{(f_i*g)'(z)}+\beta_i\frac{z\,(f_i*g)'(z)}{(f_i*g)(z)}\right)+p-\sum_{i=1}^{n}\left[(p-1)\alpha_i+p\beta_i\right]. \tag{3.3}$$

Taking real part on both sides of (3.3), we get

$$\Re\left(1+\frac{z\mathcal{J}''_{p,g}(z)}{\mathcal{J}'_{p,g}(z)}\right)=p-\sum_{i=1}^{n}[(p-1)\alpha_i+p\beta_i]+$$
$$\sum_{i=1}^{n}\Re\left(\alpha_i\frac{z(f_i*g)''(z)}{(f_i*g)'(z)}+\beta_i\frac{z(f_i*g)'(z)}{(f_i*g)(z)}\right). \tag{3.4}$$

Using (3.1) in (3.4) yields

$$\Re\left(1+\frac{z\mathcal{J}''_{p,g}(z)}{\mathcal{J}'_{p,g}(z)}\right)<p+\frac{1}{4}\quad(z\in\mathbb{U}). \tag{3.5}$$

Hence by Lemma 2.1, $\mathcal{J}_{p,g}(z)$ is p-valently starlike in $\mathbb{U}$ which implies $\mathcal{J}_{p,g}(z)\in\mathcal{S}_p^*$. This complete the proof of Theorem 3.1. □

Taking $\alpha=(0,0,...,0)$ and $g(z)=\frac{z^p}{1-z}$ $(z\in\mathbb{U})$ in Theorem 3.1, we get the following result.

Corollary 3.2. *([7], Theorem 2.1) Let $\beta_i>0$ be real numbers for all $i=1,2,3,...,n$. If $f_i\in\mathcal{A}_p$ for all $i=1,2,3,..,n$ satisfies*

$$\Re\left(\frac{zf_i'(z)}{f_i(z)}\right)<p+\frac{1}{4\sum_{i=1}^{n}\beta_i}\quad(z\in\mathbb{U})$$

then $\mathcal{F}_p$ is p-valently starlike in $\mathbb{U}$.

Remark 3.3. If we set $n=p=1,\ \beta_1=\beta,\ f_1=f$ in Corollary 3.2, then we have ([7], Corollary 2.2).

Further, taking $\beta=(0,0,...,0)$ $g(z)=\frac{z^p}{1-z}$ in Theorem 3.1 we get the following result.

Corollary 3.4. *([7], Theorem 3.1) Let $\alpha_i>0$ be real numbers for all $i=1,2,3,...,n$. If $f_i\in\mathcal{A}_p$ for all $i=1,2,3,...,n$ satisfies*

$$\Re\left(1+\frac{zf_i''(z)}{f_i'(z)}\right)<p+\frac{1}{4\sum_{i=1}^{n}\alpha_i}\quad(z\in\mathbb{U}),$$

then $\mathcal{G}_p$ is p-valently starlike in $\mathbb{U}$.

Remark 3.5. Letting $n=p=1,\ \alpha_1=\alpha$ and $f_1=f$ in Corollary 3.4 we get the following result due to Frasin ([7], Corollary 3.2).

Furthermore, taking $p=1,\ n=1,\ \alpha_1=\alpha,\ \beta_1=\beta,\ f_1=f$ and $g(z)=\frac{z}{1-z}$ in Theorem 3.1, we have the following:

Corollary 3.6. *Let $f\in\mathcal{A}$ and $\alpha,\ \beta>0$. If*

$$\Re\left\{\alpha\frac{zf''(z)}{f'(z)}+\beta\frac{zf'(z)}{f(z)}\right\}<\beta+\frac{1}{4}\quad(z\in\mathbb{U})$$

then the integral operator $\mathcal{F}_{\alpha,\beta}(z)$ defined in (1.11) belong to starlike function class $\mathcal{S}_p^*$.

The next theorem gives another sufficient condition for the integral operator $\mathcal{J}_{p,g}$ to be p-valently starlike functions in $\mathcal{U}$.

Theorem 3.7. *Let* $\alpha_i,\ \beta_i \in \mathbb{R}_+ \cup \{0\}$ *for all* $i = 1, 2, ..., n,\ \alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ *and* $\beta = (\beta_1, \beta_2, ..., \beta_n)$. *If* $f_i,\ g \in \mathcal{A}_p$ *for all* $i = 1, 2, ..., n$ *satisfies the relation*

$$\left|\alpha_i \frac{z(f_i * g)''(z)}{(f_i * g)'(z)} + \beta_i \left(\frac{z(f_i * g)'(z)}{(f_i * g)(z)} - p\right)\right| < \frac{p+1}{n} - (p-1)\alpha_i \quad (z \in \mathbb{U}) \tag{3.6}$$

where $\sum_{i=1}^{n} \alpha_i > 1$, *then* $\mathcal{J}_{p,g}$ *is p-valently starlike in* $\mathcal{U}$.

Proof. From (3.3) and applications of triangle's inequalities give

$$|1 + \frac{z\mathcal{J}''_{p,g}(z)}{\mathcal{J}'_{p,g}(z)} - p| = \left|\sum_{i=1}^{n}\left(\alpha_i \frac{z(f_i * g)''(z)}{(f_i * g)'(z)} + \beta_i \frac{z(f_i * g)'(z)}{(f_i * g)(z)}\right) - \right.$$

$$\left.\sum_{i=1}^{n}((p-1)\alpha_i + p\beta_i)\right| = \left|\sum_{i=1}^{n}\left(\alpha_i \frac{z(f_i * g)''(z)}{(f_i * g)'(z)} + \beta_i \left(\frac{z(f_i * g)'(z)}{(f_i * g)(z)} - p\right)\right) - \right.$$

$$\left.(p-1)\sum_{i=1}^{n}\alpha_i\right| < \sum_{i=1}^{n}\left|\alpha_i \frac{z(f_i * g)''(z)}{(f_i * g)'(z)} + \beta_i \left(\frac{z(f_i * g)'(z)}{(f_i * g)(z)} - p\right)\right| + (p-1)\sum_{i=1}^{n}\alpha_i \tag{3.7}$$

Making use of (3.6) in (3.7)we get

$$\left|1 + \frac{z\mathcal{J}''_{p,g}(z)}{\mathcal{J}'_{p,g}(z)} - p\right| < p + 1. \tag{3.8}$$

Therefore, the result follows by application of Lemma 2.6. The proof of Theorem 3.7 is completed. □

Remark 3.8. Putting $g(z) = \frac{z^p}{1-z},\ \alpha = (0, 0, ..., 0)$ in Theorem 3.7 we get the result of Frasin ([7], Theorem 2.3).

Remark 3.9. Putting $g(z) = \frac{z^p}{1-z},\ \beta = (0, 0, ..., 0)$ in Theorem 3.7 we get the result of Frasin ([7], Theorem 3.3).

Remark 3.10. Putting $n = 1,\ p = 1,\ \alpha_1 = 0,\ f_1 = f,\ \beta_1 = \beta > 0,\ g(z) = \frac{z}{1-z}$ in Theorem 3.7 we get the result of Frasin ([7],Corollary 2.4).

Remark 3.11. Letting $n = 1,\ p = 1,\ \alpha_1 = \alpha > 0,\ f_1 = f,\ \beta = (0, 0, ..., 0),\ g(z) = \frac{z}{1-z}$ in Theorem 3.7 we get the result of Frasin ([7],Corollary 3.4).

4. Close-to-convex function

The following theorem gives sufficient conditions for the integral operator $\mathcal{J}_{p,g}$ to be p-valently close-to-convex in $\mathbb{U}$.

Theorem 4.1. *Let* $f_i,\ g \in \mathcal{A}_p,\ \alpha_i,\ \beta_i \in \mathbb{R}_+ \cup \{0\}$ *for all* $i = 1, 2, 3, ..., n,\ \alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ *and* $\beta = (\beta_1, \beta_2, ..., \beta n)$. *For* $z \in \mathbb{U}$, *if*

$$\Re\left\{\alpha_i \frac{z\,(f_i * g)''\,(z)}{(f_i * g)'(z)} + \beta_i \frac{z\,(f_i * g)'\,(z)}{(f_i * g)(z)}\right\} < (p-1)\alpha_i + p\beta_i + \frac{a+b}{n(1+a)(1-b)}, \tag{4.1}$$

where $a > 0,\ b \geq 0$ *and* $a + 2b \leq 1$, *then* $\mathcal{J}_{p,g}(z)$ *is p-valently close-to-convex function in* $\mathbb{U}$.

Proof. Making use of (4.1) in (3.3), we get

$$\Re\left(1+\frac{z\mathcal{J}''_{p,g}(z)}{\mathcal{J}'_{p,g}(z)}\right) < p+\frac{a+b}{(1+a)(1-b)}.$$

Hence by Lemma 2.3 we conclude that $\mathcal{J}_{p,g} \in \mathcal{C}_p(\delta)$. □

Remark 4.2. Putting $g(z)=\frac{z^p}{1-z}$, $\alpha=(0,0,0,...,0)$ and $\beta_i>0$ in Theorem 4.1, we get the result due to Frasin ([7], Theorem 2.5).

Remark 4.3. Putting $g(z)=\frac{z^p}{1-z}$, $\beta=(0,0,0,...,0)$ and $\alpha_i>0$ in Theorem 4.1, we get the result due to Frasin ([7], Theorem 3.5).

Remark 4.4. Letting $n=p=1, \alpha_1=0,\ \beta_1=\beta>0, f_1=f$ and $g(z)=\frac{z}{1-z}$ in Theorem 4.1 we get the result due to Frasin ([7], Corollary 2.6).

Remark 4.5. Letting $n=p=1, \beta_1=0, \alpha_1=\alpha, f_1=f$ and $g(z)=\frac{z}{1-z}$ in Theorem 4.1 we get the result due to Frasin ([7], Corollary 3.6).

5. Uniformly close-to-convex function

In this section we give sufficient conditions for the generalize integral operator $\mathcal{J}_{p,g}(z)$ to be uniformly close-to-convex in $\mathbb{U}$.

Theorem 5.1. *Let $f_i,\ g \in \mathcal{A}_p, \alpha_i,\ \beta_i \in \mathbb{R}_+ \cup \{0\}$ for all $i=1,2,3,...,n$ and $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$ and $\beta=(\beta_1,\beta_2,...,\beta_n)$. If*

$$\Re\left(\alpha_i\frac{z\left(f_i*g\right)''(z)}{(f_i*g)'(z)}+\beta_i\frac{z\left(f_i*g\right)'(z)}{(f_i*g)(z)}\right) < (p-1)\alpha_i+p\beta_i+\frac{1}{3n} \quad (z\in\mathbb{U}), \tag{5.1}$$

then $\mathcal{J}_{p,g}(z)$ is uniformly p-valent close-to-convex in $\mathbb{U}$.

Proof. Making using (5.1) in (3.3) and an application of Lemma 2.2 give the result. The proof of Theorem 5.1 is thus completed. □

Remark 5.2. Putting $g(z)=\frac{z^p}{1-z}$, $\beta_i>0$ and $\alpha=(0,0,...,0)$ in Theorem 5.1 we get the frasin ([7], Theorem 2.7).

Remark 5.3. Letting $g(z)=\frac{z^p}{1-z}$, $\beta=(0,0,..,0)$ and $\alpha_i>0$ for $i=1,2,3,..,n$ in Theorem 5.1 we get the result of Frasin ([7], Theorem 3.7).

Remark 5.4. Taking $n=p=1, \beta_1=\beta$ and $f_1=f$ in Remark 5.2 we have ([7], Corollary 2.8).

Remark 5.5. Taking $n=p=1, \alpha_1=\alpha$ and $f_1=f$ in Remark 5.3 we have ([7], Corollary 3.8).

6. Strong starlikeness of the operators $\mathcal{J}_{p,g}$

The following theorem gives sufficient conditions for the operator $\mathcal{J}_{p,g}$ to be strongly starlike of order δ in $\mathbb{U}$.

Theorem 6.1. *Let $\alpha_i,\ \beta_i \in \mathbb{R}_+ \cup \{0\}$ for all $i = 1, 2, 3, ..., n$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)$. If $f_i, g \in \mathcal{A}_p$ for all $i = 1, 2, 3, ..., n$ satisfies*

$$\Re\left\{\alpha_i \frac{z(f_i * g)''(z)}{(f_i * g)'(z)} + \beta_i \frac{z(f_i * g)'(z)}{(f_i * g)(z)}\right\} > (p-1)\alpha_i + p\beta_i - \frac{\delta}{2n} \quad (z \in \mathbb{U}), \tag{6.1}$$

then $\mathcal{J}_{p,g}$ is strongly starlike of order δ $(0 < \delta \leq 1)$ in $\mathbb{U}$.

Proof. In view of (3.3) and (6.1) and by using Lemma 2.5, we deduce that $\mathcal{J}_{p,g}$ is strongly starlike of order δ. □

Putting $g(z) = \frac{z^p}{1-z}$, $\alpha = (0, 0, ..., 0)$ in Theorem 6.1, we get the following result:

Corollary 6.2. *([7], Theorem 4.1) Let $\beta_i > 0$ be real numbers for all $i = 1, 2, 3, .., n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, 3, .., n$ satisfies*

$$\Re\left(\frac{zf_i'(z)}{f_i(z)}\right) > p - \frac{\delta}{2\sum_{i=1}^{n}\beta_i} \quad (z \in \mathbb{U}), \tag{6.2}$$

then $\mathcal{F}_p$ is strongly starlike of order δ $(0 < \delta \leq 1)$ in $\mathbb{U}$.

Remark 6.3. Putting $n = p = 1$, $\beta_1 = \beta$ and $f_1 = f$ in the Corollary 6.2 , we get the result of Frasin ([7],Corollary 4.2).

Further, letting $g(z) = \frac{z^p}{1-z}$, $\beta = (0, 0, 0, ..., 0)$ in Theorem 6.1 we get the following result.

Corollary 6.4. *([7], Theorem 4.3) Let $\alpha_i > 0$ be the real numbers for all $i = 1, 2, 3, ..., n$. If $f_i \in \mathcal{A}_p$ for $i = 1, 2, 3, .., n$ satisfies*

$$\Re\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right) > p - \frac{\delta}{2\sum_{i=1}^{n}\alpha_i} \quad (z \in \mathbb{U}), \tag{6.3}$$

then $\mathcal{G}_p$ is strongly starlike of order δ $0 < \delta \leq 1$ in $\mathbb{U}$.

Remark 6.5. Letting $n = p = 1$, $\alpha_1 = \alpha > 0$ and $f_1 = f$ in the above Corollary, we get the result (see [7], Corollary 4.4).

Theorem 6.6. *Let $\alpha_i,\ \beta_i \in \mathbb{R}_+ \cup \{0\}$ for all $i = 1, 2, 3, ..., n$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)$. If $f_i,\ g \in \mathcal{A}_p$ for all $i = 1, 2, 3, ..., n$ satisfies*

$$\Re\left\{\alpha_i \frac{z(f_i * g)''(z)}{(f_i * g)'(z)} + \beta_i \frac{z(f_i * g)'(z)}{(f_i * g)(z)}\right\} > (p-1)\alpha_i + p\beta_i - \frac{3p+4}{4n} \quad (z \in \mathbb{U}), \tag{6.4}$$

then

$$\Re\sqrt{\frac{z\mathcal{J}_{p,g}'(z)}{\mathcal{J}_{p,g}(z)}} > \frac{\sqrt{p}}{2}. \tag{6.5}$$

Proof. Using (6.4) in (3.4), we have

$$\Re\left(1+\frac{z\mathcal{J}''_{p,g}(z)}{\mathcal{J}_{p,g}(z)}\right) > \frac{p}{4}-1. \tag{6.6}$$

The result follows in view of Lemma 2.4. □

Remark 6.7. Putting $g(z)=\frac{z^p}{1-z}$, $\alpha=(0,0,...,0)$ in the Theorem 6.6 , we get the result of Frasin ([7],Theorem 2.9)

Remark 6.8. Putting $n=p=1$, $\beta_1=1$ and $f_1=f$, $\alpha=(0,0,...,0)$ and $g(z)=\frac{z}{1-z}$ in the Theorem 6.6 , we get the result of Frasin ([7],Corollary 2.10).

Remark 6.9. Taking $\beta=(0,0,...,0)$ and $g(z)=\frac{z^p}{1-z}$ in the Theorem 6.6 , we get the result of Frasin ([7],Theorem 3.9).

Remark 6.10. Letting $n=p=1$, $\alpha_1=1$ and $f_1=f$, $\beta=(0,0,...,0)$ and $g(z)=\frac{z}{1-z}$ in the Theorem 6.6 , we get the result of Frasin ([7],Corollary 3.10).

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References

[1] Alexander, W., *Functions which map the interior of the unit circle upon simple regions*, Ann. Math., **17**(1915), no. 1, 12-22.

[2] Al-Kharsani, H.A., Al-Hajiry, S.S., *A note on certain inequalities for p-valent functions*, J. Inequal. Pure Appl. Math., **9**(2008), no. 3.

[3] Breaz, D., Breaz, N., *Two integral operators*, Stud. Univ. Babeş-Bolyai Math., Cluj-Napoca, **3**(2002), 13-19.

[4] Breaz, D., Owa, S., Breaz, N., *A new integral univalent operator*, Acta Univ. Apul., **16**(2008), 11-16.

[5] Dorf, M., Szynal, J., *Linear invariance and integral operators of univalent functions*, Demonstratio Math., **38**(2005), no. 1, 47-57.

[6] Dziok, J., *Applications of the Jack lemma*, Acta Math. Hungar., **105**(2004), no. 1-2, 93-102.

[7] Frasin, B.A., *New general integral operators of p-valent functions*, J. Inequal. Pure Appl. Math., **10**(2009), no. 4.

[8] Frasin, B.A., *Order of convexity and univalency of general integral operator*, J. Franklin Institute, **348**(2011), 1013-1019.

[9] Goodman, A.W., *On uniformly starlike functions*, J. Math. Anal. Appl., **55**(1991), 364-370.

[10] Goswami, P., Bulut, S., *Starlikeness of general integral operator for meromorphic multivalent functions*, J. Complex Anal., Vol. 2013, Art. ID 690584.

[11] Miller, S.S., Mocanu, P.T., Reade, M.O., *Starlike integral operators*, Pacific J. Math., **79**(1978), no. 1, 157-168.

[12] Nunokawa, M., *On the multivalent functions*, Indian J. Pure Appl. Math., **20**(1989), 577-582.

[13] Nunokawa, M., Owa, S., Ikeda, A., *On the strongly starlikeness of multivalently convex functions of order α*, IJMMS, **28**(2001), no. 1, 51-55.

[14] Owa, S., *On the Nunokawa's conjecture for multivalent functions*, Bull. Austral. Math. Soc., **41**(1990), 301-305.

[15] Pascu, N., Pescar, V., *On the integral operators of Kim-Merkis and Pfaltzgraff*, Mathematica, **32**(1990), no. 55, 185-192.

[16] Pfaltzgraff, J.A., *Univalence of the integral of $f'(z)^{\lambda}$*, Bull. London Math. Soc., **7**(1975), no. 3, 254-256.

[17] Raina, R.K., Bapna, I.B., *Inequalities defining certain subclasses of analytic functions involving fractional calculus operators*, J. Inequal. Pure Appl. Math., **5**(2004), no. 2.

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On sandwich theorems for some subclasses defined by generalized hypergeometric functions

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Abstract. In this paper we derive some subordination and superordination results for certain normalized analytic functions in the open unit disc, which are acted upon by a class of generalized hypergeometric function $H_{q,s}(\alpha_1)$.

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1. Introduction

Let H be the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, n]$ denotes the subclass of the functions $f \in H$ of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \quad (a \in \mathbb{C}), \tag{1.1}$$

and we let

$$A_m = \left\{ f \in H, f(z) = z + a_{m+1} z^{m+1} + a_{m+2} z^{m+2} + \dots \right\}.$$

Also, let $A_1 = A$ be the subclass of the functions $f \in H$ of the form

$$f(z) = z + a_2 z^2 + \dots \tag{1.2}$$

For $f, g \in H$, we say that the function f is subordinate to g, written symbolically as follows:

$$f \prec g \quad or \quad f(z) \prec g(z),$$

if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $(z \in U)$, such that $f(z) = g(w(z))$ for all $z \in U$. In particular, if the function $g(z)$ is univalent in U, then we have the following equivalence (cf., e.g., [15]; see also [16, p.4]):

$$f(z) \prec g\ (z) \Leftrightarrow f(0) \prec g(0) \quad and \quad f(U) \subset g(U).$$

Supposing that p and h are two analytic functions in U, let

$$\varphi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}.$$

If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if p satisfies the second-order superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then p is called to be a solution of the differential superordination (1.3). (If f is subordinate to F, then F is superordination to f). An analytic function q is called a subordinant of (1.3), if $q(z) \prec p(z)$ for all the functions p satisfying (1.3). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all of the subordinants q of (1.3), is called the best subordinant (cf., e.g.,[15], see also [16]).

Recently, Miller and Mocanu [17] obtained sufficient conditions on the functions h, q and φ for which the following implication holds:

$$k(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z). \tag{1.4}$$

Using the results Miller and Mocanu [17], Bulboaca [5] considered certain classes of first-order differential superordinations as well as superordination preserving integral operators [4]. Ali et al. [1], have used the results of Bulboaca [5] and obtained sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z), \tag{1.5}$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = 1$. Shanmugam et al. [23] obtained sufficient conditions for normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z),$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{\{f(z)\}^2} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = 1$ and $q_2(0) = 1$, while Obradovic and Owa [20] obtained subordination results with the quantity $\left(\frac{f(z)}{z}\right)^\mu$. A detailed investigation of starlike functions of complex order and convex functions of complex order using Briot–Bouquet differential subordination technique has been studied very recently by Srivastava and Lashin [26] (see also [27], [2] and [19]).

For complex parameters

$$\alpha_1, \ldots, \alpha_q \text{ and } \beta_1, \ldots, \beta_s \quad (\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}; j = 1, 2, \ldots, s),$$

we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by (see, for example, [25, p.19])

$$ {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k}{(\beta_1)_k \ldots (\beta_s)_k} \cdot \frac{z^k}{k!}$$

$$(q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\}; z \in U),$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_\nu = \frac{\Gamma(\theta+\nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}), \\ \theta(\theta-1)\ldots(\theta+\nu-1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases}$$

Corresponding to the function $h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$, defined by

$$h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z\, {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),$$

Dziok and Srivastava [9] (see also [10]) considered a linear operator

$$H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) : A \to A,$$

which is defined by the following Hadamard product (or convolution):

$$H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)f(z) = h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z). \tag{1.6}$$

We observe that, for a function $f(z) \in A_m$, we have

$$H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)f(z) = z + \sum_{k=m+1}^{\infty} \frac{(\alpha_1)_{k-1} \ldots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \ldots (\beta_s)_{k-1}(1)_{k-1}} a_k z^k.$$

For $m = 1$, we have (see [9])

$$H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)f(z) = z + \sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-1} \ldots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \ldots (\beta_s)_{k-1}(1)_{k-1}} a_k z^k. \tag{1.7}$$

For convenience, we write

$$H_{q,s}(\alpha_1) = H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s).$$

It is easily follows from (1.7) that (see [9])

$$z\left(H_{q,s}(\alpha_1)f(z)\right)' = \alpha_1 H_{q,s}(\alpha_1+1)f(z) - (\alpha_1 - 1)H_{q,s}(\alpha_1)f(z). \tag{1.8}$$

It should be remarked that the linear operator $H_{q,s}(\alpha_1)$ is a generalization of many other linear operators considered earlier. In particular for $f \in A$ we have the following observation:

(i) $H_{2,1}(a, b; c)f(z) = I_c^{a,b} f(z) \left(a, b \in \mathbb{C}; c \notin \mathbb{Z}_0^-\right)$, where the linear operator $I_c^{a,b}$ was investigated by Hohlov [12];

(ii) $H_{2,1}(\delta+1, 1; 1)f(z) = D^\delta f(z) (\delta > -1)$, where D^δ is the Ruscheweyh derivative of $f(z)$ (see [22]);

(iii) $H_{2,1}(\mu+1, 1; \mu+2)f(z) = \mathcal{F}_\mu(f)(z) = \frac{\mu+1}{z^\mu} \int_0^z t^{\mu-1} f(t)dt \ (\mu > -1)$, where $\mathcal{F}_\mu$ is the Libera integral operator (see [13], [14] and [3]);

(iv) $H_{2,1}(a, 1; c)f(z) = L(a, c)f(z) (a \in \mathbb{R}; c \in \mathbb{R}\backslash\mathbb{Z}_0^-)$, where $L(a, c)$ is the Carlson-Shaffer operator (see [6]);

(v) $H_{2,1}(\lambda+1, c; a)f(z) = I^\lambda(a, c)f(z) (a, c \in \mathbb{R}\backslash\mathbb{Z}_0^-; \lambda > -1)$, where $I^\lambda(a, c)f(z)$ is the Cho–Kwon–Srivastava operator (see [7]);

(vi) $H_{2,1}(\mu, 1; \lambda+1)f(z) = I_{\lambda,\mu} f(z) (\lambda > -1; \mu > 0)$, where $I_{\lambda,\mu} f(z)$ is the Choi–Saigo–Srivastava operator [8] which is closely related to the Carlson–Shaffer [6] operator $L(\mu, \lambda+1)f(z)$;

(vii) $H_{2,1}(1, 1; n+1)f(z) = I_n f(z) (n \in \mathbb{N}_0)$, where $I_n f(z)$ is the Noor operator of $n - th$ order (see [18]) ;

(viii) $H_{2,1}(2,1;2-\mu)f(z)=\Omega_z^{\mu}f(z)$ $(\mu\neq 2,3,4,\ldots)$, where Ω_z^{μ} is the fractional derivative operator (see Owa and Srivastava [21]).

2. Preliminaries

In order to prove our subordination and superordination results, we make use of the following known definition and lemmas.

Definition 2.1. [17] Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U}\setminus E(f)$, where

$$E(f)=\{\zeta:\zeta\in\partial U \ \ and \ \ \lim_{z\to\zeta}f(z)=\infty\}, \tag{2.1}$$

and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial U\setminus E(f)$.

Lemma 2.2. [16] *Let the function $q(z)$ be univalent in the unit disc U, and let θ and φ be analytic in a domain D containing $q(U)$, with $\varphi(w)\neq 0$ when $w\in q(U)$. Set $Q(z)=zq'(z)\varphi(q(z))$, $h(z)=\theta(q(z))+Q(z)$ and suppose that*

(i) Q is a starlike function in U,

(ii) $\operatorname{Re}\left(\dfrac{zh'(z)}{Q(z)}\right)>0$ *for* $z\in U$.

If p is analytic in U with $p(0)=q(0)$, $p(U)\subseteq D$ and

$$\theta(p(z))+zp'(z)\varphi(p(z))\prec\theta(q(z))+zq'(z)\varphi(q(z)), \tag{2.2}$$

then $p(z)\prec q(z)$, and q is the best dominant.

Lemma 2.3. [16] *Let g be a convex function in U and let*

$$h(z)=g(z)+m\alpha zg'(z),$$

where $\alpha>0$ and m is a positive integer. If

$$p(z)=g(0)+p_m z^m+\ldots.$$

is analytic in U and

$$p(z)+\alpha zp'(z)\prec h(z),$$

then

$$p(z)\prec g(z),$$

and this result is sharp.

Lemma 2.4. [11] *Let h be a convex function with $h(0)=a$ and let $\gamma\in\mathbb{C}$ with* $\operatorname{Re}(\gamma)\geq 0$. *If $p\in H$ with $p(0)=a$ and*

$$p(z)+\frac{1}{\gamma}zp'(z)\prec h(z),$$

then

$$p(z)\prec q(z)\prec h(z),$$

where

$$q(z)=\frac{\gamma}{nz^{(\gamma/n)}}\int_0^z h(t)t^{(\gamma/n)-1}dt \qquad (z\in U).$$

The function q is convex and is the best dominant.

Lemma 2.5. [4] *Let $q(z)$ be a convex univalent function in the unit disc U and let ϑ and φ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\operatorname{Re}\left\{\dfrac{\vartheta'(q(z))}{\varphi(q(z))}\right\} > 0$ *for* $z \in U$;

(ii) $zq'(z)\varphi(q(z))$ *is starlike in* U.

If $p \in H[q(0),1] \cap Q$ with $p(U) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U, and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)),$$

then $q(z) \prec p(z)$, and q is the best subordinant.

3. Subordination results for analytic functions

Unless otherwise mentioned we shall assume throughout the paper that $q \leq s+1; q, s \in \mathbb{N}_0, \mu, \beta \in \mathbb{C}^*, \eta, \alpha, \delta, \xi \in \mathbb{C}, z \in U$ and the powers understood as principle values.

Theorem 3.1. *Let the function q be analytic and univalent in U, with $q(z) \neq 0$ $(z \in U^* = U\backslash\{0\})$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let*

$$\operatorname{Re}\left\{1 + \frac{\xi}{\beta}q(z) + \frac{2\delta}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0 \quad (z \in U). \tag{3.1}$$

If $f \in A_m$ and q satisfies the following subordination:

$$\Psi(f, \alpha_1, \alpha, \delta, \xi, \beta, \mu, \eta) \prec \alpha + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)}, \tag{3.2}$$

where

$$\begin{aligned}\Psi(f, \alpha_1, \alpha, \delta, \xi, \beta, \mu) &= \alpha + \xi\left(\frac{H_{q,s}(\alpha_1)f(z)}{z}\right)^{\mu}\left(\frac{z}{H_{q,s}(\alpha_1+1)f(z)}\right)^{\eta} \\ +\delta\left(\frac{H_{q,s}(\alpha_1)f(z)}{z}\right)^{2\mu}&\left(\frac{z}{H_{q,s}(\alpha_1+1)f(z)}\right)^{2\eta} + \beta\mu\alpha_1\left[\frac{H_{q,s}(\alpha_1+1)f(z)}{H_{q,s}(\alpha_1)f(z)} - 1\right] \\ &+\beta\eta(\alpha_1+1)\left[1 - \frac{H_{q,s}(\alpha_1+2)f(z)}{H_{q,s}(\alpha_1+1)f(z)}\right],\end{aligned} \tag{3.3}$$

then

$$\left(\frac{H_{q,s}(\alpha_1)f(z)}{z}\right)^{\mu}\left(\frac{z}{H_{q,s}(\alpha_1+1)f(z)}\right)^{\eta} \prec q(z)$$

and q is the best dominant of (3.2).

Proof. Define the function p by

$$p(z) = \left(\frac{H_{q,s}(\alpha_1)f(z)}{z}\right)^{\mu}\left(\frac{z}{H_{q,s}(\alpha_1+1)f(z)}\right)^{\eta} \quad (z \in U). \tag{3.4}$$

Then the function $p(z)$ is analytic in U and $p(0) = 1$. Differentiating (3.4) logarithmically with respect to z, we have

$$\frac{zp'(z)}{p(z)} = \mu\left[\frac{z\left(H_{q,s}(\alpha_1)f(z)\right)'}{H_{q,s}(\alpha_1)f(z)} - 1\right] + \eta\left[1 - \frac{z\left(H_{q,s}(\alpha_1+1)f(z)\right)'}{H_{q,s}(\alpha_1+1)f(z)}\right].$$

By using the identity (1.8) in the resulting equation, we have

$$\frac{zp'(z)}{p(z)} = \mu\left[\frac{\alpha_1 H_{q,s}(\alpha_1+1)f(z)}{H_{q,s}(\alpha_1)f(z)} - \alpha_1\right] + \eta\left[(\alpha_1+1) - \frac{(\alpha_1+1)H_{q,s}(\alpha_1+2)f(z)}{H_{q,s}(\alpha_1+1)f(z)}\right].$$

By setting

$$\theta(w) = \alpha + \xi w(z) + \delta w^2(z) \qquad \textit{and} \qquad \phi(w) = \frac{\beta}{w},$$

it can be easily observed that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}^*$ and that $\phi(w) \neq 0 (w \in \mathbb{C}^*)$. Also, by letting

$$Q(z) = zq'(z)\phi(q(z)) = \beta\frac{zq'(z)}{q(z)} \tag{3.5}$$

and

$$h(z) = \theta\{q(z)\} + Q(z) = \alpha + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)}, \tag{3.6}$$

we find that Q is starlike univalent in U and that

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left\{1 + \frac{\xi}{\beta}q(z) + \frac{2\delta}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0.$$

The assertion (3.5) of Theorem 3.1 now follows by an application of Lemma 2.2.

Putting $q(z) = \frac{1+Az}{1+Bz}, (-1 \leq B < A \leq 1)$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Suppose that*

$$\operatorname{Re}\left\{1 + \frac{\xi}{\beta}\left(\frac{1+Az}{1+Bz}\right) + \frac{2\delta}{\beta}\left(\frac{1+Az}{1+Bz}\right)^2 - \frac{(A-B)z}{(1+Az)(1+Bz)} - \frac{2Bz}{1+Bz}\right\} > 0 \ (z \in U).$$

If $f \in A_m$ satisfies the subordination

$$\Psi(f, \alpha_1, \alpha, \delta, \xi, \beta, \mu, \eta) \prec \alpha + \xi\left(\frac{1+Az}{1+Bz}\right) + \delta\left(\frac{1+Az}{1+Bz}\right)^2 + \frac{\beta(A-B)z}{(1+Az)(1+Bz)},$$

where $\Psi(f, \alpha_1, \alpha, \delta, \xi, \beta, \mu, \eta)$ is given by (3.3), then

$$\left(\frac{H_{q,s}(\alpha_1)f(z)}{z}\right)^\mu \left(\frac{z}{H_{q,s}(\alpha_1+1)f(z)}\right)^\eta \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Putting $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma (0 < \gamma \leq 1)$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.3. *Suppose that*

$$\operatorname{Re}\left\{1 + \frac{\xi}{\beta}\left(\frac{1+z}{1-z}\right)^\gamma + \frac{2\delta}{\beta}\left(\frac{1+z}{1-z}\right)^{2\gamma} - \frac{2\gamma z}{1-z^2} + \frac{2z(\gamma+z)}{(1-z)(1+z)}\right\} > 0 \ \ (z \in U).$$

If $f \in A_m$ satisfies the subordination

$$\Psi(f, \alpha_1, \alpha, \delta, \xi, \beta, \mu, \eta) \prec \alpha + \xi\left(\frac{1+z}{1-z}\right)^\gamma + \delta\left(\frac{1+z}{1-z}\right)^{2\gamma} + \frac{2\gamma\beta z}{1-z^2},$$

where $\Psi(f,\alpha_1,\alpha,\delta,\xi,\beta,\mu,\eta)$ *is given by (3.3), then*

$$\left(\frac{H_{q,s}(\alpha_1)f(z)}{z}\right)^{\mu}\left(\frac{z}{H_{q,s}(\alpha_1+1)f(z)}\right)^{\eta}\prec\left(\frac{1+z}{1-z}\right)^{\gamma}$$

and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best dominant.*

Putting $q(z)=e^{\mu Az}$, with $|\mu A|<\pi$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.4. *Suppose that*

$$\mathrm{Re}\{1+\frac{\xi}{\beta}e^{\mu Az}+\frac{2\delta}{\beta}e^{2\mu Az}\}>0\quad(z\in U).$$

If $f(z)\in A_m$ *satisfies the subordination*

$$\Psi(f,\alpha_1,\alpha,\delta,\xi,\beta,\mu)\prec\alpha+\xi e^{\mu Az}+\delta e^{2\mu Az}+\beta\mu Az,$$

where $\Psi(f,\alpha_1,\alpha,\delta,\xi,\beta,\mu,\eta)$ *is given by (3.3), then*

$$\left(\frac{H_{q,s}(\alpha_1)f(z)}{z}\right)^{\mu}\left(\frac{z}{H_{q,s}(\alpha_1+1)f(z)}\right)^{\eta}\prec e^{\mu Az}$$

and $e^{\mu Az}$ *is the best dominant.*

Remark 3.5. (i) Putting $q=2$, $s=1$, $\alpha_1=\alpha_2=\beta_1=1$, $m=1$, $\delta=\xi=\eta=0$, $\beta=\frac{1}{\mu}$ and $q(z)=e^{\mu Az}$, with $|\mu A|<\pi$ in Theorem 3.1, we obtain the result obtained by Obradovic and Owa [20];

(ii) Putting $q(z)=\frac{1}{(1-z)^{2b}}$ $(b\in\mathbb{C}^*)$, $q=2$, $s=1$, $\alpha_1=\alpha_2=\beta_1=1$, $m=1$, $\delta=\xi=\eta=0$, $\mu=\alpha=1$ and $\beta=\frac{1}{b}$ in Theorem 3.1, we obtain the result obtained by Srivastava and Lashin [26];

(iii) Putting $q(z)=\frac{1}{(1-z)^{2ab}}$ $(a,b\in\mathbb{C}^*)$, $q=2$, $s=1$, $\alpha_1=\alpha_2=\beta_1=1$, $m=1$, $\delta=\xi=\eta=0$, $\mu=\alpha=1$ and $\beta=\frac{1}{ab}$ in Theorem 3.1, we obtain the result obtained by Obradovic et al.[19];

(iv) Putting $q(z)=(1+Bz)^{\mu(A-B)/B}$, $(\mu\in\mathbb{C}^*,\ -1\le B<A\le 1)$, $q=2$, $s=1$, $\alpha_1=\alpha_2=\beta_1=1$, $m=1$, $\beta=\frac{1}{\mu}$, $\alpha=1$ and $\delta=\xi=\eta=0$ in Theorem 3.1, we obtain the result obtained by Obradovic and Owa [20];

(v) Putting $q(z)=(1-z)^{-2ab\cos\lambda e^{-i\lambda}}$ $(a,b\in C^*,|\lambda|<\frac{\pi}{2})$, $\beta=\frac{e^{i\lambda}}{ab\cos\lambda}$, $q=2$, $s=1$, $\alpha_1=\alpha_2=\beta_1=1$, $m=1$, $\delta=\xi=\eta=0$ and $\mu=\alpha=1$ in Theorem 3.1, we obtain the result obtained by Aouf et al. [2, Theorem 1].

Theorem 3.6. *Let* $h\in H,h(0)=1,h'(0)\neq 0$ which satisfy

$$\mathrm{Re}\{1+\frac{zh''}{(z)}h'(z)\}>-\frac{1}{2}\quad(z\in U).\tag{3.7}$$

If $f(z)\in A_m$ *satisfies the differential subordination:*

$$\frac{H_{q,s}(\alpha_1+k)f(z)}{z}\prec h(z)\qquad(k\in\mathbb{Z}^+;z\in U^*),$$

then

$$\frac{H_{q,s}(\alpha_1+k-1)f(z)}{z}\prec g(z)\qquad(k\in\mathbb{Z}^+),$$

where

$$g(z)=\frac{(\alpha_1+k-1)}{mz^{(\alpha_1+k-1)/m}}\int_0^z h(t)t^{((\alpha_1+k-1)/m)-1}dt \qquad (k\in\mathbb{Z}^+).$$

The function g is convex and is the best dominant.

Proof. Let the function $p(z)$ be defined by

$$p(z)=\frac{H_{q,s}(\alpha_1+k-1)f(z)}{z} \qquad (k\in\mathbb{Z}^+). \tag{3.8}$$

Then the function p is analytic in U and $p(0)=1$. Differentiating (3.8) logarithmically with respect to z, we have

$$\frac{zp'(z)}{p(z)}=\left[\frac{z\left(H_{q,s}(\alpha_1+k-1)f(z)\right)'}{H_{q,s}(\alpha_1+k-1)f(z)}-1\right] \quad (k\in\mathbb{Z}^+).$$

By using the identity (1.8), we have

$$\frac{zp'(z)}{p(z)}=\left[\frac{(\alpha_1+k-1)H_{q,s}(\alpha_1+k)f(z)}{H_{q,s}(\alpha_1+k-1)f(z)}-(\alpha_1+k-1)\right]$$

and hence

$$p(z)+\frac{zp'(z)}{\alpha_1+k-1}=\frac{H_{q,s}(\alpha_1+k)f(z)}{z} \qquad (k\in\mathbb{Z}^+).$$

The assertion of Theorem 3.6 now follows by applying Lemma 2.4.
Putting $k=1$ in Theorem 3.6, we get.

Corollary 3.7. *If $f\in A_m$ satisfies the differential subordination:*

$$\frac{H_{q,s}(\alpha_1+1)f(z)}{z}\prec h(z),$$

then

$$\frac{H_{q,s}(\alpha_1)f(z)}{z}\prec g(z),$$

where

$$g(z)=\frac{\alpha_1}{mz^{\alpha_1/m}}\int_0^z h(t)t^{(\alpha_1/m)-1}dt \qquad (z\in U).$$

The function $g(z)$ is convex and is the best dominant.

By using Lemma 2.3 we can prove the following theorem.

Theorem 3.8. *Let g be convex function with $g(0)=1$. Let h be a function, such that*

$$h(z)\prec g(z)+\frac{m}{\lambda+1}zg'(z).$$

If $f\in A_m$ *satisfies the subordination:*

$$\frac{H_{q,s}(\alpha_1+k)f(z)}{z}\prec h(z) \qquad (k\in\mathbb{Z}^+), \tag{3.9}$$

then

$$\frac{H_{q,s}(\alpha_1+k-1)f(z)}{z}\prec g(z) \qquad (k\in\mathbb{Z}^+),$$

and is the best dominant.

Proof. The proof of Theorem 3.8 is much akin to the proof of Theorem 3.6 and hence we omit the details involved.

Next, by appealing to Lemma 2.5, we prove Theorem 3.9.

Theorem 3.9. *Let q be analytic and convex univalent in U, such that $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Further, let us assume that*

$$\mathrm{Re}\{\frac{2\delta}{\beta}(q(z))^2 + \frac{\xi}{\beta}q(z)\} > 0 \quad (z \in U). \tag{3.10}$$

If $f \in A_m$, $0 \neq \left(\frac{H_{q,s}(\alpha_1)f(z)}{z}\right)^{\mu}\left(\frac{z}{H_{q,s}(\alpha_1+1)f(z)}\right)^{\eta} \in H[q(0),1]$, $\Psi(f,\alpha_1,\alpha,\delta,\xi,\beta,\mu,\eta)$ is univalent in U, where $\Psi(f,\alpha_1,\alpha,\delta,\xi,\beta,\mu)$ is given by (3.3), and

$$\alpha + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)} \prec \Psi(f,\alpha_1,\alpha,\delta,\xi,\beta,\mu,\eta), \tag{3.11}$$

then

$$q(z) \prec \left(\frac{H_{q,s}(\alpha_1)f(z)}{z}\right)^{\mu}\left(\frac{z}{H_{q,s}(\alpha_1+1)f(z)}\right)^{\eta} \tag{3.12}$$

and q is the best subordinant of (3.11).

Proof. By setting

$$\vartheta(z) = \alpha + \xi w + \delta w^2 \quad \text{and} \quad \varphi(w) = \beta\frac{w'}{w},$$

it is easily observed that ϑ is analytic in $\mathbb{C}$. Also, φ is analytic in $\mathbb{C}^*$ and that $\varphi(w) \neq 0$ $(w \in \mathbb{C}^*)$.

Since q is convex (univalent) function it follows that,

$$\mathrm{Re}\left\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\right\} = \mathrm{Re}\left\{\frac{2\delta}{\beta}(q(z))^2 + \frac{\xi}{\beta}q(z)\right\} > 0 \quad (z \in U).$$

The assertion (3.12) of Theorem 3.9 follows by an application of Lemma 2.5.

4. Sandwich result

Combining Theorem 3.1 and Theorem 3.9, we get the following sandwich theorem.

Theorem 4.1. *Let q_1 be convex univalent and q_2* be univalent in U such that $q_1(z) \neq 0$ *and $q_2(z) \neq 0$.* Suppose q_1 *satisfies (3.10) and q_2* satisfies *(3.1). If $f \in A_m$, $0 \neq \left(\frac{H_{q,s}(\alpha_1)f(z)}{z}\right)^{\mu}\left(\frac{z}{H_{q,s}(\alpha_1+1)f(z)}\right)^{\eta} \in H[q(0),1]$, and $\Psi(f,\alpha_1,\alpha,\delta,\xi,\beta,\mu,\eta)$ is univalent in U and satisfies*

$$\alpha + \xi q_1(z) + \delta(q_1(z))^2 + \beta\frac{zq_1'(z)}{q_1(z)} \prec \Psi(f,\alpha_1,\alpha,\delta,\xi,\beta,\mu,\eta) \prec \alpha + \xi q_2(z) + \delta(q_2(z))^2 + \beta\frac{zq_2'(z)}{q_2(z)}, \tag{4.1}$$

where $\Psi(f,\alpha_1,\alpha,\delta,\xi,\beta,\mu,\eta)$ is given by (3.3), then

$$q_1(z) \prec \left(\frac{H_{q,s}(\alpha_1)f(z)}{z}\right)^{\mu}\left(\frac{z}{H_{q,s}(\alpha_1+1)f(z)}\right)^{\eta} \prec q_2(z)$$

and q_1 *and* q_2 *are, respectively, the best subordinant and best dominant.*

Remark 4.2. (i) Putting $q = 2$, $s = 1$, $\alpha_1 = a$ $(a > 0)$, $\alpha_2 = 1$ and $\beta_1 = c$ $(c > 0)$ in our results we will improve all results obtained by Shanmugam et al. [24];

(ii) By specializing the parameters $q, s, \alpha_i(\alpha_1, \ldots, \alpha_q)$ and $\beta_j(\beta_1, \ldots, \beta_s)$ in our results, we obtain the corresponding results due to various operators mentioned in the introduction.

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References

[1] Ali, R.M., Ravichandran, V., Khan, M.H., Subramaniam, K.G., *Differential sandwich theorems for certain analytic functions*, Far East J. Math. Sci., **15**(2004), no. 1, 87-94.

[2] Aouf, M.K., Al-Oboudi, F.M., Haidan, M.M., *On some results for λ-spirallike and λ-Robertson functions of complex order*, Publ. Institute Math. Belgrade, **77**(2005), no. 91, 93-98.

[3] Bernardi, S.D., *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., **135**(1969), 429-446.

[4] Bulboacă, T., *A class of superordination preserving integral operators*, Indag. Math. (New Series), **13**(2002), no. 3, 301-311.

[5] Bulboacă, T., *Classes of first-order differential subordinations*, Demonstratio Math., **35**(2002), no. 2, 287-392.

[6] Carlson, B.C., Shaffer, D.B., *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal., **15**(1984), 737-745.

[7] Cho, N.E., Kwon, O.H., Srivastava, H.M., *Inclusion and argument properties for certain subclasses of multivalent functions associated with a family of linear operators*, J. Math. Anal. Appl., **292**(2004), 470-483.

[8] Choi, J.H., Saigo, M., Srivastava, H.M., *Some inclusion properties of a certain family of integral operators*, J. Math. Anal. Appl., **276**(2002), 432-445.

[9] Dziok, J., Srivastava, H.M., *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput., **103**(1999), 1-13.

[10] Dziok, J., Srivastava, H.M., *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, Integral Transform. Spec. Funct., **14**(2003), 7-18.

[11] Hallenbeck, D.J., Ruscheweyh, St., *Subordination by convex functions*, Proc. Amer. Math. Soc., **52**(1975), 191-195.

[12] Hohlov, Yu.E., *Operators and operations in the class of univalent functions*, (in Russian), Izv. Vysšh. Učebn. Zaved. Mat., **10**(1978), 83-89.

[13] Libera, R.J., *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., **16**(1965), 755-658.

[14] Livingston, A.E., *On the radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc., **17**(1966), 352-357.

[15] Miller, S.S., Mocanu, P.T., *Differential subordinations and univalent functions*, Michigan Math. J., **28**(1981), 157-171.

[16] Miller, S.S., Mocanu, P.T., *Differential Subordinations Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Math., No. 225, Marcel Dekker Inc. New York, Basel, 2000.

[17] Miller, S.S., Mocanu, P.T., *Subordinations of differential superordinations*, Complex Variables, **48**(2003), no. 10, 815-826.

[18] Noor, K.I., *On new classes of integral operator*, J. Nat. Geom., **16**(1999), no. 1-2, 71-80.

[19] Obradovic, M., Aouf, M.K., Owa, S., *On some results for starlike functions of complex order*, Publ. Inst. Math. Belgrade, **46**(1989), no. 60, 79-85.

[20] Obradovic, M., Owa, S., *On certain properties for some classes of starlike functions*, J. Math. Anal. Appl., **145**(1990), no. 2, 357-364.

[21] Owa, S., Srivastava, H.M., *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math., **39**(1987), 1057-1077.

[22] Ruscheweyh, St., *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49**(1975), 109-115.

[23] Shanmugam, T.N., Ravichandran, V., Sivasubramanian, S., *Differential sandwich theorems for some subclasses of analyitc functions*, Austral. J. Math. Anal. Appl., **3**(2006), no. 1, 1-11.

[24] Shanmugam, T.N., Sivasubramanian, S., Owa, S., *On sandwich results for some subclasses of analytic functions involving certain linear operator*, Integral Transform. Spec. Funct., **21**(2010), no. 1, 1-11.

[25] Srivastava, H.M., Karlsson, P.W., *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Ltd., Chichester, Halsted Press (John Wiley & Sons, Inc.), New York, 1985.

[26] Srivastava, H.M., Lashin, A.Y., *Some applications of the Briot-Bouquet differential subordination*, J. Inequal. Pure Appl. Math., **6**(2005), no. 2, 1-7.

[27] Tuneski, N., *On certain sufficient conditions for starlikeness*, Internat. J. Math. Math. Sci., **23**(2000), no. 8, 521-527.

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An existence theorem for a non-autonomous second order nonlocal multivalued problem

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Abstract. In this paper we prove the existence of mild solutions for a nonlocal problem governed by an abstract semilinear non-autonomous second order differential inclusion, where the non-linear part is an upper-Caratheodory semicontinuous multimap. Our existence theorem is obtained thanks to the introduction of a fundamental Cauchy operator. Finally we apply our main result to provide the controllability of a problem involving a non-autonomous wave equation.

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1. Introduction

Recently in [8] H.R. Henríquez, V. Poblete, J.C. Pozo have studied the existence of mild solutions for a nonlocal problem governed by the following non-autonomous wave equation

$$\frac{\partial^2 w(t,\xi)}{\partial t^2} = \frac{\partial^2 w(t,\xi)}{\partial \xi^2} + b(t)\frac{\partial w(t,\xi)}{\partial \xi} + \tilde{f}(t,\ w(t,\xi)),\ t \in J = [0,a], \tag{1.1}$$

Starting from this paper, we are interested to study the following control problem

$$\begin{cases} \frac{\partial^2 w(t,\xi)}{\partial t^2} = \frac{\partial^2 w(t,\xi)}{\partial \xi^2} + b(t)\frac{\partial w(t,\xi)}{\partial \xi} + f(t,\ w(t,\xi), u(t,\xi)),\ t \in J \\ w(t,0) = w(t,\ 2\pi),\ t \in J, \\ \frac{\partial w(t,0)}{\partial \xi} = \frac{\partial w(t,\ 2\pi)}{\partial \xi},\ t \in J, \\ w(0,\xi) = \sum_{i=1}^m \frac{2\pi t_i}{\xi_i},\ \xi \in R, \\ \frac{\partial w(0,\xi)}{\partial t} = \sum_{i=1}^m \frac{2\pi}{\xi_i},\ \xi \in R, \\ u(t,\xi) \in U(t,\ w(t,\xi)) \end{cases} \tag{1.2}$$

where $0 < t_1 < ... < t_i < ... < t_m < a$ and $0 < \xi_1 < ... < \xi_i < ... < \xi_m < 2\pi$, $b : J \to R$, $f : J \times R \times R \to R$ and $U : J \times R \to \mathcal{P}(R)$.

By using the classical arguments (see, for example [15]) the controllability of (5.1) is brought back to the existence of mild solutions for a problem described by the non-autonomous semilinear second order differential inclusion with nonlocal conditions

$$\begin{cases} x''(t) \in A(t)x(t) + F(t, x(t)),\ t \in J \\ x(0) = g(x) \\ x'(0) = h(x) \end{cases} \tag{1.3}$$

where $J = [0, a]$ is an interval of the real line, $\{A(t)\}_{t\in J}$ is a family of bounded, linear operators defined in a subspace $D(A(t)) = D(A)$ dense in a real Banach space X generating a "*fundamental system*", and g, h are two operators defined on the trajectories and assuming values in X.
Recently the existence of nonlocal mild solutions in Banach space has been investigated for semilinear non-autonomous second order differential equations in [9], in [7] and in [8], while there exists an extensive literature for the autonomous case (see, for example, [6], [11], [12] , [13] and [14]).
The note is organized in the following way. We start by introducing the fundamental Cauchy operator and by characterizing some of its properties, which play a key role to prove the existence of mild nonlocal solutions for problem (1.3) in the case that the nonlinear part of the semilinear second order differential inclusion is given by an upper-Caratheodory semicontinuous multimap. In order to obtain our main existence result we use the powerful tools introduced in [9], [7] and a fixed point theorem for condensing multimaps. Our existence theorem extends in a broad sense all the existence results above mentioned. In the last section we apply our existence proposition for (1.3) in order to establish the controllability of (1.2).

2. Preliminaries

Let X, Y be topological spaces and $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y. We recall that a map $F : X \to \mathcal{P}(Y)$ is said to be *upper semicontinuous* (*lower semicontinuous*) if $F^+(V) = \{x \in X : F(x) \subset V\}$ ($F^-(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$) is an open subset of X, for every open $V \subset Y$; the multimap F is said to have *closed graph* if the set $graphF = \{(x, y) \in X \times Y : y \in F(x)\}$ is closed in $X \times Y$ (see [5]).
In this paper X is a real Banach space endowed with a norm $\|.\|$, and we will use the following notations:

$$\mathcal{P}_b(X) = \{H \in \mathcal{P}(X) : H \text{ bounded}\} \tag{2.1}$$

$$\mathcal{P}_c(X) = \{H \in \mathcal{P}(X) : H \text{ convex}\}; \tag{2.2}$$

$$\mathcal{P}_f(X) = \{H \in \mathcal{P}(X) : H \text{ closed}\}; \tag{2.3}$$

$$\mathcal{P}_k(X) = \{H \in \mathcal{P}(X) : H \text{ compact}\}; \tag{2.4}$$

$$\mathcal{P}_{fc}(X) = \mathcal{P}_f(X) \cap \mathcal{P}_c(X)... \tag{2.5}$$

Further, let $J = [0, a]$ be an interval of the real line endowed with the usual Lebesgue measure λ.
A function $f : J \to X$ is said to be *strongly measurable* if there is a sequence of simple functions $(s_n)_n$ which converges to f almost everywhere.

Moreover, we denote by $C(J;X)$ the space consisting of all continuous functions from J into X provided with the norm $\|.\|_\infty$ of uniform convergence, by $L^1(J,X)$ the space of all X-valued Bochner integrable functions on J with norm $\|u\|_1 = \int_0^a \|u(t)\|dt$ and $L^1_+(J) = \{f \in L^1(J,\mathbb{R}) : \ f(t) \geq 0, \text{ for a.e. } t \in J\}$.
A countable set $\{f_n\}_n \subset L^1(J,X)$ is said to be *semicompact* if: (i) $\{f_n\}_n$ is *integrably bounded*, i.e. there exists $\omega \in L^1_+(J)$ such that $\|f_n(t)\| \leq \omega(t)$, for a.e. $t \in J$ and for every $n \in \mathbb{N}$; (ii) the set $\{f_n(t)\}_n$ is relatively compact in X, for a.e. $t \in J$.
Now let us consider the following nonlocal problem governed by a non-autonomous abstract semilinear second order differential inclusion

$$\begin{cases} x''(t) \in A(t)x(t) + F(t,x(t)), \ t \in J \\ x(0) = g(x) \\ x'(0) = h(x). \end{cases} \tag{2.6}$$

In this problem F is an X-valued multimap defined on $J \times X$, $g, h : C(J;X) \to X$ are functions; $\{A(t)\}_{t\in J}$ is a family, generating a "*fundamental system*" $\{S(t,s)\}_{t,s\in J}$ of bounded linear operators $A(t) : D(A) \to X$ (where $D(A)$ is a dense subspace of X) such that, for each $x \in D(A)$, the function $t \mapsto A(t)x$ is continuous in J .
First we recall the concept of the "fundamental system", introduced by Kozak in [9] and recently used by H.R. Henríquez, V. Poblete, J.C. Pozo in [8].

Definition 2.1. A family $\{S(t,s)\}_{t,s\in J}$ of bounded linear operators $S(t,s) : X \to X$ is called *fundamental system* generated by the family $\{A(t)\}_{t\in J}$ if
(**S1**) for each $x \in X$, the function $S(.,.)x : J \times J \to X$ is of class C^1 and
(a) for each $t \in J$, $S(t,t)x = 0$, $\forall x \in X$;
(b) for each $t, s \in J$ and for each $x \in X$, $\frac{\partial}{\partial t}S(t,s) \mid_{t=s} x = x$ and $\frac{\partial}{\partial s}S(t,s) \mid_{t=s} x = -x$;
(**S2**) for all $t, s \in J$, if $x \in D(A)$, then $S(t,s)x \in D(A)$, the map $S(.,.)x : J \times J \to X$ is of class C^2 and
(a) $\frac{\partial^2}{\partial t^2}S(t,s)x = A(t)S(t,s)x$, $\forall (t,s) \in J \times J$, $\forall x \in D(A)$;
(b) $\frac{\partial^2}{\partial s^2}S(t,s)x = S(t,s)A(s)x$, $\forall (t,s) \in J \times J$, $\forall x \in D(A)$;
(c) $\frac{\partial^2}{\partial s \partial t}S(t,s) \mid_{t=s} x = 0$, $\forall s \in J$, $\forall x \in D(A)$;
(**S3**) for all $s, t \in J$, if $x \in D(A)$, then $\frac{\partial}{\partial s}S(t,s)x \in D(A)$. Moreover, there exist $\frac{\partial^3}{\partial t^2 \partial s}S(t,s)x$ and $\frac{\partial^3}{\partial s^2 \partial t}S(t,s)x$ and
(a) $\frac{\partial^3}{\partial t^2 \partial s}S(t,s)x = A(t)\frac{\partial}{\partial s}S(t,s)x$, $\forall (t,s) \in J \times J$, $\forall x \in D(A)$;
(b) $\frac{\partial^3}{\partial s^2 \partial t}S(t,s)x = \frac{\partial}{\partial t}S(t,s)A(s)x$, $\forall (t,s) \in J \times J$, $\forall x \in D(A)$;
and for all $x \in D(A)$ the function $(t,s) \mapsto A(t)\frac{\partial}{\partial s}S(t,s)x$ is continuous in $J \times J$.

Moreover, as in [8], a map $S : J \times J \to \mathcal{L}(X)$, where $\mathcal{L}(X)$ denote the space of all bounded linear operators in X with the norm $\|.\|_{\mathcal{L}(X)}$, is said to be a "*fundamental operator*" if $\{S(t,s)\}_{t,s\in J}$ is a fundamental system. Moreover, as in [8], we introduce, for each $(t,s) \in J \times J$ the operator

$$C(t,s) = -\frac{\partial}{\partial s}S(t,s) : X \to X. \tag{2.7}$$

By using the Banach-Steinhaus Theorem it is possible to prove that the fundamental system $\{S(t,s)\}_{t,s\in J}$ satisfies the following properties:
there exists two constants $K, K^*, K_1 > 0$ such that
$(p1)$ $\|C(t,s)\|_{\mathcal{L}(X)} \leq K,\ \forall (t,s) \in J \times J;$
$(p2)$ $\|S(t,s)\|_{\mathcal{L}(X)} \leq K \mid t-s \mid,\ \forall t,s \in J$
$(p3)$ $\|S(t,s)\|_{\mathcal{L}(X)} \leq Ka,\ \forall t,s \in J$
$(p4)$ $\|S(t_2,s) - S(t_1,s)\|_{\mathcal{L}(X)} \leq K^* \mid t_2 - t_1 \mid, \forall t_1, t_2, s \in J$
$(p5)$ $\exists K_1 > 0\ :\ \|\frac{\partial}{\partial s} S(t_2,s) - \frac{\partial}{\partial s} S(t_1,s)\|_{\mathcal{L}(X)} \leq K_1 \mid t_2 - t_1 \mid, \forall t_1, t_2, s \in J.$
Now we recall the definition of a mild solutions for the nonlocal problem (2.6)

Definition 2.2. A continuous function $u : J \to X$ is a *mild solution* for (2.6) if

$$u(t) = -\frac{\partial}{\partial s} S(t,s) \mid_{s=0} g(u) + S(t,0)h(u) + \int_0^t S(t,\xi) f(\xi) d\xi,\ \forall t \in J$$

where $f \in S^1_{F(.,u(.))} = \{f \in L^1(J;X) : f(t) \in F(t,u(t)) \text{ a.e. } t \in J\}$.

In the sequel let us denote by 0_n the zero-element of $\mathbb{R}^n$ and by $\preccurlyeq$ the partial ordering given by the standard positive cone $\mathbb{R}^n_{0,+} := (\mathbb{R}^+_0)^n$, i.e. $x \preccurlyeq y$ if and only if $y - x \in \mathbb{R}^n_{0,+}$; clearly, $x \prec y$ means that $x \preccurlyeq y$ and $x \neq y$.

Definition 2.3. A function $\beta : \mathcal{P}_b(X) \to \mathbb{R}^n_{0,+}$ is said to be a "measure of noncompactness" (MNC, for short) in the Banach space X if, for every $\Omega \in \mathcal{P}_b(X)$, the following properties are satisfied:
(β_1) $\beta(\Omega) = 0_n$ if and only if $\bar{\Omega}$ is compact;
(β_2) $\beta(\bar{co}(\Omega)) = \beta(\Omega)$.

Further, a MNC $\beta : \mathcal{P}_b(X) \to \mathbb{R}^n_{0,+}$ is said to be:
monotone if $\Omega_1, \Omega_2 \in \mathcal{P}_b(X) : \Omega_1 \subset \Omega_2$ implies $\beta(\Omega_1) \preccurlyeq \beta(\Omega_2)$;
nonsingular if $\beta(\{x\} \cup \Omega) = \beta(\Omega)$, for every $x \in X$, $\Omega \in \mathcal{P}_b(X)$;
invariant under closure if $\beta(\bar{\Omega}) = \beta(\Omega)$, $\Omega \in \mathcal{P}_b(X)$;
invariant with respect to the union with compact set if $\beta(\Omega \cup C) = \beta(\Omega)$, for every relatively compact set $C \subset X$ and $\Omega \in \mathcal{P}_b(X)$.

In this setting we provide the following definitions (see [2]).

Definition 2.4. If D is a nonempty subset of X, a map $\phi : D \to \mathcal{P}(X)$ is said to be "condensing" with respect to a MNC $\beta : \mathcal{P}_b(X) \to \mathbb{R}^n_{0,+}$ (shortly, " β-condensing ") if
(I) $\phi(D)$ is bounded
and
(II) $\beta(\Omega) \preccurlyeq \beta(\phi(\Omega))$ implies $\beta(\Omega) = 0_n$, $\Omega \in \mathcal{P}_b(D)$
or equivalently
(II)' $0_n \prec \beta(\Omega)$ implies $\beta(\Omega) \not\preccurlyeq \beta(\phi(\Omega))$, $\Omega \in \mathcal{P}_b(D)$ (i.e. $\beta(\phi(\Omega)) \prec \beta(\Omega)$ is true or $\beta(\phi(\Omega))$ and $\beta(\Omega)$ are not comparable).

We can now recall the following Sadowskii type fixed point theorem for multimaps condensing with respect to a vectorial measure of noncompactness (see [2], Theorem 2.2).

Theorem 2.5. *Let $\beta : \mathcal{P}_b(X) \to \mathbb{R}^n_{0,+}$ be a nonsingular MNC, D be a closed convex subset of a Banach space X and $\phi : D \to \mathcal{P}_{fc}(D)$ be a map such that*
(ϕ1) ϕ has weakly closed graph in $D \times X$, i.e. for every sequence $(x_n)_n$ in D, $x_n \to x$, $x \in D$, and for every sequence $(y_n)_n$, $y_n \in \phi(x_n)$, $y_n \to y$, then $S(x,y) \cap \phi(x) \neq 0$, where $S(x,y) = \{x + \lambda(y-x) : \lambda \in [0,1]\}$;
(ϕ2) ϕ is β-condensing.
Then there exists $x \in D$ with $x \in \phi(x)$.

Next, we consider the set $\mathbb{R}^2_{0,+} = \mathbb{R}^+_0 \times \mathbb{R}^+_0$ endowed with the partial ordering $\preccurlyeq$ before introduced. Fixed a constant $L \geq 0$ we can introduce the function $\nu_L : \mathcal{P}_b(C(J;X)) \to \mathbb{R}^2_{0,+}$ defined by

$$\nu_L(\Omega) = \max_{\{w_n\}_n \subset \Omega} (\tau(\{w_n\}_n), \lambda(\{w_n\}_n)),\ \forall \Omega \in \mathcal{P}_b(C(J;X)), \tag{2.8}$$

being

$$\tau(\{w_n\}_n) = \sup_{t \in J} e^{-Lt} \eta(\{w_n(t)\}_n); \tag{2.9}$$

and

$$\lambda(\{w_n\}_n) = mod_C(\{w_n\}_n) \tag{2.10}$$

where η is the Hausdorff MNC in the Banach space X and mod_C is the modulus of continuity in $C(J;X)$ (see [3]).

3. The fundamental Cauchy operator

To study the problem (2.6) we introduce the following operator, which will play a key role in our next existence result.

Definition 3.1. Let $\{S(t,s)\}_{t,s \in J}$ be the fundamental system generated by the family $\{A(t)\}_{t \in J}$ of bounded linear operators in the Banach space X, presented in (2.6). We will call the operator $G_S : L^1(J;X) \to C(J;X)$ defined by

$$G_S f(t) = \int_0^t S(t,s) f(s) ds,\ t \in J = [0,a],\ f \in L^1(J;X) \tag{3.1}$$

the fundamental Cauchy operator.

First we present the following result, which is analogous to the one proved in [10] or in [4], respectively for the Cauchy operator and for the generalized Cauchy operator.

Theorem 3.2. *The fundamental Cauchy operator G_S satisfies the following properties:*
(G_S1) $\|G_S f(t) - G_S g(t)\| \leq Ka \int_0^t \|f(s) - g(s)\| ds$, $t \in J$, $f, g \in L^1(J;X)$;
where Ka is the constant presented in (p3);
(G_S2) for any compact $H \subset X$ and sequence $(f_n)_n$, $f_n \in L^1(J;X)$, such that $\{f_n(t)\}_n \subset H$ for a.e. $t \in J$, the weak convergence $f_n \rightharpoonup f_0$ implies the convergence $G_S f_n \to G_S f_0$.

Proof. Let $t \in J$ and $f, g \in L^1(J;X)$ be fixed. Thanks to the definition of the fundamental Cauchy operator and to the property $(p3)$ we have:

$$\|G_S f(t) - G_S g(t)\| = \|\int_0^t S(t,s)(f(s) - g(s))ds\| \leq Ka \int_0^t \|f(s) - g(s)\|ds \quad (3.2)$$

Therefore we can deduce that G_S satisfies (G_S1).
Let us prove the property (G_S2).
Fix a compact set $H \subset X$ and $t \in J$, let us consider the set $Q_t \subseteq X$ defined as folllows:

$$Q_t = \bigcup_{s\in[0,t]} S(t,s)H. \quad (3.3)$$

Now we show that Q_t is compact. Let us fix the map $q_H : J \times J \times H \to X$ defined by $q_H(t,s,x) = S(t,s)x$, $(t,s,x) \in J \times J \times H$. Then, for each $\varepsilon > 0$, there exist finitely many $x_1, \ldots, x_p \in X$ such that:

$$H \subset \cup_{i=1}^p (x_i + \frac{\varepsilon}{4Ka} B_1(0)),$$

where $B_1(0)$ is the open unit ball in X.
Now, by the strongly continuity of S, there exists $\eta_H(\varepsilon) > 0$ such that for every $(t_1,s_1),(t_2,s_2) \in \Delta$ with $\max\{|t_1 - t_2|, |s_1 - s_2|\} < \eta_H(\varepsilon)$ we have

$$\|S(t_1,s_1)x_i - S(t_2,s_2)x_i\| < \frac{\varepsilon}{4}\ , \text{ for every } i = 1,\ldots,p. \quad (3.4)$$

Put $\tilde{\eta}_H(\varepsilon) = \min\{\eta_H(\varepsilon), \frac{\varepsilon}{4Ka}\}$, for arbitrary $(t_1,s_1,z_1),(t_2,s_2,z_2) \in J \times J \times H$ such that $\max\{|t_1-t_2|, |s_1-s_2|, \|z_1-z_2\|\} < \tilde{\eta}_H(\varepsilon)$, since there exists $j \in \{1,\ldots,p\}$ such that $\|z_1 - x_j\| < \frac{\varepsilon}{4Ka}$, by $(p3)$ and (3.4) we get

$$\begin{aligned}
&\|q_H(t_1,s_1,z_1) - q_H(t_2,s_2,z_2)\| = \|S(t_1,s_1)z_1 - S(t_2,s_2)z_2\| \\
&\leq \|S(t_1,s_1)z_1 - S(t_1,s_1)x_j\| + \|S(t_1,s_1)x_j - S(t_2,s_2)x_j\| \\
&+ \|S(t_2,s_2)x_j - S(t_2,s_2)z_1\| + \|S(t_2,s_2)z_1 - S(t_2,s_2)z_2\| \\
&\leq \|S(t_1,s_1)(z_1 - x_j)\| + \|[S(t_1,s_1) - S(t_2,s_2)]x_j\| \\
&+ \|S(t_2,s_2)(x_j - z_1)\| + \|S(t_2,s_2)(z_1 - z_2)\| \\
&\leq Ka\|z_1 - x_j\| + \frac{\varepsilon}{4} + Ka\|x_j - z_1\| + Ka\|z_1 - z_2\| < \varepsilon.
\end{aligned}$$

Therefore the map q_H is uniformly continuous. Hence, being true that

$$Q_t = q_H(\{t\} \times [0,t] \times H)\ ,$$

we can say that Q_t is compact.

Now, we show that, for every sequence $(f_n)_n$, $f_n \in L^1(J;X)$, such that $\{f_n(t)\}_n \subset H$ a.e. $t \in J$ we have that the set $\{G_S f_n(t)\}_n$ is relatively compact in X, for every $t \in J$. To this end, fixed $t \in J$, it is enough to prove that

$$\{G_S f_n(t)\}_n \subset t\bar{co}(Q_t). \quad (3.5)$$

Let us associate to $(f_n)_{n=1}^{+\infty}$ a sequence $(\tilde{f}_n)_{n=1}^{+\infty}$ such that $\tilde{f}_n(t) \in H$ for every $t \in J$ and $\tilde{f}_n = f_n$ a.e. in J. By applying [5, Corollary 3.10.19], we obtain

$$G_S f_n(t) = \int_0^t S(t,s)\tilde{f}_n(s)\,ds \in t\,\overline{co}\,\{S(t,s)\tilde{f}_n(s) : s \in [0,t]\} \subset t\,\overline{co}\,Q_t,\ \forall n \in \mathbb{N}\ .$$

Next, let us show now that $\{G_S f_n\}_{n=1}^{+\infty} \subset C(J;X)$ is (uniformly) equicontinuous in J. To see this, fixed $\varepsilon > 0$, we can choose $\delta(\varepsilon) = \frac{\varepsilon}{aK_2(K+K^*)}$, where the constants K and K^* are respectively that from properties $(p1)$ and $(p4)$ in Section 2, while K_2 is a constant such that

$$\|f_n(t)\| \leq K_2,\ \text{a.e. } t \in J,\ \forall n \in \mathbb{N}.$$

Then, for every $t_1, t_2 \in J$, $|t_2 - t_1| \leq \delta(\varepsilon)$, and w.l.o.g. $t_2 > t_1$, we have (see $(p3)$ and $(p4)$ in Section 2):

$$\begin{aligned}
\|G_S f_n(t_2) - G_S f_n(t_1)\| &\leq \int_0^{t_1} \|[S(t_2,s) - S(t_1,s)]f_n(s)\|ds \\
&\quad + \int_{t_1}^{t_2} \|S(t_2,s)\|_{\mathcal{L}(X)}\|f_n(s)\|ds \\
&\leq \int_0^{t_1} \|S(t_2,s) - S(t_1,s)\|_{\mathcal{L}(X)}\|f_n(s)\|ds + K_2Ka(t_2 - t_1) \\
&\leq \int_0^{a} K^*(t_2 - t_1)K_2 ds + K_2Ka(t_2 - t_1) \leq \delta(\varepsilon)aK_2(K + K^*) = \varepsilon.
\end{aligned}$$

Hence we have the equicontinuity in J of the set $\{G_S f_n\}_n$.
Furthermore, the condition (G_S1) implies that the linear operator G_S is bounded, hence G_S is weakly continuous, i.e. if $f_n \rightharpoonup f_0$ then $G_S f_n \rightharpoonup G_S f_0$. Now, by applying a generalized version of the Ascoli-Arzelà criterion obtained by Ambrosetti in [1], we get the relative compactness of the set $\{G_S f_n\}_n$.
The relative compactness of $\{G_S f_n\}_{n=1}^{+\infty}$ provides that the last convergence is in the norm of the space $C(J;X)$. So also (G_S2) is stated.

Remark 3.3. Condition (G_S1) obviously implies the Lipschitz condition
$(G_S1)'$ $\|G_S f - G_S g\|_\infty \leq Ka\|f - g\|_1,$ for all $f, g \in L^1(J;X)$.

4. Existence results

Thanks to the properties of the fundamental Cauchy operator we are able to prove our main existence result.

Theorem 4.1. *Let $J = [0,a]$, X a Banach space and $\{A(t)\}_{t\in J}$ a family which satisfies the property:*

(A) $\{A(t)\}_{t\in J}$ is a family of bounded linear operators, defined in a subspace $D(A)$ dense in X and taking values in X, generating a fundamental system $\{S(t,s)\}_{(t,s)\in J\times J}$ such that, for each $x \in D(A)$, the function $t \mapsto A(t)x$ is continuous in J.

Let $F : J \times X \to \mathcal{P}_{kc}(X)$ a multimap which satisfies the following hypothesis:
(F1) for every $x \in X$, $F(.,x)$ admits a B-measurable selector;
(F2) for a.e. $t \in J$, $F(t,.)$ is upper semicontinuous;

(F3) there exists a function $\alpha \in L^1_+(J)$ such that

$$\|F(t,x)\| = \sup_{z \in F(t,x)} \|z\| \leq \alpha(t)(1 + \|x\|)$$

for a.e. $t \in J$ and for all $x \in X$.
(F4) there exists a function $m \in L^1_+(J)$ such that

$$\eta(F(t,B)) \leq m(t)\eta(B)$$

for a.e. $t \in J$ and for every $B \in \mathcal{P}_b(X)$ (where η is the Hausdorff MNC in X).

Let $g, h : C(J;X) \to X$ be two functions which satisfy the following properties:
(gh1) g, h are compact, i.e. they are continuous and map bounded sets into relatively compact sets;
(gh2) there exists $Q > 0$: $\|g(u)\| \leq Q$, $\|h(u)\| \leq Q$ for every $u \in C(J;X)$.

Then there exists at least one mild solution for the nonlocal problem (2.6).

Proof. We consider the integral multioperator $\Gamma : C(J;X) \to \mathcal{P}_c(C(J;X))$ defined as

$$\Gamma(u) = \{y \in C(J;X) : y(t) = C(t,0)g(u) + S(t,0)h(u) + \int_0^t S(t,\xi)f(\xi)d\xi,\ t \in J,\ f \in S^1_{F(.,u(.))}\} \tag{4.1}$$

for all $u \in C(J;X)$.
Note that, for all $u \in C(J;X)$, since $S_{F(.,u(.))} \neq \emptyset$ (see [10], Lemma 5.1.1) we have $\Gamma(u) \neq \emptyset$. Moreover Γ takes convex values thanks the convexity of the values of F.
From now on we proceed by steps.
Step 1: There exists a set which is invariant under the action of the operator Γ.
Step 1a: We put

$$q_n = \max_{t \in J}\{\int_0^t Kae^{-n(t-s)}\alpha(s)ds\} \tag{4.2}$$

for all $n \in \mathbb{N}$, where K, α are respectively from $(p1)$ and $(F3)$ and a is the size of J .
Let us show that

$$\inf_{n \in \mathbb{N}} q_n = 0. \tag{4.3}$$

From definition of maximum, for all $n \in \mathbb{N}$, there exists $t_n \in J$ such that

$$q_n - \frac{1}{n} < \int_0^{t_n} Kae^{-n(t_n-s)}\alpha(s)ds = \int_0^a \psi_n(s)ds \tag{4.4}$$

being $\psi_n : J \to \mathbb{R}$ the function defined as follows

$$\psi_n(s) = Kae^{-n(t_n-s)}\chi_{[0,t_n]}(s)\alpha(s), \text{ for all } s \in J,$$

where $\chi_{[0,t_n]}$ is the characteristic function of the interval $[0,t_n]$. Eventually passing to a subsequence, the sequence $(\psi_n)_n$ is such that

$$\lim_{n\to\infty} \psi_n(s) = 0, \text{ for all } s \in J$$

$$|\ \psi_n(s)\ | \leq Ka\alpha(s) =: \alpha^*(s),\ \forall s \in J,\ \forall n \in \mathbb{N},$$

where $\alpha^* \in L^1_+(J)$. So the Dominated Convergence Theorem implies that

$$\lim_{n\to\infty} \int_0^a \psi_n(s)ds = 0.$$

Since $q_n \geq 0$, by (4.4) we obtain $\lim_{n\to\infty} q_n = 0$. Hence (4.3) is proved.
Therefore, there exists $N \in \mathbb{N}$ such that

$$q_N < 1. \tag{4.5}$$

Now, let us consider the norm $\|.\|_N : C(J;X) \to \mathbb{R}_0^+$ defined by:

$$\|u\|_N = \max_{t\in J} e^{-Nt}\|u(t)\|, \ \forall u \in C(J;X), \tag{4.6}$$

which is equivalent to the usual norm $\|.\|_\infty$ in $C(J;X)$ (cfr. [10], (5.2.7)).
Let us fix

$$R \geq \frac{K(Q + aQ + a\|\alpha\|_1)}{1 - q_N} \tag{4.7}$$

where K is the constant in $(p1)$, a is the size of the interval J, Q and q_N are respectively from $(gh2)$ and (4.5).

Step 1b: Now, we consider (see (4.7))

$$H_R = \{u \in C(J;X) : \|u\|_N \leq R\}, \tag{4.8}$$

the closed ball in $(C(J;X), \|.\|_N)$.
We show that

$$\Gamma(H_R) \subset H_R. \tag{4.9}$$

Fixed $u \in H_R$ and $y \in \Gamma(u)$, for every $t \in J$, we have

$$e^{-Nt}\|y(t)\| \leq e^{-Nt}\|C(t,0)g(u)\| + e^{-Nt}\|S(t,0)h(u)\| +$$
$$+ e^{-Nt}\int_0^t \|S(t,\xi)f(\xi)\|d\xi$$

Then by using $(p1)$, $(p2)$, $(F3)$, $(gh2)$ and (4.6) we have

$$e^{-Nt}\|y(t)\| \leq \|C(t,0)\|_{\mathcal{L}(X)}\|g(u)\| + \|S(t,0)\|_{\mathcal{L}(X)}\|h(u)\| +$$
$$+ e^{-Nt}\int_0^t \|S(t,\xi)\|_{\mathcal{L}(X)}\|f(\xi)\|d\xi \leq$$
$$\leq K(Q + aQ + a\|\alpha\|_1) + e^{-Nt}Ka\int_0^t \alpha(\xi)\|u(\xi)\|d\xi =$$
$$= K(Q + aQ + a\|\alpha\|_1) + e^{-Nt}Ka\int_0^t e^{N\xi}\alpha(\xi)e^{-N\xi}\|u(\xi)\|d\xi \leq$$
$$\leq K(Q + aQ + a\|\alpha\|_1) + \|u\|_N Ka\int_0^t e^{-N(t-\xi)}\alpha(\xi)d\xi.$$

So, recalling that $u \in H_R$, by (4.8), (4.7) and ((4.2) for n=N) we obtain

$$e^{-Nt}\|y(t)\| \leq K(Q + aQ + a\|\alpha\|_1) + R\int_0^t Ka\, e^{-N(t-\xi)}\alpha(\xi)d\xi \leq$$
$$\leq K(Q + aQ + a\|\alpha\|_1) + Rq_N \leq R.$$

Now, by (4.6), we have

$$\|y\|_N \leq R,$$

hence $y \in H_R$. Therefore (4.9) is true.

Step 2: In order to prove the existence of a mild solution for (2.6) it is enough to have the existence of a fixed point for the restriction $\Gamma_{|_{H_R}}$ (shortly Γ_R), i.e. (see (4.9)) for the map

$$\Gamma_R : H_R \to \mathcal{P}_c(H_R). \tag{4.10}$$

To this aim, we will show that Γ_R satisfies all the hypotheses of Theorem 2.5, where the Banach space considered is $(C(J;X), \|.\|_\infty)$ (shortly $C(J;X)$). Obviously H_R, which is a closed ball in the space $(C(J;X), \|.\|_N)$, is a closed and convex subset of $C(J;X)$.

Step 2a: The integral multioperator Γ_R has closed graph.

Let $(u_n)_n$ be a sequence in H_R such that $u_n \to \bar{u}$ and let $(z_n)_n$ be a sequence such that $z_n \in \Gamma(u_n)$, $\forall n \in \mathbb{N}$, and $z_n \to \bar{z}$ in $C(J;X)$.

Moreover, let $(f_n)_n$ be a sequence such that, for every $n \in \mathbb{N}$, $f_n \in S^1_{F(.,u_n(.))}$, and

$$z_n(t) = C(t,0)g(u_n) + S(t,0)h(u_n) + \int_0^t S(t,\xi)f_n(\xi)d\xi, \text{ for all } t \in J \tag{4.11}$$

Now, let us note that the set $\{f_n\}_n$ is integrably bounded. This follows from the boundness of the set $\{u_n\}_n$ in $C(J,X)$ and from $(F3)$.

Furthermore, let us show that the set $\{f_n(t)\}_n$ is relatively compact in X for a.e. $t \in J$. Indeed, by using $(F4)$ and the monotonicity of the Hausdorff MNC, for a.e. $t \in J$, being $\{u_n(t)\}_n \in \mathcal{P}_b(X)$, we can write the extimate

$$\eta(\{f_n(t)\}_n) \le \eta(F(t,\{u_n(t)\}_n)) \le m(t)\eta(\{u_n(t)\}_n). \tag{4.12}$$

Next, since the set $\{u_n(t)\}_n$ is relatively compact in X, from (4.12), we have $\eta(\{f_n(t)\}_n) = 0$, i.e. the set $\{f_n(t)\}_n$ is relatively compact.

Now, we can use [[10], Proposition 4.2.1] to conclude that the set $\{f_n\}_n$ is weakly compact in $L^1(J;X)$, so w.l.o.g. we can assume $f_n \rightharpoonup \bar{f}$ in $L^1(J;X)$.

Then, in virtue of Theorem 3.2 and Remark 3.3 we can say that the fundamental Cauchy operator satisfies $(G_S1)'$ and (G_S2). Therefore, since the set $\{f_n\}_n$ is semicompact we can apply [[10], Theorem 5.1.1] and deduce

$$G_S f_n \to G_S \bar{f} \text{ in } C(J;X). \tag{4.13}$$

Moreover, fixed $t \in J$, since $C(t,0), S(t,0) \in \mathcal{L}(X)$ and g, h are continuous in $C(J;X)$ (see hypothesis $(gh1)$), we have:

$$C(t,0)g(u_n) \to C(t,0)g(\bar{u}), \text{ per } n \to \infty \tag{4.14}$$

$$S(t,0)h(u_n) \to S(t,0)h(\bar{u}), \text{ per } n \to \infty \tag{4.15}$$

Hence, by passing to the limit in (4.11), from (4.13), (4.14)), (4.15) we have

$$\lim_{n\to\infty} z_n(t) = \lim_{n\to\infty} [C(t,0)g(u_n) + S(t,0)h(u_n) + G_S f_n(t)] =$$

$$= C(t,0)g(\bar{u}) + S(t,0)h(\bar{u}) + G_S\bar{f}(t)$$

Now, the uniqueness of the limit algorithm guarantees that (see (3.1)):

$$\bar{z}(t) = C(t,0)g(\bar{u}) + S(t,0)h(\bar{u}) + \int_0^t S(t,\xi)\bar{f}(\xi)d\xi, \text{ for every } t \in J.$$

By [[10], Lemma 5.1.1], we have that $\bar{f} \in S^1_{F(.,\bar{u}(.))}$, hence we can conclude that $\bar{z} \in \Gamma_R(\bar{u})$. Therefore, Γ_R has closed graph.
Step 2b: For every $l \in \mathbb{N}$ we can consider the real number

$$p_l := \max_{t\in J} \int_0^t 2Kae^{-l(t-s)}m(s)ds \tag{4.16}$$

where K, a, m are respectively from $(p1)$, (2.6) and $(F4)$. By means of similar arguments as the ones used to prove (4.5), we can choose $l = L$ large enough so that

$$p_L < 1. \tag{4.17}$$

In correspondence to such an L we consider the vectorial MNC ν_L on $C(J;X)$ defined in (2.8).
Next, we prove that the integral multioperator Γ_R is ν_L-condensing.
First, by (4.9) the equivalence of the norms $\|.\|_\infty$ and $\|.\|_N$ implies the boundness of the set $\Gamma_R(H_R)$ in $(C(J;X), \|.\|_\infty)$ Therefore, condition (I) of ν_L-consensivity holds. Now we show that condition (II) is satisfied too. So let $\Omega \subset H_R$ be a bounded set such that

$$\nu_L(\Omega) \preccurlyeq \nu_L(\Gamma_R(\Omega)), \tag{4.18}$$

we will prove that $\nu_L(\Omega) = 0_2$.
Recalling that $\nu_L(\Gamma_R(\Omega))$ is a maximum (see (2.8)), we consider the countable set $\{y_n\}_n \subset \Gamma_R(\Omega)$ which achieves that maximum. Let now $\{u_n\}_n \subset \Omega$ be a set such that $y_n \in \Gamma_R(u_n)$, $n \in \mathbb{N}$. Moreover, for every $n \in \mathbb{N}$, by (4.1), (4.10), (3.1) there exists $f_n \in S^1_{F(.,u_n(.))}$ such that

$$y_n(t) = C(t,0)g(u_n) + S(t,0)h(u_n) + G_S f_n(t),\ t \in J. \tag{4.19}$$

Of course, since (4.18) holds, we have (see (2.8))

$$(\tau(\{u_n\}_n), \lambda(\{u_n\}_n)) \preccurlyeq \nu_L(\Omega) \preccurlyeq \nu_L(\Gamma_R(\Omega)) = (\tau(\{y_n\}_n), \lambda(\{y_n\}_n)). \tag{4.20}$$

First of all, from the above relation we have the inequality

$$\tau(\{u_n\}_n) \leq \tau(\{y_n\}_n). \tag{4.21}$$

Let us estimate (cf. (2.9))

$$\tau(\{y_n\}_n) = \sup_{t\in J} e^{-Lt}\eta(\{y_n(t)\}_n). \tag{4.22}$$

Fixed $t \in J$, by using (4.19), $(p1)$ and $(p2)$ of the fundamental system and the properties of η, we have

$$\eta(\{y_n(t)\}_n) \leq \eta(\{C(t,0)g(u_n)\}_n) + \eta(\{S(t,0)h(u_n)\}_n) + \eta(\{G_S f_n(t)\}_n) \leq$$

$$\leq K\eta(g(\{u_n\}_n)) + Ka\eta(h(\{u_n\}_n)) + \eta(\{G_S f_n(t)\}_n). \tag{4.23}$$

Being $\{u_n\}_n$ a bounded set, from hypothesis $(gh1)$ we can deduce that $\eta(g(\{u_n\})_n) = 0$ and $\eta(h(\{u_n\})_n) = 0$. Therefore, by (4.23) we have

$$\eta(\{y_n(t)\}_n) \leq \eta(\{G_S f_n(t)\}_n). \tag{4.24}$$

In order to apply Theorem 4.2.2 of [10], we first note that the boundness of $\{u_n\}_n$ in $C(J;X)$ and $(F3)$ imply that the set $\{f_n\}_n$ is integrably bounded. Moreover by $(F4)$, for a.e. $s \in J$, we have

$$\eta(\{f_n(s)\}_n) \leq \eta(F(s,\{u_n(s)\}_n)) \leq m(s)\eta(\{u_n(s)\}_n) \leq$$
$$\leq e^{Ls}m(s)\sup_{\xi\in J} e^{-L\xi}\eta(\{u_n(\xi)\}_n) = e^{Ls}m(s)\tau(\{u_n\}_n) =: v(s) \tag{4.25}$$

where obviously $v \in L^1_+(J)$.
On the other hand, by using Theorem 3.2 we know that the fundamental Cauchy operator G_S satisfies (G_S1) and (G_S2). Now we are in the position to apply Theorem 4.2.2 of [10], so we get (cf. (4.25)):

$$\eta(\{G_S f_n(t)\}_n) \leq 2Ka\int_0^t v(s)ds = 2Ka\tau(\{u_n\}_n)\int_0^t e^{Ls}m(s)ds,\ \forall t\in J, \tag{4.26}$$

hence, by (4.24) and (4.26) we have

$$\eta(\{y_n(t)\}_n) \leq 2Ka\tau(\{u_n\}_n)\int_0^t e^{Ls}m(s)ds,\ \forall t\in J.$$

From this last inequality, remembering (2.9) and (4.16) with $l = L$, we obtain

$$\tau(\{y_n\}_n) \leq \sup_{t\in J}\,[2Ka\tau(\{u_n\}_n)\int_0^t e^{-L(t-s)}m(s)ds] \leq p_L\tau(\{u_n\}_n) \tag{4.27}$$

Therefore (4.21) and (4.27) imply

$$\tau(\{u_n\}_n) \leq \tau(\{y_n\}_n) \leq p_L\tau(\{u_n\}_n), \tag{4.28}$$

and so, since $p_L < 1$ (4.17), we achieve

$$\tau(\{u_n\}_n) = 0. \tag{4.29}$$

By (4.28) we also deduce

$$\tau(\{y_n\}_n) = 0. \tag{4.30}$$

Now we show that (cf. (2.10))

$$\lambda(\{y_n\}_n) = mod_C(\{y_n\}_n) = 0 \tag{4.31}$$

To this aim, we prove that $mod_C(\{y_n\}_n) = 0$. Indeed, from (4.28) and (2.9), we have that

$$\eta(\{u_n(t)\}_n) = 0,\ \forall t\in J.$$

Moreover, the set $\{f_n\}_n$ is semicompact since it is integrably bounded and $\eta(\{f_n(t)\}_n) = 0$, for a.e. $t \in J$ (see (4.25) and (4.29)). Therefore, recalling again that G_S satisfies properties (G_S1) and (G_S2), we can apply Theorem 5.1.1 of [10] so that the set $\{G_S f_n\}_n$ is relatively compact in $C(J;X)$. Clearly, if a subset of $C(J;X)$ is relatively compact, then its elements constitute an equicontinuous family on J. Hence, fixed $\varepsilon > 0$, there exists $\delta_1(\varepsilon) = \delta_1(\frac{\varepsilon}{3}) > 0$ such that for every $t_1, t_2 \in J$, $\mid t_1 - t_2 \mid < \delta_1(\varepsilon)$ we have

$$\|G_S f_n(t_2) - G_S f_n(t_1)\| < \frac{\varepsilon}{3},\ \forall n\in\mathbb{N}. \tag{4.32}$$

In addition, put $\delta_2(\varepsilon) =: \frac{\varepsilon}{3Qmax\{K^*,K_1\}}$ we have (see $(p4)$, $(p5)$, (2.7) and $(gh2)$)

$$\|C(t_2,0)g(u_n) - C(t_1,0)g(u_n)\| \leq Q\ max\{K^*, K_1\}|t_2 - t_1| < \frac{\varepsilon}{3}, \tag{4.33}$$

$$\|S(t_2,0)h(u_n) - S(t_1,0)h(u_n)\| \leq Q\ max\{K^*, K_1\}|t_2 - t_1| < \frac{\varepsilon}{3}, \tag{4.34}$$

for all $t_1, t_2 \in J$, $| t_2 - t_1 |< \delta_2(\varepsilon)$, $\forall n \in \mathbb{N}$.
Now, fixed $\delta(\varepsilon) = min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\} > 0$, by (4.32), (4.33) and (4.34) we can deduce that, for every $t_1, t_2 \in J$ such that $| t_1 - t_2 |< \delta(\varepsilon)$ we can say

$$\|y_n(t_2) - y_n(t_1)\| \leq \|C(t_2,0)g(u_n) - C(t_1,0)g(u_n)\| +$$
$$+\|S(t_2,0)h(u_n) - S(t_1,0)h(u_n)\| + \|G_S f_n(t_2) - G_S f_n(t_1)\| < \varepsilon,$$

for all $n \in \mathbb{N}$, i.e., the set $\{y_n\}_n$ is equicontinuous on J. So we conclude (see (2.10)):

$$\lambda(\{y_n\}_n) = mod_C(\{y_n\}_n) = 0 \tag{4.35}$$

From (4.20), by using (4.30) and (4.35) we deduce $\nu_L(\Omega) = 0_2$.
Hence, condition (II) of ν_L-condensity is verified too, therefore Γ_R is ν_L-condensing.
Step 3: Finally we are in the position to apply Theorem 2.5. Hence the multioperator Γ_R has a fixed point in H_R, i.e. there exists $x \in H_R$ such that

$$x(t) = C(t,0)g(x) + S(t,0)h(x) + \int_0^t S(t,s)f(s)ds,\ t \in J$$

where $f \in S^1_{F(.,x(.))}$. Of course, x is a mild solution for (2.6).

5. An application

Now we apply the result established in the preceding section to study the controllability of the following non-autonomous wave equation with initial conditions

$$\begin{cases} \frac{\partial^2 w(t,\xi)}{\partial t^2} = \frac{\partial^2 w(t,\xi)}{\partial \xi^2} + b(t)\frac{\partial w(t,\xi)}{\partial \xi} + f(t,\ w(t,\xi), u(t,\xi)),\ t \in J \\ w(t,0) = w(t,\ 2\pi),\ t \in J, \\ \frac{\partial w(t,0)}{\partial \xi} = \frac{\partial w(t,\ 2\pi)}{\partial \xi},\ t \in J, \\ w(0,\xi) = \sum_{i=1}^m \frac{2\pi t_i}{\xi_i},\ \xi \in \mathbb{R}, \\ \frac{\partial w(0,\xi)}{\partial t} = \sum_{i=1}^m \frac{2\pi}{\xi_i},\ \xi \in \mathbb{R}, \\ u(t,\xi) \in U(t,\ w(t,\xi)),\ t \in J,\ \xi \in \mathbb{R} \end{cases} \tag{5.1}$$

where $0 < t_1 < ... < t_i < ... < t_m < a$ and $0 < \xi_1 < ... < \xi_i < ... < \xi_m < 2\pi$, $f : J \times \mathbb{C} \times \mathbb{C} \to \mathbb{R}$, $b : J \to \mathbb{R}$ and $U : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$.

First, we fix the Banach space $X = L^2(\mathbb{T}, \mathbb{C})$, where $\mathbb{T}$ is the quotient group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ of all $2\pi-$periodic $2-$integrable functions. As in [8], we will use the identification between the functions defined on $\mathbb{T}$ and the $2\pi-$periodic functions from $\mathbb{R}$ to $\mathbb{C}$. Next, put the Sobolev space

$$H^2(\mathbb{T},\mathbb{C}) = \{X_x : \mathbb{T} \to \mathbb{C} : X_x([\xi]) = x(\xi),\ x : \mathbb{R} \to \mathbb{C}\ is\ 2\pi - \text{periodic} :$$
$$\exists\ x'_{[0,2\pi]}, x''_{[0,2\pi]} \in L^2([0,2\pi],\mathbb{C})\},$$

provided by the norm

$$\|X_x\|_{H^2(\mathbb{T},\mathbb{C})} = \|x\|_{L^2([0,2\pi],\mathbb{C})} + \|x'\|_{L^2([0,2\pi],\mathbb{C})} + \|x''\|_{L^2([0,2\pi],\mathbb{C})}, \tag{5.2}$$

we consider the operator $A_0 : D(A_0) = H^2(\mathbb{T},\mathbb{C}) \to L^2(\mathbb{T},\mathbb{C})$ so defined

$$A_0 X_x = \frac{d^2}{d\xi} x, \quad X_x \in H^2(\mathbb{T},\mathbb{C})$$

which is the infinitesimal generator of a strongly continuous cosine family $\{C_0(t)\}_{t\in\mathbb{R}}$, where $C_0(t) : L^2(\mathbb{T},\mathbb{C}) \to L^2(\mathbb{T},\mathbb{C})$, for every $t \in \mathbb{R}$ (see [8]). Moreover, we fix the function $P : J \to \mathcal{L}(H^1(\mathbb{T},\mathbb{C}), L^2(\mathbb{T},\mathbb{C}))$ defined in this way

$$P(t)X_x = b(t)\frac{dX_x}{d\xi}, \quad t \in J, \;\; X_x \in H^1(\mathbb{T},\mathbb{C}).$$

where we assume that the function $b : J \to \mathbb{R}$ of (5.1) is C^1 on J. Now we are in the position to define the family $\{A(t) : t \in J\}$, where, for every $t \in J$, $A(t) : H^2(\mathbb{T},\mathbb{C}) \to L^2(\mathbb{T},\mathbb{C})$, is an operator so defined

$$A(t) = A_0 + P(t), \quad t \in J. \tag{5.3}$$

In [7] Henriquez has proved that this family generates a fundamental system $\{S(t,s)\}_{t,s\in J}$, which is compact (see [8], Lemma 4.1).

On the function f we assume that $\tilde{f} : J \times H^2(\mathbb{T},\mathbb{C}) \times L^2(\mathbb{T},\mathbb{C}) \to H^2(\mathbb{T},\mathbb{C})$ so defined

$$\tilde{f}(t,\ x,\ u)([\xi]) = \ f(t,\ x(\xi),\ u(\xi)),\ t \in J,\ x \in H^2(\mathbb{T},\mathbb{C}),\ u \in L^2(\mathbb{T},\mathbb{C})\ ,\ [\xi] \in \mathbb{T}, \tag{5.4}$$

satisfies the following properties:

$(f1)$ for every $x \in H^2(\mathbb{T},\mathbb{C})$, $\;u \in L^2(\mathbb{T},\mathbb{C})$, $\tilde{f}(.,x,u)$ is B-measurable;

$(f2)$ for a.e. $t \in J$, $\tilde{f}(t,.,.)$ is continuous;

$(f3)$ there exists $k \in L^1_+(J)$:

$$\|\tilde{f}(t,x_1,u) - \tilde{f}(t,x_2,u)\|_{H^2([0,2\pi],\mathbb{C})} \leq \ k(t)\|x_1 - x_2\|_{H^2([0,2\pi],\mathbb{C})},$$

for a.e. $t \in J,\ x_1, x_2 \in H^2(\mathbb{T},\mathbb{C}),\ u \in L^2(\mathbb{T},\mathbb{C})$;

Moreover we require that the multimap $\tilde{U} : J \times H^2(\mathbb{T},\mathbb{C}) \to \mathcal{P}(L^2(\mathbb{T},\mathbb{C}))$ so defined

$$\tilde{U}(t,\ x)([\xi]) = \ U(t,\ x(\xi)),\ t \in J,\ x \in H^2(\mathbb{T},\mathbb{C}),\ [\xi] \in \mathbb{T}, \tag{5.5}$$

satisfies the conditions

(U0) for every $t \in J$, $\;x \in H^2(\mathbb{T},\mathbb{C})$, $\tilde{U}(t,x)$ is compact;

(U1) for every $x \in H^2(\mathbb{T},\mathbb{C})$, $\tilde{U}(.,x)$ is measurable;

(U2) for a.e. $t \in J$, $\tilde{U}(t,.)$ is upper semicontinuous;

(U3) $\tilde{U}$ is superpositionally measurable, i.e. for every measurable multimap V : $H^2(\mathbb{T},\mathbb{C}) \to \mathcal{P}_k(L^2(\mathbb{T},\mathbb{C}))$ the multimap $\tilde{U}(.,V(.))$ is measurable;

(U4) $\tilde{f}(t,\ x,\ \tilde{U}(t,x))$ is convex, $\;t \in J,\ x \in H^2(\mathbb{T},\mathbb{C})$;

(U5) there exists $\alpha \in L^1_+(J)$ such that

$$\|\tilde{f}(t,x,\tilde{U}(t,x))\|_{L^2([0,2\pi],\mathbb{C})} \ \leq \ \alpha(t)(1 + \|x\|_{H^2([0,2\pi],\mathbb{C})}),$$

for a.e. $t \in J$, for all $x \in H^2(\mathbb{T},\mathbb{C})$;

(U6) for every $t \in J$, $x \in H^2(\mathbb{T},\mathbb{C})$ and for any bounded $\Omega \subset H^2(\mathbb{T},\mathbb{C})$, the set $\tilde{f}(t,\ x,\ \tilde{U}(t,\Omega))$ is compact in $H^2(\mathbb{T},\mathbb{C})$;

Now we introduce the map $F : J \times H^2(\mathbb{T},\mathbb{C}) \to \mathcal{P}(H^2(\mathbb{T},\mathbb{C}))$ so defined

$$F(t,\ x) = \tilde{f}(t,\ x,\ \tilde{U}(t,x)), \quad t \in J,\ \ x \in H^2(\mathbb{T},\mathbb{C}), \tag{5.6}$$

and the maps $g : C(J; H^2(\mathbb{T},\mathbb{C})) \to H^2(\mathbb{T},\mathbb{C})$ and $h : C(J; H^2(\mathbb{T},\mathbb{C})) \to H^2(\mathbb{T},\mathbb{C})$ respectively defined in the following way

$$g(x)([\xi]) = \sum_{i=0}^{m} \frac{2\pi t_i}{\xi_i}, \quad [\xi] \in \mathbb{T},\ x \in C(J; H^2(\mathbb{T},\mathbb{C})); \tag{5.7}$$

$$h(x)([\xi]) = \sum_{i=0}^{m} \frac{2\pi}{\xi_i}, \ \ [\xi] \in \mathbb{T}, \quad x \in C(J; H^2(\mathbb{T},\mathbb{C})). \tag{5.8}$$

The previous arguments lead to revise a function $w : J \times \mathbb{R} \to \mathbb{C}$ such that

$$w(t,.)\ \ 2\pi - \text{periodic}, \quad t \in J$$
$$w(t,.)\ |_{[0,2\pi]} \in L^2([0,2\pi],\mathbb{C}), \quad t \in J$$

as $x : J \to L^2(\mathbb{T},\mathbb{C})$ so defined

$$x(t)([\xi]) = w(t,\xi), \quad t \in J, \ \ [\xi] \in \mathbb{T}. \tag{5.9}$$

Hence we can rewrite problem (5.1) in the form

$$\begin{cases} x''(t) \in [x(t)]'' + b(t)[x(t)]' + F(t,x(t)),\ t \in J \\ x(t)([0]) = x(t)([2\pi]), \ \ [x(t)]'([0]) = [x(t)]'([2\pi]), \quad t \in J \\ x(0) = g(x) \\ x'(0) = h(x) \end{cases} \tag{5.10}$$

First, we note that the conditions $(U4)$ and $(U6)$ imply respectively that the multimap F (see (5.6)) takes convex and compact values. Moreover, by $(U1)$ we can say that, for every $x \in H^2(\mathbb{T},\mathbb{C})$, the multimap $Q_x : J \to \mathcal{P}(H^2(\mathbb{T},\mathbb{C}) \times L^2(\mathbb{T},\mathbb{C}))$ defined as $Q_x(t) = \{x\} \times \tilde{U}(t,x)$, $t \in J$ is measurable. Hence from conditions $(f1)$, $(f2)$ we have that $F(.,x) = \tilde{f}(.,\ Q_x(.))$ is measurable. Now by using the classical Kuratowski Ryll-Nardzewski measurable selection theorem we can conclude that the hypothesis $(F1)$ is fulfilled. On the other hand, from $(U2)$ we have that, for a.e. $t \in J$, the multimap $V_t : H^2(\mathbb{T},\mathbb{C}) \to \mathcal{P}(H^2(\mathbb{T},\mathbb{C}) \times L^2(\mathbb{T},\mathbb{C}))$, $V_t(x) = \{x\} \times \tilde{U}(t,x)$, $x \in H^2(\mathbb{T},\mathbb{C})$, is upper semicontinuous and so, taking into account of $(f2)$, Theorem 1.2.8 of [10] implies that $(F2)$ holds. Moreover, by using $(U5)$ we deduce that F has the property $(F3)$.

Next we prove that the multimap F satisfies the condition $(F4)$. Fixed $t \in J$ such that the property expressed in $(f3)$ holds, we consider the multimap $B_t : H^2(\mathbb{T},\mathbb{C}) \times H^2(\mathbb{T},\mathbb{C}) \to \mathcal{P}(H^2(\mathbb{T},\mathbb{C}))$ defined as $B_t(x,y) = \tilde{f}(t,\ y,\ \tilde{U}(t,x))$, $x, y \in H^2(\mathbb{T},\mathbb{C})$. Then, we fix $x,\ y_1,\ y_2 \in H^2(\mathbb{T},\mathbb{C})$ and let $b_1 = \tilde{f}(t,\ y_1,\ u)$ and $b_2 = \tilde{f}(t,\ y_2,\ u) \in B_t(x,y_2)$, where $u \in \tilde{U}(t,x)$. From $(f3)$ there exists $k \in L^1_+(J)$:

$$\begin{aligned} \|b_2 - b_1\|_{H^2([0,2\pi],\mathbb{C})} &= \|\tilde{f}(t,y_2,u) - \tilde{f}(t,y_1,u)\|_{H^2([0,2\pi],\mathbb{C})} \\ &\le k(t)\|y_2 - y_1\|_{H^2([0,2\pi],\mathbb{C})}, \end{aligned}$$

by which we can deduce that the multimap $B_t(x,.)$ is $k(t)$-Lipsichitz with respect to the Haudorff metric. Moreover we also can note that $(U6)$ allows to say that, for every bounded subset Ω of $H^2(\mathbb{T},\mathbb{C})$, the set $B_t(\Omega \times \{y\}) = \tilde{f}(t,\ y,\ \tilde{U}(t,\ \Omega))$ is compact in $H^2(\mathbb{T},\mathbb{C})$. Therefore all hypotheses of Proposition 2.2.2 of [10] are satisfied, hence we have

$$\eta(F(t,\ \Omega)) = \eta(\tilde{f}(t,\ \Omega \times\ \tilde{U}(t,\Omega)) = \eta(B_t(\Omega \times \Omega)) \leq k(t)\eta(\Omega)$$

where η is the Hausdorff MNC in $H^2(\mathbb{T},\mathbb{C})$.
Then we can conclude that $(F4)$ holds.
Finally, obviously the maps g and h have the properties $(gh1)$ and $(gh2)$ required in our existence theorem. Then from Theorem 4.1 we can deduce that there exists a continuous function $\hat{x} : J \to H^2(\mathbb{T},\mathbb{C})$ that is a *mild solution* for (5.10), i.e.

$$\hat{x}(t) = -\frac{\partial}{\partial s}S(t,s)\mid_{s=0} g(u) + S(t,0)h(u) + \int_0^t S(t,\xi)q(\xi)d\xi, \quad t \in J, \tag{5.11}$$

where $q \in S^1_{F(.,\hat{x}(.))} = \{p \in L^1(J;X) : p(t) \in F(t,\hat{x}(t)) \text{ a.e. } t \in J\}$.
Now, since $\tilde{U}$ is superpositionally B-measurable (see (U3)), the multimap $Q : J \to \mathcal{P}(H^2(\mathbb{T},\mathbb{C}) \times L^2(\mathbb{T},\mathbb{C}))$ so defined $Q(t) = \{(\hat{x}(t),\ \tilde{U}(t,\hat{x}(t))\}$, $t \in J$, having compact values (see $(U0)$), is strongly measurable. Moreover, we recall that the multimap F takes compact values in $H^2(\mathbb{T},\mathbb{C})$ and that it has the properties $(F1)$ and $(F2)$. Hence, we are in the position to apply the Filippov implicit function lemma in the version furnished in ([15], Corollary 1.15). Then we can say that there exists a Bochner-measurable selection $\hat{u} : J \to L^2(\mathbb{T},\mathbb{C})$ of the multimap $F(.,Q(.)) = \tilde{f}(.,\ \hat{x}(.),\ U(t,\hat{x}(.)))$. At this point, by considering the following functions $w : J \times \mathbb{R} \to \mathbb{C}$ and $u : J \times \mathbb{R} \to \mathbb{C}$ so defined

$$w(t,\xi) = \hat{x}(t)([\xi]), \quad t \in J, \quad \xi \in \mathbb{R}$$
$$u(t,\xi) = \hat{u}(t)([\xi]), \quad t \in J, \quad \xi \in \mathbb{R},$$

we can conclude that $\{w,u\}$ is an admissible mild-pair for the problem (5.1).

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References

[1] Ambrosetti, A., *Un teorema di esistenza per le equazioni differenziali negli spazi di Banach*, Rend. Sem. Univ. Padova, **39**(1967), 349-360.

[2] Bungardi, S., Cardinali, T., Rubbioni, P., *Semilinear nonlocal integro-differential inclusions via vectorial measures of noncompactness*, In press on Appl. Anal. (2016), http://dx.doi.org/10.1080/00036811.2016.1227969.

[3] Couchouron, J.F., Kamenski, M., *An abstract topological point of view averaging principle in the theory of differential inclusions*, Nonlinear Analysis, **42**(2000), 1101-1129.

[4] Cardinali, T., Rubbioni, P., *On the existence of mild solutions of semilinear evolution differential inclusions*, J. Math. Anal. Appl., **308**(2005), no. 2, 620-635.

[5] Denkowski, Z., Migorski, S., Papageorgiou, N.S., *An Introduction to Nonlinear Analysis*, Theory, Kluwer Acad. Publ. Boston/ Dordrecht/ London, 2003.

[6] Fattorini, H.O., *Second Order Linear Differential Equations in Banach Spaces*, North-Holland Publishing Co., Amsterdam, 1985.

[7] Henríquez, H.R., *Existence of solutions of non-autonomous second order differential equations with infinite delay*, Nonlinear Anal., **74**(10)(2011), 3333-3352.

[8] Henríquez, H.R., Poblete, V., Pozo, J.C., *Mild solutions of nonautonomous second order problems with nonlocal initial conditions*, J. Math. Anal. Appl., **412**(2014), 1064-1083.

[9] Kozak, M., *A fundamental solution of a second-order differential equation in Banach space*, Univ. Iagel. Acta Math., **32**(1995), 275-289.

[10] Kamenskii, M., Obukhovskii, V., Zecca, P., *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, De Gruyter Ser. Nonlinear Anal. Appl. 7, Walter de Gruyter, Berlin, 2001.

[11] Kisynski, J., *On cosine operator functions and one-parameter groups of operators*, Studia Math., **44**(1972), 93-105.

[12] Serizawa, H., Watanabe, M., *Time-dependent perturbation for cosine families in Banach spaces*, J. Math. 12, Houston, (1986), 579-586.

[13] Travis, C.C., Webb, G.F., *Second order differential equations in Banach space*, Nonlinear Equations in Abstract Spaces, Academic Press, New York, 1978.

[14] Vasilev, V.V., Piskarev, S.I., *Differential equations in Banach spaces, II, Theory of cosine operator functions*, J. Math. Sci. (N.Y.), **122**(2004), 7047-7060.

[15] Zvyagin, V., Obukhovskii, V., Zvyagin, A., *On inclusions with multivalued operators and their applications to some optimization problem*, J. Fixed Point Theory Appl., **16**(2014), 27-82.

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Determinantal inequalities for J-accretive dissipative matrices

Natália Bebiano and João da Providência

Abstract. In this note we determine bounds for the determinant of the sum of two J-accretive dissipative matrices with prescribed spectra.

Mathematics Subject Classification (2010): 46C20, 47A12.

Keywords: J-accretive dissipative matrix, J-selfadjoint matrix, indefinite inner norm.

1. Results

Consider the complex n-dimensional space $\mathbf{C}^n$ endowed with the indefinite inner product

$$[x, y]_J = y^* Jx, \quad x, y \in \mathbf{C}^n,$$

where $J = I_r \oplus -I_{n-r}$, and corresponding J-norm

$$[x, x]_J = |x_1|^2 + \ldots + |x_r|^2 - |x_{r+1}|^2 - \ldots - |x_n|^2.$$

In the sequel we shall assume that $0 < r < n$, except where otherwise stated.

The J-*adjoint* of $A \in \mathbf{C}^{n\times n}$ is defined and denoted as

$$[A^{\#}x, x] = [x, Ax]$$

or, equivalently, $A^{\#} := JA^*J$, [4]. The matrix A is said to be J-*Hermitian* if $A^{\#} = A$, and is J-*positive definite* (*semi-definite*) if JA is positive definite (semi-definite). This kind of matrices appears on Quantum Physics and in Symplectic Geometry [10]. An arbitrary matrix $A \in \mathbf{C}^{n\times n}$ may be uniquely written in the form

$$A = \mathrm{Re}^J A + i\mathrm{Im}^J A,$$

where

$$\mathrm{Re}^J A = (A + A^{\#})/2, \ \mathrm{Im}^J A = (A - A^{\#})/(2i)$$

are J-Hermitian. This is the so-called J-Cartesian decomposition of A. J-Hermitian matrices share properties with Hermitian matrices, but they also have important differences. For instance, they have real and complex eigenvalues, these occurring in conjugate pairs. Nevertheless, the eigenvalues of a J-positive matrix are all real, being r positive and $n-r$ negative, according to the J-norm of the associated eigenvectors being positive or negative. A matrix A is said to be *J-accretive* (resp. *J-dissipative*) if $J\mathrm{Re}^J A$ (resp. $J\mathrm{Im}^J A$) is positive definite. If both matrices $J\mathrm{Re}^J A$ and $J\mathrm{Im}^J A$ are positive definite the matrix is said to be *J-accretive dissipative.* We are interested in obtaining determinantal inequalities for J-accretive dissipative matrices. Determinantal inequalities have deserved the attention of researchers, [2], [3], [5]-[9], [11].

Throughout, we shall be concerned with the set

$$D^J(A,C)=\{\det(A+VCV^{\#}):V\in\mathcal{U}(r,n-r)\},$$

where $A,C\in\mathbf{C}^{n\times n}$ are J-unitarily diagonalizable with prescribed eigenvalues and $\mathcal{U}(r,n-r)$ is the group of J-unitary transformations in $\mathbf{C}^n$ (V is *J-unitary* if $VV^{\#}=I$), [12]. The so-called J-unitary group is connected, nevertheless it is not compact. As a consequence, $D^J(A,C)$ is connected. This set is invariant under the transformation $C\to UCU^{\#}$ for every J-unitary matrix U, and, for short, $D^J(A,C)$ is said to be J-unitarily invariant.

In the sequel we use the following notation. By S_n we denote the symmetric group of degree n, and we shall also consider

$$S_n^r=\{\sigma\in S_n:\sigma(j)=j,\ j=r+1,\dots,n\},\tag{1.1}$$

$$\hat{S}_n^r=\{\sigma\in S_n:\sigma(j)=j,\ j=1,\dots,r\}.\tag{1.2}$$

Let $\alpha_j,\ \gamma_j\in\mathbf{C}$, $j=1,\dots,n$ denote the eigenvalues of A and C, respectively. The $r!(n-r)!$ points

$$z_\sigma=z_{\xi\tau}=\prod_{j=1}^{r}(\alpha_j+\gamma_{\xi(j)})\prod_{j=r+1}^{n}(\alpha_j+\gamma_{\tau(j)}),\ \xi\in S_n^r,\ \tau\in\hat{S}_n^r.\tag{1.3}$$

belong to $D^J(A,C)$.

The purpose of this note, which is in the continuation of [1], is to establish the following results.

Theorem 1.1. *Let $J=I_r\oplus -I_{n-r}$, and A and C be J- positive matrices with prescribed real eigenvalues*

$$\alpha_1\geq\ldots\geq\alpha_r>0>\alpha_{r+1}\geq\ldots\geq\alpha_n\tag{1.4}$$

and

$$\gamma_1\geq\ldots\geq\gamma_r>0>\gamma_{r+1}\geq\ldots\geq\gamma_n,\tag{1.5}$$

respectively. Then

$$|\det(A+iC)|\geq\left((\alpha_1^2+\gamma_1^2)\ldots(\alpha_n^2+\gamma_n^2)\right)^{1/2}.$$

Corollary 1.2. *Let $J=I_r\oplus -I_{n-r}$, and B be a J-accretive dissipative matrix. Assume that the eigenvalues of $\mathrm{Re}^J B$ and $\mathrm{Im}^J B$ satisfy (1.4) and (1.5), respectively. Then,*

$$|\det(B)|\geq\left((\alpha_1^2+\gamma_1^2)\ldots(\alpha_n^2+\gamma_n^2)\right)^{1/2}.$$

Example 1.3. In order to illustrate the necessity of A and C to be J-positive matrices in Theorem 1.1, let $A = \mathrm{diag}(\alpha_1, \alpha_2)$, $C = \mathrm{diag}(\gamma_1, \gamma_2)$, with $\alpha_1 = \gamma_1 = 1$, $\alpha_2 = 3/2$, $\gamma_2 = -2$, and $J = \mathrm{diag}(1, -1)$. We find $(\alpha_1^2 + \gamma_1^2)(\alpha_2^2 + \gamma_2^2) = 27/2$. However, the minimum of $|\det(A + iVBV^{\#}|^2$, for V ranging over the J-unitary group , is 49/4.

Theorem 1.4. *Let $J = I_r \oplus -I_{n-r}$, and A and C be J-unitary matrices with prescribed eigenvalues*

$$\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_n$$

and

$$\gamma_1 \ldots, \gamma_r, \gamma_{r+1}, \ldots. \gamma_n,$$

respectively. Assume moreover that

$$\frac{\Im\alpha_1}{2(1+\Re\alpha_1)} \leq \ldots \leq \frac{\Im\alpha_r}{2(1+\Re\alpha_r)} < 0 < \frac{\Im\alpha_{r+1}}{2(1+\Re\alpha_{r+1})} \leq \ldots \leq \frac{\Im\alpha_n}{2(1+\Re\alpha_n)} \tag{1.6}$$

and

$$\frac{\Im\gamma_1}{2(1-\Re\gamma_1)} \leq \ldots \leq \frac{\Im\gamma_r}{2(1-\Re\gamma_r)} < 0 < \frac{\Im\gamma_{r+1}}{2(1-\Re\gamma_{r+1})} \leq \ldots \leq \frac{\Im\gamma_n}{2(1-\Re\gamma_n)}. \tag{1.7}$$

Then

$$D^J(A, C) = (\alpha_1 + \gamma_1) \ldots (\alpha_n + \gamma_n)[1, +\infty[\ .$$

We shall present the proofs of the above results in the next section.

2. Proofs

Lemma 2.1. *Let $g : \mathcal{U}(r, n-r) \to \mathbf{R}$ be the real valued function defined by*

$$g(U) = \det(I + A_0^{-1}UC_0JU^*JA_0^{-1}UC_0JU^*J),$$

where $A_0 = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$, $C_0 = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$ and α_i, γ_j satisfy (1.4) and (1.5). Then the set

$$\{U \in \mathcal{U}(r, n-r) : g(U) \leq a\},$$

where

$$a > \prod_{j=1}^{n} \left(1 + \frac{\gamma_j^2}{\alpha_j^2}\right),$$

is compact.

Proof. Notice that $JA_0 > 0,\ JC_0 > 0$, so we may write

$$g(U) = \det(I + WW^*WW^*),$$

where

$$W = (JA_0)^{-1/2}U(JC_0)^{1/2}.$$

The condition $g(U) \leq a$ implies that W is bounded, and is satisfied if we require that $WW^* \leq \kappa I$, for $\kappa > 0$ such that $(1 + \kappa^2)^n \leq a$. Thus, also U is bounded. The result follows by Heine-Borel Theorem. □

Proof of Theorem 1.1

Under the hypothesis, A is nonsingular. Since the determinant is J-unitarily invariant and C is J-unitarily diagonalizable, we may consider $C = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$. We observe that

$$|\det(A+iC)|^2 = \det\left((A+iC)(A-iC)\right) = \left(\prod_{i=1}^{n} \alpha_i\right)^2 \det\left((I+iA^{-1}C)(I-iA^{-1}C)\right)$$

Clearly,

$$\det\left((I+iA^{-1}C)(I-iA^{-1}C)\right) = \det(I + A^{-1}CA^{-1}C).$$

The set of values attained by $|\det(A+iC)|^2$ is an unbounded connected subset of the positive real line. In order to prove the unboundedness, let us consider the J-unitary matrix V obtained from the identity matrix I through the replacement of the entries $(r,r),\ (r+1,r+1)$ by $\cosh\ u$, and the replacement of the entries $(r,r+1),\ (r+1,r)$ by $\sinh\ u$, $u \in R$. We may assume that $A_0 = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$. A simple computation shows that

$$\begin{aligned} |\det(A_0 + iVCV^{\#})|^2 &= \prod_{j=1}^{n} (\alpha_j^2 + \gamma_j^2) \\ &- 2(\alpha_r - \alpha_{r+1})(\gamma_r - \gamma_{r+1})(\alpha_{r+1}\gamma_r + \alpha_r\gamma_{r+1})(\sinh\ u)^2 \\ &+ (\alpha_r - \alpha_{r+1})^2(\gamma_r - \gamma_{r+1})^2(\sinh\ u)^4. \end{aligned}$$

Thus, the set of values attained by $|\det(A_0 + iVCV^{\#})|$ is given by

$$[(\alpha_1^2 + \gamma_1^2)^{1/2} \ldots (\alpha_n^2 + \gamma_n^2)^{1/2}, +\infty[\ .$$

As a consequence of Lemma 2.1, the set of values attained by $|\det(A+iC)|^2$ is closed and a half-ray in the positive real line. So, there exist matrices A, C such that the endpoint of the half-ray is given by $|\det(A+iC)|^2$. Let us assume that the endpoint of this half-ray is attained at $|\det(A+iC)|^2$. We prove that A commutes with C. Indeed, for $\epsilon \in \mathbf{R}$ and an arbitrary J-Hermitian X, let us consider the J-unitary matrix given as

$$\mathrm{e}^{iX} = i + i\epsilon X - \frac{\epsilon^2}{2}X^2 + \ldots.$$

We obtain by some computations

$$\begin{aligned} f(\epsilon) &:= \det(I + A^{-1}\mathrm{e}^{-i\epsilon X}C\mathrm{e}^{i\epsilon X}A^{-1}\mathrm{e}^{-i\epsilon X}C\mathrm{e}^{i\epsilon X}) \\ &= \det(I + A^{-1}CA^{-1}C - i\epsilon(A^{-1}[X,C]A^{-1}C + A^{-1}CA^{-1}[X,C]) + \mathcal{O}(\epsilon^2) \\ &= \det(I + A^{-1}CA^{-1}C) \\ &\times \det\left(I - i\epsilon(I + A^{-1}CA^{-1}C)^{-1}(A^{-1}[X,C]A^{-1}C + A^{-1}CA^{-1}[X,C])\right) + \mathcal{O}(\epsilon^2) \\ &= \det(I + A^{-1}CA^{-1}C) \\ &\times \exp\left(-i\epsilon \mathrm{tr}((I + A^{-1}CA^{-1}C)^{-1}(A^{-1}[X,C]A^{-1}C + A^{-1}CA^{-1}[X,C]))\right) + \mathcal{O}(\epsilon^2), \end{aligned}$$

where $[X,Y]=XY-YX$ denotes the commutator of the matrices X and Y. The function $f(\epsilon)$ attains its minimum at $\det(I+A^{-1}CA^{-1}C)$, if

$$\left.\frac{\mathrm{d}f}{\mathrm{d}\epsilon}\right|_{\epsilon=0}=0.$$

Then we must have

$$\operatorname{tr}\left((I+A^{-1}CA^{-1}C)^{-1}(A^{-1}[X,C]A^{-1}C+A^{-1}CA^{-1}[X,C])\right)=0,$$

for every J-Hermitian X. That is

$$[C,(A^{-1}C(I+A^{-1}CA^{-1}C)^{-1}A^{-1}+(I+A^{-1}CA^{-1}C)^{-1}A^{-1}CA^{-1})]=0,$$

and so, performing some computations, we find

$$\begin{aligned}&[C,(A^{-1}C(I+A^{-1}CA^{-1}C)^{-1}A^{-1}C+(I+A^{-1}CA^{-1}C)^{-1}A^{-1}CA^{-1}C)]\\ &=2\left[C,\ \frac{A^{-1}CA^{-1}C}{I+A^{-1}CA^{-1}C}\right]=2\left[C,\ I-\frac{I}{I+A^{-1}CA^{-1}C}\right]\\ &=-2\left[C,\ \frac{I}{I+A^{-1}CA^{-1}C}\right]=\frac{2I}{I+(A^{-1}C)^2}\left[C,\ (A^{-1}C)^2\right]\frac{I}{I+(A^{-1}C)^2}=0.\end{aligned}$$

Thus

$$[C,(A^{-1}C)^2]=0.$$

Assume that C, which is in diagonal form, has distinct eigenvalues. Then $(A^{-1}C)^2$ is a diagonal matrix as well as $((JA)^{-1}JC)^2$. Furthermore, $((JC)^{1/2}(JA)^{-1}(JC)^{1/2})^2$ is diagonal. Since $(JC)^{1/2}(JA)^{-1}(JC)^{1/2}$ is positive definite, it is also diagonal, and so are $(JA)^{-1}JC$ and $A^{-1}C$. Henceforth, A is also a diagonal matrix and commutes with C. (If C has multiple eigenvalues we can apply a perturbative technique and use a continuity argument).

For $\sigma\in S_n$, such that $\sigma(1),\ldots,\sigma(r)\leq r$, we have

$$(\alpha_1^2+\gamma_{\sigma(1)}^2)\ldots(\alpha_n^2+\gamma_{\sigma(n)}^2)\geq(\alpha_1^2+\gamma_1^2)\ldots(\alpha_n^2+\gamma_n^2).$$

Thus, the result follows. □

In the proof of Theorem 1.4, the following lemma is used (cf. [1, Theorem 1.1]).

Lemma 2.2. *Let B,D be J-positive matrices with eigenvalues satisfying*

$$\beta_1\geq\ldots\geq\beta_r>0>\beta_{r+1}\geq\ldots>\beta_n,$$

and

$$\delta_1\geq\ldots\geq\delta_r>0>\delta_{r+1}\geq\ldots>\delta_n.$$

Then

$$D^J(B,D)=\{(\beta_1+\delta_1)\ldots(\beta_n+\delta_n)\ t:t\geq1\}.$$

Proof of Theorem 1.4

Since, by hypothesis, A,C, are J-unitary matrices, considering convenient Möbius transformations, it follows that

$$B=\frac{i}{2}\frac{A-I}{A+I},\ D=-\frac{i}{2}\frac{C+I}{C-I}\qquad(2.1)$$

are J-Hermitian matrices. Since

$$B + D = -i(A + I)^{-1}(C + A)(C - I)^{-1},$$

we obtain

$$\det(B + D) = i^n \frac{\det(A + C)}{\prod_{j=1}^{n}(1 + \alpha_j)(1 - \gamma_j)}.$$

Assume that the eigenvalues of B and D are

$$\sigma(B) = \{\beta_1, \ldots, \beta_n\}, \ \sigma(D) = \{\delta_1, \ldots, \delta_n\},$$

respectively. From (2.1) we get,

$$\beta_j = -\frac{\Im\alpha_j}{2(1 + \Re\alpha_j)}, \ \delta_j = -\frac{\Im\gamma_j}{2(1 - \Re\gamma_j)}.$$

From (1.6) and (1.7) we conclude that

$$\beta_1 \geq \ldots \geq \beta_r > 0 > \beta_{r+1} \geq \ldots > \beta_n,$$

and

$$\delta_1 \geq \ldots \geq \delta_r > 0 > \delta_{r+1} \geq \ldots > \delta_n,$$

so that the matrices B and D are J-positive. From Lemma 2.2 it follows that

$$D^J(B, D) = (\beta_1 + \delta_1) \ldots (\beta_n + \delta_n)[1, +\infty[\ .$$

Thus, $D^J(A, C)$ is a half-line with endpoint at

$$(\alpha_1 + \gamma_1) \ldots (\alpha_n + \gamma_n),$$

or, more precisely,

$$D^J(A, C) = \{(\alpha_1 + \gamma_1) \ldots (\alpha_n + \gamma_n) \ t : t \geq 1\}.$$

□

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References

[1] Bebiano, N., da Providência, J., *A Fiedler-type Theorem for the Determinant of J-Positive matrices,* Math. Inequal. Appl., (to appear).

[2] Bebiano, N., Kovacec, A., da Providência, J., *The validity of Marcus-de Oliveira Conjecture for essentially Hermitian matrices*, Linear Algebra Appl., **197**(1994), 411-427.

[3] Bebiano, N., Lemos, R., da Providência, J., Soares, G., *Further developments of Furuta inequality of indefinite type*, Mathematical Inequalities and Applications, **13**(2010), 523-535.

[4] Bognar, J., *Indefinite Inner Product Spaces*, Springer, 1974.

[5] Drury, S.W., Cload, B., *On the determinantal conjecture of Marcus and de Oliveira*, Linear Algebra Appl., **177**(1992), 105-109.

[6] Drury, S.W., *Essentially Hermitian matrices revisited*, Electronic Journal of Linear Algebra, **15**(2006), 285-296.

[7] Fiedler, M., *The determinant of the sum of Hermitian matrices*, Proc. Amer. Math. Soc., **30**(1971), 27-31.

[8] Li, C.-K., Poon, Y.-T., Sze, N.-S., *Ranks and determinants of the sum of matrices from unitary orbits*, Linear and Multilinear Algebra, **56**(2008), 108-130.

[9] Marcus, M., *Plucker relations and the numerical range*, Indiana Univ. Math. J., **22**(1973), 1137-1149.

[10] Mc Duff, D., Salamon, D., *Introduction to Symplectic Topology*, Oxford Matematical Monographs, Clarendon Press, 1998.

[11] Mehta, M.L., *Matrix Theory, Selected Topics and Useful Results*, Industan Publishing Corporation, New Delhi, 1971.

[12] Nakazato, H., Bebiano, N., da Providência, J., *J-orthostochastic matrices of size* 3×3 *and numerical ranges of Krein space operators*, Linear Algebra Appl., **407**(2005), 211-232.

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Ball convergence of a stable fourth-order family for solving nonlinear systems under weak conditions

Ioannis K. Argyros, Munish Kansal and Vinay Kanwar

Abstract. We present a local convergence analysis of fourth-order methods in order to approximate a locally unique solution of a nonlinear equation in Banach space setting. Earlier studies have shown convergence using Taylor expansions and hypotheses reaching up to the fifth derivative although only the first derivative appears in these methods. We only show convergence using hypotheses on the first derivative. We also provide computable: error bounds, radii of convergence as well as uniqueness of the solution with results based on Lipschitz constants not given in earlier studies. The computational order of convergence is also used to determine the order of convergence. Finally, numerical examples are also provided to show that our results apply to solve equations in cases where earlier studies cannot apply.

Mathematics Subject Classification (2010): 65D10, 65D99.

Keywords: Local convergence, nonlinear equation, Lipschitz condition, Fréchet derivative.

1. Introduction

Let B_1, B_2 be Banach spaces and D be a convex subset of B_1. Let also $L(B_1, B_2)$ denote the space of bounded linear operators from B_1 into B_2.

In the present paper, we deal with the problem of approximating a locally unique solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where $F : D \subseteq B_1 \to B_2$ is a Fréchet-differentiable operator.

Numerous problems can be written in the form of (1.1) using Mathematical Modelling [3, 5, 8, 9, 12, 13, 18, 19, 22, 26, 28, 29, 30]. Analytical methods for solving such problems are almost non-existent and therefore, it is only possible to obtain approximate solutions by relying on numerical methods based on iterative procedure

[1-24]. In particular, we present the local convergence of the methods studied in [14] and defined for each $n = 0, 1, 2, 3, \ldots$ by

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ z_n = y_n - \dfrac{1}{\beta}F'(x_n)^{-1}F(y_n), \\ x_{n+1} = z_n - F'(x_n)^{-1}\left(\alpha F(y_n) + \beta F(z_n)\right), \end{cases} \tag{1.2}$$

where $\alpha = 2 - \frac{1}{\beta} - \beta,\ \beta \in \mathbb{R}\backslash\{0\}$ and $\alpha \in \mathbb{R}$.

Method (1.2) has fourth-order of convergence, except for $\beta = 1/5$. For this particular value, method attains fifth-order of convergence. The fourth order of convergence was based on Taylor expansions and hypotheses reaching up to the fifth derivative of function F although only the first derivative appears in these methods. Moreover, no computable error bounds on the distances $\|x_n - x^*\|$ or uniqueness results or compuatble radius of convergence were given. These problems reduce the applicability of these methods.

As a motivational example, define function F on $D = [\frac{-1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} F'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \\ F''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x, \\ F'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, the results in [14] cannot be used to solve the equation $F(x) = 0$, since function F''' is unbounded on D.

In the present study, we only use hypotheses on the first derivative and find error bounds, radii of convergence and uniqueness results based on Lipschitz constants. Moreover, since we avoid derivatives of order higher than one, we compute the computational order of convergence which does not require the knowledge of x^* or the existence of high order derivatives. This way we expand the applicability of these methods.

The rest of the paper is organized as follows: The local convergence of both methods is given in Section 2, whereas numerical examples are provided in the concluding Section 3.

2. Local convergence

We present the local convergence analysis of method (1.2) in this section.

The local convergence analysis is based on some scalar functions and parameters. Let $L_0 > 0,\ L > 0,\ M \geq 1,\ \beta \in \mathbb{R}\backslash\{0\}$ and $\alpha \in \mathbb{R}$ be given parameters. Define function

g_1, g_2, h_2, g_3 and h_3 on the interval $[0, \frac{1}{L_0})$ by

$$\begin{aligned}
g_1(t) =& \frac{Lt}{2(1-L_0t)},\\
g_2(t) =& \Big(1+\frac{M}{|\beta|(1-L_0t)}\Big)g_1(t),\\
h_2(t) =& g_2(t)-1,\\
g_3(t) =& g_2(t)+\frac{M}{1-L_0t}\left(|\alpha|g_1(t)+|\beta|g_2(t)\right),\\
h_3(t) =& g_3(t)-1
\end{aligned}$$

and parameter r_A by

$$r_A = \frac{2}{2L_0+L}.$$

We have that $g_1(r_A) = 1$ and $0 \le g_1(t) < 1$ for each $t \in [0, r_A)$.

We also get that $h_2(0) = h_3(t) = -1 < 0$ and $h_2(t) \to +\infty,\ h_3(t) \to +\infty$ as $t \to \frac{1^-}{L_0}$. It follows from intermediate value theorem that functions h_2 and h_3 have zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_2 and r_3 the smallest such zeros.
Define the convergence radius r by

$$r = min\{r_A, r_2, r_3\}. \tag{2.1}$$

Then, we have that

$$0 < r \le r_A \tag{2.2}$$

and

$$0 \le g_i(t) < 1,\ i = 1,2,3. \tag{2.3}$$

Let $U(v,\rho)$ and $\bar{U}(v,\rho)$ stand, respectively for the open and closed balls in B_1 with center $v \in B_1$ and of radius $\rho > 0$. Next, we present the local convergence analysis of method (1.2) using the preceding notation.

Theorem 2.1. *Let $F : D \subseteq B_1 \to B_2$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$ and $L_0 > 0$ such that for each $x \in D$*

$$F(x^*) = 0,\ F'(x^*)^{-1} \in L(B_2, B_1), \tag{2.4}$$

and

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le L_0\|x - x^*\|. \tag{2.5}$$

Moreover, suppose that there exist constants $L > 0$ and $M \ge 1$ such that for each $x, y \in D_0 := D \cap U\left(x^*, \frac{1}{L_0}\right)$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le L\|x - y\|, \tag{2.6}$$

$$\|F'(x^*)^{-1}F'(x)\| \le M \tag{2.7}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{2.8}$$

where the radius of convergence r is defined by (2.1). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r)$-$\{x^*\}$ by method (1.2) is well defined, remains in $U(x^*, r)$ and *converges to the solution x^* of equation $F(x) = 0$. Moreover, the following estimates hold*

$$\|y_n - x^*\| \le g_1(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < r, \tag{2.9}$$

$$\|z_n - x^*\| \le g_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| \tag{2.10}$$

and

$$\|x_{n+1} - x^*\| \le g_3(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|, \tag{2.11}$$

where the "g" functions are defined previously. Furthermore, for $T \in [r, \frac{2}{L_0})$, the limit point x^* is the only solution of $F(x) = 0$ in $D_1 := U(x^*, T) \cap D$.

Proof. We shall show estimates (2.9)–(2.11) using mathematical induction. By hypothesis $x_0 \in U(x^*, r)$-$\{x^*\}$, (2.1), (2.4) and (2.5), we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \le L_0\|x_0 - x^*\| < L_0 r < 1. \tag{2.12}$$

It follows from (2.12) and the Banach lemma on invertible functions [7, 26, 28, 30] that $F'(x_0)^{-1} \in L(B_2, B_1)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \le \frac{1}{1 - L_0\|x_0 - x^*\|}. \tag{2.13}$$

Hence, y_0, z_0, x_1 are well defined by method (1.2) for $n = 0$. We can have that

$$y_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0). \tag{2.14}$$

Using (2.1), (2.2), (2.3) (for $i = 1$), (2.6), (2.13) and (2.14), we obtain in turn that

$$\begin{aligned}\|y_0 - x^*\| &= \|x_0 - x^* - F'(x_0)^{-1}F(x_0)\| \le \|F'(x_0)^{-1}F'(x^*)\| \\ &\quad \|\int_0^1 F'(x^*)^{-1}\Big(F'(x^* + \theta(x_0 - x^*)) - F'(x_0)\Big)(x_0 - x^*)d\theta\| \\ &\le \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\| < r,\end{aligned} \tag{2.15}$$

which shows (2.9) for $n = 0$ and $y_0 \in U(x^*, r)$. We also have that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.16}$$

Notice that $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| < r$, so $x^* + \theta(x_0 - x^*) \in U(x^*, r)$. Then, by (2.7) and (2.16), we get that

$$\|F'(x^*)^{-1}F(x_0)\| \le M\|x_0 - x^*\|. \tag{2.17}$$

In view of (2.1), (2.2), (2.3) (for $i = 2$), (2.13), (2.15) and (2.17) (for $x_0 = y_0$), we get that

$$\begin{aligned}\|z_0 - x^*\| &\le \|y_0 - x^*\| + \frac{M\|y_0 - x^*\|}{|\beta|(1 - L_0\|x_0 - x^*\|)} \\ &\le \Big(1 + \frac{M}{|\beta|(1 - L_0\|x_0 - x^*\|)}\Big)g_1(\|x_0 - x^*\|))\|x_0 - x^*\| \\ &= g_2(\|x_0 - x^*\|))\|x_0 - x^*\| \le \|x_0 - x^*\| < r,\end{aligned} \tag{2.18}$$

which shows (2.10) for $n = 0$ and $z_0 \in U(x^*, r)$. By (2.1), (2.2), (2.3) (for $i = 3$), (2.13), (2.15) and (2.17) (for $x_0 = y_0$), we get that

$$\begin{aligned} \|x_1 - x^*\| &\le \|z_0 - x^*\| + \frac{M}{1 - L_0(\|x_0 - x^*\|)}\big(|\alpha|\|y_0 - x^*\| + |\beta|\|z_0 - x^*\|\big) \\ &\le \Big[g_2(\|x_0 - x^*\|) + \frac{M}{1 - L_0(\|x_0 - x^*\|)}\big(|\alpha| g_1(\|x_0 - x^*\|) \\ &+ |\beta| g_2(\|x_0 - x^*\|)\big)\Big]\|x_0 - x^*\| \\ &= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{2.19}$$

which shows (2.11) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, x_1 by x_n, y_n, x_{n+1} in the preceding estimates, we complete the induction for estimates (2.9)–(2.11). Then, in view of the estimate

$$\|x_{n+1} - x^*\| \le c\|x_n - x^*\| < r, \qquad c = g_3(\|x_0 - x^*\|) \in [0, 1),$$

we deduce that $\lim_{n\to\infty} x_n = x^*$ and $x_{n+1} \in U(x^*, r)$. Finally, to show the uniqueness part, let $y^* \in D_1$ with $F(y^*) = 0$. Define

$$Q = \int_0^1 F'(x^* + \theta(y^* - x^*)d\theta.$$

Using (2.5), we get that

$$\|F'(x^*)^{-1}(Q - F'(x^*))\| \le \frac{L_0}{2}\|x^* - y^*\| \le \frac{L_0}{2} T < 1. \tag{2.20}$$

Hence, $Q^{-1} \in L(B_2, B_1)$. Then, by the identity $0 = F(y^*) - F(x^*) = Q(y^* - x^*)$, we conclude that $x^* = y^*$. □

Remark 2.2. 1. The condition (2.7) can be dropped, since this condition follows from (2.5), if we set

$$M(t) = 1 + L_0 t$$

or

$$M(t) = M = 2,$$

since $t \in [0, \frac{1}{L_0})$.

2. The results obtained here can also be used for operators F satisfying autonomous differential equations [5, 7] of the form:

$$F'(x) = P(F(x)),$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$.

3. The radius $\bar{r_A} = \frac{2}{2L_0 + L_1}$ was shown by us to be the convergence radius of Newton's method [5]

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n), \text{ for each } n = 0, 1, 2, \dots \tag{2.21}$$

provided the conditions (2.4)–(2.6) hold on D. Let L_1 be the corresponding to L constant. It follows from the definition of r that the convergence radius r of the method (1.2) cannot be larger than the convergence radius $\bar{r_A}$ of the second order Newton's method (3.3). As already noted in [5], $\bar{r_A}$ is at least as large as the convergence ball given by Rheinboldt [28]

$$r_R = \frac{2}{3L_1}.$$

In particular, for $L_0 < L_1$, we have that

$$r_R < r_1$$

and

$$\frac{r_R}{\bar{r_A}} \to \frac{1}{3} \text{ as } \frac{L_0}{L_1} \to 0.$$

That is our convergence ball $\bar{r_A}$ is atmost three times larger than Rheinboldt's. The same value of r_R was given by Traub [30]. Notice that $L \le L_1$, since $D_0 \subseteq D$. Therefore, $\bar{r_A} \le r_A$.

4. It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of stronger conditions used in [14]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi^* = \sup \frac{\ln\left(\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}\right)}{\ln\left(\frac{\|x_n-x^*\|}{\|x_{n-1}-x^*\|}\right)},$$

or the approximate computational order of convergence (ACOC) defined by

$$\xi = \sup \frac{\ln\left(\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}\right)}{\ln\left(\frac{\|x_n-x_{n-1}\|}{\|x_{n-1}-x_{n-2}\|}\right)}.$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F. Notice also that the computation of ξ does not require knowledge of x^*.

3. Numerical examples

We present numerical examples in this section.

Example 3.1. Let $X = Y = \mathbb{R}^3$, $D = \bar{U}(0,1)$, $x^* = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet derivative is given by

$$F'(w) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have that $L_0 = e - 1$, $L = e^{\frac{1}{L_0}} = 1.789572397$, $M = e^{\frac{1}{L_0}} = 1.7896$ and $L_1 = e$. The parameters using method (1.2) are:

$$r_A = 0.382692,\ r_2 = 0.145318,\ r_3 = 0.0826175,\ r = 0.0826175,\ \bar{r_A} = 0.324947.$$

Example 3.2. Let $B_1 = B_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $D = \bar{U}(0,1)$ and $B(x) = F''(x)$ for each $x \in D$. Define function F on D by

$$F(\phi)(x) = \phi(x) - 5\int_0^1 x\theta\phi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\phi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\phi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D. \tag{3.2}$$

Then, we get that $x^* = 0, L_0 = 7.5, L_1 = 15, L = 15, M = 2$. The parameters using method (1.2) are:

$$r_A = 0.0666667,\ r_2 = 0.0198959,\ r_3 = 0.0101189,\ r = 0.0101189,\ \bar{r_A} = 0.0666667.$$

Example 3.3. Let $B_1 = B_2 = \mathbb{R}$, $D = \bar{U}(0,1)$. Define F on D by

$$F(x) = e^x - 1.$$

Then, $F'(x) = e^x$ and $\xi = 0$. We get that $L_0 = e - 1 < L = e^{\frac{1}{L_0}} < L_1 = e$ and $M = 2$. Then, for method (1.2) the parameters are:

$$r_A = 0.382692,\ r_2 = 0.13708,\ r_3 = 0.0742433,$$

$$r = 0.0742433,\ \bar{r_A} = 0.324947,\ \xi = 3.8732.$$

Example 3.4. Let $B_1 = B_2 = \mathbb{R}$ and define function F on $D = \mathbb{R}$ by

$$F(x) = \beta x - \gamma \sin(x) - \delta, \tag{3.3}$$

where $\beta,\ \gamma,\ \delta$ are given real numbers. Suppose that there exists a solution ξ of $F(x) = 0$ with $F'(\xi) \neq 0$. Then, we have

$$L_1 = L_0 = L = \frac{|\gamma|}{|\beta - \gamma\cos\xi|},\ M = \frac{|\gamma| + |\beta|}{|\beta - \gamma\cos\xi|}.$$

Then one can find the convergence radii for different values of $\beta,\ \gamma$ and δ. As a specific example, let us consider Kepler's equation (3.3) with $\beta = 1$, $0 \leq \gamma < 1$ and $0 \leq \delta \leq \pi$. A numerical study was presented in [15] for different values of γ and δ. Let us take $\gamma = 0.9$ and $\delta = 0.1$. Then the solution is given by $x^* = 0.6308435$.
Hence, for method (1.2) the parameters are:

$$r_A = 0.202387,\ r_2 = 0.032669,\ r_3 = 0.00804637,$$

$$r = 0.00804637,\ \bar{r_A} = 0.202387,\ \xi = 4.0398.$$

Example 3.5. Returning back to the motivational example at the introduction of this paper, we have that $L = L_0 = 146.6629073$, $M = 2$, $L_1 = L$. The parameters using method (1.2) are:

$$r_A = 0.00689682,\ r_2 = 0.0033639187,\ r_3 = 0.00230533728667086,$$
$$r = 0.00230533728667086,\ \bar{r_A} = 0.00689682 \text{ and } \xi = 3.4324.$$

References

[1] Adomian, G., *Solving Frontier problem of physics: The decomposition method*, Kluwer Academic Publishers, Dordrechet, 1994.

[2] Amat, S., Busquier, S., Guttiérrez, J.M., *Geometric constructions of iterative functions to solve nonlinear equations*, J. Comput. Appl. Math., **157**(2003), 197–205.

[3] Amat, S., Busquier, S., Plaza, S., *Dynamics of the King's and Jarratt iterations*, Aequationes. Math., **69**(2005), 212–213.

[4] Amat, S., Hernández, M.A., Romero, N., *A modified Chebyshev's iterative method with at least sixth order of convergence*, Appl. Math. Comput., **206**(1)(2008), 164–174.

[5] Argyros, I.K., *Convergence and Applications of Newton-type Iterations*, Springer, 2008.

[6] Argyros, I.K., Chen D., Quian, Q., *The Jarratt method in Banach space setting*, J. Comput. Appl. Math., **51**(1994), 103–106.

[7] Argyros, I.K., Hilout S., *Computational Methods in Nonlinear Analysis*, World Scientific Publ. Comp. New Jersey, 2013.

[8] Argyros, I.K., Magreñán, A.A., *Ball convergence theorems and the convergence planes of an iterative method for nonlinear equations*, SeMA, **71**(1)(2015), 39–55.

[9] Argyros, I.K., George, S., *Local convergence of some high-order Newton-like method with frozen derivatives*, SeMA. doi: 10.1007/s40324-015-00398-8.

[10] Babolian, E., Biazar, J., Vahidi, A.R., *Solution of a system of nonlinear equations by Adomian decomposition method*, Appl. Math. Comput., **150**(2004), 847–854.

[11] Cordero, A., Torregrosa, J.R., *Variants of Newton's method using fifth-order quadrature formulas*, Appl. Math. Comput., **190**(2007), 686–698.

[12] Cordero, A., Torregrosa, J.R., *Variants of Newton's method for functions of several variables*, Appl. Math. Comput., **183**(2006), 199–208.

[13] Cordero, A., Torregrosa, J.R., Vassileva, M.P., *Increasing the order of convergence of iterative schemes for solving nonlinear systems*, J. Comput. Appl. Math., **252**(2012), 86–94.

[14] Cordero, A., Gutiérrez, J.M., Magreñán, A.A., Torregrosa, J.R., *Stability analysis of a parametric family of iterative methods for solving nonlinear models*, Appl. Math. Comput. (to appear).

[15] Danby, J.M.A., Burkardt, T.M., *The solution of Kepler's equation*, I. Celest. Mech., **31**(1983), 95–107.

[16] Darvishi, M.T., Barati, A., *A third-order Newton-type method to solve systems of nonlinear equations*, Appl. Math. Comput., **187**(2007), 630–635.

[17] Darvishi, M.T., Barati, A., *Super cubic iterative methods to solve systems of nonlinear equations*, Appl. Math. Comput., **188**(2007), 1678–1685.

[18] Ezquerro, J.A., Hernández, M.A., *New iterations of R-order four with reduced computational cost*, BIT Numer Math., **49**(2009), 325–342.

[19] Ezquerro, J.A., Hernández, M.A., *A uniparametric halley type iteration with free second derivative*, Int. J. Pure and Appl. Math., **6**(1)(2003), 99–110.

[20] Golbabai, A., Javidi, M., *A new family of iterative methods for solving system of nonlinear algebraic equations*, Appl. Math. Comput., **190**(2007), 1717–1722.

[21] Gutiérrez, J.M., Hernández, M.A., *Recurrence relations for the super-Halley method*, Comput. Math. Appl., **36**(1998), 1–8.

[22] Kou, J., *A third-order modification of Newton method for systems of nonlinear equations*, Appl. Math. Comput., **191**(2007), 117–121.

[23] Montazeri, H., Soleymani, F., Shateyi, S., Motsa, S.S., *On a new method for computing the numerical solution of systems of nonlinear equations*, J. Appl. Math. 2012, Article ID 751975.

[24] Noor, M.A., Waseem M., *Some iterative methods for solving a system of nonlinear equations*, Comput. Math. Appl., **57**(2009), 101–106.

[25] Petković, M.S., Neta, B., Petković, L.D., Dzunić, J., *Multipoint Methods for Solving Nonlinear Equations*, Elsevier, Amsterdam, 2013.

[26] Potra, F.A., Pták, V., *Nondiscrete introduction and iterative processes*, Research Notes in Mathematics, Pitman, Boston, MA, **103**(1984).

[27] Qifang, Su., *A unified model for solving a system of nonlinear equations*, Appl. Math. Comput. (to appear).

[28] Rheinboldt, W.C., *An adaptive continuation process for solving systems of nonlinear equations*, Mathematical Models and Numerical Methods (A.N. Tikhonov et al. eds.), Banach Center, Warsaw, Poland, 3(19), 129–142.

[29] Sharma, J.R., Guha, R.K., Sharma, R., *An efficient fourth-order weighted-Newton method for systems of nonlinear equations*, Numer. Algor., **62**(2013), 307–323.

[30] Traub, J.F., *Iterative Methods for the Solution of Equations*, Prentice-Hall, New Jersey, 1964.

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Book reviews

Aram V. Arutyunov and Valeri Obukhovskii; Convex and set-valued analysis. Selected topics, De Gruyter Graduate, De Gruyter, Berlin 2017, viii+201 p., ISBN: 978-3-11-046028-5/pbk; 978-3-11-046030-8/ebook.

The book, consisting of two relatively independent parts, is based on courses taught by the first author at the Moscow State University and by the second one at the Voronezh University. A preliminary Russian version, written by the first author, was published in 2014 with Fizmatlit Editors, Moscow, but the present book contains many additions and extensions.

The first part of the book is devoted to convex analysis - convex sets, separation of convex sets, convex functions, continuity and differentiability properties of convex functions, the Young-Fenchel conjugate, convex cones. Although, for more clarity and accessibility, the presentation is done, in general, in the finite dimensional Euclidean case, some topics are treated in a more general context - the separation of convex sets in a normed space, the existence of some positive functionals on normed spaces ordered by closed convex cones, and the Young-Fenchel conjugate in a Hilbert space.

The second part is devoted to set-valued analysis. After a detailed introduction to Hausdorff metric and its essential properties, one passes to the study of continuity (upper and lower) of set-valued maps. Measurable set-valued maps and measurable selections, with applications to set-valued superposition operators (satisfying a Carathéodori-type condition), are included as well. An important part of the book is devoted to fixed point and coincidence point theorems for set-valued maps (mainly), with applications to differential inclusions. Several nice results of the authors, involving metric regularity and covering maps theory, are presented. A proof of the Brouwer fixed point theorem based on the degree theory for single-valued maps is given, while the degree theory for set-valued maps is applied to fixed point results for this kind of maps.

Numerous examples and exercises complete the main text. The prerequisites are minimal: basic topology, some linear algebra and rudiments of functional analysis.

Written by two experts in these areas and based on their teaching experience, the book contains a clear and accessible introduction to convex and set-valued analysis. It can be used for courses on these topics or for self-study.

Adrian Petruşel

Vidyadhar S. Mandrekar; Weak convergence of stochastic processes. With applications to statistical limit theorems, De Gruyter Graduate, De Gruyter, Berlin 2016, vi+141 p., ISBN: 978-3-11-047542-5/pbk; 978-3-11-047631-6/ebook).

The book is devoted to a detailed study of weak convergence in probability theory with applications to Brownian motion, inference in statistics and convergence in martingale theory.

As the first chapter contains only the Introduction, the effective presentation starts in Ch. 2, *Weak convergence in metric spaces*, with the introduction of cylindrical measures as a tool for the study of Brownian motion. Sections 2.10 and 2.11 of this chapter are concerned with the weak convergence of probability measures on complete separable metric spaces (Polish spaces) - Portmanteau Theorem, tightness and Prokhorov's compactness criterium. It is worth to mention that extensions of these results to non-separable metric spaces are given in Ch. 6, *Empirical processes*, where, with a suitable definition of the weak convergence of nets of random variables, one obtains analogs of the results from the separable case - Portmanteau Theorem, tightness, asymptotic tightness and compactness.

Ch. 3, *Weak convergence on* $C[0,1]$ *and* $D[0,1]$, is dealing with the distributional counterpart of weak convergence. The techniques developed in Sections 2.10 and 2.11 are applied to the space $C[0,1]$, one introduces the Skorokhod topology and the Skorokhod metric on the spaces $D[0,T]$ and $D[0,\infty)$ of functions having only discontinuities of the first kind. Compact sets in $C[0,1]$ and $D[0,1]$ are characterized - Arzela-Ascoli in the first case and in terms of tightness in the second one. The chapter ends with Aldous' tightness criterium, characterizing compactness in terms of stopping times.

Chapters 4. *Central limit theorem for semi-martingales and applications*, and 5. *Central limit theorems for dependent variables, are devoted to applications*, as, e.g., statistical limit theorems for censored data that arise in clinical trials.

Written by an expert in probability theory and stochastic processes, the book succeeds to present, in a relatively small number of pages, some fundamental results on weak convergence in probability theory and stochastic process and applications.

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