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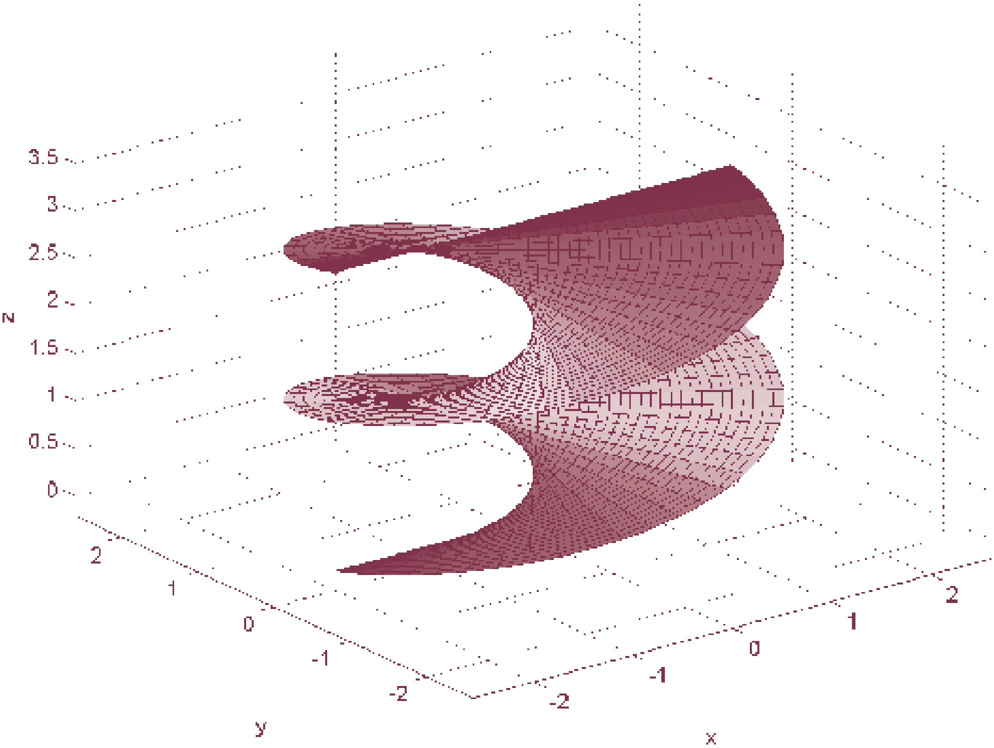

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Global smoothness preservation and simultaneous approximation by multivariate discrete operators

George A. Anastassiou and Merve Kester

Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. In this article we study the multivariate generalized discrete singular operators defined on $\mathbb{R}^N$, $N \geq 1$, regarding their simultaneus global smoothness preservation property with respect to L_p norm for $1 \leq p \leq \infty$, by using higher order moduli of smoothness. Furthermore, we study their simultaneous approximation properties.

Mathematics Subject Classification (2010): 26A15, 26D15, 41A17, 41A25, 41A28, 41A35, 41A80.

Keywords: Simultaneous global smoothness, simultaneous approximation with rates, multivariate generalized discrete singular operators, modulus of smoothness.

1. Background

In [1], Chapter 3, the author defined

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1,2,...,r, \\ 1 - \sum\limits_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 0, \end{cases} \tag{1.1}$$

for $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$ and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1,2,...,m \in \mathbb{N}. \tag{1.2}$$

See that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1. \tag{1.3}$$

Additionally,in [1], the author used

Definition 1.1. *Let $f \in C\left(\mathbb{R}^N\right)$, $N \geq 1$, $m \in \mathbb{N}$, the mth modulus of smoothness for $1 \leq p \leq \infty$, is given by*

$$\omega_m (f; h)_p := \sup_{\|t\|_2 \leq h} \|\Delta_t^m (f)\|_{p,x}, \tag{1.4}$$

$h > 0$, where

$$\Delta_t^m f(x) := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + jt). \tag{1.5}$$

Denote

$$\omega_m (f; h)_\infty = \omega_m (f, h). \tag{1.6}$$

Above, $x, t \in \mathbb{R}^N$.

Additionally, in [4], the authors defined the following operators:

Let μ_{ξ_n} be a Borel measure on $\mathbb{R}^N$, $N \geq 1$, $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. Assume that $\nu := (\nu_1, ..., \nu_N)$, $x := (x_{1,...,}x_N) \in \mathbb{R}^N$ and $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function.

i) When

$$\mu_{\xi_n}(\nu) = \frac{e^{-\frac{\sum_{i=1}^{N} |\nu_i|}{\xi_n}}}{\sum_{\nu_1=-\infty}^{\infty} ... \sum_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N} |\nu_i|}{\xi_n}}}, \tag{1.7}$$

they defined generalized multiple discrete Picard operators as:

$$P_{r,n}^{*\,[m]}(f; x_1, ..., x_N) \tag{1.8}$$

$$= \frac{\sum_{\nu_1=-\infty}^{\infty} ... \sum_{\nu_N=-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_{j,r}^{[m]} f(x_1 + j\nu_1, ..., x_N + j\nu_N) \right) e^{-\frac{\sum_{i=1}^{N} |\nu_i|}{\xi_n}}}{\sum_{\nu_1=-\infty}^{\infty} ... \sum_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N} |\nu_i|}{\xi_n}}}.$$

ii) When

$$\mu_{\xi_n}(\nu) = \frac{e^{-\frac{\sum_{i=1}^{N} \nu_i^2}{\xi_n}}}{\sum_{\nu_1=-\infty}^{\infty} ... \sum_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N} \nu_i^2}{\xi_n}}}, \tag{1.9}$$

they defined generalized multiple discrete Gauss-Weierstrass operators as:

$$W_{r,n}^{*\,[m]}\left(f;x_1,...,x_N\right) \tag{1.10}$$
$$= \frac{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\left(\sum\limits_{j=0}^{r}\alpha_{j,r}^{[m]}f\left(x_1+j\nu_1,...,x_N+j\nu_N\right)\right)e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}.$$

iii) Let $\hat{\alpha}\in\mathbb{N}$ and $\beta>\frac{1}{\hat{\alpha}}$. When

$$\mu_{\xi_n}\left(\nu\right)=\frac{\prod\limits_{i=1}^{N}\left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}}{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\prod\limits_{i=1}^{N}\left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}}, \tag{1.11}$$

they defined the generalized multiple discrete Poisson-Cauchy operators as:

$$Q_{r,n}^{*\,[m]}\left(f;x_1,...,x_N\right) \tag{1.12}$$
$$= \frac{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\left(\sum\limits_{j=0}^{r}\alpha_{j,r}^{[m]}f\left(x_1+j\nu_1,...,x_N+j\nu_N\right)\right)\prod\limits_{i=1}^{N}\left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}}{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\left(\prod\limits_{i=1}^{N}\left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}\right)}.$$

iv) When

$$\mu_{\xi_n}\left(\nu\right)=\frac{e^{-\frac{\sum_{i=1}^{N}\left|\nu_i\right|}{\xi_n}}}{\left(1+2\xi_n e^{-\frac{1}{\xi_n}}\right)^N}, \tag{1.13}$$

they defined the generalized multiple discrete non-unitary Picard operators as:

$$P_{r,n}^{[m]}\left(f;x_1,...,x_N\right) \tag{1.14}$$
$$= \frac{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\left(\sum\limits_{j=0}^{r}\alpha_{j,r}^{[m]}f\left(x_1+j\nu_1,...,x_N+j\nu_N\right)\right)e^{-\frac{\sum_{i=1}^{N}\left|\nu_i\right|}{\xi_n}}}{\left(1+2\xi_n e^{-\frac{1}{\xi_n}}\right)^N}.$$

v) When

$$\mu_{\xi_n}\left(\nu\right)=\frac{e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\left(\sqrt{\pi\xi_n}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi_n}}\right)\right)+1\right)^N}, \tag{1.15}$$

they defined the generalized multiple discrete non-unitary Gauss-Weierstrass operators as:

$$W_{r,n}^{[m]}\left(f;x_1,...,x_N\right) \tag{1.16}$$
$$= \frac{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_{j,r}^{[m]} f\left(x_1+j\nu_1,...,x_N+j\nu_N\right)\right) e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\left(\sqrt{\pi}\xi_n\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi_n}}\right)\right)+1\right)^N},$$

where $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int\limits_0^x e^{-t^2}dt$ with $\operatorname{erf}(\infty)=1$.

Additionally, in [4], article they assumed that $0^0=1$.

In [4], for $\alpha_i \in \mathbb{N}$, the authors defined the sums

$$c_{\alpha,n,\tilde{j}} := \frac{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \left(\prod\limits_{i=1}^{N} \nu_i^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}, \tag{1.17}$$

$$p_{\alpha,n,\tilde{j}} := \frac{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \left(\prod\limits_{i=1}^{N} \nu_i^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}, \tag{1.18}$$

and for $\hat{\alpha} \in \mathbb{N}$ and $\beta > \frac{\alpha_i+r+1}{2\hat{\alpha}}$, they introduced

$$q_{\alpha,n,\tilde{j}} := \frac{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \left(\prod\limits_{i=1}^{N} \nu_i^{\alpha_i}\left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}\right)}{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \prod\limits_{i=1}^{N}\left(\nu_i^{2\hat{\alpha}}+\xi_n^{2\hat{\alpha}}\right)^{-\beta}}. \tag{1.19}$$

Furthermore, they proved that

$$c_{\alpha,n,\tilde{j}},\ p_{\alpha,n,\tilde{j}},\ q_{\alpha,n,\tilde{j}} < \infty,\ \forall \xi_n \in (0,1], \tag{1.20}$$

and for $\alpha_i \in \mathbb{N}$, as $\xi_n \to 0$ when $n \to \infty$, the authors showed that

$$c_{\alpha,n,\tilde{j}},\ p_{\alpha,n,\tilde{j}}, \text{and } q_{\alpha,n,\tilde{j}} \to 0. \tag{1.21}$$

In [4], they also proved

$$m_{\xi_n,P} = \prod_{i=1}^{N}\left(\frac{\sum\limits_{\nu_i=-\infty}^{\infty} e^{-\frac{|\nu_i|}{\xi_n}}}{1+2\xi_n e^{-\frac{1}{\xi_n}}}\right) \to 1 \text{ as } \xi_n \to 0^+, \tag{1.22}$$

and

$$m_{\xi_n,W}=\prod_{i=1}^{N}\left(\frac{\sum\limits_{\nu_i=-\infty}^{\infty}e^{-\frac{\nu_i^2}{\xi}}}{1+\sqrt{\pi\xi_n}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi_n}}\right)\right)}\right)\to 1 \text{ as } \xi_n\to 0^+. \tag{1.23}$$

Moreover, in [4], the authors defined the following error quantities:

$$E_{n,P}^{[0]}(f;x):=P_{r,n}^{[0]}(f;x)-f(x), \tag{1.24}$$

$$E_{n,W}^{[0]}(f;x):=W_{r,n}^{[0]}(f;x)-f(x).$$

Furthermore, they introduced the errors ($n\in\mathbb{N}$):

$$E_{n,P}^{[m]}(f;x) \tag{1.25}$$
$$:=P_{r,n}^{[m]}(f;x)-f(x)-\sum_{\widetilde{j}=1}^{m}\delta_{\widetilde{j},r}^{[m]}\left(\sum_{\substack{\alpha_1,\dots,\alpha_N\geq 0:\\ |\alpha|=\widetilde{j}}}\frac{\tilde{c}_{\alpha,n,\widetilde{j}}f_\alpha(x)}{\prod\limits_{i=1}^{N}\alpha_i!}\right),$$

and

$$E_{n,W}^{[m]}(f;x) \tag{1.26}$$
$$:=W_{r,n}^{[m]}(f;x)-f(x)-\sum_{\widetilde{j}=1}^{m}\delta_{\widetilde{j},r}^{[m]}\left(\sum_{\substack{\alpha_1,\dots,\alpha_N\geq 0:\\ |\alpha|=\widetilde{j}}}\frac{\tilde{p}_{\alpha,n,\widetilde{j}}f_\alpha(x)}{\prod\limits_{i=1}^{N}\alpha_i!}\right),$$

where

$$\tilde{c}_{\alpha,n,\widetilde{j}}:=\frac{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\left(\prod\limits_{i=1}^{N}\nu_i^{\alpha_i}\right)e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}{\left(1+2\xi_n e^{-\frac{1}{\xi_n}}\right)^N} \tag{1.27}$$

and

$$\tilde{p}_{\alpha,n,\widetilde{j}}:=\frac{\sum\limits_{\nu_1=-\infty}^{\infty}\cdots\sum\limits_{\nu_N=-\infty}^{\infty}\left(\prod\limits_{i=1}^{N}\nu_i^{\alpha_i}\right)e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\left(\sqrt{\pi\xi_n}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi_n}}\right)\right)+1\right)^N}. \tag{1.28}$$

In [4], the authors proved

Proposition 1.2. *Let* $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N \in \mathbb{N}$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = m \in \mathbb{N}$. *Then, there exist* K_1, K_2, $K_3 > 0$ *such that*

$$\begin{aligned} & u^*_{P,\xi_n} \\ = \; & \frac{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} \left(\prod_{i=1}^{N} |\nu_i|^{\alpha_i} \right) \left(1 + \frac{\|\nu\|_2}{\xi_n}\right)^r e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}} \\ \leq \; & K_1 < \infty, \end{aligned} \tag{1.29}$$

$$\begin{aligned} & u^*_{W,\xi_n} \\ = \; & \frac{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} \left(\prod_{i=1}^{N} |\nu_i|^{\alpha_i} \right) \left(1 + \frac{\|\nu\|_2}{\xi_n}\right)^r e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}} \\ \leq \; & K_2 < \infty, \end{aligned} \tag{1.30}$$

and

$$\begin{aligned} & u^*_{Q,\xi_n} \\ = \; & \frac{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} \left(\prod_{i=1}^{N} |\nu_i|^{\alpha_i} \right) \left(1 + \frac{\|\nu\|_2}{\xi_n}\right)^r \left(\prod_{i=1}^{N} \left(\nu_i^{2\hat{\alpha}} + \xi_n^{2\hat{\alpha}}\right)^{-\beta} \right)}{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} \left(\prod_{i=1}^{N} \left(\nu_i^{2\hat{\alpha}} + \xi_n^{2\hat{\alpha}}\right)^{-\beta} \right)} \\ \leq \; & K_3 < \infty, \end{aligned} \tag{1.31}$$

for all $\xi_n \in (0, 1]$ *where* $\hat{\alpha}$, $n \in \mathbb{N}$, $\beta > \max\left\{\frac{1+r+\alpha_i}{2\hat{\alpha}}, \ \frac{r+2}{2\hat{\alpha}}\right\}$ *for all* $i = 1, ..., N$, *and* $\nu = (\nu_1, ..., \nu_N)$.

Additionally, in [4], the authors defined

$$\Phi^*_{P,\xi_n} := \frac{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} \left(1 + \frac{\|\nu\|_2}{\xi_n}\right)^r e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}, \tag{1.32}$$

$$\Phi^*_{W,\xi_n} := \frac{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} \left(1 + \frac{\|\nu\|_2}{\xi_n}\right)^r e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}, \tag{1.33}$$

and

$$\Phi^*_{Q,\xi_n} := \frac{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \left(1+\frac{\|\nu\|_2}{\xi_n}\right)^r \prod\limits_{i=1}^{N} \left(\nu_i^{2\hat{\alpha}} + \xi_n^{2\hat{\alpha}}\right)^{-\beta}}{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \prod\limits_{i=1}^{N} \left(\nu_i^{2\hat{\alpha}} + \xi_n^{2\hat{\alpha}}\right)^{-\beta}}. \tag{1.34}$$

They also showed Φ^*_{P,ξ_n}, Φ^*_{W,ξ_n}, and Φ^*_{Q,ξ_n} are uniformly bounded for all $\xi_n \in (0,1]$, where $\hat{\alpha} \in \mathbb{N}$, $\beta > \frac{r+2}{2\hat{\alpha}}$.

On the other hand, in [5], the authors proved

Proposition 1.3. *Let* $\nu := (\nu_1, ..., \nu_N)$, $\alpha := (\alpha_{1,...,}\alpha_N) \in \mathbb{R}^N$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N \in \mathbb{N}$, $|\alpha| := \sum\limits_{i=1}^{N} \alpha_i = m \in \mathbb{Z}^+$, *and* $p \geq 1$. *Then,*

$$S^{p,m}_{P^*,\xi_n} = \frac{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \left(\prod\limits_{i=1}^{N} |\nu_i|^{\alpha_i}\right)^p \left(1+\frac{\|\nu\|_2}{\xi_n}\right)^{rp} e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}, \tag{1.35}$$

$$S^{p,m}_{W^*,\xi_n} = \frac{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \left(\prod\limits_{i=1}^{N} |\nu_i|^{\alpha_i}\right)^p \left(1+\frac{\|\nu\|_2}{\xi_n}\right)^{rp} e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}, \tag{1.36}$$

and

$$S^{p,m}_{Q^*,\xi_n} = \frac{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \sum\limits_{\nu_N=-\infty}^{\infty} \left(\prod\limits_{i=1}^{N} |\nu_i|^{\alpha_i}\right)^p \left(1+\frac{\|\nu\|_2}{\xi_n}\right)^{rp} \left(\prod\limits_{i=1}^{N} \left(\nu_i^{2\hat{\alpha}} + \xi_n^{2\hat{\alpha}}\right)^{-\beta}\right)}{\sum\limits_{\nu_1=-\infty}^{\infty} \cdots \sum\limits_{\nu_N=-\infty}^{\infty} \left(\prod\limits_{i=1}^{N} \left(\nu_i^{2\hat{\alpha}} + \xi_n^{2\hat{\alpha}}\right)^{-\beta}\right)}, \tag{1.37}$$

are uniformly bounded for all $\xi_n \in (0,1]$ *where* $\hat{\alpha}$, $n \in \mathbb{N}$,

$$\beta > \max\left\{\frac{1+\lceil \alpha_i p\rceil + \lceil rp\rceil}{2\hat{\alpha}}, \frac{2+\lceil rp\rceil}{2\hat{\alpha}}\right\}$$

for all $i = 1, ..., N$, *and* $\nu = (\nu_1, ..., \nu_N)$.

Finally, in [5], when $p \geq 1$, they obtained the following inequalities for the error quantities $E^{[0]}_{n,P}(f;x)$, $E^{[0]}_{n,P}(f;x)$, and the errors $E^{[m]}_{n,P}(f;x)$, $E^{[m]}_{n,P}(f;x)$:

$$\left\|E^{[0]}_{n,P}(f)\right\|_p \leq m_{\xi_n,P} \left\|P^{*[0]}_{r,n}(f) - f\right\|_p + \|f\|_p \left|m_{\xi_n,P} - 1\right|. \tag{1.38}$$

$$\left\| E_{n,W}^{[0]}(f) \right\|_p \leq m_{\xi_n,W} \left\| W_{r,n}^{*[0]}(f) - f \right\|_p + \|f\|_p \left| m_{\xi_n,W} - 1 \right|, \tag{1.39}$$

$$\begin{aligned} \left\| E_{n,P}^{[m]}(f;x) \right\|_p &\leq m_{\xi_n,P} \left\| P_{r,n}^{*\,[m]}(f) - f - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \right. \\ &\quad \times \left. \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{c_{\alpha,n,\widetilde{j}} f_\alpha}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_p \\ &\quad + \|f\|_p \left| m_{\xi_n,P} - 1 \right|, \end{aligned} \tag{1.40}$$

and

$$\begin{aligned} \left\| E_{n,W}^{[m]}(f) \right\|_p &\leq m_{\xi_n,W} \left\| W_{r,n}^{*\,[m]}(f) - f - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \right. \\ &\quad \times \left. \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{p_{\alpha,n,\widetilde{j}} f_\alpha}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_p \\ &\quad + \|f\|_p \left| m_{\xi_n,W} - 1 \right|. \end{aligned} \tag{1.41}$$

2. Main Results

We start with the general global smoothness preservation results for the operators $P_{r,n}^{*\,[m]}$, $W_{r,n}^{*\,[m]}$, and $Q_{r,n}^{*\,[m]}$, defined as in (1.8), (1.10), and (1.12).

Theorem 2.1. *Let $h > 0$, $f \in C\left(\mathbb{R}^N\right)$, $N \geq 1$.*

i) Assume $\omega_{\bar{m}}(f,h) < \infty$. Then

$$\omega_{\bar{m}} \left(P_{r,n}^{*\,[m]} f, h \right) \leq \left(\sum_{j=0}^{r} \left| \alpha_{j,r}^{[\bar{m}]} \right| \right) \omega_{\bar{m}}(f,h), \tag{2.1}$$

$$\omega_{\bar{m}} \left(W_{r,n}^{*\,[m]} f, h \right) \leq \left(\sum_{j=0}^{r} \left| \alpha_{j,r}^{[\bar{m}]} \right| \right) \omega_{\bar{m}}(f,h), \tag{2.2}$$

$$\omega_{\bar{m}} \left(Q_{r,n}^{*\,[m]} f, h \right) \leq \left(\sum_{j=0}^{r} \left| \alpha_{j,r}^{[\bar{m}]} \right| \right) \omega_{\bar{m}}(f,h). \tag{2.3}$$

ii) Assume $f \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right) \right)$, $p \geq 1$. Then

$$\omega_{\bar{m}} \left(P_{r,n}^{*\,[m]} f, h \right)_p \leq \left(\sum_{j=0}^{r} \left| \alpha_{j,r}^{[\bar{m}]} \right| \right) \omega_{\bar{m}}(f,h)_p, \tag{2.4}$$

$$\omega_{\bar{m}}\left(W_{r,n}^{*\,[m]}f,h\right)_p \leq \left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f,h)_p, \tag{2.5}$$

$$\omega_{\bar{m}}\left(Q_{r,n}^{*\,[m]}f,h\right)_p \leq \left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f,h)_p. \tag{2.6}$$

Proof. By [1], Chapter 3. □

Next, we give

Remark 2.2. Let $r=1$, then we calculate that $\alpha_{0,1}^{[m]}=0$, $\alpha_{1,1}^{[m]}=1$. Now, denote

$$P_{1,n}^{*\,[m]}(f;x) := P_n^{*\,[m]}(f;x), \tag{2.7}$$

$$W_{1,n}^{*\,[m]}(f;x) := W_n^{*\,[m]}(f;x), \tag{2.8}$$

$$Q_{1,n}^{*\,[m]}(f;x) := Q_n^{*\,[m]}(f;x). \tag{2.9}$$

By Theorem 2.1 and Remark 2.2, we obtain

Theorem 2.3. *Let $h>0$, $f\in C\left(\mathbb{R}^N\right)$, $N\geq 1$.*
i) Assume $\omega_{\bar{m}}(f,h)<\infty$. Then

$$\omega_{\bar{m}}\left(P_n^{*\,[m]}f,h\right) \leq \omega_{\bar{m}}(f,h), \tag{2.10}$$

$$\omega_{\bar{m}}\left(W_n^{*\,[m]}f,h\right) \leq \omega_{\bar{m}}(f,h), \tag{2.11}$$

$$\omega_{\bar{m}}\left(Q_n^{*\,[m]}f,h\right) \leq \omega_{\bar{m}}(f,h). \tag{2.12}$$

ii) Assume $f\in\left(C\left(\mathbb{R}^N\right)\cap L_p\left(\mathbb{R}^N\right)\right)$, $p\geq 1$. Then

$$\omega_{\bar{m}}\left(P_n^{*\,[m]}f,h\right)_p \leq \omega_{\bar{m}}(f,h)_p, \tag{2.13}$$

$$\omega_{\bar{m}}\left(W_n^{*\,[m]}f,h\right)_p \leq \omega_{\bar{m}}(f,h)_p, \tag{2.14}$$

$$\omega_{\bar{m}}\left(Q_n^{*\,[m]}f,h\right)_p \leq \omega_{\bar{m}}(f,h)_p. \tag{2.15}$$

We present the our general global smoothness preservation results for the non-unitary operators $P_{r,n}^{[m]}$ and $W_{r,n}^{[m]}$ as follows

Theorem 2.4. *Let $h>0$, $f\in C\left(\mathbb{R}^N\right)$, $N\geq 1$.*
i) Assume $\omega_{\bar{m}}(f,h)<\infty$. Then

$$\omega_{\bar{m}}\left(P_{r,n}^{[m]}f,h\right) \tag{2.16}$$

$$\leq \left(\frac{1+2e^{-1/\xi_n}(\xi_n+1)}{1+2\xi_n e^{-1/\xi_n}}\right)^N\left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f,h),$$

$$\omega_{\bar{m}}\left(W_{r,n}^{[m]}f,h\right) \tag{2.17}$$
$$\leq \left\{1+\frac{2e^{-1/\xi_n}}{\sqrt{\pi\xi_n}\left[1-\operatorname{erf}\left(\frac{1}{\xi_n}\right)\right]+1}\right\}^N\left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}\left(f,h\right).$$

ii) Assume $f\in\left(C\left(\mathbb{R}^N\right)\cap L_p\left(\mathbb{R}^N\right)\right)$, $p\geq 1$. *Then*

$$\omega_{\bar{m}}\left(P_{r,n}^{[m]}f,h\right)_p \tag{2.18}$$
$$\leq \left(\frac{1+2e^{-1/\xi_n}\left(\xi_n+1\right)}{1+2\xi_n e^{-1/\xi_n}}\right)^N\left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}\left(f,h\right)_p,$$

$$\omega_{\bar{m}}\left(W_{r,n}^{[m]}f,h\right)_p \tag{2.19}$$
$$\leq \left\{1+\frac{2e^{-1/\xi_n}}{\sqrt{\pi\xi_n}\left[1-\operatorname{erf}\left(\frac{1}{\xi_n}\right)\right]+1}\right\}^N\left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}\left(f,h\right)_p.$$

Proof. We see that

$$P_{r,n}^{[m]}\left(f;x\right)=\lambda_1\left(\xi_n\right)P_{r,n}^{*[m]}\left(f;x\right), \tag{2.20}$$

and

$$W_{r,n}^{[m]}\left(f;x\right)=\lambda_2\left(\xi_n\right)W_{r,n}^{*[m]}\left(f;x\right), \tag{2.21}$$

where

$$\lambda_1\left(\xi_n\right) := \frac{\sum_{\nu_1=-\infty}^{\infty}\cdots\sum_{\nu_N=-\infty}^{\infty}e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}{\left(1+2\xi_n e^{-1/\xi_n}\right)^N} = \prod_{i=1}^{N}\left(\frac{\sum_{\nu_i=-\infty}^{\infty}e^{-\frac{|\nu_i|}{\xi_n}}}{1+2\xi_n e^{-1/\xi_n}}\right), \tag{2.22}$$

and

$$\lambda_2\left(\xi_n\right) := \frac{\sum_{\nu_1=-\infty}^{\infty}\cdots\sum_{\nu_N=-\infty}^{\infty}e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\left[\sqrt{\pi\xi_n}\left(1-\operatorname{erf}\left(\frac{1}{\xi_n}\right)\right)+1\right]^N} = \prod_{i=1}^{N}\left(\frac{\sum_{\nu_i=-\infty}^{\infty}e^{-\frac{\nu_i^2}{\xi_n}}}{\sqrt{\pi\xi_n}\left[1-\operatorname{erf}\left(\frac{1}{\xi_n}\right)\right]+1}\right). \tag{2.23}$$

Additionally, in [2], the author showed that

$$\frac{\sum\limits_{\nu_i=-\infty}^{\infty} e^{-\frac{|\nu_i|}{\xi_n}}}{1+2\xi_n e^{-1/\xi_n}} \leq \frac{1+2e^{\frac{-1}{\xi}}(\xi+1)}{1+2\xi e^{\frac{-1}{\xi}}}, \tag{2.24}$$

and

$$\frac{\sum\limits_{\nu_i=-\infty}^{\infty} e^{-\frac{\nu_i^2}{\xi_n}}}{\sqrt{\pi\xi_n}\left[1-\operatorname{erf}\left(\frac{1}{\xi_n}\right)\right]+1} \leq 1+\frac{2e^{\frac{-1}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}. \tag{2.25}$$

Thus, by (2.22), (2.23), (2.24), (2.25), and Theorem 2.1 the proof is complete. □

Now, we demonstrate the following optimality result

Proposition 2.5. *Above inequalities* (2.10)-(2.12) *are sharp. The equalities are attained by any*

$$g_j(x) = x_j^{\overline{m}}, \quad j=1,...,N, \quad x=(x_1,...,x_j,...,x_N) \in \mathbb{R}^N.$$

Proof. By [1], Chapter 3. □

In [6], the authors observed

Theorem 2.6. *Let* $f \in C^l(\mathbb{R}^N)$, $l, N \in \mathbb{N}$. *Here* μ_{ξ_n} *is a Borel probability measure on* $\mathbb{R}^N$, $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *a bounded sequence. Let* $\tilde{\beta} := \left(\tilde{\beta}_1, ..., \tilde{\beta}_N\right)$, $\tilde{\beta}_i \in \mathbb{Z}^+$, $i=1,...,N$; $\left|\tilde{\beta}\right| := \sum\limits_{i=1}^{N} \tilde{\beta}_i = l$. *Here* $f(x+\nu j)$, $x \in \mathbb{R}^N$, $\nu \in \mathbb{Z}^N$, *is* μ_{ξ_n}*-integrable with respect to* ν, *for* $j=1,...,r$. *There exist* μ_{ξ_n}*-integrable functions* $h_{i_1,j}$, $h_{\tilde{\beta}_1,i_2,j}$, $h_{\tilde{\beta}_1,\tilde{\beta}_2,i_3,j}, ..., h_{\tilde{\beta}_1,\tilde{\beta}_2,...,\tilde{\beta}_{N-1},i_N,j} \geq 0$ $(j=1,...,r)$ *on* $\mathbb{R}^N$ *such that*

$$\left|\frac{\partial^{i_1} f(x+\nu j)}{\partial x_1^{i_1}}\right| \leq h_{i_1,j}(\nu), \quad i_1 = 1,...,\tilde{\beta}_1, \tag{2.26}$$

$$\left|\frac{\partial^{\tilde{\beta}_1+i_2} f(x+\nu j)}{\partial x_2^{i_2}\partial x_1^{\tilde{\beta}_1}}\right| \leq h_{\tilde{\beta}_1,i_2,j}(\nu), \quad i_2 = 1,...,\tilde{\beta}_2,$$

$$\vdots$$

$$\left|\frac{\partial^{\tilde{\beta}_1+\tilde{\beta}_2+...+\tilde{\beta}_{N-1}+i_N} f(x+\nu j)}{\partial x_N^{i_N}\partial x_{N-1}^{\tilde{\beta}_{N-1}}...\partial x_2^{\tilde{\beta}_2}\partial x_1^{\tilde{\beta}_1}}\right| \leq h_{\tilde{\beta}_1,\tilde{\beta}_2,...,\tilde{\beta}_{N-1},i_N,j}(\nu), \quad i_N = 1,...,\tilde{\beta}_N,$$

$\forall\ x \in \mathbb{R}^N$, $\nu \in \mathbb{Z}^N$.

i) When

$$\mu_{\xi_n}(\nu) = \frac{e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}{\sum\limits_{\nu_1=-\infty}^{\infty} ... \sum\limits_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N}|\nu_i|}{\xi_n}}}, \tag{2.27}$$

then both of the next exist and

$$\left(P_{r,n}^{*\,[m]}(f;x)\right)_{\tilde{\beta}} = P_{r,n}^{*\,[m]}\left(f_{\tilde{\beta}};x\right). \tag{2.28}$$

ii) When

$$\mu_{\xi_n}(\nu) = \frac{e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}{\sum_{\nu_1=-\infty}^{\infty}\cdots\sum_{\nu_N=-\infty}^{\infty} e^{-\frac{\sum_{i=1}^{N}\nu_i^2}{\xi_n}}}, \tag{2.29}$$

then both of the next exist and

$$\left(W_{r,n}^{*\,[m]}(f;x)\right)_{\tilde{\beta}} = W_{r,n}^{*\,[m]}\left(f_{\tilde{\beta}};x\right). \tag{2.30}$$

iii) Let $\hat{\alpha} \in \mathbb{N}$ *and* $\beta > \frac{1}{\hat{\alpha}}$. *When*

$$\mu_{\xi_n}(\nu) = \frac{\prod_{i=1}^{N}\left(\nu_i^{2\hat{\alpha}} + \xi_n^{2\hat{\alpha}}\right)^{-\beta}}{\sum_{\nu_1=-\infty}^{\infty}\cdots\sum_{\nu_N=-\infty}^{\infty}\prod_{i=1}^{N}\left(\nu_i^{2\hat{\alpha}} + \xi_n^{2\hat{\alpha}}\right)^{-\beta}}, \tag{2.31}$$

then both of the next exist and

$$\left(Q_{r,n}^{*\,[m]}(f;x)\right)_{\tilde{\beta}} = Q_{r,n}^{*\,[m]}\left(f_{\tilde{\beta}};x\right). \tag{2.32}$$

Corollary 2.7. *When* $r = 1$, *by the Theorem* 2.6, *we observe that*

$$\left(P_{n}^{*\,[m]}(f;x)\right)_{\tilde{\beta}} = P_{n}^{*\,[m]}\left(f_{\tilde{\beta}};x\right), \tag{2.33}$$

$$\left(W_{n}^{*\,[m]}(f;x)\right)_{\tilde{\beta}} = W_{n}^{*\,[m]}\left(f_{\tilde{\beta}};x\right), \tag{2.34}$$

and

$$\left(Q_{n}^{*\,[m]}(f;x)\right)_{\tilde{\beta}} = Q_{n}^{*\,[m]}\left(f_{\tilde{\beta}};x\right). \tag{2.35}$$

For the non-unitary operators $P_{r,n}^{[m]}$ and $W_{r,n}^{[m]}$ we have

Theorem 2.8. *Let the assumption of Theorem* 2.6 *be true. Then we have*

$$\left(P_{r,n}^{[m]}(f;x)\right)_{\tilde{\beta}} = P_{r,n}^{[m]}\left(f_{\tilde{\beta}};x\right), \tag{2.36}$$

and

$$\left(W_{r,n}^{[m]}(f;x)\right)_{\tilde{\beta}} = W_{r,n}^{[m]}\left(f_{\tilde{\beta}};x\right). \tag{2.37}$$

Proof. By (2.20), (2.21) , and Theorem 2.6, we obtain

$$\begin{aligned}\left(P_{r,n}^{[m]}(f;x)\right)_{\tilde{\beta}} &= \lambda_1(\xi_n)\left(P_{r,n}^{*[m]}(f;x)\right)_{\tilde{\beta}} \\ &= \lambda_1(\xi_n)\,P_{r,n}^{*[m]}\left(f_{\tilde{\beta}};x\right) = P_{r,n}^{[m]}\left(f_{\tilde{\beta}};x\right),\end{aligned} \tag{2.38}$$

and

$$\left(W_{r,n}^{[m]}(f;x)\right)_{\tilde{\beta}} = \lambda_2(\xi_n)\left(W_{r,n}^{*[m]}(f;x)\right)_{\tilde{\beta}} = \lambda_2(\xi_n) W_{r,n}^{*[m]}\left(f_{\tilde{\beta}};x\right) = W_{r,n}^{[m]}\left(f_{\tilde{\beta}};x\right). \tag{2.39}$$

□

Next, we get

Theorem 2.9. *Let $h > 0$, $\gamma = 0, \tilde{\beta}$, and the assumptions of the Theorem 2.6 be true.*
i) Assume $\omega_{\bar{m}}(f_\gamma, h) < \infty$. Then

$$\omega_{\bar{m}}\left(\left(P_{r,n}^{*\,[m]}f\right)_\gamma, h\right) \leq \left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f_\gamma, h), \tag{2.40}$$

$$\omega_{\bar{m}}\left(\left(W_{r,n}^{*\,[m]}f\right)_\gamma, h\right) \leq \left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f_\gamma, h), \tag{2.41}$$

$$\omega_{\bar{m}}\left(\left(Q_{r,n}^{*\,[m]}f\right)_\gamma, h\right) \leq \left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f_\gamma, h). \tag{2.42}$$

ii) Assume $f_\gamma \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$, $p \geq 1$. Then

$$\omega_{\bar{m}}\left(\left(P_{r,n}^{*\,[m]}f\right)_\gamma, h\right)_p \leq \left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f_\gamma, h)_p, \tag{2.43}$$

$$\omega_{\bar{m}}\left(\left(W_{r,n}^{*\,[m]}f\right)_\gamma, h\right)_p \leq \left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f_\gamma, h)_p, \tag{2.44}$$

$$\omega_{\bar{m}}\left(\left(Q_{r,n}^{*\,[m]}f\right)_\gamma, h\right)_p \leq \left(\sum_{j=0}^{r}\left|\alpha_{j,r}^{[\bar{m}]}\right|\right)\omega_{\bar{m}}(f_\gamma, h)_p. \tag{2.45}$$

Proof. By Theorem 2.1 and Theorem 2.6. □

Additionally, as a quick result of Theorem 2.3 and Theorem 2.6, we have

Corollary 2.10. *Let $h > 0$, $\gamma = 0, \tilde{\beta}$, and the assumptions of the Theorem 2.6 be true.*
i) Assume $\omega_{\bar{m}}(f, h) < \infty$. Then

$$\omega_{\bar{m}}\left(\left(P_n^{*\,[m]}f\right)_\gamma, h\right) \leq \omega_{\bar{m}}(f_\gamma, h), \tag{2.46}$$

$$\omega_{\bar{m}}\left(\left(W_n^{*\,[m]}f\right)_\gamma, h\right) \leq \omega_{\bar{m}}(f_\gamma, h), \tag{2.47}$$

$$\omega_{\bar{m}}\left(\left(Q_n^{*\,[m]}f\right)_\gamma, h\right) \leq \omega_{\bar{m}}(f_\gamma, h). \tag{2.48}$$

ii) Assume $f \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$, $p \geq 1$. *Then*

$$\omega_{\bar{m}}\left(\left(P_n^{*\,[m]} f\right)_\gamma, h\right)_p \leq \omega_{\bar{m}}\left(f_\gamma, h\right)_p, \tag{2.49}$$

$$\omega_{\bar{m}}\left(\left(W_n^{*\,[m]} f\right)_\gamma, h\right)_p \leq \omega_{\bar{m}}\left(f_\gamma, h\right)_p, \tag{2.50}$$

$$\omega_{\bar{m}}\left(\left(Q_n^{*\,[m]} f\right)_\gamma, h\right)_p \leq \omega_{\bar{m}}\left(f_\gamma, h\right)_p. \tag{2.51}$$

Additionally for the non-unitary operators, $P_{r,n}^{[m]}$ and $W_{r,n}^{[m]}$, we obtain

Theorem 2.11. *Let* $h > 0$, $\gamma = 0, \tilde{\beta}$, *and the assumptions of the Theorem* 2.6 *be true.*

i) Assume $\omega_{\bar{m}}\left(f_\gamma, h\right) < \infty$. *Then*

$$\omega_{\bar{m}}\left(\left(P_{r,n}^{[m]} f\right)_\gamma, h\right) \tag{2.52}$$

$$\leq \left(\frac{1 + 2e^{-1/\xi_n}\left(\xi_n + 1\right)}{1 + 2\xi_n e^{-1/\xi_n}}\right)^N \left(\sum_{j=0}^r \left|\alpha_{j,r}^{[\bar{m}]}\right|\right) \omega_{\bar{m}}\left(f_\gamma, h\right),$$

$$\omega_{\bar{m}}\left(\left(W_{r,n}^{[m]} f\right)_\gamma, h\right) \tag{2.53}$$

$$\leq \left\{1 + \frac{2e^{-1/\xi_n}}{\sqrt{\pi}\xi_n\left[1 - \operatorname{erf}\left(\frac{1}{\xi_n}\right)\right] + 1}\right\}^N \left(\sum_{j=0}^r \left|\alpha_{j,r}^{[\bar{m}]}\right|\right) \omega_{\bar{m}}\left(f_\gamma, h\right).$$

ii) Assume $f_\gamma \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$, $p \geq 1$. *Then*

$$\omega_{\bar{m}}\left(\left(P_{r,n}^{[m]} f\right)_\gamma, h\right)_p \tag{2.54}$$

$$\leq \left(\frac{1 + 2e^{-1/\xi_n}\left(\xi_n + 1\right)}{1 + 2\xi_n e^{-1/\xi_n}}\right)^N \left(\sum_{j=0}^r \left|\alpha_{j,r}^{[\bar{m}]}\right|\right) \omega_{\bar{m}}\left(f_\gamma, h\right)_p,$$

$$\omega_{\bar{m}}\left(\left(W_{r,n}^{[m]} f\right)_\gamma, h\right)_p \tag{2.55}$$

$$\leq \left\{1 + \frac{2e^{-1/\xi_n}}{\sqrt{\pi}\xi_n\left[1 - \operatorname{erf}\left(\frac{1}{\xi_n}\right)\right] + 1}\right\}^N \left(\sum_{j=0}^r \left|\alpha_{j,r}^{[\bar{m}]}\right|\right) \omega_{\bar{m}}\left(f_\gamma, h\right)_p.$$

Proof. By Theorem 2.4 and Theorem 2.8. □

Now we show our simultaneous approximation results.

We start with

Theorem 2.12. *Let $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $m, l \in \mathbb{N}$, $N \geq 1$, $x \in \mathbb{R}^N$. Let the assumptions of Theorem 2.6 is true and $\gamma = 0, \tilde{\beta}$. Assume $\left\| f_{\gamma+\alpha} \right\|_\infty < \infty$. Then for all $x \in \mathbb{R}^N$, we have*

i)

$$\left\| \left(P_{r,n}^{*\,[m]} f \right)_\gamma - f_\gamma - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{c_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_\infty \tag{2.56}$$

$$\leq \sum_{\left(\substack{\alpha_1,\ldots,\alpha_N \geq 0 \\ |\alpha| = m}\right)} \frac{\left(\omega_r\left(f_{\gamma+\alpha}, \xi_n\right)\right)}{\left(\prod_{i=1}^{N} \alpha_i!\right)} u_{P,\xi_n}^{*},$$

for $\xi_n \in (0,1]$.

ii)

$$\left\| \left(W_{r,n}^{*\,[m]} f \right)_\gamma - f_\gamma - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{p_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_\infty \tag{2.57}$$

$$\leq \sum_{\left(\substack{\alpha_1,\ldots,\alpha_N \geq 0 \\ |\alpha| = m}\right)} \frac{\left(\omega_r\left(f_{\gamma+\alpha}, \xi_n\right)\right)}{\left(\prod_{i=1}^{N} \alpha_i!\right)} u_{W,\xi_n}^{*},$$

for $\xi_n \in (0,1]$.

iii)

$$\left\| \left(Q_{r,n}^{*\,[m]} f \right)_\gamma - f_\gamma - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{q_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_\infty \tag{2.58}$$

$$\leq \sum_{\left(\substack{\alpha_1,\ldots,\alpha_N \geq 0 \\ |\alpha| = m}\right)} \frac{\left(\omega_r\left(f_{\gamma+\alpha}, \xi_n\right)\right)}{\left(\prod_{i=1}^{N} \alpha_i!\right)} u_{Q,\xi_n}^{*},$$

for $\xi_n \in (0,1]$, and $\hat{\alpha} \in \mathbb{N}$, $\beta > \max\left\{\frac{1+r+\alpha_i}{2\hat{\alpha}},\ \frac{r+2}{2\hat{\alpha}}\right\}$.

Proof. By [4] and Theorem 2.6. □

Next, when $m = 0$, we obtain

Theorem 2.13. *Let $f \in C_B^l\left(\mathbb{R}^N\right)$, $l \in \mathbb{N}$, $N \geq 1$. Let the assumptions of Theorem 2.6 is true and $\gamma = 0, \tilde{\beta}$. Then for all $x \in \mathbb{R}^N$, we have* *i)*

$$\left\| \left(P^{*}{}_{r,n}^{[0]} f \right)_{\gamma} - f_{\gamma} \right\|_{\infty} \leq \Phi^{*}_{P,\xi_n} \omega_r(f_{\gamma}, \xi_n), \tag{2.59}$$

for $\xi_n \in (0, 1]$.

ii)

$$\left\| \left(W^{*}{}_{r,n}^{[0]} f \right)_{\gamma} - f_{\gamma} \right\|_{\infty} \leq \Phi^{*}_{W,\xi_n} \omega_r(f_{\gamma}, \xi_n), \tag{2.60}$$

for $\xi_n \in (0, 1]$.

iii)

$$\left\| \left(Q^{*}{}_{r,n}^{[0]} f \right)_{\gamma} - f_{\gamma} \right\|_{\infty} \leq \Phi^{*}_{Q,\xi_n} \omega_r(f_{\gamma}, \xi_n), \tag{2.61}$$

for $\xi_n \in (0, 1]$, *and* $\hat{\alpha} \in \mathbb{N}$, $\beta > \frac{r+2}{2\hat{\alpha}}$.

Proof. By [4] and Theorem 2.6. □

For the non-unitary cases we have

Theorem 2.14. *Let* $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $m, l \in \mathbb{N}$, $N \geq 1$. *Let the assumptions of Theorem* 2.6 *is true and* $\gamma = 0, \tilde{\beta}$. *Assume* $\|f_{\gamma+\alpha}\|_{\infty} < \infty$. *Then for all* $x \in \mathbb{R}^N$, *we have*

i)

$$\begin{aligned} &\left\| \left(E^{[m]}_{n,P}(f) \right)_{\gamma} \right\|_{\infty} \\ \leq\ & m_{\xi_n,P} \sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0 \\ |\alpha| = m}} \frac{\omega_r\left(f_{\alpha+\gamma}, \xi_n\right)}{\left(\prod_{i=1}^{N} \alpha_i!\right)} u^{*}_{P,\xi_n} \\ &+ \|f_{\gamma}\|_{\infty} \left| m_{\xi_n,P} - 1 \right|, \end{aligned} \tag{2.62}$$

and

$$\begin{aligned} &\left\| \left(E^{[m]}_{n,W}(f) \right)_{\gamma} \right\|_{\infty} \\ \leq\ & m_{\xi_n,W} \sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0 \\ |\alpha| = m}} \frac{\omega_r\left(f_{\alpha+\gamma}, \xi_n\right)}{\left(\prod_{i=1}^{N} \alpha_i!\right)} u^{*}_{W,\xi_n} \\ &+ \|f_{\gamma}\|_{\infty} \left| m_{\xi_n,W} - 1 \right|. \end{aligned} \tag{2.63}$$

ii) Let $f \in C^l_B\left(\mathbb{R}^N\right)$, $l \in \mathbb{N}$, $N \geq 1$. *Let the assumptions of Theorem* 2.6 *is true and* $\gamma = 0, \tilde{\beta}$. *Then for all* $x \in \mathbb{R}^N$, *we have*

$$\begin{aligned} &\left\| \left(E^{[0]}_{n,P}(f) \right)_{\gamma} \right\|_{\infty} \\ \leq\ & m_{\xi_n,P} \Phi^{*}_{P,\xi_n} \omega_r(f_{\gamma}, \xi_n) + \|f_{\gamma}\|_{\infty} \left| m_{\xi_n,P} - 1 \right|, \end{aligned} \tag{2.64}$$

and

$$\left\| \left(E_{n,W}^{[0]}(f;x) \right)_{\gamma} \right\|_{\infty} \leq m_{\xi_n,W} \Phi_{W,\xi_n}^{*} \omega_r(f_\gamma,\xi_n) + \|f_\gamma\|_{\infty} |m_{\xi_n,W} - 1| . \tag{2.65}$$

Proof. By [4], (1.38) – (1.41), and by the equalities $\left(E_{n,P}^{[m]}(f,x)\right)_{\gamma} = E_{n,P}^{[m]}(f_\gamma,x)$ and $\left(E_{n,W}^{[m]}(f,x)\right)_{\gamma} = E_{n,W}^{[m]}(f_\gamma,x)$ for $m \in \mathbb{Z}^+$. □

Now, we give our L_p results. We begin with

Theorem 2.15. *Let* $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $m,l \in \mathbb{N}$, $N \geq 1$, $\gamma = 0, \tilde{\beta}$, $f_{\gamma+\alpha} \in L_p\left(\mathbb{R}^N\right)$, $|\alpha| = m$, $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. *Let the assumptions of Theorem* 2.6 *be true. Then*

i)

$$\left\| \left(P_{r,n}^{*\,[m]} f\right)_{\gamma} - f_\gamma - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{c_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_p \tag{2.66}$$

$$\leq \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \left(S_{P^*,\xi_n}^{p,m} \right)^{\frac{1}{p}} \omega_r\left(f_{\gamma+\alpha},\xi_n\right)_p .$$

ii)

$$\left\| \left(W_{r,n}^{*\,[m]} f\right)_{\gamma} - f_\gamma - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{p_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_p \tag{2.67}$$

$$\leq \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \left(S_{W^*,\xi_n}^{p,m} \right)^{\frac{1}{p}} \omega_r\left(f_{\gamma+\alpha},\xi_n\right)_p .$$

iii)

$$\left\| \left(Q_{r,n}^{*\,[m]} f\right)_{\gamma} - f_\gamma - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{q_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_p \tag{2.68}$$

$$\leq \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \left(S_{Q^*,\xi_n}^{p,m} \right)^{\frac{1}{p}} \omega_r\left(f_{\gamma+\alpha},\xi_n\right)_p ,$$

where $\hat{\alpha} \in \mathbb{N}$, $\beta > \max\left\{\frac{1+\lceil \alpha_i p\rceil + \lceil rp\rceil}{2\hat{\alpha}}, \ \frac{2+\lceil rp\rceil}{2\hat{\alpha}}\right\}$ *for all* $i = 1, ..., N$.

Proof. By [5] and Theorem 2.6. □

Next, we present our results for the case of $m = 0$ and $p > 1$.

Theorem 2.16. *Let* $f \in C^l\left(\mathbb{R}^N\right)$, $l \in \mathbb{N}$, $N \geq 1$, $\gamma = 0, \tilde{\beta}$, $f_\gamma \in L_p\left(\mathbb{R}^N\right), x \in \mathbb{R}^N$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. *Let the assumptions of Theorem* 2.6 *be true. Then*

i)

$$\left\| \left(P_{r,n}^{*\,[0]} f\right)_\gamma - f_\gamma \right\|_p \leq \left(S_{P^*,\xi_n}^{p,0}\right)^{\frac{1}{p}} \omega_r\left(f_\gamma, \xi_n\right)_p. \tag{2.69}$$

ii)

$$\left\| \left(W_{r,n}^{*\,[0]} f\right)_\gamma - f_\gamma \right\|_p \leq \left(S_{W^*,\xi_n}^{p,0}\right)^{\frac{1}{p}} \omega_r\left(f_\gamma, \xi_n\right)_p. \tag{2.70}$$

iii)

$$\left\| \left(Q_{r,n}^{*\,[0]} f\right)_\gamma - f_\gamma \right\|_p \leq \left(S_{Q^*,\xi_n}^{p,0}\right)^{\frac{1}{p}} \omega_r\left(f_\gamma, \xi_n\right)_p, \tag{2.71}$$

where $\hat{\alpha} \in \mathbb{N}$, $\beta > \frac{2+\lceil rp\rceil}{2\hat{\alpha}}$.

Proof. By [5] and Theorem 2.6. □

For the case of $m = 0$ and $p = 1$, we have

Theorem 2.17. *Let* $f \in C^l\left(\mathbb{R}^N\right)$, $l \in \mathbb{N}$, $N \geq 1$, $\gamma = 0, \tilde{\beta}$, $f_\gamma \in L_1\left(\mathbb{R}^N\right), x \in \mathbb{R}^N$, *and* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. *Let the assumptions of Theorem* 2.6 *be true.*

i)

$$\left\| \left(P_{r,n}^{*\,[0]} f\right)_\gamma - f_\gamma \right\|_1 \leq S_{P^*,\xi_n}^{1,0} \omega_r\left(f_\gamma, \xi_n\right)_1. \tag{2.72}$$

ii)

$$\left\| \left(W_{r,n}^{*\,[0]} f\right)_\gamma - f_\gamma \right\|_1 \leq S_{W^*,\xi_n}^{1,0} \omega_r\left(f_\gamma, \xi_n\right)_1. \tag{2.73}$$

iii)

$$\left\| \left(Q_{r,n}^{*\,[0]} f\right)_\gamma - f_\gamma \right\|_1 \leq S_{Q^*,\xi_n}^{1,0} \omega_r\left(f_\gamma, \xi_n\right)_1, \tag{2.74}$$

where $\hat{\alpha} \in \mathbb{N}$, $\beta > \frac{2+r}{2\hat{\alpha}}$.

Proof. By [5] and Theorem 2.6. □

Next, we give the case of $m \in \mathbb{N}$ and $p = 1$ as

Theorem 2.18. *Let* $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $m, l \in \mathbb{N}$, $N \geq 1$, $\gamma = 0, \tilde{\beta}$, $f_{\gamma+\alpha} \in L_1\left(\mathbb{R}^N\right)$, $|\alpha| = m$, $x \in \mathbb{R}$, *and* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. *Let the assumptions of Theorem* 2.6 *be true. Then* *i)*

$$\left\| \left(P^{*\,[m]}_{r,n} f \right)_\gamma - f_\gamma - \sum_{\widetilde{j}=1}^{m} \delta^{[m]}_{\widetilde{j},r} \left(\sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{c_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_1 \tag{2.75}$$

$$\leq \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) S^{1,m}_{P^*,\xi_n} \omega_r \left(f_{\gamma+\alpha}, \xi_n \right)_1 .$$

ii)

$$\left\| \left(W^{*\,[m]}_{r,n} f \right)_\gamma - f_\gamma - \sum_{\widetilde{j}=1}^{m} \delta^{[m]}_{\widetilde{j},r} \left(\sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{p_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_1 \tag{2.76}$$

$$\leq \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) S^{1,m}_{W^*,\xi_n} \omega_r \left(f_{\gamma+\alpha}, \xi_n \right)_1 .$$

iii)

$$\left\| \left(Q^{*\,[m]}_{r,n} f \right)_\gamma - f_\gamma - \sum_{\widetilde{j}=1}^{m} \delta^{[m]}_{\widetilde{j},r} \left(\sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{q_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_1 \tag{2.77}$$

$$\leq \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) S^{1,m}_{Q^*,\xi_n} \omega_r \left(f_{\gamma+\alpha}, \xi_n \right)_1 ,$$

where $\hat{\alpha} \in \mathbb{N}$ *,* $\beta > \max\left\{ \frac{1+\alpha_i+r}{2\hat{\alpha}},\ \frac{2+r}{2\hat{\alpha}} \right\}$ *for all* i*.*

Proof. By [5] and Theorem 2.6. □

Finally, we give our L_p results for the error quantities $E^{[0]}_{n,P}(f;x)$, $E^{[0]}_{n,P}(f;x)$, and the errors $E^{[m]}_{n,P}(f;x)$, $E^{[m]}_{n,P}(f;x)$. We begin with

Theorem 2.19. *Let* $f \in C^{m+l}\left(\mathbb{R}^N\right)$*,* $m, l \in \mathbb{N}$*,* $N \geq 1$*,* $\gamma = 0, \tilde{\beta}$*,* $f_{\gamma+\alpha} \in L_p\left(\mathbb{R}^N\right)$*,* $|\alpha| = m$*,* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*, and* $0 < \xi_n \leq 1$*,* $n \in \mathbb{N}$*. Let the assumptions of Theorem*

2.6 *be true. Then*

$$\left\| \left(E_{n,P}^{[m]}(f) \right)_\gamma \right\|_p \tag{2.78}$$
$$\leq m_{\xi_n,P} \left(\frac{m \left(S_{P^*,\xi_n}^{p,m} \right)^{\frac{1}{p}} \omega_r \left(f_{\alpha+\gamma}, \xi_n \right)_p}{\left(q\left(m-1 \right)+1 \right)^{\frac{1}{q}}} \right) \sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!}$$
$$+ \left\| f_\gamma \right\|_p \left| m_{\xi_n,P} - 1 \right|,$$

and

$$\left\| \left(E_{n,W}^{[m]}(f) \right)_\gamma \right\|_p \tag{2.79}$$
$$\leq m_{\xi_n,W} \left(\frac{m \left(S_{W^*,\xi_n}^{p,m} \right)^{\frac{1}{p}} \omega_r \left(f_{\alpha+\gamma}, \xi_n \right)_p}{\left(q\left(m-1 \right)+1 \right)^{\frac{1}{q}}} \right) \sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!}$$
$$+ \left\| f_\gamma \right\|_p \left| m_{\xi_n,W} - 1 \right|.$$

Proof. By [5], (1.40), (1.41), and by the equalities $\left(E_{n,P}^{[m]}(f,x) \right)_\gamma = E_{n,P}^{[m]}(f_\gamma,x)$ and $\left(E_{n,W}^{[m]}(f,x) \right)_\gamma = E_{n,W}^{[m]}(f_\gamma,x)$ for $m \in \mathbb{Z}^+$. □

Next, we present the following results for the case of $m = 0$ and $p > 1$ as

Theorem 2.20. *Let* $f \in C^l\left(\mathbb{R}^N\right)$*,* $l \in \mathbb{N}$*,* $N \geq 1$*,* $\gamma = 0, \tilde{\beta}$*,* $f_\gamma \in L_p\left(\mathbb{R}^N\right)$*,* $p, q > 1$ *:* $\frac{1}{p} + \frac{1}{q} = 1$*, and* $0 < \xi_n \leq 1$*,* $n \in \mathbb{N}$*. Let the assumptions of Theorem* 2.6 *be true. Then*

$$\left\| \left(E_{n,P}^{[0]}(f) \right)_\gamma \right\|_p \leq m_{\xi_n,P} \left(S_{P^*,\xi_n}^{p,0} \right)^{\frac{1}{p}} \omega_r \left(f_\gamma, \xi_n \right)_p + \left\| f_\gamma \right\|_p \left| m_{\xi_n,P} - 1 \right|, \tag{2.80}$$

and

$$\left\| \left(E_{n,W}^{[0]}(f) \right)_\gamma \right\|_p \leq m_{\xi_n,W} \left(S_{W^*,\xi_n}^{p,0} \right)^{\frac{1}{p}} \omega_r \left(f_\gamma, \xi_n \right)_p + \left\| f_\gamma \right\|_p \left| m_{\xi_n,W} - 1 \right|. \tag{2.81}$$

Proof. By [5], (1.38), (1.39), and by the equalities $\left(E_{n,P}^{[m]}(f,x) \right)_\gamma = E_{n,P}^{[m]}(f_\gamma,x)$ and $\left(E_{n,W}^{[m]}(f,x) \right)_\gamma = E_{n,W}^{[m]}(f_\gamma,x)$ for $m \in \mathbb{Z}^+$. □

For the case of $m = 0$ and $p = 1$, we obtain

Theorem 2.21. *Let* $f \in C^l\left(\mathbb{R}^N\right)$*,* $l \in \mathbb{N}$*,* $N \geq 1$*,* $\gamma = 0, \tilde{\beta}$*,* $f_\gamma \in L_1\left(\mathbb{R}^N\right)$*, and* $0 < \xi_n \leq 1$*,* $n \in \mathbb{N}$*. Let the assumptions of Theorem* 2.6 *be true. Then*

$$\left\| \left(E_{n,P}^{[0]}(f) \right)_\gamma \right\|_1 \leq m_{\xi_n,P} S_{P^*,\xi_n}^{1,0} \omega_r \left(f_\gamma, \xi_n \right)_1 + \left\| f_\gamma \right\|_1 \left| m_{\xi_n,P} - 1 \right|, \tag{2.82}$$

and

$$\left\| \left(E_{n,W}^{[0]}(f) \right)_{\gamma} \right\|_1 \leq m_{\xi_n,W} S_{W^*,\xi_n}^{1,0} \omega_r \left(f_\gamma, \xi_n \right)_1 + \| f_\gamma \|_1 \left| m_{\xi_n,W} - 1 \right|. \tag{2.83}$$

Proof. By [5], (1.38), (1.39), and by the equalities $\left(E_{n,P}^{[m]}(f,x) \right)_{\gamma} = E_{n,P}^{[m]}(f_\gamma, x)$ and $\left(E_{n,W}^{[m]}(f,x) \right)_{\gamma} = E_{n,W}^{[m]}(f_\gamma, x)$ for $m \in \mathbb{Z}^+$. □

Our final result is for the case of $m \in \mathbb{N}$ and $p = 1$

Theorem 2.22. *Let* $f \in C^{m+l}\left(\mathbb{R}^N\right)$*,* $m, l \in \mathbb{N}$*,* $N \geq 1$*,* $\gamma = 0, \tilde{\beta}$*,* $f_{\gamma+\alpha} \in L_1\left(\mathbb{R}^N\right)$*,* $|\alpha| = m$*, and* $0 < \xi_n \leq 1$*,* $n \in \mathbb{N}$*. Let the assumptions of Theorem* 2.6 *be true. Then*

$$\begin{aligned} \left\| \left(E_{n,P}^{[m]}(f) \right)_{\gamma} \right\|_1 \leq \; & m_{\xi_n,P} \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) S_{P^*,\xi_n}^{1,m} \omega_r \left(f_{\alpha+\gamma}, \xi_n \right)_1 \\ & + \| f_\gamma \|_1 \left| m_{\xi_n,P} - 1 \right|, \end{aligned} \tag{2.84}$$

and

$$\begin{aligned} \left\| \left(E_{n,W}^{[m]}(f) \right)_{\gamma} \right\|_1 \leq \; & m_{\xi_n,W} \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) S_{W^*,\xi_n}^{1,m} \omega_r \left(f_{\alpha+\gamma}, \xi_n \right)_1 \\ & + \| f_\gamma \|_1 \left| m_{\xi_n,W} - 1 \right|. \end{aligned} \tag{2.85}$$

Proof. By [5], (1.40), (1.41), and by the equalities $\left(E_{n,P}^{[m]}(f,x) \right)_{\gamma} = E_{n,P}^{[m]}(f_\gamma, x)$ and $\left(E_{n,W}^{[m]}(f,x) \right)_{\gamma} = E_{n,W}^{[m]}(f_\gamma, x)$ for $m \in \mathbb{Z}^+$. □

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A characterization of relatively compact sets in $L^p(\Omega, B)$

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Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. We give a characterization of relatively compact sets F in $L^p(\Omega, B)$ for $p \in [1, \infty)$, B a Banach-space, and $\Omega \subset \mathbb{R}^n$. This is a generalization of the results obtained in [12] for the space $L^p((0,T), B)$ with $T > 0$, first to rectangles $\Omega = (a, b) \subset \mathbb{R}^n$ and, under additional conditions, to arbitrary open and bounded subsets of $\mathbb{R}^n$. An application of the main compactness result to a problem arising in homogenization of processes on periodic surfaces is given.

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1. Introduction

In this paper, we prove a Kolmogorov-Riesz-type compactness result for the space $L^p(\Omega, B)$ with $p \in [1, \infty)$, $\Omega \subset \mathbb{R}^n$ open and bounded, and B a Banach space. Such a result was proved in [12] for $\Omega = (0, T)$ with $T > 0$. We generalize this result to rectangles Ω in $\mathbb{R}^n$, see Theorem 2.2, and under additional assumptions to arbitrary open and bounded domains $\Omega \subset \mathbb{R}^n$, see Corollary 2.5.

Similar results in the framework of vector-valued Sobolev and Besov spaces can also be found in [2], see Theorem 5.2 and the proof of Theorem 1.1. There, the compactness result is obtained under the assumption that there exists $\theta > 0$, such that

$$\sup_{h \in \mathbb{R}^n \setminus \{0\}} \frac{\|f(\cdot + h) - f\|_{L^p(\Omega_h, B)}}{|h|^\theta} < \infty.$$

However, our results are proven under the weaker assumption (ii) in Theorem 2.2.

In the homogenization theory, we are often concerned with sequences of functions in the space $L^p((0,T) \times \Omega, B)$, for which we have to show strong convergence. Here, due to lack of regularity, classical results like e. g., the Aubin-Lions Lemma cannot be applied, and the compactness result derived in this paper is an appropriate alternative. In Section 3, we give an application of our main compactness result for a problem arising in homogenization of processes on periodic surfaces.

2. Main result

In this section, we prove our main compactness theorem and related results. The proof is based on the Arzelà-Ascoli theorem, which for the sake of completness is repeated below, and uses similar arguments as in [12].

Lemma 2.1 (Arzelà-Ascoli). *Let T be a compact Hausdorff space and B be a Banach-space. A subset $F \subset C(T,B)$ is relatively compact in $C(T,B)$ iff the following conditions hold:*

(i) For every $x \in T$, the set $F(x) := \{f(x) : f \in F\}$ is relatively compact in B.
(ii) F is uniformly equicontinuous, i. e., for all $\epsilon > 0$ there exists $\eta > 0$ such that

$$\|f(x_2) - f(x_1)\|_B < \epsilon \quad \textit{for all } f \in F,\ x_1, x_2 \in T \textit{ with } \|x_2 - x_1\| < \eta.$$

Proof. See e. g., [4, Theorem 0.4.11]. □

For an arbitrary set $\Omega \subset \mathbb{R}^n$ and a vector $\xi \in \mathbb{R}^n$, we define

$$\Omega_\xi := \Omega \cap (\Omega - \xi).$$

Further, for $a, b \in \mathbb{R}^n$ we define

$$(a,b) := (a_1, b_1) \times \ldots \times (a_n, b_n),$$

with $(a_i, b_i) := (b_i, a_i)$ if $b_i < a_i$. For $f : \Omega \to B$ and $h \in \mathbb{R}^n$ we define

$$\tau_h f : (\Omega - h) \to B, \quad \tau_h f(x) = f(x+h).$$

We now state our main theorem:

Theorem 2.2. *Let $p \in [1,\infty)$, B be a Banach-space, $\Omega = (a,b)$ with $a, b \in \mathbb{R}^n$ $(a_i < b_i)$, and $F \subset L^p(\Omega, B)$. Then F is relatively compact in $L^p(\Omega, B)$ iff*

(i) for every rectangle $C \subset \Omega$ the set $\left\{\int_C f dx : f \in F\right\}$ is relatively compact in B,
(ii) for $z \in \mathbb{R}^n$ with $0 \le z_i < b_i - a_i$, $i = 1, \ldots, n$ it holds

$$\sup_{f \in F} \|\tau_z f - f\|_{L^p(\Omega_z, B)} \to 0 \quad \textit{for} \quad z \to 0.$$

Proposition 2.3. *The condition (ii) in Theorem 2.2 is equivalent to the following one:*

$(ii)'$ For $i = 1, \ldots, n$ and $s > 0$ it holds

$$\sup_{f \in F} \|\tau_{se_i} f - f\|_{L^p(\Omega_{se_i}, B)} \to 0 \quad \textit{for} \quad s \to 0,$$

where e_i is the i-th unit normal vector.

Proof of the Proposition 2.3. It is straightforward that (ii) implies $(ii)'$. For the other implication, we choose $z \in \mathbb{R}^n$ with $z_i \geq 0$ small. Then we have $z = \sum_{i=1}^n z_i e_i$ and we define $z^0 := 0 \in \mathbb{R}^n$ and $z^j := \sum_{i=1}^j z_i e_i$ for $j \in \{1, \ldots, n\}$. Of course $z^n = z$. Now, we use the triangle inequalitys to obtain

$$\begin{aligned} \|\tau_z f - f\|_{L^p(\Omega_z, B)} &\leq \sum_{j=0}^{n-1} \|\tau_{z^{j+1}} f - \tau_{z^j} f\|_{L^p(\Omega_z, B)} \\ &\leq \sum_{i=1}^{n} \|\tau_{z_i e_i} f - f\|_{L^p(\Omega_{z_i e_i}, B)}, \end{aligned}$$

where for the last inequality we used for $j = 0, \ldots, n-1$

$$\begin{aligned} \|\tau_{z^{j+1}} f - \tau_{z^j} f\|^p_{L^p(\Omega_z, B)} &= \int_{\Omega_z} \left\| f\left(x + \sum_{i=1}^{j+1} z_i e_i\right) - f\left(x + \sum_{i=1}^{j} z_i e_i\right)\right\|_B^p dx \\ &= \int_{\Omega_z + \sum_{i=1}^j z_i e_i} \big\| f(x + z_{j+1} e_{j+1}) - f(x) \big\|_B^p dx \\ &\leq \int_{\Omega_{z_{j+1} e_{j+1}}} \big\| f(x + z_{j+1} e_{j+1}) - f(x) \big\|_B^p dx. \end{aligned}$$

In the last inequality, we used the inclusion $\Omega_z + \sum_{i=1}^j z_i e_i \subset \Omega_{z_{j+1} e_{j+1}}$. In fact,

$$\Omega_{z_{j+1} e_{j+1}} = \{y \in \mathbb{R}^n : y_{j+1} \in (a_{j+1}, b_{j+1} - z_{j+1}),\ y_i \in (a_i, b_i) \text{ for } i \neq j+1\}$$

and for $y \in \Omega_z + \sum_{i=1}^j z_i e_i = [a, b - z] + \sum_{i=1}^j z_i e_i$, we have

$$\begin{aligned} y_i &\in (a_i + z_i, b_i), \text{ for } i = 1, \ldots, j \\ y_i &\in (a_i, b_i - z_i), \text{ for } i = j+1, \ldots, n. \end{aligned}$$

The claim follows. □

Proof of Theorem 2.2. Assume first that F is relatively compact in $L^p(\Omega, B)$. Then, we can use exactly the same arguments as in the proof of [12, Theorem 1]. In fact, (i) follows from the continuity of the mapping $f \mapsto \int_C f dx$ from $L^p(\Omega, B)$ into B, and (ii) follows, since in metric spaces, relatively compact sets are totally bounded, and the density of $C^0(\overline{\Omega}, B)$ in $L^p(\Omega, B)$.

Conversely, assume that (i) and (ii) hold. Let $f \in F$, and $h \in \mathbb{R}^n$ with $h_i > 0$ for $i = 1, \ldots, n$ (for example choose $h = s\frac{b-a}{2}$ with $s > 0$). Set

$$V_h := |(0, h)| > 0,$$

the measure of $(0, h)$. For $x \in \overline{\Omega_h}$, we have $(x, x+h) \subset \Omega$, and we define the function

$$(M_h f)(x) := \frac{1}{V_h} \int\limits_{(x, x+h)} f(z) dz.$$

We first show that $M_h f \in C\left(\overline{\Omega_h}, B\right)$, and the set

$$M_h F := \{M_h f : f \in F\}$$

is relatively compact in $C(\overline{\Omega_h}, B)$. Let $\epsilon > 0$, and choose $\delta > 0$ so small, that for $i = 1, \dots, n$ it holds

$$\|\tau_{\tilde{\delta} e_i} f - f\|_{L^1(\Omega_{\tilde{\delta} e_i}, B)} < \frac{V_h \epsilon}{n}, \text{ for all } \tilde{\delta} \leq \delta.$$

This is possible due to (ii) and the continuity of the embedding $L^p(\Omega_{\tilde{\delta} e_i}, B)$ into $L^1(\Omega_{\tilde{\delta} e_i}, B)$. Let $x_0 \in \overline{\Omega_h}$ and $x \in B_\delta(x_0) \cap \overline{\Omega_h}$, where the ball is taken with respect to the $\|\cdot\|_\infty$-norm on $\mathbb{R}^n$. Then $x = x_0 + \sum_{i=1}^n \delta_i e_i$ with $\delta_i \in (-\delta, \delta)$. For $j = 1, \dots, n$, we define the vector

$$x^j = x_0 + \sum_{i=1}^j \delta_i e_i.$$

Thus, we have $x^n = x$. Now, as in the proof of Proposition 2, we obtain

$$\|M_h f(x) - M_h f(x_0)\|_B \leq \sum_{j=0}^{n-1} \|M_h f(x^{j+1}) - M_h f(x^j)\|_B, \tag{2.1}$$

and we have $x^{j+1} - x^j = \delta_{j+1} e_{j+1}$ for $j = 0, \dots, n-1$. Without loss of generality, we assume that $\delta_i > 0$ for $i = 1, \dots, n$. Otherwise, i. e., for $\delta_i < 0$, we change the role of x^{j+1} and x^j in the following argumentation and for $\delta_i = 0$ it is trivial. It holds that

$$\begin{aligned} \|M_h f(x^{j+1}) - M_h f(x^j)\|_B &= \frac{1}{V_h} \left\| \int_{(x^j, x^j+h)} \left(\tau_{x^{j+1}-x^j} f - f\right)(z) dz \right\|_B \\ &\leq \frac{1}{V_h} \int_{(x^j, x^j+h)} \|\tau_{\delta_{j+1} e_{j+1}} f - f\|_B dz \\ &\overset{(*)}{\leq} \frac{1}{V_h} \|\tau_{\delta_{j+1} e_{j+1}} f - f\|_{L^1(\Omega_{\delta_{j+1} e_{j+1}}, B)} < \frac{\epsilon}{n}, \end{aligned} \tag{2.2}$$

where in $(*)$ we used $(x^j, x^j + h) \subset \Omega_{\delta_{j+1} e_{j+1}}$. In fact, from $x, x_0 \in \overline{\Omega_h}$, it follows by contradiction, that $x^i \in \overline{\Omega_h}$ for $i = 1, \dots, n$. This implies that

$$a_{j+1} \leq x^j_{j+1} \quad \text{and} \quad x^j_{j+1} + h_{j+1} = x^{j+1}_{j+1} - \delta_{j+1} + h_{j+1} \leq b_{j+1} - \delta_{j+1},$$

for $j = 1, \dots, n-1$, and hence, the inclusion $(x^j, x^j + h) \subset \Omega_{\delta_{j+1} e_{j+1}}$. From (2.1) and (2.2), we obtain that $M_h f \in C(\overline{\Omega_h}, B)$, and especially the set $M_h F$ is uniformly equicontinuous in $C(\overline{\Omega_h}, B)$.

For $x \in \Omega_h$ we obtain from the assumption (i) that the set

$$(M_h F)(x) := \left\{ \frac{1}{V_h} \int_{(x, x+h)} f dy : f \in F \right\}$$

is relatively compact in B. From Lemma 2.1 it follows that $M_h F$ is relatively compact in $C(\overline{\Omega_h}, B)$.

The next step in the proof is to show that F is the uniform limit of $M_h F$ in $L^p(\Omega_\xi, B)$ for $h \to 0$, see also [12, (2.2)]. We start from the following relation which holds for $x \in \Omega_h$:

$$(M_h f - f)(x) = \frac{1}{V_h}\int_{(x,x+h)} f(z) - f(x)dz = \frac{1}{V_h}\int_{(0,h)} (\tau_z f - f)(x)dz.$$

With the Jensen-inequality and the Fubini-Theorem we get

$$\begin{aligned}\|M_h f - f\|^p_{L^p(\Omega_h,B)} &= \int_{\Omega_h} \left\| \frac{1}{V_h}\int_{(0,h)} \tau_z f(x) - f(x)dz \right\|^p_B dx \\ &\leq \frac{1}{V_h}\int_{\Omega_h}\int_{(0,h)} \|\tau_z f(x) - f(x)\|^p_B \, dzdx \\ &\leq \sup_{z\in(0,h)} \|\tau_z f - f\|^p_{L^p(\Omega_h,B)},\end{aligned}$$

and therefore

$$\|M_h f - f\|_{L^p(\Omega_h,B)} \leq \sup_{z\in(0,h)} \|\tau_z f - f\|_{L^p(\Omega_h,B)}.$$

Due to assumption (ii), for every $\epsilon > 0$ we can choose h so small that for every $z \in (0,h)$ and every $f \in F$ we have

$$\|\tau_z f - f\|_{L^p(\Omega_h,B)} \leq \|\tau_z f - f\|_{L^p(\Omega_z,B)} < \epsilon,$$

and we obtain

$$\|M_h f - f\|_{L^p(\Omega_h,B)} < \epsilon.$$

Hence, F is the uniform limit of $M_h F$ in $L^p(\Omega_\xi, B)$ with $\xi = \frac{b-a}{2}$ for $h \to 0$. Since $M_h F$ is relatively compact in $C(\overline{\Omega_\xi}, B)$, it is also relatively compact in $L^p(\Omega_\xi, B)$, because the embedding $C(\overline{\Omega_\xi}, B) \hookrightarrow L^p(\Omega_\xi, B)$ is continuous. From [12, (2.2)] it follows that F is relatively compact in $L^p(\Omega_\xi, B)$.

Until now we have only established that F is relatively compact in $L^p(\Omega_\xi, B)$, but we have to show the result for the whole domain Ω. Let $\Sigma := \{-1, 1\}^n$ and for $z \in \mathbb{R}^n$ we define $z_\sigma := (\sigma_1 z_1, \dots, \sigma_n z_n)$. Of course, we have $\#\Sigma = 2^n$ and $\overline{\Omega} = \bigcup_{\sigma\in\Sigma} \overline{\Omega_{\xi_\sigma}}$. If additionally $z_i \geq 0$ for $i = 1, \dots, n$, then we write $z^+_\sigma := \frac{z_\sigma + z}{2}$ (positive components of z_σ) and $z^-_\sigma := \frac{z_\sigma - z}{2}$ (the negative components of z_σ), such that $z_\sigma = z^+_\sigma + z^-_\sigma$. For $h \in \mathbb{R}^n$ we write $(x, x + h_\sigma) := (x + h^-_\sigma, x + h^+_\sigma)$.
We define the function $M_{h_\sigma} f$ in the same way as $M_h f$, i. e.,

$$M_{h_\sigma} f(x) := \frac{1}{V_h}\int_{(x,x+h_\sigma)} f(z)dz \quad \text{for} \quad x \in \overline{\Omega_{h_\sigma}},$$

and for all $x \in \overline{\Omega_{h_\sigma}}$ we obtain with the transformation formula

$$(M_{h_\sigma} f - f)(x) = \frac{1}{V_h}\int_{(x,x+h_\sigma)} f(z) - f(x)dz = \frac{1}{V_h}\int_{(0,h_\sigma)} (\tau_z f - f)(x)dz.$$

With Fubini's Theorem, the Jensen-inequality and again by integration by substitution, we get

$$\begin{aligned}\|M_{h_\sigma}f-f\|^p_{L^p(\Omega_{h_\sigma},B)}&\le\frac{1}{V_h}\int_{\Omega_{h_\sigma}}\int_{(0,h_\sigma)}\|(\tau_z f-f)(x)\|^p_B dzdx\\&=\frac{1}{V_h}\int_{\Omega_{h_\sigma}}\int_{(0,h)}\|f(x+z_\sigma)-f(x)\|^p_B dzdx\\&=\frac{1}{V_h}\int_{(0,h)}\int_{\Omega_{h_\sigma}}\|f(x+z^+_\sigma+z^-_\sigma)-f(x)\|^p_B dxdz\\&=\frac{1}{V_h}\int_{(0,h)}\int_{\Omega_{h_\sigma}+z^-_\sigma}\|f(x+z^+_\sigma)-f(x-z^-_\sigma)\|^p_B dxdz\\&\le\frac{1}{V_h}\int_{(0,h)}\int_{\Omega_z}\|f(x+z^+_\sigma)-f(x-z^-_\sigma)\|^p_B dxdz,\end{aligned}$$

where in the last inequality we used $\Omega_{h_\sigma}+z^-_\sigma\subset\Omega_z$ for $z\in(0,h)$. To show this, we consider for $y\in\Omega_{h_\sigma}+z^-_\sigma$, and for $i=1,\dots,n$ the following two cases:

1) $\sigma_i=1$: Then $(h_\sigma)_i=h_i$ and $(z^-_\sigma)_i=0$ and therefore $y_i\in(a_i,b_i-h_i)\subset(a_i,b_i-z_i)$.
2) $\sigma_i=-1$: Then $(h_\sigma)_i=-h_i$ and $(z^-_\sigma)_i=-z_i$ and therefore $y_i\in(a_i+h_i-z_i,b_i-z_i)\subset(a_i,b_i-z_i)$.

Thus, $y_i\in(a_i,b_i-z_i)$ for $i=1,\dots,n$, i.e., $y\in\Omega_z$. Hence,

$$\begin{aligned}\|M_{h_\sigma}f-f\|_{L^p(\Omega_{h_\sigma},B)}&\le\sup_{z\in(0,h)}\|\tau_{z^+_\sigma}f-\tau_{-z^-_\sigma}f\|_{L^p(\Omega_z,B)}\\&\le\sup_{z\in(0,h)}\|\tau_{z^+_\sigma}f-f\|_{L^p(\Omega_z,B)}+\sup_{z\in(0,h)}\|\tau_{-z^-_\sigma}f-f\|_{L^p(\Omega_z,B)}\\&\le\sup_{z\in(0,h)}\|\tau_{z^+_\sigma}f-f\|_{L^p(\Omega_{z^+_\sigma},B)}+\sup_{z\in(0,h)}\|\tau_{-z^-_\sigma}f-f\|_{L^p(\Omega_{-z^-_\sigma},B)}.\end{aligned}$$

With the same arguments as above we obtain that F is relatively compact in $L^p(\Omega_{\xi_\sigma},B)$ for all $\sigma\in\Sigma$. Hence, $\overline{F}$ is sequentially compact in $L^p(\Omega,B)$ and therefore F is relatively compact in $L^p(\Omega,B)$. □

The next proposition gives us a further characterization of the condition (ii) in Theorem 2.2, where we use a special decomposition of the domain Ω, and consider the shifts on fixed domains. We use the same notation as in the proof of Theorem 2.2, especially we have $\xi=\frac{b-a}{2}$.

Proposition 2.4. *The condition (ii) in Theorem* 2.2 *is equivalent to the following one:*

(ii)″ For $z\in\mathbb{R}^n$ and $z_i\ge 0$ it holds

$$\sup_{f\in F}\|\tau_{z_\sigma}f-f\|_{L^p(\Omega_{\xi_\sigma},B)}\to 0 \textit{ for } z\to 0$$

for all $\sigma\in\Sigma$.

Proof. Let (ii) from Theorem 2.2 be true. We use similar arguments as in the last part of the proof of Theorem 2.2. Let $\epsilon > 0$ and $\delta > 0$ so small that for all $h \in [0,\delta]^n$ the following holds

$$\|\tau_h f - f\|_{L^p(\Omega_h, B)} < \frac{\epsilon}{2}.$$

Now, for $z \in [0,\delta]^n$ it follows by substitution and from

$$\Omega_{\xi_\sigma} + z_\sigma^- \subset \Omega_z \tag{2.3}$$

(which is proved below) that

$$\begin{aligned} \|\tau_{z_\sigma} f - f\|^p_{L^p(\Omega_{\xi_\sigma}, B)} &= \int_{\Omega_{\xi_\sigma}} \|f(x + z_\sigma) - f(x)\|^p_B dx \\ &= \int_{\Omega_{\xi_\sigma} + z_\sigma^-} \|f(x + z_\sigma^+) - f(x - z_\sigma^-)\|^p_B dx \\ &\le \int_{\Omega_z} \|f(x + z_\sigma^+) - f(x - z_\sigma^-)\|^p_B dx \\ &= \|\tau_{z_\sigma^+} f - \tau_{-z_\sigma^-} f\|^p_{L^p(\Omega_z, B)}. \end{aligned}$$

Since $z_\sigma^+, -z_\sigma^- \in [0,\delta]^n$, it follows that

$$\begin{aligned} \|\tau_{z_\sigma} f - f\|_{L^p(\Omega_{\xi_\sigma}, B)} &\le \|\tau_{z_\sigma^+} f - f\|_{L^p(\Omega_z, B)} + \|\tau_{-z_\sigma^-} f - f\|_{L^p(\Omega_z, B)} \\ &\le \|\tau_{z_\sigma^+} f - f\|_{L^p(\Omega_{z_\sigma^+}, B)} + \|\tau_{-z_\sigma^-} f - f\|_{L^p(\Omega_{-z_\sigma^-}, B)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Let us now give the proof of (2.3): For $x \in \Omega_{\xi_\sigma} + z_\sigma^-$ exists $\bar{x} \in \Omega_{\xi_\sigma}$ with $x = \bar{x} + z_\sigma^-$, i. e.,

$$\bar{x}_i \in \begin{cases} \left(a_i, \frac{b_i - a_i}{2}\right) & \text{for } \sigma_i = 1 \\ \left(\frac{b_i - a_i}{2}, b_i\right) & \text{for } \sigma_i = -1 \end{cases}, \quad \text{and} \quad (z_\sigma^-)_i = \begin{cases} 0 & \text{for } \sigma_i = 1 \\ -z_i & \text{for } \sigma_i = -1 \end{cases},$$

for $i = 1, \ldots, n$. Hence, we obtain

$$x_i \in \begin{cases} \left(a_i, \frac{b_i - a_i}{2}\right) & \text{for } \sigma_i = 1 \\ \left(\frac{b_i - a_i}{2} - z_i, b_i - z_i\right) & \text{for } \sigma_i = -1. \end{cases}$$

Since $\Omega_z = \prod_{i=1}^n (a_i, b_i - z_i)$, we obtain $x \in \Omega_z$.

Conversely, let $(ii)''$ hold. For $\epsilon > 0$ choose $\delta > 0$ so small that for all $\sigma \in \Sigma$ and all $h \in [0,\delta]^n$, we have

$$\|\tau_{h_\sigma} f - f\|_{L^p(\Omega_{\xi_\sigma}, B)} < \frac{\epsilon}{2\sqrt[p]{2^n}}.$$

Let $z \in [0,\delta]^n$, then we obtain for $\sigma \in \Sigma$

$$\begin{aligned}\|\tau_z f - f\|^p_{L^p(\Omega_{\xi_\sigma}\cap\Omega_z,B)} &= \int_{\Omega_{\xi_\sigma}\cap\Omega_z} \|f(x+z)-f(x)\|^p_B dx \\ &= \int_{\Omega_{\xi_\sigma}\cap\Omega_z} \|f(x+z^+_\sigma - z^-_\sigma)-f(x)\|^p_B dx \\ &= \int_{(\Omega_{\xi_\sigma}\cap\Omega_z)-z^-_\sigma} \|f(x+z^+_\sigma)-f(x+z^-_\sigma)\|^p_B dx \\ &\le \|\tau_{z^+_\sigma} f - \tau_{z^-_\sigma} f\|^p_{L^p(\Omega_{\xi_\sigma},B)}.\end{aligned}$$

Further, we have $z^+_\sigma, -z^-_\sigma \in [0,\delta]^n$ and $z^+_\sigma = (z^+_\sigma)_\sigma$ and $z^-_\sigma = (-z^-_\sigma)_\sigma$, what implies

$$\begin{aligned}&\|\tau_z f - f\|_{L^p(\Omega_{\xi_\sigma}\cap\Omega_h,B)} \\ &\qquad \le \|\tau_{z^+_\sigma} f - f\|_{L^p(\Omega_{\xi_\sigma},B)} + \|\tau_{z^-_\sigma} f - f\|_{L^p(\Omega_{\xi_\sigma},B)} < \frac{\epsilon}{\sqrt[p]{2^n}},\end{aligned}$$

i. e.,

$$\|\tau_z f - f\|^p_{L^p(\Omega_z,B)} = \sum_{\sigma\in\Sigma} \|\tau_z f - f\|^p_{L^p(\Omega_{\xi_\sigma}\cap\Omega_z,B)} < \epsilon^p. \qquad \square$$

Until now we have only considered rectangular domains in $\mathbb{R}^n$. Now we extend our result to more general domains. However, we need an additional assumption to control the functions near the boundary. We use the same notation as above and define for $\delta > 0$ the set $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > \delta\}$ and for $z \in \mathbb{R}^n$ the set

$$\Omega^z_\delta := \{x \in \Omega_\delta : x + z \in \Omega_\delta\} = \{x, x+z \in \Omega_\delta\}.$$

Corollary 2.5. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Let $F \subset L^p(\Omega, B)$ for a Banach space B and $p \in [1,\infty)$. Then F is relatively compact in $L^p(\Omega,B)$ iff*

(i) for every measurable set $C \subset \Omega$ the sequence $\left\{\int_C f dx : f \in F\right\}$ is relatively compact in B,

(ii) for all $\delta > 0$ it holds that $\sup_{f\in F} \|\tau_z f - f\|_{L^p(\Omega^z_\delta,B)} \to 0$ for $z \to 0$,

(iii) for $\delta > 0$ it holds that $\sup_{f\in F} \int_{\Omega\setminus\Omega_\delta} |f(x)|^p dx \to 0$ for $\delta \to 0$.

Proof. For F relatively compact in $L^p(\Omega,B)$ the statements (i) - (iii) can be established in a similar way as in Theorem 2.2.

Now assume, that (i) - (iii) hold. Since Ω is bounded, there exists a rectangle $W \subset \mathbb{R}^n$ with $\Omega \subset\subset W$. Extend every function $f \in F$ by zero to a function $\tilde{f} \in L^p(W,B)$ and obtain a set $\tilde{F} \subset L^p(W,B)$. Using the same arguments as in [1, U2.21], we can show that the assumptions of Theorem 2.2 are fulfilled and the claim follows. $\square$

3. Application

We consider an application of the compactness criterion derived din Section 2 to the homogenization of a nonlinear reaction-diffusion-problem on a rapidly oscillating periodic surface. Such problems arise in the mathematical modelling of processes in porous catalysts, see e.g. [7, 9], in biological structures, like e.g. biochemical processes in cells and tissue, see e.g. [6, 8, 11]. The periodically oscillating surface and the so called microscopic or ϵ-problem are given in the following.

Let $Y = (0,1)^n$ with $n \in \mathbb{N}$, $n \geq 3$, and $\Omega = (a,b) \subset \mathbb{R}^n$ with $a, b \in \mathbb{Z}^n$ and $a_i < b_i$ for $i = 1, \ldots, n$. We assume that the sequence ϵ fulfills $\epsilon^{-1} \in \mathbb{N}$. Further, let $\Gamma \subset Y$ be a $C^{1,1}$-submanifold, such that

$$\Gamma_\epsilon := \left\{ x \in \Omega : x = \epsilon(k+y) \text{ for some } k \in \mathbb{Z}^n,\, y \in \overline{\Gamma} \right\}$$

is connected and of class $C^{1,1}$. Especially, we have $\partial\Gamma_\epsilon \subset \partial\Omega$. On Γ_ϵ we consider the following problem:

$$\begin{aligned} \partial_t u_\epsilon - \Delta_{\Gamma_\epsilon} u_\epsilon &= f(u_\epsilon) && \text{in } (0,T) \times \Gamma_\epsilon, \\ -\nabla_{\Gamma_\epsilon} u_\epsilon \cdot \nu_{\Gamma_\epsilon} &= 0 && \text{on } (0,T) \times \partial\Gamma_\epsilon, \\ u_\epsilon(0) &= u^0 && \text{in } \Gamma. \end{aligned} \tag{3.1}$$

Here, Δ_{Γ_ϵ} denotes the Laplace-Beltrami-operator, $f \in C^{0,1}(\mathbb{R})$, i. e., f is globally Lipschitz-continuous, and $u^0 \in C^1\big(\overline{\Omega}\big)$. For the sake of simplicity the diffusion-coefficient is equal to 1, the nonlinearity f does not depend on a macroscopic or oscillating variable, and on the boundary $\partial\Gamma_\epsilon$, we consider a Neumann-zero condition. However, the following method can easily be generalized to more general problems, e. g., systems of equations and general diffusion-tensors. We are looking for a weak solution of Problem (3.1), i. e., $u_\epsilon \in L^2((0,T), H^1(\Gamma_\epsilon)) \cap H^1((0,T), L^2(\Gamma_\epsilon))$, such that for all $\phi \in H^1(\Gamma_\epsilon)$ we have

$$\int_{\Gamma_\epsilon} \partial_t u_\epsilon \phi d\sigma + \int_{\Gamma_\epsilon} \nabla_{\Gamma_\epsilon} u_\epsilon \cdot \nabla_{\Gamma_\epsilon} \phi d\sigma = \int_{\Gamma_\epsilon} f(u_\epsilon) \phi d\sigma \tag{3.2}$$

almost everywhere in $(0,T)$. With the Galerkin-method we obtain:

Proposition 3.1. *There exists a unique weak solution u_ϵ of Problem* (3.1)*, such that*

$$\|u_\epsilon\|_{L^\infty((0,T),L^2(\Gamma_\epsilon))} + \left\|\nabla_{\Gamma_\epsilon} u_\epsilon\right\|_{L^2((0,T),L^2(\Gamma_\epsilon))} + \|\partial_t u_\epsilon\|_{L^2((0,T)\times\Gamma_\epsilon)} \leq C\epsilon^{-\frac{1}{2}}.$$

This (microscopic) model describes the processes and the medium in a very detailed way. However, due to its high complexity it is not appropriate for practical applications, especially it is not amenable to numerical computations. Therefore, an effective (macroscopic, homogenized) model is needed, which is an approximation of the microscopic one, and consists of equations formulated on a macroscopic scale. The effective model is derived by using methods of periodic homogenization. This consists in showing that for $\epsilon \to 0$, the sequence of solutions (u_ϵ) converges to a limit function u_0, and in the derivation of the limit problem satisfied by u_0.

The appropriate techniques to be used for the derivation of the effective model in our application are the method of two-scale convergence for functions on periodic surfaces introduced in [9], and its equivalent characterisation with the help of the unfolding operator, see e.g. [3, 6]. Based on the estimates (3.1), passing to the limit in the linear terms in the equation (3.2) can be performed like in [5], where a linear problem was considered. Taking the limit in the nonlinear term is however more challenging. To achieve this, we make use of the unfolding operator

$$\mathcal{T}_\epsilon^b : L^2((0,T) \times \Gamma_\epsilon) \to L^2((0,T) \times \Omega \times \Gamma),$$

see [3, 6], defined via

$$\mathcal{T}_\epsilon^b \phi(t,x,y) := \phi\left(t, \epsilon\left[\frac{x}{\epsilon}\right] + \epsilon y\right).$$

Here $[\cdot]$ denotes the Gauß-bracket. Thus, from the theory developed in [5], we obtain the existence of a limit function $u_0 \in L^2((0,T),H^1(\Omega)) \cap H^1((0,T),L^2(\Omega))$ with $u_0(0) = u^0$, such that for all $\phi \in C_0^\infty\big((0,T)\times\overline{\Omega}\big)$ it holds that

$$\begin{aligned}|\Gamma|\int_0^T\int_\Omega \partial_t u_0 \phi dx dt + \int_0^T\int_\Omega D^*\nabla u_0 \cdot \nabla\phi dx dt = \\ \lim_{\epsilon\to 0}\int_0^T\int_\Omega\int_\Gamma f\big(\mathcal{T}_\epsilon^b u_\epsilon\big)\mathcal{T}_\epsilon^b \phi d\sigma_y dx dt.\end{aligned} \tag{3.3}$$

The homogenized diffusion-coefficient $D^* \in \mathbb{R}^{n\times n}$ is given by

$$D_{ij}^* = \int_\Gamma \big(\nabla_\Gamma w_i + \nabla_\Gamma y_i\big)\cdot\nabla_\Gamma y_j d\sigma,$$

where w_i for $i \in \{1,\ldots,n\}$ are the solutions of the following so called cell problems:

$$\begin{aligned} -\nabla_\Gamma\cdot\big(\nabla_\Gamma w_i + \nabla_\Gamma y_i\big) &= 0 \quad \text{in } \Gamma,\\ -(\nabla_\Gamma w_i + \nabla_\Gamma y_i\big)\cdot\nu &= 0 \quad \text{on } \partial\Gamma,\\ w_i \text{ is } Y\text{-periodic and } \int_\Gamma w_i d\sigma &= 0. \end{aligned}$$

To show the convergence of the nonlinear term we use the fact that

$$\mathcal{T}_\epsilon^b\phi \to \phi \text{ strongly in } L^2((0,T)\times\Omega\times\Gamma),$$

due to the regularity of ϕ. Hence, to go to the limit on the right-hand side in (3.3), it remains to show the weak convergence of $f\big(\mathcal{T}_\epsilon^b u_\epsilon\big)$ to $f(u_0)$ in $L^2((0,T)\times\Omega\times\Gamma)$. Therefore, we show the strong convergence of $\mathcal{T}_\epsilon^b u_\epsilon$ to u_0 in $L^2((0,T)\times\Omega\times\Gamma)$. Then, due to the Lipschitz-regularity of f, we actually obtain the strong convergence of $f\big(\mathcal{T}_\epsilon^b u_\epsilon\big)$ to $f(u_0)$ in $L^2((0,T)\times\Omega\times\Gamma)$. In [11] such a result was proved by showing that $\mathcal{T}_\epsilon^b u_\epsilon$ is a Cauchy-sequence. However, this result strongly relied on the fact, that the diffusion coefficient in the microscopic problem was of order ϵ^2, which led to an equation for $\mathcal{T}_\epsilon^b u_\epsilon$ where all coefficients were of order one. In our paper this is not the case, and the argument with the Cauchy-sequence cannot be applied. Instead, we use the compactness criterion from Section 2. A similar approach was used in [10], where the classical compactness criterion by Kolmogorov, see e. g., [13], for the space $L^2((0,T)\times\Omega\times Z)$, with $Z = (0,1)^{n-1}\times(-1,1)$, was employed. This is not appropriate for the situation in our application since shifts with respect to the surface-variable y make no sense.

In the following, we use the same notations as in Section 2, especially $\xi = \frac{b-a}{2}$.

Lemma 3.2. *Let $l \in \mathbb{N}_0^n$. Then, for all $\epsilon > 0$, such that $|l_i\epsilon| < \left|\frac{b_i - a_i}{2}\right|$ the following estimate holds for all $\sigma \in \Sigma$*

$$\|\tau_{\epsilon l_\sigma} u_\epsilon - u_\epsilon\|_{L^2((0,T)\times(\Gamma_\epsilon)_{\xi_\sigma})} \leq C|l|\sqrt{\epsilon}.$$

Proof. We test the variational equation for $\tau_{\epsilon l_\sigma} u_\epsilon - u_\epsilon$ with $\eta^2(\tau_{\epsilon l_\sigma} u_\epsilon - u_\epsilon)$, where $\eta \in C_0^\infty(\mathbb{R}^n)$ is a cut-off function with $0 \leq \eta \leq 1$, $\eta = 1$ in Ω_{ξ_σ}, and zero outside a small neighbourhood of Ω_{ξ_σ}. Then, Gronwall's inequality and the Lipschitz-continuity of u^0 give the desired result. □

Theorem 3.3. *For $\epsilon \to 0$, we have*

$$\mathcal{T}_\epsilon^b u_\epsilon \to u_0 \ \textit{strongly in } L^2((0,T) \times \Omega \times \Gamma).$$

Especially, we obtain

$$f\big(\mathcal{T}_\epsilon^b u_\epsilon\big) \to f(u_0) \ \textit{strongly in } L^2((0,T) \times \Omega \times \Gamma).$$

Proof. We consider $\mathcal{T}_\epsilon^b u_\epsilon$ as a function from $L^2(\Omega, L^2((0,T) \times \Gamma))$ and prove the condition (i) in Theorem 2.2 and $(ii)''$ in Proposition 2.4. Let $A \subset \Omega$ measurable, and define $v_A^\epsilon := \int_A \mathcal{T}_\epsilon^b u_\epsilon(\cdot_t, x, \cdot_y)dx$. The a priori estimate in Proposition 3.1 imply that v_A^ϵ is bounded in $L^2((0,T), H^1(\Gamma)) \cap H^1((0,T), L^2(\Gamma))$, and due to the Aubin-Lions Lemma the sequence is relatively compact in $L^2((0,T), L^2(\Gamma))$. It remains to check condition $(ii)''$. For $z \in \mathbb{R}^n$ with $z_i \geq 0$ small, we obtain as in the proof of [10, Theorem 2.3, page 700] for $l(\epsilon, z, m) := m + \left[\frac{z}{\epsilon}\right]$

$$\begin{aligned}
&\left\|\tau_{z_\sigma} \mathcal{T}_\epsilon^b u_\epsilon - \mathcal{T}_\epsilon^b u_\epsilon\right\|^2_{L^2(\Omega_{\xi_\sigma}, L^2((0,T)\times\Gamma))} \\
&\qquad \leq \epsilon \sum_{m \in \{0,1\}^n} \left\|\tau_{\epsilon l(\epsilon,z,m)_\sigma} u_\epsilon - u_\epsilon\right\|^2_{L^2((0,T)\times(\Gamma_\epsilon)_{\xi_\sigma})} \leq C\epsilon^2 |l(\epsilon, z, m)|^2.
\end{aligned}$$

Since $|l(\epsilon, z, m)|\epsilon \to 0$ for $\epsilon \to 0$ and $z \to 0$, condition $(ii)''$ is valid. Hence, Theorem 2.2 and Proposition 2.4 imply the desired result. □

Altogether, we immediately obtain that u_0 fulfills the following variational equation:

$$|\Gamma| \int_\Omega \partial_t u_0 \phi dx + \int_\Omega D^* \nabla u_0 \cdot \nabla \phi dx = |\Gamma| \int_\Omega f(u_0) \phi dx,$$

for all $\phi \in H^1(\Omega)$ and almost everywhere in $(0,T)$. The corresponding inital and boundary value problem is

$$\begin{aligned}
|\Gamma|\partial_t u_0 - \nabla \cdot (D^* \nabla u_0) &= |\Gamma| f(u_0) && \text{in } (0,T) \times \Omega \\
-D^* \nabla u_0 \cdot \nu &= 0 && \text{on } (0,T) \times \partial\Omega \\
u_0(0) &= u^0 && \text{in } \Omega.
\end{aligned}$$

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On some classes of Fleming-Viot type differential operators on the unit interval

Francesco Altomare

Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. Of concern are some classes of initial-boundary value differential problems associated with one-dimensional Fleming-Viot differential operators. Among other things, these operators occur in some models from population genetics to study the fluctuation of gene frequency under the influence of mutation and selection. The main aim of this survey paper is to discuss old and more recent results about the existence, uniqueness and continuous dependence from initial data of the solutions to these problems through the theory of the C_0-semigroups of operators. Other additional aspects which will be highlighted, concern the approximation of the relevant semigroups in terms of positive linear operators. The given approximation formulae allow to infer several preservation properties of the semigroups together with their asymptotic behavior. The analysis is carried out in the context of the space $C([0,1])$ as well as, in some particular cases, in $L^p([0,1])$ spaces, $1 \leq p < +\infty$. Finally, some open problems are also discussed.

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1. Introduction

In the present paper we shall discuss initial-boundary value differential problems associated with differential operators of the form

$$A(u)(x) := \frac{\alpha(x)}{2} u''(x) + (p(1-x) - qx)u'(x) \quad (0 < x < 1) \tag{1.1}$$

acting on suitable subspaces of $C^2_*([0,1])$, the linear space of all real-valued continuous functions on $[0,1]$ which are twice continuously differentiable on $]0,1[$.
Here, $\alpha \in C([0,1]), 0 \leq \alpha(x)$ for every $x \in [0,1]$, $p \geq 0$ and $q \geq 0$.

The differential operators (1.1) are referred to as the one-dimensional Fleming-Viot operators and they occur in some models from population genetics to study the fluctuation of gene frequency under the influence of mutation and selection ([15]).

Setting

$$a := p + q \text{ and } b := \begin{cases} 1 & \text{if } p = q = 0, \\ p/(p+q) & \text{if } p + q > 0, \end{cases}$$

the operator (1.1) turns into the operator

$$A(u)(x) := \frac{\alpha(x)}{2} u''(x) + a(b - x)u'(x) \quad (0 < x < 1) \tag{1.2}$$

with $a \geq 0$ and $0 \leq b \leq 1$, which, to our purposes, is more convenient to handle.

We begin to state the first main problem we shall deal with.

Problem 1.1. Determine a linear subspace $D(A)$ of $C^2_*([0, 1])$ such that

(i) For every $u \in D(A)$, $A(u)$ continuously extends to $[0, 1]$.
(ii) The operator $A : D(A) \to C([0, 1])$ generates a strongly continuous Markov semigroup $(T(t))_{t \geq 0}$ on $C([0, 1])$.

For some details concerning the theory of strongly continuous (Markov) semigroup and for unexplained terminology the reader is referred, e.g., to [6, Chapter 2]. If $(A, D(A))$ generates a strongly continuous semigroup, then, given $u_0 \in C([0, 1])$, the following Cauchy problem is well-posed

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \alpha(x) \frac{\partial^2 u(x,t)}{\partial x^2} + a(b - x) \frac{\partial u(x,t)}{\partial x} & 0 < x < 1, t \geq 0, \\ u(\cdot, t) \in D(A) & t \geq 0, \\ \lim_{t \to 0^+} u(x, t) = u_0(x) & \text{uniformly w.r. to } 0 \leq x \leq 1, \end{cases} \tag{1.3}$$

if and only if $u_0 \in D(A)$.

Moreover, the unique solution to (1.3) is given by

$$u(x, t) = T(t)u_0(x) \quad (0 \leq x \leq 1, \ t \geq 0). \tag{1.4}$$

and it continuously depends on the initial datum u_0.
The subspace $D(A)$ (if any) is referred to as a well-posed domain for A.

Note also that (1.3) is, indeed, an initial-boundary value problem since the boundary conditions are usually included in the definition of $D(A)$.

The partial differential equation which appears in (1.3) is the so-called *backward equation* of a normal Markov process

$$(\Omega, \mathcal{U}, (P^x)_{x \in [0,1]}, (Z_t)_{t \geq 0})$$

having $[0, 1]$ as state space, with mean instantaneous velocity $a(b - x)$ and variance instantaneous velocity $\alpha(x)$ at the position $x \in [0, 1]$ (see, e.g., [6, Section 2.3.2]

Having determined $D(A)$, we shall discuss the next subsequent problem:

Problem 1.2. Introduce (if any) a sequence of positive linear operators $(L_n)_{n\geq 1}$ on $C([0,1])$ such that for every $t \geq 0$, for some sequence $(k(n))_{n\geq 1}$ of positive integers and for every $f \in C([0,1])$,

$$T(t)f = \lim_{n\to\infty} L_n^{k(n)} f \quad \text{uniformly on } [0,1]. \tag{1.5}$$

In such a case, we say that the sequence $(L_n)_{n\geq 1}$ is an admissible sequence for the semigroup $(T(t))_{t\geq 0}$.

In principle, from formula (1.5) it is possible to infer some preservation properties of the semigroup which have their counterparts in terms of regularity properties (with respect to the spatial variable $x \in [0,1]$) of the solutions

$$u(x,t) = T(t)u_0(x)$$

to the initial-boundary value problems (1.3).

Moreover, estimates of the quantities $\|T(t)f - L_n^{k(n)} f\|$ could give numerical approximations of the solutions themselves.

According to a theorem of H. F. Trotter ([6, Corollary 2.2.3]), a natural way to get the approximation formula (1.5), is to show that

(i) $\|L_n^k\| \leq M \exp(\omega nk)$ for some $M \geq 1$ and $\omega \in \boldsymbol{R}$, and for every $n, k \geq 1$,
and, in addition, to determine (if any) a linear subspace D_0 of $D(A)$ such that

(ii) D_0 is a core for $D(A)$, i.e., D_0 is dense in $D(A)$ for the graph norm

$$\|u\|_A := \|u\| + \|A(u)\| \ (u \in D(A)),$$

and

(iii) For every $u \in D_0$,

$$\lim_{n\to\infty} n(L_n(u) - u) = A(u) \text{ uniformly on } [0,1].$$

In such a case, formula (1.5) holds true for every $t \geq 0$, for every sequence $(k(n))_{n\geq 1}$ of positive integers such that $k(n)/n \to t$ and for every $f \in C([0,1])$.
Moreover, $\|T(t)\| \leq M \exp(\omega t)$ for every $t \geq 0$.

When conditions (i)–(iii) are satisfied, we say that the sequence $(L_n)_{n\geq 1}$ is a strong admissible sequence for the semigroup $(T(t))_{t\geq 0}$.

In the subsequent section we shall survey some old and more recent results about these two problems. However, we point out that for the case $\alpha(x) = x(1-x)$ $(x \in [0,1])$, rather satisfactory results have been obtained (see, e.g., [11], [12], [13], [16] and the references therein). For the general case, parts of the results we are discussing in the present paper are taken from [8].

We also point out that in the paper [8] as well as in the monograph [6], similar problems have been treated for general convex compact subsets K of $\boldsymbol{R}^d, d \geq 1$, having non-empty interior.

In these contexts the differential operators are of the form

$$A(u)(x) := \frac{1}{2}\sum_{i,j=1}^{d} \alpha_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} a(b_i - x_i)\frac{\partial u}{\partial x_i}(x). \tag{1.6}$$

$(u \in C^2(K),\ x \in K)$.

However, in the framework of the unit interval more complete results can be shown. For additional results concerning Fleming-Viot type differential operators we refer, e.g., to [1], [14] and the references therein.

2. Generation results and approximation

On account of the Feller theory developed in the 1950s (see, e.g., [6, Section 2.3.3]), we shall describe four groups of boundary conditions which allow to determine well-posed domains for A.

From now on we shall assume that

(i) $0 < \alpha(x)$ for each $0 < x < 1$ and $\alpha(0) = \alpha(1) = 0$.
(ii) α is differentiable at 0 and at 1 and $\alpha'(0) \neq 0 \neq \alpha'(1)$;

From conditions (i) and (ii) it follows that

$$0 < \alpha(x) \leq Mx(1-x) \text{ for each } 0 < x < 1 \text{ and for some } M > 0.$$

There is no loss of generality in assuming that $M = 1$ because, if $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ denote the Markov semigroups generated by the differential operators associated with α, a and b, and $\frac{\alpha}{M}$, $\frac{a}{M}$ and b respectively, then

$$T(t) = S(Mt) \quad \text{for every } t \geq 0.$$

Thus, from now on we shall assume that

(iii) $0 < \alpha(x) \leq x(1-x)$ for each $0 < x < 1$.

The special case $\alpha(x) = x(1-x)$ for every $x \in [0,1]$, will be referred to as the maximal case.

Finally, we also assume that

(iv) the function

$$r(x) := \begin{cases} \dfrac{ab}{2\alpha'(0)} & \text{if } x = 0, \\ \dfrac{a(b-x)x(1-x)}{2\alpha(x)} & \text{if } 0 < x < 1, \\ \dfrac{a(1-b)}{2\alpha'(1)} & \text{if } x = 1, \end{cases} \tag{2.1}$$

is Hölder continuous at 0 and at 1.

Condition (iv) is satisfied, for instance, if α is differentiable in [0, 1]. Moreover, note also that $\alpha'(0) \leq 1$ and $-1 \leq \alpha'(1)$.

It is also useful to consider the function

$$\lambda(x) := \frac{a(b-x)}{r(x)} = \begin{cases} 2\alpha'(0) & \text{if } x = 0, \\ \dfrac{x(1-x)}{2\alpha(x)} & \text{if } 0 < x < 1, \\ -2\alpha'(1) & \text{if } x = 1. \end{cases} \tag{2.2}$$

Then, $A = \lambda B$, where B denotes the differential operator

$$B(u)(x) := \frac{x(1-x)}{2} u''(x) + r(x)u'(x) \qquad (0 < x < 1) \tag{2.3}$$

$(u \in C_*^2([0,1]))$.

Thus, on account of a well-known multiplicative perturbation generation result (see, e.g., [6, Theorem 2.3.11]) the generation problems for A can be solved by studying similar ones for B.

2.1. The case $a = 0$ and the case $0 < ab < \alpha'(0)/2$ and $0 < a(1-b) < -\alpha'(1)/2$

In these cases a well-posed domain for A is the so-called Ventcel' domain of A. For a proof of the next result it is enough to combine [6, Theorem 5.7.2] and [13, pp. 120-121, item (2)], taking the formula $A = \lambda B$ into account.

Theorem 2.1. *If $a = 0$ or if $0 < ab < \alpha'(0)/2$ and $0 < a(1-b) < -\alpha'(1)/2$, then a well-posed domain for A is*

$$D_V(A) := \left\{ u \in C_*^2([0,1]) \;\middle|\; \lim_{x\to 0^+} A(u)(x) = 0 = \lim_{x\to 1^-} A(u)(x) = 0 \right\}, \tag{2.4}$$

The capital letter V refers to the mathematician Ventcel' who extended the Feller work to multidimensional settings. Moreover, the Ventcel' conditions, i.e., the boundary conditions incorporated in $D_V(A)$, imply that, once the Markov process reaches 0 or 1, then it stops there for ever ([6, Subsection 2.3.3].

As regards the construction of a strong admissible sequence for the semigroup, we are able to provide a solution for the case $a = 0$ only and we leave as an open problem the second subcase $0 < ab < \alpha'(0)/2$ *and* $0 < a(1-b) < -\alpha'(1)/2$. However, at least in the maximal case $\alpha(x) = x(1-x)$ $(x \in [0,1])$, it is possible to describe the asymptotic behaviour of the semigroup also for the second subcase (for the case $a = 0$ see the subsequent results).

We have indeed (see [11, Theorem 4.2]) that for every $f \in C([0,1])$

$$\lim_{t\to+\infty} T(t)f = f(0)(1-\varphi) + f(1)\varphi \quad \textit{uniformly on } [0,1],$$

where, for every $x \in [0,1]$,

$$\varphi(x) := \frac{\int_0^x t^{-2ab}(1-t)^{-2a(1-b)}dt}{\int_0^1 t^{-2ab}(1-t)^{-2a(1-b)}dt}.$$

In particular,

$$\lim_{t\to\infty} T(t)(f) = 0 \quad \text{uniformly on } [0,1]$$

if and only if $f(0) = f(1) = 0$.

We proceed now to study the case $a = 0$. According to [6, Remark 4.5.5], consider a Markov operator T on $C([0,1])$ such that $T(e_1) = e_1$ and

$$\alpha = T(e_2) - e_2,$$

where $e_1(x) = x$ and $e_2(x) = x^2$ $(x \in [0,1])$.

By appealing to the Riesz representation theorem, consider the family $(\mu_x)_{0\leq x\leq 1}$ of probability Borel measures on $[0,1]$ such that

$$T(f)(x) := \int_0^1 f d\mu_x, \quad (0 \leq x \leq 1 \text{ and } f \in C([0,1])). \tag{2.5}$$

Definition 2.2. *For every $n \geq 1$, the n-th Bernstein-Schnabl operator associated with T is the positive linear operator $B_n : C([0,1]) \longrightarrow C([0,1])$ defined for every $f \in C([0,1])$ and $x \in [0,1]$ as*

$$B_n(f)(x) := \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + \cdots + x_n}{n}\right) d\mu_x(x_1) \cdots d\mu_x(x_n). \tag{2.6}$$

For a detailed analysis on these operators and for a proof of the results below we refer to the monographs [3, Chapter 6] and [6, Chapter 3] and the references therein.

Theorem 2.3. *The sequence $(B_n)_{n\geq 1}$ of Bernstein-Schnabl operators associated with T is a strong admissible sequence for the semigroup generated by the operator $(A, D_V(A))$ for the case $a = 0$, i.e.,*

$$A(u)(x) := \frac{\alpha(x)}{2} u''(x) \qquad (0 < x < 1)$$

and

$$D_V(A) := \left\{ u \in C^2_*([0,1]) \mid \lim_{x\to 0^+} A(u)(x) = 0 = \lim_{x\to 1^-} A(u)(x) = 0 \right\}.$$

Moreover, $C^2([0,1])$ is a core for $D_V(A)$.

From the theorem above it is possible to infer some preservation properties of the semigroup which have their counterparts in terms of regularity properties (with respect to the spatial variable $x \in [0,1]$) of the solutions

$$u(x,t) = T(t)u_0(x)$$

to the relevant initial-boundary value problems.

For given $M > 0$ and $0 < \sigma \leq 1$ we set

$$\begin{aligned} &Lip(M,\sigma) := \\ &\{f \in C([0,1]) \mid\mid f(x) - f(y) \mid\leq M \mid x - y \mid^\sigma \text{ for every } x, y \in [0,1]\}. \end{aligned} \tag{2.7}$$

Corollary 2.4. *The following statements hold true:*

1. *$T(t)f = f$ on 0 and 1 for every $f \in C([0,1])$.*
2. *If the operator T maps continuous increasing functions into (continuous) increasing functions, then each $T(t)$ maps continuous increasing functions into increasing functions.*
3. *If $T(Lip(1,1)) \subset Lip(1,1)$, then for every $M > 0$, $0 < \sigma \leq 1$ and $t \geq 0$,*
$$T(t)(Lip(M,\sigma)) \subset Lip(M,\sigma).$$
4. *If $f \in C([0,1])$, the following statements are equivalent:*
   - (i) *f is convex;*
   - (ii) *$B_{n+1}(f) \leq B_n(f)$ for every $n \geq 1$;*
   - (iii) *$f \leq B_n(f)$ for every $n \geq 1$;*

(iv) $f \leq T(t)f$ *for every* $t \geq 0$.

(5) *For every* $f \in C([0,1])$

$$\lim_{t\to\infty} T(t)(f) = (1-e_1)f(0) + e_1 f(1)$$

uniformly on $[0,1]$ *and hence*

$$\lim_{t\to\infty} T(t)(f) = 0 \qquad \text{uniformly on } [0,1]$$

if and only if $f(0) = f(1) = 0$.

In order to show the behaviour of the semigroup $(T(t))_{t\geq 0}$ on convex functions, for every $f \in C([0,1])$ and $x, y \in [0,1]$, consider

$$\Delta(f;x,y) := B_2(f)(x) + B_2(f)(y) - 2\iint_{[0,1]^2} f\left(\frac{s+t}{2}\right) d\mu_x(s)d\mu_y(t),$$

where the operators B_2 is the Bernstein-Schnabl operator of order 2.

Theorem 2.5. *Suppose that*

(i) T *maps continuous convex functions into (continuous) convex functions;*

(ii) $\Delta(f;x,y) \geq 0$ *for every convex function* $f \in C([0,1])$ *and for every* $x, y \in [0,1]$.

If $f \in C([0,1])$ *is convex, then* $T(t)f$ *is convex for every* $t \geq 0$ *and* $(T(t)f)_{t\geq 0}$ *is increasing.*

For additional results concerning Bernstein-Schnabl operators we also refer to [2].

2.2. The case $ab \geq \alpha'(0)/2$ **and** $a(1-b) \geq -\alpha'(1)/2$

For all the results shown in this subsection the reader is referred to [8, Sections 3 and 4]

Theorem 2.6. *If* $ab \geq \alpha' 0)/2$ *and* $a(1-b) \geq -\alpha'(1)/2$, *then a well-posed domain for* A *is*

$$D_M(A) := \left\{ u \in C^2_*([0,1]) \ \mid \ \lim_{x\to 0^+} A(u)(x) \in \mathbf{R} \text{ and } \lim_{x\to 1^-} A(u)(x) \in \mathbf{R} \right\}.$$

The domain $D_M(A)$ is also referred to as the maximal domain for A. Moreover, the maximal boundary conditions incorporated in the domain $D_M(A)$, imply that the probability that the Markov process reaches 0 or 1 in a finite time is zero ([6, Subsection 2.3.3].

As regards the construction of a strong admissible sequence for the semigroup, consider again a Markov operator T on $C([0,1])$ such that $T(e_1) = e_1$ and $\alpha = T(e_2) - e_2$, along with the family $(\mu_x)_{0\leq x\leq 1}$ of probability Borel measures on $[0,1]$ representing T (see (2.5)). Finally let μ be a probability Borel measure on $[0,1]$.

Then, for every $n \geq 1$, consider the positive linear operator C_n defined by setting

$$C_n(f)(x) = \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + \ldots + x_n + a x_{n+1}}{n+a}\right) d\mu_x(x_1)\cdots d\mu_x(x_n) d\mu(x_{n+1}) \tag{2.8}$$

for every $x \in [0,1]$ and for every $f \in C([0,1])$.

The germ of the idea of the above definition goes back to [9] (see also [10]). Subsequently, in [4] (see also [5]) the authors considered a natural generalization to multidimensional settings such as hypercubes and simplices, obtaining, as a particular case, the multidimensional Kantorovich operators on these frameworks.

The general definition (introduced in the context of general convex compact subsets) has been set in the recent paper [7], obtaining a new class of positive linear operators which encompasses not only several well-known approximation processes both in univariate and multivariate settings, but also new ones in finite and infinite dimensional frameworks as well.

Clearly, in the special case $a = 0$, the operators C_n correspond to the B_n ones. Moreover, introducing the auxiliary continuous function

$$I_n(f)(x) := \int_0^1 f\left(\frac{n}{n+a}x + \frac{a}{n+a}t\right) d\mu(t)$$

($f \in C([0,1])$, $x \in [0,1]$, $n \geq 1$), then

$$C_n(f) = B_n(I_n(f)).$$

Therefore $C_n(f) \in C([0,1])$ and the operator $C_n : C([0,1]) \to C([0,1])$, being linear and positive, is continuous with norm equal to 1, because $C_n(\mathbf{1}) = \mathbf{1}$.

We proceed to show some specific examples.

Example 2.7. Consider the maximal case $\alpha(x) = x(1-x)$ which corresponds to the Markov operator $T_1 : C([0,1]) \longrightarrow C([0,1])$ defined, for every $f \in C([0,1])$ and $0 \leq x \leq 1$, by

$$T_1(f)(x) := (1-x)f(0) + xf(1).$$

Then, the Bernstein-Schnabl operators associated with T_1 are the classical Bernstein operators

$$B_n(f)(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

($n \geq 1$, $f \in C([0,1])$, $x \in [0,1]$).

Considering, as above, $a \geq 0$ along with a probability Borel measure μ on $[0,1]$, we get

$$C_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_0^1 f\left(\frac{k+at}{n+a}\right) d\mu(t)$$

($n \geq 1$, $f \in C([0,1])$, $x \in [0,1]$).

In particular, if μ is the Borel-Lebesgue measure λ_1 on $[0,1]$, then we get

$$C_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_0^1 f\left(\frac{k+at}{n+a}\right) dt$$

For $a = 1$, this formula gives the classical Kantorovich operators. Moreover, as already remarked, for $a = 0$ we obtain the Bernstein operators; thus, by means of the previous formula, we obtain a link between these fundamental sequences of approximating operators in terms of a continuous parameter $a \in [0,1]$. For other examples we refer to [7].

Theorem 2.8. *Assume that b is the baricenter of the measure μ. If $ab \geq \alpha'(0)/2$ and $a(1-b) \geq -\alpha'(1)/2$, then the sequence $(C_n)_{n\geq 1}$ is a strong admissible sequence for the semigroup generated by the operator $(A, D_M(A))$.*
Moreover, $C^2([0,1])$ is a core for $D_M(A)$.

Now we proceed to show some properties of the semigroup $(T(t))_{t\geq 0}$ generated by $(A, D_M(A))$ which have their counterparts in terms of regularity properties (with respect to the spatial variable $x \in [0,1]$) of the solutions

$$u(x,t) = T(t)u_0(x)$$

to the initial-boundary value problems (1.1).

However, the next property concerns the sequence $(C_n)_{n\geq 1}$ and it seems to be not devoid of interest. It is related to some saturation aspects for these operators (see [6, Remark 2.2.12])

Theorem 2.9. *If $u, v \in C([0,1])$ and if $\lim_{n\to\infty} n(C_n(u)-u) = v$ uniformly on $[0,1]$, then $u \in D_M(A)$ and $A(u) = v$.*
In particular, if $\lim_{n\to\infty} n(C_n(u)-u) = 0$ uniformly on $[0,1]$, then $u \in D_M(A)$ and $A(u) = 0$, i.e.,

$$\frac{\alpha(x)}{2}u''(x) + a(b-x)u'(x) = 0 \ (x \in]0,1[).$$

From now on we refer again to a Markov operator T on $C([0,1])$ generating the coefficient α, i.e., $T(e_1) = e_1$ and $\alpha = T(e_2) - e_2$. For every $m \geq 1$, denote by $P_m([0,1])$ the subset of all polynomials on $[0,1]$ of degree no greater than m.

Theorem 2.10. *If $T(P_m([0,1])) \subset P_m([0,1])$ for every $m \geq 1$, then*

$$T(t)(P_m([0,1])) \subset P_m([0,1]) \text{ for every } m \geq 1 \text{ and } t \geq 0.$$

Theorem 2.11. *If $T(\mathrm{Lip}(1,1)) \subset \mathrm{Lip}(1,1)$, then*

$$T(t)(\mathrm{Lip}(M,1)) \subset \mathrm{Lip}(M,1) \text{ for every } t \geq 0 \quad \text{and } M \geq 0.$$

(see (2.7)).

Theorem 2.12. *Suppose that conditions (i) and (ii) of Theorem 2.5 are satisfied. If $f \in C([0,1])$ is convex, then $T(t)f$ is convex for every $t \geq 0$.*

Additional results can be shown for the maximal case $\alpha(x) = x(1-x)$ $(x \in [0,1])$. Thus, $\alpha'(0) = 1$ and $\alpha'(1) = -1$ so that

$$ab \geq 1/2 \text{ and } a(1-b) \geq 1/2.$$

Combining results of [11] and [12], it is possible to show that the semigroup $(T(t))_{t\geq 0}$ can be also expressed as a limit of iterates of Bernstein-Durrmeyer operators with Jacobi weights which are defined as

$$M_n(f)(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} a_{n,k}(f) \tag{2.9}$$

$(n \geq 1,\ f \in C([0,1]),\ x \in [0,1])$, where

$$a_{n,k}(f) := \frac{1}{B(k+\gamma+1, n-k+\delta+1)} \int_0^1 t^{k+\gamma}(1-t)^{n-k+\delta} f(t)dt,$$
$$\gamma = 2ab-1 \text{ and } \delta = 2a(1-b)-1,$$

and B denotes the usual Euler's *Beta* function.

By means of such operators it is possible to show that (see [12, Section 3.2]

Theorem 2.13. *For every $p \geq 1$, $(T(t))_{t\geq 0}$ extends to a positive contraction C_0-semigroup $(\widetilde{T}(t))_{t\geq 0}$ on $L^p([0,1],\mu)$, where μ is the absolutely continuous measure having the normalized Jacobi weight*

$$w_{\gamma,\delta} := \frac{x^\gamma(1-x)^\delta}{\int_0^1 t^\gamma(1-t)^\delta f(t)dt}$$

as density with respect to the Borel-Lebesgue measure on $[0,1]$.
Moreover, the generator $(\widetilde{A}, D(\widetilde{A}))$ of the semigroup $(\widetilde{T}(t))_{t\geq 0}$ is an extension of $(A, D_M(A))$ and $C^2([0,1])$ is a core for $(\widetilde{A}, D(\widetilde{A}))$.
Therefore, $(\widetilde{A}, D(\widetilde{A}))$ is the closure of $(A, D_M(A))$ in $L^p([0,1],\mu)$ as well.
Furthermore, if $t \geq 0$ and if $(k(n)_{n\geq 1}$ is a sequence of positive integers such that $\lim_{n\to\infty} k(n)/n = t$, then for every $f \in L^p([0,1],\mu)$,

$$\widetilde{T}(t)(f) = \lim_{n\to\infty} M_n^{k(n)}(f) \qquad \text{in } L^p([0,1],\mu).$$

Finally, for every $f \in C([0,1])$,

$$\lim_{t\to+\infty} T(t)(f) = \int_0^1 f(x)\, d\mu(x)$$

uniformly on $[0,1]$, and for every $f \in L^p([0,1],\mu), 1 \leq p < +\infty)$

$$\lim_{t\to+\infty} \widetilde{T}(t)(f) = \int_0^1 f(x)\, d\mu(x) \quad \text{in } L^p([0,1],\mu).$$

We also point out that, in the particular case $b = 1/2$ (and hence $a \geq 1$), then the previous results continue to hold true in the space $L^p([0,1])$, $(1 \leq p < +\infty)$, and with the generalized Kantorovich operators as strong admissible sequence (see [8, Section 4]).

2.3. The case $ab < \alpha'(0)/2$ and $a(1-b) \geq -\alpha'(1)/2$ and the case $ab \geq \alpha'(0)/2$ and $a(1-b) < -\alpha'(1)/2$

In these cases the well-posed domain for A are the so-called mixed domain of A. For a proof of the next generation results we refer to [6, Theorem 5.7.7] and [13, pp. 120-121, item (2)], taking the formula $A = \lambda B$ into account.

Theorem 2.14.

(i) *If $ab < \alpha'(0)/2$ and $a(1-b) \geq -\alpha'(1)/2$, then a well-posed domain for A is*

$$D_{VM}(A) := \left\{ u \in C^2_*([0,1]) \;\middle|\; \lim_{x\to 0^+} A(u)(x) = 0 \text{ and } \lim_{x\to 1^-} A(u)(x) \in \mathbf{R} \right\}.$$

(ii) *If* $ab \geq \alpha'(0)/2$ *and* $a(1-b) < -\alpha'(1)/2$*, then a well-posed domain for* A *is*

$$D_{MV}(A) := \left\{ u \in C_*^2([0,1]) \;\mid\; \lim_{x\to 0^+} A(u)(x) \in \mathbf{R} \text{ and } \lim_{x\to 1^-} A(u)(x) = 0 \right\}.$$

The domains $D_{VM}(A)$ and $D_{MV}(A)$ are referred to as the mixed domains for A. The relevant boundary conditions imply that the probability that the Markov process reaches 1, resp. 0, in a finite time is zero whereas the probability that the Markov process reaches 0, resp. 1, in a finite time is strictly positive and, when it reaches that point, then it remains there for ever.

As regards the construction of a strong admissible sequence for the semigroup generated by the mixed domains $D_{VM}(A)$ and $D_{MV}(A)$, we are able to provide a solution only for the case

$$b = 0 \text{ and } a \geq -\alpha'(1)/2,$$

as well as for the case

$$b = 1 \text{ and } a \geq \alpha'(0)/2,$$

and we leave the remaining cases as an open problem.

In both the previous special cases, a strong admissible sequence is given by particular generalized Kantorovich operators (2.8) obtained with μ being the unit mass concentrated at 0, resp. at 1, namely,

$$C_n(f)(x) = \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + \ldots + x_n}{n+a}\right) d\mu_x(x_1)\cdots d\mu_x(x_n)$$

and, respectively,

$$C_n(f)(x) = \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + \ldots + x_n + a}{n+a}\right) d\mu_x(x_1)\cdots d\mu_x(x_n)$$

for every $n \geq 1$, $x \in [0,1]$ and $f \in C([0,1])$.
Moreover, $C^2([0,1]) \bigcap D_{VM}(A)$, resp. $C^2([0,1]) \bigcap D_{MV}(A)$, is a core for $D_{VM}(A)$, resp. $D_{MV}(A)$.

All the shape preserving properties described for the maximal domains continue to hold true in these case, except that the asymptotic behaviour of the semigroup (see [8, Theorem 3.9]).

To this respect we have indeed (see [11, Theorem 4.2]) that, in the maximal case $\alpha(x) = x(1-x)$ $(x \in [0,1])$, for every $f \in C([0,1])$

$$\lim_{t\to+\infty} T(t)f := \begin{cases} f(0) & if \;\; ab < \frac{1}{2} \;\; and \;\; a(1-b) \geq \frac{1}{2}, \\ f(1) & if \;\; ab \geq \frac{1}{2} \;\; and \;\; a(1-b) < \frac{1}{2}. \end{cases}$$

Remarks 2.15. 1. It is worth pointing out that, because of Theorem 2.13, an initial-boundary value problem like (1.3) also holds true in the setting of $L^p([0,1],\mu)$ spaces other than in the space $C([0,1])$. Accordingly, it would be desirable to investigate whether a result similar to Theorem 2.13 holds true also when α is not maximal.

Perhaps, the analysis of such a problem might lead to the need to introduce a new sequence of positive linear operators generalizing Bernstein-Durrmeyer operators (2.9).

2. Apart from the Ventcel' domain (with $a = 0$) (see Corollary 2.4, statement (v)), all the results concerning the asymptotic behaviour of the semigroups have been established when α is maximal. It should be interesting to get similar results in the non-maximal case.

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Some classes of surfaces generated by Nielson and Marshall type operators on the triangle with one curved side

Teodora Cătinaş

Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. We construct some classes of surfaces which satisfy some given conditions, using some Hermite, Nielson and Marshall type interpolation operators defined on a triangle with one curved side.

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Keywords: Blending interpolation, Hermite, Nielson and Marshall interpolation operators, surfaces generation.

1. Introduction

In some recent papers, we have introduced and studied some interpolation operators for the functions defined on triangles with curved sides (see, e.g., [8]-[11], [13], [14], [16], [17]). They permit essential boundary conditions to be satisfied exactly and they come as an extension of the interpolation operators on triangles with all straight edges, introduced and studied for example in [1], [3]-[7], [12], [22]-[28].

We consider here a standard triangle, $\tilde{T}$, having the vertices $V_1 = (1, 0)$, $V_2 = (0, 1)$ and $V_3 = (0, 0)$, two straight sides Γ_1, Γ_2, along the coordinate axes, and the third side Γ_3 (opposite to the vertex V_3), which is defined by the one-to-one functions f and g, where g is the inverse of the function f, i.e., $y = f(x)$ and $x = g(y)$, with $f(0) = g(0) = 1$. There is no restriction to consider this standard triangle $\tilde{T}$, since any triangle with one curved side can be obtained from this standard triangle by an affine transformation which preserves the form and order of accuracy of the interpolant [4].

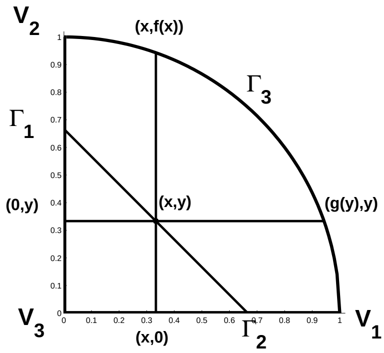

Figure 1: Triangle $\tilde{T}$.

The bending interpolants interpolate on an infinite set of points (segments, curves, etc.), so having such element as a boundary of an object, we may generate surfaces that contain the given boundary (see, e.g., [2], [18]-[21]). The aim of this paper is to construct some surfaces which satisfy some given conditions on the boundary of a domain that can be decomposed in triangles with one curved side. We construct some new surfaces using some Hermite, Nielson and Marshall type interpolation operators introduced in [13] and [14]. These operators come as extensions to triangle $\tilde{T}$, of some interpolation operators for triangles, given, for example, in [4], [5], [25].

2. Surfaces generation by Hermite, Nielson and Marshall type operators

Suppose that F is a real-valued function defined on $\tilde{T}$, and that it has all partial derivatives needed. We consider three types of interpolation operators defined on $\tilde{T}$:

- the Hermite interpolation operators H_1 and H_2 defined by [13]:

$$\begin{aligned}(H_1F)(x,y) =& \tfrac{[2x+g(y)][x-g(y)]^2}{g^3(y)}F(0,y) + \tfrac{x[x-g(y)]^2}{g^2(y)}F^{(1,0)}(0,y) \\ &+ \tfrac{x^2[-2x+3g(y)]}{g^3(y)}F(g(y),y) + \tfrac{x^2[x-g(y)]}{g^2(y)}F^{(1,0)}(g(y),y), \\ (H_2F)(x,y) =& \tfrac{[2y+f(x)][y-f(x)]^2}{f^3(x)}F(x,0) + \tfrac{y[y-f(x)]^2}{f^2(x)}F^{(0,1)}(x,0) \\ &+ \tfrac{y^2[-2y+3f(x)]}{f^3(x)}F(x,f(x)) + \tfrac{y^2[y-f(x)]}{f^2(x)}F^{(0,1)}(x,f(x)),\end{aligned} \tag{2.1}$$

- the Nielson type interpolation operators given by [14]:

$$\begin{aligned}(N_1F)(x,y) &= yF(x,f(x)) + (1-f(x))F(g(y),y), \\ (N_2F)(x,y) &= F(0,y) + F(x,0) - F(0,0),\end{aligned} \tag{2.2}$$

- the Marshall type operators defined by [14]:

$$
\begin{aligned}
(Q_1F)(x,y) =& yF(0,1) + g(y)F(\tfrac{x}{g(y)},0), \qquad (2.3)\\
(Q_2F)(x,y) =& xF(1,0) + f(x)F(0,\tfrac{y}{f(x)}),\\
(Q_3F)(x,y) =& (f(x)-y)F(0,0) + (1-f(x)+y)F\left(\tfrac{x}{1-f(x)+y},\tfrac{y}{1-f(x)+y}\right).
\end{aligned}
$$

For obtaining the first class of surfaces, we consider the boolean sum of the Nielson type operators N_1 and N_2, given in (2.2), namely,

$$
\begin{aligned}
((N_1\oplus N_2)F)(x,y) =& y[F(x,f(x)) - F(0,f(x)) - F(x,0) + F(0,0)] \qquad (2.4)\\
&+ (1-f(x))[F(g(y),y) - F(0,y) - F(g(y),0)]\\
&+ F(0,y) + F(x,0) - f(x)F(0,0),
\end{aligned}
$$

and we apply the condition that the roof stays on its support, i.e., $F|_{\Gamma_3} = 0$. We get

$$
\begin{aligned}
S_N :=& -yF(0,f(x)) + (1-y)F(x,0) + [y-f(x)]F(0,0) \qquad (2.5)\\
&+ f(x)F(0,y) + [f(x)-1]F(g(y),0).
\end{aligned}
$$

Theorem 2.1. *If $F|_{\Gamma_3} = 0$, then we have the following properties of the operator S_N :*

$$
\begin{aligned}
(S_NF)(x,0) &= F(x,0).\\
(S_NF)(0,y) &= F(0,y).\\
(S_NF)(x,f(x)) &= 0.
\end{aligned}
$$

Proof. The proof follows directly by the expression of S_N from (2.5). □

In the second level of approximation we use the Hermite interpolation operators, given in (2.1), taking into account the condition $F|_{\Gamma_3} = 0$, i.e.,

$$
\begin{aligned}
(H_1^1F)(x,y) :=& \tfrac{[2x+g(y)][x-g(y)]^2}{g^3(y)}F(0,y) + \tfrac{x[x-g(y)]^2}{g^2(y)}F^{(1,0)}(0,y) \qquad (2.6)\\
&+ \tfrac{x^2[x-g(y)]}{g^2(y)}F^{(1,0)}(g(y),y),\\
(H_2^1F)(x,y) :=& \tfrac{[2y+f(x)][y-f(x)]^2}{f^3(x)}F(x,0) + \tfrac{y[y-f(x)]^2}{f^2(x)}F^{(0,1)}(x,0)\\
&+ \tfrac{y^2[y-f(x)]}{f^2(x)}F^{(0,1)}(x,f(x)).
\end{aligned}
$$

We apply the following approximations:

$$
\begin{aligned}
F(x,0) \approx (H_1^1F)(x,0) =& (2x+1)(x-1)^2F(0,0) + x(x-1)^2F^{(1,0)}(0,0) \qquad (2.7)\\
&+ x^2(x-1)F^{(1,0)}(1,0),
\end{aligned}
$$

$$
\begin{aligned}
F(0,y) \approx (H_2^1F)(0,y) =& (2y+1)(y-1)^2F(0,0) + y(y-1)^2F^{(0,1)}(1,0) \qquad (2.8)\\
&+ y^2(y-1)F^{(0,1)}(0,1),
\end{aligned}
$$

and by (2.5), we obtain the following class of surfaces:

$$
\begin{aligned}
(S_1F)(x,y) =& -y(H_2^1F)(0,f(x)) + (1-y)(H_1^1F)(x,0) + [y-f(x)]F(0,0)\\
&+ f(x)(H_2^1F)(0,y) + [f(x)-1](H_1^1F)(g(y),0),
\end{aligned}
$$

i.e.,

$$\begin{aligned}(S_1F)(x,y) =& - y\{[2f(x)+1][f(x)-1]^2F(0,0)+f(x)[f(x)-1]^2F^{(0,1)}(1,0) \\ &+ f(x)^2[f(x)-1]F^{(0,1)}(0,1)\} + (1-y)[(2x+1)(x-1)^2F(0,0) \\ &+ x(x-1)^2F^{(1,0)}(0,0)+x^2(x-1)F^{(1,0)}(1,0)] + [y-f(x)]F(0,0) \\ &+ f(x)[(2y+1)(y-1)^2F(0,0)+y(y-1)^2F^{(0,1)}(1,0) \\ &+ y^2(y-1)F^{(0,1)}(0,1)] + [f(x)-1]\{[2g(y)+1][g(y)-1]^2F(0,0) \\ &+ g(y)[g(y)-1]^2F^{(1,0)}(0,0)+g(y)^2[g(y)-1]F^{(1,0)}(1,0)\}.\end{aligned} \tag{2.9}$$

For obtaining the second class of surfaces, we consider the boolean sum of the Marshall type operators Q_1, Q_2 and Q_3, given in (2.3):

$$\begin{aligned}((Q_1\oplus Q_2\oplus Q_3)F)(x,y) =& g(y)F(\tfrac{x}{g(y)},0)+f(x)F(0,\tfrac{y}{f(x)})+(1-f(x)+y)\cdot \\ &\cdot F(\tfrac{x}{1-f(x)+y},\tfrac{y}{1-f(x)+y}) - yF(0,1)-g(y)f(\tfrac{x}{g(y)})F(0,0) \\ &- g(y)\left[1-f(\tfrac{x}{g(y)})\right]F\left(\tfrac{x}{g(y)}\Big/(1-f(\tfrac{x}{g(y)})),0\right),\end{aligned} \tag{2.10}$$

and supposing that the roof stays on its support we set the condition $F|_{\Gamma_3}=0$, hence we obtain

$$\begin{aligned}S_Q :=& g(y)F(\tfrac{x}{g(y)},0)+f(x)F(0,\tfrac{y}{f(x)}) \\ &- yF(0,1)-g(y)f(\tfrac{x}{g(y)})F(0,0) \\ &- g(y)\left[1-f(\tfrac{x}{g(y)})\right]F\left(\tfrac{x}{g(y)}\Big/(1-f(\tfrac{x}{g(y)})),0\right).\end{aligned} \tag{2.11}$$

Theorem 2.2. *If $F|_{\Gamma_3}=0$, then we have*

$$(S_QF)(x,f(x))=0.$$

Proof. The proof follows directly replacing in (2.11). □

In the second level we use the Hermite interpolation operators, given in (2.6), and the approximations (2.7) and (2.8), and we obtain the following class of surfaces:

$$\begin{aligned}(S_2F)(x,y) =& g(y)(H_1^1F)(\tfrac{x}{g(y)},0)+f(x)(H_2^1F)(0,\tfrac{y}{f(x)}) \\ &- yF(0,1)-g(y)f(\tfrac{x}{g(y)})F(0,0) \\ &- g(y)\left[1-f(\tfrac{x}{g(y)})\right](H_1^1F)\left(\tfrac{x}{g(y)}\Big/(1-f(\tfrac{x}{g(y)})),0\right),\end{aligned}$$

given below by

$$\begin{aligned}(S_2F)(x,y) =& \tfrac{[2x+g(y)][x-g(y)]^2}{g^2(y)}F(0,0) + \tfrac{x[x-g(y)]}{g^2(y)}^2 F^{(1,0)}(0,0) \\ &+ \tfrac{x^2[x-g(y)]}{g^2(y)}F^{(1,0)}(1,0) \\ &+ \tfrac{[2y+f(x)][y-f(x)]^2}{f^2(x)}F(0,0) + \tfrac{y[y-f(x)]^2}{f^2(x)}F^{(0,1)}(1,0) \\ &+ \tfrac{y^2[y-f(x)]}{f^2(x)}F^{(0,1)}(0,1) \\ &- yF(0,1) - g(y)f(\tfrac{x}{g(y)})F(0,0) \\ &- g(y)\left[1 - f(\tfrac{x}{g(y)})\right]\cdot \\ &\cdot \{[2m(x,y)+1][m(x,y)-1]^2F(0,0) \\ &+ m(x,y)[m(x,y)-1]^2F^{(1,0)}(0,0) \\ &+ m^2(x,y)[m(x,y)-1]F^{(1,0)}(1,0)\},\end{aligned} \tag{2.12, 2.13}$$

where $m(x,y)$ denotes $\frac{x}{g(y)}\Big/(1-f(\frac{x}{g(y)}))$.

Other classes of surfaces may be obtained using the conditions

$$F|_{\Gamma_3} = F^{(0,1)}\Big|_{\Gamma_3} = F^{(1,0)}\Big|_{\Gamma_3} = 0. \tag{2.14}$$

We consider the boolean sum of the Nielson type operators N_1 and N_2, given in (2.4), taking into account the conditions (2.14), and we get the operator S_N given in (2.5).

In the second level we use the Hermite interpolation operators, given in (2.1), taking into account the conditions (2.14), so we have

$$\begin{aligned}(H_1^2F)(x,y) :=& \tfrac{[2x+g(y)][x-g(y)]^2}{g^3(y)}F(0,y) + \tfrac{x[x-g(y)]^2}{g^2(y)}F^{(1,0)}(0,y), \\ (H_2^2F)(x,y) :=& \tfrac{[2y+f(x)][y-f(x)]^2}{f^3(x)}F(x,0) + \tfrac{y[y-f(x)]^2}{f^2(x)}F^{(0,1)}(x,0).\end{aligned} \tag{2.15}$$

Using the following approximations:

$$F(x,0) \approx (H_1^2F)(x,0) = (2x+1)(x-1)^2F(0,0) + x(x-1)^2F^{(1,0)}(0,0),$$

$$F(0,y) \approx (H_2^2F)(0,y) = (2y+1)(y-1)^2F(0,0) + y(y-1)^2F^{(0,1)}(1,0),$$

by (2.5), we obtain the following class of surfaces:

$$\begin{aligned}(S_3F)(x,y) =& -y(H_2^2F)(0,f(x)) + (1-y)(H_1^2F)(x,0) + [y-f(x)]F(0,0) \\ &+ f(x)(H_2^2F)(0,y) + [f(x)-1](H_1^2F)(g(y),0),\end{aligned}$$

further given as

$$
\begin{aligned}
(S_3F)(x,y) =& - y\{[2f(x)+1][f(x)-1]^2F(0,0) + f(x)[f(x)-1]^2F^{(0,1)}(1,0)\} \qquad (2.16)\\
&+ (1-y)[(2x+1)(x-1)^2F(0,0) + x(x-1)^2F^{(1,0)}(0,0)]\\
&+ [y-f(x)]F(0,0) + f(x)[(2y+1)(y-1)^2F(0,0)\\
&+ y(y-1)^2F^{(0,1)}(1,0)] + [f(x)-1]\{[2g(y)+1][g(y)-1]^2F(0,0)\\
&+ g(y)[g(y)-1]^2F^{(1,0)}(0,0)\}.
\end{aligned}
$$

Next we consider the boolean sum of the Marshall type operators, given in (2.10), taking into account the conditions (2.14), and we get S_Q given in (2.11).

Further, we apply the Hermite interpolation operators H_1^2 and H_2^2, given in (2.15), and we get the following class of surfaces:

$$
\begin{aligned}
(S_4F)(x,y) =& g(y)(H_1^2F)(\tfrac{x}{g(y)},0) + f(x)(H_2^2F)(0,\tfrac{y}{f(x)})\\
&- yF(0,1) - g(y)f(\tfrac{x}{g(y)})F(0,0)\\
&- g(y)\left[1 - f(\tfrac{x}{g(y)})\right](H_1^2F)\left(\tfrac{x}{g(y)}\Big/(1-f(\tfrac{x}{g(y)})),0\right),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
(S_4F)(x,y) =& \tfrac{[2x+g(y)][x-g(y)]^2}{g^2(y)}F(0,0) + \tfrac{x[x-g(y)]^2}{g^2(y)}F^{(1,0)}(0,0) \qquad (2.17)\\
&+ \tfrac{[2y+f(x)][y-f(x)]^2}{f^2(x)}F(0,0) + \tfrac{y[y-f(x)]^2}{f^2(x)}F^{(0,1)}(1,0)\\
&- yF(0,1) - g(y)f(\tfrac{x}{g(y)})F(0,0)\\
&- g(y)\left[1 - f(\tfrac{x}{g(y)})\right]\cdot\\
&\cdot\{[2m(x,y)+1][m(x,y)-1]^2F(0,0)\\
&+ m(x,y)[m(x,y)-1]^2F^{(1,0)}(0,0)\},
\end{aligned}
$$

where $m(x,y)$ denotes $\frac{x}{g(y)}\Big/\left(1-f\left(\frac{x}{g(y)}\right)\right)$.

3. Numerical examples

Example 3.1. Consider $F : \tilde{T} \to \mathbb{R}$,

$$F(x,y) = \frac{(x^2+y^2-h^2)^2}{x^2+y^2+1} \quad \text{and} \quad f(x) = \sqrt{1-x^2}.$$

In Figure 2 we plot the graphs of the surface S_1F, given in (2.9).

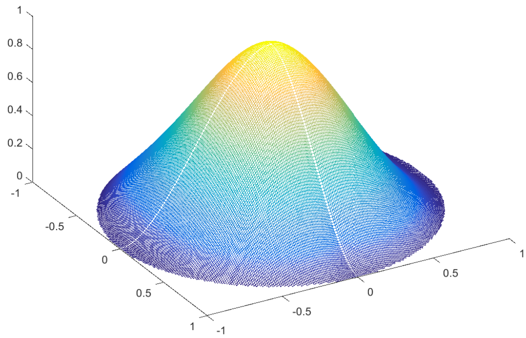

Figure 2: The surface S_1.

Example 3.2. Consider the function $f(x) = \sqrt{1-x^2}$ and $F : \tilde{T} \to \mathbb{R}$.
In Figure 3 we plot the graphs of surface S_2F, given in (2.12), assigning to the data $(F(0,0), F(0,1), F^{(1,0)}(0,0),\ F^{(1,0)}(1,0), F^{(0,1)}(1,0), F^{(0,1)}(0,1))$ the values $(-1/4, -1, 1, -1, 1)$.

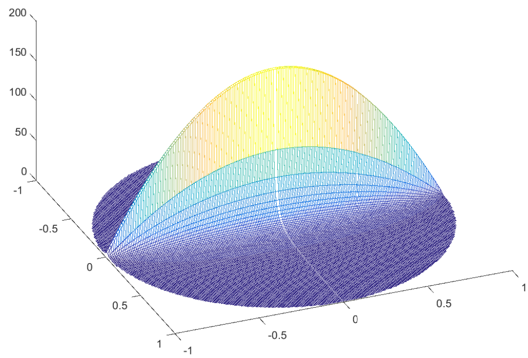

Figure 3: The surface S_2.

Example 3.3. Consider the data from Example 3.1. In Figure 4 we plot the graphs of the surface S_3F, given in (2.16).

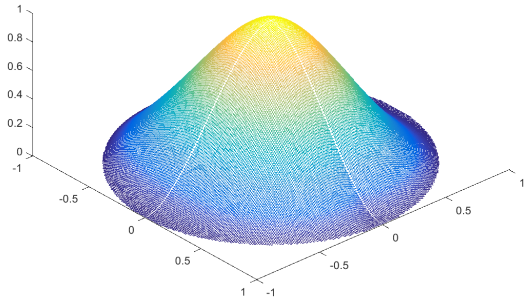

Figure 4: The surface S_3.

Example 3.4. Consider same data as in Example 3.2. In Figure 5 we plot the graphs of the surface S_4F, given in (2.17).

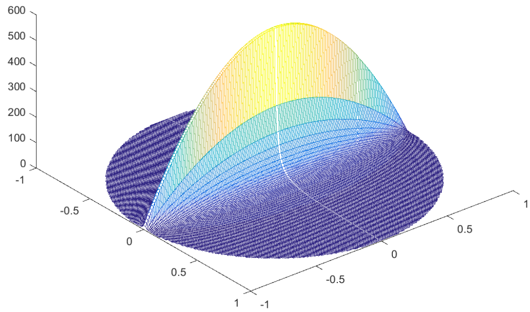

Figure 5: The surface S_4.

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Superdense unbounded divergence of a class of interpolatory product quadrature formulas

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Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. The aim of this paper is to highlight the superdense unbounded divergence of a class of product quadrature formulas of interpolatory type on Jacobi nodes, associated to the Banach space of all real continuous functions defined on $[-1,1]$, and to a Banach space of measurable and essentially bounded functions $g : [-1,1] \to \mathbb{R}$. Some aspects regarding the convergence of these formulas are pointed out, too.

Mathematics Subject Classification (2010): 41A10, 65D32.

Keywords: Superdense set, unbounded divergence, product quadrature formulas, Dini-Lipschitz convergence.

1. Introduction

This paper deals with a class of interpolatory product quadrature formulas, regarding their divergence and the convergence rate, as follows. Let C be the Banach space of all continuous functions $f : [-1,1] \to \mathbb{R}$, endowed with the supremum norm $\|\cdot\|$. Denoting by μ the Lebesgue measure on the interval $[-1,1]$, let $(L_p, \|\cdot\|_p)$, $1 \le p \le \infty$, be the Banach space of all measurable functions (equivalence classes of functions, with respect to the equality μ-a.e.) $g : [-1,1] \to \mathbb{R}$, normed by

$$\|g\|_p = \left(\int_{-1}^{1} |g(x)|^p dx \right)^{1/p}, \text{ if } 1 \le p < \infty, \text{ and } \|g\|_\infty = \text{esssup}|g|.$$

According to [7], [8], if $p \in [1,\infty]$ and $\rho \in L_q$ (with $p^{-1} + q^{-1} = 1$) are given such that $\rho(x) > 0$ μ-a.e. on $[-1,1]$, the notation $(L_p^{(1/\rho)}, \|\cdot\|_p^{(1/\rho)})$ stands for the Banach space of all measurable functions g for which $g/\rho \in L_p$ and $\|g\|_p^{(1/\rho)} = \|g/\rho\|_p$.

Further, let consider an arbitrary triangular node matrix

$$\mathcal{M} = \{x_{kn} : \ n \ge 1, \ 1 \le k \le n\}$$

so that the n-th row of $\mathcal{M}$, $n \geq 1$, contains n distinct nodes of $[-1, 1]$, then let us denote, as usual, by $\mathcal{L}_n f \in \mathcal{P}_{n-1}$ (the space of all polynomials of degree at most $n-1$) and λ_n the Lagrange interpolation polynomial and the Lebesgue function associated to the n-th row of $\mathcal{M}$, respectively, i.e.,

$$(\mathcal{L}_n f)(x) = \sum_{k=1}^{n} f(x_{kn}) l_{kn}(x),\ f \in C,\ \lambda_n(x) = \sum_{k=1}^{n} |l_{kn}(x)|,$$

where l_{kn} are the fundamental Lagrange interpolation polynomials, [2], [10].
The equalities

$$\int_{-1}^{1} g(x)f(x)dx = \int_{-1}^{1} g(x)(\mathcal{L}_n f)(x)dx + R_n(f; g),\ f \in C,\ g \in L_p^{(1/\rho)},\ n \geq 1 \quad (1.1)$$

involving

$$R_n(P, g) = 0,\ \forall\ f \in \mathcal{P}_{n-1} \text{ and } g \in L_p^{(1/\rho)},\ n \geq 1 \quad (1.2)$$

describe *product quadrature formulas of interpolatory type*, associated to the spaces C and $L_p^{(1/\rho)}$.

If $p = 1$, these product quadrature formulas were intensively studied, in their convergence aspects, for various functions $g \in L_1$, $\rho \in L_\infty$ (including $\rho(x) = (1-x)^a(1+x)^b$, $a, b \geq 0$) and node matrices $\mathcal{M}$, [1], [3], [4], [7], [8]. We notice, also, the divergence result obtained by I.H. Sloan and W.E. Smith, for arbitrary node matrices $\mathcal{M}$ and $\rho(x) = 1$, $-1 \leq x \leq 1$, [8, Th. 6]. A recent result, [5], refers to more general product quadrature formulas of interpolatory type, involving polynomial projection operators $\mathcal{L}_n : C \to \mathcal{P}_{n-1}$ (namely $\mathcal{L}_n f \in \mathcal{P}_{n-1}$, $\forall\ f \in C$, and $\mathcal{L}_n f = f$ if and only if $f \in \mathcal{P}_{n-1}$) instead of Lagrange projections in (1.1) and highlights the phenomenon of double condensation of singularities for the corresponding product quadrature formulas (1.1), meaning unbounded divergence on superdense sets belonging to the spaces C and $L_1^{(1/\rho)}$, for arbitrary node matrices $\mathcal{M}$ and $\rho \in L_\infty$, with $\rho(x) > 0$ μ-a.e. on $[-1, 1]$.

The aim of this paper is to point out the superdense unbounded divergence of the product quadrature formulas described by (1.1) and (1.2) for $p = \infty$, $\rho(x) = (1-x)^a(1+x)^b$, with $a, b > -1$, and $\mathcal{M} = \mathcal{M}^{(\alpha,\beta)}$, $\alpha > -1$, $\beta > -1$, where $\mathcal{M}^{(\alpha,\beta)}$ is the Jacobi node matrix (namely, its n-th row contains the roots $x_n^{(\alpha,\beta)}$, $1 \leq k \leq n$, of the Jacobi polynomial $P_n^{(\alpha,\beta)}$, $n \geq 1$). Moreover, some aspects regarding the convergence of these formulas (for functions $f \in C$ satisfying a Dini-Lipschitz condition and arbitrary $g \in L_\infty^{(1/\rho)}$) will be presented in the last section.

In this paper, the notation M_k, $k \geq 1$, stands for some positive constants which do not depend on n. Also, we denote by $\omega(f, \cdot)$ the modulus of continuity associated to a function $f \in C$.

2. Unbounded divergence on superdense sets

Suppose that $\rho(x) = (1-x)^a(1+x)^b$, $a, b > -1$ and $\mathcal{M} = \mathcal{M}^{(\alpha,\beta)}$, $\alpha > -1$, $\beta > -1$. Let U_n, $n \geq 1$, be the continuous linear operators defined as

$$\begin{cases} U_n : C \to (L_\infty^{(1/\rho)})^*;\ f \mapsto U_n f \\ (U_n f)(g) = \displaystyle\int_{-1}^{1} g(x)(\mathcal{L}_n f)(x)dx;\ f \in C,\ g \in L_\infty^{(1/\rho)}, \end{cases} \tag{2.1}$$

where Y^* is the Banach space of all continuous linear functionals defined on the normed space Y.

Using standard arguments and classic results of Functional Analysis, we obtain (see also [8]):

$$\|U_n\| = \sup\{\|U_n f\| :\ f \in C,\ \|f\| \leq 1\}$$

and

$$\begin{aligned} \|U_n f\| &= \sup\left\{\left|\int_{-1}^{1} g(x)(\mathcal{L}_n f)(x)dx\right| :\ g/\rho \in L_\infty,\ \|g/\rho\|_\infty \leq 1\right\} \\ &= \sup\left\{\left|\int_{-1}^{1} \rho(x)g(x)(\mathcal{L}_n f)(x)dx\right| :\ g \in L_\infty,\ \|g\|_\infty \leq 1\right\}, \end{aligned}$$

so we get

$$\|U_n\| = \sup\{\|\rho\mathcal{L}_n f\|_1 :\ f \in C,\ \|f\| \leq 1\},\ n \geq 1. \tag{2.2}$$

Now, we can state:

Theorem 2.1. *Suppose that $\alpha \geq 2a + 3/2$ or $\beta \geq 2b + 3/2$. Then, a superdense set X_0 in the Banach space $L_\infty^{(1/\rho)}$ exists such that for every g in X_0, the set of C consisting of all functions for which the product quadrature formulas described by (1.1) and (1.2) are unbounded divergent, namely*

$$Y_0(g) = \left\{f \in C :\ \limsup_{n\to\infty}\left|\int_{-1}^{1} g(x)(\mathcal{L}_n f)(x)dx\right| = \infty\right\},$$

is superdense in the Banach space C.

Proof. First, we show that the set $\{\|U_n\| :\ n \geq 1\}$ is unbounded. Similarly to [9], let consider the function $f_n \in C$, $n \geq 1$, defined by

$$f_n(x) = \begin{cases} (-1)^k, & \text{if} \quad x = x_{kn}^{(\alpha,\beta)},\ 0 \leq k \leq n+1 \\ \text{linear}, & \text{if} \quad x \in [x_{kn}^{(\alpha,\beta)}, x_{k,n-1}^{(\alpha,\beta)}],\ 1 \leq k \leq n+1, \end{cases}$$

where $x_{0n}^{(\alpha,\beta)} = 1$ and $x_{n+1,n}^{(\alpha,\beta)} = -1$.

It follows from (2.2):

$$\|U_n\| \geq \|\rho\mathcal{L}_n f_n\|_1 = \int_{-1}^{1}(1-x)^a(1+x)^b|(\mathcal{L}_n f_n)(x)|dx. \tag{2.3}$$

Next, let us suppose that $\alpha \geq 2a + 3/2 > -1/2$ and set $q_0 = 1 - \dfrac{4(a+1)}{2\alpha+1}$ (so, $0 \leq q_0 < 1$). Using the estimation of [9, formula (3.3), with $p = 1$ and $q = q_0$], we get:

$$\begin{cases} \|U_n\| \geq M_1 \log n, & \text{if} \quad q_0 = 0 \\ \|U_n\| \geq M_2 n^{q_0(\alpha+1/2)}, & \text{if} \quad q_0 > 0. \end{cases} \tag{2.4}$$

The relations (2.3) and (2.4) prove the unboundedness of the set $\{\|U_n\| :\ n \geq 1\}$, if $\alpha \geq 2a + 3/2$; similarly, the same assertion is true for $\beta \geq 2b + 3/2 > -\dfrac{1}{2}$.

Now, we apply the principle of condensation of singularities [3, Theorem 5.2], with $X = L_\infty^{(1/\rho)}$, $T = C$, $Y = \mathbb{R}$, $J = \mathbb{N}^*$ and $A_n(g; f) = (U_n f)(g)$. It is easily seen that the hypotheses 1° and 2° of this principle are fulfilled. In order to show the validity of the hypothesis 3°, denote by $\mathcal{U} = \{U_n :\ n \geq 1\}$ the family of the operators defined by (2.1). Using the principle of condensation of singularities, [3, Th. 5.4], with respect to the family $\mathcal{U}$ and taking into account the unboundedness of the set $\{\|U_n\| :\ n \geq 1\}$, we infer that the set of the singularities of $\mathcal{U}$, namely

$$\mathcal{S}(U) = \{f \in C :\ \sup\{\|U_n f\| :\ n \geq 1\} = \infty\}, \tag{2.5}$$

is superdense in C. Now, take $T_0 = \mathcal{S}(U)$ from (2.5) and remark that

$$\sup\{\|A_n f\| :\ n \geq 1\} = \sup\{\|U_n f\| :\ n \geq 1\} = \infty,$$

for every $f \in T_0$, therefore the hypothesis 3° of [3, Theorem 5.2] holds, too. Finally, denote by $Y_0(g)$ the set of singularities of the family $\mathcal{A}(g) = \{A_n(g, \cdot) :\ n \geq 1\}$, which completes the proof. □

3. Dini-Lipschitz convergence

Let us estimate the quadrature errors $R_n(f; g)$ of (1.1), see also [7], [8]. Denoting by $I : C \to (L_\infty^{(1/\rho)})^*$, the operator given by $(If)(g) = \displaystyle\int_{-1}^{1} g(x) f(x) dx$ and taking into account the interpolatory condition (1.2), we get:

$$|R_n(f; g)| = |(U_n - I)(f - p)(g)| \leq \|U_1 - I\| \cdot \|f - p\| \cdot \|g\|_\infty^{(1/\rho)}. \tag{3.1}$$

Further, we obtain, for every $f \in C$:

$$\|\rho \mathcal{L}_n f\|_1 = \int_{-1}^{1} \rho(x) |(\mathcal{L}_n f)(x)| dx \leq \left(\int_{-1}^{1} \rho(x) \lambda_n(x) dx \right) \|f\|,$$

so, (2.2) leads to:

$$\|U_n\| \leq \|\rho \lambda_n\|_1. \tag{3.2}$$

Similarly, we get

$$\|I\| \leq \|\rho\|_1. \tag{3.3}$$

Now, combining the relations (3.1), (3.2) and (3.3), the estimation

$$\|R_n(f; g)\| \leq M_3 (\|\rho\|_1 + \|\rho \lambda_n\|_1) \cdot \|g/\rho\|_\infty \cdot \omega\left(f; \frac{1}{n}\right) \tag{3.4}$$

holds for sufficient large $n \geq 1$.

The following step is to estimate $\|\rho\lambda_n\|_1$. We have:

$$\begin{cases} \|\rho\lambda_n\|_1 = \displaystyle\int_{-1}^{1}(1-x)^a(1+x)^b\lambda_n(x)dx = I_n^{(1)} + I_n^{(2)}, \text{ with} \\ I_n^{(1)} = \displaystyle\int_{-1}^{0}(1-x)^a(1+x)^b\lambda_n(x)dx \text{ and} \\ I_n^{(2)} = \displaystyle\int_{0}^{1}(1-x)^a(1+x)^b\lambda_n(x)dx. \end{cases} \tag{3.5}$$

Using the estimation

$$\lambda_n(x) - 1 \sim |P_n^{(\alpha,\beta)}|\sqrt{n}[1 + (1-x)^{(2\alpha+1)/4}\log n], \; 0 \le x \le 1, \; [6],$$

we obtain

$$I_n^{(2)} \sim \int_0^1 (1-x)^a dx + \sqrt{n}\int_0^1 (1-x)^a|P_n^{(\alpha,\beta)}(x)|dx$$
$$+\sqrt{n}(\log n)\int_0^1 (1-x)^{a+\alpha/2+1/4}|P_n^{(\alpha,\beta)}(x)|dx. \tag{3.6}$$

Next, the estimation [10, formula (7.34.1)]

$$\int_0^1 (1-x)^\mu|P_n^{(\alpha,\beta)}(x)|dx \sim \begin{cases} n^{\alpha-2\mu-2}, & \alpha > 2\mu+3/2 \\ n^{-1/2}\log n, & \alpha = 2\mu+3/2 \\ n^{-1/2}, & \alpha < 2\mu+\dfrac{3}{2} \end{cases} ; \; \alpha,\beta,\mu > -1,$$

gives for $\mu = a$ and $\mu = a + \alpha/2 + 1/4$, respectively:

$$\int_0^1 (1-x)^a|P_n^{(\alpha,\beta)}(x)|dx \sim \begin{cases} n^{\alpha-2a-2}, & \alpha > 2a+3/2 \\ n^{-1/2}\log n, & \alpha = 2a+3/2 \\ n^{-1/2}, & \alpha < 2a+3/2 \end{cases} \tag{3.7}$$

$$\int_0^1 (1-x)^{a+\alpha/2+1/4}|P_n^{(\alpha,\beta)}|dx \sim n^{-1/2}. \tag{3.8}$$

Finally, a combination of (3.6), (3.7) and (3.8) yields:

$$I_n^{(2)} \sim 1 + \log n + \begin{cases} n^{\alpha-2a-3/2}, & \alpha > 2a+3/2 \\ \log n, & \alpha = 2a+3/2 \\ 1, & \alpha < 2a+3/2. \end{cases} \tag{3.9}$$

A similar estimation holds for $I_n^{(1)}$ of (3.5), namely:

$$I_n^{(1)} \sim 1 + \log n + \begin{cases} n^{\beta-2b-3/2}, & \beta > 2b+3/2 \\ \log n, & \beta = 2b+3/2 \\ 1, & \beta < 2b+3/2. \end{cases} \tag{3.10}$$

Now, we prove the following statement.

Theorem 3.1. *If $\rho(x) = (1-x)^a(1+x)^b$, $-1 < \alpha \le 2a + 3/2$ and $-1 < \beta \le 2b + 3/2$, then the product quadrature formulas given by* (1.1) *and* (1.2) *are convergent for each $g \in L_\infty^{(1/\rho)}$ and for each $f \in C$ satisfying a Dini-Lipschitz condition*

$$\lim_{\delta \searrow 0} \omega(f; \delta) \log \delta = 0.$$

Proof. The relations (3.5), (3.9) and (3.10) lead to the estimation $\|\rho\lambda_n\|_1 \sim \log n$. which combined with (3.4) gives:

$$|R_n(f; g)| \le M_4 \cdot \|g/\rho\|_\infty \cdot \omega\left(f; \frac{1}{n}\right) \log n,$$

for sufficient large $n \ge 1$, which completes the proof. □

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Korovkin type theorem in the space $\tilde{C}_b[0,\infty)$

Zoltán Finta

Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. A Korovkin type theorem is established in the space $\tilde{C}_b[0,\infty)$ of all uniformly continuous and bounded functions on $[0,\infty)$ for a sequence of positive linear operators, the approximation error being estimated with the aid of the usual modulus of continuity. As applications we obtain quantitative results for q-Baskakov operators.

Mathematics Subject Classification (2010): 41A36, 41A25.

Keywords: Korovkin theorem, modulus of continuity, K-functional, q-integers, q-Baskakov operators.

1. Introduction

The well-known Korovkin's theorem ensures the convergence of sequences of positive linear operators to the identity operator in the strong operator topology. For $C[0,1]$ the Banach space of all continuous functions f on $[0,1]$ equipped with the norm $\|f\| = \sup\{|f(x)| : x \in [0,1]\}$, and for the test-functions $e_i(x) = x^i$, $x \in [0,1]$, $i \in \{0,1,2\}$, Korovkin's theorem is the following (see [5, p. 8]): *let* $(L_n)_{n\geq 1}$ *be a sequence of positive linear operators such that* $L_n : C[0,1] \to C[0,1]$. *Then* $\|L_n f - f\| \to 0$ *as* $n \to \infty$ *for all* $f \in C[0,1]$ *if and only if* $\|L_n e_i - e_i\| \to 0$ *as* $n \to \infty$ *for* $i \in \{0,1,2\}$. Specifically we recover Weierstrass' approximation theorem if we choose for L_n the Bernstein operators $B_n : C[0,1] \to C[0,1]$ defined by

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right). \tag{1.1}$$

The so-called q-Bernstein operators were introduced by Phillips [12], and they are generalization of (1.1) based on q-integers. To present these operators we recall some notions of the q-calculus (see e.g. [11]). Let $q > 0$. For each non-negative integer n, the q-integers $[n]_q$ and the q-factorials $[n]_q!$ are defined by

$$[n]_q = \begin{cases} 1+q+\ldots+q^{n-1}, & \text{if} \quad n \geq 1 \\ 0, & \text{if} \quad n = 0 \end{cases}$$

and

$$[n]_q! = \begin{cases} [1]_q[2]_q \ldots [n]_q, & \text{if} \quad n \geq 1 \\ 1, & \text{if} \quad n = 0. \end{cases}$$

For integers $0 \leq k \leq n$, the q-binomial coefficients are defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Then the q-Bernstein operators $B_{n,q} : C[0,1] \to C[0,1]$ are introduced as

$$(B_{n,q}f)(x) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q x^k (1-x)(1-qx)\ldots(1-q^{n-k-1}x) f\left(\frac{[k]_q}{[n]_q}\right). \tag{1.2}$$

For $q = 1$, we recover the operators (1.1). If $0 < q < 1$, then $B_{n,q}$ are positive linear operators. However, they do not satisfy the conditions of Korovkin's theorem, because $(B_{n,q}e_0)(x) = 1$, $(B_{n,q}e_1)(x) = x$ and

$$(B_{n,q}e_2)(x) = x^2 + \frac{1}{[n]_q} x(1-x) \to x^2 + (1-q)x(1-x) \neq x^2,$$

as $n \to \infty$ (see [12, pp. 513-514]). The investigation of convergence of operators (1.2) for $0 < q < 1$ fixed has resulted in the discovery of a Korovkin type theorem in $C[0,1]$ due to Wang [14]. Applying Wang's result to (1.2), there exists a limit operator $B_{\infty,q}$ on $C[0,1]$ such that $(B_{n,q}f)_{n\geq 1}$ converges to $B_{\infty,q}f$ uniformly on $[0,1]$ as $n \to \infty$. The operator $B_{\infty,q}$ was introduced by Il'inskii and Ostrovska [10], and it is called the limit q-Bernstein operator. Furthermore, in [6] and [7], we established new Korovkin type theorems for parameter depending sequences of operators defined on $C[0,1]$; these results are different from Wang's result.

On the other hand, in [8] and [9], Korovkin type theorems are studied in weighted spaces, showing that the direct analogue of Korovkin's theorem is not valid in spaces of functions defined on the semi-axis $[0,\infty)$ or on the whole real line, but under additional conditions can be obtained analogous theorem to Korovkin's theorem. Let φ be a strictly increasing continuous function on $[0,\infty)$ such that $\lim_{x\to\infty} \varphi(x) = +\infty$ and $\rho(x) = (1+\varphi^2(x))^{-1}$, $x \geq 0$. Further, let $B_\rho[0,\infty)$ be the set of all functions f satisfying the condition $\rho(x)|f(x)| \leq M_f$ for $x \geq 0$, where M_f is a positive constant depending only on f. We denote by $C_\rho[0,\infty)$ the space $C[0,\infty) \cap B_\rho[0,\infty)$ with the norm $\|f\|_\rho = \sup\{\rho(x)|f(x)| : x \geq 0\}$, and $C_\rho^*[0,\infty) = \{f \in C_\rho[0,\infty) : \lim_{x\to\infty} \rho(x)|f(x)| < \infty\}$. Gadjiev was the first in noticing the relevance of the spaces $C_\rho^*[0,\infty)$ in proving Korovkin type theorems. We have the following result [8]: *let* $(A_n)_{n\geq 1}$ *be a sequence of positive linear operators acting from* $C_\rho[0,\infty)$ *to* $B_\rho[0,\infty)$ *satisfying the conditions* $\lim_{n\to\infty} \|A_n\varphi^i - \varphi^i\|_\rho = 0$ *for* $i \in \{0,1,2\}$. *Then* $\lim_{n\to\infty} \|A_nf - f\|_\rho = 0$ *for any* $f \in C_\rho^*[0,\infty)$.

In what follows, let $C_b[0,\infty)$ be the space of all continuous and bounded functions f on $[0,\infty)$, equipped with the norm $\|f\| = \sup\{|f(x)| : x \geq 0\}$. Further, we set $\tilde{C}_b[0,\infty) = \{f \in C_b[0,\infty) : f$ is uniformly continuous on $[0,\infty)\}$. We consider the function $\rho \in C_b[0,\infty)$ such that $\rho(x) > 0$ for all $x \geq 0$, and the space $C_\rho[0,\infty) = \{f \in C[0,\infty) : \rho f$ is bounded on $[0,\infty)\}$ equipped with the norm $\|f\|_\rho = \sup\{\rho(x)|f(x)| : x \geq 0\}$. Obviously $C_\rho[0,\infty)$ is a Banach space, and for $\rho(x) = 1$, $x \geq 0$, we have $C_\rho[0,\infty) = C_b[0,\infty)$. The goal of the paper is to establish a Korovkin type theorem for a sequence of positive linear operators $(L_n)_{n\geq 1}$, where $L_n : \tilde{C}_b[0,\infty) \to C_\rho[0,\infty)$ and $(L_n)_{n\geq 1}$ converges to its limit operator $L_\infty : \tilde{C}_b[0,\infty) \to C_\rho[0,\infty)$, which is not necessarily the identity operator. The approximation error $\|L_n f - L_\infty f\|_\rho$ will be estimated with the aid of the usual modulus of continuity of $f \in \tilde{C}_b[0,\infty)$ defined by

$$\omega(f;\delta) = \sup\{|f(x) - f(y)| : x, y \in [0,\infty), |x - y| \leq \delta\}, \quad \delta > 0. \tag{1.3}$$

As applications we obtain quantitative estimates for some q-Baskakov operators.

2. Main result

For $W = \{g \in C_b[0,\infty) : g' \in C_b[0,\infty)\}$, $f \in C_b[0,\infty)$ and $\delta > 0$, the K-functional defined by $K(f;\delta) = \inf\{\|f - g\| + \delta\|g'\| : g \in W\}$ and the modulus of continuity (1.3) are equivalent (see [5, p. 177, Theorem 2.4]), i.e. there exists $C > 0$ such that

$$C^{-1}\omega(f;\delta) \leq K(f;\delta) \leq C\omega(f;\delta). \tag{2.1}$$

Throughout this paper C denotes positive constant independent of n and x, but not necessarily the same in different cases.

The next theorem is our Korovkin type theorem.

Theorem 2.1. *Let $(L_n)_{n\geq 1}$, $L_n : \tilde{C}_b[0,\infty) \to C_\rho[0,\infty)$ be a sequence of positive linear operators, and let $(\alpha_n)_{n\geq 1}$ be a positive sequence with $\alpha_n \to 0$ as $n \to \infty$. If the sequence $(\beta_n)_{n\geq 1}$ satisfies the conditions*

(*i*) $\beta_n + \beta_{n+1} + \ldots + \beta_{n+p-1} \leq C\alpha_n$ *for all* $n, p \geq 1$,
(*ii*) $\|L_n g - L_{n+1} g\|_\rho \leq C\beta_n \|g'\|$ *for all* $g \in W$ *and* $n \geq 1$,

then there exists a positive linear operator $L_\infty : \tilde{C}_b[0,\infty) \to C_\rho[0,\infty)$ such that $\|L_n f - L_\infty f\|_\rho \to 0$ as $n \to \infty$, where $f \in \tilde{C}_b[0,\infty)$ is arbitrary. Moreover

$$\|L_n f - L_\infty f\|_\rho \leq c\,\omega(f;\alpha_n) \tag{2.2}$$

for all $f \in \tilde{C}_b[0,\infty)$ and $n \geq 1$; c is a constant depending only on $\|L_1 e_0\|_\rho$.

Proof. By (i) and (ii), we have

$$\begin{aligned}
\|L_n g - L_{n+p} g\|_\rho &\leq \|L_n g - L_{n+1} g\|_\rho + \|L_{n+1} g - L_{n+2} g\|_\rho + \ldots \\
&\quad + \|L_{n+p-1} g - L_{n+p} g\|_\rho \\
&\leq C(\beta_n + \beta_{n+1} + \ldots + \beta_{n+p-1})\|g'\| \\
&\leq C\alpha_n \|g'\|
\end{aligned} \tag{2.3}$$

for all $g \in W$ and $n, p \geq 1$. Because $e_0 \in W$, we find, in view of (2.3), that $L_n e_0 = L_{n+p} e_0$ for $n, p \geq 1$. Hence

$$L_n e_0 = L_1 e_0 \tag{2.4}$$

for all $n \geq 1$. Further, $e_0 \in \tilde{C}_b[0, \infty)$ implies that $L_1 e_0 \in C_\rho[0, \infty)$, i.e.

$$\|L_1 e_0\|_\rho < \infty. \tag{2.5}$$

Taking into account that L_n are positive linear operators and (2.4) is satisfied, we obtain

$$\begin{aligned} \rho(x)|(L_n f)(x)| &\equiv \rho(x)|L_n(f, x)| \leq \rho(x) L_n(|f|, x) \leq \rho(x) L_n(\|f\| e_0, x) \\ &= \rho(x)\|f\| L_n(e_0, x) = \rho(x)\|f\|(L_n e_0)(x) \\ &= \rho(x)\|f\|(L_1 e_0)(x), \end{aligned}$$

where $f \in \tilde{C}_b[0, \infty)$ and $x \in [0, \infty)$. Hence, by (2.5),

$$\|L_n f\|_\rho \leq \|L_1 e_0\|_\rho \|f\| \tag{2.6}$$

for every $f \in \tilde{C}_b[0, \infty)$. Using (2.3) and (2.6), we find for arbitrary $g \in W$ that

$$\begin{aligned} \|L_n f - L_{n+p} f\|_\rho &\leq \|L_n f - L_n g\|_\rho + \|L_n g - L_{n+p} g\|_\rho \\ &\quad + \|L_{n+p} g - L_{n+p} f\|_\rho \\ &\leq 2\|L_1 e_0\|_\rho \|f - g\| + C\alpha_n \|g'\| \\ &\leq \max\{2\|L_1 e_0\|_\rho, C\}\{\|f - g\| + \alpha_n \|g'\|\}. \end{aligned}$$

Taking the infimum on the right hand side over all $g \in W$, we get

$$\|L_n f - L_{n+p} f\|_\rho \leq \max\{2\|L_1 e_0\|_\rho, C\} K(f; \alpha_n).$$

Hence, by (2.1),

$$\|L_n f - L_{n+p} f\|_\rho \leq c\,\omega(f; \alpha_n), \tag{2.7}$$

where c depends on $\|L_1 e_0\|_\rho$. Further, for $f \in \tilde{C}_b[0, \infty)$ and $\alpha_n \to 0$ as $n \to \infty$, we have $\omega(f; \alpha_n) \to 0$ as $n \to \infty$. Thus, by (2.7), we obtain that $(L_n f)_{n \geq 1}$ is a Cauchy sequence in the Banach space $C_\rho[0, \infty)$. Therefore there exists an operator L_∞ on $\tilde{C}_b[0, \infty)$ such that $\|L_n f - L_\infty f\|_\rho \to 0$ for every $f \in \tilde{C}_b[0, \infty)$. This also implies that L_∞ is a positive linear operator on $\tilde{C}_b[0, \infty)$, because $L_n : \tilde{C}_b[0, \infty) \to C_\rho[0, \infty)$ are positive linear operators, $n \geq 1$. Now let $p \to \infty$ in (2.7), then we obtain the estimation (2.2), which completes the proof of the theorem. □

3. Applications

In what follows we shall use the following notation:

$$(z; q)_n = (1 - z)(1 - qz) \dots (1 - q^{n-1} z),$$

where z is a real number, $0 < q < 1$ and $n = 1, 2, \dots$ Then

$$\left(\frac{q^2 x}{1 + x}; q\right)_n = \left(1 - \frac{q^2 x}{1 + x}\right)\left(1 - \frac{q^3 x}{1 + x}\right) \dots \left(1 - \frac{q^{n+1} x}{1 + x}\right)$$

and

$$(-qx; q)_{n+k} = (1 + qx)(1 + q^2 x) \dots (1 + q^{n+k} x)$$

for $x \geq 0$ and $k = 0, 1, 2, \ldots$

In [2], Aral and Gupta introduced the operators $B^*_{n,q} : C_b[0,\infty) \to C[0,\infty)$, where $n = 1, 2, \ldots$ and $0 < q < 1$, given by

$$(B^*_{n,q} f)(x) = \left(\frac{q^2 x}{1+x}; q\right)_n \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{q^{k+1}[n]_q}\right) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{q^2 x}{1+x}\right)^k. \qquad (3.1)$$

In [13], C. Radu defined the operators $V^*_{n,q} : C_b[0,\infty) \to C[0,\infty)$,

$$(V^*_{n,q} f)(x) = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{k(k-1)/2} \frac{(qx)^k}{(-qx;q)_{n+k}} f\left(\frac{[k]_q}{[n]_q q^{k-1}}\right), \qquad (3.2)$$

where $n = 1, 2, \ldots$ and $0 < q < 1$ (see also [3, (2.1)]). When $q = 1$, the operators $B^*_{n,q}$ and $V^*_{n,q}$ become the classical Baskakov operator [4].

For (3.1) we compute the difference $(B^*_{n,q} g)(x) - (B^*_{n+1,q} g)(x)$, where $g \in W$ and $x \geq 0$. We have

$$\begin{aligned}
&(B^*_{n,q} g)(x) - (B^*_{n+1,q} g)(x) \\
&= \left(\frac{q^2 x}{1+x}; q\right)_n \sum_{k=0}^{\infty} g\left(\frac{[k]_q}{q^{k+1}[n]_q}\right) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{q^2 x}{1+x}\right)^k \\
&\quad - \left(\frac{q^2 x}{1+x}; q\right)_{n+1} \sum_{k=0}^{\infty} g\left(\frac{[k]_q}{q^{k+1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{q^2 x}{1+x}\right)^k \\
&= \left(\frac{q^2 x}{1+x}; q\right)_n \sum_{k=0}^{\infty} \left\{ g\left(\frac{[k]_q}{q^{k+1}[n]_q}\right) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \right. \\
&\quad \left. - \frac{1 + x(1-q^{n+2})}{1+x} g\left(\frac{[k]_q}{q^{k+1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q \right\} \left(\frac{q^2 x}{1+x}\right)^k \\
&= \left(\frac{q^2 x}{1+x}; q\right)_n \sum_{k=1}^{\infty} \left\{ g\left(\frac{[k]_q}{q^{k+1}[n]_q}\right) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \right. \\
&\quad \left. - g\left(\frac{[k]_q}{q^{k+1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q \right\} \left(\frac{q^2 x}{1+x}\right)^k + \left(\frac{q^2 x}{1+x}; q\right)_n \\
&\quad \times \sum_{k=0}^{\infty} \left(1 - \frac{1 + x(1-q^{n+2})}{1+x}\right) g\left(\frac{[k]_q}{q^{k+1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{q^2 x}{1+x}\right)^k \\
&= \left(\frac{q^2 x}{1+x}; q\right)_n \sum_{k=0}^{\infty} \left\{ g\left(\frac{[k+1]_q}{q^{k+2}[n]_q}\right) \begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q - g\left(\frac{[k+1]_q}{q^{k+2}[n+1]_q}\right) \right. \\
&\quad \left. \times \begin{bmatrix} n+k+1 \\ k+1 \end{bmatrix}_q + g\left(\frac{[k]_q}{q^{k+1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q q^n \right\} \left(\frac{q^2 x}{1+x}\right)^{k+1}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{q^2x}{1+x};q\right)_n \sum_{k=0}^{\infty} \left\{ \left[\begin{array}{c} n+k \\ k+1 \end{array} \right]_q \left(g\left(\frac{[k+1]_q}{q^{k+2}[n]_q}\right) - g\left(\frac{[k+1]_q}{q^{k+2}[n+1]_q}\right)\right)\right. \\
&\quad \left. + q^n \left[\begin{array}{c} n+k \\ k \end{array} \right]_q \left(g\left(\frac{[k]_q}{q^{k+1}[n+1]_q}\right) - g\left(\frac{[k+1]_q}{q^{k+2}[n+1]_q}\right)\right)\right\} \left(\frac{q^2x}{1+x}\right)^{k+1} \\
&= \left(\frac{q^2x}{1+x};q\right)_n \sum_{k=0}^{\infty} \left\{ \left[\begin{array}{c} n+k \\ k+1 \end{array} \right]_q \int_{[k+1]_q/q^{k+2}[n+1]_q}^{[k+1]_q/q^{k+2}[n]_q} g'(u)\,du \right. \\
&\quad \left. + q^n \left[\begin{array}{c} n+k \\ k \end{array} \right]_q \int_{[k+1]_q/q^{k+2}[n+1]_q}^{[k]_q/q^{k+1}[n+1]_q} g'(u)\,du \right\} \left(\frac{q^2x}{1+x}\right)^{k+1},
\end{aligned}
$$

where we have used

$$
\left[\begin{array}{c} n+k \\ k+1 \end{array} \right]_q + q^n \left[\begin{array}{c} n+k \\ k \end{array} \right]_q = \left[\begin{array}{c} n+k+1 \\ k+1 \end{array} \right]_q .
$$

Hence

$$
\begin{aligned}
&|(B^*_{n,q}g)(x) - (B^*_{n+1,q}g)(x)| \\
&\quad \leq \left(\frac{q^2x}{1+x};q\right)_n \sum_{k=0}^{\infty} \left\{ \left[\begin{array}{c} n+k \\ k+1 \end{array} \right]_q \left| \frac{[k+1]_q}{q^{k+2}[n]_q} - \frac{[k+1]_q}{q^{k+2}[n+1]_q} \right| \right. \\
&\quad\quad \left. + q^n \left[\begin{array}{c} n+k \\ k \end{array} \right]_q \left| \frac{[k]_q}{q^{k+1}[n+1]_q} - \frac{[k+1]_q}{q^{k+2}[n+1]_q} \right| \right\} \left(\frac{q^2x}{1+x}\right)^{k+1} \|g'\| \\
&\quad = 2\|g'\| \left(\frac{q^2x}{1+x};q\right)_n \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k \\ k \end{array} \right]_q \frac{q^n}{[n+1]_q} \frac{1}{q^{k+2}} \left(\frac{q^2x}{1+x}\right)^{k+1} \\
&\quad = \frac{2q^{n-1}}{[n+1]_q} \|g'\| \left(\frac{q^2x}{1+x};q\right)_n \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k \\ k \end{array} \right]_q \left(\frac{qx}{1+x}\right)^{k+1}. \qquad (3.3)
\end{aligned}
$$

Because (see [1, p. 420])

$$
\sum_{k=0}^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array} \right]_q z^k = (1-z)^{-1}(1-qz)^{-1}\dots(1-q^{n-1}z)^{-1}, \quad |z|<1,
$$

we have, by (3.3),

$$
\begin{aligned}
&|(B^*_{n,q}g)(x) - (B^*_{n+1,q}g)(x)| \\
&\quad \leq \frac{2q^{n-1}}{[n+1]_q} \|g'\| \left(1-\frac{q^2x}{1+x}\right)\left(1-\frac{q^3x}{1+x}\right)\dots\left(1-\frac{q^{n+1}x}{1+x}\right) \\
&\quad\quad \times \frac{qx}{1+x}\left(1-\frac{qx}{1+x}\right)^{-1}\left(1-\frac{q^2x}{1+x}\right)^{-1}\dots\left(1-\frac{q^{n+1}x}{1+x}\right)^{-1} \\
&\quad = \frac{2q^{n-1}}{[n+1]_q} \|g'\| \frac{qx}{1+x} \frac{1+x}{1+x(1-q)} \\
&\quad \leq \frac{2q^{n-1}}{[n+1]_q} \|g'\| \frac{q}{1-q} = \frac{2q^n}{1-q^{n+1}} \|g'\|. \qquad (3.4)
\end{aligned}
$$

We set $\beta_n = q^n/(1-q^{n+1})$, $n = 1, 2, \ldots$ Then

$$\begin{aligned}\beta_n + \beta_{n+1} + \ldots + \beta_{n+p-1} &= \frac{q^n}{1-q^{n+1}} + \frac{q^{n+1}}{1-q^{n+2}} + \ldots + \frac{q^{n+p-1}}{1-q^{n+p}} \\ &\le \frac{q^n}{1-q^{n+1}}(1+q+\ldots+q^{p-1}) \\ &\le \frac{q^n}{(1-q)(1-q^{n+1})} \qquad (3.5)\end{aligned}$$

for all $n, p = 1, 2, \ldots$ Due to (3.4) and (3.5), we can apply Theorem 2.1 (case $\rho(x) = 1$, $x \ge 0$) with $\alpha_n = q^n/(1-q)(1-q^{n+1})$, $n = 1, 2, \ldots$ Thus we obtain the following

Theorem 3.1. *For the operators $B^*_{n,q}$ defined by (3.1) and $q \in (0,1)$ given, there exists a positive linear operator $B^*_{\infty,q} : \tilde{C}_b[0,\infty) \to C_b[0,\infty)$ such that*

$$\|B^*_{n,q}f - B^*_{\infty,q}f\| \le C\,\omega(f; q^n/(1-q)(1-q^{n+1}))$$

for all $f \in \tilde{C}_b[0,\infty)$ and $n = 1, 2, \ldots$

Here C is independent of $\|B^*_{1,q}e_0\|$, because $B^*_{n,q}e_0 = e_0$ (see [2, Lemma 2]) implies that $\|B^*_{n,q}f\| \le \|f\|$, $f \in \tilde{C}_b[0,\infty)$. This justifies that $B^*_{n,q}f \in C_b[0,\infty)$ for $f \in \tilde{C}_b[0,\infty)$.

Now we shall study the sequence $(V^*_{n,q})_{n\ge1}$ defined by (3.2). In the same way as above, we obtain the following representation for $(V^*_{n,q}g)(x) - (V^*_{n+1,q}g)(x)$, where $g \in W$ and $x \ge 0$:

$$\begin{aligned}&(V^*_{n,q}g)(x) - (V^*_{n+1,q}g)(x) \\ &\quad = \sum_{k=0}^{\infty} q^{k(k+1)/2} \frac{(qx)^{k+1}}{(-qx;q)_{n+k+1}} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left\{ q^n g\left(\frac{[k]_q}{[n+1]_q q^{k-1}}\right) \right. \\ &\qquad \left. - \frac{[n+k+1]_q}{[k+1]_q} g\left(\frac{[k+1]_q}{[n+1]_q q^k}\right) + \frac{[n]_q}{[k+1]_q} g\left(\frac{[k+1]_q}{[n]_q q^k}\right)\right\}\end{aligned}$$

(see also [3, Theorem 6]). Hence, by $[n+k+1]_q = [n]_q + q^n[k+1]_q$, we get

$$\begin{aligned}&(V^*_{n,q}g)(x) - (V^*_{n+1,q}g)(x) \\ &= \sum_{k=0}^{\infty} q^{k(k+1)/2} \frac{(qx)^{k+1}}{(-qx;q)_{n+k+1}} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left\{ q^n \left(g\left(\frac{[k]_q}{[n+1]_q q^{k-1}}\right) \right.\right. \\ &\quad \left.\left. - g\left(\frac{[k+1]_q}{[n+1]_q q^k}\right)\right) + \frac{[n]_q}{[k+1]_q}\left(g\left(\frac{[k+1]_q}{[n]_q q^k}\right) - g\left(\frac{[k+1]_q}{[n+1]_q q^k}\right)\right)\right\} \\ &= \sum_{k=0}^{\infty} q^{k(k+1)/2} \frac{(qx)^{k+1}}{(-qx;q)_{n+k+1}} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left\{ q^n \int_{[k+1]_q/[n+1]_q q^k}^{[k]_q/[n+1]_q q^{k-1}} g'(u)\,du \right. \\ &\quad \left. + \frac{[n]_q}{[k+1]_q} \int_{[k+1]_q/[n+1]_q q^k}^{[k+1]_q/[n]_q q^k} g'(u)\,du \right\}.\end{aligned}$$

Then

$$\begin{aligned}
&|(V^*_{n,q}g)(x) - (V^*_{n+1,q}g)(x)| \\
&\le \sum_{k=0}^{\infty} q^{k(k+1)/2} \frac{(qx)^{k+1}}{(-qx;q)_{n+k+1}} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \Bigg\{ q^n \left| \frac{[k]_q}{[n+1]_q q^{k-1}} - \frac{[k+1]_q}{[n+1]_q q^k} \right| \\
&\quad + \frac{[n]_q}{[k+1]_q} \left| \frac{[k+1]_q}{[n]_q q^k} - \frac{[k+1]_q}{[n+1]_q q^k} \right| \Bigg\} \|g'\| \\
&= \frac{2q^n}{[n+1]_q} \|g'\| \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{(qx)^{k+1}}{(-qx;q)_{n+k+1}} \begin{bmatrix} n+k \\ k \end{bmatrix}_q . \qquad (3.6)
\end{aligned}$$

Because of [13, Remark 4], we have

$$(V^*_{n+1,q}e_0)(x) = \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q q^{k(k-1)/2} \frac{(qx)^k}{(-qx;q)_{n+k+1}} = 1.$$

Therefore, by (3.6), we obtain

$$|(V^*_{n,q}g)(x) - (V^*_{n+1,q}g)(x)| \le \frac{2q^{n+1}x}{[n+1]_q} \|g'\|$$

or

$$\frac{1}{1+qx} |(V^*_{n,q}g)(x) - (V^*_{n+1,q}g)(x)| \le \frac{2q^n}{[n+1]_q} \|g'\|.$$

With the notation $\rho(x) = 1/(1+qx)$, $x \ge 0$, we have

$$\|V^*_{n,q}g - V^*_{n+1,q}g\|_\rho \le \frac{2q^n}{[n+1]_q} \|g'\|. \qquad (3.7)$$

Now we set $\beta_n = q^n/[n+1]_q$, $n = 1, 2, \ldots$ Then

$$\begin{aligned}
\beta_n + \beta_{n+1} + \ldots + \beta_{n+p-1} &\le \frac{q^n}{[n+1]_q}(1 + q + \ldots + q^{p-1}) \\
&\le \frac{q^n}{1-q^{n+1}} \qquad (3.8)
\end{aligned}$$

for all $n, p = 1, 2, \ldots$ Due to (3.7) and (3.8), we can apply Theorem 2.1 with $\alpha_n = q^n/(1-q^{n+1})$, $n = 1, 2, \ldots$ In conclusion we obtain the following

Theorem 3.2. *For the operators $V^*_{n,q}$ defined by (3.2), $q \in (0,1)$ given and $\rho(x) = 1/(1+qx)$, $x \ge 0$, there exists a positive linear operator $V^*_{\infty,q} : \tilde{C}_b[0,\infty) \to C_\rho[0,\infty)$ such that*

$$\|V^*_{n,q}f - V^*_{\infty,q}f\|_\rho \le C\,\omega(f; q^n/(1-q^{n+1}))$$

for all $f \in \tilde{C}_b[0,\infty)$ and $n = 1, 2, \ldots$

The constant C is independent of $\|V^*_{1,q}e_0\|_\rho$, because

$$\begin{aligned}
\|V^*_{n,q}f\|_\rho &= \sup\{\rho(x)|(V^*_{n,q}f)(x)| : x \ge 0\} \le \sup\{|(V^*_{n,q}f)(x)| : x \ge 0\} \\
&\le \|f\| \sup\{(V^*_{n,q}e_0)(x) : x \ge 0\} = \|f\| \sup\{e_0(x) : x \ge 0\} = \|f\|,
\end{aligned}$$

where $f \in \tilde{C}_b[0,\infty)$ (see [13, Remark 4]).

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On some numerical iterative methods for Fredholm integral equations with deviating arguments

Sanda Micula

Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. In this paper we develop iterative methods for nonlinear Fredholm integral equations of the second kind with deviating arguments, by applying Mann's iterative algorithm. This proves the existence and the uniqueness of the solution and gives a better error estimate than the classical Banach Fixed Point Theorem. The iterates are then approximated using a suitable quadrature formula. The paper concludes with numerical examples.

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Keywords: Fredholm integral equations, deviating arguments, numerical approximations, Altman's algorithm, Mann's iterative algorithm.

1. Preliminaries

Integral equations arise in many fields of mathematics, engineering, physics, etc., as they provide a strong tool for modeling various applications, phenomena and processes occurring in actuarial sciences, statistical study of dynamic living population, elasticity theory, diffraction problems, quantum mechanics, etc. In addition, a large class of initial and boundary value problems can be reformulated as integral equations. Thus, many researchers aim to find efficient and rapidly convergent algorithms for the numerical solution of Fredholm integral equations (see e.g. [2], [10], [11], [9]).

In this paper, we consider a Fredholm integral equation of the type

$$x(t) \quad = \quad \int_a^b K\big(t, s, x(s), x(g(s))\big)\, ds + f(t), \quad t \in [a, b], \tag{1.1}$$

where $K \in C\left([a, b] \times [a, b] \times \mathbb{R}^2\right)$, $f \in C[a, b]$ and $g \in C\left([a, b], [a, b]\right)$.
Other assumptions will be made on K, g and f later on.

As is well known, the solvability of (1.1) is based on fixed point theory. We define the operator $F : C[a,b] \to C[a,b]$ by

$$Fx(t) \quad = \quad \int_a^b K\big(t, s, x(s), x(g(s))\big)\, ds + f(t). \tag{1.2}$$

Then finding a solution of the integral equation (1.1) is equivalent to finding a fixed point for the operator F:

$$x = Fx. \tag{1.3}$$

We recall the main results of fixed point theory on a Banach space.

Definition 1.1. *Let $(X, ||\cdot||)$ be a Banach space. A mapping $F : X \to X$ is called a* $\mathbf{q-contraction}$ *if there exists a constant $0 \le q < 1$ such that*

$$||Fx - Fy|| \le q||x - y||, \tag{1.4}$$

for all $x, y \in X$.

We have the classical result, the contraction principle on a Banach space.

Theorem 1.2. *Let $(X, ||\cdot||)$ be a Banach space and $F : X \to X$ be a $q-$contraction. Then*

(a) *F has exactly one fixed point $x^* \in X$;*
(b) *the sequence of successive approximations $x_{n+1} = Fx_n, n \in \mathbb{N}$, converges to the solution x^*, for any arbitrary choice of initial point $x_0 \in X$;*
(c) *the error estimates*

$$\begin{aligned} ||x_n - x^*|| &\le \frac{q^n}{1-q}\ ||x_1 - x_0||, \\ ||x_n - x^*|| &\le \frac{q}{1-q}\ ||x_n - x_{n-1}|| \end{aligned} \tag{1.5}$$

hold for every $n \in \mathbb{N}$.

This result can be improved, using Mann iteration (Altman's algorithm) instead of Picard iteration. We recall the main results (see [1], [4]).

Theorem 1.3. *Let $(X, ||\cdot||)$ be a Banach space and $F : X \to X$ be a $q-$contraction. Let $0 < \varepsilon_n \le 1$ be a sequence of numbers satisfying*

$$\sum_{n=0}^{\infty} \varepsilon_n = \infty. \tag{1.6}$$

Then

(a) *equation $x = Fx$ has exactly one solution $x^* \in X$;*
(b) *the sequence of successive approximations*

$$x_{n+1} = (1 - \varepsilon_n)x_n + \varepsilon_n F x_n, \quad n = 0, 1, \ldots \tag{1.7}$$

converges to the solution x^*, for any arbitrary choice of initial point $x_0 \in X$;

(c) *for every $n \in \mathbb{N}$, there holds the error estimate*

$$||x_n - x^*|| \quad \leq \quad \frac{e^{1-q}}{1-q}\ ||x_0 - Fx_0||\ e^{-(1-q)y_n}, \tag{1.8}$$

where $y_0 = 0,\ y_n = \sum_{i=0}^{n-1} \varepsilon_i$, *for* $n \geq 1$.

Remark 1.4. Theorem 1.3 still holds true if X is replaced by any closed convex subset $Y \subseteq X$.

Most of the times (for suitable choices of ε_n and q), the error estimate in (1.8) is better than the one in (1.5) and the iterative method (1.7) converges faster than the classical one.

For more considerations on iterative algorithms, see e.g. [4], [7], [8]. The aim of this paper is to apply Altman's Theorem 1.3 to Fredholm integral equations of the second kind with deviating arguments.

2. Existence and uniqueness of the solution

We want to apply Altman's iterative algorithm to the operator equation (1.3). To this end, we consider the space $X = C[a, b]$ equipped with the Chebyshev norm

$$||x|| := \max_{t\in[a,b]} |x(t)|, \quad x \in X \tag{2.1}$$

and the ball $B_R := \{x \in C[a, b] \mid ||x - f|| \leq R\}$, for some $R > 0$. Then $(X, ||\cdot||)$ is a Banach space and $B_R \subseteq X$ is a closed convex subset.

Theorem 2.1. *Let* $F : (X, ||\cdot||) \to (X, ||\cdot||)$ *be defined by* (1.2). *Assume that*

(i) *there exist constants* $l_1, l_2 > 0$ *such that*

$$|K(t, s, u_1, v_1) - K(t, s, u_2, v_2)| \leq l_1|u_1 - u_2| + l_2|v_1 - v_2|, \tag{2.2}$$

for all $t, s \in [a, b]$ *and all* $u_1, u_2, v_1, v_2 \in [R_1 - R, R_2 + R]$, *where*

$$R_1 := \min_{t\in[a,b]} f(t), \quad R_2 := \max_{t\in[a,b]} f(t);$$

(ii)

$$q \quad := \quad (b - a)(l_1 + l_2) \quad < \quad 1; \tag{2.3}$$

(iii)

$$M_K(b - a) \quad \leq \quad R, \tag{2.4}$$

where $M_K := \max |K(t, s, u, v)|$ *over all* $t, s \in [a, b]$ *and all* $u, v \in [R_1 - R, R_2 + R]$.

Then

(a) *equation* (1.3) *has exactly one solution* $x^* \in X$;

(b) *the sequence of successive approximations*

$$x_{n+1} = \left(1 - \frac{1}{n+1}\right) x_n + \frac{1}{n+1} F x_n, \quad n = 0, 1, \ldots \tag{2.5}$$

converges to the solution x^*, *for any arbitrary initial point* $x_0 \in X$;

(c) *for every $n \in \mathbb{N}$, the error estimate*

$$||x_n - x^*|| \leq \frac{e^{1-q}}{1-q} ||x_0 - Fx_0|| \, e^{-(1-q)y_n} \tag{2.6}$$

holds, where $y_0 = 0$, $y_n = \sum_{i=0}^{n-1} \frac{1}{i+1}$, for $n \geq 1$.

Proof. We want to use Theorem 1.3 for $\varepsilon_n = \frac{1}{n+1}$, which obviously satisfies the conditions of Theorem 1.3.
Let $t \in [a, b]$ be fixed. By (2.2), we have

$$\begin{aligned}
|(Fx - Fy)(t)| &\leq \int_a^b \Big| K\big(t, s, x(s), x(g(s))\big) - K\big(t, s, y(s), y(g(s))\big)\Big| \, ds \\
&\leq l_1 \int_a^b |x(s) - y(s)| \, ds + l_2 \int_a^b |x(g(s)) - y(g(s))| \, ds \\
&\leq l_1(b-a)||x - y|| + l_2(b-a) \max_{g(s)\in[a,b]} \big|x(g(s)) - y(g(s))\big| \\
&\leq (b-a)(l_1 + l_2)||x - y||,
\end{aligned}$$

since $\max_{g(s)\in[a,b]} \big|x(g(s)) - y(g(s))\big| \leq \max_{s\in[a,b]} |x(s) - y(s)|$. Hence,

$$||Fx - Fy|| = \max_{t\in[a,b]} |(Fx - Fy)(t)| \leq q||x - y||$$

and by (2.3), it follows that F is a $q-$contraction.

Next, for every fixed $t \in [a, b]$, we have

$$\begin{aligned}
|Fx(t) - f(t)| &\leq \int_a^b \Big| K\big(t, s, x(s), x(g(s))\big)\Big| \, ds \\
&\leq M_K(b-a).
\end{aligned}$$

Thus, by (2.4), we have $F(B_R) \subseteq B_R$. Now our result follows from Theorem 1.3 and Remark 1.4. □

For more considerations on Mann iteration, see e.g. [4].

3. Numerical iterative methods

Altman's fixed point theorem provides iterative methods for solving equation (1.3). But, obviously, the iterates in (2.5) cannot be computed analytically, they have to be approximated numerically.

Consider a quadrature formula

$$\int_a^b \varphi(s)ds = \sum_{i=0}^m a_i\varphi(s_i) + R_\varphi, \tag{3.1}$$

with nodes $a = s_0 < s_1 < \cdots < s_m = b$, coefficients $a_i \in \mathbb{R}, i = 0, 1, \ldots, m$ and for which the remainder satisfies

$$|R_\varphi| \le M, \tag{3.2}$$

for some $M > 0$, with $M \to 0$ as $m \to \infty$.

Let $a = t_0 < t_1 < \cdots < t_m = b$ be the nodes and let $x_0 = \tilde{x}_0 \equiv f$ be the initial approximation. Then we use the iteration (2.5) and the numerical integration scheme (3.1) to approximate $x_n(t_k)$ and $x_n(g(t_k))$ with $\tilde{x}_n(t_k)$ and $\tilde{x}_n(g(t_k))$, respectively, for $k = \overline{0, m}$ and $n = 0, 1, \ldots$. For simplicity, we make the following notations:

$$\begin{aligned}
K_{k,i,n} &:= K\big(t_k, t_i, x_n(t_i), x_n(g(t_i))\big),\\
K_{g,k,i,n} &:= K\big(g(t_k), t_i, x_n(t_i), x_n(g(t_i))\big),\\
\tilde{K}_{k,i,n} &:= K\big(t_k, t_i, \tilde{x}_n(t_i), \tilde{x}_n(g(t_i))\big),\\
\tilde{K}_{g,k,i,n} &:= K\big(g(t_k), t_i, \tilde{x}_n(t_i), \tilde{x}_n(g(t_i))\big),\\
\tilde{x}_{n+1}(t_k) &:= \left[\left(1 - \frac{1}{n+1}\right)\tilde{x}_n(t_k) + \frac{1}{n+1}\left(\sum_{i=0}^m a_i\tilde{K}_{k,i,n} + f(t_k)\right)\right],\\
\tilde{x}_{n+1}(g(t_k)) &:= \left[\left(1 - \frac{1}{n+1}\right)\tilde{x}_n(g(t_k))\right.\\
&+ \left.\frac{1}{n+1}\left(\sum_{i=0}^m a_i\tilde{K}_{g,k,i,n} + f(g(t_k))\right)\right],\\
\tilde{R}_{n,k} &:= x_n(t_k) - \tilde{x}_n(t_k),\\
\tilde{R}_{g,n,k} &:= x_n(g(t_k)) - \tilde{x}_n(g(t_k)).
\end{aligned}$$

We have:

$$\begin{aligned}
x_{n+1}(t_k) &= \left(1 - \frac{1}{n+1}\right)x_n(t_k) + \frac{1}{n+1}\left(\int_a^b K\left(t_k, s, x_n(s), x_n(g(s))\right)ds + f(t_k)\right)\\
&= \left(1 - \frac{1}{n+1}\right)\left(\tilde{x}_n(t_k) + \tilde{R}_{n,k}\right) + \frac{1}{n+1}\left(\sum_{i=0}^m a_iK_{k,i,n} + R_K + f(t_k)\right) \qquad (3.3)\\
&= \left(1 - \frac{1}{n+1}\right)\left(\tilde{x}_n(t_k) + \tilde{R}_{n,k}\right)\\
&+ \frac{1}{n+1}\left(\sum_{i=0}^m a_i\tilde{K}_{k,i,n} + \sum_{i=0}^m a_i(K_{k,i,n} - \tilde{K}_{k,i,n}) + R_K + f(t_k)\right)\\
&= \left[\left(1 - \frac{1}{n+1}\right)\tilde{x}_n(t_k) + \frac{1}{n+1}\left(\sum_{i=0}^m a_i\tilde{K}_{k,i,n} + f(t_k)\right)\right] + \tilde{R}_{n+1,k} = \tilde{x}_{n+1}(t_k) + \tilde{R}_{n+1,k}.
\end{aligned}$$

Similarly, we derive

$$\begin{aligned}
\tilde{x}_{n+1}(g(t_k)) &= \left(1 - \frac{1}{n+1}\right)\tilde{x}_n(g(t_k)) \\
&+ \frac{1}{n+1}\left(\sum_{i=0}^{m} a_i \tilde{K}_{g,k,i,n} + f(g(t_k))\right) + \tilde{R}_{g,n+1,k} \qquad (3.4) \\
&= \tilde{x}_{n+1}(g(t_k)) + \tilde{R}_{g,n+1,k}.
\end{aligned}$$

Let

$$\tilde{R}^{(n,m)} = \max_{t_k \in [a,b]} \{ |x_n(t_k) - \tilde{x}_n(t_k)|, |x_n(g(t_k)) - \tilde{x}_n(g(t_k))| \}. \qquad (3.5)$$

Suppose that for the quadrature formula (3.1), condition (3.2) ensures the fact that $\tilde{R}^{(n,m)}$ defined above depends only on m and that $\tilde{R}^{(n,m)} = \tilde{R}^{(m)} \to 0$, as $m \to \infty$. Then the exact solution x^* can be approximated by the iterates $\tilde{x}_n$ at the nodes t_k and $g(t_k)$ and we can give an error estimate for our numerical iterative method. To better illustrate the approximations, we consider below one of the most popular numerical integration schemes, the trapezoidal rule.

3.1. Approximation using the trapezoidal rule

As in [5], [6], consider the composite trapezoidal rule

$$\int_a^b \varphi(s)\, ds = \frac{b-a}{2m}\left[\varphi(a) + 2\sum_{j=1}^{m-1}\varphi(s_j) + \varphi(b)\right] + R_\varphi,$$

where the $m+1$ nodes are $s_j = a + \dfrac{b-a}{m} j,\ j = \overline{0,m}$ and the remainder is given by

$$R_\varphi = -\frac{(b-a)^3}{12m^2}\varphi''(\eta), \quad \eta \in (a,b).$$

We use it to approximate the integrals in (2.5), as in (3.3) and (3.4), with initial approximation $x_0 = \tilde{x}_0 \equiv f$. For the error, we need the second derivative $[K(t_k, s, x_n(s), x_n(g(s)))]_s''$. We have

$$\begin{aligned}
[K(t_k, s, u, v)]_s' &= \frac{\partial K}{\partial s} + \frac{\partial K}{\partial u}u' + \frac{\partial K}{\partial v}v' \\
[K(t_k, s, u, v)]_s'' &= \frac{\partial^2 K}{\partial s^2} + 2\frac{\partial^2 K}{\partial s \partial u}u' + 2\frac{\partial^2 K}{\partial s \partial v}v' + 2\frac{\partial^2 K}{\partial u \partial v}u'v' \\
&+ \frac{\partial^2 K}{\partial u^2}(u')^2 + \frac{\partial^2 K}{\partial v^2}(v')^2 + \frac{\partial K}{\partial u}u'' + \frac{\partial K}{\partial v}v''
\end{aligned}$$

i.e.

$$\begin{aligned}
\left[K(t_k,s,x_n(s),x_n(g(s)))\right]_s'' &= \frac{\partial^2 K}{\partial s^2} + 2\frac{\partial^2 K}{\partial s \partial u} x_n'(s) + 2\frac{\partial^2 K}{\partial s \partial v} x_n'(g(s))g'(s) \\
&+ 2\frac{\partial^2 K}{\partial u \partial v} x_n'(s) x_n'(g(s))g'(s) + \frac{\partial^2 K}{\partial u^2}(x_n'(s))^2 \\
&+ \frac{\partial^2 K}{\partial v^2}\left[x_n'(g(s))g'(s)\right]^2 + \frac{\partial K}{\partial u} x_n''(s) \qquad (3.6)\\
&+ \frac{\partial K}{\partial v}\left(x_n''(g(s))(g'(s))^2 + x_n'(g(s))g''(s)\right)
\end{aligned}$$

For any $t \in [a,b]$,

$$\begin{aligned}
x_n(t) &= \left(1-\frac{1}{n}\right) x_{n-1}(t) \\
&+ \frac{1}{n}\left(\int_a^b K\left(t,s,x_{n-1}(s),x_{n-1}(g(s)\right)\, ds + f(t))\right), \\
x_n'(t) &= \left(1-\frac{1}{n}\right) x_{n-1}'(t) \\
&+ \frac{1}{n}\left(\int_a^b \frac{\partial K}{\partial t}\left(t,s,x_{n-1}(s),x_{n-1}(g(s)\right)\, ds + f'(t))\right), \\
x_n''(t) &= \left(1-\frac{1}{n}\right) x_{n-1}''(t) \\
&+ \frac{1}{n}\left(\int_a^b \frac{\partial^2 K}{\partial t^2}\left(t,s,x_{n-1}(s),x_{n-1}(g(s)\right)\, ds + f''(t))\right).
\end{aligned}$$

It is clear from our work so far, that if the functions K, g and f are C^2 functions with bounded second order derivatives, then for $\tilde{R}^{(n,m)}$ defined in (3.5), we have

$$\tilde{R}^{(n,m)} \leq \frac{(b-a)^3}{12m^2} M_0, \qquad (3.7)$$

where M_0 depends on a, b, l_1, l_2 and the functions K, g and f, but *not* on n or m.

We can now give an error estimate for our approximation.

Theorem 3.1. *Assume the conditions of Theorem* 2.1 *hold. Further, assume that* K, g *and* f *are* C^2 *functions with bounded second order derivatives. Then for the true solution* x^* *of* (1.3) *and the approximations* $\tilde{x}_n$ *given by* (3.3) – (3.4), *the error estimate*

$$||x^* - \tilde{x}_n|| \leq \frac{e^{1-q}}{1-q} ||x_0 - Fx_0||\, e^{-(1-q)y_n} + \frac{(b-a)^3}{12m^2} M_0 \qquad (3.8)$$

holds for every $n \in \mathbb{N}$, *where* $y_0 = 0$, $y_n = \sum_{i=0}^{n-1} \frac{1}{i+1}$, *for* $n \geq 1$, $||x^* - \tilde{x}_n||$ *denotes* $\max_{t_k \in [a,b]} \left\{\left|x^*(t_k) - \tilde{x}_n(t_k)\right|, \left|x^*(g(t_k)) - \tilde{x}_n(g(t_k))\right|\right\}$ *and* M_0 *is described in* (3.7).

Proof. Since

$$\begin{aligned}\left|x^*(t_k)-\tilde{x}_n(t_k)\right| &\le \left|x^*(t_k)-x_n(t_k)\right|+\left|x_n(t_k)-\tilde{x}_n(t_k)\right|,\\ \left|x^*(g(t_k))-\tilde{x}_n(g(t_k))\right| &\le \left|x^*(g(t_k))-x_n(g(t_k))\right|+\left|x_n(g(t_k))-\tilde{x}_n(g(t_k))\right|,\end{aligned}$$

the assertion follows from (3.7) and Theorem 2.1. □

4. Numerical examples

Example 4.1. Consider the nonlinear Fredholm integral equation

$$\begin{aligned}x(t) &= \frac{3}{64}\int_0^\pi x(s)\left(\frac{1}{2}\cos t\cos\frac{s}{2}+x\left(\frac{s}{2}\right)\sin t\right)ds\\ &+ \frac{1}{64}(31\sin t-\cos t), \qquad (4.1)\end{aligned}$$

for $t\in[0,\pi]$.

The exact solution of (4.1) is $x^*(t)=\frac{1}{2}\sin t$.
Here,

$$\begin{aligned}K(t,s,u,v) &= \frac{3}{64}u\left(\frac{1}{2}\cos t\cos\frac{s}{2}+v\sin t\right),\\ g(t) &= \frac{t}{2},\\ f(t) &= \frac{1}{64}(31\sin t-\cos t).\end{aligned}$$

Let $R=1$. We have $R_1=-\frac{1}{64}$ and $R_2=\frac{\sqrt{962}}{64}$.
Then, on $[a,b]\times[a,b]\times[R_1-R,R_2+R]^2=[0,\pi]\times[0,\pi]\times[-65/64,1+\sqrt{962}/64]^2$, we have

$$M_K\le\frac{3}{64}(R_2+R)\left(\frac{1}{2}+R_2+R\right)$$

and, so,

$$M_K(b-a)\le 0.434<1=R.$$

Also, on $[0,\pi]\times[0,\pi]\times[-65/64,1+\sqrt{962}/64]^2$,

$$\frac{\partial K}{\partial u}=\frac{3}{64}\left(\frac{1}{2}\cos t\cos\frac{s}{2}+v\sin t\right),$$

so $l_1\le\frac{3}{64}\left(\frac{1}{2}+R_2+R\right)$ and

$$\frac{\partial K}{\partial v}=\frac{3}{64}u\sin t,$$

so $l_2\le\frac{3}{64}(R_2+R)$. Hence,

$$q=(b-a)(l_1+l_2)\approx 0.551<1.$$

Thus, conditions (2.2), (2.3) and (2.4) hold, which means the hypotheses of Theorem 3.1 are satisfied. Also, for $R = 1$, we have that $x^* \in B_R$.

We consider the trapezoidal rule with $m = 12$, $m = 16$ and $m = 24$, with the corresponding nodes $t_k = \frac{\pi}{m}k, k = \overline{0, m}$. Table 1 contains the errors

$$||x^* - \tilde{x}_n|| = \max_{t_k \in [a,b]} \{|x^*(t_k) - \tilde{x}_n(t_k)|, |x^*(g(t_k)) - \tilde{x}_n(g(t_k))|\},$$

with initial approximation $x_0(t) = f(t) = \frac{1}{64}(31 \sin t - \cos t)$.

Table 1. Error estimates $||x^* - \tilde{x}_n||$ for Example 4.1

n \ m	12	16	24
1	$1.942720e - 00$	$1.354476e - 00$	$4.983236e - 01$
2	$8.223781e - 01$	$4.405026e - 01$	$6.338715e - 02$
3	$3.015578e - 01$	$9.174332e - 02$	$7.990126e - 03$
4	$7.997435e - 02$	$1.989751e - 02$	$8.986247e - 04$
5	$1.963239e - 02$	$7.428768e - 03$	$1.422981e - 04$
10	$9.795423e - 04$	$8.012446e - 05$	$3.116458e - 06$

Example 4.2. Next, consider the nonlinear two-point boundary-value problem

$$x''(t) - e^{x(t)} = 0, \quad t \in [0, 1]; \quad x(0) = x(1) = 0, \tag{4.2}$$

which is used in magnetohydrodynamics (see [3]). The unique solution of (4.2) is given by

$$x^*(t) = -\ln(2) + 2\ln\left(\frac{c}{\cos(c(t - 1/2)/2)}\right),$$

where c is the only solution of $c/\cos(c/4) = \sqrt{2}$.

Problem (4.2) can be reformulated as the Fredholm integral equation

$$x(t) = \int_0^1 k(t, s)e^{x(s)}\, ds, \quad t \in [0, 1], \tag{4.3}$$

where the kernel

$$k(t, s) = -\min\{t, s\}(1 - \max\{t, s\}) = \begin{cases} -s(1 - t), & s \le t, \\ -t(1 - s), & s > t \end{cases} \tag{4.4}$$

is Green's function for the homogeneous problem

$$x''(t) = 0, \quad t \in [0, 1]; \quad x(0) = x(1) = 0.$$

We have

$$\begin{aligned} K(t, s, u, v) &= k(t, s)e^u, \\ g(t) &= f(t) \equiv 0. \end{aligned}$$

Again, we take $R = 1$. In this case, $R_1 = R_2 = 0$ and $\max|K| = \max\left|\frac{\partial K}{\partial u}\right| = \frac{1}{4}\cdot e$, for $(t,s,u,v) \in [0,1]^2 \times [-1,1]^2$. Thus,

$$\begin{aligned} q &= (b-a)(l_1+l_2) = l_1 = \frac{1}{4}\cdot e < 1, \\ M_K(b-a) &= M_K = \frac{1}{4}\cdot e < 1 = R, \end{aligned}$$

so the hypotheses of Theorem 3.1 are satisfied.

As before, we use the trapezoidal rule with $m = 12$, $m = 16$ and $m = 24$ and nodes $t_k = \frac{1}{m}k, k = \overline{0,m}$. The errors $||x^* - \tilde{x}_n|| = \max\limits_{k=\overline{0,m}} |\tilde{x}_n(t_k) - x^*(t_k)|$ are given in Table 2, with initial approximation $x_0 \equiv 0$.

Table 2. Error estimates $||x^* - \tilde{x}_n||$ for Example 4.2

n \ m	12	16	24
1	$1.080564e-02$	$1.080564e-02$	$1.080564e-02$
2	$1.094821e-03$	$1.066866e-03$	$1.023419e-03$
3	$4.890231e-04$	$4.178235e-04$	$6.098823e-05$
4	$5.712236e-05$	$5.014429e-05$	$2.082737e-05$
5	$2.034852e-05$	$9.640748e-06$	$6.161384e-06$
10	$2.026459e-07$	$1.678721e-07$	$8.890239e-08$

5. Conclusions and future work

We have developed a numerical iterative method for approximating solutions of Fredholm integral equations of the second kind, with deviating arguments, using a combination of successive approximations (Mann iteration) for fixed points of integral operators and quadrature formulas (the trapezoidal rule). Compared to other recent numerical methods for solving these integral equations – such as collocation, Galerkin, Nyström or other projection methods, wavelets-based approximations methods, Adomian decomposition, etc – the present method has two major advantages, the relative simplicity in proving the convergence of the approximate solutions to the exact solution (using fixed point theory) and the low cost of implementation (as it uses a well known quadrature formula, which is already implemented in most mathematical software). Yet, as the examples show, it gives a good approximation even with a relatively small number of iterations and of quadrature nodes. In the examples chosen, the numerical results are quite good and the errors decrease rapidly as n (the number of iterations) and/or m (the number of quadrature nodes) increase.

As for future work, similar ideas to the ones described in this paper can be applied to other types of integral equations, integral equations with more complicated kernels, or kernels satisfying other conditions than the ones considered in this work. Also, other fixed point successive approximations can be considered, which, under certain conditions, may converge faster. Last, but not least, more accurate numerical integration schemes can be employed in order to increase the speed of convergence of the method.

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Some variants of contraction principle, generalizations and applications

Ioan A. Rus

Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. In this paper we present the following variant of contraction principle: *Saturated principle of contraction.* Let (X, d) be a complete metric space and $f : X \to X$ be an l-contraction. Then we have:

(*i*) $F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}^*$.

(*ii*) $f^n(x) \to x^*$ as $n \to \infty$, $\forall\ x \in X$.

(*iii*) $d(x, x^*) \leq \psi(d(x, f(x)))$, $\forall\ x \in X$ where $\psi(t) = \frac{t}{1-l}$, $t \geq 0$.

(*iv*) $y_n \in X$, $d(y_n, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$.

(*v*) $y_n \in X$, $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty \Rightarrow y_n \to x^*$ as $n \to \infty$.

(*vi*) If $Y \subset X$ is a nonempty bounded and closed subset with $f(Y) \subset Y$, then $x^* \in Y$ and $\bigcap_{n\in\mathbb{N}} f^n(Y) = \{x^*\}$.

The basic problem is: which other metric conditions imply the conclusions of this variant ? We give some answers for this problem. Some applications and open problems are also presented.

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Keywords: Metric space, generalized metric space, contraction, quasicontraction, generalized contraction, Bessaga operator, Janos operator, Picard operator, ψ-Picard operator, well-posedness of fixed point problem, Ostrowski property, data dependence, Ulam stability, iterative algorithm, iterative algorithm stability, stability under operator perturbation, functional differential equation, functional integral equation.

1. Introduction and preliminaries

The number of papers on fixed point theory in which appear metric conditions is a large one (see: [20], [49], [32], [48], [4], [9], [16], [17], [24], [25], [26], [34], [36], [52], [64], [68],. . .). In these papers two fixed point theorems appear under the same name, *contraction principle*:

(1) Let (X,d) be a complete metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point (i.e., $F_f = \{x^*\}$).
(2) Let (X,d) be a complete metric space and $f : X \to X$ be an l-contraction. Then we have:
   (i) $F_f = \{x^*\}$.
   (ii) $f^n(x) \to x^*$ as $n \to \infty$, $\forall\ x \in X$.

By Contraction Principle (CP) we understand this (2) variant.

On the other hand, in many papers appear some properties of fixed point equations, where the corresponding operator is a contraction (see [56], [65], [40], [4], [6], [32], [25], [47], [49], [50], [51], [52], [53], [54], [56], [57], [64], [65], [72], [73],...). So, in this paper we present a new variant of contraction principle, a variant with generous conclusions. This variant is the following:

Theorem 1.1 (Saturated principle of contraction (SPC)). *Let (X,d) be a complete metric space and $f : X \to X$ be an l-contraction. Then we have:*

(i) *There exists $x^* \in X$ such that,*

$$F_{f^n} = \{x^*\},\ \forall\ n \in \mathbb{N}.$$

(ii) *For all $x \in X$, $f^n(x) \to x^*$ as $n \to \infty$.*
(iii) $d(x,x^*) \leq \psi(d(x,f(x)))$, $\forall\ x \in X$, *where* $\psi(t) = \frac{t}{1-l}$, $t \geq 0$.
(iv) *If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X such that*

$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty,$$

then, $y_n \to x^*$ as $n \to \infty$.
(v) *If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X such that*

$$d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty,$$

then, $y_n \to x^*$ as $n \to \infty$.
(vi) *If $Y \subset X$ is a closed subset such that $f(Y) \subset Y$, then $x^* \in Y$. Moreover, if in addition Y is bounded, then*

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = \{x^*\}.$$

It is clear that, all conclusions in this theorem are well known. For a better understanding of this variant of contraction principle, some remarks and commentaries are necessary.

Conclusion (i) is a set-theoretical one. If X is a nonempty set and $f : X \to X$ is an operator such that, $F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$, then by definition we call f a Bessaga operator.

Conclusion (ii) is a topological one. All Picard iterations converge to the unique fixed point of the operator. If $(X,\to)$ is an L-space and $f : X \to X$ is an operator such that we have (i) and (ii), then by definition f is a Picard operator.

Conclusion (iii) is a metrical one and is very important in the theory of fixed point equations. We obtain from this estimate, for example, a data dependence of the fixed point under operator perturbation.

If in a metric space an operator f satisfies (i), (ii) and (iii), then by definition the operator f is a ψ-Picard operator and the estimation in (iii) is called retraction-displacement estimation. In this definition, ψ is a function, $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, increasing and continuous in 0 with $\psi(0) = 0$.

If in a metric space (X, d) an operator $f : X \to X$ satisfies (i) and (iv) then by definition the fixed point problem for f is well posed. We remark that we can present this notion in a linear L-space. Let $(X, +, \mathbb{R}, \to)$ be a linear L-space and $f : X \to X$ be an operator. By definition the fixed point problem for f is well posed if:

- (i) $F_f = \{x^*\}$.
- (ii) If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $y_n - f(y_n) \to 0$ as $n \to \infty$, then $y_n \to x^*$ as $n \to \infty$.

If in a metric space (X, d) an operator satisfies (i) and (v), then by definition the operator f has the Ostrowski property. We remark that we can present this notion in a linear L-space.

Let $(X, +, \mathbb{R}, \to)$ be a linear L-space and $f : X \to X$ be an operator. By definition the operator f has the Ostrowski property if:

- (i) $F_f = \{x^*\}$.
- (ii) If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $y_{n+1} - f(y_n) \to 0$ as $n \to \infty$, then $y_n \to x^*$ as $n \to \infty$.

First part of conclusion (vi) is useful for the localization of the fixed point. Second part is a set-theoretical one, under metrical conditions. If X is a nonempty set and $f : X \to X$ is an operator such that

$$\bigcap_{n\in\mathbb{N}} f^n(X) = \{x^*\},$$

then by definition f is a Janos operator. On the other hand, from (vi) we have the following property of a contraction:

If (X, d) is a complete metric space and $f : X \to X$ is a contraction with $F_f = \{x^*\}$ and $\{y_n\}_{n\in\mathbb{N}}$ is a bounded sequence in X, then $f^n(y_n) \to x^*$ as $n \to \infty$.

It is well known that there is an extensive bibliography of the generalized contractions (see Ortega and Rheinboldt [32], Istrăţescu [20], Rhoades [48], Krasnoselskii and Zabrejko [26], Kirk and Sims [25], Granas and Dugundji [17], Goebel [16], Berinde [4], Rus [49], Rus [52], Rus, Petruşel and Petruşel [64], Rus and Şerban [65], Petruşel, Rus and Şerban [41], Rus and Şerban [65], Kirk and Shahzad [24],...). The problem is which metrical conditions which appear in the metrical fixed point theorems imply conclusions in the SPC ? We shall consider the problem in this paper. Some applications are given and open problems are presented.

Throughout this paper the notations and terminologies in [56], [65] and [40] are used. Moreover we consider these references as starting papers for our study.

The structure of the paper is the following:

2. Some variants of SPC
3. Examples of relevant metrical conditions
4. The case of generalized metric spaces
5. Applications

6. Other research directions

2. Some variants of SPC

We start our considerations with the following useful variant.

Theorem 2.1 (SPC with respect to a strongly equivalent metric). *Let X be a nonempty set, d and ρ be two metrics on X and $f : X \to X$ be an operator. We suppose that:*

(a) *(X, ρ) is a complete metric space.*
(b) *There exist $c_1, c_2 > 0$ such that*

$$c_1 d(x, y) \leq \rho(x, y) \leq c_2 d(x, y),\ \forall\ x, y \in X.$$

(c) *f is an l-contraction with respect to the metric ρ.*

Then we have:

(i) *$F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}^*$.*
(ii) *$f^n(x) \overset{d}{\to} x^*$ as $n \to \infty$.*
(iii) *$d(x, x^*) \leq \psi(d(x, f(x)))$, $\forall\ x \in X$, where*

$$\psi(t) = \frac{c_2 t}{c_1(1-l)},\ t \geq 0.$$

(iv) *The fixed point problem for f is well posed with respect to the metric d.*
(v) *The operator f has the Ostrowski property with respect to the metric d.*
(vi) *If $Y \subset X$ is a bounded and close subset in (X, d) with $f(Y) \subset Y$, then $x^* \in Y$ and*

$$\bigcap_{n \in \mathbb{N}} f^n(Y) = \{x^*\}.$$

Proof. The proof follows from SPC in (X, ρ) and the condition (b) of strongly equivalence of the metrics d and ρ (see [40]). □

An other variant is the following.

Theorem 2.2 (Saturated Principle of Quasicontraction (SPQC)). *Let (X, d) be a metric space and $f : X \to X$ be an operator. We suppose that there exists a fixed point x^* of f and $0 < l < 1$ such that:*

$$d(f(x), x^*) \leq l d(x, x^*),\ \forall\ x \in X.$$

Then, we have (i)-(vi) in Theorem 1.1.

Proof. For (i)-(v) the proofs are similar with the proofs in Theorem 1.1.

(vi) Let $x \in Y$. Then $f^n(x) \in Y$, $\forall\ n \in \mathbb{N}$. But, $f^n(x) \to x^*$ as $n \to \infty$. Since Y is closed, it follows that $x^* \in Y$. In fact, we have (i)-(vi) in Theorem 1.1, with respect to Y. Indeed, for the second part of (vi), we have $\delta(f(Y), \{x^*\}) \leq \delta(Y, \{x^*\})$, where δ is the diameter functional with respect to d. Moreover, $\delta(f^n(Y), \{x^*\}) \leq l^n \delta(Y, \{x^*\}) \to 0$ as $n \to \infty$. So, $\bigcap_{n \in \mathbb{N}} f^n(Y) = \{x^*\}$. □

We also have the following result.

Theorem 2.3 (SPQC with respect to a strongly equivalent metric). *Let X be a nonempty set, d and ρ be two metrics on X and $f : X \to X$ be an operator. We suppose that:*

(a) *There exist $c_1, c_2 > 0$ such that*

$$c_1 d(x, y) \leq \rho(x, y) \leq c_2 d(x, y),\ \forall\ x, y \in X.$$

(b) *There exists a fixed point x^* of f and $0 < l < 1$ such that*

$$\rho(f(x), x^*) \leq l\rho(x, x^*),\ \forall\ x \in X.$$

Then we have (i)-(vi) in Theorem 2.1

Proof. The proof follows from Theorem 2.2 in (X, ρ) and condition (a). □

Now, we finish this section with the following definition.

Definition 2.4. *Let (X, d) be a complete metric space and $f : X \to X$ be an operator. We call relevant a metric condition on f which implies the uniqueness of fixed point if it implies also, conclusions such as in SPC.*

3. Examples of relevant metrical conditions

We start with the following remark.

Lemma 3.1. *Let (X, d) be a complete metric space and $f : X \to X$ be an operator. If a metrical condition on f implies the conclusions in CP and if in addition f is an l-quasicontraction, then we have for f the conclusions in SPC with*

(iii) $d(x, x^*) \leq \frac{1}{1-l} d(x, f(x)),\ \forall\ x \in X.$

Proof. The proof follows from SPQC. □

From this Lemma the following question rises.

Problem 3.2. Which metric conditions on f imply that f is a quasicontraction ?

From Lemma 3.1, we have, as examples, the following results.

Theorem 3.3. *Let (X, d) be a complete metric space and $f : X \to X$ be such that there exists $0 < l < \frac{1}{2}$, with*

$$d(f(x), f(y)) \leq l[d(x, f(x)) + d(y, f(y))],\ \forall\ x, y \in X.$$

Then we have the conclusions in SPC, with

(iii) $d(x, x^*) \leq \frac{1}{1-2l} d(x, f(x)),\ \forall\ x \in X.$

Proof. (i)-(ii). This is Kannan's theorem. Kannan's theorem is not a generalization of CP, but implies conclusions in CP.

(iii)-(vi). From Kannan's metrical condition it follows that f is a $2l$-quasicontraction. From SPQC we have (iii)-(vi). □

Theorem 3.4. (see [4], [52]) *Let (X, d) be a complete metric space and $f : X \to X$ be an operator. We suppose that there exist $a, b, c \in \mathbb{R}_+$, $a < 1$, b and $c < \frac{1}{2}$, such that for each $x, y \in X$ at least one of the following conditions is true:*

(1) $d(f(x), f(y)) \leq ad(x, y)$,
(2) $d(f(x), f(y)) \leq b[d(x, f(x)) + d(y, f(y))]$,
(3) $d(f(x), f(y)) \leq c[d(x, f(y)) + d(y, f(x))]$.

Then we have the conclusions in SPC with

(iii) $d(x, x^*) \leq \frac{1}{1-l}d(x, f(x))$, $\forall\, x \in X$,

where $l = \max\{a, 2b, 2c\}$.

Proof. (i)-(ii). This is Zamfirescu's theorem. It is a generalization of CP.

(iii)-(vi). From Zamfirescu's metrical conditions it follows that f is an l-quasicontraction with $l = \max\{a, 2b, 2c\}$. The proof follows from SPQC. □

Theorem 3.5. (see [35]) *Let* (X, d) *be a complete metric space,* m *be a positive integer,* $A_1,\ldots, A_m \in P_{cl}(X)$, $Y := \bigcup_{i=1}^{m} A_i$, *and* $f : Y \to Y$ *be an operator. We suppose that:*

(a) $\bigcup_{i=1}^{m} A_i$ *is a cyclic representation of* Y *with respect to* f.
(b) f *is a cyclic* l*-contraction.*

Then we have the conclusions in SPC with

(iii) $d(x, x^*) \leq \frac{1}{1-l}d(x, f(x))$, $\forall\, x \in Y$.

Proof. (i)-(ii). This is Kirk-Srinivasan-Veeramany's theorem. It is a generalization of CP.

(iii)-(vi). From the definition of cyclic representation it follows that $x^* \in \bigcap_{i=1}^{m} A_i$. From the definition of cyclic l-contraction it follows that $f : Y \to Y$ is an l-quasicontraction. The proof follows from SPQC. □

Theorem 3.6. (see [71]) *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be an operator. Let* $\theta : [0, 1[\to]\frac{1}{2}, 1]$ *be defined by*

$$\theta(t) := \begin{cases} 1 & \text{if } 0 \leq t \leq (\sqrt{5} - 1)/2, \\ (1 - t)t^{-2} & \text{if } (\sqrt{5} - 1)/2 \leq t \leq 2^{-\frac{1}{2}}, \\ (1 + t)^{-1} & \text{if } 2^{-\frac{1}{2}} \leq t < 1. \end{cases}$$

We suppose that there exists $l \in [0, 1[$ *such that*

$$x, y \in X,\ \theta(l)d(x, f(x)) \leq d(x, y) \Rightarrow d(f(x), f(y)) \leq ld(x, y).$$

Then, we have the conclusions in SPC with

(iii) $d(x, x^*) \leq \frac{1}{1-l}d(x, f(x))$, $\forall\, x \in X$.

Proof. (i)-(ii). This is Suzuki's theorem. It is a generalization of CP.

(iii)-(vi). From the Suzuki's metrical condition we have that f is an l-quasicontraction. The proof follows from SPQC. □

Now, we give an example in a set with two metrics.

Theorem 3.7. (see [64], p. 40; see also [15], [43], [51]) *Let X be a nonempty set, d and ρ be two metrics on X and $f : X \to X$ be an operator. We suppose that:*

(a) $d(x, y) \leq \rho(x, y)$, $\forall\ x, y \in X$.
(b) (X, d) *is a complete metric space.*
(c) f *is an l-contraction with respect to* ρ.
(d) f *is continuous with respect to* d.

Then we have:

(i) $F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}^*$.
(ii) $f^n(x) \xrightarrow{d} x^*$ *as* $n \to \infty$, *and* $f^n(x) \xrightarrow{\rho} x^*$ *as* $n \to \infty$.
(iii) $\rho(x, x^*) \leq \frac{1}{1-l}\rho(x, f(x))$, $\forall\ x \in X$.
(iv) *The fixed point problem for f is well posed with respect to* ρ.
(v) *The operator f has the Ostrowski property with respect to the metric* ρ.
(vi) *If $Y \subset X$ is a bounded and closed subset in (X, ρ) with $f(Y) \subset Y$, then $x^* \in Y$ and*

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = \{x^*\}.$$

Proof. (*i*)-(*ii*). This is Maia's fixed point theorem. It is a generalization of CP.

(*iii*)-(*vi*). We remark that f is an l-quasicontraction. The proof follows from SPQC. □

In what follows we shall present an example from asymptotical fixed point theorems.

There are many asymptotic metrical fixed point results. We mention the contributions made by R. Caccioppoli (1930), J. Weisinger (1952), A.N. Kolmogorov and S.V. Fomin (1957), I.I. Kolodner (1964), S.C. Chy and J.B. Diaz (1965), V.W. Bryant (1968), V.M. Sehgal (1969), L.F. Guseman (1970), V.I. Istrăţescu (1973), W. Walter (1970, 1981), F. Browder (1979), I.A. Rus (1980), J.D. Stein (1998 (2000)), J. Jachymski and J.D. Stein (1999), K. Goebel (2002), W.A. Kirk (2003), S. Andras (2003), A.D. Arvanitakis (2003), A.S. Mureşan (2014) (see [13], [20], [72], [50], [73], [66], [70](this paper of Stein has no references on asymptotic conditions!), [23], [1], [2], [16],...).

Our example in this direction is the following.

Theorem 3.8. (see [72]) *Let (X, d) be a complete metric space and $f : X \to X$ be such that there exists $k \in \mathbb{N}^*$ for which f^k is an l-contraction. Then we have:*

(i) $F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}$.
(ii) $f^n(x) \to x^*$ *as* $n \to \infty$, $\forall\ x \in X$.

If in addition, f^s is l_s-Lipschitz, $s \in \mathbb{N}^*$, then:

(iii) $d(x, x^*) \leq \frac{c_2}{1-l^{\frac{1}{k}}} d(x, f(x))$, $\forall\ x \in X$,
where $c_2 = 1 + l_1 l^{-\frac{1}{k}} + \ldots + l_{k-1} l^{\frac{1-k}{k}}$.
(iv) *The fixed point problem for f is well posed.*
(v) *The operator f has the Ostrowski property.*

(*vi*) *If* $Y \subset X$ *is a bounded and closed subset with* $f(Y) \subset Y$*, then* $x^* \in Y$ *and*

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = \{x^*\}.$$

Proof. (*i*)-(*ii*). It follows from the following remark:

If (X, τ) is a Hausdorff topological space and $f : X \to X$ is an operator, then the following statements are equivalent:

(1) f is a Picard operator
(2) There exists $k \in \mathbb{N}^*$ such that f^k is a Picard operator.

(*iii*)-(*vi*). The functional, $\rho : X \times X \to \mathbb{R}_+$, defined by

$$\rho(x, y) := d(x, y) + l^{-\frac{1}{k}} d(f(x), f(y)) + \ldots + l^{\frac{1-k}{k}} d(f^{k-1}(x), f^{k-1}(y))$$

is a metric on X which is strongly equivalent with the metric d (see, for example [72]), with $c_1 = 1$ and $c_2 = 1 + l_1 l^{-\frac{1}{k}} + \ldots + l_{k-1} l^{\frac{1-k}{k}}$. Moreover the operator f is an $l^{\frac{1}{k}}$-contraction with respect to ρ. The proof follows from Theorem 2.1. □

In order to present the next example we need some preliminaries.

Let $(X, +, \mathbb{R}, \|\cdot\|, K)$ be an ordered Banach space. By definition the cone K is normal if there exists $c_N > 0$ such that,

$$x, y \in X,\ 0 \le x \le y \Rightarrow \|x\| \le c_N \|y\|.$$

The cone K is reproducing if, $X = K - K$. So, each element $x \in X$ admits a presentation, $x = u - v$, where $u, v \in K$. Moreover each element $x \in X$ admits a presentation, $x = u - v$ such that, $\|u\|, \|v\| \le c_r \|x\|$, where c_r does not depend of x.

In an ordered Banach space with reproducing cone, the functional, $\|\cdot\|_r : X \to \mathbb{R}_+$, defined by, $\|x\|_r := \inf\{\|y\| \mid -y \le x \le y\}$, is a norm on X. For this norm we have (see [26], p. 320),

$$(2c_N + 1)^{-1} \|x\| \le \|x\|_r \le 2c_g \|x\|,\ \forall\ x \in X.$$

Our example in an ordered Banach space is the following.

Theorem 3.9. (see [26]) *Let* X *be an ordered Banach space with a reproducing and normal cone* K *and* $g : X \to X$ *be a positive linear operator with,* $\|g\| < 1$*. If an operator* $f : X \to X$ *satisfies the condition*

$$-g(x - y) \le f(x) - f(y) \le g(x - y),\ \forall\ x, y \in X,\ x \ge y$$

then f *satisfies the conclusions in SPC, with*

(*iii*) $\|x - x^*\| \le \frac{2c_g(2c_N+1)}{1-\|g\|} \|x - f(x)\|,\ \forall\ x \in X.$

Proof. (*i*)-(*ii*). It is the Krasnoselskii's theorem. From the Krasnoselskii's proof it follows that the operator f is a $\|g\|$-contraction with respect to the strongly equivalent norm, $\|\cdot\|_r$. So, the conclusions (*iii*)-(*vi*), follows from Theorem 2.1. □

4. The case of generalized metric spaces

The universe of generalized metric spaces is a very large one (see, for example, [5], [15], [22], [24], [43], [44], [46], [55], [64],...). In what follows we shall present only some examples for our problem in some generalized metric spaces.

Theorem 4.1 (Saturated principle of contraction in a partial metric space (see [55]**).** *Let (X,p) be a complete partial metric space and $f : X \to X$ be an l-contraction. Then we have:*

(i) $F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}^*$.
(ii) $p(f^n(x), x^*) \to 0$ *as* $n \to \infty$, $\forall\ x \in X$.
(iii) $p(x, x^*) \le \frac{1}{1-l} p(x, f(x))$, $\forall\ x \in X$.
(iv) $\{y_n\}_{n\in\mathbb{N}} \subset X$, $p(y_n, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow p(y_n, x^*) \to 0$ *as* $n \to \infty$.
(v) $\{y_n\}_{n\in\mathbb{N}} \subset X$, $p(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow p(y_n, x^*) \to 0$ *as* $n \to \infty$.
(vi) *Let* $Y \subset X$ *be a nonempty subset such that* $f(Y) \subset Y$, $x^* \in Y$ *and* $\sup\{p(x,y) \mid x,y \in Y\} < +\infty$. *Then,*

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = \{x^*\}.$$

Proof. (i)-(ii). This is Matthews' theorem.
(iii)-(v). See [55]. See also [64], pp. 53-58.
(vi) It is clear that $x^* \in \bigcap_{n\in\mathbb{N}} f^n(Y)$. Let $u \in \bigcap_{n\in\mathbb{N}} f^n(Y)$. Then there exists $x_n \in Y$ such that $u = f^n(x_n)$.
We have, $p(u,x^*) = p(f^n(x_n), x^*) = p(f^n(x_n), f^n(x^*)) \le l^n p(x_n, x^*) \le l^n \delta_p(Y) \to 0$ as $n \to \infty$. So, $u = x^*$. □

In a similar way we have

Theorem 4.2 (Saturated principle of quasicontraction in a partial metric space). *Let (X,p) be a partial metric space and $f : X \to X$ be an operator. We suppose that:*

(a) *There exists an $x^* \in X$, fixed point of f.*
(b) *f is an l-quasicontraction.*

Then we have the conclusions in Theorem 4.1.

There are examples of saturate principle of generalized contractions in $\mathbb{R}^m_+$-metric spaces. For the example corresponding to Perov's fixed point principle, see [64], pp. 82-85.

Now we give an example in a gauge space. Let (X,d) be a generalized metric space with $d(x,y) \in s(\mathbb{R}_+)$. So, $d(x,y) = \{d_k(x,y)\}_{k\in\mathbb{N}^*}$ where d_k is a pseudometric, for all $k \in \mathbb{N}^*$ and for each $(x,y) \in X \times X$ there exists $k \in \mathbb{N}^*$ such that $d_k(x,y) \neq 0$. Let $l = (l_1, \dots, l_n, \dots)$ be such that $0 \le l_k < 1$, $\forall\ k \in \mathbb{N}^*$. By definition, an operator $f : X \to X$ is an l-contraction if

$$d_k(f(x), f(y)) \le l_k d_k(x,y),\ \forall\ x,y \in X,\ \forall\ k \in \mathbb{N}^*.$$

For the basic notions in a generalized metric space with $d(x,y) \in s(\mathbb{R}_+)$, see [64], [21], [56],...

We have

Theorem 4.3 (Saturated principle of contraction in a $s(\mathbb{R}_+)$-metric space). *Let* (X, d), $d(x, y) \in s(\mathbb{R}_+)$, *be a complete metric space and* $f : X \to X$ *be an* l*-contraction. Then we have:*

(i) $F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}^*$.
(ii) $f^n(x) \to x^*$ *as* $n \to \infty$, $\forall\ x \in X$.
(iii) $d_k(x, x^*) \leq \frac{1}{1-l_k} d(x, f(x))$, $\forall\ x \in X$.
(iv) $\{y_n\}_{n\in\mathbb{N}} \subset X$, $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$.
(v) $\{y_n\}_{n\in\mathbb{N}} \subset X$, $d(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty \Rightarrow y_n \to x^*$ *as* $n \to \infty$.
(vi) *Let* $Y \subset X$ *be a bounded and closed subset with* $f(Y) \subset Y$. *Then* $x^* \in Y$ *and*

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = \{x^*\}.$$

Proof. (i)-(ii). This is the Cain and Nashed's fixed point theorem. The Cain-Nashed's theorem is a generalization of CP.

(iii). From the definition of l-contraction we have that f is an l_k-contraction with respect to the pseudometric d_k. Now the proof is standard.

(iv). $d(y_n, f(y_n)) \to 0$ as $n \to \infty$ implies that $d_k(y_n, f(y_n)) \to 0$ as $n \to \infty$. From (iii) we have (iv).

(v). $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty \Rightarrow d_k(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty$. But,

$$d_k(y_{n+1}, x^*) \leq d_k(y_{n+1}, f(y_n)) + d_k(f(y_n), x^*) \leq$$
$$\leq d_k(y_{n+1}, f(y_n)) + l d(y_n, f(y_{n-1})) + \ldots + l^{n+1} d_k(y_0, x^*).$$

Now the proof follows from a Cauchy lemma.

(vi). Let $y \in Y$. Then $f^n(y) \in Y$, $\forall\ n \in \mathbb{N}$, and $f^n(y) \to x^*$ as $n \to \infty$. Since Y is closed it follows that $x^* \in Y$. It is clear that $x^* \in \bigcap_{n\in\mathbb{N}} f^n(Y)$. Let $u \in \bigcap_{n\in\mathbb{N}} f^n(Y)$. Then there exists $x_n \in Y$ such that $u = f^n(x_n)$. We have that

$$d_k(u, x^*) = d_k(f^n(x_n), x^*) \leq l_k^n \delta_{d_k}(Y) \to 0 \text{ as } n \to \infty,\ \forall\ k \in \mathbb{N}^*.$$

This implies, $u = x^*$. □

We also have

Theorem 4.4 (Saturated principle of quasicontraction in a $s(\mathbb{R}_+)$-metric space). *Let* (X, d), $d(x, y) \in s(\mathbb{R}_+)$, *be a generalized metric space and* $f : X \to X$ *be an operator. We suppose that:*

(a) *There exists an* $x^* \in X$, *a fixed point of* f.
(b) f *is an* l*-quasicontraction.*

Then we have the conclusions in Theorem 4.3.

From the above considerations the following questions rise:

Problem 4.5. Which metric conditions in a $s(\mathbb{R}_+)$-metric space (Colojoară, Gheorghiu,. . .) imply the conclusions in Theorem 4.3 ?

Problem 4.6. Let (X,d) be a generalized metric space with $d(x,y) \in s(\mathbb{R}_+)$. Let $f : X \to X$ be an operator. Let $M(\mathbb{R}_+)$ be the set of infinite matrices with elements in $\mathbb{R}_+$ and let I be the identity matrix in $M(\mathbb{R}_+)$. Our definition of l-contraction reads as follows

$$d(f(x), f(y)) \leq lId(x,y),\ \forall\ x,y \in X.$$

For a good definition for contractions in a such generalized metric space it is necessarily to put a more general matrix instead of lI (see [56]).

In the above setting, which are the contractions with the properties (i)-(vi) ?

5. Applications

5.1. More applications of SPC appear as applications of Picard operators. Let us mention abstract applications to: data dependence of fixed point under the operator perturbation ([49], [52], [64], [3], [4], [15], [12], [34], [56], [68]), Ulam stability of fixed point equations ([60], [65],...), abstract Gronwall lemmas ([57], [9], [15], [27], [53],...). For concrete applications to functional differential equations and to functional integral equations, see: [1], [3], [9], [14], [15], [19], [27], [30], [31], [33], [34], [39], [53], [58], [68], [74], [75],...

5.2. An other application is concerning iterated Picard operator systems. Let (X,d) be a complete metric space and $f_1,\dots,f_m : X \to X$ be some Picard operators. These operators generate the following operator on $P(X)$,

$$T_f : P(X) \to P(X),\ T_f(A) := f_1(A) \cup \ldots \cup f_m(A),\ \forall\ A \in P(X).$$

The problem is to study the properties of T_f in terms of properties of $f_1,\dots,f_m$. This problem is a particular case of the following Nadler problem:

Let (X,d) be a complete metric space and $f : X \to P(X)$ be a multivalued operator. Let $T_f : P(X) \to P(X)$ be the operator defined by, $T_f(A) = \bigcup_{a\in A} f(a)$. The problem is to study the properties of T_f in terms of properties of f.

For example it is well known the following result:

Theorem 5.1 (Nadler (1969), Hutchinson (1981)). *Let (X,d) be a complete metric space and $f_i : X \to X$ be an l-contraction, $i = \overline{1,m}$. Then the set-to-set operator, $T_f : P_{cp}(X) \to P_{cp}(X)$ is well defined and it is an l-contraction in $(P_{cp}(X), H_d)$. Here, H_d is the Pompeiu-Hausdorff metric corresponding to d.*

From the SPC we have:

Theorem 5.2. *Let T_f be as in Theorem 5.1. Then we have:*

(i) $F_{T_f} = \{A^*\}$.

(ii) $T_f^n(A) \overset{H_d}{\to} A^*$ *as* $n \to \infty$, $\forall\ A \in P_{cp}(X)$.

(iii) $H_d(A, A^*) \leq \frac{1}{1-l} H_d(A, T_f(A))$, $\forall\ A \in P_{cp}(X)$.

(iv) *If* $A_n \in P_{cp}(X)$, $n \in \mathbb{N}$ *are such that*

$$H_d(A_n, T_f(A_n)) \to 0 \text{ as } n \to \infty,$$

then, $A_n \overset{H_d}{\to} A^*$ *as* $n \to \infty$.

(v) *If* $A_n \in P_{cp}(X)$, $n \in \mathbb{N}$ *are such that*

$$H_d(A_{n+1}, T_f(A_n)) \to 0 \text{ as } n \to \infty,$$

then, $A_n \overset{H_d}{\to} A^*$ *as* $n \to \infty$.

(vi) *Let* $U \subset P_{cp}(X)$ *be a bounded and closed subset such that* $f(U) \subset U$. *Then,* $A^* \in U$ *and*

$$\bigcap_{n \in \mathbb{N}} T_f^n(U) = \{A^*\}.$$

5.3. The SPC has applications in the variational theory of differential equations. Let us consider the following example.

In [45] (see also [8]), Radu Precup presents the following interesting result:

Theorem 5.3. *Let* X *be a Hilbert space,* $N : X \to X$ *be a contraction with the unique fixed point* u^*. *If there exists a* C^1*-functional,* $E : X \to \mathbb{R}$, *bounded from below such that*

$$E'(u) = u - N(u), \text{ for all } u \in X,$$

then u^* *minimizes the functional, i.e.,*

$$E(u^*) = \inf_X E.$$

The Precup proof for this theorem can be read as follows.

As a consequence of Bishop-Phelps' theorem, there is a sequence (u_n) with

$$E(u_n) \to \inf_X E \text{ and } E'(u_n) \to 0.$$

Since, $E'(u_n) = u_n - N(u_n) \to 0$ and N is a contraction, from conclusion (v) in SPC we have that, $u_n \to u^*$.

From this proof the following remark follows:

Remark 5.4. In Theorem 5.3 we can put instead of the operator N, an operator for which the fixed point problem is well posed.

6. Other research directions

6.1. What does it mean Saturated principle of fiber contraction ?

References: [69], [68], [67], [1].

6.2. To extend the results in sec. 2 to the case of nonself operators.

References: [6], [12], [4], [64], [17],...

6.3. To extend the results in sec. 2 to the case of multivalued operators.

References: [37], [41], [64],...

6.4. To extend the results in sec. 2 to the case of nonself multivalued operators.

References: [42], [64],...

6.5. Let (X, d) be a complete metric space, $f : X \to X$ be an operator with $F_f = \{x^*\}$ and $f_n : X \to X$ be a sequence which converges in some sense to f. Consider the iterative algorithm

$$x_{n+1} = f_n(x_n).$$

In which conditions on f and f_n this algorithm is convergent to x^* ? What estimate we have for $d(x, x^*)$?

References: [32], [4], [7], [11], [34], [59].

6.6. To extend the results in this paper to the weakly Picard operators.

References: [62], [53], [63], [64], [65], [4],...

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On summation/integration methods for slowly convergent series

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Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. A survey on summation/integration methods for computation of slowly convergent series is presented. Methods are based on some transformations of series to integrals, with respect to certain nonclassical weight functions over $\mathbb{R}_+$, and an application of suitable quadratures of Gaussian type for numerical calculating of such integrals with a high accuracy. In particular, applications to some series with irrational terms are considered. Several numerical examples are included in order to illustrate the efficiency of these methods.

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1. Introduction

In this paper we give an account on the so-called *summation/integration* methods for fast summation of slowly convergent series and present their application, including series with irrational terms. We consider convergent series of the form

$$\sum_{k=1}^{+\infty}(\pm 1)^k f(k), \tag{1.1}$$

with a given function $z \mapsto f(z)$, with certain properties with respect to the variable z. Here, the function f can depend on several other parameters, e.g., $f(z; x, y, \ldots)$, so that these summation processes can be applied also to some classes of functional series, not only to numerical series. Regarding the properties of the function f is often appropriate to extract a finite number of first terms in (1.1), e.g.,

$$\sum_{k=1}^{+\infty}(\pm 1)^k f(k) = \sum_{k=1}^{m-1}(\pm 1)^k f(k) + \sum_{k=m}^{+\infty}(\pm 1)^k f(k), \tag{1.2}$$

and then apply the procedure to the series starting with the index $k = m$.

The basic idea of these methods is to transform the second series in (1.2) (or directly the series (1.1) if $m = 1$) to an integral with respect to certain weight function w on $\mathbb{R}_+$, and then to approximate this weighted integral by a quadrature sum,

$$\sum_{k=m}^{+\infty} (\pm 1)^k f(k) = \int_0^{+\infty} g(t)w(t)\,\mathrm{d}t \approx \sum_{\nu=1}^{N} A_\nu g(\tau_\nu), \tag{1.3}$$

where the function g is connected with the original function f in some way.

Thus, such summation/integration methods need two steps: (1) transformation "sum to integral"; (2) construction of the quadrature rules

$$\int_0^{+\infty} g(t)w(t)\,\mathrm{d}t = \sum_{\nu=1}^{N} A_\nu g(\tau_\nu) + R_N(g;w), \tag{1.4}$$

with respect to the weight function w.

In our approach in (1.4) we take the Gaussian quadrature formulas, where the nodes $\tau_\nu \equiv \tau_\nu^{(n)}$ and the weight coefficients (Christoffel numbers) $A_\nu \equiv A_\nu^{(n)}$, $\nu = 1, \ldots, N$, can be determined by the well-known Golub-Welsch algorithm [6] if we know the coefficients in the three-term recurrence relation of the corresponding polynomials orthogonal with respect to the weight function w. Usually the weight function w is *strong non-classical* and those recursive coefficients must be constructed numerically. Basic procedures for generating these coefficients are the *method of* (*modified*) *moments*, the *discretized Stieltjes–Gautschi procedure*, and the *Lanczos algorithm* and they play a central role in the so-called *constructive theory of orthogonal polynomials*, which was developed by Walter Gautschi in the eighties on the last century (cf. [2]). The problem is very sensitive with respect to small perturbations in the data. The basic references are [2], [4], [8], and [10].

For the construction of Gaussian quadrature rules (1.4) with respect to the strong non-classical weight functions w on $\mathbb{R}_+$ today we use a recent progress in symbolic computation and variable-precision arithmetic, as well as our Mathematica package `OrthogonalPolynomials` (see [1], [12]). The package is downloadable from Web Site: `http://www.mi.sanu.ac.rs/ gvm/`. The approach enables us to overcome the numerical instability in generating coefficients of the three-term recurrence relation for the corresponding orthogonal polynomials with respect to the weight function w (cf. [2], [4], [8], [10]). In this construction we need only a procedure for the symbolic calculation of moments or their calculation in variable-precision arithmetic.

In the sequel, we mention only two methods for such kind of transformations: *Laplace transform method* and *Contour integration over a rectangle*, including a similar procedure for series with irrational terms. Several numerical examples are given in order to illustrate the efficiency of these methods.

2. Laplace transform method

In this section we present the basic idea of the *Laplace transform method* and give some considerations about applicability of this method. For details and several examples see [5], [8, pp. 398–401] and [11].

Suppose that the general term of series is expressible in terms of the Laplace transform, or its derivative, of a known function.

Consider only the case when

$$f(s) = \mathcal{L}[g(t)] = \int_0^{+\infty} \mathrm{e}^{-st} g(t)\,\mathrm{d}t, \quad \mathrm{Re}\, s \geq 1.$$

Then

$$\begin{aligned}\sum_{k=1}^{+\infty} (\pm 1)^k f(k) &= \sum_{k=1}^{+\infty} (\pm 1)^k \int_0^{+\infty} \mathrm{e}^{-kt} g(t)\,\mathrm{d}t \\ &= \int_0^{+\infty} \left(\sum_{k=1}^{+\infty} \left(\pm \mathrm{e}^{-t}\right)^k \right) g(t)\,\mathrm{d}t,\end{aligned}$$

i.e.,

$$\sum_{k=1}^{+\infty} (\pm 1)^k f(k) = \int_0^{+\infty} \frac{\pm \mathrm{e}^{-t}}{1 \mp \mathrm{e}^{-t}}\, g(t)\,\mathrm{d}t = \pm \int_0^{+\infty} \frac{1}{\mathrm{e}^t \mp 1}\, g(t)\,\mathrm{d}t.$$

In this way, the *summation of series* (1.1) is transformed to *integration problems*

$$T = \sum_{k=1}^{+\infty} f(k) = \int_0^{+\infty} \mathrm{e}^{-t} \frac{g(t)}{1 - \mathrm{e}^{-t}}\,\mathrm{d}t = \int_0^{+\infty} \frac{t}{\mathrm{e}^t - 1} \frac{g(t)}{t}\,\mathrm{d}t \tag{2.1}$$

and

$$S = \sum_{k=1}^{+\infty} (-1)^k f(k) = \int_0^{+\infty} \frac{1}{\mathrm{e}^t + 1} (-g(t))\,\mathrm{d}t. \tag{2.2}$$

The first integral representation (2.1) for the series T suggests an application of the Gauss-Laguerre quadrature rule (with respect to the exponential weight $w(t) = \mathrm{e}^{-t}$) to the function

$$\frac{g(t)}{1 - \mathrm{e}^{-t}} = \frac{t}{1 - \mathrm{e}^{-t}} \frac{g(t)}{t},$$

supposing that $g(t)/t$ is a smooth function. However, the convergence of these Gauss-Laguerre rules can be very slow, according to the presence of poles on the imaginary axis at the points $2k\pi\mathrm{i}$ ($k = \pm 1, \pm 2, \ldots$).

Therefore, a better choice is the second integral representation in (2.1), with the Bose-Einstein weight function $\varepsilon(t) = t/(\mathrm{e}^t - 1)$ on $\mathbb{R}^+$. Supposing again that $t \mapsto g(t)/t$ is a smooth function, the corresponding Gauss-Bose-Einstein quadrature formula converges rapidly.

In the case of "alternating" series, the obtained integral representation (2.2) needs a construction of Gaussian quadrature rule with respect to the Fermi-Dirac weight function $\varphi(t) = 1/(\mathrm{e}^t + 1)$ on $\mathbb{R}^+$.

Thus, for computing series T and S we need the Gauss-Bose-Einstein quadrature rule

$$\int_0^{+\infty} \varepsilon(t) u(t)\,\mathrm{d}t = \sum_{k=1}^{N} A_k u(\xi_k) + R_N(u; \varepsilon) \tag{2.3}$$

and the Gauss-Fermi-Dirac quadrature rule

$$\int_0^{+\infty} \varphi(t)u(t)\,\mathrm{d}t = \sum_{k=1}^{N} B_k u(\eta_k) + R_N(u;\varphi), \tag{2.4}$$

respectively, whose parameters, nodes (ξ_k and η_k) and weight coefficients (A_k and B_k), for each $N \le n$, can be calculated by the Mathematica package `OrthogonalPolynomials`, starting from the corresponding moments of the weight functions, $\mu_k(\varepsilon) = \int_0^{+\infty} x^k \varepsilon(t)\,\mathrm{d}t$ and $\mu_k(\varphi) = \int_0^{+\infty} x^k \varphi(t)\,\mathrm{d}t$, $k = 0, 1, \ldots, 2n-1$. The convergence of the quadrature formulas (2.3) and (2.4) is fast for smooth functions $t \mapsto u(t)$ ($u(t) = g(t)/t$ and $g(t)$), so that low-order Gaussian rules provide one possible summation procedure.

However, if g is no longer smooth function, for example, if its behaviour as $t \to 0$ is such that $g(t) = t^\gamma h(t)$, where $0 < \gamma < 1$ and $h(0)$ is a constant, then the previous formulas for series T and S should be reduced to the following forms

$$T = \int_0^{+\infty} \frac{t^\gamma}{\mathrm{e}^t - 1}\, h(t)\,\mathrm{d}t \tag{2.5}$$

and

$$S = \int_0^{+\infty} \frac{t^\gamma}{\mathrm{e}^t + 1}(-h(t))\,\mathrm{d}t, \tag{2.6}$$

respectively.

Introducing the weight functions from (2.5) and (2.6) as $\varepsilon_\gamma(t)$ and $\varphi_\gamma(t)$, respectively, then their moments are

$$\mu_k(\varepsilon_\gamma) = \zeta(k+\gamma+1)\Gamma(k+\gamma+1), \quad k \ge 0, \tag{2.7}$$

and

$$\begin{aligned} \mu_k(\varphi_\gamma) &= \left(1 - 2^{-k-\gamma}\right)\zeta(k+\gamma+1)\Gamma(k+\gamma+1) \\ &= \left(1 - 2^{-k-\gamma}\right)\mu_k(\varepsilon_\gamma), \quad k \ge 0, \end{aligned} \tag{2.8}$$

where $\Gamma(z)$ is gamma function and $\zeta(z)$ is the Riemann zeta function.

Evidently, the moments for the Bose-Einstein weight are

$$\mu_k(\varepsilon) = \mu_k(\varepsilon_1) = (k+1)!\zeta(k+2), \quad k \ge 0,$$

while for the Fermi-Dirac weight these moments are $\mu_k(\varphi) = \mu_k(\varphi_0)$, except $k = 0$, i.e.,

$$\mu_k(\varphi) = \begin{cases} \log 2, & k = 0, \\ (1 - 2^{-k})k!\zeta(k+1), & k > 0. \end{cases}$$

Example 2.1. For the series

$$\sum_{k=1}^{+\infty} \frac{(\pm 1)^k}{k\sqrt{k+1}}$$

we put

$$f(s) = \frac{1}{s\sqrt{s+1}} = \int_0^{+\infty} \mathrm{e}^{-st}\mathrm{erf}(\sqrt{t})\,\mathrm{d}t, \quad \mathrm{Re}\, s > 0,$$

i.e., $g(t) = \mathrm{erf}(\sqrt{t})$, where $\mathrm{erf}(z)$ is the error function (the integral of the Gaussian distribution), given by

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-t^2}\,\mathrm{d}t.$$

According to the first integral representation in (2.1), we can apply the Gauss-Laguerre quadrature rule

$$T = \sum_{k=1}^{+\infty} \frac{1}{k\sqrt{k+1}} = \int_0^{+\infty} \mathrm{e}^{-t}\Psi(t)\,\mathrm{d}t = \sum_{\nu=1}^{N} A_\nu^L \Psi(\tau_\nu^L) + R_N^L(\Psi), \tag{2.9}$$

where

$$\Psi(t) = \frac{\mathrm{erf}(\sqrt{t})}{1-\mathrm{e}^{-t}} = \frac{1}{\sqrt{\pi t}}\left(2 + \frac{1}{3}t + \frac{1}{30}t^2 - \frac{1}{315}t^3\right) + O\big(t^{7/2}\big). \tag{2.10}$$

Otherwise, the exact value of T is

$$T = 2.184009470267851952894734157852949070443908406263229420200\ldots$$

(see Example 3.1).

Here we have an example in which the function g is no longer smooth, having a square root singularity at $t = 0$. Relative errors in the Gauss-Laguerre approximations

$$Q_N^{\mathrm{Lag}} = \sum_{\nu=1}^{N} A_\nu^L \Psi(\tau_\nu^L)$$

are given in Table 1. Numbers in parentheses indicate decimal exponents, e.g. 5.01(−3) means 5.03×10^{-3}.

TABLE 1. Relative errors in different quadrature sums in Example 2.1

N	Q_N^{Lag}	Q_N^{BE}	Q_N^{genBE}
10	1.40(−1)	1.57(−1)	6.58(−11)
20	9.98(−2)	1.09(−1)	1.65(−20)
30	8.17(−2)	8.76(−2)	4.46(−30)
40	7.09(−2)	7.53(−2)	1.23(−39)
50	6.34(−2)	6.70(−2)	3.45(−49)

Another way for calculating the value of T is to apply the Gauss-Bose-Einstein quadrature rule (2.3) to the last integral in (2.1), where $u(t) = \mathrm{erf}(\sqrt{t})/t$. The corresponding relative errors in the Bose-Einstein approximations Q_N^{BE} are presented in the same table.

As we can see, these two quadrature sums are quite inefficient. In order to get a quadrature sequence with a fast convergence, we note first that

$$\mathrm{erf}(\sqrt{t}) = \sqrt{\frac{t}{\pi}}\left(2 - \frac{2}{3}t + \frac{1}{5}t^2 - \frac{1}{21}t^3\right) + O\big(t^{9/2}\big).$$

This means that we should take the integral (2.1) in the form

$$T=\sum_{k=1}^{+\infty}\frac{1}{k\sqrt{k+1}}=\int_0^{+\infty}\frac{\sqrt{t}}{\mathrm{e}^t-1}\,\frac{\operatorname{erf}(\sqrt{t})}{\sqrt{t}}\,\mathrm{d}t,$$

and then apply the Gaussian rule with respect to the generalized Bose-Einstein weight $t^{-1/2}\varepsilon(t)=\sqrt{t}/(\mathrm{e}^t-1)$ (see [5] and [3]). In the last column of Table 1 we give the corresponding quadrature approximations Q_N^{genBE}. The fast convergence of Q_N^{genBE} is evident!

3. Method of contour integration over a rectangle

As we have seen in the previous section, the function g in (1.3) is connected with the original function f over its Laplace transform, while the weight functions are $\varepsilon(t)=t/(\mathrm{e}^t-1)$ and $\varphi(t)=1/(\mathrm{e}^t+1)$ (or their generalized forms).

In 1994 we developed a method based on a contour integration over a rectangle Γ in the complex plane [9], in which the weight w in (1.3) is one of the hyperbolic functions

$$w_1(t)=\frac{1}{\cosh^2 t}\quad\text{and}\quad w_2(t)=\frac{\sinh t}{\cosh^2 t},\tag{3.1}$$

and the function g can be expressed in terms of the indefinite integral F of f chosen so as to satisfy the following decay properties: (see [7], [9], [8]):

(C1) *F is a holomorphic function in the region*

$$\{z\in\mathbb{C}\mid \operatorname{Re} z\ge\alpha,\ m-1<\alpha<m\},\tag{3.2}$$

where $m,n\in\mathbb{Z}$ $(m<n\le+\infty)$;

(C2) $\displaystyle\lim_{|t|\to+\infty}\mathrm{e}^{-c|t|}F(x+\mathrm{i}t/\pi)=0,$ *uniformly for $x\ge\alpha$;*

(C3) $\displaystyle\lim_{x\to+\infty}\int_{\mathbb{R}}\mathrm{e}^{-c|t|}\,\big|F(x+\mathrm{i}t/\pi)\big|\,\mathrm{d}t=0,$

where $c=2$ (or $c=1$ for "alternating" series).

Taking $\Gamma=\partial G$ and $G=\{z\in\mathbb{C}\,:\,\alpha\le\operatorname{Re} z\le\beta,\ |\operatorname{Im} z|\le\delta/\pi\}$ with $m-1<\alpha<m$, $n<\beta<n+1$, and $\delta>0$, we proved in [9] (see also [8]) that

$$T_{m,n}=\sum_{k=m}^{n}f(k)=\frac{1}{2\pi i}\oint_\Gamma\Big(\frac{\pi}{\sin\pi z}\Big)^2F(z)\,\mathrm{d}z\tag{3.3}$$

and

$$S_{m,n}=\sum_{k=m}^{n}(-1)^kf(k)=\frac{1}{2\pi i}\oint_\Gamma\Big(\frac{\pi}{\sin\pi z}\Big)^2\cos\pi z\,F(z)\,\mathrm{d}z,\tag{3.4}$$

where F is an integral of f.

Setting $\alpha=m-1/2$, $\beta=n+1/2$, and letting $\delta\to+\infty$, under conditions (C1), (C2), and (C3), the previous integrals over Γ reduce to the weighted integrals over $(0,+\infty)$,

$$\sum_{k=m}^{+\infty}f(k)=\int_0^{+\infty}w_1(t)\Phi\left(m-\frac{1}{2},\frac{t}{\pi}\right)\,\mathrm{d}t\tag{3.5}$$

and

$$\sum_{k=m}^{+\infty}(-1)^k f(k) = (-1)^m \int_0^{+\infty} w_2(t)\Psi\left(m-\frac{1}{2},\frac{t}{\pi}\right)\mathrm{d}t, \tag{3.6}$$

where the weight functions w_1 and w_2 are given by (3.1), and Φ and Ψ by

$$\Phi(x,y) = -\frac{1}{2}\left[F(x+\mathrm{i}y)+F(x-\mathrm{i}y)\right] = -\mathrm{Re}\,F(x+\mathrm{i}y)$$

and

$$\Psi(x,y) = \frac{1}{2\mathrm{i}}\left[F(x+\mathrm{i}y)-F(x-\mathrm{i}y)\right] = \mathrm{Im}\,F(x+\mathrm{i}y).$$

The integrals (3.5) and (3.6) can be calculated by using the N-point Gaussian quadratures with respect to the hyperbolic weights w_1 and w_2,

$$\int_0^{+\infty} g(t)w_s(t)\,\mathrm{d}t = \sum_{\nu=1}^{N} A_{\nu,s}^N g(\tau_{\nu,s}^N) + R_{N,s}(g) \quad (s=1,2), \tag{3.7}$$

with weights $A_{\nu,s}^N$ and nodes $\tau_{\nu,s}^N$, $\nu=1,\dots,N$ $(s=1,2)$. Such quadratures are exact for all polynomials of degree at most $2N-1$ $(g\in\mathcal{P}_{2N-1})$ and their numerical construction is given in [9] and [11]. For example, for constructing Gaussian quadratures for $s=1$ and $N\le 50$, we use the first $2N=100$ moments (in symbolic form) and then we construct the recursion coefficients in the three-term recurrence relation for orthogonal polynomials with respect to the hyperbolic weight function w_1 on $(0,+\infty)$. The following procedure in the MATHEMATICA package `OrthogonalPolynomials` provides Gaussian quadratures (with `Precision->60`) for each $N = 5(5)50$ (i.e., `{n,5,50,5}`):

```
<<orthogonalPolynomials`
  f[s_]:=1/(s(s+1)^(1/2));
  F[z_]:=Log[(Sqrt[1+z]-1)/(1+Sqrt[1+z])];
  Phi[x_,y_]:=-Re[F[x+I y]]; w1[x_]:=1/Cosh[x]^2;
  mom=Join[{1,Log[2]},Table[(2^(k-1)-1)k!/4^(k-1)Zeta[k],
      {k,2,99}]];
  {al,be}=aChebyshevAlgorithm[mom,WorkingPrecision->100];
(* {al1,be1}=aChebyshevAlgorithm[mom,WorkingPrecision->130];
   N[Max[Abs[al/al1-1],Abs[be/be1-1]],3] *)
  pq[n_]:=aGaussianNodesWeights[n,al,be,WorkingPrecision->65,
      Precision->60];
  nw=Table[pq[n],{n,5,50,5}];
```

The part between the comment signs (`(*` and `*)`) is used only to determine the maximal relative error in the recursive coefficients, which is, in our case, 4.16×10^{-63}. Therefore, the precision of Gaussian parameters (nodes and weights) is at least 60 decimal digits!

Example 3.1. We again consider the series from Example 2.1.

Here, $f(z) = 1/(z\sqrt{1+z})$, and $F(z) = \log\left(\frac{\sqrt{z+1}-1}{\sqrt{z+1}+1}\right)$, the integration constant being zero on account of the condition (C3).

Thus, using the Gaussian quadrature (3.7) (for $s = 1$), we approximate the series T by

$$T = \sum_{k=1}^{+\infty} \frac{1}{k\sqrt{k+1}} \approx Q_{N,m} = \sum_{k=1}^{m-1} \frac{1}{k\sqrt{k+1}} + \sum_{\nu=1}^{N} A_{\nu,1}^{N} \Phi\left(m - \frac{1}{2}, \frac{\tau_{\nu,1}^{N}}{\pi}\right).$$

For $m = 1$ the first sum on the right side is empty. The corresponding code in Mathematica is:

```
Q[m_]:=If[m==1,0,Sum[f[j],{j,1,m-1}]] +
     Table[nw[[k]][[2]].Phi[m-1/2,nw[[k]][[1]]/Pi],{k,1,10}];
```

The quadrature sums $Q_{N,1}$ and $Q_{N,3}$ are presented in Table 2, and $Q_{N,15}$ in Table 3. Digits in error are underlined.

Table 2. Quadrature sums $Q_{N,m}$ for $m = 1$ and $m = 3$

N	$Q_{N,1}$	$Q_{N,3}$
5	$2.18\underline{399979}$	$2.184009\underline{469}$
10	$2.184009\underline{183}$	$2.1840094702678\underline{658}$
15	$2.18400947\underline{764}$	$2.1840094702678519\underline{550}$
20	$2.18400946\underline{996}$	$2.1840094702678519528\underline{9639}$
25	$2.184009470\underline{281}$	$2.18400947026785195289473\underline{9799}$
30	$2.18400947026\underline{793}$	$2.1840094702678519528947341\underline{7553}$
35	$2.18400947026\underline{770}$	$2.1840094702678519528947341\underline{57762}$
40	$2.1840094702678\underline{697}$	$2.1840094702678519528947341578 52\underline{089}$

Table 3. Quadrature sums $Q_{N,15}$

N	$Q_{N,15}$
5	$2.1840094702678519\underline{8767}$
10	$2.18400947026785195289473415\underline{81999}$
15	$2.18400947026785195289473415785294907\underline{06127}$
20	$2.1840094702678519528947341578529490704439084\underline{170}$
25	$2.184009470267851952894734157852949070443908406263\underline{233}$
30	$2.1840094702678519528947341578529490704439084062632294 20\underline{199}$

As we can see, the sequence of quadrature sums $\{Q_{N,m}\}_N$ converges faster for larger m. This rapidly increasing of convergence of the summation process as m increases in due to the logarithmic singularities $\pm \mathrm{i}\pi\left(m - \frac{1}{2}\right)$ of the function

$$z \mapsto \Phi\left(m - \frac{1}{2}, \frac{z}{\pi}\right), \ z = t + \mathrm{i}s,$$

moving away from the real line. In Figure 1 we present the function

$$(t, s) \mapsto \left|\Phi\left(m - \frac{1}{2}, \frac{1}{\pi}(t + \mathrm{i}s)\right)\right|,$$

when $m = 1$ and $m = 5$.

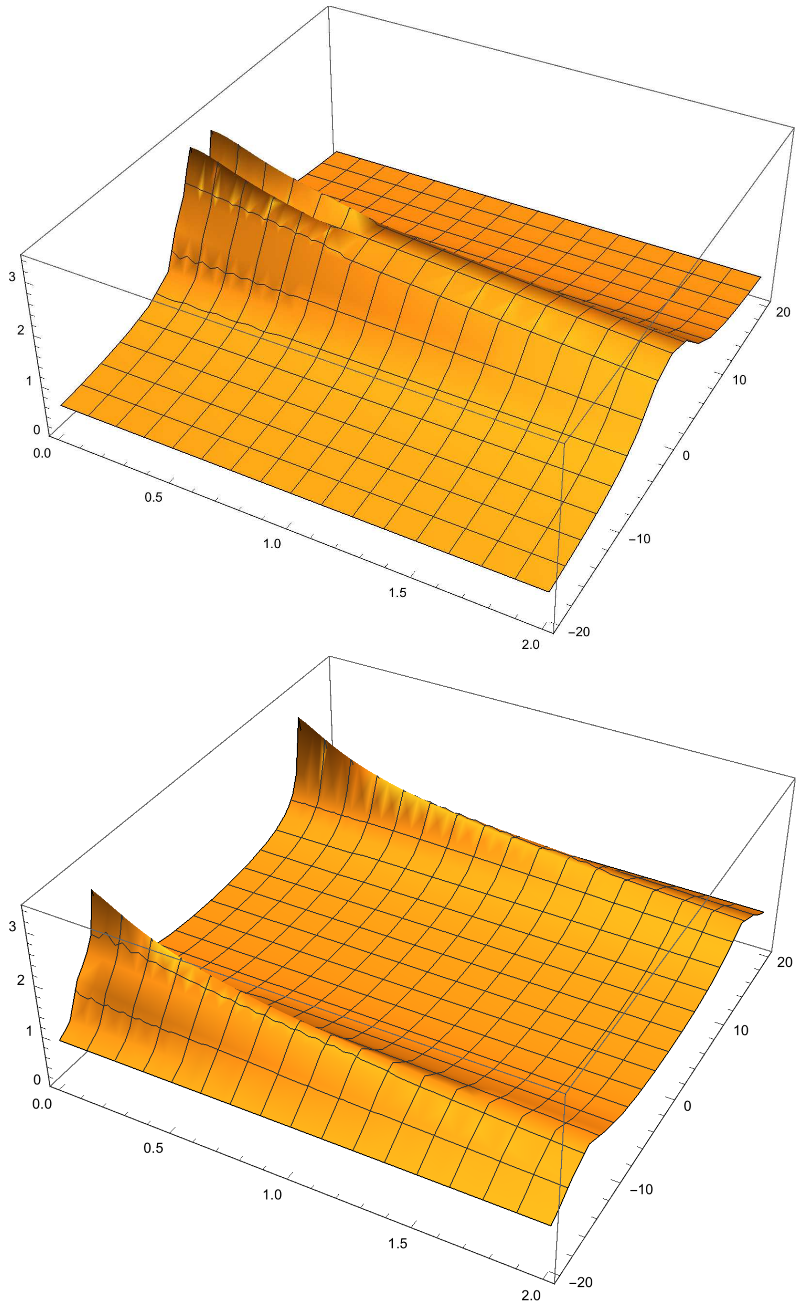

Figure 1. The function $(t, s) \mapsto \left|\Phi\left(m - \frac{1}{2}, \frac{1}{\pi}(t + \mathrm{i}s)\right)\right|$ for $m = 1$ (top) and $m = 5$ (bottom)

For example, for $m = 15$ we obtained the same values of $Q_{N,15}$ for $N = 40(5)50$ and it can be taken as an exact value of the sum,

$$T = 2.18400947026785195289473415785294907044390840626322942020025 1.$$

for calculating the relative errors,

$$\mathrm{err}_{N,m} = \left|\frac{Q_{N,m} - T}{T}\right|,$$

in other quadrature sums $Q_{N,m}$ for smaller $m < 15$. These relative errors for some selected m are presented in Table 4.

TABLE 4. Relative errors $\mathrm{err}_{N,m}$ in the quadrature sums $Q_{N,m}$

N	$m=1$	$m=2$	$m=3$	$m=5$	$m=10$
5	4.43(−6)	3.59(−9)	4.65(−10)	8.05(−13)	8.28(−16)
10	1.31(−7)	5.02(−12)	6.34(−15)	6.25(−19)	8.30(−25)
15	3.38(−9)	6.19(−15)	9.81(−19)	6.88(−24)	2.77(−32)
20	1.39(−10)	1.31(−17)	7.59(−22)	1.14(−28)	2.84(−38)
25	6.15(−12)	2.57(−19)	2.58(−24)	1.15(−31)	6.01(−44)
30	3.73(−14)	4.17(−21)	8.10(−27)	6.13(−35)	1.09(−48)

4. Series with irrational terms

In this section we consider some important series of the form

$$U_{\pm}(a,\nu) = \sum_{k=1}^{+\infty} \frac{(\pm 1)^{k-1}}{(k^2 + a^2)^{\nu+1/2}}\,.$$

In 1916 Kapteyn (see [14, p. 386]) proved the formula

$$U_{+}(a,\nu) = \sum_{k=1}^{+\infty} \frac{1}{(k^2 + a^2)^{\nu+1/2}} = \frac{\sqrt{\pi}}{(2a)^{\nu}\Gamma(\nu+1/2)} \int_0^{+\infty} \frac{t^{\nu}}{\mathrm{e}^t - 1} J_{\nu}(at)\,\mathrm{d}t$$

which is valid when $\mathrm{Re}\,\nu > 0$ and $|\,\mathrm{Im}\,a| < 1$. Here, J_{ν} is the Bessel function of the order ν, defined by

$$J_{\nu}(t) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{t}{2}\right)^{2k+\nu}. \tag{4.1}$$

Since for $F(p) = 1/(p^2 + a^2)^{\nu+1/2}$ $\big(\mathrm{Re}\,\nu > -1/2,\ \mathrm{Re}\,p > |\,\mathrm{Im}\,a|\big)$, using the method of Laplace transform, we find the original function

$$f(t) = \frac{\sqrt{\pi}}{(2a)^{\nu}\Gamma(\nu+1/2)}\, t^{\nu} J_{\nu}(at),$$

as well as

$$U_-(a,\nu)=\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{(k^2+a^2)^{\nu+1/2}}=\frac{\sqrt{\pi}}{(2a)^\nu\Gamma(\nu+1/2)}\int_0^{+\infty}\frac{t^\nu}{\mathrm{e}^t+1}J_\nu(at)\,\mathrm{d}t.$$

Thus, this method leads to an integration of the Bessel function $t\mapsto J_\nu(at)$ with Einstein's weight $\varepsilon(t)$ or Fermi's weight $\varphi(t)$. For some special values of ν, we can use also quadratures with respect to the weights $t^{\pm 1/2}\varepsilon(t)$ and $t^{\pm 1/2}\varphi(t)$ (see [5] and [3]).

In the following example we show how to compute $U_\pm(a,\nu)$, $0<\nu<1$, with a high accuracy.

Example 4.1. According to the expansion of the Bessel function (4.1), we can consider $U_\pm(a,\nu)$ in the form (as the corresponding series in Example 2.1)

$$U_\pm(a,\nu)=\sum_{k=1}^{+\infty}\frac{(\pm 1)^{k-1}}{(k^2+a^2)^{\nu+1/2}}=\frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+1/2)}\int_0^{+\infty}\frac{t^{2\nu}}{\mathrm{e}^t\mp 1}\frac{J_\nu(at)}{(at)^\nu}\,\mathrm{d}t,$$

and then construct Gaussian quadratures with respect to the (generalized) Einstein and Fermi weights $\varepsilon_{2\nu}(t)$ and $\varphi_{2\nu}(t)$ on $(0,+\infty)$, respectively. Their moments are given by (2.7) and (2.8), respectively, where $\gamma=2\nu$.

These series are slowly convergent for small ν. For example, for the remainder $R_n(a,\nu)$ of the series $U_+(a,\nu)$, we have

$$R_n(a,\nu)=\sum_{k=n+1}^{+\infty}\frac{1}{(k^2+a^2)^{\nu+1/2}}<\int_n^{+\infty}\frac{\mathrm{d}x}{(x^2+a^2)^{\nu+1/2}}.$$

For $n\gg a$ the right hand side in the previous inequality can be simplified as

$$\int_n^{+\infty}\frac{\mathrm{d}x}{x^{2\nu+1}}=\frac{1}{2\nu n^{2\nu}},$$

so that we can roughly conclude that for a small ε, the remainder $R_n(a,\nu)<\varepsilon$ if $n>n_\varepsilon=\left[(2\nu\varepsilon)^{-1/(2\nu)}\right]$. The values of n_ε for $\varepsilon=10^{-3}$ and some given values of ν are presented in Table 5.

TABLE 5. The values of n_ε for $\varepsilon=10^{-3}$ and some values of ν

ν	5×10^{-1}	10^{-1}	10^{-2}	10^{-3}	10^{-4}
n_ε	10^3	3×10^{18}	9×10^{234}	3×10^{2849}	7×10^{33494}

Using the MATHEMATICA package `OrthogonalPolynomials` and e.g. the first 100 moments $\mu_k(\varepsilon_{2\nu})$, $k=0,1,\ldots,99$, in the symbolic form (2.7), we can construct for a given $\nu=10^{-4}$ the first 50 recursive coefficients in the three-term recurrence relation with the maximal relative errors of about 6.09×10^{-53} if we use the `WorkingPrecision -> 95` in the Chebyshev method of moments, implemented in this package by the command

```
<<orthogonalPolynomials‘
  moments=Table[Gamma[1+k+2v]Zeta[1+k+2v], {k,0,99}];
  mv=moments/.{v -> 1/10000};
  {alfaE,betaE}=aChebyshevAlgorithm[mv,WorkingPrecision->95];
```

These coefficients enables us to construct the corresponding Gaussian rules for any $N \le 50$,

$$\int_0^{+\infty} \frac{t^{2\nu}}{\mathrm{e}^t \mp 1} u(t)\,\mathrm{d}t \approx Q_N(u;\varepsilon_{2\nu}) = \sum_{k=1}^{N} A_k u(\xi_k), \tag{4.2}$$

where ξ_k and A_k, $k = 1, \ldots, N$, are nodes and weight coefficients. Corresponding Gaussian approximations $Q_N(u;\varepsilon_{2\nu})$, for

$$u(t) = u(t;a,\nu) = \frac{\sqrt{\pi}}{2^\nu \Gamma(\nu+1/2)} \cdot \frac{J_\nu(at)}{(at)^\nu}, \tag{4.3}$$

$a = 1/4$ and $\nu = 10^{-4}$, are presented in Table 6. Digits in error are underlined. In the same table we give also the relative errors $\mathrm{err}_N(a,\nu)$ in these approximations, taking $Q_{50}(u;\varepsilon_{2\nu})$ as the exact value of the sum.

TABLE 6. Gaussian approximations $Q_N(u;\varepsilon_{2\nu})$ and relative errors $\mathrm{err}_N(a,\nu)$ for $u(t)$ given by (4.3)

N	$Q_N(u;\varepsilon_{2\nu})$	$\mathrm{err}_N(a,\nu)$
5	5000.541106014918	8.29(−14)
10	5000.54110601450371233515	1.53(−22)
15	5000.541106014503712334387545429320	1.29(−31)
20	5000.541106014503712334387545429967462497083	3.71(−41)
25	5000.5411060145037123343875454299674624972689174559	1.11(−49)

The relative errors $\mathrm{err}_N(a,\nu)$ for $0 < a < 1$ and $\nu = 10^{-4}$ in log-scale are displayed in Figure 2 for $N = 5(5)15$ nodes in the quadrature formula (4.2).

Remark 4.2. When $a \to 0$ the function $u(t)$, defined by (4.3), tends to the constant $2^{-v}/\Gamma(v+1)$. Then the quadrature sums in (4.2) give the same value for each N,

$$U_+(0,10^{-4}) = 5000.577230227876819593803166655352232742180084 7082,$$

which is, in fact, an approximative value of the well-known ζ function at the point $2\nu + 1 = 1.0002$.

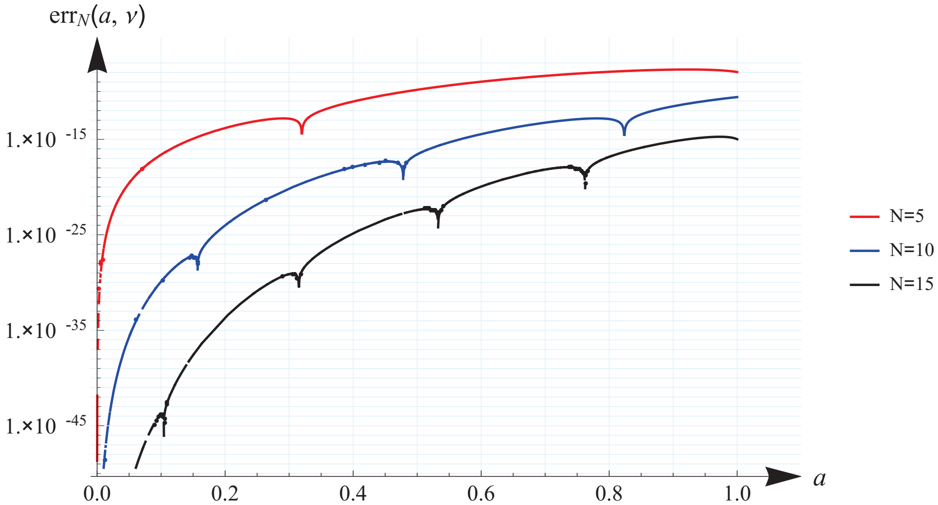

Figure 2. Relative errors in quadrature sums $Q_N(u;\varepsilon_{2\nu})$ for $N=5$ (red line), $N=10$ (blue line), and $N=15$ nodes (black line), when $\nu=10^{-4}$

Finally, we consider an alternative method for the series of the form

$$\sum_{k=-\infty}^{+\infty} f(k,\sqrt{k^2+a^2}) \quad \text{and} \quad \sum_{k=-\infty}^{+\infty} (-1)^k f(k,\sqrt{k^2+a^2}) \quad (a>0),$$

where f is a rational function. Such series can be reduced to some appropriate integrals, by integrating the corresponding function $z \mapsto F(z) = f(z,\sqrt{z^2+a^2})g(z)$, with $g(z) = \pi/\tan \pi z$ and $g(z) = \pi/\sin \pi z$, respectively, over certain circle C_n with the cuts.

In the sequel we illustrate this alternative method in the simplest case when $f(z,w) = 1/w$, i.e., to summation of the series

$$U_-(a,0) = \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{\sqrt{k^2+a^2}}, \quad a>0. \tag{4.4}$$

Thus, we integrate the function $z \mapsto F(z) = g(z)/\sqrt{z^2+a^2}$, with $g(z) = \pi/\sin \pi z$, over the circle

$$C_n = \left\{ z \in \mathbb{C} \;\middle|\; |z| = n + \frac{1}{2} \right\}, \qquad n > a,$$

with cuts along the imaginary axis, so that the critical singularities $\mathrm{i}a$ and $-\mathrm{i}a$ are eliminated (cf. [13, p. 217]). Precisely, the contour of integration Γ is given by $\Gamma = C_n^1 \cup l_1 \cup \gamma_1 \cup l_2 \cup C_n^2 \cup l_3 \cup \gamma_2 \cup l_4$, where C_n^1 and C_n^2 are parts of the circle C_n, γ_1 and γ_2 are small circular parts of radius ε and centres at $\pm\mathrm{i}a$, and l_k $(k=1,2,3,4)$ are the corresponding line segments (see Figure 3).

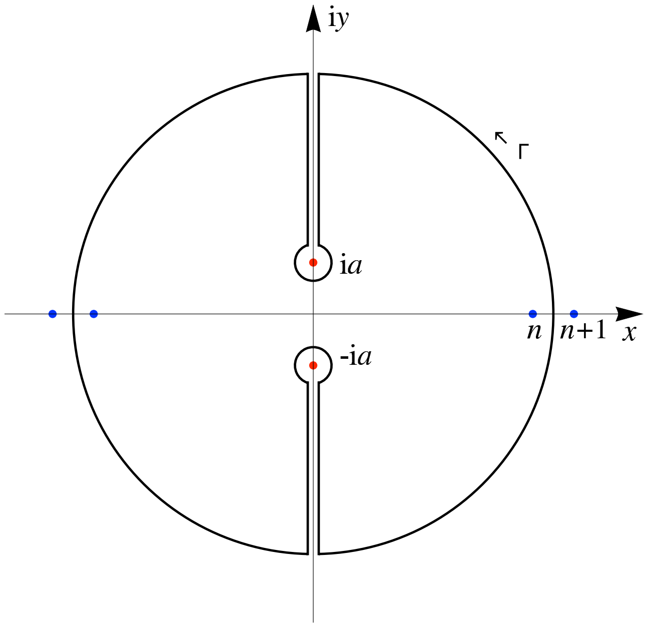

Figure 3. The contour of integration Γ

Let $F^*(z)$ be the branch of $F(z)$ which corresponds to the value of the square root which is positive for $z = 1$. Since

$$\oint_\Gamma F^*(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=-n}^{n}\frac{(-1)^k}{\sqrt{k^2+a^2}},$$

and $\int_{\gamma_1} \to 0$, $\int_{\gamma_2} \to 0$, when $\varepsilon \to +0$, and $\int_{C_n^1 \cup C_n^2} \to 0$, when $n \to +\infty$, we obtain

$$\sum_{k=1}^{+\infty}\frac{(-1)^k}{\sqrt{k^2+a^2}} = -\frac{1}{2a} + \int_a^{+\infty}\frac{\mathrm{d}u}{\sinh \pi u\sqrt{u^2-a^2}},$$

i.e.,

$$\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{\sqrt{k^2+a^2}} = \frac{1}{2a} - \frac{1}{2}\int_{-1}^{+1}\Big(t\sinh\frac{\pi a}{t}\Big)^{-1}\frac{\mathrm{d}t}{\sqrt{1-t^2}}.$$

Thus, we have reduced $U_-(a,0)$ to a problem of Gauss-Chebyshev quadrature. Since $t \mapsto \big(t\sinh(\pi a/t)\big)^{-1}$ is an even function we can apply the $(2n)$-point Gauss-Chebyshev approximations with only n functional evaluations, so that we have

$$U_-(a,0) \approx GC(N;a) = \frac{1}{2a} - \frac{\pi}{2N}\sum_{k=1}^{N}\Big(\tau_k\sinh\frac{\pi a}{\tau_k}\Big)^{-1}, \tag{4.5}$$

where $\tau_k = \cos\big((2k-1)\pi/(4N)\big)$, $k = 1,\ldots,N$.

Example 4.3. We consider now the series

$$U_-(a,0)=\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{\sqrt{k^2+a^2}}$$

for two different values of a, $a=1/4$ and $a=10$, whose exact values are

$$U_-\Big(\frac{1}{4},0\Big)=0.66632618906466580605283262942098890800417625596202\ldots$$

and

$$U_-(10,0)=0.04999999999998988303868784011212623150952067574918\ldots,$$

respectively.

For calculating the sum $U_-(a,0)$ we apply the Gauss-Chebyshev quadrature approximation (4.5), the Gauss-Fermi-Dirac rule (2.4), as well as the quadrature rule (3.7) for $s=2$.

The relative errors in the Gauss-Chebyshev quadrature sums $GC(N;a)$ for small value $a=1/4$ are given in Table 7, and for $a=10$ in Table 8. In these tables we also present the corresponding relative errors for the Gauss-Fermi-Dirac quadrature sums

$$GFD(N;a)=\sum_{k=1}^{N}B_k u(\eta_k),$$

obtained by (2.4), where $u(t)=J_0(at)$.

TABLE 7. Relative errors in the quadrature sums $GC(N;a)$, $GFD(N;a)$ and $Q_{N,m}(a)$ for $a=1/4$

N	$GC(N;a)$	$GFD(N;a)$	$Q_{N,1}(a)$	$Q_{N,5}(a)$	$Q_{N,10}(a)$
10	7.15(−4)	8.78(−19)	2.68(−5)	5.77(−14)	4.72(−20)
20	2.28(−5)	2.64(−37)	2.59(−7)	3.15(−23)	1.69(−28)
30	8.99(−8)	1.85(−55)	1.82(−8)	4.46(−25)	2.22(−35)
40	3.54(−7)		7.46(−10)	4.00(−29)	2.65(−41)
50	4.40(−8)		4.93(−11)	5.11(−33)	6.60(−47)

Finally, we apply the quadrature rule (3.7) for $s=2$ to compute the weighted integral (3.6). The construction of this quadrature we need the moments (cf. [11])

$$\mu_k^{(2)}=\int_0^{+\infty}t^k w_2(t)\,\mathrm{d}t=\begin{cases}1, & k=0,\\[4pt] k\Big(\dfrac{\pi}{2}\Big)^k|E_{k-1}|, & k\,(\text{odd})\ge 1,\\[4pt] \dfrac{2k}{4^k}\left[\psi^{(k-1)}(\tfrac14)-\psi^{(k-1)}(\tfrac34)\right], & k\,(\text{even})\ge 2,\end{cases}$$

where $\zeta(k)$ is the Riemann zeta function, E_k are Euler's numbers, and $\psi(z)$ is the logarithmic derivative of the gamma function, i.e., $\psi(z)=\Gamma'(z)/\Gamma(z)$.

TABLE 8. Relative errors in the quadrature sums $GC(N;a)$, $GFD(N;a)$ and $Q_{N,m}(a)$ for $a = 10$

N	$GC(N;a)$	$GFD(N;a)$	$Q_{N,1}(a)$	$Q_{N,5}(a)$	$Q_{N,10}(a)$
10	1.86(−22)	2.26	4.86(−14)	8.68(−17)	6.60(−20)
20	3.38(−31)	1.98	2.70(−14)	7.06(−21)	7.69(−28)
30	4.83(−38)	1.05	3.38(−15)	1.00(−23)	3.77(−33)
40	5.38(−45)	4.94(−2)	4.65(−15)	7.95(−26)	1.50(−37)
50	3.70(−49)	4.42(−1)	1.05(−15)	1.02(−27)	2.15(−41)

As in Example 3.1 we consider quadrature sums in the form

$$Q_{N,m}(a) = \sum_{k=1}^{m-1} \frac{(-1)^{k-1}}{\sqrt{k^2+a^2}} + (-1)^{m-1} \sum_{\nu=1}^{N} A_{\nu,2}^N \Psi\left(m - \frac{1}{2}, \frac{\tau_{\nu,2}^N}{\pi}\right),$$

where $\Psi(x,y) = \operatorname{Im} F(x+\mathrm{i}y)$ and $F(z) = \log(z+\sqrt{z^2+a^2})$. Although condition (C3), in this case, is not satisfied the sequence of quadrature sums $Q_{N,m}(a)$ converges. This means that this requirement can be weakened, but it will be studied elsewhere.

As we can see, the convergence of Gauss-Chebyshev approximations $GC(N;a)$ is faster if the parameter a is larger. However, the Laplace transform method ($GFD(N;a)$) is very efficient for a small parameter a, but, when a increases, the integrand $J_0(at)$ becomes a highly oscillatory function and the convergence of the quadrature process slows down considerably.

Also, we can see a rapidly increasing of convergence of the summation process $Q_{N,m}(a)$ as m increases.

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Interpolation methods for multivalued functions

Ildiko Somogyi and Anna Soós

Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. The aim of these article is to study the interpolation problem for multivalued functions. We give some methods for the approximation of these functions.

Mathematics Subject Classification (2010): 65D05, 65D07.

Keywords: Numerical interpolation, spline functions.

1. Introduction

The notion of multivalued functions appeared in the first half of the twentieth century. A multivalued function also known as multi-function, multimap, set-valued function. This is a "function" that assume two or more values for each point from the domain. These functions are not functions in the classical way because for each point assign a set of points, so there is not a one-to-one correspondence. The term of "multivalued function" is not correct, but became very popular. Multivalued functions often arise as inverse of functions which are non-injective. For example the inverse of the trigonometric, exponential, power or hyperbolic functions are multivalued functions. Also the indefinite integral can be considered as a multivalued function. These functions appears in many areas, for example in physics in the theory of defects of crystals, for vortices in superfluids and superconductors but also in optimal control theory or game theory in mathematics.

2. Interpolation problem

Let $[a, b] \subseteq \mathbb{R}$ and $f : [a, b] \to \mathbb{P}(\mathbb{R})$ be a multivalued function, where $\mathbb{P}(\mathbb{R})$ is the power set of $\mathbb{R}$, and $f(x)$ is nonempty for every $x \in [a, b]$. We say that a multivalued function is single-valued if, $f(x)$ contain only one element for every $x \in [a, b]$. Thus a common function can be considered as single-valued multifunction. Furthermore, we suppose that for each $x \in [a, b]$, $\text{card}(f(x)) < \infty$. We suppose that the points $x_i \in [a, b], i = 1, 2, \dots, l$ are given and also the set of function values on this points are known $y_{ij} \in \mathbb{R}, i = 1, 2, \dots, l, j = 1, 2, \dots, k$. We will interpolate the sets of points $M_j = \{(x_i, y_{ij}), i = 1, 2, \ldots, l\}, j = 1, 2, \ldots, k$ using an interpolation operator $P_j : C[a, b] \longrightarrow \mathbb{R}, j = 1, 2, \ldots, k$ and the remainder operator R_j.

Definition 2.1. *If $x \in [a, b], x \neq x_i, i = 1, 2, \ldots, k$, the value of the multivalued function in x is approximated by the following set $\{P_1(x), \ldots, P_k(x)\}$. The approximation error on the point x is given by $R_1(x) + \ldots + R_k(x)$.*

Definition 2.2. *We have the following interpolation formula:*

$$f(x) = (P_1 \cup P_2 \ldots \cup P_k)(x) + (R_1 + R_2 + \ldots + R_k)(x) \tag{2.1}$$

where $P_1 \cup \ldots \cup P_k$ is the interpolation operator and $R_1 + \ldots + R_k$ is the remainder operator.

Remark 2.3. The $P_1 \cup \ldots \cup P_k$ is an interpolation operator because the following interpolation condition are satisfied: $P_i(x_j) = y_{ji}$.

Theorem 2.4. *The interpolation operator $P_1 \cup \ldots \cup P_k$ exists and is unique.*

Proof. It is obvious, because at each set $M_j, j = 1, 2, \ldots, k$ the interpolation operators $P_j, j = 1, 2, \ldots, k$ exists and are unique. □

Furthermore let's consider the case when we have the following type of data $\{(x_i, y_{n_j j}), i = 1, 2, \ldots, l, j = 1, 2, \ldots, k, n_i \in \mathbb{N}, n_i < \infty\}$.
Let be $m = \min\{n_j, j = 1, 2, \ldots, k\}$, then we will consider the following set of data $\{(x_m, y_{mi}, i = 1, 2, \ldots, k)\}$, in this way we reduce the problem to the previous case.

3. Lagrange-type multivalued interpolation

If we considering the case when at each set M_j, the points (x_i, y_{ij}) are interpolated using Lagrange type interpolation, then the interpolation operator is $L_{l_1} \cup \ldots \cup L_{l_k}$, where $L_{l_i}, i = 1, 2, \ldots, k$ are $l - 1$ degree Lagrange polynomials, and the remainder is equal to $R_{l_1} + R_{l_2} + \ldots + R_{l_k}$ where R_{l_i} are the corresponding remainder operators.

Theorem 3.1. *The value of the multivalued Lagrange type interpolation function on the point $x \in [a, b]$, $x \neq x_i, i = 1, 2, \ldots, l$ is given by*

$$L_{l_1} \cup \ldots \cup L_{l_k}(x) = \bigcup_{i=1}^{k} \sum_{j=1}^{l-1} l_{ij}(x) y_{ij} \tag{3.1}$$

where l_{ij} are the basic Lagrange polynomials with degree $l - 1$.

Proof. From Theorem 2.4 we have that the value of the multivalued function on the point x is approximated by the following values $\{P_1(x), \ldots, P_k(x)\}$, where P_i are the corresponding interpolation operators for the data (x_i, y_{ij}), $j = 1, 2, \ldots, l$. Because now we use Lagrange-type interpolation to approximate these data, we have

$$P_i(x) = L_{l-1}(x) = \sum_{i=1}^{l} l_{ij}(x) y_{ij},$$

where l_{ij} are the corresponding basic Lagrange polynomials. □

We suppose that $y_{ij} = f_j(x_i)$ where $f_j \in C[a,b], j = 1,2,\ldots,k$.

Theorem 3.2. *If $f_j \in C^{l-1}[a,b], j = 1,2,\ldots,k,$ and $\exists f_j^{(l)} j = 1,2,\ldots,k$ on $[a,b]$ then the remainder of the multivalued interpolation formula is*

$$(R_1 + R_2 + \ldots + R_k)(x) = \sum_{j=1}^{k} \frac{u(x)}{l!} f_j^{(l)}(\xi_j) \tag{3.2}$$

were $\xi_j \in (a,b)$ and $u(x) = (x - x_1)(x - x_2)\ldots(x - x_l)$.

Proof. If we consider the Lagrange interpolation formula for each set M_j

$$f_j(x) = L_j(x) + R_j(x), j = 1,2,\ldots,k$$

where if $f_j \in C^{l-1}[a,b]$ and $\exists f_j^{(l)}$ on $[a,b]$ then there $\exists \xi_j \in (a,b), j = 1,2,\ldots,k$ such that

$$R_j(x) = \frac{u(x)}{(l!)} f_j^{(l)}(\xi_j), j = 1,2,\ldots,k$$

□

Example 3.3. If consider the multivalued function, obtained as the inverse of the function $g(x) = \sin(x)$, on the interval $[a,b] = [-1,1]$, using the method described below with Lagrange type interpolation operators on each set of points M_j, we obtain the graph from figure 1, where the dotted line is the graph of the multivalued function and the continuous line is the graph of the multivalued function obtained by interpolation.

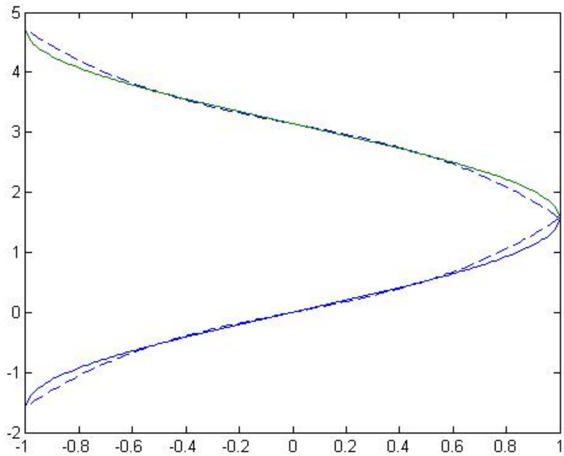

Figure 1. Interpolation of multivalued function with Lagrange operators

4. Shepard-type multivalued interpolation

We suppose that the points $x_i \in [a,b]$, $i = 1,2,\ldots,l$ are given and also the set of function values on this points are known $y_{ij} \in \mathbb{R}, i = 1,\ldots,l, j = 1,\ldots,k$. We will interpolate the sets of points $M_j = \{(x_i, y_{ij}), i = 1, \ldots, l\}, j = 1, \ldots, k$ using Shepard interpolation studied also in [2], [1], [4] and [6].

Theorem 4.1. *The Shepard-type multivalued interpolation operator is*

$$\bigcup_{i=1}^{k} S_i(x) = \bigcup_{i=1}^{k} \sum_{j=1}^{l} A_j(x) y_{ij}, \tag{4.1}$$

where S_i are the univariate Shepard operators and

$$A_j(x) = \frac{\prod_{i=1, i \neq j} l|x - x_i|^{\mu}}{\sum_{t=1}^{l} \prod_{i=1, i \neq t} l|x - x_i|^{\mu}}$$

and $\mu \in \mathbb{R}_+$.

Remark 4.2. The basis functions A_j can be also written in the following barycentric form

$$A_j(x) = \frac{|x - x_j|^{-\mu}}{\sum_{i=1}^{l} |x - x_k|^{-\mu}}, j = 1, 2, \ldots, l,$$

and they satisfy

$$\sum_{j=1}^{l} A_j(x) = 1, A_j(x_p) = \delta_{jp}, j, p = 1, 2, \ldots, l.$$

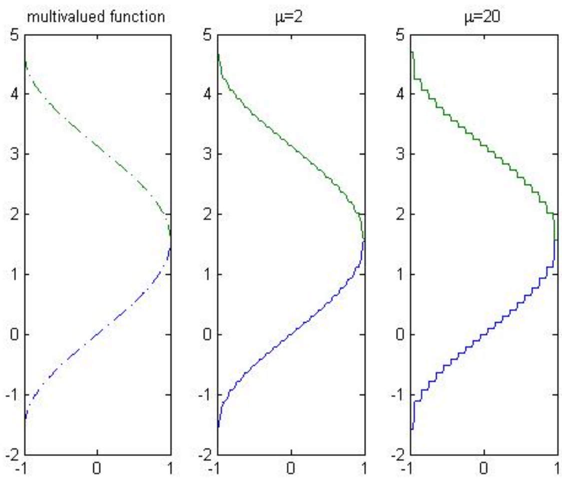

Figure 2. Interpolation of multivalued function with Shepard operators

From the remark it follows that the Shepard operators has the following properties: first of all they have the interpolation conditions $S_i(x_j) = y_{ij}$, $j = 1, 2, \dots, l$, $i = 1, 2, \dots, k$, and they have the degree of exactness $dex(S_i) = 0, i = 1, 2, \dots, k$. The graph of the function from the previous example in the case when we use Shepard operators with different parameters, is given in Figure 2.

The major disadvantage of the Shepard operator is the low degree of exactness, but this can be overcome combining the Shepard operator with another interpolation operators, for example Lagrange, Hermite, Birkhoff or other interpolation operators.

5. Spline-type multivalued interpolation

We will consider again the points $x_i \in [a, b], i = 1, 2, \dots, l$ and also the set of function values on this points $y_{ij} \in \mathbb{R}, i = 1, 2, \dots, l, j = 1, 2, \dots, k$ which are known, let $M_j = \{(x_i, y_{ij}), i = 1, 2, \dots, l\}, j = 1, 2, \dots, k$ be the set of interpolation points. In this section we will interpolate the multivalued function given by the set of points from M_j with spline interpolation function.
We suppose that the values $y_{ij} = f_j(x_i)$, where $f_j \in H^{m,2}[a, b]$ is the set of functions with $f_j \in C^{m-1}[a, b]$, $f^{(m-1)}$ absolute continuous on $[a, b]$ and $f^{(m)} \in L^2[a, b]$.

Theorem 5.1. *The multivalued interpolation operator in the case of spline interpolation is*

$$\bigcup_{i=1}^{l} S_i(x) = \bigcup_{i=1}^{l} \sum_{j=1}^{k} s_{ij}(x) y_{ij} \tag{5.1}$$

where s_{ij} are the fundamental spline interpolation functions.

Remark 5.2. The fundamental spline functions satisfies the following minimum properties $\|S_i^{(m)}\|_2 \longrightarrow \min$, in the set of all functions which satisfies the interpolation conditions.

To determine the fundamental spline functions we can use the structural characterization theorem of spline functions given also in [3] and we have

$$s_{ij}(x) = \sum_{t=0}^{m-1} a_t^{ij} x^t + \sum_{p=1}^{l} b_p^{ij} (x - x_p)_+^{2m-1}, i = 1, 2, \dots, l, j = 1, 2, \dots, k$$

with a_t^{ij}, and b_p^{ij} obtained as the solution of the following systems:

$$\begin{aligned} s_{ij}^{(r)}(\alpha) &= 0,\ r = m, \dots, 2m-1, \ \texttt{and}\ \alpha > x_l \\ s_{ij}(x_\nu) &= \delta_{j\nu},\ \nu = 1, 2, , \dots, l \end{aligned}$$

for $j = 1, 2, \dots, k, i = 1, 2, \dots, l$.

Theorem 5.3. *If $f_j \in H^{m,2}[a, b], j = 1, 2, \dots, k$ then the remainder term of the spline-type multivalued interpolation formula is*

$$\sum_{j=1}^{k} R_j(x) = \sum_{j=1}^{k} \int_a^b \varphi_j(x, t) f_j^{(m)}(t) dt \tag{5.2}$$

where

$$\varphi_j(x,t) = \frac{(x-t)_+^{m-1}}{(m-1)!} - \sum_{i=1}^{l} s_{ij}(x)(x_i - t)_+^{m-1}, j = 1, 2, \ldots, k.$$

This follows from the representation of the error using the Peano theorem.

The graph of the function from the previous example using third degree natural spline interpolation operators is given in Figure 3.

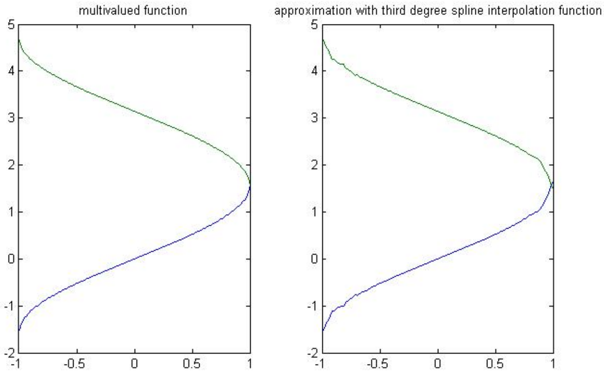

Figure 3. Interpolation of multivalued function with spline operators

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Eigenstructure of the genuine Beta operators of Lupaş and Mühlbach

Heiner Gonska, Margareta Heilmann and Ioan Raşa

Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. The eigenstructure of genuine Beta operators is described, a limiting case of Beta-Jacobi operators. Its similarity to that of the classical Bernstein operators is emphasized. The significance of the mappings considered here comes, among others, from their role as a building block in genuine Bernstein-Durrmeyer operators.

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1. Introduction and historical notes

The present note deals with the eigenstructure of certain Beta-type operators introduced independently by Mühlbach and Lupaş in the early seventies of the last century (see [10],[11],[9]).

Mühlbach's definition is the more general one. For $\lambda > 0$ he defined mappings T_λ, given for $f \in C[0,1]$, $x \in [0,1]$ by

$$T_\lambda(f;x) = \begin{cases} f(0), & x = 0, \\ \displaystyle\int_0^1 f(t)K_\lambda(t,x)dt, & x \in (0,1), \\ f(1), & x = 1. \end{cases}$$

The kernel is given by

$$K_\lambda(t,x) = \frac{1}{B(\frac{x}{\lambda}, \frac{1-x}{\lambda})} t^{\frac{x}{\lambda}-1}(1-t)^{\frac{1-x}{\lambda}-1},$$

where $B(\cdot,*)$ is the Beta function, a.k.a. Euler's integral of the first kind. For more on this function see, e.g., MathWorld [16] and the references given there. Mühlbach's work was motivated by three earlier papers of Stancu, see [12] , [13], [14].

If $1/\lambda = n$ is a natural number, then we arrive at Lupaş' version of the operator, given for strictly positive integers n by

$$\bar{\mathbb{B}}_n(f;x) = \begin{cases} f(0), & x=0,\\ \dfrac{1}{B(nx,n(1-x))}\displaystyle\int_0^1 t^{nx-1}(1-t)^{n(1-x)-1}f(t)dt, & x\in(0,1),\\ f(1), & x=1. \end{cases}$$

The $\bar{\mathbb{B}}_n$ are positive linear endomorphisms of $C[0,1]$; they reproduce linear functions and have second moments smaller than the classical Bernstein operators B_n. More precisely, see [9, Satz 2.28],

$$\bar{\mathbb{B}}_n((e_1-x)^2;x) = \frac{x(1-x)}{n+1} \le \frac{x(1-x)}{n} = B_n((e_1-x)^2;x).$$

The restrictions $\bar{\mathbb{B}}_n : \Pi_n \to \Pi_n$ and $\bar{\mathbb{B}}_n : \Pi \to \Pi$ are bijective, and $\bar{\mathbb{B}}_n : C[0,1] \to C[0,1]$ is injective. Moreover, it is known from [2] and [3] that $\bar{\mathbb{B}}_n$ preserves monotonicity and (ordinary) convexity.

Our reason to call them *genuine* Beta operators is due to the facts that they are the limiting cases of Beta operators with Jacobi weights and unique in the sense that they are the only ones among them which reproduce linear functions. Calling them *genuine* is also justified by the decomposition $B_n \circ \bar{\mathbb{B}}_n = U_n$; here U_n is the so-called genuine Bernstein-Durrmeyer operator which has been attracting much attention. Much more on Beta-Jacobi operators can be found in [6], [15], [7].

The genuine operators $\bar{\mathbb{B}}_n$ were also used in attempts to decompose the classical Bernstein operators into non-trivial building blocks. Reports on these were given by Gonska et al. [5] and by Heilmann and Rasa [8]. Aspects concerning their power series are described in [1].

2. The eigenstructure of $\bar{\mathbb{B}}_n$

The purpose of this article is to give a concise description of the eigenstructure of the Beta operators considered here. By direct computation it is easy to find the first eigenvalues and eigenpolynomials of $\bar{\mathbb{B}}_n$:

$$\begin{aligned}
\eta_0^{(n)} &= 1, & q_0^{(n)}(x) &= 1,\\
\eta_1^{(n)} &= 1, & q_1^{(n)}(x) &= x-\frac{1}{2},\\
\eta_2^{(n)} &= \frac{n}{n+1}, & q_2^{(n)}(x) &= x(x-1),\\
\eta_3^{(n)} &= \frac{n^2}{(n+1)(n+2)}, & q_3^{(n)}(x) &= x(x-1)\left(x-\frac{1}{2}\right),\\
\eta_4^{(n)} &= \frac{n^3}{(n+1)(n+2)(n+3)}, & q_4^{(n)}(x) &= x(x-1)\left(x(x-1)+\frac{n+1}{5n+6}\right).
\end{aligned}$$

As

$$\begin{aligned} \bar{\mathbb{B}}_n e_0 &= e_0, \\ \bar{\mathbb{B}}_n e_k(x) &= \frac{nx(nx+1)\dots(nx+k-1)}{n(n+1)\dots(n+k-1)},\ k \geq 1, \end{aligned} \tag{2.1}$$

following directly from the definition of $\bar{\mathbb{B}}_n$, we conclude that the eigenvalues of $\bar{\mathbb{B}}_n : \Pi \longrightarrow \Pi$ are the numbers

$$\eta_k^{(n)} = \frac{(n-1)!}{(n+k-1)!} n^k,\ k \geq 0. \tag{2.2}$$

Let us denote by $p_k^{(n)}$ the eigenpolynomials of B_n (see [4]). Here are some examples (see [4, (9.1)]).

$$\begin{aligned} p_0^{(n)}(x) &= 1, \\ p_1^{(n)}(x) &= x - \frac{1}{2}, \\ p_2^{(n)}(x) &= x(x-1), \\ p_3^{(n)}(x) &= x(x-1)\left(x - \frac{1}{2}\right), \\ p_4^{(n)}(x) &= x(x-1)\left(x(x-1) + \frac{n-1}{5n-6}\right). \end{aligned}$$

Thus we have

$$q_k^{(n)} = p_k^{(n)},\ 0 \leq k \leq 3$$

and

$$\lim_{n\to\infty} q_k^{(n)}(x) = \lim_{n\to\infty} p_k^{(n)}(x),\ k = 4, \tag{2.3}$$

uniformly in $[0,1]$. We shall show that the eigenstructure of $\bar{\mathbb{B}}_n$ is similar to that of B_n; in particular, that (2.3) holds for all $k \geq 0$. Since the polynomials

$$\lim_{n\to\infty} p_k^{(n)}(x) := p_k^*(x),\ k \geq 0,$$

are completely described in [4], we get the same information about $\lim_{n\to\infty} q_k^{(n)}(x)$. Let $k \geq 2$ and $n \geq 1$. We want to determine $q_k^{(n)} \in \Pi_k$ such that

$$\bar{\mathbb{B}}_n q_k^{(n)} = \eta_k^{(n)} q_k^{(n)}. \tag{2.4}$$

We put

$$q_k^{(n)}(x) = \sum_{j=0}^{k} a(n,k,j)x^j, \quad \text{with} \quad a(n,k,k) = 1. \tag{2.5}$$

Hence

$$\bar{\mathbb{B}}_n(q_k^{(n)};x) = \sum_{j=0}^{k} a(n,k,j)\bar{\mathbb{B}}_n(e_j;x).$$

With (2.1) we derive

$$\bar{\mathbb{B}}_n(q_k^{(n)};x) = \sum_{j=0}^{k} a(n,k,j)\frac{nx(nx+1)\dots(nx+j-1)}{n(n+1)\dots(n+j-1)} = \frac{n^k}{n(n+1)\dots(n+k-1)}\sum_{j=0}^{k} a(n,k,j)x^j. \tag{2.6}$$

From the definition of the Stirling numbers of first kind $s(j,i)$, we obtain immediately

$$nx(nx+1)\dots(nx+j-1) = \sum_{i=0}^{j} s(j,i)(-1)^{j-i}n^i x^i,$$

so that (2.6) becomes, after some manipulation,

$$\sum_{i=0}^{k}\left\{\sum_{j=i}^{k}\frac{s(j,i)(-1)^{j-i}n^i}{n(n+1)\dots(n+j-1)}a(n,k,j)\right\}x^i = \sum_{i=0}^{k}\frac{a(n,k,i)n^k}{n(n+1)\dots(n+k-1)}x^i.$$

This leads to

$$\sum_{j=i}^{k}\frac{s(j,i)(-1)^{j-i}}{n(n+1)\dots(n+j-1)}a(n,k,j) = \frac{n^{k-i}}{n(n+1)\dots(n+k-1)}a(n,k,i), \tag{2.7}$$

for all $i=0,1,\dots,k$. Since $s(i,i)=1$, we can solve (2.7) for $a(n,k,i)$ getting

$$a(n,k,i) = \frac{\sum_{j=i+1}^{k}(-1)^{j-i-1}s(j,i)(n+j)(n+j+1)\dots(n+k-1)a(n,k,j)}{(n+i)(n+i+1)\dots(n+k-1)-n^{k-i}}, \tag{2.8}$$

for all $i\in\{k-1,k-2,\dots,0\}$. Recalling that n and k are given, and $a(n,k,k)=1$, (2.8) represents a recurrence relation for computing $a(n,k,i)$, $i=k-1,k-2,\dots,0$. In particular, using $s(k,k-1)=-\frac{k(k-1)}{2}$, $s(k,k-2)=\frac{k(k-1)(k-2)(3k-1)}{24}$, we get

$$a(n,k,k-1) = -\frac{k}{2}, \tag{2.9}$$

$$a(n,k,k-2) = \frac{k(k-1)(k-2)}{24}\cdot\frac{6n+3k-5}{(2k-3)n+(k-1)(k-2)}. \tag{2.10}$$

Let us prove by induction that

$$a^*(k,j) := \lim_{n\to\infty} a(n,k,j) = \prod_{l=1}^{k-j}\frac{(k+1-l)(k-l)}{l(l-2k+1)}. \tag{2.11}$$

For $j=k$ (2.11) is verified because $a(n,k,k)=1$. Due to (2.9), (2.11) is verified also for $j=k-1$. Suppose now that (2.11) is true for $j=i+1$, and let's prove it for $j=i$. From (2.8) we infer

$$\begin{aligned} a(n,k,i) &= \left\{(i+(i+1)+\dots+(k-1))n^{k-i-1} + \text{ terms of lower degree}\right\}^{-1} \\ &\quad \times s(i+1,i)\left(n^{k-i-1} + \text{ terms of lower degree}\right)a(n,k,i+1), \end{aligned}$$

so that, by the induction hypothesis,

$$\begin{aligned}
a^*(k,i) &= \frac{s(i+1,i)}{i+(i+1)+\cdots+(k-1)}a^*(k,i+1)\\
&= -\frac{i(i+1)}{(k-i)(k+i-1)}\prod_{l=1}^{k-i-1}\frac{(k+1-l)(k-l)}{l(l-2k+1)}\\
&= \prod_{l=1}^{k-i}\frac{(k+1-l)(k-l)}{l(l-2k+1)},
\end{aligned}$$

and this completes the proof of (2.11).

It follows that

$$\lim_{n\to\infty} q_k^{(n)}(x) = \sum_{j=0}^{k} a^*(k,j)x^j,$$

and the coefficients $a^*(k,j)$ are equal to the coefficients $c^*(j,k)$ from [4, Theorem 4.1]. This leads to

$$\lim_{n\to\infty} q_k^{(n)}(x) = \lim_{n\to\infty} p_k^{(n)}(x) =: p_k^*(x),\ k\geq 0, \tag{2.12}$$

where (see [4, Theorem 4.5]) $p_0^*(x)=1$, $p_1^*(x)=x-\frac{1}{2}$, and

$$p_k^*(x) = \frac{k!(k-2)!}{(2k-2)!}x(x-1)P_{k-2}^{(1,1)}(2x-1),\ k\geq 2. \tag{2.13}$$

($P_m^{(1,1)}$ are the Jacobi polynomials, orthogonal with respect to the weight $(1-t)(1+t)$ on the interval $[-1,1]$.)

Summarizing, we have proved the following

Theorem 2.1. (i) *The eigenvalues of* $\bar{\mathbb{B}}_n:\Pi\to\Pi$ *are the numbers given by* (2.2)*.*

(ii) *The corresponding monic eigenpolynomials are described by* (2.5)*, where the coefficients* $a(n,k,j)$ *satisfy the recurrence relation* (2.8)*.*

(iii) *The eigenpolynomials satisfy the asymptotic relation* (2.12)*.*

So the eigenstructure of the genuine Beta operators is similar to that of the classical Bernstein operators.

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Book reviews

Ravi P. Agarwal, Erdal Karapinar, Donal O'Regan and Antonio Francisco Roldán-López-de-Hierro; Fixed Point Theory in Metric Type Spaces, Cham: Springer, 2015, xvii+385 p. ISBN 978-3-319-24080-0/hbk; 978-3-319-24082-4/ebook).

The book is devoted to fixed points in generalized metric (G-metric) spaces. A G-metric on a set X is a function $G : X^3 \to [0,\infty)$ satisfying the following conditions:

(G1) $G(x,y,z) = 0 \iff x = y = z$;

(G2) $G(x,x,y) > 0)$ for all $x, y \in X$ with $x \neq y$;

(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $x \neq y$;

(G4) $G(x,y,z) = G(\pi(x,y,z))$ for every pemutation π of x,y,z;

(G5) $G(x,y,z) \leq G(x,w,w) + G(w,y,z)$ for all $x,y,z,w \in X$.

A typical example of a G-metric is the perimeter of a triangle in $\mathbb{R}^n$.

Generalized metric spaces were introduced by Dhage in the nineteens of the last century, but his papers contained some flaws (mainly concerning the topological properties of these spaces), which were fixed by Z. Mustafa and B. Sims in J. Nonlinear Convex Anal. (2006).

As the authors consider fixed point results for mappings T on G-metric spaces satisfying a condition of the type

$$G(Tx,Ty,Tz) \leq G(x,y,y) - \phi(G(,x,y,y)),$$

where $\phi : [0,\infty) \to [0,\infty)$ is called a control function, the second chapter, *Preliminaries*, contains a detailed study of these control functions.

One considers 18 properties that could be satisfied by a control function (like being monotone, subadditive, semi-continuous, or satisfying for all $t > 0$, $\phi(t) < t$, $\lim_n \phi^n(t) = 0$, or the convergence of the series $\sum_n \phi^n(t)$, etc). Combining some of these properties (usually three of them) one obtains various control functions considered in the fixed point theory of mapping on metric spaces as Matkowski comparison function, Geraghty, Boyd-Wong, Ćirić, Krasnoselski functions, etc. This chapter contains a detailed study of the relations between these properties and between the classes of functions they define.

The basic properties of G-metric spaces are studied in Ch. 3, *G-Metric Spaces*, while the rest of the book is devoted to various fixed point results in this class of spaces: Ch. 4, *Basic Fixed Point Results in the Setting of G-Metric Spaces*, Ch. 5 *Fixed Point Theorems in Partially Ordered G-Metric Spaces*, Ch. 6, *Further Fixed Point Results on*

G-Metric Spaces, Ch. 7, *Fixed Point Theorems via Admissible Mappings*, Ch. 8, *New Approaches to Fixed Point Results on G-Metric Spaces,* Ch. 9, *Expansive Mappings*, Ch. 10, *Reconstruction of G-Metrics: G^*-Metrics*, Ch. 11, *Multidimensional Fixed Point Theorems on G-Metric Spaces*, Ch. 12, *Recent Motivating Fixed Point Theory.*

Some supplementary material is collected in an Appendix, *Some Basic Definitions and Results in Metric Spaces.*

The book, including many contributions of its authors, provides an accessible and up-to-date source of information for researchers in fixed point theory in metric spaces and in various of their generalizations, for mappings satisfying some very general conditions.

S. Cobzaş

Afif Ben Amar and Donal O'Regan; Topological Fixed Point Theory for Singlevalued and Multivalued Mappings and Applications, Cham: Springer, 2016, , x+194 p.
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The present book on fixed points focusses on applications to integral equations and to nonlinear eigenvalue problems. Since the main context is functional analytic, the authors devoted the first chapter of the book, *Basic concepts*, to the presentation of some basic notions and results in functional analysis – normed spaces, ordered vector spaces and ordered normed spaces, locally convex spaces, weak topologies (weak compactness, Dunford-Pettis property), compact and weakly compact operators. The chapter ends with some fixed point theorems – Krasnoselskii's, Leray-Scahuder theory, and fixed points for multivalued maps. Although some proofs are included, the majority of the results are presented without proofs.

The second chapter is dedicated to nonlinear eigenvalue problems in Banach spaces satisfying the Dunford-Pettis property. The third chapter is concerned with Leray-Schauder type theorems for mappings which are condensing with respect to De Blasi measure of weak noncompactness. This study is continued in the fourth chapter for mappings with sequentially closed graph. Applications are given to Volterra-type integral equations under Henstock-Kurzweil-Pettis integrability and to integral equations in Lebesgue spaces.

The fifth chapter is devoted to fixed points for applications of the form $AxBx + Cx$, $x \in X$, defined on a Banach algebras X, under appropriate conditions on the operators A, B, C and on the Banach algebra X. Applications are given to some nonlinear functional integral equations.

The sixth chapter is concerned with a class of operators $F : D \to X$, X a Banach space and $D \subset X$, introduced by Gowda and Isac in 1993, called by them (ws)-compact and meaning that F is $\|\cdot\|$-continuous and $(F(x_n))$ contains a $\|\cdot\|$-convergent sequence for every weakly convergent sequence (x_n) in D.

The last chapter of the book (Ch. 7) presents some results on approximate fixed point sequences (i.e. sequences satisfying $x_n - Fx_n \to 0$) for multivalued mappings with applications to Nash equilibrium in noncooperative games.

The book, including original contributions of the authors, is addressed to researchers interested in applications of fixed point results (in functional analytic context) to integral equations, ordinary and partial differential equations, game theory, etc. The detailed exposition of the subject and the prerequisites make it appropriate for graduate courses in linear and in nonlinear functional analysis.

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